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Out-of-equilibrium economic dynamics and persistent polarisation

by

James Porter

Thesis

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for the degree of

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Declarations

I declare that any material contained in this thesis has not been submitted for a degree to any other university.

A paper, co-authored with Prof. Sayantan Ghosal, entitled: "Decentralized Exchange, Out-of-Equilibrium Dynamics, and Convergence to Efficiency", drawn from chapters 3 and the earlier parts of 4 of this thesis has been accepted for publication by Mathematical Social Sciences.

Some of the material from chapter 2 was published by the UK Collaborative on Development Sciences[1] entitled "Complexity Science and International Development".

James Porter

2012

[1]The final published verison can be found via `http://www.ukcds.org.uk/`; the full url is not included as it doesn't fit in this footnote.

Abstract

Most of economics is equilibrium economics of one sort or another. The study of out-of-equilibrium economics has largely been neglected. This thesis, engaging with ideas and techniques from complexity science, develops frameworks and tools for out-of-equilibrium modelling. We initially focus our attention on models of exchange before examining methods of agent-based modelling. Finally we look at a set of models for social dynamics with non-trivial micro-macro interrelationships.

Chapter 2 introduces complexity science and relevant economic concepts. In particular we examine the idea of complex adaptive systems, the application of complexity to economics, some key ideas from microeconomics, agent-based modelling and models of segregation and/or polarisation.

Chapter 3 develops an out-of-equilibrium, fully decentralised model of bilateral exchange. Initially we study the limiting properties of our out-of-equilibrium dynamic, characterising the conditions required for convergence to pairwise and Pareto optimal allocation sets. We illustrate problems that can arise for a rigid version of the model and show how even a small amount of experimentation can overcome these. We investigate the model numerically characterising the speed of convergence and changes in ex post wealth.

In chapter 4 we now explicitly model the trading structure on a network. We derive analytical results for this general network case. We investigate the effect of network structure on outcomes numerically and contrast the results with the fully connected case of chapter 3. We look at extensions of the model including a version with an endogenous network structure and a versions where agents can learn to accept a 'worthless' but widely available good in exchanges.

Chapter 5 outlines and demonstrates a new approach to agent-based modelling which draws on a number techniques from contemporary software engineering. We develop a prototype framework to illustrate how the ideas might be applied in practice in order to address methodological gaps in many current approaches. We develop example agent-based models and contrast the approach with existing agent-based modelling approaches and the kind of purpose built models which were used for the numerical results in chapters 3 and 4.

Chapter 6 develops a new set of models for thinking about a wide range of social dynamics issues including human capital acquisition and migration. We analyse the models initially from a Nash equilibrium perspective. Both continuum and finite versions of the model are developed and related. Using the criterion of stochastic stability we think about the long run behaviour of a version of the model. We introduce agent heterogeneity into the model. We conclude with a fully dynamic version of the model (using techniques from chapter 5) which looks at endogenous segregation.

Chapter 1

Introduction

Most of economic theory is equilibrium economics of one sort or another. An economic equilibrium is not necessarily, as might be assumed from a mathematics or physics viewpoint, a stationary point of a dynamical system (or even a stationary distribution) but is (roughly) a state of affairs where people know what to expect, where demand and supply meet, where prices are stable and so on. But even after we make such allowances we are still left with the conclusion that this is clearly not the only or perhaps even the typical state of affairs in a "real" economy.

This thesis takes a number of steps towards thinking about economics in a more dynamic way. I look at decentralised versions of economic models of exchange, which do not require many of the typical strong assumptions of standard models, particularly with respect to information. I identify several key ideas which I argue should be included within agent-based modelling frameworks and create a prototype framework which demonstrates the feasibility and effectiveness of these ideas. Finally I create a set of economic models for thinking about certain kinds of complex social dynamic issues.

In the next chapter I introduce complexity science in a very broad sense. I look at the key idea of a complex adaptive system and some of the key work in applying this idea to economics (I return to several of the ideas in more detail later on). Next, I look at more work which connects complexity to economics from economists and from other researchers such as physicists, including for example Axtell [2005]; Kirman [2010].

I then take a different angle, examining a few key pieces of work related to ideas of out-of-equilibrium economics. These include some of the work of Fisher [1983] in his attempt to formulate an out-of-equilibrium framework, rational expectations and rationalisability via Guesnerie [1992] and some contemporary accounts of microeconomics including Bowles [2006] which formulates a general approach to microeconomics which is more in keeping with dynamical approaches. Focusing on these three strands of work is obviously not an attempt at a comprehensive survey, which would be impossibly broad, but hopefully gives the reader some sense of the context in which this PhD is situated.

Following these introductory accounts I examine key topics in microeconomics which are either directly utilised by this thesis or which are alternative methods to, or context for, the work contained here. Included here is an overview of bargaining, fictitious play and adaptive heuristics. I examine in detail a foundational strategic model of general equilibrium Gale [2000] for which this thesis in part provides an alternative. The final topic in this microeconomic section is the idea of conventions and related solution concept of stochastic stability, a method I will use in chapter 6.

Continuing on from these microeconomic concepts I survey work in agent-based modelling in economics and other contexts (though the most directly relevant background for this topic can be found in chapter 5). I then bring together the strands of economic theory and complexity and/or agent-based modelling, looking at Gintis [2007a] and other works.

The final part of the literature review surveys conceptual issues, particularly in microeconomics, of relevance to social dynamics, in particular to models of culture, aspirations, polarisation and decision making. I pick up on many of these themes in chapter 6.

In chapter 3 I look at a dynamic model of pairwise exchange which jettisons many of the typical assumptions such as global knowledge, foresight and agent homogeneity. I consider a model with agents, an arbitrary number of goods, heterogeneous utility functions and random matching of pairs for exchange. One agent makes an offer. This offer can then either be accepted, if it is improving, or rejected, if it is not. I examine this initially analytically, showing that under certain conditions we will asymptotically obtain an outcome which is Pareto efficient. I allow for a weakening of the initial formulation, in particular for experimentation (or noise) and show that we can still asymptotically obtain the same outcomes, subject to restrictions on the form of experimentation or noise. The model is extended to production using the process of Rader [1976].

I then examine the model numerically using standard models of preferences (the focus is mostly on Cobb-Douglas preferences, though more awkward cases are considered). The numerical results obtained show that convergence in this kind of model is surprisingly rapid given the lack of knowledge on the part of the agents.

I also look numerically at an ex post measure of wealth change which shows an approximately linear relationship between initial and final wealth, but with considerable noise. I show, numerically, how noise can both improve or dis-improve the speed of convergence and how for certain cases it makes convergence possible at all.

In chapter 4 I extend the model from chapter 3 to an explicit market structure. Each agent is connected to some subset of the entire economy and we assume that there is some path from each agent to every other agent. Versions of the analytical results presented in the previous chapter are derived which take into account the network structure (essentially the previous chapter assumed the case of a fully connected network).

Numerically I show that for random networks, beyond a certain (surprisingly low) threshold level of connectivity the results are very similar to those for the fully connected network of chapter 3. When I re-examine wealth changes I show that the wealth changes may be highly dependent on network structure and that even for a simple model where more central agents are making no explicit attempt to exploit their position, they may end up, probabilistically, with a disproportionately high wealth level.

This framework is extended in two ways. Firstly, I consider how making the network formation endogenous (via selection of neighbours based on past performance) changes the network structure. Secondly, I consider a version of the model with a 'worthless' good which all agents have access to; this can play the part of money as agents learn to value this and use it to trade where they may not otherwise have a mutual co-incidence of wants.

Chapter 5 surveys some of the major general approaches to agent-based modelling which are popular in the socio-economic agent-based modelling community. I look at some of the most popular general purpose frameworks such as SWARM, Repast and Netlogo and split approaches into roughly three groups: those which provide a full agent-based modelling language; those which provide libraries, extensions or frameworks for existing general purpose languages; and finally visual tools. I argue that visual tools, while being ideal for many particular tasks, haven't really established themselves for more general purpose tasks. Setting those aside leaves two major approaches.

I develop an alternative to these two major branches which draws on the idea of an internal domain specific language from Fowler [2005], that is customising and building a framework for a general purpose language for a specific task, in this case agent-based modelling. I apply this idea of an internal domain specific language to a multi-levelled conception of agent-based modelling which provides a kind of scaffolding for four level agent-based models including an ‘agent level’ at which individual classes of agents are specified, a ‘world level’ in which agent instances exists and carry out actions in time periods, a ‘simulation level’ at which runs of the world can be specified and finally an ‘experiment’ level at which we specify the investigation we want to carry out (parameters to be varied or simulations to be compared).

I also incorporate a number of ideas from contemporary software engineering practice which, while can be (and to some extent are) utilised in agent-based modelling, do not seem to have established themselves. In particular I focus on automated testing and version control. While these could be utilised by anyone developing agent-based models in general purpose languages there seems to little existing emphasis or expectation of doing so. The framework which I developed for this chapter both facilitates version control and includes tests for its basic functionality (while allowing users to easily add their own).

Projects such as Netlogo have some support for the idea of agent-based modelling within an explicitly broader context of a ’project’. I extend the scope of this beyond documentation and model to include output generation: the framework I implement generates a number of outputs such as LaTeX files, HTML files (which can be imported into Word, Pages and so on) visualisations, tables and a simple presentation ready form. This can greatly accelerate work by automatically generating at least an early version of the final outputs.

The framework is illustrated through examples including a simple example model, a model of an App Store and through a recreation of a model from earlier in the thesis within the framework. I evaluate the framework considering issues such as performance and ease of use. I conclude by outlining the lessons learned for agent-based modelling and the next steps that could be taken to build a full framework based on this prototype (it is currently capable of substantial real work but probably requires too much technical expertise for general use).

In chapter 6 I explore a set of strategic models of social dynamics which take into account both the complementarity and competitive effects of agents action on one another. I consider an economy in which agents make a choice of whether to put in effort, with some cost, or not. If they put in effort then there is a probability of success which depends on how many other agents put in effort. I analyse purely competitive and complementary version of this model (where increasing the proportion of agents who put in effort strictly decreases or increases the probability of success) along with a benchmark case which includes elements of both competition and cooperation.

I look at the finite version of the model and characterise the long run behaviour of the economy with a simple dynamic using the criteria of stochastic stability. I extend the model to a case with heterogeneous agents, that is each agent has a skill level (or cost of human capital acquisition) which determines their individual expected utility upon effort. Finally, I explore the issue of persistent endogenous segregation in human capital acquisition via an agent-based model based on this framework.

Chapter 7 brings together the conclusions of preceding chapters and summarises some possibilities for future work building on this thesis. Hopefully this has given you a good idea of the contents of this thesis. Most of the major chapters, with the exception of chapter 4 which follows on directly from 3, are relatively self-contained.

Chapter 2

Background and Literature Review

This chapter gives a broad introduction to complexity science before focusing on complex adaptive systems, and in particular some research looking at economies as complex adaptive systems. We then look in more detail at economically focused research from complexity science and research from economists and physicists which link complexity to economics.

We then look at a highly selective historical account of some economic topics of relevance to the thesis looking at three key areas: early attempts at out-of-equilibrium economics, the alternative approach of rational expectations and rationalisability and finally the general subject of microeconomic research. In each case the focus is on a key piece of work. This should give a flavour of the research that has been undertaken for those not familiar with the fields.

The focus then turns to surveying microeconomic approaches and models of relevance to the thesis, such as bargaining and fictitious play. We then look at some agent-based modelling work, before concluding with a survey of work relating economics and complexity science. Later chapters give additional background as appropriate.

2.1 Complexity Science

2.1.1 Introduction

Complexity Science is an imprecise term, covering an array of disciplines, methodologies and ideas. Defining it along the lines of the scientific study of complex systems still leaves one with an impossibly wide range of topics to cover. One can characterise a complex systems as one featuring some or all of the following properties: interconnected components, nonlinearity, self-organisation/self-organised criticality, adaptive agents, evolution and emergent properties.

A system is complex if it has many interacting components; so one cannot understand it as merely the sum of its parts and study each part in isolation one must pay attention to the way in which different components interact. Furthermore these interactions may be nonlinear: given inputs to a system the output may not simply be the sum of some linear scaling of inputs, but some more sensitive process based on the input. But while this process may be more sensitive to input it may also be more robust, self-organising to classes or states; that is states with particular overall qualitative properties, without fine tuning of inputs to the process or initial conditions. But it may self-organise to a critical state; one where a small change in a particular input may have dramatic outcomes. Clear examples of this kind of phenomenon are avalanches or earthquakes, in both these cases the system slowly changes over time, but the overall state is apparently stable until a catastrophic event occurs, as the result of a small change in input.

There may be adaptive agents whose behaviour changes depending on their local environment and in doing so may regulate the overall behaviour of a system. This system may evolve over time, perhaps with co-evolution of individual components. And these various behaviours may result in the emergence of an outcome at a higher level that was not intended or explicitly aimed for at a lower level.

In general for systems with the above properties exact mathematical solutions are impossible to obtain, so one must utilise other techniques. Complexity science draws on and combines a range of methodologies from disparate disciplines including physics, ecology, mathematics, computer science, economics and others. These include nonlinear dynamics, stochastic processes, statistical physics, agent-based modelling, evolutionary game theory, machine learning/statistical inference and decision theory.

In calling a piece of work complexity science, in contrast to just pursuing say evolutionary game theory within a biological context, one is typically doing two things:

1. Emphasising the interdisciplinary aspects of the work. That is to say, applying work from one field to a seemingly unrelated field.

2. Emphasising the complex nature of the problem one is facing as work in each of the above fields could be split, albeit imprecisely, into complex and non-complex.

There are also a wide variety of general introductions to complexity science for a general audience, such as Strogatz [2003] and for a more technical audience Bar-Yam [1997]; Erdi [2007]; Miller and Page [2007]. The ideas from complexity science have been applied to many disparate fields, one example is the relationship to international development for which the Overseas Development Institute's recent working paper Ramalingam et al. [2008] is an excellent starting point.

2.1.2 Complex Adaptive Systems

Complex adaptive systems are those with large numbers of agents which interact and adapt or learn. To be considered a complex adaptive system (as opposed to a 'mere' multi-agent system) features such as emergence or self-organisation may be observed. This kind of complex system is of particular interest for economic considerations where we may be particularly interested in the macroeconomic behaviour of a system, where we have a model of microeconomic components (agents, firms and so on).

A key early researcher in applying ideas from complex systems to economics is Peter Albin. In Albin [1998] several major pieces of this research are brought together along with an extended introduction Foley [1998]. This introduction does a good job of synthesising much of this research summarising ideas including linear dynamical systems, oscillators, nonlinear dynamical systems and chaos. Cellular automata are identified as a good laboratory for thinking about nonlinear dynamical systems questions as they can show single state stability, oscillatory stability, chaos and complex evolutions[1].

Of particular interest for this thesis is chapter 5, originally published as Albin and Foley [1992], which looks at a model of dispersed exchange (100 agents on a lattice with two goods), symmetric Cobb-Douglas utility functions (each agent values each good equally) for final holdings. The sum of endowments to agents is equal to 100 and the total endowment over both goods is 10000. This formulation has a straightforward equilibrium solution, where prices are set equally and each agent would consume an equal amount of both goods. Into this basic framework an advertising cost (broadcast of willingness to trade) is introduced such that agents must both advertise to be able to trade at all. The key feature considered is agents forming an estimation of the Walrasian equilibrium price from their local data (the neighbourhood size is a parameter). After a few rounds this can result in highly

[1]This last category, or class 4 cellular automata, are those where nearly all initial patterns eventually evolve into interesting (local) structures which interact in complex ways and may be remain in some form for many periods.

efficient outcomes, without central coordination. Below some related models are considered, particularly in section 2.7.4 and in chapter 3 we investigate a considerably less homogeneous decentralised system.

In Baumol [2004] the idea of a "Red-Queen game" is described: a competitive scenario where each player must match or exceed the efforts of all other rivals, so each must bid higher and higher. In this kind of scenario moving ahead of rivals simply induces them to put in more effort, meaning everyone is perpetually short of resources. Several historical examples are cited, where the necessity to fund armies led to kings turning to "despised economic activities such as commerce". The final example cited is of innovation arms races in high technology: companies must continually innovate to stay ahead of rivals, which means that the market mechanism has provided incentives which have increased growth in resources directed towards research.

A relatively recent account of markets considered as complex adaptive systems (CAS) can be found in Markose [2005][2] which identifies three main aspects of markets as CAS:

1. formal complexity issues of equilibria,
2. the dynamic behind growth of adaption and
3. the empirical or testable elements of CAS in economic systems.

A detailed historical account is given of the development of ideas relating to CAS which identifies three stages in the development of the theoretical ideas. These ideas are associated with methods which have been applied to economic modelling with varying consequences. The first stage is formalist/deductive methods, method include classical optimisation and probability and classical economic models. The consequences for economic modelling include perfect competition, homogeneous rational expectations and complete markets. The second stage results from mathematical ideas related to incompleteness and NP-hard problems and the associated computability methods. The economic consequences include ideas such as Hayek's limits on constructivist reasoning and various un-solvability and incompleteness results. The third stage is that including inductive methods and self organising dynamics with associated methods of adaptive computing, such as genetic algorithms, cellular automata and agent-based models. The consequential economic models include ideas such as power laws, scaling, market crashes, evolution of norms and so

[2]This is part of a special issue, we look at two other papers from it, Axtell [2005] and Durlauf [2005], in more detail below.

on. We return to many of these and related issues, in particular see Axtell [2005] and Durlauf [2005] in section 2.2.3 and an examination of the theme of evolution in section 2.9.

2.2 Economics and Complexity (Science)

2.2.1 Complex Economics

Alan Kirman is a prominent critic of much of mainstream theoretical economics, see for example Kirman [2009a]; Colander et al. [2009]; Kirman [2009b]. In his recent book on Complex Economics Kirman [2010] looks at his alternative to the main paradigms of contemporary economic theory. He questions the key mainstream economic ideas of rationality and equilibrium by pointing out their lack of realism. He highlights the stability problems with existing equilibrium theories. His alternative is a simpler, but more heterogeneous world, with emergent coordination. The structure of markets should be a key concern and macroeconomics demands new theory. His work is particularly interesting with respect to this thesis as it engages both with economics (while a critic of much of economics he is very much an economist) and complexity.

He argues for the centrality of networks for economics (we look in more detail at networks in the next section of this chapter) and observes that the relative lack of economic interest until recently in networks is odd when compared to other disciplines, such as sociology where networks have long been recognised as important and the typical economic assumptions of "ultra rational" behaviour have been rejected. Network interactions can lie between the two extremes of independent agents in markets and game theoretic interactions.

Kirman looks at Fish markets as they provide an example where aggregation of behaviour may lead to rationality at a collective level. First looks at a simple theoretical model, then uses simulations to extend it. Detailed data is available for Marseille[3], a market where sellers name a price, buyers can reject or not (more or less). Kirman argues that what agents do depends on who they are in contact with; gradually a trading network will have built up from experience of who will sell or buy certain kinds of fish at certain prices. There is significant dispersion of prices suggesting this market is not a good approximation of a competitive one; also typical equilibrium concepts with price dispersion don't work well (as the price dispersion

[3] *1.* The name of the buyer *2.* The name of the seller *3.* The type of fish *4.* The weight of the lot *5.* The price per kilo at which it was sold *6.* The order of the transaction in the daily sales of the seller. The total number of transactions for which Kirman has data is 237,162.

isn't fixed over time).

The traditional notion of competitive demand, the amount an individual would buy if were only constrained by income, is questioned in the context of the fish market and implicitly more broadly. A traditional model is proposed, then one with less rationality. Using mean field analysis Kirman suggests we are likely to see behaviour splitting into two categories: buying from random sellers and repeated buying from the same sellers, with little in behaviour in between. Simulations for larger models give the same outcome. This conclusion is supported by the empirical data. Another fish market (Ancona) which is based on an auction system is analysed. Again the suggested result is that agents don't behave rationally at an individual level; however thanks to market structure aggregate outcome is rational.

In looking at financial markets Kirman covers the various forms of the efficient markets hypothesis (EMH) along with their testable implications. Models which assume EMH must then assume that any volatile behaviour such as a crash is the result of exogenous factors; it cannot be explained within the model. The problem is that often there is no clear evidence for significant exogenous events driving volatile behaviour. And there is much historical evidence for endogenously generate bubbles. Of course most fundamentally a crash may be the very thing we wish to model.

The concept of herding is introduced with a simple example showing when players act sequentially the outcome may be worse when *more* information is available to them. Another type of herding may be via a recruiting process where popular resource use is reinforced via use by others.

If expectations are heterogeneous, as Kirman argues for, then should move away from rational expectation framework. May be able to think in terms of an *equilibrium distribution* of a market[4]. While these the kinds of approaches used in this chapter (simple rules which produce time series with plausible characteristics) are suggestive, it would be interesting to see if the results still hold in a more realistic (perhaps sophisticated agent-based?) framework.

Finally Kirman presents a physical model related to Shelling's segregation model. By looking at a continuous variant of the model can characterise the macro results of a wide variety of micro behaviours. Simulations show interesting results for the case where equal numbers of neighbours of each type are most desired: macro segregation, but with constant micro movement due to dissatisfaction.

In Colander et al. [2004] Colander, Holt and Rosser give their view of the

[4] Subject to formulating the market in a particular way it can be proven that the distribution of prices will be ergodic and hence has a unique limit distribution.

"changing face of mainstream economics" arguing that it has moved away from the holy trinity (rationality, selfishness and equilibrium) to purposeful behaviour, enlightened self-interest and sustainability. They view economics as a 'complex system' for which categories like *orthodox* are backward looking and of limited usefulness. They highlight the role played by what they call the edge of economics: "the edge of economics is that part of mainstream economics that is critical of orthodoxy, and that part of heterodox economics that is taken seriously by the elite of the profession. Our argument is that modern mainstream economics is open to new approaches, as long as they are done with a careful understanding of the strengths of the recent orthodox approach and with a modelling methodology acceptable to the mainstream."

They discuss the time lag in the latest research becoming part of (under)graduate syllabus and how this can be useful (they give the example of the reluctance to incorporate macroeconomic dynamic stochastic general equilibrium models). They highlight several areas as examples of the edge of economics, to quote from Colander et al. [2004]:

1. "Evolutionary game theory is redefining how institutions are integrated into the analysis."

2. "Ecological economics is redefining how nature and the economy are viewed as interrelating."

3. "Psychological economics is redefining how rationality is treated."

4. "Econometric work dealing with the limitations of classical statistics is redefining how economists think of empirical proof."

5. "Complexity theory is offering a way of redefining how we conceive of general equilibrium."

6. "Computer simulations are offering a way of redefining models and how they are used."

7. "Experimental economics is changing the way economists think about empirical work."

2.2.2 Economics and Physics

Econophysics is the application of methods from physics to the study of financial markets and economies. It has mostly focused on areas like stock markets for which large amounts of data are available, though some recent work has been of a more theoretical nature. Rosser Jr. [a, 2007, b] offer a physicist's take on dynamic conceptions of complexity and how they could be applied to economics. A contrasting perspective can be found in Gallegati et al. [2006], an assessment of recent work in econophysics is given by several economists. They feel that there are problems with: a lack of awareness of existing work in economics, resistance to more rigorous and robust statistical methodologies: visual methods may be misleading e.g. power law fits; limited range for scaling effects[5], the belief that that universal regularities can be found in many areas (in economics relationships may change over time) and the particular theoretical models used to explain empirical phenomena which may for example restrict study to exchange only models due to conserved quantities. Also thinks that many models not realistic and suggests that econophysics needs to go beyond exchange only models.

A response by McCauley McCauley [2006] agrees with much of the criticism in Gallegati et al. [2006] including the criticism of conservation laws, but points out that the neo-classical economic model has a global conservation law assumption and that many econophysics researchers do not make these kinds of assumptions. McCauley is critical of making use of the economics literature and combatively suggests it *"should be either abandoned or tested empirically ... never accepted as a basis for modelling"*. He concludes with a strong critique of the content of economic courses, suggesting alternatives for the future.

In Yegorov [2007] Yegerov looks at econophysics (or "socio-physics") in a general sense, attempting to approach the subject from an "economist friendly" perspective. He keeps many economic assumptions while introducing a physics-based methodology, optimistically speculating about a unified science. In terms of future research the following are identified: analytical descriptions of transaction processes, understanding historical paths, understanding stability of regimes and role of state in economy. Economics, it is argued, should broaden to consider other forces alongside the "invisible hand", such as spatial forces, scale economies and deviations from self-regarding preferences (see section 2.5 for a more substantial account of similar ideas).

2.2.3 Further Economics and Complexity

In section 2.4 we look at a model where the rational expectations equilibrium is a key concept. But first we note earlier research which asks when such equilibria are

[5] Many consider this a major problem in much 'complexity' work.

computable[6]. In Spear [1989] two results are reached:

- In a two stage learning model with full information on the state of the world it is possible for agents to learn this equilibrium (that is, it is computable).
- When agents have incomplete information this is no longer the case.

Even if a solution is computable, the difficulty (or computational complexity) in computing it may be prohibitive. Axtell [2005] offers an attractive formulation of a decentralised economy from the perspective of computational complexity, dropping much of the typical Walrasian framework, to produce a model which can account for the endogenous emergence of prices. It links economic exchange with computational complexity, and is draw on agent-based modelling. In complexity terms the job of the Walrasian auctioneer (or whatever alternative mechanism is in place) is extremely hard. One can argue that it is so difficult to compute Walrasian picture is not reasonable and that it is not (empirically) reasonable anyway. Axtell proposes a model of decentralised $k-$lateral exchange and shows that it converges geometrically in total utility. Under conditions (see Goldman and Starr [1982]) it will converge to Pareto optimal allocations. Some numerical examples are given of heterogeneous Cobb-Douglas agents.

Foley [1999] introduces the idea of statistical equilibrium, instead of a traditional equilibrium balancing supply and demand, here we have flows in a market with an equilibrium distribution of prices subject to classes of traders who are replaced having made their target trades.

In Crockett et al. [2008] a model of a pure exchange economy is developed where agents use the common normalised global utility gradient as a vector of prices to obtain an implied wealth distribution.

A 'complexity' approach to macroeconomics is adopted in the book Gatti and et al. [2006]. First it analyses many of the stylised facts of macroeconomies. Then, in contrast with more traditional work, presents an agent based model (chapter four). The overarching conceptualisation is that as in statistical physics we may obtain universal laws which do not depend on regularity at the mirco-level. The authors argue that economists must adopt an approach based on interacting heterogeneous agents. One particular model is that of a sequential economy, with many firms and banks which is able to capture many "empirical" properties of genuine economies.

A model of local interaction with choices of ranked actions, with infrequent upward shifts and rapid imitation (or diffusion) is investigated in Arenas et al. [2002].

[6] Whether there exists an algorithm which will run in finite time and produce the result.

If the costs of not coordinating are high then the system can behave critically[7]. The system is optimal at the frontier of the critical region[8]. The model has n agents on a ring, at every time step each agent i adopts an action $a_i(t) \in \mathbb{R}_+$ or technology level. Agents interact with neighbours and obtain payoffs

$$\psi(a, a') = f(a) - g(a, a')$$

where $g(a, a') = 0$ when $a = a'$ (so highest payoff when equal). The payoff will be lower if one is less technologically developed rather more for the same difference in development. The economy updates via diffusion, a best response to the actions of neighbours, and an occasional exogenous perturbation or 'innovation' which shift technology level up by some random amount.

This model is simulated numerically. Depending on a parameter level we may see avalanche sizes following a power law or where they are all of the system size. Similar results are obtained for the advances (technological increases). Instead of the synchronised situations expected by evolutionary game theory we have heterogenous waves. One can also look at the long range behaviour and in particular at the performance which seems to be optimal at the 'lower edge' of the critical region. One key question the paper leaves us with is what networks might be conducive to critical behaviour and when might such networks be formed endogenously.

Much research connecting complexity and economic is theoretical. In contrast Durlauf [2005] explicitly connects complexity to empirical economics. Durlauf identifies four properties of complexity of particular relevance to social science contexts:

1. Nonergodicity
2. Phase transition
3. Emergent properties
4. Universality

An example which has been proposed of the first property and third property is the adoption of the QWERTY keyboard. A more efficient alternative was available (DVORAK), that is the emergence, out of decentralised decisions, of QWERTY was only one of many possible long term outcomes. Durlauf is somewhat skeptical (he doesn't think these kind of historical examples are persuasive evidence in support of

[7]

[8] This is explicitly linked to the notion of "edge of chaos".

a complex systems perspective). Various power laws have been identified in econometric data. These stylised facts are helpful for theoretically modelling. Durlauf questions whether these observations can allow the inference that a given economic situation is complex as there are interpretative problems. Durlauf concludes with suggestion that economic complexity will not become a major part of economic reasoning until theoretical and empirical work is better connected.

2.3 Disequilibrium Foundations of General Equilibrium

In Fisher [1983] we find an early exploration of out-of-equilibrium dynamics. Fisher questions assumption of stability, or quick enough convergence to equilibrium to make disequilibrium study irrelevant. He critiques Lucas' sense of stability as always clearing markets, "essentially a tâtonnement process", as begging the question. Having omniscient agents who can calculate general equilibrium prices is absurd[9]. He argues that stability is also an important issue for macroeconomics.

Perhaps most relevantly he asks the key question: what should a satisfactory dynamic theory have? His answer: dynamics and adjustment to equilibrium over time. While a closed form solution would be ideal it is too much to hope for. We must ground assumptions of functional forms of agents in theories of behaviour. Convergence speeds should be determined endogenously and not simply assumed to be high. Adjustment processes must arise from optimising behaviour of agents; not via a direct computational algorithm for equilibria.

Fisher highlights four major developments of the stability literature: the realisation that subject worth looking at, the realisation that global results could be obtained, the introduction of non-tâtonnement process and the realisation that specifying disequilibrium processes leads to more satisfactory results. He covers early results such as Samuelson's formulation of a non-equilibrium system and a formalisation of Walras' Tâtonnement process which is now considered anachronistic and which required very strong assumptions. He observes that there is a "present action postulate" implicit in all stability literature that does not allow for development over time.

This and related early work was superseded by global results such as Arrow and Hurwicz [1958]. These results use Lyapounov's second method; roughly we show the existence of a fixed point, we show it is globally stable and show adjustment process is quasi-stable (there is a limit point for all initial conditions). The method

[9] See also section 2.2 where we cover Axtell's work which looks at the computational complexity of economic coordination.

used to do this is to find a Lyapounov function $L(P)$ which is continuous, bounded below and strictly decreasing except at fixed points. In this work the aggregate excess demand functions must satisfy weak axiom of revealed preference[10]. They conjectured general global stability, but a counterexample was soon found, see Scarf [1959].

Fisher continues his study in Fisher [1983], looking at Disequilibrium with Arbitrage, some refinements to his model and further issues concerning equilibrium and stability; though doesn't really succeed in his original, ambitious objectives.

Later in section 2.6.4 we look at a more recent attempt to build a general equilibrium model on strategic foundations, but for now we jump forwards by about ten years as we turn to a key piece of work on rational expectations.

2.4 Rational Expectations

An alternative to a dynamic model of coordination is to use a rational expectations model. Here we think about forecasting activity and the concept of eductive learning stability based on the game-theoretic conception of rationalizability. We look at Guesnerie [1992] as a clear example of this work. It shows that in a particular model the conditions for coordination of beliefs are robust to noise and that stability increases with differentiation of products and when decisions are sequential and observable.

Guesnerie introduces an adaptation of a model from Muth with the following basic features: a continuum of producers (farmers), indexed by $f \in [0, 1]$. A cost function $C_f(q)0$ for quantity q, so when sold at price p price taking leads to maximising $pq - C_f(q)$, where C convex and differentiable, supply function $S(p, f) = (\partial_q C_f)^{-1}$. The aggregate supply is $\int S(p, f)df$, but a priori will depend on expected p, so

$$\bar{S}(d\mu(p, f)) = S(E(p), f)).$$

We have a aggregate, downward slopping demand $D(p)$. In basic version of the model we have $C(q, f) = q^2/2C(f)$, from which we can derive aggregate supply and demand. Producers must decide at time t what they will produce to sell at time $t + 1$.

[10]The weak axiom of revealed preference is a characteristic of the behaviour of economic agents. If an individual chooses choice A and never choice B when faced with a choice of both alternatives, they should never choose B when faced with a choice of A,B and additional options.

Let strategy set of farmer be S_f, then payoff to f is

$$\{D^{-1}(\int s_{f'})\}s_f - C(s_f, f).$$

Roughly speaking rationaizability is dervived from the rationality of the farmer – the farmer uses best responses to some strategy profile of other farmers, which may actually be played, who are rational so some possible best responses not best responses and are eliminated. More formally we iteratively removed strategies via rule (where τ is iterations of rule not time in any meaningful sense in the game):

$$S(\tau, f) = S(\tau-1, f)/\{\bar{s} \in S(\tau-1, f) | \bar{s} \text{ is not best response to any}$$

$$\prod_{f' \neq f} s_{f'} \text{ where } s_{f'} \in S(\tau-1, f')\}$$

leaving us with definition: a *rationalizable strategy* is one in $\Re = \prod_f (\bigcap_{\tau=0}^{\infty} S(\tau, f))$. A *rationalizable-expectations equilibrium* is a function $Q(f)$ of producers supplies where each such strategy is rationalizable as defined above. We can then find rationalizable-expectations equilibrium price $p = D^{-1}(\int Q(f)df)$ or the market clearing price. A *competitive equilibrium* consists of price p such that $S(p) = D(p)$, whereas a *rational expectations equilibrium* is a distribution on $\{p\}$ and is generated by market clearing condition as $t+1$ when believed by all agents at time t. As there is no noise the rational-expectations equilibrium is a perfect foresight equilibrium (and thus coincides with competitive equilibrium which is unique given some assumptions). It follows that every Nash Equilibrium is rationalizable (as never eliminated in iterative process), so the rational-expectations equilibrium is necessarily a rationalizable-expectations equilibrium. Will say equilibrium is associated with *strongly* rational expectations if converse true, calling it a strongly rationalizable-expectations equilibrium which is a rationalizable-expectations equilibrium which is unique.

For a linear model Guesnerie characterises the equilibrium (or set equilibria) using these formulations (proposition 1 in Guesnerie [1992]). We basically have a notional time tâtonnement process that (in this case) will converge. But what about evolving approaches? These require repetition rather than notional time[11], but if we disallow inventories, we could use this approach to obtain a similar result to the static rationalizable-expectations equilibria case. Guesnerie obtains the result that "success" of eductive learning implies "success" of adaptive learning. The converse

[11]Guesnerie is happier with notional time than a repeating, evolving process.

does not hold.

This eductive process can be viewed as ultimate phases of demanding evolutionary models. It is idealised but Guesnerie claims that something like it is likely to take place in actual decisions and forces at work likely to at least go in same direction as above analysis (that is a few steps of this notional tâtonnement). He is able to identify several consequences of this process:

1. Coordination improved when decisions are sequential and multi-commodity. The effect of noise is ambiguous.
2. There might be a general connection between conditions of success for eductive/notional learning and evolutionary learning.
3. Market structures can reflect the conditions of eductive coordination.

2.5 Microeconomics: contemporary perspectives

The textbook Microeconomic Theory, Mas-Colell et al. [1995], (hereafter MT to distinguish it from the general term) is the standard microeconomics theory textbook for postgraduate students at top universities around the world. In this section I summarise the key elements of the this kind of approach relevant to my work, though my focus is on a more recent approach perhaps best exemplified by Bowles [2006] which attempts to move on from that of Mas-Colell et al. [1995]. A considerably more extensive and detailed look at selected microeconomic topics can be found in the next chapter.

MT covers many standard microeconomic theory topics but at least in terms of relating it to this thesis the most relevant material is probably the substantial coverage of general equilibrium theory. We see example equilibrium models, pure exchange economies and examination of properties such as the welfare properties of equilibrium and the existence and uniqueness of (general) equilibrium for certain economic models.

Many criticisms have been made of general equilibrium economics, or rather the central emphasis which has historically been placed on the framework. One significant early critique (or perhaps more accurately limitation to the ambitions of it) was as remarked above Scarf [1959]. He proposed very simple examples where prices were globally unstable (they do not tend towards an equilibrium for non-equilibrium initial conditions)[12]. The simplest example has a cycle of indifference/desire which

[12] In Gintis [2007a] one such example is used as a test case for a different price coordination mechanism. See section 2.7.4.

results in cyclical price movement. One more recent, and at least from a certain perspective quite damning, criticism has been made by Herbert Gintis, who contrasting the text with an equivalent one from Physics, notes the absence of a single 'economic fact' in the entire book[13].

In the remainder of this section I focus on Bowles [2006] as a non-quite-traditional but still mainstream, recently published, graduate text in microeconomics that explores many ideas of relevance to my PhD. It covers many standard topics, but generally from a somewhat unconventional angle. I will refer to later sections of the thesis where wider and more extensive accounts can be found. The work has many connections with that of Herbert Gintis who features prominently in section 2.7.4. In addition Bowles co-authored, with Gintis, an interdisciplinary text covering many related themes Bowles and Gintis [2010].

Bowles highlights the importance to both classical and contemporary economists of the subject of the wealth of nations and people; reversals of fortune and what he considers the often overlooked importance of institutions. He then asks *"What can modern economics say about the wealth and poverty of nations and people? No less important, what can it do?"* claiming that *"contrary to its conservative reputation economics has always been about changing the way the world works"*. Bowles thinks that the neoclassical paradigm is ill-suited to the tasks that interested its co-founder Marshall as it typically omits many key aspects of economic importance: power, experience, out-of-equilibrium dynamics and institutional persistence and change.

He will retain the purposeful actions of agents, constraints on actions by competition and unintended aggregate outcomes; but other aspects of what he calls the Walrasian paradigm will be replaced. He argues that his caricature of the Walrasian paradigm [14] is recognisable of the form of economics taught in the 1980s and although it has been somewhat displaced by the use of new tools like Game

[13] The observation is made in a review of Rationality in Economics (and elsewhere) and though MT is not explicitly identified it is clear this is the book he is referring to.

[14] To quote: *"The Walrasian approach represents economic behaviour as the solution to a constrained optimization problem faced by a fully informed individual in a virtually institution-free environment. Robbins celebrated definition of the subject (in the epigraph) reflects this equation of economics with constrained optimization. The passage of time is represented simply by a discount rate; people do not learn or acquire new preferences over time; institutions do not evolve. The actions of others are represented by nothing more complicated than a given vector of market-clearing prices, while proximity is captured by a cost of transportation. Property rights and other economic institutions are represented simply by a budget constraint. An economic actor in this model is roughly Robinson Crusoe, with prices standing in for nature. The economists Crusoes inhabit a world in which goods are scarce, but whatever institutions are necessary to coordinate their activities in an optimal manner are freely available. The supply of optimal institutions can thus be ignored for the same reason that Adam Smith used to explain why economists need not theorize about the value of water: they are free goods." Bowles [2006]*

Theory the core assumptions of "equilibrium, greed and rationality" still remain. In their place he wants to include: non-contractural social interactions, adaptive and other-regarding behaviours and "generalised increasing returns"[15] and as results may have multiple equilibria we should must consider equilibria selection.

Bowles outlines some general principles for Microeconomics, namely that it must draw on insights of all behavioural sciences and must use empirical restrictions (otherwise relaxation of Walrasian assumptions leads to vacuousness). This is especially important for individual behaviour. Theory, he argues, still provides useful restrictions on sets of plausible assumptions and outcomes. He further suggests that specific models should be used in place of attempts, which are typical in economic theory, to model general markets (Alan Kirman, among others, have stressed this point, see section 2.2.1.)

Bowles first looks at coordination problems and the *classical constitutional conundrum*: how can social interactions be structured to allow people free choice but avoiding aggregate outcomes none would have chosen. Bowles argures Pareto standard is too strong (unrealisable) and too weak (it abstracts from other goals, particularly fairness). In any case, alternatives may not be Pareto rankable in practice.

He highlights several fields such as *Implementation theory, mechanism design* and *optimal contract theory* which attempt to answer problem of inducing appropriate behaviour where well being of other is affected. Both traditional and evolutionary game theory is introduced and split into two key groups: pure common interest games (only one Pareto optimal) and pure conflict games (all outcomes Pareto optimal), for example all zero sum games. He points out that games may have path dependent outcomes with convention outcomes, I focus on this topic in section 2.6.5.

A, perhaps the, key issue of economic study (in particular for anything claiming both the title economics and complexity science) is the relation of aggregate outcomes to individual intentions; this is a key issue for this thesis (I devote section 2.2 to surveying this issue and much of the thesis is at least implicitly about it). Bowles contrasts two main classes of population-level models that address this question: the general equilibrium models and evolutionary dynamic models of biological systems under chance, inheritance and natural selection.

Bowles considers concept of replicator dynamic and replicator equations observing that Robert May, in an evolutionary biology setting, discovered that evo-

[15] This is a misleading name, positive feedback would have been a more helpful term to use; as he is not just increasing returns to scale.

lutionarily irrelevant equilibria and/or nonequilibrium states can be of substantial importance. Though for economic purposes he argues we should go further to deal with hierarchical populations where differential replication may take place at various levels. In a basic model we can look at stationary states and their stability but this kind of basic model doesn't allow us to consider innovation.

He places an emphasis on social preferences or taking account of consequences for others. He distinguishes between reciprocity and strong reciprocity (expecting future reward to self). While self-interest is not implied by rationality it is generally assumed. Bowles argues that we should conceive of individuals as *rule-following, adaptive agents* with preferences which are situationally specific and endogenous. Bowles argues that no satisfactory conception of behaviour has yet been devised. He identifies the heterogeneity of individuals as something which frameworks should be able to take into account, something we explicitly include in chapters 3, 4 and 6.

The issue of bargaining is covered, including that using the Nash model[16]; both simple and corresponds to intuitions, but not really account of how bargaining takes place. Alternating offers bargaining is then examined, which does explicitly model a process whereby an agreement will be reached and several issues identified:

1. relative costs of bargaining even if trivial will determine result,
2. bargaining never breaks down and outcomes always Pareto optimal,
3. empirical evidence suggests people don't behave in these ways. Furthermore people will try to conceal desires in bargaining; the above approaches require knowledge of utility functions.

I look at this in more detail in section 2.6.1 (from a more sympathetic perspective). Bowles identifies four main problems with many economic models of markets:

1. "Describes nobody's actual behaviour in most markets"; in a sense it is not really about a market system.
2. Needs quasi-global stability (to give convergence from initial allocation) so much stricter requirements on utility than previously demanded.
3. Incomplete; lack of out-of-equilibrium account.
4. Market completeness assumptions are generally false.

[16]We require: 1) Invariant to linear transformations of utility functions, 2) Pareto optimal, 3) Independence of irrelevant alternatives, 4) Symmetry.

Alternative approaches, which at least attempt to address some of these issues, are surveyed later in this chapter. See also chapters 3 and 4 for the out-of-equilibrium approach to bargaining taken in this thesis.

2.6 Microeconomics

2.6.1 Bargaining

The standard text in this area is Rubenstein and Osbourne [1990] which looks principally at strategic, sequential, decentralised bargaining. The basic axiomatic approach (Nash's Solution) is formulated along the following lines

1. Have N bargainers/players.
2. Reach agreement in set A or disagree D.
3. Each player has preference relation over set $A \cup \{D\}$.
4. A bargaining problem is a set S, d, $S \in \mathbb{R}^2$, $d \in S$ where S is set of feasible utility pairs and d is disagreement with (in Nash's formulation) four axioms:
   (a) Invariance to Equivalent Utility Representations
   (b) Symmetry
   (c) Independence of Irrelevant Alternatives
   (d) Pareto Efficiency

   and a solution is a function f on bargaining problems to $\mathbb{R}^2$ that assigns a unique element of S.

Given this kind of formulation it can be shown that there is a unique bargaining solution satisfying Nash's axioms given by

$$f^N(S, d) = \arg \max_{(d_1,d_2) \leq (s_1,s_2) \in S} (s_1 - d_1)(s_2 - d_2).$$

We can formulate this within a strategic framework and Rubenstein develops a number of results for models with decentralised bargaining with one-time entry to the market. He goes on to look at trading procedures and anonymity assumptions.

2.6.2 Fictitious Play

In a fictitious play model, Fudenberg and Levine [1998], each player assumes that all other players have an underlying, constant distribution of strategies. The players then adopt a best response to what they perceive as the fixed strategy of the other players. Over time the strategies evolve, perhaps reaching an equilibrium. One can model this using discrete or continuous time. There are many variations on fictitious play and it is considered a standard technique within the area.

To formally define fictitous play, for each player i let κ_0^i be the prior "count", that is function from strategies of the other players to a count in $\mathbb{Z}^+$. Every time another player plays a strategy that count in incremented and we have the corresponding κ_t^i at time step t. From this we obtain a naïve probability distribution on strategies defined by

$$\gamma_t^i = \frac{\kappa_t^i}{\sum_j \kappa_t^j}.$$

We then map this probability distribution on strategies to a best response via a rule ρ_t which maps the probability distribution obtained via γ_t^i to a best response. For a basic form of fictitious play this best response will take the form of a pure strategy. For non-degenerate games the process will approach an equilibrium, see early chapters of Fudenberg and Levine [1998] for a full account.

While this approach described above has benefits: it is deterministic and easy to work with, we may be able to obtain more satisfactory results if we adopt a stochastic scheme. One such possibility is that of a smoothed best responses. We will no longer choose the single pure strategy but a mixed strategy which weights the strategy possibilities according to their expected payoffs. It is argued in Fudenberg and Levine [1998] that this closely models human behaviour when a decision is uncertain. One scheme is by using the function

$$P(\text{Choosing strategy } s_j) = \frac{exp(E_j/\gamma)}{\sum_i exp(E_i/\gamma)}$$

where s_j are the pure strategy choices, E_j is the expected payoff for strategy s_j and γ is a smoothing term that captures the "probability of making a mistake"[17].

[17] We can formulate this entire approach in an explicitly Bayesian way modelling the prior belief via a Dirichlet distribution which is a conjugate prior for the multinomial distribution which we can use to model underlying strategy choices. This approach is useful for looking at learning in games with more than two players.

2.6.3 Adaptive Heuristics

We look at Hart [2005], but some of Hart's earlier work with Mas-Colell in Hart and Mas-Colell [2000] is also explored. When thinking about learning we could classify dynamic models into three main categories. Learning Dynamics are an essentially a Bayesian framework with prior beliefs about the game, updated as game is played; Hart thinks of this as 'high rationality'. Evolutionary Dynamics have a population of players in each role who plays same one-shot action; selection and mutation take place, with resultant mixed strategies perhaps derived from relative surviving proportions of population; Hart thinks of this as 'no rationality'. Finally, he considers adaptive heuristics which deals with simple, myopic, unsophisticated rules, which are "adaptive", that is they induce behaviour that reacts to play of game in ways that seem better; Hart considers this 'in-between rationality'.

For adaptive heuristics we assume we have a game Γ, N players, each with action set S^i and utility function $u^i : S \to \mathbb{R}$. Γ is played repeatedly in periods $t = 1, 2, \ldots$ with action played denoted S^i_t. *Regret Matching* is defined by: switch in next period to an action with probability proportion to regret (the increase in payoff had that change always been made in past). More precisely let U be average payoff so far, j the action in previous period, then for each alternative action k let $V(k)$ be average payoff he would have obtained by playing this instead of j in each case where he played j. Now regret $R(k) := V(k) - U$ if positive, 0 otherwise. Now play k at $T + 1$ with probability

$$\sigma_{T+1}(k) = \begin{cases} cR(k) & \text{if } k \neq j, \\ 1 - \sum_{l \neq j} cR(l) & \text{if } k = j. \end{cases}$$

The main result of Hart and Mas-Colell [2000] is that if each player uses regret matching then joint distribution of play converges to *set* of correlated equilibria[18] of the game.

Regret matching has nice behavioural features, as it captures a sort of "inertia" and commonly used rules of behaviour. We can generalise the regret matching to a Lipschitz continuous sign-preserving real function of the regret and show that it doesn't matter which particular form each player adopts. Can also show that there

[18]The idea is that each player chooses their action based on the value of a public signal. A strategy assigns an action to every possible observation of the public signal a player can make. If there does not exist a player who would want to deviate from the strategy determined by the public signal (assuming the other players do not deviate), this distribution of strategies is a correlated equilibrium.

exist no uncoupled dynamics[19] that guarantee Nash convergence.

2.6.4 Strategic Foundations of (General) Equilibrium

The goal of Gale [2000] is to find strategic foundations of perfect competition. This kind of program requires:

1. A description of market or economy.
2. An extensive form game describing behaviour of agents in above.
3. An analysis showing under what conditions a perfectly competitive equilibrium is reached.

Gale will focus on Dynamic Matching and Bargaining Games (DMBG). In many classic models prices set exogeneously by market clearing condition and where agents believe they can buy or sell arbitrary amount at current prices. A fully strategic approach on other hand will allow us to ask when perfect competition is appropriate and could have a normative role in telling us how to achieve it. There are many results mapping from cooperative games to competitive equilibria.

In the Cournot model as suppliers tend to infinity, we tend to a competitive equilibrium. The Cournot-Shubik market game is a centralised attempt at what Gale will attempt with his *decentralised* DMBG formulation. Some related examples are given like Rubenstein's[20] alternating offers bargaining model Rubinstein [1982] which has unique subgame perfect equilibrium, the formulation of this bargaining in Marshallian markets with a continuum of traders and the associated notion of quasi-stationary subgame perfect equilibrium which although an equilibrium with instant agreement is not a competitive equilibrium.

Gale's DMBG model has one period matches, complete information and a limiting outcome which is more like competitive equilibrium than the above Rubenstein model. Production is ignored to achieve tractability in Gale's model. In pure exchange economy competitive equilibrium is characterised by efficiency and budget balance; loosely speaking a competitive equilibrium is an efficient allocation which satisfies budget balance. In order to prove a competitive limit theorem we need to make some assumptions which are not endogenous to either agents or the equilibrium state. The key assumption is that any agent has negligible effect on equilibrium. A *pure exchange economy* consists of

[19] Strategies may depend on actions taken by other players, but not why they do it that is on their utility functions. This condition is common to a wide range of approaches such as fictitious play, best-reply and so on

[20] See also a brief introduction to this kind of approach above in section 2.6.1.

1. Commodities: indexed $h = 1, \ldots, l$ homogeneous, perfectly divisible; commodity bundle $x = (x_1, \ldots, x_l)$.

2. Agents: indexed $1 \ldots m$

3. Consumption sets: set of feasible consumption bundles for i, $X_i \subset \mathbb{R}$, each assumed to be non-empty, convex, bounded below and closed.

4. Endowment: each agent has endowment $e_i \in X_i$.

5. Preferences: have utility function $u_i : X_i \to \mathbb{R}$ which is concave (i.e. agents risk adverse), continuous and increasing.

So the whole exchange economy defined by a $\Xi = (X_i, e_i, u_i)_{i=1}^m$. What way can we define efficiency? An attainable allocation is a bundle x such that

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i$$

and furthermore is *(strongly) Pareto Efficient* if there does not exist another allocation which makes some agents better off without making any worse off; it is *pairwise-efficient* if no two agents can improve situation by trading. The non-standard assumptions made are: smooth preferences and the indifference surface through initial endowment doesn't intersect boundary of consumption set[21]. We only consider individually rational allocations, that is $u_i(x_i) \geq u_i(e_i)$. Using the Kuhn-Tucker Theorem[22] We can prove that an attainable and individually rational allocation x is Pareto efficient if and only if it is pairwise-efficient.

To formulate this strategically we want to restrict agent's behaviour via a non-cooperative game. Gale represents trade by a Rubenstein style (alternating offers) bargaining:

1. Time: countable number of dates, indexed $t = 1, 2 \ldots$.

[21] On page 48 an example is given that shows how we run into problems without this assumption.

[22] The **Kuhn-Tucker theorem** roughly states that: given we want to optimise $f(x_1, \ldots x_n)$ subject to constraints $g(x_1, \ldots, x_n) \geq 0$ and $x_i \geq 0$; form Langrangian function $F(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n)$ and solve:

1. $\frac{\partial F}{\partial x_i} \leq 0$
2. $x_i \geq 0$
3. $x_i \frac{\partial F}{\partial x_i} = 0$
4. $\frac{\partial F}{\partial \lambda} \geq 0$
5. $\lambda \geq 0$
6. $\lambda \frac{\partial F}{\partial \lambda} = 0$

2. Matching: at each date a pair (i_t, j_t) is matched (and assumed infinite number of dates at which any pair matched).

3. Bargaining: i is proposer, j is responder. Proposer chooses vector z of feasible trades (for himself and receiver), responder chooses "yes" or "no" and trade happens or go to next date, with no change in allocations.

The play is described by path $\{a_t\}_{t=1}^{\infty}$ where $a_t = (x_t, z_t, r_t)$, roughly "(allocation, proposal, response)". We call history h_t and let H_i be information sets at which player i controls play. A strategy is a decision rule that specifies action for player at every information set where he moves; formally it is a function $f_i : H_i \to \mathbb{R}^l$. Define path $a^f = a_t^f$ and outcome $\Xi(a^f)$. We define payoff for agent as

$$\lim\ \inf_{n\to\infty} u_i(x_{it}) = \sup_{T\geq 1}\{\inf\{u_i(x_{it}|t \geq T\}\}.$$

Though often the sequence of utilities will itself have a limit. Let payoff for agent i be

$$v_i(f) = \lim\ \inf_{t\to\infty} u_i(\xi_{it}(a^f))$$

So have a normal form game Γ with m players, strategy profiles $F = \times_{i=1}^{m} F_i$ and payoff function v.

The Nash equilbrium of the game will be the strategy profile f^* such that each f_i^* maximises payoff to i given f_{-i}^*. But this is too weak a concept for our purposes. Could perhaps use *subgame perfect equilibrium*[23] (SPE) but Gale will actually go further and restrict memory; we restrict use to *Markov Strategies*[24] and use *Markov Perfect Equilibrium (MPE)* as equilibrium concept, where choice is restricted to SPEs, but MPEs are optimal.

An *Edgeworth Process* is evolution of state/utility process which is non-decreasing over time: non-deceasing vector of scalars rather than single objective function. Gale proposes "Edgeworth Property", though in his DMBG version agents are far-sighted. As an agent can never become worse off (offer $z = 0$ and reject trade), $v_i(f^*) = \lim_{t\to\infty} u_i(x_{it})$. We can show that if f^* be MPE of Γ and $\{x(t)\}_{t=1}^{\infty}$ be equilibrium outcome. Then $v_i(f^*) = \lim_{t\to\infty} u_i(x_{it}), \forall i$. And if we further assume each u_i is strictly concave (ensures allocations converge if utilities do). Then the MPE outcome $\{x_t\}$ converges to $\lim_{t\to\infty} x_t \in P$.

Now define sequence of economies (E^m) (i.e. each of size m) and matching probabilities π^m, can define DMBG on these. Want to look at (f^m) where $f^m \in$

[23]Every player must use optimal action in every situation not just along equilibrium paths

[24]For any information set same z chosen and same response made.

MPE(Γ^m). If $y^m = \lim_{t\to\infty} x_t^m$, then by Second Welfare Theorem (and concavity) there exists a supporting price vector p^msuch that:

$$u_i(y_i) \geq u_i(p_i^m) \text{ and } y_i \neq y_i^m \implies p^m \cdot y_i > p^m \cdot y_i^m.$$

We can normalise these price vectors and get bounded sequence; which thus has a subsequence converging to p^0, a limiting price vector.

Even for a very large number of agents, players could condition responses on the actions of a single agent; something we wish to avoid. Introduce a continuity assumption: for fixed $\{f^m\}$ (sequence of MPE), $\forall\epsilon$, there exists M and T such that $\forall j$ and strategy $f_j \in F_j^m$ we have

$$\frac{1}{m}\sum_{i=1}^{m} ||\xi_{it}(a(f_{-j}^m, f_j)) - y_i^m|| < \epsilon, \forall m > M, \forall t > T$$

where $\xi_t(a(f_{-j}, \hat{f}_j))$ is the allocation given a deviation by agent j to $\hat{f}_j$. That is for large enough m, t one player cannot change allocations by large amount. This isn't a desirable assumption but makes things tractable. We also make a curvature assumption.

We want to prove that any net trade lying in competitive budget set $\{ \in \mathbb{R}|p^0 \cdot z \leq 0\}$ is achievable in asymptotically pure MPE for m large enough, that is will have shown each agent can obtain same utility as in competitive equilibrium with p_0. First note that as limiting allocation is efficient and has supporting price vector for any net trade z, can find n such that $-z/n$ is preferred trade for any agent j holding limiting bundle y_j^m (from curvature assumption). Then can achieve net trade z by offering z/n when possible and rejecting anything else.

Define E^m as exchange economy with first m agents; define game Γ^m by specifying order of play. π^m. $\{\Gamma^m\}$ is a *competitive sequence of games* if for every i:

1. X_i non-empty and closed.
2. $e_i \in X_i$
3. u_i is strictly concave, increasing and C^1 on open superset of X_i
4. We have curvature assumption
5. $\{X_i\}$ uniformly bounded below and mean endowments are uniformly bounded above.

If $\{f^m\}$ is a sequence of equilibrium strategy profiles such that $\{f^m\} \in$ MPE(Γ^m)$\forall m$, then a subsequence of $\mathscr{M}$ is a *competitive sequence of equilibria* if:

1. for every m, f^m is MPE and $x_t^m \to y^m$ for almost every ω.
2. for every $m \in \mathscr{M}$, there exists p^m such that if $||p^m|| = 1$ and $u_i(x_i) > u_i(y_i^m)$, then $p^m \cdot x_i > p^m \cdot y_i^m \forall i = 1, \ldots, m$.
3. $\lim_{m\in\mathscr{M}} y_i^m = y_i^0$ and $\lim_{m\in\mathscr{M}} p^m = p^0$.
4. subsequence satisfies continuity assumption.

Call (y^0, p^0) a *(strong) competitive limit equilibrium* if

$$\forall i, y_i^0 \in \arg\max\{u_i(x_i)|x_i \in X_i, p^0 \cdot x_i \leq p^0 \cdot e_i\}$$

(and if Y^0 is attainable). Let $\{\Gamma^m\}$ be a competitive sequence of games with $\{f^m\}_{m\in\mathscr{M}}$ a corresponding competitive sequence of equilibria. Let y^0 and p^0 be limiting allocation and price system. These are a (strong) competitive limit equilibrium if convergence of $\{y_i^m\}$ is uniform.

It is possible to derive several more substantial results within in this framework including results for introducing stochasticity into matching process. There is also a imitation model which will converge to the equilibrium outcome with a profit maximising price maker. For these and further results see Gale [2000].

Gale draws his basic formulation of a DMBG from Goldman and Starr [1982] which we now turn to. In this paper be look at t-wise optimal allocations. A $t-$wise optimal allocation is one which cannot be improved via trading taking place in groups of size t. The key question is when such an allocation is Pareto optimal. Have pure exchange economy of M consumers, N commodities, with an allocation an $M \times N$ non-negative matrix $A = (a_{ij})$ where a_{ij} is i's holding of good j. Utility functions are a vector of C^2, quasi-concave functions $u^i : \mathbb{R}_+^N \to \mathbb{R}$. Also have vector of linearised utility functions about A. Due to nature of utility functions we can characterise state of economy by these linearised utility functions.

A t-wise reallocation Z such that $z_i = 0$ for all but $t \in 1, \ldots, M$. It is a t-wise improvement, if $p_i z_i \geqq 0$, with strict inequality for at least one i. Make obvious definition of t-wise optimal. Say has t-wise equivalence property if set of t-wise optimal is set of Pareto optimal. A state is a pair (A, P), let $\Pi^t(A), \Pi^*(A)$ be the set of t-wise, Pareto Optimal states respectively.

In a result due to Rader it can be shown that ff A has strictly positive row, then $\Pi^2(A) = \Pi^*(A)$. This means there is a trader with all goods. The corresponding price system will be his marginal utilities. If we have pairwise optimality then marginal rates of substitution coincide with the strictly positive row. And in a theorem due to Feldman can show if A has strictly positive column, then $\Pi^2(A) = \Pi^*(A)$.

So every trader has some of one good. We can use this good as numeraire and get price for other goods. Pairwise optimality guarantees marginal rates of substitution of goods held by two traders are equal. We can then generalise results to if A has some row (or column) with $t-2$ or fewer zero entries then $\Pi^t(A) = \Pi^*(A)$. And with some additional conditions: $\Pi^t(A) = \Pi^*(A) \Leftrightarrow \Pi^t(A) = \Pi^{t+1}(A)$.

2.6.5 Conventions

A *convention*, see Young [1993], is an expected, possibly non-symmetric equilibrium. Young [1993] looks at an evolutionary framework to examine how these arise and how stable they are. The basic model consists of an n person game, with players drawn from large but finite population, players basing actions on sample of events from recent past and occasionally mistakes/experiments.

This is similar to fictitious play (see section 2.6.2) but with limited memory. For the basic form of adaptive play there need not be convergence to a Nash equilibrium (pure or mixed); but when we have stochasticity and finite memory we have a stationary distribution in the long run; if all of the weight of this distribution is on one equilibrium we call it *stochastically stable.*

Formally let Γ be n person game in strategic form, with S_i the finite set of strategies available to player i. N is population of individuals partitioned into n non-empty C_j. Each member of C_j could play role i in game. All individuals in a class have same utility function $u_i(s)$ for $s \in S$, set of strategy tuples. $t = 1, 2, \ldots$ are successive time periods. Player i chooses pure strategy $s_i(t)$. Let combined tuple be $s(t)$ and history up to t, $h(t)$. Each player takes k samples from m most recent periods to determine strategy. So now think of general history h as a state in Markov chain, with successor obtained by deleting leftmost element and adjoining new rightmost element. We have transition rule which is a *best reply distribution*, that is give positive probability to strategy if and only if $\exists$ sample to which it is best reply (and also is independent of t). Define:

$$P^0_{hh'} = \prod_i p_i(s_i|h).$$

where $P^0_{hh'} = 0$ if h' not a successor state of h. Clearly a strict, pure Nash equilibrium. m times in succession is an absorbing state, call this a *convention.* Convergence to absorbing state implies a strict, pure Nash equilibrium.

We can define a *best reply graph* of Γ such that each vertex s is n-tuple of strategies and there is directed edge $s \to s'$ if and only if $s \neq s'$ and exists an agent i such that s'_i is best reply to s_{-i} and $s_{-i} = s_{-i}$. A game is acyclic if best reply graph has no directed cycles. *Weakly acyclic* if for all initial vertices s, there is a directed path to s^* from which there is no exiting edge. $L(s)$ is length of shortest path from s to a strict Nash equilibrium Let $L_\Gamma = \max_s L(s)$.

Young obtains the result that if Γ a weakly acyclic n-person game. If $k \leq m/(L_\Gamma + 2)$ then adaptive play converges almost surely to a convention[25].

Now we introduce noise term - probability a player in role i experiments $\epsilon\lambda_i$. Now let $q_i(s|h)$ be conditional probability that i chooses s given that he experiments and history h. It turns out that subject to suitable conditions selected equilibria are independent of precise q, λ. The conditional transition probability for a subset of players J is

$$Q^J_{hh'} = \prod_{j \in J} q_j(s_j|h) \prod_j \notin J p_j(s_j|h)$$

or

$$Q^J_{hh'} = 0 - \text{if } h' \text{ not successor}$$

and new perturbed Markov process has transition function:

$$P^\epsilon_{hh'} = (\prod_{i=1}^n (1 - \epsilon\lambda_i))P^0_{hh'} + \sum_{J \subseteq N, J \neq \emptyset} \epsilon^{|J|} (\prod_{j \in J} \lambda_j)((\prod_{j \notin J} (1 - \epsilon\lambda_j))Q^J_{hh'}.$$

We call this process *adaptive play with memory m, sample size k, experimentation probabilities* $\epsilon\lambda_i$ *and experimentation distributions* q_i. P^0 is the unperturbed process.

Now consider P^ϵ where ϵ is small. This is irreducible and aperiodic, so has unique stationary distribution μ^ϵ (that is $\mu^\epsilon P^\epsilon = \mu^\epsilon$). Let μ^ϵ_h be probability that h is observed at any sufficiently large t. A state is *stochastically stable* relative to P^ϵ if $\lim_{\epsilon \to 0} \mu^\epsilon_h > 0$. Define a *mistake* for h' successor of h as component of s_i of s not an optimal response by i to any sample of size k from h. The *resistance* $r(h, h')$ is total number of mistakes in transition $h \to h'$ (if h' successor) or ∞ (otherwise).

Now view H as vertices of directed graph. For every h, h' insert directed edge if $r(h, h')$ is finite and let this be it's weight. Let $H_1, H_2 \ldots, H_J$ be recurrent communication classes of P^ϵ. Now look at graph $\mathscr{G}$ with vertices H_k and directed edges with weight $r_{i,j}$, the path of least total resistance between classes.

Finally for every vertex let $\mathscr{T}$ be set off all i-trees[26] on $\mathscr{G}$. The resistance of i-tree is sum of resistances of edges. The *stochastic potential* of H_i is least resistance among all i-trees. Can obtain the result that the stochastically stable states of adaptive play P^ϵ are states in recurrent communication classes of P^0 with minimum stochastic potential.

[25] This bound might not be tight.

[26] An i-tree is spanning tree such that for every vertex j there is a unique directed path $j \to i$.

This kind of model can be developed and applied in many directions. In Young [1998] the above ideas are developed further via the introduction of a spatial model (in Chapter 6), looking at more general n-person games (in Chapter 7) and bargaining (in Chapter 8). which includes the introduction of a limited sort of heterogeneity. In Axtell et al. [2001] Axtel, Epstein and Young explore a model of the evolution of conventions where certain members of the population have "tags" (e.g. black, white) but are otherwise identical. While an equitable norm is the only stochastically stable norm, a discriminatory norm may persist for many periods. The concept they use to describe this is that of *broken ergodicity*: the process is technically ergodic but may spend a lot of time in a far from asymptotic state. In Young [2009] a simple framework for diffusion (where agents have a propensity to adopt) is explored via three mechanisms: contagion, social influence and social learning. This method will be used in chapter 6 in order to analyse the long run behaviour of a set of economic models.

2.7 Agent-Based Modelling

Agent-Based Modelling is the creation of a computational model which simulate the interactions of, potentially many, independent agents and their interactions in an attempt to characterise the resulting behaviour of the systems as a whole. It is a technique used within many disciplines including economics, computer science and sociology. Agent-Based Computational approaches have been widely recognised as a promising technique for general economic and organisational modelling, having been featured recently in high profile publications such as The Economist (Economist [2010]), the Proceedings of the National Academy of Sciences Bonabeau [2002] and Nature (Farmer and Foley [2009]). Agent-based models have been applied to a wide range of practical problems from logistics to traffic modelling. The resource 50 Facts About Agent-Based Computing[27], a commercial promotional document, gives an attractive overview of many of the success stories and statistics related to real applications of agent-based modelling.

The classic example of Agent-Based Modelling is the Schelling Segregation model Schelling [1971]. In it there are two colours of agents. Each agent weakly prefers to live in a neighbourhood consisting mostly of his own colour. If these agents are initially randomly scattered across a landscape. When agents are unhappy they move to a new location where they will be happy. Even though they would be happy

[27]From Agentlink, available from `http://www2.econ.iastate.edu/tesfatsi/AgentLink.50CommercialApplic.MLuck.pdf`, covers many commercial applications of agent-based modelling.

to live in a mixed neighbourhood, they quickly segregate into blocks of their own colour.

In this section we look at some agent-based modelling work particularly in relation to economics, though we also consider a few examples from further afield.

2.7.1 Early Agent-Based Models

The challenge of designing economic agents which behave the way actual humans do is set out in Arthur [1991]; the idea being that we could represent agents via parameterised decision algorithms chosen and calibrated such that their behaviour matches human behaviour. It is accepted that a universal model is unlikely, but it may be possible to achieve the goal for contexts. For the parameterised learning an automaton which uses a form of reinforcement learning is suggested with a tuneable rate of learning. This automaton can be calibrated via experiments with human subjects. Arthur considers the traditional economic expectation of convergence to equilibrium and suggests that his automaton (and humans) may not always converge.

While not, in its original conception, a model for financial markets Arthur [1994] introduces a model which in a simplified form has become a major research field for agent-based modelling of financial markets. The model introduced is the 'El Farol Bar Problem', where there are a set of agents who will enjoy going to the El Farol bar, but only if less than sixty percent of all agents go; otherwise they would prefer to stay at home. In this model traditional economic approaches do not work well, as there is no 'rational' solution[28]. The approach taken is to assume that agents individually form k predictions of next week's attendance from the last d weeks. Examples of predictions given are:

- Same as last week
- Mirror image around 50 of last week's attendance
- Bounded trend of last 8 weeks
- Same as m weeks ago (periodic)

Agents act on their most accurate predictor and update each week given the resulting situations. This is then enacted as a computational experiment and it is observed that mean attendance always converges to 60. The predictors self-organise

[28]There are mixed equilibrium solutions; however Arthur argues against this approach, as it misses the point.

into a kind of equilibrium where of those active around 40 percent predict above 60 attendance and around 60 percent predict below 60 attendance. This model has subsequently been simplified in the form of the Minority Game: now an odd number of agents choose between two options independently and want to be in the minority. There have been literally hundreds of research papers on this topic and it is widely seen to be ; Challet et al. [2005] includes both an introduction to and a comprehensive collection of many major papers on the Minority Game.

In Arthur et al. [1997] an influential agent-based model of a stock market is presented. We look at this work via the retrospective LeBaron [2002]. This examines the development of the Santa Fe artificial stock market, one of the first major attempts to build a (somewhat) detailed agent-based model of a financial market. The initial artificial stock market model was one with a risk free asset and a risky stock paying a dividend

$$d_t = d + \rho(d_{t-1} - d) + \mu_t$$

where d, ρ are fixed parameters and $\mu \sim \mathcal{N}(0, \sigma_\mu^2)$. Agents have individual expectations for the price change of the stock and for its variance. Using a classifier system where agents' decisions are formed from a series of bits, each corresponding to a property of the market, such as a rule relating price, risk free interest and dividend. Each agent has a set of 100 of such rules and there is no sharing of rules between agents. A genetic algorithm is used *by each agent* such that periodically the twenty worst performing rules are removed and replaced with new rules via both crossover and mutation from their existing rules. This kind of individual based selection between rules is similar in style to Arthur [1994]. The model generates many features of real financial data, specifically excess kurtosis in returns, low linear autocorrelation and persistent volatility.

2.7.2 Agent-Based Modelling in Economics

In Tesfatsion [2006] a framework ("constructive approach") is formulated for Agent-Based Computational Economics (ACE). This is derived from the Walrasian equilibrium framework, but with the "auctioneer" removed. Four major strands of ACE work are identified:

1. Empirical understanding: why do global regularities persist despite lack of central control and planning.

2. Normative understanding: can we discover good economic designs.

3. Qualitative insight and theory generation: economic systems can be more fully understood through systematic examination of potential dynamical behaviours.

4. Methodological advancement: programming, visualisation and validation tools will have to be carefully considered.

Tesfatsion argues that agent-based tools allow for more realistic models of cognitive agents than in 'homo economus' models. Events are wholly endogenous and driven purely by agent interactions. But we need to construct dynamically complete economic model, that is start with full initial conditions and then continue without further exogenous events; to do this we must look at wide range of plausible initial specifications. Models are difficult to verify against empirical data due to the contingency of empirical events.

Tesfatsion describes the general process by which we may replace Walrasian "auctioneer" (or equivalents) by agent-based process. Something like the following circular flow required:

1. Terms of Trade

2. Buyer-Seller matching

3. Rationing

4. Trade

5. Settlement

6. Shake-Out

We then must consider what concept of equilibrium or solution concept should be used. A stationary-state is likely to be too strong for interesting models. Possible alternatives include: unchanging carrying capacity, continual market clearing, unchanging structure, fixed trade network, unchanging belief pattern and steady-state growth pattern (constant rates of growth). However we may want a weakened form of the one or more of the above.

This kind of constructive approach will have several key steps (an example of procurement processes is given in section 4 of Tesfatsion [2006]):

1. Select as benchmark case an equilibrium economic model from literature that is clearly presented and addresses some important issue

2. Remove every assumption which externally entails equilibrium condition

3. Dynamically complete model by introducing trading/interactions between agents in model

4. Define "equilibrium" for this complete model

We don't have to impose rational expectations, but can incorporate realistic models of learning. May introduce conventions and organisations to provide "scaffolding".

In Axelrod [2006] examples are given where Agent-Based Modelling has been used to facilitate interdisciplinary work, to look at problems which are mathematically intractable and where it has revealed unity across disciplines. Of particular interest are remarks on the difficulties of "selling" Agent-Based Modelling and tentative standards for using this kind of technique, such as parameter values to investigate.

In Judd [2006] some computational issues related to ACE are explored. The main appeal of computational approaches is outlined: the elements previously sacrificed for simplicity can be investigated. He looks at two objections to numerical approaches. Lack of generality: to which he argues that theories (really) look at a "continuum of examples" (but perhaps a measure zero set of plausible/interesting examples). Viewed this way, Judd argues that the *relevance* and *robustness* of examples is more important than the number. A second common objection, errors, is dismissed, as when handled carefully these are negligible. But how can we systematically do computational (economic) research? We can't (currently at least) prove theorems using computers, but we can:

1. Search for counter examples to a proposition.

2. Use Monte Carlo sampling methods (which can be clearly expressed in terms of classical or Bayesian statistics)

3. Use regression methods to obtain "shape" of some distribution.

4. Replication and generalisation – can perhaps straightforwardly adapt a computer model to a new case, something which is often not at all straightforward for a theorem.

How agents learn is a key aspect of Agent-Based Modelling. In Brenner [2006] various learning processes are surveyed. The assumption is that we wish to model, in an agent-based way, human behaviour as closely as possible; with a categorisation adapted from the psychological literature. Economists are typically divided into two camps vis à vis learning: a desire that learning should converge towards optimal behaviour and those not so interested in optimality. Human learning, it is claimed, can be split roughly into reinforcement learning and cognitive learning.

Three frequently used reinforcement learning models are: Bush-Moteller model, the principle of melioration and the Roth-Erev model. In all models the frequency of behaviours that lead to better results increases. Melioration works by average past behaviour. The others assume change is determined by current outcome together with previously determined frequency distribution; they only store frequencies, not the entire past.

We can model routine based learning using techniques such as experimentation (trial and error), melioration/experience collection, imitation, satisficing (meet an aspirational level), replicator dynamics and selection mutation equation, evolutionary algorithms and combined models (for example experience weighted attraction, variation-imitation-decision).

And we can model beliefs via Fictitious Play (see section 2.6.2), Bayesian Learning (the oldest and most prominent 'optimal' model), Least Squares Learning, Genetic Programming, Classifier Systems, Neural Networks, Rule Learning (i.e. cognitive learning following same kind of process as reinforcement learning) and Stochastic Belief Learning (information from own experience or communication, when contradictory information obtained stochastically stick to old belief or not).

Brenner argues that in relation to routine versus belief learning the choice is difficult and related to choice of realistic versus simple model which may be more tractable. In choosing a model we should be capturing real learning process, obtaining an outcome matching stylised facts and, perhaps, converging to equilibrium. In assessing validity he argues experimental studies should provide main criteria, but evidence not always available and laboratory results may not apply to real world. We want effective description of resultant behaviour, even if details contradict psychology. Simple models are generally good (not over-fitted) but bad (as learning processes can be complicated) and unnecessary as current computer power means that we may be able to look at large number of parameters.

His practical recommendations include the suggestion that evolutionary approaches are good for population level results though perhaps not individual dynamics. Fictitious Play (see above) is both simple and supported by evidence and thus recommended. In terms of belief learning he suggests that more research needed. It seems like Fictitious Play and rule learning recommended as simpler solutions; stochastic belief learning is good if more information about beliefs is available.

In LeBaron and Tesfatsion [2008] the idea of modelling macroeconomic processes via systems of interacting agents is surveyed. Emphasis is placed on work which carries out some kind of empirical validation, through for example matching agent-level behaviour with people's behaviour.

In LeBaron [2001] specific guidance is given for the construction of models of financial markets; these are well organised, centralised and dynamic. To design a model one must think about the key areas of trade (do we assume simple response to excess demand, build market such that local equilibrium price can be found easily or explicitly model dynamics of trading in attempt to approximate real world?), securities, evolution, calibration and time (there are problems with how to deal with the past, the relative rates of change and the issue of concurrency).

2.7.3 Agent-Based Modelling in other disciplines

I have looked at some approaches to agent-based modelling in other disciplines such as computer science and sociology. Often the goals and basic questions are quite different to those we may wish to ask in an economic context, however, some of the methods used may be similar.

Some of the most relevant work in computer science looks at issues of trust and information exchange, such as Nowak and Sigmund [2005], which offers an indirect explanation for reciprocal behaviour without individual benefit, and Griffiths [2006], which looks at the idea of multidimensional trust in a peer to peer agent-based system where production takes place in a cooperate way. The issues of decentralisation and information are typically given a more *explicit* treatment than in economics models.

Looking at sociology and picking two illustrative examples we can clearly see the contrast with computer science. Edmonds et al. [2009] use the idea of "tags" (a socially distinctive mark not hardwired to any particular trait) and shows how reasonably robust cooperation can emerge in various systems. Yang and Gilbert [2008] is an example of an attempt at quantitative Agent-Based Modelling which connects an ethnographic investigation to agent-based modelling highlighting danger of adding concretisation to model and suggesting that networks of agents may prove to be useful class of models.

Another key distinction to draw from the above examples, and which holds more generally, is that agent-based work in computer science typically has a much stronger focus on the design of efficient, autonomous systems of various kinds; whereas in economics and sociology the interest is typically in building models of societies or markets. This kind of distinction in approach is important when considering the kind building agent-based modelling methodologies as in chapter **??**: the kind of tools suitable for economics or social agent-based modelling are quite different to those which might be suitable for most computer science agent-based modelling (or multi-agent systems) work.

2.7.4 Economic theory and Agent-Based Modelling

Herbert Gintis has developed and brought together two areas of work relevant to my research. The first is his agent-based modelling work, which is quite basic, though does ask interesting economic questions. The second is his theoretical work which attempts among other objectives to unify the behaviour sciences(!)

In Gintis [2006, 2007a] the dynamical properties of variants of classical models of economies are examined using an agent-based modelling approach. The key innovation of this work is the proposal of a decentralised mechanism which leads to convergence of pricing from a far from equilibrium situation. How convincing and substantial Gintis' framework really is will be examined, but his approach is very relevant to this thesis and some of the work is in some sense parallel to that of chapter 3.

The first of his papers Gintis [2006] focuses on a barter[29] economy. This provides a simple setting in which to examine and compare approaches. Gintis' basic barter economy consists of n goods and $N >> n$ agents each of whom produces one of the goods and consumes all of the goods. The utility function of an agent i who consumes goods $\mathbf{x}^i = (x_1^i, \ldots, x_n^i)$ is $u^i = \min_j \frac{x_j^i}{o_j}$ where $\mathbf{o} = (o_1, \ldots, o_n)$ is constant for all agents. That is, with respect to utility agents are homogeneous. Each agent has a "private" price vector, giving his relative evaluation of each good, $\mathbf{p}^i = (p_1^i, \ldots, p_n^i)$ which he uses to make and accept or decline offers of trade made by other agents. In each period each agent produces n units of his production good.

Given this specification, bartering takes place in a series of periods in a process quite different to the major bartering formulations such as Rubenstein bargaining which introduces the notion of time and a penalty for delaying agreement. However, an alternative approach is demanded by the modelling approach Gintis adopts. His bargaining process is basically to randomly order each good, then each agent; in this order agents visit producers of a good attempting to carry out trades until some limit on offers has been reached. The agent consumes or holds onto the good he has obtained, and a score is calculated using the utility function given above. This is repeated for every agent.

[29] A barter economy is an economy where exchanges of goods or services take place simultaneously, without using money. The key consideration here is the relative evaluation agents, or some central coordinating body places on goods. Bartering is generally divided into axiomatic and strategic theory: the former is a more general set of results, the later focusing on more specific kinds of scenarios.

The key question here is how trades are made. If offers are made in terms of the agent's "private" price vector (its strategy) then the change in goods held must meet the constraint $M_0 = \mathbf{p}^i \cdot \mathbf{x}^i(0) = \mathbf{p}^i \cdot \mathbf{x}^i$ (where $\mathbf{x}^i(0)$ is the amount of each good an agent holds before trading). So the agents should attempt to trade for the quantities that would optimise their utility given this constraint. The optimal amount to attempt to trade for will be

$$x_j^i = \lambda^* o_j$$

where

$$\lambda^* = \frac{M^0}{\sum_j p_j^i o_j}.$$

Given the above, the agent producing good a and trading for b will make an offer of quantities x_a, x_b of his good and good to trade for via the ratio obtained by setting $p_a^i x_a = p_b^i x_b$. This offer will either be rejected, partially fulfilled or entirely fulfilled - which occur respectively when it fails to meet the evaluation of the receiver, when the receiver does not have the requested quantity and when he does. Offers are repeatedly made as outlined previously and the success of the trading is evaluated via the utility function. The economy which has been describe above does not remain static. The 'private' prices evolve via a mutation-imitation evolutionary dynamic. After a fixed number of periods a fraction of low scoring agents copy the strategies of high scoring agents with a small mutation. For fuller details of the model see Gintis [2006].

There seems to be fundamental problems with Gintis' proposed framework. The key idea is that of private prices, but on close inspection this notion is quite weak: the suggestion that they represent 'private information' is misleading. In formulating bids the prices must be revealed, but even more problematically they are copied by other weakly performing agents. Furthermore to be able to formulate bids the agents must 'know' enough to derive the equilibrium prices in any case (at least in the most basic form of the model). A better description would be 'individual's price'. The evolutionary framework is in some ways standard but it is not convincingly justified in the context of this work; in particular imitation conflicts with the notion of privacy of prices. It is difficult to see how one could approach Gintis' framework in an analytic way. While one can derive the optimal pricing; this is only the case for the basic framework. It would be ideal if an alternative framework could be formulated that was more analytically tractable, though this is perhaps too ambitious a goal.

In Bilancini and Petri [2008] two key findings of Gintis [2007a] are criticised:

1. the claim that the results justify "the importance of the Walrasian model in contemporary economic theory"

2. "models which allow [agents] to imitate successful others lead to an economy with a reasonable level of stability and efficiency."

One objection – the possibility of multiplicity of equilibria – is noted, but a more fundamental objection, namely that Gintis does not really include capital goods (but rather some "indestructible, non-produced factor"), forms the focus of Bilancini and Petri [2008]. That Gintis' results are not generalisable to models including capital goods – something which for all their problems, Walrasian models are, moreover it is in these generalisations we are typically interested in – means that claim (1) doesn't really hold. It is argued claim (2) will hold, but only in economies without capital. Furthermore the difficulty of introducing investment decisions to a Gintis-style model is pointed out.

Using the notion of stochastic stability Young [2008] Mandel and Botta Mandel and Botta [2009] explore equilibrium selection in a simplified form of Gintis' model. As in Gintis' formulation each agent is homogeneous (at least in terms of utility function). Noting that in this economy any price is an equilibrium price (no excess demand), they propose a *minimal trading equilibrium.* They consider instead of price pricing a selection of market institutions within which to perform exchange. Gintis [2010] is a refined version of his two previous work in Gintis [2006, 2007a], though many of the same weaknesses are still present.

In his text Gintis [2009a] (see also the abridged form in paper Gintis [2007b]) Gintis develops a five fold framework to unify the behavioural sciences: gene-culture coevolution, sociopsychological theory of norms, game theory, rational actor model (or Beliefs, Preferences, Constraints model) and complexity theory. The book is very much an economist's take on offering a unified framework for the social sciences, it would most likely not be accepted by other disciplines, much fully less understood. Together with Samuel Bowles he has produced another text Bowles and Gintis [2010] which approaches things from a more biological/sociological angle.

2.8 Networks

The use of a network as a modelling tool for societies and complex organisations is becoming increasingly common. In a network model there are a set of vertices, which could represent for example people, and a set of edges representing connections, for example social relations. Networks provide a simple, computationally workable[30] way of modelling a wide range of phenomena, but one which allows modellers to keep the important details in the model. By explicitly capturing interactions between parts of a system, they are a natural tool to use for the modelling of complex systems.

A wide variety of sources exist for the well established mathematical theory of networks, see for example Newman [2008] for a starting point. Most of the relevant definitions are standard to network theory. So we draw on typical definitions of graphs, edges, nodes; represent them via adjacency matrices, have weighted networks, directed networks; nodes have degrees (number of edges), clustering (proportion of "friends who are also friends"). One key one which comes from Economics is the notion of *pairwise stability*: no player would be better off if she severed one of her links and no pair would benefit from adding a link. The relationship between stability defined in this way and efficiency varies.

Jackson [2010c] offers a recent overview of work in social and economic networks. Networks, he argues, provide a more realistic framework for markets that might otherwise be conceived of as centralised. Labour markets are one area where networks have been extensively applied. Networks have also been used for the modelling of learning and diffusion. He reviews the main mathematical concepts and introduces small world networks; clustering is a key way of looking at structure and homophily a key way of looking at how links depend on other characteristics of nodes. The study of network formation is a key part of the economic study of networks.

Jackson [2008] provides a concise overview of the main ideas from the perspective of Economics. Random networks studied typically of form of Poisson random networks: a link between i and j is formed with probability independent of all other such links. The resultant degree of any given node is thus binomial and well approximated by the Poisson distribution. Small world networks also a key area of study, here a regular network is rewired randomly and there is large range of parameters where network is both highly clustered but also has low average distance. Other models often considered include preferential attachment models and richer sequential link formation models.

Jackson highlights several areas where networks have an impact: diffusion, learning and bargaining/trade – where we introduce a cost to establishing trading relationships and so we no longer get Pareto optimal outcomes. There may be a

[30] `http://en.wikipedia.org/wiki/Social_network_analysis_software` provides a comprehensive, though far from exhaustive, list of software packages for network modelling and analysis of social networks.

significant *Price of Anarchy*[31].

Diffusion from a strategic perspective is focused on in Jackson and Yariv [2010], where models for marketing, collective action and fads are surveyed. They move on to look at a variety of models embedded in a social setting utilising a unified framework where we have a finite set of agents connected by a network with binary actions (e.g. infected or not); they look at infection modelling, graphical games (games where payoff depends on their own actions and those of direct neighbours) and perhaps most relevantly for the work here dynamics, albeit via the study of equilibria.

The survey paper Jackson [2010b] looks at both epidemiological-style diffusion and recent strategic diffusion models. It first examines empirical work, summarising many of the interesting results in terms of correlation with ones social network and identifying the challenges of disentangling the effects of homophily and in unobserved commonalities in nodes (typically individuals). Various asymptotic results are summarised for networks and diffusion. Then strategic results are introduced. Section 4.5 looks at dynamic results on networks, though this is a little developed area within Economics.

In Kirman et al. [2007] network formation is examined using a boundedly (spatially myopic) self interested evaluation of the marginal contribution an agent will make; but the key feature is that this calculation is limited to some level of neighbours i.e. just immediate potential neighbours (level 1), also to neighbour's neighbours (level 2) and so on. It turns out agents only need to look at level 2 as no further levels will improve upon outcomes. (Here we use a directed network and consider networks $g \subset g^N$, where g^N is set of networks ignoring links which are only unidirectional.)

Section 10.3 of Jackson [2010a] the work on markets on networks is surveyed. Bilateral trading networks have been constructed Corominas-Bosch [2004] using the idea of alternating offers bargaining in two-sided markets (buyers and sellers on a bipartite network) and finding an equilibrium where agents are very patient (that is as temporal discounting tends to zero ($\delta \to 1$)). In simple cases gains from trade are competed away to zero by the side with more agents. An inductive decomposition algorithm can be used to find limiting payoffs for more complex network structures which seemingly gives a good prediction of results from experiments Charness et al. [2007]. Jackson [2010a] also gives further examples of networked trading.

[31] The game theoretic notion of the price of anarchy, that is of the ratio of global utility between the worst Nash equilibrium outcome and social optimal, is a major concept with the computer science study of game theory, in particular on networks. It is surprisingly rare in the economics literature.

A graph-theoretic version of the classical Arrow-Debreu General Equilibrium model is formulated in Kakade et al. [2004]. There are a set of commodities, consumers (with endowments and utility functions) and a vector of prices. We have goods good(i, h) which is traditional good h sold by consumer i. We have trade restrictions modelled via an undirected graph over consumers. Each consumer has a local price vector $\mathbf{p}^i$, the prices at which each consumer is selling. Trade is budget constrained if the endowment is completely sold and value is less than sum of purchases. They look for two properties at equilibrium: consumer rationality and market clearance. Assuming utilities are continuous, strictly monotonic in every good and quasi-concave[32] and we have non-zero endowments then a "graphical equilibrium" exists.

We consider use explicit network models in chapter 4. An explicit network model may be one way of considering the kind of communities/economies we consider in chapter 6, though we do not adopt this approach, though in the penultimate section of chapter 6 we do consider agents on a lattice, which is a particular form of regularly connected network.

2.9 Evolution and Economics

An empirical approach to trying to identify what evolutionary economic research activity has taken place is carried out by Silva and Teixeira in Silva and Teixeira [2009] who use bibliometic methods to tackle the problem. The identify two extreme positions: *"history of economic thought and methodology"* and *"games"*. They highlight the lack of empirically-related work (1.4% from 1982-1991, 3.1% from 2002-2005). They claim that the use of evolutionary concepts has been increasingly associated with formal theorising; behind this "stands [...] a considerable amount of work on complex dynamical systems through computed simulation". Evolutionary publications in top rated journals restricted to *'games'* work.

A survey of some of the varying techniques used by 'evolutionary' economics is given in Safarzynska and van den Bergh [2009]. Various modelling techniques have been used. *Evolutionary game theory* originates from the work of Maynard Smith and Price; with the interactions of boundedly rational players, random selection of individuals, little or no informations about game. Here evolutionarily stable strategies is typically the central concept. Various selection dynamics, but replicator dynamics dominates (population frequency varies with payoffs). Other options include best response, imitation and mutator dynamics. Mutation can offer

[32] If $u_i(y) > u_i(x)$ then $u_i(\alpha y + (1 - \alpha)x) > u_i(x)$.

stochastic perturbation. *Evolutionary Computation* for example genetic algorithms, genetic programming, evolutionary programming, learning classifier systems. There is some kind of search process for better solutions. *Multiagent Models* See elsewhere in this document for a thorough introduction. The paper also outlines some "building blocks". *Diversity* is central to evolutionary modelling: we must have a heterogeneous population. We may also have *innovation*. *Selection* in some form is also fundamental. *Bounded rationality* and explicit modelling of decision processes often included. We may also have (related to selection) *diffusion* of behaviours/technologies/... already present in the economy. Evolutionary processes may be *path dependent* and lock in to trajectories. *Coevolution* e.g. of demand and supply may be modelled along with *multi-level selection*. The paper concludes that there is unlikely to be convergence in building blocks and modelling techniques for evolutionary economics in the near future.

In Laslier a simple "Walrasian" market is examined. The key changes to the traditional model are decentralisation, "adaptive rationality" and sequentiality. Agents assess performance on comparison with previous results, reasoning on the basis of information available to them from environment ("situated rationality"). Simulation results are given for a labour market. And this framework generalised to a wider class of models. The main difficulty reported is in generalising to groups of markets.

In Mirowski [2007] a way of thinking about markets is explicated. Mirowski defines a market as software which carries out functions like: data assimilation and communication, order routing, queuing and execution, price discovery and assignment, record keeping and so on. He lists five areas which have looked at markets as comptutational entities: mechanism design, zero-Intelligence Agent canon[33], "market microstructure" within finance, "engineering economics" and the artificial intelligence literature where it deals with markets. He argues that all these approaches fail to properly apprehend the evolutionary nature of markets and then proposes his idea of the *markomata*, rejecting the call for a model arguing that it would be equivalent to giving a general model for evolution or a computer, though he does provide a rough framework with reference to hierarchies of languages/automata. If we take his theory seriously Mirowski identifies a number of empirical consequences. In summary Mirowski [2007] is an interesting if highly speculative paper. A fuller history is offered in the (rather long, initially highly readable) Machine Dreams Mirowski [2002].

An account of evolutionary microeconomics is given in Jacques Lesourne

[33] "Canon" is exaggeration or irony.

and Walliser [2006]. Of particular interest is chapter five on mimetic interactions. Imitation is classified into

1. Informational: A imitates B who he assumes knows more.
2. Normative: conformance because of threat of punishment or existence of established norm.
3. Preferential: utility function of both individual and collective (of group) utility.

A model of binary choices is developed and numerical results presented. Some grand claims about mimetic scenarios conclude the chapter: multiple equilibria the rule rather than exception, frontiers between disciplines questioned, socioeconomics rejuvenated and more need for empirical work.

In the final two chapters of Gintis [2009b] Gintis examines dynamical frameworks for evolutionary game theory. The first introduces the idea of a *replicator dynamic*; essentially the relative frequency of a strategy increases if it has an above average payoff. One can generalise this idea to multiple dimensions.

2.10 Polarisation and Segregation

In chapter 6 I create a set of models for polarisation and/or segregation, applying many of the insights and methods from earlier chapters. In this section we look at some of the existing related work in this area. We focus on work related to microeconomic modelling of development, particularly complexity issues such as the influence of neighbours (local interactions).

2.10.1 Concepts and Context

Durlauf and Young [2001] brings together a wide variety of perspectives and approaches to the general area of social dynamics including attempts to describe general approaches, work from H. Peyton Young (we have already described the general approach featured in this volume) and Samuel Bowles (another researcher we have already featured previously in this chapter). Though it also includes ideas ranging from econometrics to a sort of game-theoretic approach to political philosophy.

Fernandez [2008] offers a survey of some of the various ways to think about culture and economics, defining *culture* as the systematic variation across groups of individuals separated by space. Neoclassical economics has largely ignored culture but some recent efforts have been made by economists to answer the question of "whether culture matters?" There is some empirical evidence (e.g. difference in economic experimental results across countries) and varying approaches survey, epidemiological approach (based on immigrants) and historical case studies.

So how does one incorporate culture into standard models? Fernandez [2008] suggests that it is "definitely" not necessary to modify the standard economic model; we can instead think of societal preferences as a choice of equilibrium strategy[34]. Though this claim seems inconsistent with later comments about the blurring of preferences and beliefs. The survey concludes with a list of difficulties with empirical work on culture. The volume Rao and Walton [2004] offers and interdisciplinary perspectives on the role culture plays in development.

In Appadurai [2004] the question of why culture matters for development and the reduction of poverty is examined. The key idea of this essay is that ideas of *the future* are embedded and nourished in culture, whereas the typical approaches focus on the "pastness" of culture. The key question is then how to build the capacity to aspire, considered as a cultural capacity.

We can think of capacity to aspire as a dense combination of nodes and pathways in a map of aspirations nurtured by real world conjectures and refutations. An extensive case study is given of a group which has managed to mobilise its capacity to aspire and in doing so has changed the terms of recognition (an alliance of housing activists in Mumbai). The essay also offers concrete steps in how to build the capacity to aspire.

The influential Wilkinson and Pickett [2009] is a recent accessible argument that inequality is responsible for a multitude of social problems; of particular interest is the third chapter which summarises work looking at how inequality may have dramatic psychological effects on individuals. This is focused on the idea that we have a certain relative status or class position and the dramatic psychological effects which this can have. Many studies have shown that we are more anxious than we used to be, for example Twenge [2000] which summarises work on changes in overall threat levels, economic conditions and social connectedness over time. Meta-analyses of existing studies showed a substantial linear increase in anxiety, with the cohort effect explaining a large proportion of the variance. While many statistics for social conditions correlated well with anxiety, economic conditions did not once other major factors were controlled for. The key conclusion is that people must feel safe and connected to others if they are have low levels of anxiety.

A key aspect of feeling secure is the lack of "social evaluative threat"; apparently the three most powerful sources of stress which effect health are *"low social*

[34]Gintis adopts a somewhat similar position vis a vis what culture might be i.e. different selected equilibria, but offers a convincing case that the "standard model" needs to be modified.

status, lack of friends and stress in early life". The sociologist Thomas Scheff goes as far as call shame the social emotion, where shame is *"the self's perception of the evaluation of the self by other(s)*. In Scheff [1988] he presents a theory of how this leads to compulsion to obey exterior norms.

The work in the above subsection is a few highlights of features which tend not to be included in economic models. While chapter 7 by no means addresses these issues in full if provides a framework which is more amenable to modelling this kind of phenomena.

2.10.2 Growth, innovation and technological change

Neoclassical macroeconomic approaches to growth assume growth levels are determined by an exogenous scientific process which runs independently of economic activity. The basic form is *AK theory* which in its simplest form (where the marginal product of capital is constant) has aggregate output

$$Y = AK$$

where A is constant, K capital level. In this formulation what matters is the saving rate s; growth rate will be

$$g = \frac{1}{Y}\frac{dY}{dt} = sA - \delta$$

where δ is a depreciation constant.

Many more sophisticated models have followed under the titles like innovation-based theory in the early nineties from economists such as Romer, Aghion and Howitt. For a fuller account Howitt [2008] is a good starting point. But typically this kind of work has been in a sense quite simplistic, particularly with respect to micro-macro relationships.

The recent Arthur [2009] explores the (surprisingly) little studied explored notion of a general theory of technology and technological change. *Technologies* are combinations of elements which are themselves technologies; they use phenomena for purposes, they 'program' nature (phenomena) for a goal. From technologies arise other technologies via *combinatorial evolution*. Arthur connects his concepts of evolving technology to an evolving economy and develops a conceptual framework for thinking about technology.

2.10.3 Aspiration and decisions

Nowhere, one might argue, is a traditional economic approach, with its emphasis on equilibrium solutions, less suited that in the modelling of fundamental socio-economic changes. Below we explore some of the existing work in economics which build (at least somewhat) dynamic models for this purpose or which include features such as segregation, polarisation and aspiration.

Genicot and Ray examine how individuals react to aspirations and how aspirations are formed in Genicot and Ray [2009]. An infinite number of families with one individual per generations which has utilities round a reference point (an "aspiration"). Income $y_t = c_t + k_t$ (consumption plus investment in child), such that $y_{t+1} = f(k_t)$. This is standard; the key idea is that utility is given by $u(c_t) + \Omega(y_{t+1}, a_t)$ where a_t is the aspiration at time t and

$$\Omega(z, a) = v(z) + w(z/a)$$

where v is intrinsic utility of target income and w depends increasingly on ration of income to aspirations.

They assume that without aspirations welfare of children not fully internalised and obtain results:

1. Target ration is increasing in aspirations ration
2. If aspirations unattained growth rates locally increasing in aspirations ratio

The really interesting question is how aspirations are formed. They consider aspirations formed as a result of ambient income distribution perhaps a prediction of the next generation's incomes or perhaps from neighbourhoods.

In Esteban and Ray [2008] a model is proposed with m groups, n_i the share of the population in each group and a distance δ_{ij} between groups i and j. In equilibrium the level of conflict can be very well represented by a linear function of inequality, fractionalisation and polarisation. The more recent Esteban and Ray [2010] explores similar ideas in this case where we have *contests*: agents can challenge system with probability of winning given by normalisation of contributions from each group, or accept the probabilistic payoff given by the system. Conflict occurs when one group receives less than it could by precipitating non-cooperation; however once we consider both occurrence and intensity of conflict, the relationship between polarisation, fractionalisation and conflict is non-linear as costly conflict may not be undertaken even under undesirable systems.

The more recent Mookherjee et al. [2010] brings together some of the authors' previous work and work from Bowles et al. [2009]. Both present *equilibrium* models of segregation and the resultant outcomes for society; both have some kind of skill accumulation which is a rational decision in equilibrium about whether to acquire high skills, where the cost of attaining these depends on an individual (intrinsic) skill factor and a local (geographic) factor. In Bowles et al. [2009] their are a set of groups (or equivalently regions; two is the case most extensively considered) whereas in Mookherjee et al. [2010] there is a geographic continuum; both consider models with a continuum of agents. We look in more detail at Bowles et al. [2009] below.

In Dalton and Ghosal [2010a] a model for decisions is proposed which attempts to deal with situations where individual's choices do not reflect what is best for them. Decisions are made relative to a reference point; an individual may internalise the effect of her choices on her psychological state, or may not. In the former case she choose an action and psychological state which maximises true best interest, a *standard decision problem*, or she may take as given her psychological state, a *behaviour decision problem*. Examples where these diverge including a student and motivation to study, default options and a smoker are given.

In this work we have a decision scenario $D = (A, P, \pi)$ with A actions, P psychological states and map π from $A \to P$ of feedback from A to P; where agents can rank over $A \times P$. Various analytical results can be derived characterising the testability and other properties of solutions, the policy implications are discussed and the model is related to literature within social psychology and philosophy.

In Dalton and Ghosal [2010b] this kind of model is used to frame poverty and aspirations. Instead of exogenous poverty traps, endogenous factors play an important role, perhaps along with exogenous factors such as initial circumstances. Aspirations are a function of chosen actions; this may be internalised or not. They may use interpersonal comparisons (perhaps only with similar individuals) in setting aspirations.

Intergenerational transfer of wealth is investigated for "small-scale" societies in Mulder et al. [2009]. The transmission varies greatly with agricultural/pastoral communities being comparable to those developed countries with strongest inheritance of inequalities such as the USA and Italy, whereas hunter-gatherer/horticultural communities are comparable to Nordic social democratic countries.

If we are thinking about how agents might determine an action via a learning process a natural technique which goes beyond the typical mean field approaches is to consider learning weighted by similarity. An objective formulation based on the notion of empirical similarity is outlined in Gilboa et al. [2010]. Probabilities are estimated empirically based on the similarity of an event to past events. The model uses a weighted euclidean distance; using the negative exponential of this as the corresponding similarity. This method is ill suited for identifying trends in data or where deduction could be used. Another key issue is if and when the similarity function should be updated.

In chapter 6 we consider in detail some work specifically relevant to the approaches adopted there but the above provides a general introduction and context for the work in chapter 6.

Chapter 3

Decentralised Exchange

3.1 Introduction

This chapter studies the limiting properties of out-of-equilibrium dynamics with decentralized exchange, that is bilateral bargaining between randomly matched pairs of agents[1]. When agents have perfect foresight, the equilibrium outcomes of decentralized exchange have been used to provide strategic foundations for competitive equilibria, see for example Rubinstein and Wolinsky [1985]; Gale [1986a,b]; McLennan and Sonnenschein [1991]; Gale and Sabourian [2005]. In this chapter, starting from an out-of-equilibrium trading scenario, the conditions under which out-of-equilibrium trading convergences to efficient allocations are characterised, and, numerically, the rate of convergence to efficient allocations is examined.

Agents are assumed to be myopic, have limited information about other agents and trading histories and engage in experimentation entailing utility losses relative to current holdings. Under assumptions on preferences that ensure pair wise optimal allocations are also Pareto optimal, it is shown that limit allocations must be efficient as long as traders trade cautiously (are aware they propose and accept trades that improve their utility evaluated at their current holdings subject to a small utility loss which is almost surely bounded over time), the proposals made are drawn from a distribution that satisfies a minimum probability weight condition and the underlying trading process is connected (any pair of agents meet with positive probability after any history of matches). Two examples are constructed to show that trade may not converge to an efficient allocation if either the minimum probability weight condition fails to be satisfied or traders do not experiment[2].

[1] Our set-up could be interpreted as modelling exchange in barter economies where the underlying fundamentals are stationary.

[2] An example of such a setting is the exchange economy studied by Scarf [1959] with a unique

Straightforwardly, the results extend to the case of production once Rader's principle of equivalence Rader [1976] is invoked.

Numerically, in economies where agents' preferences can be represented by Cobb-Douglass utility functions, we show that the rate of convergence to efficient allocations is exponential even as both the number of agents and the number of commodities are varied. We are also able to show, numerically, that the distribution of initial wealth and final wealth (initial and final endowments evaluated at limit prices) have a linear relationship with each other.

In our model of decentralised exchange, the map from action profiles to prices and allocations is well-defined out-of-equilibrium; in fact there is no 'equilibrium path' or equivalent concept as for this kind of model to be considered strategically we would have to make onerous and implausible demands on information and coordination. So properties of the dynamics studied by us can be explicitly related to the behaviour of agents. In contrast, classical approaches, for example Arrow and Hahn [1971], whether tâtonnement (without explicit out-of-equilibrium trading) or non-tâtonnement (with explicit out-of-equilibrium trading)- suffer from the problem that the price adjustment and allocation dynamics isn't explicitly based on the behaviour of agents. Such conceptual problems have important consequences. For example, tâtonnement dynamics may not always converge. Moreover, to construct a convergent non-tâtonnement dynamics typically requires that the preferences of agents be known.

Various attempts have been made to model trade in decentralised economies. Early results Feldman [1973]; Rader [1976] characterise the conditions required for for a decentralised bilateral exchange economy to converge to a Pareto Optimal allocation. Goldman and Starr [1982] derives generalised versions of these results for k-lateral exchange where exchange happens between groups of k agents. An alternative approach is the assumption of "zero intelligence" Gode and Sunder [1993]. Here there are a variety of computer agents, one form of which simply makes random offers subject to a budget constraint. They speculate that the "efficient" outcomes are due to the double auction market structure under investigation. Another angle is taken by Foley's work on statistical equilibrium, for example Foley [1999], which models an economy via discrete flows of classes, that is homogeneous classes of traders entering a market who have discrete sets of trades they wish to carry out. The result is probability distributions over trades, so as in our process agents

competitive equilibrium that is globally unstable under tâtonnement dynamics. It is shown that in an example adapted from Scarf [1959], if traders do not experiment, trading fails to converge to efficient allocations. However, once trading is augmented to allow agents to experiment and such experimentation is almost surely bounded, convergence to an efficient allocation occurs.

with identical initial endowments may end up with different final allocations, but as in the many Walrasian frameworks, but unlike our approach, the trading process remains an unspecified black box. More recently Gale [2000] has approached an out of equilibrium economy with a model with decentralized exchange in the special case with two commodities and quasi-linear utility functions.

Axtell [2005] has explored decentralised exchange from a computational complexity perspective. He argues that the Walrasian auctioneer picture of exchange is not computationally feasible, while decentralised exchange is. While this adopts a somewhat decentralised (possibly bilateral perspective) it assumes a high level of information in the groups which are bargaining (essentially a Pareto optimal outcome for that group is directly calculated) and seems to sidestep the issue of coordinating the matching of these groups.

In a related contribution Fisher [1981] studied a model of general equilibrium stability in which agents are aware they are not at equilibrium. In this chapter, in contrast to Fisher [1981] we do not require agents to hold their expectations with certainty and we allow for price setting by individual agents. Herbert Gintis has looked at an agent-based model of both an exchange economy Gintis [2006] and general equilibrium economy Gintis [2007a] although the dynamics in his models, driven by evolutionary selection, are limited to quite homogeneous agents (for example, in his exchange economy agents all have the same linear utility functions).

3.2 The model

We consider individuals who are aware they are in an out-of-equilibrium state and thus realise they may make mistakes if they were to attempt to condition their current trade based on their future expectations. In response to this agents may only accept trades which improve upon their current holdings or which disimprove in a limited way. We assume that the process is connected, that is at every time any given pair of agents will attempt exchange at some point in the future. We call this process, in a connected exchange economy, *cautious trading* and specify fully below. After the specification of the model, analytical results are presented. Examples clarifying the key conditions follow. The end of this section presents proofs of the previously presented results and shows how the model can incorporate production.

3.2.1 Specification of model

There are individuals $I = \{1, \ldots, i, \ldots, I\}$, commodities $J = \{1, \ldots, j, \ldots, J\}$ and endowments $e_i^j \in \mathbb{R}, e_i^j > 0$ of commodity j for individual i. Trade takes place in periods $t \in T = \{1, 2, \ldots\}$ and we write the bundle of commodities belonging to individual i at time t as $\mathbf{x}_{it}$ and restrict these to positive bundles (you can only trade what you currently have). Agents have strictly increasing real valued utility functions $u_i(\mathbf{x}_{it})$ which are defined for all non-negative consumption bundles[3].

In each period t two agents are matched at random with equal probability that any particular pair will be selected. We will assume that once a pair is matched the two agents put up all their current holdings for exchange. One agent, the proposer, which without loss of generality is m, proposes a non-positive[4] trade $\mathbf{z}_t$ to a responder n such that:

$$x^j_{mt} > -z^j_t > -x^j_{nt} \quad \forall j \tag{3.1}$$

and

$$u_m(\mathbf{x}_{mt} + \mathbf{z}_t) > u_m(\mathbf{x}_{mt}) - h_m. \tag{3.2}$$

The first condition is just that the trade would leave m and n with positive quantities of each good. The second condition is that the trade is utility increasing for m or disimproves by at most h_m. The responder, n, will accept the trade if it improves his utility or disimproves by at most h_n, that is

$$u_n(\mathbf{x}_{nt} - \mathbf{z}_t) > u_n(\mathbf{x}_{nt}) - h_n. \tag{3.3}$$

but reject it otherwise (in which case no trade takes place)[5]. The maximum disimprovement terms h_m, h_n will be fully specified shortly.

Note that the requirement that agents put up all their current holdings applies to a wide variety of cases. Firstly no agent is likely to have an incentive to conceal his holdings; an agent is free to reject any offer that is put on the table and by concealing some of his holdings the agent reduces the probability of generating a mutually improving trade[6]. Secondly a wide variety of conceptions of markets

[3]Formally, the trading dynamics we study in this chapter has the feature that agents do not consume till trade stops. However, following Ghosal and Morelli [2004], note that a reinterpretation of our model so that agents trade durable goods that generate consumption flows within each period will allow for both consumption and trade.

[4]That is not simply proposing a gift: it must be an actual exchange.

[5]The critical requirements are that two agents are randomly matched and a non-negative trading proposal is generated. We adopt a proposer-responder formalisation as for the later numerical work a concrete process must be specified, but a wide variety of processes are essentially equivalent to the process above. For example, the proposal could be generated by a third party.

[6]Analytically for the below convergence results we could work with a weaker condition, an upper bound on trade proposals, but the form presented here will turn out to be numerically convenient, something we will return to in section 3.3.

would make an agent's holding's public knowledge and this is a stronger requirement than the framework presented here which only requires that the proposer knows his current holdings, his utility function and the responder's holdings.

For the analytical results presented in this section attention is focused on the solution concept of $k-$wise optimality. An allocation $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_I)$ is *pairwise optimal* if there exists no way of redistributing bundles between any pair m, n that would make at least one strictly better off, while making the other at least as well off. This notion can be generalised to $k-$wise optimality in the obvious way. If $k = I$ (that is if it equals the total number of agents in the economy) then one would be considering Pareto optimality. The formulation of these concepts used here is drawn from Goldman and Starr [1982].

Let us assume that the proposals are drawn at random from the set of all proposals Z that meet the conditions specified in equations 3.1 and 3.2, such that there is a strictly positive probability of choosing a proposal within any open set $X \subset Z$. Furthermore we will assume that the random choice of a new proposal will satisfy the following *minimal probability weight condition*: there exists some $c \in (0, 1]$ such that for all periods t the probability of choosing a proposal from any open subset X of Z is greater than cp where p is the probability of choosing a proposal in X if we choose from a multivariate uniform random distribution over Z. We could actually use the weaker condition that for some strictly positive proportion of periods the original condition holds, however (at least with respect to analytical results) this would in effect mean ignoring the other periods. In section 3.2.3 we show how that without this condition it is possible to take any well behaved process and map it to one which has a strictly positive probability of never converging, despite possibly having positive probability of making a successful trade in every period.

A benchmark process that satisfies the above conditions and which will satisfy our solution concepts is a set of n agents, with Cobb-Douglas utility functions, interior endowments and uniformly random proposals over the set of improving trades (here we set $h_i = 0 \, \forall i \in I$). In Lemma 1 this is proven for a more general set of scenarios and in section 3.3 further results are obtained numerically for this and more sophisticated cases.

So far a basic model of bilateral trade has been introduced; however there is one more major element to fully specify for the model. In section 3.2.3 it is shown how the process that has been defined along the lines proposed above may under certain circumstances fail to carry out *any* trades, even where it would benefit all individuals in such an economy to do so which suggests that some kind of experimentation may be necessary for trade to occur. It also makes sense that the model can include some notion of noise, as this suggests greater robustness. It turns out one can include both ideas in a straightforward way by including an "experimentation" process in the model.

We specify a very general form of experimentation in a relatively straightforward way which can include both making mistakes and a (heterogeneous), limited amount of experimentation on the part of agents. For each agent an experimentation function f_{it} from current period in $t \in \mathbb{N}$ to a probability in $[0, 1]$ is required. Furthermore the limit as $t \to \infty$ should be 0; this is the probability in a given period that experimentation will take place. Also required is a further function h_{it} which determines the loss in utility that is acceptable in a given period (that is loss in utility for each agent engaged in trade), subject to a similar condition that the limit a $t \to \infty$ is 0, that is no loss is deemed acceptable at the limit. So now the h_m, h_n terms in equations 3.2 and 3.3 could be replaced with this more general process:

With probability f_{mt}

$$u_m(\mathbf{x}_{mt} + \mathbf{z}_t) > u_m(\mathbf{x}_{mt}) - h_{mt}. \tag{3.4}$$

and with probability f_{nt}

$$u_n(\mathbf{x}_{nt} - \mathbf{z}_t) > u_n(\mathbf{x}_{nt}) - h_{nt}. \tag{3.5}$$

otherwise as before.

However for most of the analytical results in this section the heterogeneity is unimportant as we are looking at asymptotic properties. The realised loss by agent i in period t is $\mathscr{E}(x)_{it}$ where is there is a gain in utility by i we set $\mathscr{E}(x)_{it} = 0$.

Total experimentation is almost surely bounded if the composite process described above leads to a total loss across periods t to all agents that is bounded with probability one. Formally, with probability one, the sum over losses

$$\sum_{i=1}^{I} \sum_{t=1}^{\infty} \epsilon_{it} \leq H$$

for some finite H. Any form of experimentation which ceases in finite time will trivially satisfy this condition. This is our final restriction on Cautious Trading.

To clarify these concepts consider the following examples:

1. Let $f(t) = \tilde{p}^t$ where $\tilde{p} < 1$ and fix experimentation level $h(t) = \tilde{\epsilon}$ for all agents for all t. This is almost surely bounded.

2. Fix the experimentation probability at $f(t) = \tilde{p} > 0$ for all agents, for all t and set experimentation level $h(t) = 1/t^2$. This is also almost surely bounded.

3. Fix experimentation level at some positive ϵ and fix probability of experimentation at probability $\tilde{p} > 0$. This is not almost surely bounded.

All of the key concepts have now been introduced. So to state in full, a set of n agents with strictly positive endowments carry out *Cautious Trading* if:

1. In each period t two agents m, n are matched at random (with equal probability of any particular pair being selected).

2. Agent m proposes a non-positive trade such that $x^j_{mt} > -z^j_t > -x^j_{nt} \;\; \forall j$.

3. The trade either improves m's utility level or with probability f_{mt} might reduce it by at most h_{mt}.

4. The trade is accepted if it either improves n's utility level or with probability $f_n(t)$ might reduce it by at most h_{nt}.

5. The experimentation process defined by the collection of experimentation levels $\{f_{it}\}_{i \in I, t \in T}$ and experimentation probabilities $\{h_{it}\}_{i \in I, t \in T}$ which is an experimentation process which is almost surely bounded.

6. The proposals satisfy the minimal probability weight condition.

3.2.2 Results

Below the key analytical results are presented for Cautious Trading. Proofs are given in section 3.2.4.

Lemma 1. *The Cautious Trading, with $h_{it} = 0 \, \forall i \in I, \forall t \in T$, converges in utility and the allocations converge to a set of pairwise optimal utility-identical allocations.*

It should be noted that even in this special case the limit allocation is path dependent, there is no unique pairwise optimal allocation; though it would be possible to define a process which had such a feature, it would necessitate greatly constraining possible exchanges. While the limit allocation is path dependent, it does have qualitative features which are not.

Corollary 1. *If after some finite time an exchange process begins cautious trading as in Lemma 1, then it will converge to a pairwise optimal allocation.*

So we could have any kind of initial experimentation process or trading conditioned on future expectations based on empirical distribution of trades and still obtain the same result if eventually cautious trading with no experimentation permitted commences. However it is possible to prove more general results.

Proposition 1. *Cautious trading converges with probability one to a set of Pairwise optimal allocations.*

The following proposition shows that under two additional assumptions cautious trading will get arbitrarily close to the Pareto frontier in finite time.

Proposition 2. *(i) If the utility functions are continuously differentiable on the interior of the consumption set a Pairwise optimal allocation is Pareto optimal. (ii) If indifference surfaces through the interior of the allocation set do not intersect the boundary of the allocation set then if one agent has some of all goods and others have some of at least one good then cautious trading without experimentation converges to a Pareto optimal set of allocations.*

Even if we augment the trading process with the possibility that agents may trade to boundary allocations, subject to the conditions of continuity and strict monotonicity this will never occur under the conditions specified below.

Corollary 2 (First Welfare Theorem for Cautious Trading)**.** *If utility functions are continuously differentiable on the interior of the allocation set and indifference curves in the interior of the allocation set do not intersect the boundary, the process of Cautious Trading will with probability one both*

1. *not go to an allocation on the boundary*
2. *and will converge to a set of Pareto Optimal allocations.*

3.2.3 Examples

This section presents two examples which make clear the crucial role of the *minimal probability weight condition* in obtaining convergence to pairwise optimal allocations and show how even in an example with particularly bad prospects for convergence a very low level of experimentation (or noise) may allow it to happen[7].

Example Suppose there are two agents i and j. We will set up our example such that there is a non-zero probability that trade will never occur. Consider i's proposals to j; assuming that no trade occurs the set, Z, that these are drawn from

[7] In figure 3.8 in section 3.3 this is numerically illustrated for low levels of experimentation.

will not vary with time. Furthermore we can partition the set of improving trades Z into Z_i the set of individually improving but not improving to j trades and $Z_{(i,j)}$ the set of mutually improving trades. Assume we are not at a pairwise (in this example trivially Pareto) optimal allocation and that Z_i is also non-empty. Now assume that i draws its proposals from a fixed probability distribution for each proposal in this particular state. That is it picks a $z \in Z_i \cup Z_{(i,j)}$. Now let p^i be the probability it picks a proposal in Z_i and $p^{(i,j)}$ be the probability it picks a proposal in $Z_{(i,j)}$. It has been assumed that there is a strictly positive probability of choosing a proposal within any open set $X \subset Z$, so this applies in particular to Z_i and $Z_{(i,j)}$.

Now consider a new process where we transform the probability distributions over the disjoint sets Z_i and $Z_{(i,j)}$ by a constant scaling such that $p^i_t = (1 - \tau_t)p^i$ and $p^{(i,j)}_t = \tau_t p^{(i,j)}$ where τ_t is given by the sequence $\tau_t = \frac{1}{2^{t+1}}$ for time periods $t = 1, 2, \ldots$. We make no restrictions on the behaviour if we were to leave the initial state and claim that there is now a positive probability that trade will never occur so a fortiori we will not converge to a pairwise/Pareto optimal.

To see this consider the probability of at some point proposing a trade in $Z_{(i,j)}$, that is one which will be accepted. This is strictly less than $p^{(i,j)} \sum_t \tau_t = \frac{p^{(i,j)}}{2}$, which implies there is a non-zero probability that trade will never occur. Actually to complete this argument we need j to propose in the same way. If both agents are proposing in this fashion then there is a non-zero probability that trade, and hence any kind of convergence, will never occur. Note that is is possible to generalise this to a larger number of agents by using the same weights on each distribution of proposals of i to any agent k.

While this example is somewhat pathological it illustrates an important point. For cautious trade to work we can't have agents conditioning their actions on the period in a way which essentially rules out trade at all, or via a limiting process[8].

Example One famous class of examples that show non-convergence and instability in a global competitive equilibrium is presented in Scarf [1959]. This example can be adapted for our model in a similar way to Gintis [2007a]: the basic idea is that there are three classes of agents each of whom has a utility function which is the minimum of the good it has and one other; but no agent, at least initially, can find an agent

[8]Note that in this example we have assumed that agent i needs to know the utility function of agent j. One could weaken this to an assumption of the knowledge of the forms of utility functions over an economy as a whole.

with whom a mutually improving trade can take place. To specify precisely:

$$u_1 = \min(x^1, x^2) \text{ with endowment } e_1 = (1, 0, 0)$$
$$u_2 = \min(x^2, x^3) \text{ with endowment } e_2 = (0, 1, 0)$$
$$u_3 = \min(x^1, x^3) \text{ with endowment } e_3 = (0, 0, 1)$$

This means that in the model proposed above, and similar models, no trade will ever take place.

However once one introduces a small probability ϵ of experimenting, that is proposing or accepting a disimproving trade (either deliberately or through making a mistake) then trade will take place and outcomes which are Pareto improvements over the initial state can be attained.

3.2.4 Proofs of results

Below the proofs of the results presented in section 3.2.2 are given.

Proof of Lemma 1. We know that $u_i^{t+1} \geq u_i^t$ for any agent i as only mutually utility increasing trades will be made as $F_{it} = 0 \,\forall i, t$. Furthermore the sequence of utility values is bounded as the set of feasible allocations is compact and as $u_{it} \leq \tilde{u}_i$, where $\tilde{u}_i = u_i(\sum_{k=1}^{I} \mathbf{e}_k)$. So for each agent i the sequence of utility values $u_i(\mathbf{x}_{it})$ converges to its supremum; call the vector of these $\bar{\mathbf{u}}$.

Now consider the sequence of allocations $\mathbf{X}_t$ generated by cautious trading. We claim that any limit points of such a sequence must be pairwise optimal allocations with utilities $\bar{\mathbf{u}}$. Suppose such a point wasn't then by definition there would exist a pair of agents i, j and trade vector $\mathbf{z}$ such that

$$u_i(\mathbf{x}_i^t + \mathbf{z}) > u_i(\mathbf{x}_i^t)$$

and

$$u_j(\mathbf{x}_i^t - \mathbf{z}) > u_j(\mathbf{x}_i^t)$$

But by assumption there is a strictly positive lower bound on the probability of picking a trade within every neighbourhood of $\mathbf{z}$ in every period. By continuity there exists some such neighbourhood of $\mathbf{z}$, for example an ϵ-ball around $\mathbf{z}$, $B_\epsilon(\mathbf{z})$ for sufficiently small ϵ , which pairwise improves (there may in fact be additional regions of our allocation space where this holds) so we know that a trade will almost surely happen at some point in the future between these two agents and so this cannot be an allocation at $\bar{\mathbf{u}}$. $\square$

Proof of Proposition 1. For a particular realisation let $\mathbf{x}_i^t$ be the current allocation of agent i at time t, u_i^t the utility of agent i at time t. Let $\mathscr{E}(x)_{it}$ be the loss in utility to agent i in period t. (If no experimentation occurs in period t for agent i then $\mathscr{E}(x)_{it} = 0$ as before.) By assumption the total amount of experimentation of all agents is almost surely bounded, so for any particular agent $\sum_{t=o} \mathscr{E}(x)_{it}$ is also bounded.

Let the sequence v_i, indexed by t, be given by $v_i^t = u_i^t + \sum_{k=1}^{t} \mathscr{E}(x)_{it}$. Then this new sequence v_i is increasing. It is also bounded as it is the sum of two bounded sequences. Therefore it converges to a limit, say $\tilde{v}_i$. But this implies that u_i also converges to some limit $\tilde{u}_i$.

Now consider once more allocations at this limit $\tilde{u}_i$. They must be pairwise optimal as if they weren't then a pairwise improving trade would be made at some point in the future, even without experimentation to perturb the state. □

Proof of Proposition 2. Let $\mathbf{X}$ be a pairwise optimal allocation in the interior of the allocation set. If we are in the interior of the allocation set then by assumption marginal rates of substitution exist for each agent i and for each pair of goods m, n. Let i, j be any two agents and m, n and two goods. Now consider the marginal rates of substitution for agents i, j of goods m, n, that is $MRS_i(m, n)$ and $MRS_j(m, n)$. If $MRS_i(m, n) \neq MRS_j(m, n)$ then a pairwise improvement is possible via an exchange of the two goods m, n between i, j. So we much have $MRS_i(m, n) = MRS_j(m, n)$ for all agents i, j and goods m, n. But if this holds then the current allocations $\mathbf{X}$ is Pareto optimal. From proposition 1 we know that the sequence converges to a set of Pairwise optimal states, so under the extra conditions imposed above it converges to a set of Pareto optimal states.

Now consider the case where one agent has some of all goods, without loss of generality let this be agent 1 and others have some of at least one good, without loss of generality let this be good 1. We need to establish that the process reaches the interior of the the allocation set then the result follows by the above argument.

Consider an agent $i \neq 1$ and the non-empty set $M = \{m \in J | x_i^m = 0\}$. As the indifference curves through the interior of the allocation set for all agents do not intersect the boundary of the set it is always in the agent's interest to accept a trade away from the boundary, that is a trade z such that for each $i \in M$, $z_i \neq 0$. When paired with agent 1 there exists an open set of trades Z which leaves him with some of all goods and improves the utility of agent 1. So eventually such a trade will happen. This argument trivially extends to all agents on the boundary, so with probability one in finite time we will reach an allocation in the interior of the allocation set. □

Proof of Corollary 2. To go to an allocation on the boundary with the above conditions an agent must in effect accept an infinite loss in utility; but with probability one this will not occur as it is assumed that the total amount of experimentation is almost surely bounded, so the loss in any particular period must also be bounded.

If the utility functions are continuously differentiable then any pairwise optimal allocation is a Pareto optimal allocation as the marginal rates of substitution of goods for each agent must be equal. By proposition 1 the process converges to a set of pairwise optimal allocation allocations, so with the additional assumption this is Pareto optimal. □

3.2.5 Extension to Production

Our convergence results for exchange can be extended to economies with production using the process described in Rader [1964, 1976]. Formally an exchange economy is an array $\{(u_i, \mathbf{e}_i, \mathbb{R}^J_+)\colon i \in I\}$. An economy with production is an array $\{(u_i, \mathbf{e}_i, \mathbb{R}^J_+) : i \in I; (Y^f) : f \in F, \theta_{if} : f \in F, i \in I\}$ where $f \in F = \{1 \ldots F\}$ is the set of firms and θ_{if} is individual i's share in firm f with $\sum_i \theta_{if} = 1, \forall f$. Assume that the production set Y^f of firm f is convex, non-empty, closed, satisfies the no free lunch condition ($Y^f \cap \mathbb{R}^J_+ \subset \{0\}$), allows for inaction (that is $0 \in Y^f$), satisfies free disposal and irreversibility (that is if $y \in Y^f$ and $y \neq 0$ then $-y \notin Y^f$). We can convert an economy with production to an economy with household production by endowing each individual i with a production set $\tilde{Y}_i = \sum_f \theta_{if} Y^f$. Next, by using Rader's principle of equivalence Rader [1976], an economy with household production can be associated with an equivalent economy with pure exchange with indirect preferences defined on trades. The conditions under which pairwise optimality implies Pareto optimality with such indirect preferences follow directly from Theorem 2 and its applications, also in Rader [1976].

3.3 Numerical Results

While we have shown that the sequence of allocations will converge to a Pareto optimal set, this does not answer the question of how long such a process will take to get close to Pareto optimal. This section examines this question via a numerical approach, showing that for a common class of utility functions, the average speed of convergence is, in a sense to be specified shortly, good[9]. This section also examines

[9]This section has been written so as to be as accessible as possible to the non-programmer. Those with experience of programming may wish to skim this section, while consulting the source code directly, the key sections of which are included in the appendix.

the question of how the cautious trading process affects the wealth of agents.

3.3.1 Numerical Model

Attention is focused on sets of heterogeneous agents with Cobb-Douglas preferences and random initial endowments as a benchmark case. We can represent the preferences by utility functions:

$$u_i(\mathbf{x_i}) = \sum_j \alpha_i^j \ln(x_i^j).$$

One can of course represent Cobb Douglas utilities by $u_i(\mathbf{x_i}) = \prod_j (x_i^j)^{\lambda_i^j}$. However, the logarithmic representation is preferred for numerical work because it has a considerably lower computational cost. We have initial endowments, $\mathbf{e}_i^j$, of each commodity drawn from a uniform distribution over $(0, 1]$ and parameters α_i^j of the functions are again drawn from $(0, 1]$ uniformly, then normalised such that the sum, $\sum_j \alpha_i^j = 1$. They are normalised to a fixed value so as to make talking about global utility as the sum of agent's utilities more meaningful; this does not change the preferences which they represent.

As before trades are restricted to the set of all trades which leave both proposer i and responder j with positive quantities of each good, that is:

$$-x_{mt}^j < z_t^i < -x_{nt}^i$$

as to actually implement the trading process it is necessary to fix some boundary values[10]

The key objects we need in our computational model is an agent and a collection of agents. The former implements agents with Cobb-Douglas utility functions as specified above, random initial endowments and importantly specifies the actual mechanics of trade proposals, acceptance or rejection and trades. The later creates a collection of these agents and carries out realisations of the economy. A schematic representation of these classes can be found in figure 3.1. Utilising these we can obtain various numerical results via processes such as that illustrated in figure 3.2.

We make one further major assumption: each agent makes one proposal per

[10] An alternative, and in some ways more satisfying alternative (as it limits required information), might be to restrict trades to within the total endowment of the economy. While analytically we would obtain the same asymptotic results, numerically it would simply lead to many rejected proposal and vastly longer running times if these were simulated directly. One could try and simulate the proposal process indirectly if one could formulate joint probability distributions over improving offers, over improving proposals and over agent pairing. However for anything other than trivial economies this is extremely difficult due to the number of dimensions and changing state when proposals accepted.

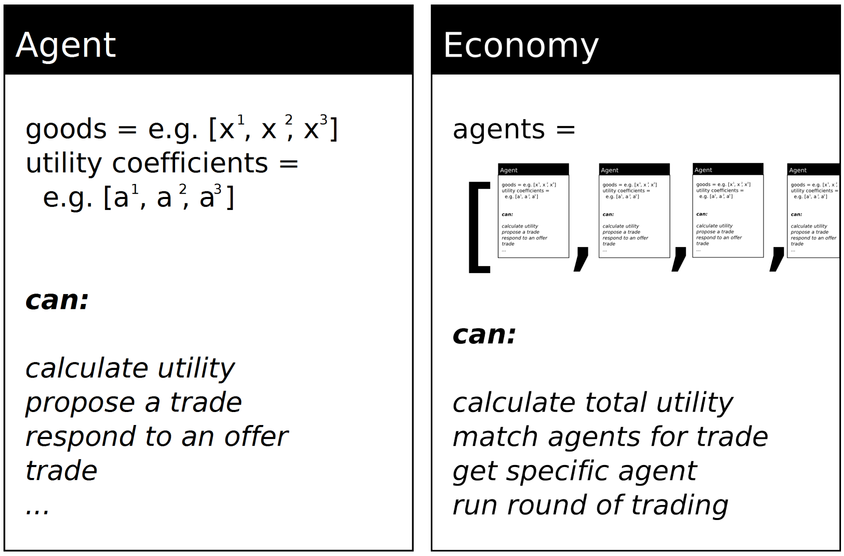

Figure 3.1: An outline of the main attributes and methods of the *Agent* and *Economy* objects.

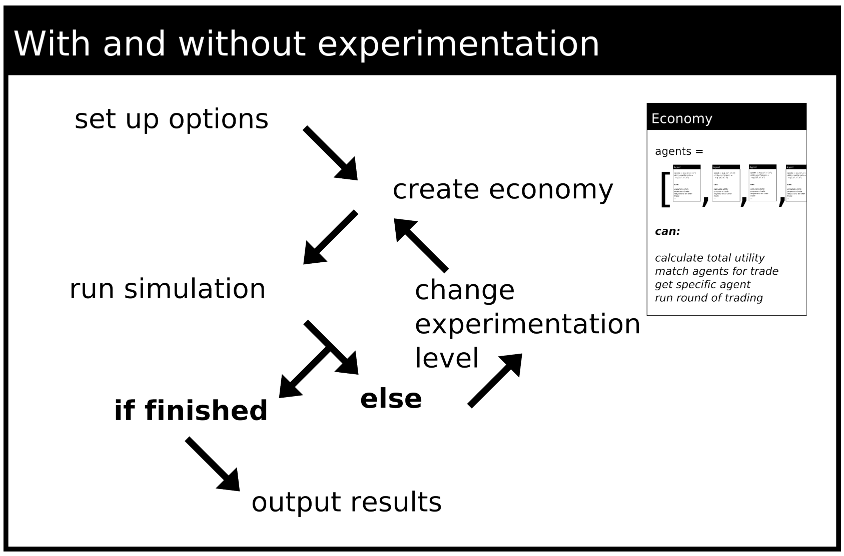

Figure 3.2: An example algorithm of a numerical simulation of Cautious Trading. The precise details vary depending the experiment being carried out but this example gives an overview of the kind of algorithm used to generate the data for most of the figures in this chapter.

round, irrespective of the size of the economy. An alternative way to approach implementing this model might be to fix some n, perhaps $n = 1$ as the total number of proposals per round, with agents drawn at random in each round. However, if one takes seriously the decentralisation of the economy, then one should assume that the agents actions are unconstrained by the size of the global economy.

3.3.2 Results

Attention was focused on estimated convergence in average global utility to assess the performance of cautious trading. To estimate this we calculate global utility by summing across utility for all agents in the economy, then take an average over many runs as the process is stochastic. One can then use the final value as an estimate of limiting utility and calculate how far away earlier values are. The last few hundred values are discarded as for them this estimate of limiting utility is not, relatively speaking, as good. The analysis depends on the increasing nature of sequences of utility values for agents this analysis to make sense.

In figure 3.3 one can see how varying the total number of agents effects the average speed of convergence. As one can see there is in fact very little qualitative effect. There is some increase in time taken, however when one plots the log of average convergence as in figure 3.4 one can see that we get a close approximation to a straight line after an initial faster period; suggesting an exponential speed of convergence, at least over the these time periods.

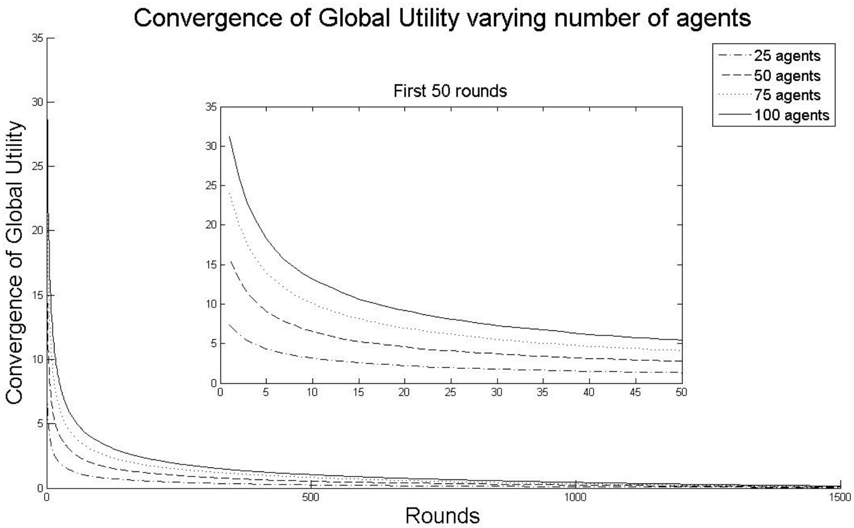

Figure 3.3: Average over many runs of global utility convergence when varying the total number of agents in the economy. *Parameters: 5 goods, 25-100 agents.*

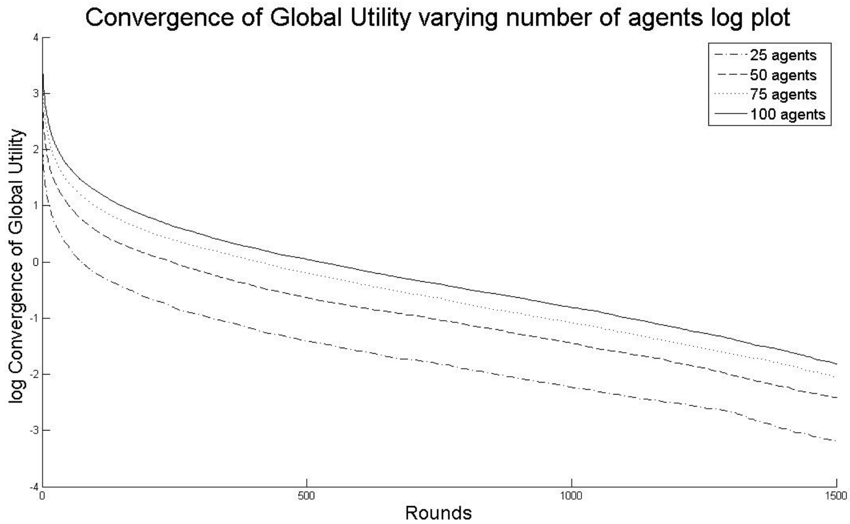

Figure 3.4: Log of average global utility convergence when varying the total number of agents in the economy. *Parameters: 5 goods, 25-100 agents.*

We also examined the effect of the number of goods via similar analysis. In figure 3.5 one can see how the speed of convergence varies with the total number of goods in the economy. There is a similar result of little qualitative change. This is more surprising as we have the same number of proposals taking place as before over larger increasing numbers of goods. When one examines the the log plot in figure 3.6 one gets the same kind of result as for varying agents.

One can fit an exponential function, via regression on the log of the values, to these average utility paths in order to obtain a numerical estimate for the average speed of convergence. In tables 3.1 and 3.2 we present such results for a range of model sizes. The important point to note is the approximately exponential convergence in global utility for a range of sizes of economy, both in terms of number of goods and number of agents, rather than the actual fitted parameters. Note that the p-values for the regressions are less than 0.0005 indicating an extremely high level of confidence in the fit of the model.

Goods	4	5	6	7
linear coefficient:	-0.0026	-0.0019	-0.0021	-0.0021
constant term:	6.3642	5.6688	5.9079	6.3833

Table 3.1: Fitting exponential function to average convergence for varying numbers of goods. The values given are for linear fit of log of convergence. For every fit the p-values for the regressions are less than 0.0005 indicating an extremely high level of confidence in the fit of the model.

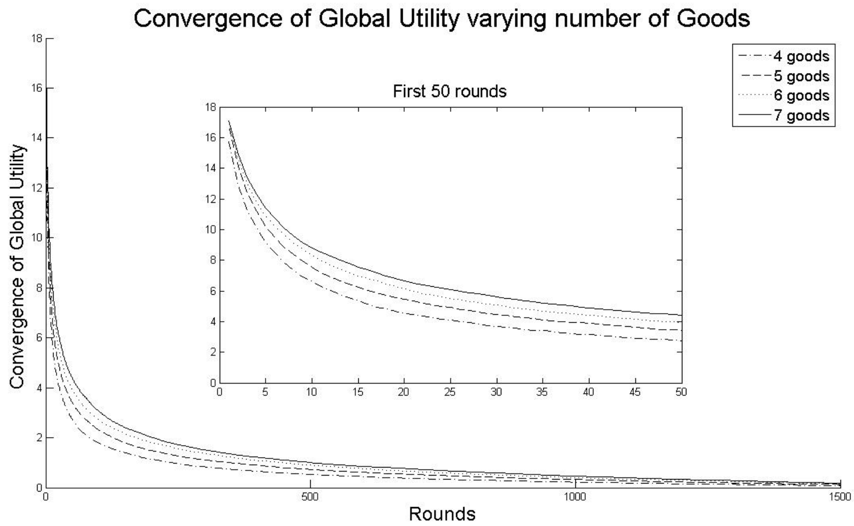

Figure 3.5: Average global utility convergence when varying the total number of goods in the economy. *Parameters: 4-7 goods, 50 agents.*

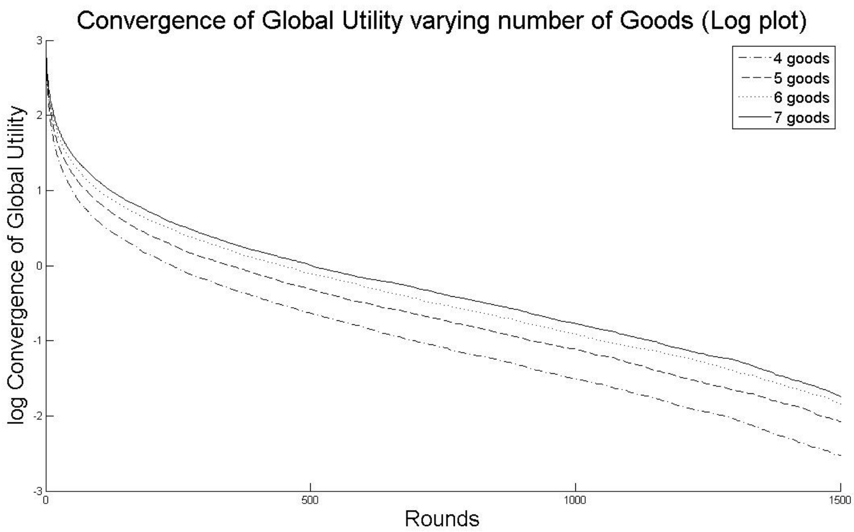

Figure 3.6: Log of average global utility convergence when varying the total number of goods in the economy. *Parameters: 4-7 goods, 50 agents.*

Agents	25	50	75	100
linear coefficient:	-0.0023	-0.0022	-0.0022	-0.0023
constant term:	4.4728	4.3750	5.3190	5.2299

Table 3.2: Fitting exponential function to average convergence for varying numbers of agents. The values given are for linear fit of log of convergence. For every fit the p-values for the regressions are less than 0.0005 indicating an extremely high level of confidence in the fit of the model.

Another aspect of the exchange process we can analyse numerically is that of wealth dynamics or change. If we had a single set of prices or relative valuations $\mathbf{p}$ in the economy, then we could obtain the wealth of an agent i, simply be calculating $\mathbf{p}\mathbf{x}_i$. In our out-of-equilibrium scenario there is no single set of prices, however given that the marginal rates of substitution converge, this implies a convergence to a uniform set of relative evaluations (in terms of changes in utility); in effect a common set of prices. From this set of prices we can calculate a value for each agent's bundle. But if we know their original endowment we can calculate their initial wealth using these prices, so we can obtain a set of relative *ex post* wealth values.

In figure 3.7 the density of wealth change is plotted. We see a linear relationship of final to original wealth, but with a very high level of noise, as one might expect. It should be emphasised that in all cases utility for every agent increases over time, however 'wealth' may change in either direction.

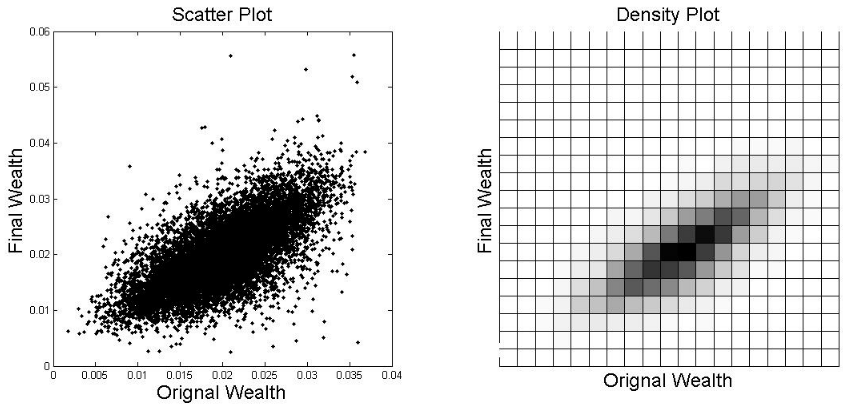

Figure 3.7: This uses the average of the final marginal rates of substitution to obtain an estimate for initial and final wealth. The darker the square the more likely such a transition from initial to final wealth in that square is; this heatmap was included as the scatter plot is difficult to read due to the high number of samples with similar changes in wealth *12500 samples (or 250 realisations of 2000 periods, with 50 agents; normalised per realisation.*

In figure 3.8 we take the example outlined above in section 3.2.1 and examine what happens numerically, introducing a small probability ϵ of making a mistake, that is proposing or accepting a disimproving trade. If no experimentation takes place no trade ever happens and global utility remains at 0. As we increase the level of experimentation short term global utility improves (rises more steeply) at the cost of a lower level of long term convergence. In cautious trading form nothing happens, but with experimentation trade happens.

For high values of experimentation faster initial improvement than low values, but longer term global utility is slightly lower and the economy more volatile. This suggests that in selecting the level of experimentation there is a trade off between convergent level of utility and speed of convergence.

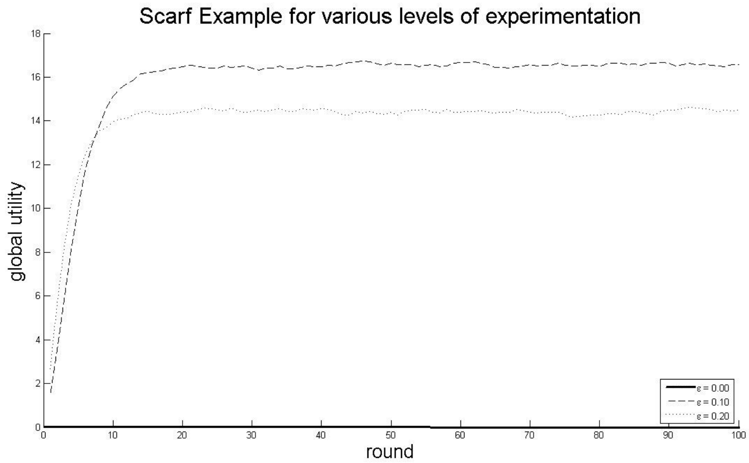

Figure 3.8: Without experimentation no trade takes place in this model adapted from a model of Scarf. Notice how long term utility appears lower for a higher level of experimentation.

Above an example adapted from Scarf was presented which showed how experimentation could lead to a better outcome than before, however, this example is a very special case. An interesting question we can ask numerically is how experimentation effects the speed of convergence in a larger, more heterogeneous example such as the Cobb Douglas utility function economy we looked at previously. In fact there is qualitatively similar long term behaviour when experimentation is included as can be seen in figure 6 where experimentation is introduced into the original model from section 3.3. For certain values of experimentation we even see slightly better overall performance with experimentation.

So we have seen that cautious trading allows trade to occur when it would not have otherwise happened. Furthermore, in economies where experimentation is not required, such as the Cobb-Douglas economy in figure 3.9, experimentation does not appear to have a qualitatively detrimental effect.

3.4 Conclusions

Even with "zero information" an exchange economy with typical assumptions will converge to a Pareto optimal outcome purely through bilateral exchange among

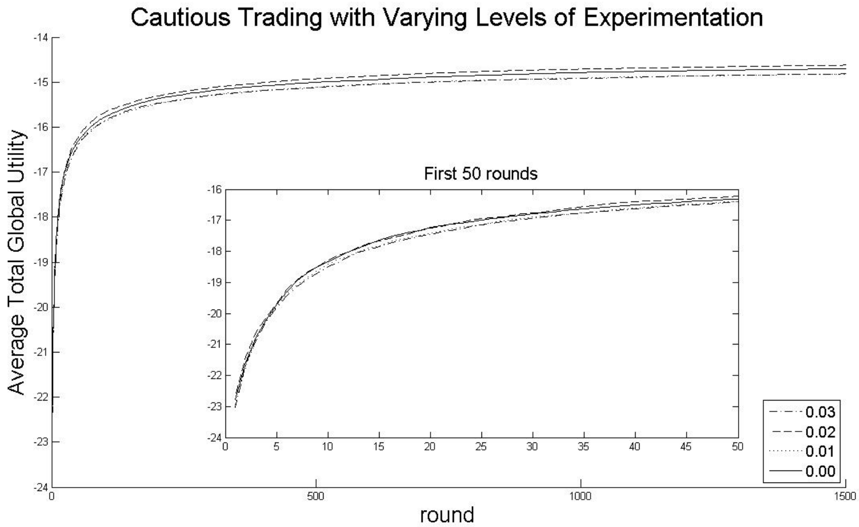

Figure 3.9: Low levels of experimentation have little effect in the Cobb Douglas economy we looked at before.

uninformed partners. It is possible to numerically examine the speed of convergence which turns out to be exponential for a typical class of utility function. Augmenting this process with experimentation leads to both convergence in some examples where it did not previously occur and potentially faster convergence in cases which did converge previously.

One can conceive of this 'zero information' as a worst case assumption. In 'real' markets one presumably has more to work with but almost never the kind of complete information that is typically assumed in comparable models of exchange. The dual discipline of having to deal with decentralisation and its resultant lack of information (not simply uncertainty over a small number of possible states of the world) and having to explicitly implement the models for numerical investigation has proved useful. In the next chapter we look at the structure of trade by explicitly modelling trading networks, contrasting the results with those of this chapter where in effect we assume a fully connected network.

Chapter 4

Exchange on Networks

4.1 Networks

In 2.8 we examined general background and related literature on networks. A network approach is the natural model for a situation with connected agents and in this chapter we extend the model from 3 to an explicit exchange network model. While this is by no means a realistic model of exchange, it does allow us to consider how network structure can affect even idealised trading. Figure 4.1 shows some simple example of network 'types'; in this chapter we will look at how these differing structures can effect exchange, but first we extend our model to networks and examine the new model analytically.

4.2 Trading on Networks

In the previous chapter we have assumed a anonymous, fully connected economy, with trading partners picked at random from all agents in the economy. But an attempt to investigate decentralised economies would be incomplete without a consideration of if and how the structure of that economy affects outcomes. The natural way to think about this is in terms of a network of agents with edges representing potential trading partners[1]. So we have a undirected graph $G = (V, E)$, where V is the set of vertices (agents) and E the set of edges (potential trading links). For a more general survey of network concepts see section 2.8.

Agent i has endowment e_i as before, however we now restrict offers and

[1]We consider only static networks, however if one is conceiving of an exchange economy where the cautious trading process is one step repeated with changes to the fundamentals of the economy at each step, then there is no reason why the network topology couldn't be considered a fundamental to be altered at each step.

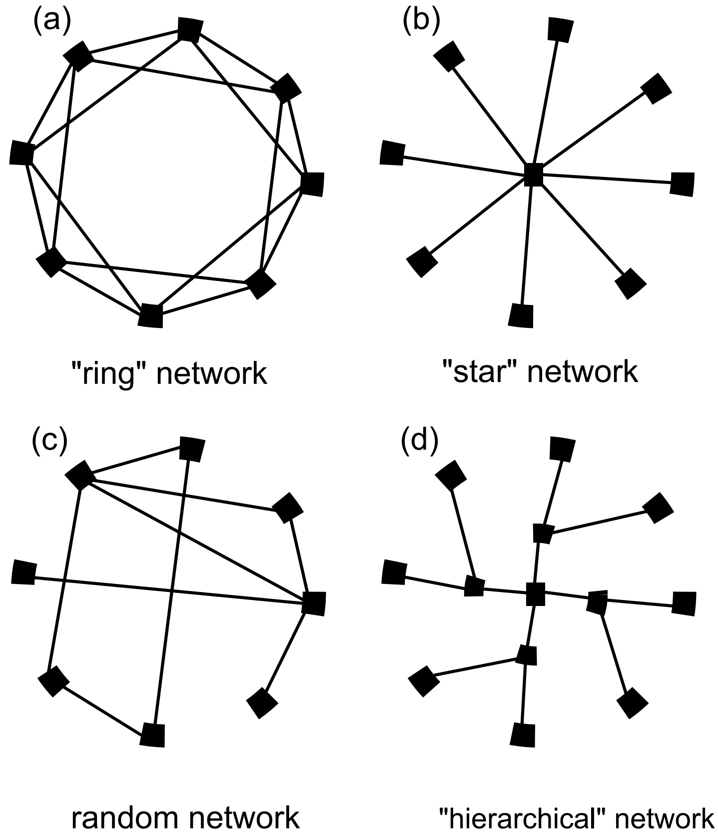

Figure 4.1: This figure illustrates some network structures. Note that simulations were run on networks of much larger size.

trade to pairs of agents connected by an edge $d \in E$. We call this *Networked Cautious Trading.* We can use the same very general formulations for proposals and acceptance as before, however our *"local"* optimality will have to be redefined as follows: an allocation $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_I)$ is *connected-pairwise optimal* if there exists no way of redistributing bundles between any connected pair m, n that would make at least one strictly better off, while making the other at least as well off. If we have a fully connected[2] graph then we would be considering *pairwise* optimality, as in the previous chapter. We will once more consider *Pareto optimality* as a solution concept, where we allow redistribution over all agents irrespective of location on the network.

4.3 Analytical Results

We can reformulate the analytical results for Cautious Trading in the context of networks, as below. We need to replace the notion of pairwise optimality with connected-pairwise optimality and take account of various networks specific issues.

Lemma 2. *Networked Cautious Trading, with $h_{it} = 0\, \forall i \in I, \forall t \in T$, converges in utility and the allocations converge to a set of connected-pairwise optimal utility-identical allocations.*

Proof. We know that $u_i^{t+1} \geq u_i^t$ for any agent i as only mutually utility increasing trades will be made as $h_{it} = 0\, \forall i, t$. Furthermore the sequence of utility values is bounded as the set of feasible allocations is compact and as $u_{it} \leq \tilde{u_i}$, where $\tilde{u_i} = u_i(\sum_{k=1}^{I} \mathbf{e}_k)$. So for each agent i the sequence of utility values $u_i(\mathbf{x}_{it})$ converges to its supremum; call the vector of these $\bar{\mathbf{u}}$.

Now consider the sequence of allocations $\mathbf{X}_t$ generated by networked cautious trading. We claim that any limit points of such a sequence must be connected-pairwise optimal allocations with utilities $\bar{\mathbf{u}}$. Suppose such a point wasn't then by definition there would exist a connected pair of agents i, j and trade vector $\mathbf{z}$ such that

$$u_i(\mathbf{x}_i^t + \mathbf{z}) > u_i(\mathbf{x}_i^t)$$

and

$$u_j(\mathbf{x}_i^t - \mathbf{z}) > u_j(\mathbf{x}_i^t)$$

But by assumption there is a strictly positive lower bound on the probability of picking a trade within every neighbourhood of $\mathbf{z}$ in every period. By continuity

[2] There is an edge linking every agent to all all other agents.

there exists some such neighbourhood of $\mathbf{z}$, for example an ϵ-ball around $\mathbf{z}$, $B_\epsilon(\mathbf{z})$ for sufficiently small ϵ , which pairwise improves (there may in fact be additional regions of our allocation space where this holds) so we know that a trade will almost surely happen at some point in the future between this pair of connected agents and so this cannot be an allocation at $\bar{\mathbf{u}}$. □

Proposition 3. *Networked Cautious Trading converges with probability one to a set of connected-pairwise optimal allocations.*

The proof here is omitted as the experimentation is assumed to be independent of network structure, so the argument in the proof of proposition 1 still holds, reinterpreting in the light of Lemma 2.

Proposition 4. *Let us assume our network is connected. (i) If the utility functions are continuously differentiable on the interior of the consumption set a connected-pairwise optimal allocation is Pareto optimal. (ii) If indifference surfaces through the interior of the allocation set do not intersect the boundary of the allocation set then if one agent has some of all goods and others have some of at least one good then Networked Cautious Trading without experimentation converges to a Pareto optimal set of allocations.*

Proof. Let $\mathbf{X}$ be a connected-pairwise optimal allocation in the interior of the allocation set. If we are in the interior of the allocation set then by assumption marginal rates of substitution exist for each agent i and for each pair of goods m, n. Let i, j be any two agents and m, n and two goods. Now consider the marginal rates of substitution for any set of connected agents i, j of goods m, n, that is $MRS_i(m, n)$ and $MRS_j(m, n)$.

If $MRS_i(m, n) \neq MRS_j(m, n)$ then a pairwise improvement is possible via an exchange of the two goods m, n between i, j. So we much have $MRS_i(m, n) = MRS_j(m, n)$ for all pairs of connected agents i, j and goods m, n. But if this holds then the current allocations $\mathbf{X}$ is Pareto optimal if the network is fully connected. From proposition 1 we know that the sequence converges to a set of Pairwise optimal states, so under the extra conditions imposed above it converges to a set of Pareto optimal states.

Now consider the case where one agent has some of all goods, without loss of generality let this be agent 1 and others have some of at least one good, without loss of generality let this be good 1. We need to establish that the process reaches the interior of the the allocation set then the result follows by the above argument.

Consider an agent $i \neq 1$ and the non-empty set $M = \{m \in J | x_i^m = 0\}$. As the indifference curves through the interior of the allocation set for all agents do not intersect the boundary of the set it is always in the agent's interest to accept a trade away from the boundary, that is a trade z such that for each $i \in M$, $z_i \neq 0$.

Agent 1 must be paired with at least one other agent, without loss of generality agent 2. For agents $1, 2$ there exists an open set of trades Z which leaves 2 him with some of all goods and improves the utility of agent 1. So eventually such a trade will happen between these agents. So with probability one in finite time we will reach an allocation for agents 1 and 2 in the interior of the allocation set. This argument extends inductively to any finite size of connected network, so we have our result. □

Corollary 3. *If utility functions are continuously differentiable on the interior of the allocation set and indifference curves in the interior of the allocation set do not intersect the boundary, then Networked Cautious Trading will, with probability one, both not go to an allocation on the boundary and will converge to a set of Pareto Optimal allocations.*

The same argument for this holds as for corollary 2 as the potential move to boundary is independent of the network (consider the sub-network of an agent and its immediate neighbours, the original argument holds; and as trade only takes place with neighbours, this gives us the result).

The result of Pareto optimality is in general a weak optimality condition, it explicitly takes no account of issues relating to equity of distribution; in a world with initial uniformly random endowments and symmetric trading potential, this is trivially equitable (at least in terms of expectations of final utility). However, if we have an asymmetric network structure of trading this is more obviously a fundamental concern, in contrast to the Cautious Trading case where we assume symmetry. There is little to say in general about such a network as if the structure matters for outcomes then we presumably will not obtain general results on outcomes. By adopting a numerical approach we can attempt to characterise the effect of network structure on expected outcome.

4.4 Numerical Results

Asymptotically Cautious Trading on a network has similar properties to anonymous, pairwise matching; though the question we want to examine here is how network structure effects outcomes. There may also be significant differences in properties like initial convergence which we are able to examine numerically. In figure 4.1 some different structures of networks were illustrated: a ring network, where each vertex has an edge joining it to each of its nearest neighbours; a star network where one vertex is joined to all others; a random network and a hierarchical network which has a hierarchy of star-like components. All these networks are connected, but far from fully connected as was assumed to be the case for the original formulation of cautious trading in chapter 3. In any case if an economy were not connected then we would really be dealing with two or more economies.

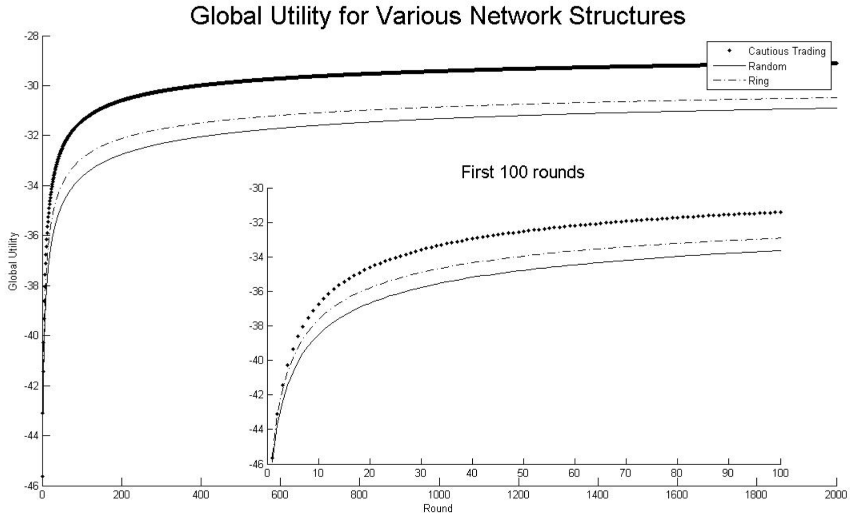

Figure 4.2: Here we have three quite different structures for our economy; however the average path of the economy if very similar. We have the original Cautious Trading, a uniformly randomly connected graph (with an average of four edges per vertex) and a ring graph (each agent is connected to it four nearest neighbours). These all have quite different properties but seemingly due to the reasonably high level of connectivity result in the same average behaviour. *Averaging over* 2000 *realisations,* 100000 *proposals per realisation,* 50 *agents.*

The key issue for our model is the level of connectivity as illustrated in figures 4.2 and 4.3. In the first figure extremely similar results are obtained for the original cautious trading process, cautious trading on a “ring” network and cautious trading on a random network. In figure 4.3 the results are quite different for networks with lower levels of connectivity. In summary, if connectivity is low, as is the case for a “star” network the speed of convergence is reduced; if it is sufficiently high (for the sizes of networks examined in this chapter this means around four edges per vertex) then the results are similar to the fully connected scenario, with the actual structure of connection having little effect.

We looked at varying levels of clustering using the Watts and Strogatz model for small world graph generation Watts and Strogatz [1998]; basically we start off

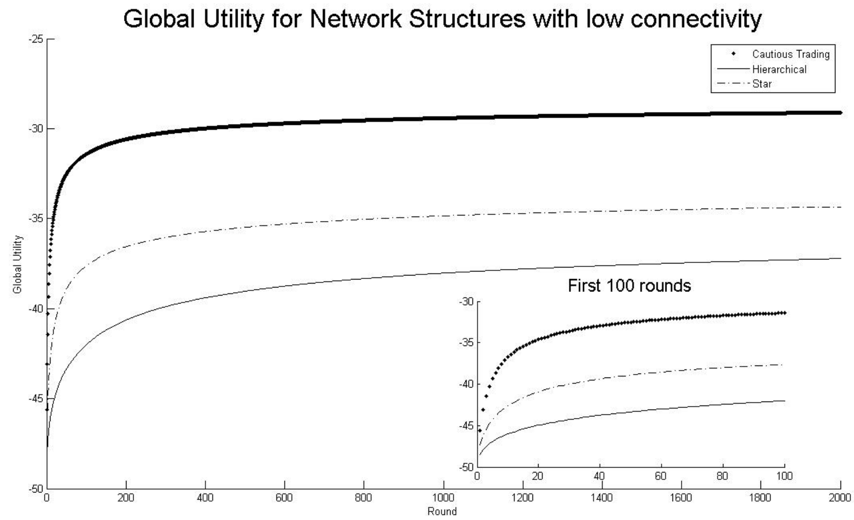

Figure 4.3: In contrast to figure 4.2 we see quite different average behaviours for these networks versus the original cautious trading. These networks are a star network in which one agent is connected to all others (an idealisation of a central market in which all agents exchange goods) and a hierarchical network (where there is a 'central market' which is connected to a small number of other 'markets', in turn connected to all the other agents in the economy). These networks have low levels of connectivity (roughly one edge per agent) and this seemingly restricts the speed of converge and lowers global utility. *Averaging over* 2000 *realisations,* 100000 *proposals per realisation,* 50 *agents.*

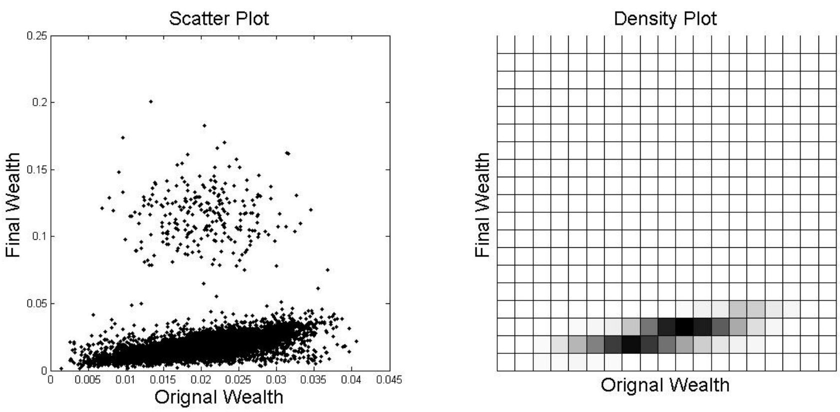

Figure 4.4: These plots show the change in wealth, using the average of the final marginal rates of substitution to obtain an estimate of changes in wealth for a networked (star network) economy *12500 samples (or 250 realisations of 2000 periods, with 50 agents; normalised on a per realisation basis such that the total wealth sums to one.*

with a ring network (agents are connected to their nearest neighbours) and rewire each edge with a fixed probability β. We obtained similar results to figure 4.2, that is there was little effect at a global level if we kept connectivity constant (as the Watts-Strogatz model does by construction).

Examining wealth change in a centralised economy (a star network) there is a substantial contrast with the original cautious trading model, now final wealth appears to be more random, with less correlation with initial wealth as can be seen in figure 4.4. Again there would appear to be a high level of noise in the system, with many outliers.

It might typically be expected that a more centralised system would be more efficient as for example the shortest path length between any two agents with the potential for mutually beneficial trade is shortened. For example Markose et al. [2004] with a simpler form of trading but with endogenous learning finds a much more structured outcome. However, in the case of a model with direct, bilateral trading it seems that this may not be the case as intermediary agents extract the gains from trade that might otherwise be more equitably shared[3]. We address this issue in the next section where we look at a version of cautious trading where the network structure is itself endogenous.

4.5 Endogenous Network Formation

There are many analytical results for network formation, see for example Jackson [2008] or for a general overview Jackson [2010a], these kind of approaches typically require stronger assumptions than it is reasonable to make for our kind of highly decentralised, limited information, dynamic model. Instead we examine numerically an extension of the cautious trading model whereby agents can break-off their low performing edges and connect to new agents.

Each agent i has a set of edges $D \subseteq E$ connecting it to a set of neighbours (potential trading partners) N_i. In this section we consider a version of cautious trading on a directed network with multiple rounds of trading. Each agent commences each round with an individual, but constant across rounds[4], endowment. Intuitively we can think of farms producing a certain type of produce each year or factories producing certain goods each day; this amount is assumed to be constant for a set of heterogeneous individuals. They start with a fixed number of randomly assigned edges which are directed in the proposal direction. They track the utility

[3]Of course a trading structure explicitly structured with social objectives should do better.

[4]Otherwise any learning or adaption in behaviour wouldn't make much sense.

gained from trades as a result of their proposals over each edge and at the end of a round they disconnect their lowest performing edge, averaged over all rounds in which they have been connected to that agent[5], and reconnect to another agent at random. While the out-degree of each agent remains constant, the in-degrees can vary and it is this which allows us to think about how agents adapting their local network structure in a self-interested way might determine global network properties.

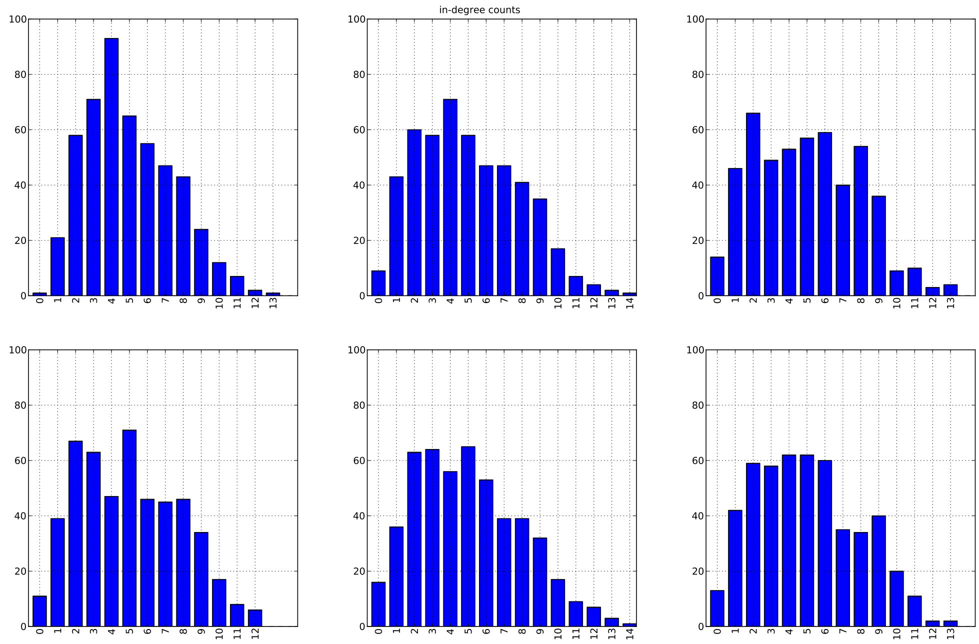

Figure 4.5: The in-degree counts for the first sixty rounds. Each round consists of 200 periods, where 50 agents propose a trade to each of their 5 current neighbours, at the end of each round the lowest performing neighbour (lowest utility gain from trades averaged over all rounds) is replaced by a new neighbour. The network structure shifts from the initial configuration (as randomly assigned) to one with a greater variance in in-degree (out-degree is kept constant and link formation is unilateral).

We find, as shown in figure 4.5 how the in-degree counts (aggregated over windows) quickly shift to a broader structure (than the initial random distribution of links). High in-degree counts are avoided as they seem to be too competitive (opportunities for the proposer to gain from the responder are reduced by the larger number of competing offers). If we look at the variance of in-degree, as in 4.6 we

[5] I also looked at a version where only the last round is taken into consideration; this results in a considerably noisier outcome.

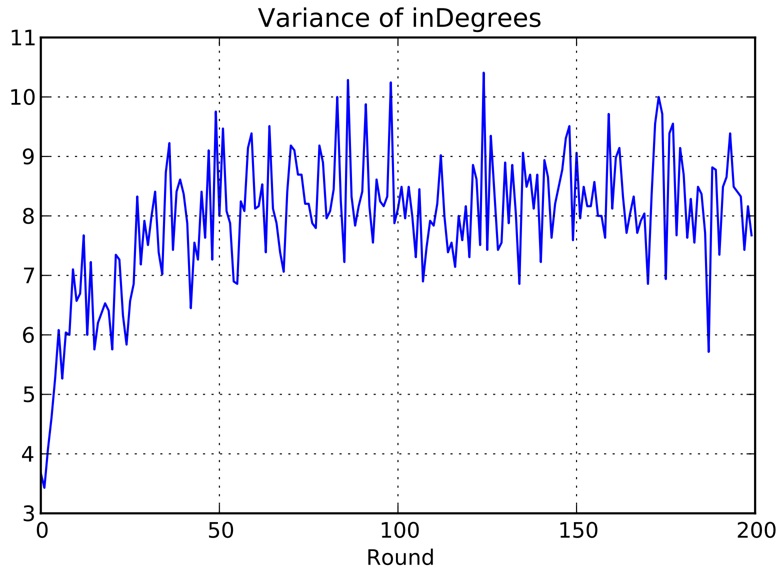

Figure 4.6: This plot shows how the variance of in-degree count by agent starts out low, then within around 50 periods converges on a value that will stay more of less constant (with substantial noise).

can see that it rapidly increases initially, reflecting the endogenous shift in network structure as seen in figure 4.5, and then remains at a roughly constant level (though with substantial noise). Similarly the aggregate in-degree counts remain roughly constant after this initial phase.

Of course alternative network formation mechanisms could be considered, in particular making link formation some kind of bilateral process. The above mechanism seems to be one of the most straightforward mechanisms which makes sense with our process of cautious trading; it is hard to see how one could advance on it without substantial ad hoc configuration or assumptions which don't really fit in with our model.

This endogenous network formation model suggests a solution to the counter-intuitive result from the previous section where centralisation didn't seem to improve outcomes and slowed conversion: while being central is good for the individual agent it appears to disadvantage its trading partners resulting in fewer opportunities for substantively beneficial trade to occur.

4.6 Learning and the Emergence of Money Within the Network

Cautious trading works well if agents are (reasonably) well connected and have a wide variety of goods. It was shown in the previous chapter how a small amount of experimentation might resolve convergence issues where the initial state or form of utility was unfavourable to trade. In this section we look specifically at a version of cautious trading where agents learn how much experimentation to engage in; in particular they can modify their decision process to accept some amount of loss in utility in order to gain a better final outcome.

Consider an economy with $n+1$ goods and agents with a linear utility function $u_i = \sum_{j=0}^{n} \alpha_i^j x_i^j$ where for all agents i, we set $\alpha_i^0 = 0$, that is we have a good which is worthless to all agents; however, all agents have some of this good.

We introduce a diversification level $\delta \in [0, 1]$ which determines the balance agents place between a diversification or experimental valuation of potential trades and a purely utility determined one. Specifically we set the valuation of a bundle of goods to be:

$$v_i(\mathbf{x}) = \sum_{j=0}^{n} (\delta \frac{1}{n+1} + (1-\delta)\alpha_i^j) x_i^j$$

so where $\delta = 0$ we have pure cautious trading and where $\delta = 1$ utility is entirely ignored and agents value goods equally.

Using this form of evaluating bundles of goods and successive rounds of trading (again each agent starting from the same initial state in each round) a simple reinforcement learning algorithm is introduced for the level of diversification. An agent picks the best level of diversification from a finite subset of $[0, 1]$ (based on the average level form previous rounds) but with a small probability chooses a level at random (ensuring that the full state space is explored).

In figure 4.7 we can see the diversification level counts at intervals of twenty rounds. We can see how agents move from not diversifying (except with their small probability of experimenting) towards engaging in moderate levels of diversification. Figure 4.8 shows the round by round mean of the diversification level clearly showing how this increases (with some noise due to the experimentation and the stochastic nature of the trading process).

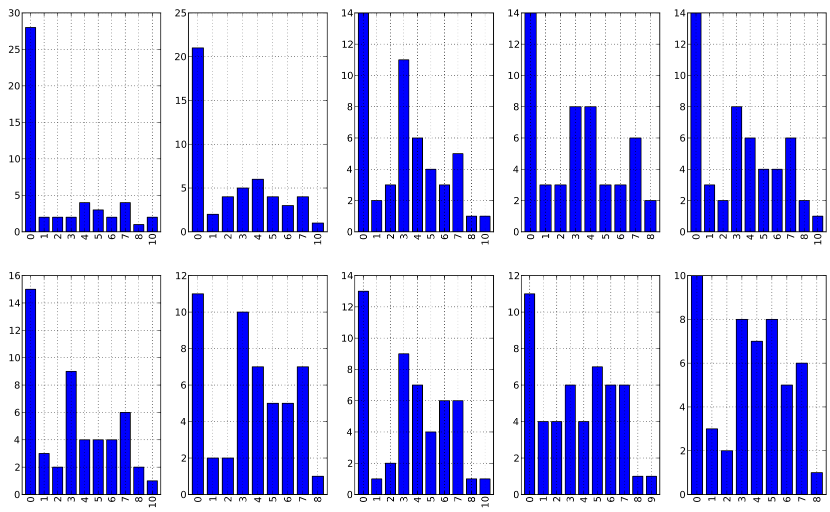

Figure 4.7: This plot shows the diversification level every 20 rounds. We can see that from an initial position of little diversification agents split into (roughly) two groups: those which do not engage in diversification and those which engage in a moderate level of diversification and so use the ‘worthless’ good (or ‘money’) in order to improve their performance. In figure 4.8 the increasing mean diversification level is made clear, though of course this misses the bimodal nature of the diversification.

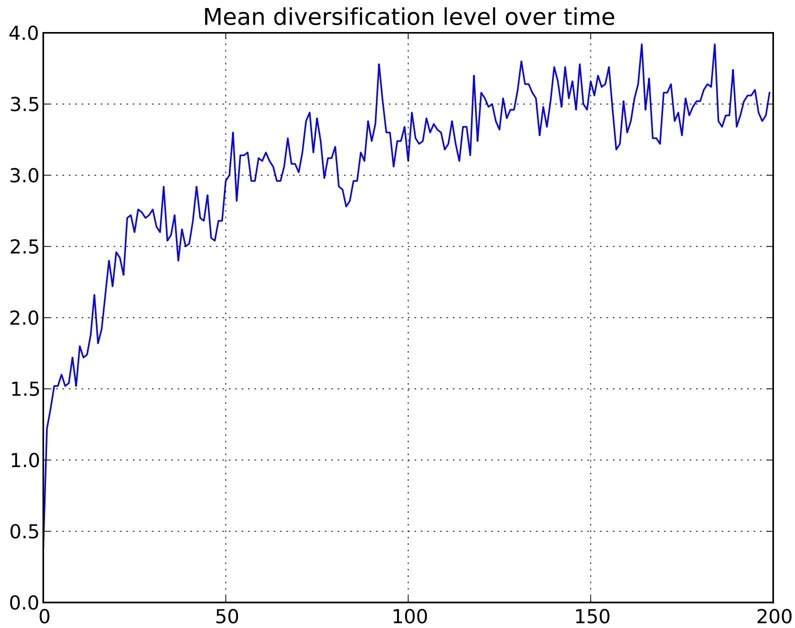

Figure 4.8: This plot shows how the mean diversification level endogenously increases as a subset of the agents learn to engage in diversification. Figure 4.7 shows the counts of agents at each level at intervals.

4.7 Conclusions

We see in figures 4.2, 4.3 and 4.4 how network structure may be important. The first shows how global utility (the sum of individual agent utilities) converges more quickly and to a persistently higher level than for the network models, though in the first few periods they are very similar; this is not surprising as a uniform random matching process allows an agent to be matched with many more potential trading partners so should allow gains from trade to be more quickly found and the initial behaviour should be roughly the same as the proportion of repeated matches will be small so it should be roughly the same. In figure 4.3 we see a much more dramatic result of a trading network, here the networks are structured in a hierarchical way which seems to be much less efficient. In figure 4.4 we see a flatter distribution of wealth, but more interestingly the ability of agents, depending on their network position, to exploit most of the gains from trade. While this is a simple model, the idea of agents being able to gain disproportionately from trade, depending on their position may be a highly significant one for economic activity in general. When we introduce endogenous network formation we see, dynamically, a preference for avoiding connecting to these highly connected nodes.

Chapter 5

Towards a new approach for Agent-Based Modelling

5.1 Introduction

This chapter looks at tools and methods for agent-based modelling of socio-economic systems. We begin with a survey of the more popular agent-based modelling frameworks such as SWARM, Repast and Netlogo. We categorise the approaches into three broad groups: visual programming systems, agent-based modelling libraries for general purpose programming languages and purpose built agent-based modelling languages (or domain specific languages). Arguing that the last two are of most interest we identify a key set of ideas for building a framework that sits in between these areas, an "internal domain specific language", and which draws on other ideas from contemporary software engineering which are of general applicability to agent-based modelling, in particular version control and testing. For each idea we identify the key elements and identify the features required to properly support it.

Having identified a set of key ideas we introduce a new framework built on these ideas, along with some additional ideas inspired by frameworks from other domains and a hierarchical way of thinking about agent-based modelling. We further remark on each of key ideas and on our choice of implementation method. We look at a very simple example using this framework to illustrate the key components of our framework which splits agent-based modelling into a five level approach with explicit consideration of project, experiment, simulation, world and agent levels. We then turn to the more substantial problem of modelling an software 'App Store', showing how we can quickly build a model and create an experiment which investigates different strategies a seller might adopt (while allowing for substantial future development of this model).

We then go on to evaluate our framework from a variety of angles, looking at issues such as performance and ease of use. We conclude by discussing the steps required to make this into a full framework[1], in particular we look at some of the major features that aren't currently included, the ways in which current features could be expanded and improved and issues relating to making it more user friendly.

5.2 Current approaches and possibilities

This section surveys some of the major current approaches to agent-based modelling before considering future possibilities. It does not even attempt to be a complete survey as even the existing surveys are somewhat incomplete given the wide range of approaches and frameworks adopted. The Wikipedia page[2] for agent-based modelling software in late 2011 lists over 70 different pieces of agent-based modelling software; and is itself an incomplete list..

A recent, broad survey of Agent-Based modelling platforms can be found in Nikolai and Madey [2009] and, aside from the Wikipedia list[3] mentioned above, there probably does not exist another more comprehensive collection. Fifty-three platforms are included and commonalities identified. In terms of programming language the popularity of Java, C++ and C is noted, at 43%, 17% and 11% respectively; alongside those frameworks which have their own special purpose language – around 28% (in contemporary software engineering this would typically be referred to as a domain specific language, see below). There are additional frameworks which use some kind of visual programming, something that we will not really focus on in this chapter (again, see below for further context). In Nikolai and Madey [2009] there is further analysis of operating system support and the kind of licences they are available under (open-source, closed-source but free to use, commercial and so on).

There are some surveys which take a narrower approach to focus on a smaller subset of the more popular frameworks. One such work is Railsback et al. [2006] which looks at the SWARM, Java SWARM, Repast, MASON and Netlogo frameworks. Within the socio-economics agent-based modelling community these are probably the mostly commonly used (to this list one might want to tentatively add Repast Simphony[4] which confusingly is not a new version of Repast, but a different,

[1] Though it is already possible to do 'real' work with it.
[2] `http://en.wikipedia.org/wiki/Comparison_of_agent-based_modeling_software`
[3] It was in fact this article which formed the original basis of the Wikipedia page.
[4] The spelling here is correct.

newer, easier to use, more graphical system). SWARM and Java SWARM are supported by the Santa Fe Institute and take a somewhat unconventional approach. In at SWARM model, the model and a separate laboratory for observing and conducting experiments. There is the additional hierarchical idea of a swarm: a group of objects and schedule of actions, which may contain sub-swarms. It uses Objective-C, which is quite unusual outside the Mac OS operating system (though SWARM does work on Windows and Linux). Java SWARM isn't really a Java framework per se, but is a cross platform Java framework for passing messages to the Objective-C SWARM. Respast is a more traditional Java approach, but attempts to replicate most of SWARM's features. MASON focuses on performance and is, in the authors' view, somewhat simpler that Repast.

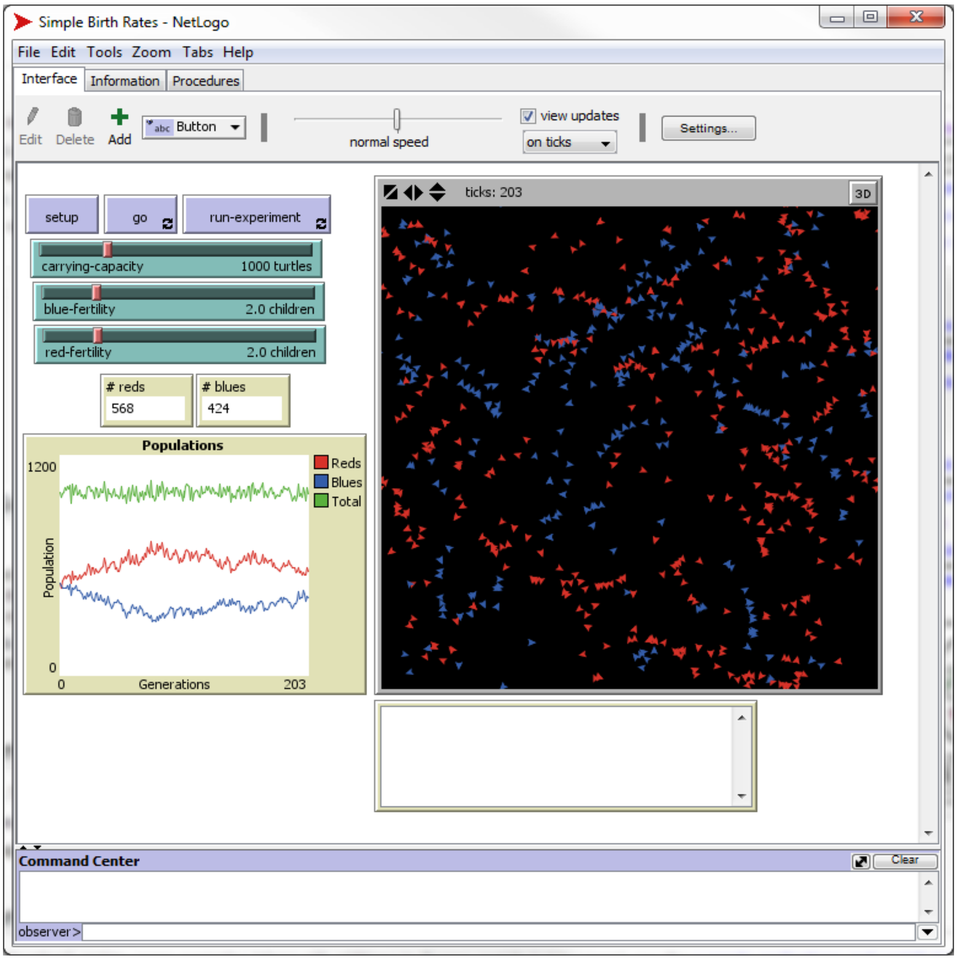

Figure 5.1: A screenshot of Netlogo. Notice the three tab interface, with graphical view/configuration, documentation and code kept separate.

Netlogo is quite distinct from the others; it uses its own language (a version of

Logo), and has an easy to use environment that can be seen in figure 5.1. A project in Netlogo contains three tabs, a set of visual configuration and observation interfaces, a documentation tab (where you can write an explanation of the project) and a tab where you write the procedures in Netlogo's Logo language which run your model. It includes an extensive collection of example projects that range from social science models to games and from particle models to 'music' generating models. If you are new to programming and agent-based modelling, or just want to play around with some pre-built models it is an excellent place to start. However, for many purposes, in particular, larger, more complex models it is less suitable.

I first implemented a version of the model in chapter 3 using Netlogo; but ended up switching to a Java model. I found there were two main problems with Netlogo for that project. Firstly, while Netlogo has a large number of built in objects [5], it isn't so easy to create substantial custom models. The Logo language is old fashioned, with an unusual syntax, limited abilities to do the kind of object orientated programming which naturally fits the idea of agent-based modelling. There are also specific issues when models become larger (everything is in one file so it is difficult to decompose code into independent parts or using libraries). While in principle this is possible I found it easier just to switch out of using Netlogo entirely. The second major issue I found arose when trying to run the kind of repetitions of experiments and collecting results in a form that could be processed which I wanted for that project. Netlogo does have a tool called Behaviour Space which can automate this, but for one of things I wanted to look at there were performance issues in calculating certain values[6].

The work in this chapter is influenced by Isaac [2011] which takes the 'reference implementations' of Railsback et al. [2006] then refines specifications of the examples and implements them in Python[7] to add to the existing framework implementations. The sixteen examples are implemented in 'pure' Python, though with the use of several libraries including a custom 'grid world', MatPlotLib[8] and NumPy[9]. The work convincing demonstrates that Python is a suitable language for agent-based modelling with many advantages over more popular approaches. It has also clarified the reference implementations from Railsback et al. [2006] and associated work. Another pertinent observation relates to the over emphasis on real-time

[5] If for example you want to create a model on a lattice, network or 2D space a lot of this has been created for you saving much time.

[6] Recent releases of Netlogo have included various performance improvements and bug fixes for this tool so it possible that the performance issues may no longer arise.

[7] `http://python.org/`

[8] A collection of Python libraries that offers analogous features to Matlab.

[9] A numerical library for Python.

visualisation in many agent-based frameworks; while this may be of pedagogical value, it isn't really particularly important for more serious work (or, of course, for preparing work for publication or presentation).

5.2.1 The two approaches

In summary, broadly speaking we can split agent-based modelling frameworks into three groups: visual programming systems, agent-based modelling libraries and agent-based modelling languages. We do not remark visual programming systems further; while they are extremely useful for learning and for allowing those unfamiliar with computer programming to engage with agent-based modelling I do not think they represent a productive way for those with the requisite computer skills to engage with general agent-based modelling. As an argument I offer two observations: firstly, aside from for some very specific purposes, visual programming tools remain relatively unused for any kind of general problem solving and the kind of approached covered above seem to be currently dominant in published agent-based modelling work. On the other hand, visual tools may be ideal for modelling in a specific domains.

So that leaves two general approaches, though the distinction is not always particularly neat as agent-based modelling libraries may range from useful components you might wish to use when doing agent-based modelling to a particular scheme for doing modelling, such as SWARM's split into model and laboratory. The final approach is of a special purpose language, taken most prominently by Netlogo. This should have many advantages as argued in Gilbert and Bankes [2002] but despite that fact that this paper is nearly a decade old, many of the challenges they note such as calibration of model and automatic case generation, still have not really been satisfactorily addressed.

5.2.2 A third way and other ideas

In this chapter we argue for a 'third way' which will draw on many ideas from contemporary software engineering practice to formulate a multi-part proposal for how agent-based modelling might be done. There is relatively little work in the field which examines these kinds of issues. An exception is Ropella et al. [2002] which identifies several key issues with particular attention paid to working with SWARM models, it is now a little dated; in particular there is now a more established understanding of testing in software engineering practice.

Internal Domain Specific Language

There are many benefits to purpose made programming language as can be seen in Netlogo and other such projects; however, there are major drawbacks. Any improvements or changes to the language will have to be made by the developers of the framework, rather, than is typically the case with a standard language, a much larger pool of developers with much greater expertise and funding. It also means that it is likely to be difficult to integrate external libraries; rather than using a standard language's standard approach there will most likely be additional hoops to jump through, if it is even possible to use external libraries.

The idea we pursue instead is an internal domain specific language, see Fowler [2005]. Whereas a domain specific language just means a specialised (programming) language, an internal (or embedded) domain specific language is using particular forms of a general purpose language to include domain specific elements, so when considering agent-based models we may have a specific language for defining and manipulating agents. But alongside theses domain specific features we have access to the full power of the base language. This is great for when it is difficult to tell ahead of time what precise functionality you will need (very much the case with agent-based modelling) but has the drawback of being less accessible to those who don't already know the base language.

Project not program

One of the nice features of Netlogo that we highlighted above was the integrated documentation tab; it means that in a limited way a visual control of program, documentation and program script are packaged together. The idea that what we are really interested in is an agent-based modelling project (or even collection of related projects) rather than a mere agent-based model will be a key idea in our approach. A project will bring together consideration of some or all of the following: a model containing (many) agents in a environment, simulations of this model, experiments which compare or otherwise run simulations, visualisations of results, presentation of the model, additional written work, additional mathematical analysis and bringing various elements together in one document. There are of course many ways in which this could be done, including considering most of it outside the scope of the framework and leaving it up to the researcher (the standard approach).

Version control

An accepted part of modern software engineering is version control[10]. This in essence means keeping track (and storing) changes to a computer program[11] and is particularly useful when multiple programmers or authors are involved as each can work independently and combine changes at a later date (distributed version control). There are three main issues vis a vis agent-based modelling frameworks: the main content should ideally be stored in text files of some sort (such as a program language's source files) as this allows for easy comparison between versions, there should be a distinction between source files (which are tracked) and generated files (which may be temporary or depend on the particular computer on which the file exists and so should not be tracked) and finally the actual configuration of the version control system.

Testing

The other major element that one might argue is missing from most approaches to agent-based modelling is automated testing. It has become an accepted part of contemporary software engineering; using frameworks such as JUnit[12] programmers can write programs which test parts of their code in an automated fashion. While any agent-based modelling framework which is a library for a standard computer programming language makes automated testing possible, there seems to be little explicit support for it. It is possible to identify three main issues. Firstly, does a framework provide examples of tests of models on which a user can base their own. Secondly, is the model itself explicitly tested and are these tests (along with the source code) open to users so they can understand and modify them. Finally, how the tests are run; most testing frameworks have specific ways of being run, that can often be integrated into development tools. The issue of testing is vitally important for addressing concerns about program correctness for agent-based modelling. While there are more formal approaches to proving software correctness these are often impractical for real projects. Simply making code open source for peer review is insufficient as it may be extremely difficult to verify it is functioning properly, ideally a comprehensive set of tests should accompany scientific work.

[10] Alternatively revision control or source control

[11] Or indeed any files; I use a version control program to keep track of changes to this thesis.

[12] `http://www.junit.org/`

5.3 A prototype framework: Ambl

In this section we describe a prototype framework[13] called Ambl which is built on the key ideas identified in the previous section. It serves as an illustration and test bed for the ideas and potentially a foundation on which a fully fledged platform could be built. We comment on additional features which could be added to Ambl towards the end of this chapter.

Project not program

Ambl conceives of an agent-based model broadly. Not only does it include specific support for *experiments*, which contain multiple *simulations* of *worlds* containing *agents*, it explicitly models the idea of a *project* which can include multiple experiments. These projects can be exported in a variety of forms, the prototype includes examples of export to LaTeX, to HTML[14] and to a simple slide based presentation format; and many more formats could easily be added and can contain things like automatically generated diagrams, tables, visualisations of results, descriptions of model components, simulations and experiments run and so on.

Internal Domain Specific Language

Ambl is part library and part internal domain specific language. For the basic building blocks of an agent-based model Ambl has particular syntax, but it is syntax within a standard programming language. As we mentioned above Ambl splits a model into five basic building blocks:

1. Agent
2. World
3. Simulation
4. Experiment
5. Project

Which are arranged as figure 5.2 illustrates. So for example to take the idea of an agent. Ambl has special syntax for adding a description to an agent, having variables and parameters, declaring things that we are interested in observing for the agent and documenting methods of interest (while keeping 'internal' methods internal).

[13] In fact both a prototype of a framework and a framework for prototyping models.
[14] Which will work in Word, Pages, OpenOffice Writer.

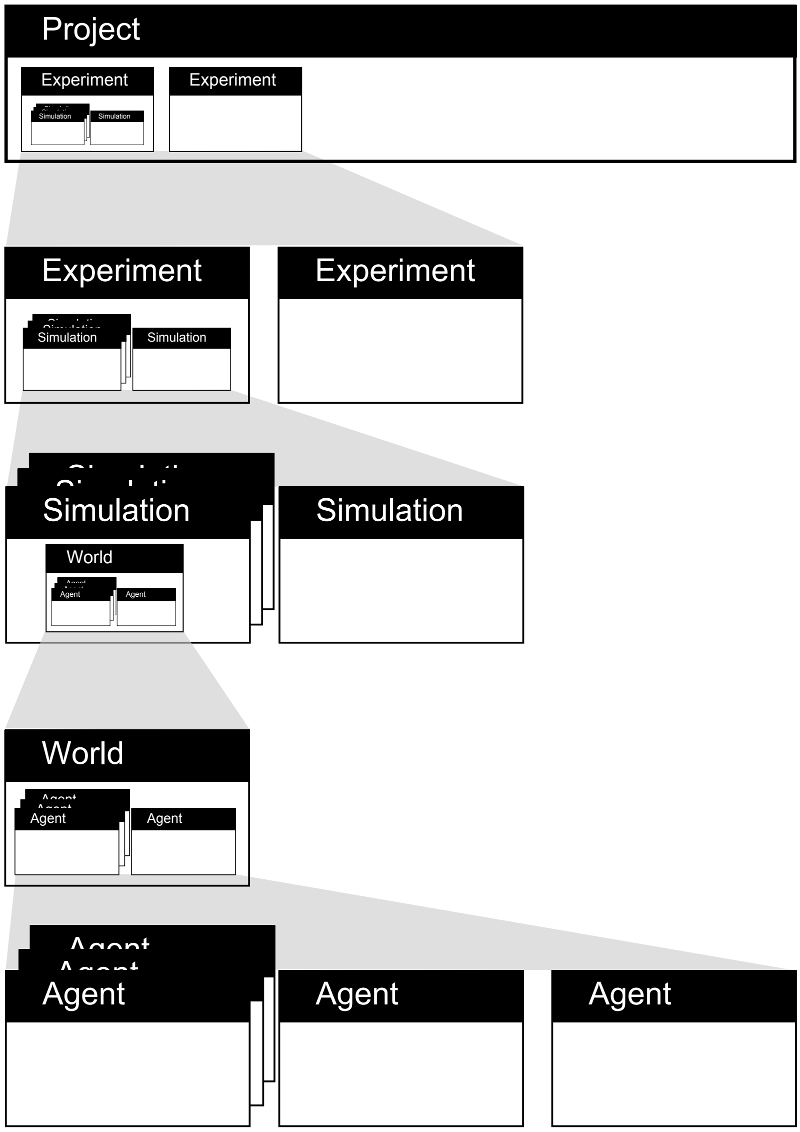

Figure 5.2: A schematic overview of the main components of an agent-based model in the Ambl framework. A project contains one or more experiments, each of which contain one or more simulations, some of which may be repeated. Each simulation takes place in a world, which has sets of agents (possibly of size one). The framework supports multi-directional relationships between agents.

In the next section we give a simple example that will make the capabilities and particular kind of implementation clearer.

Testing

The key parts of Ambl include automated tests. These serve a dual purpose: firstly they ensure that the various parts work as expected and secondly the user can see clear examples of the usage of Ambl (which helps with their understanding, gives them examples to base their projects on and gives them examples on which to base their own tests for their projects). Ambl uses the RSpec[15] framework, which is a "behaviour driven development" framework; basically the idea is that you (can) write examples of how a program should work and *then* write the program. Ambl includes a command line driven way of running all tests in a project which is compatible with integrated development environment tools which allow for features like automated testing[16].

Version control

While Ambl doesn't explicitly include version control, it includes a configuration file for the popular open source distributed version control program Git[17] telling the program which files should be tracked. All the major files are text based so work well with such version control programs. Users may wish to customise on a per project basis whether results and other outputs from runs of experiments are stored in their version control system as either option could make sense.

5.4 Implementation

Up until this point I've managed to avoid referring specifically to the language Ambl is implemented in. While appendix B and indeed the source code and documentation of Ambl give full details, we'll look at a self-contained summary here. Ambl is implemented in the Ruby[18] programming language, specifically the JRuby[19] variant[20].

[15] `http://rspec.info/`

[16] I used Rubymine (`http://www.jetbrains.com/ruby/`) for most development of Ambl and it has an 'auto-test' toggle which means that tests are run on every modification of a source file, so you get immediate feedback on whether you have broken something or can see if you have successfully fixed something).

[17] `http://git-scm.com/`

[18] `http://ruby-lang.org`

[19] `http://jruby.org`

[20] Actually most of Ambl should work, possibly with minor modifications, with any recent version of Ruby, see appendix C.

In order to create the kind of internal domain specific language which advocated above a language with flexible syntax is required. Popular choices for this kind of task include languages such as Ruby and Scala[21], but for agent-based modelling we want both flexibility in running things (or describing how they should be run) and flexibility over defining them. There are few languages that offer the kind of flexibility Ruby offers for both areas. One can endlessly debate choices of programming languages, but Ruby is a reasonable and perhaps currently the best choice for this application, even if it has some drawbacks (see section 5.7.2). But fundamentally we are addressing the methodological ideas and issues, the choice of language while relevant, particularly if one is planning on developing the particular implementation further, is not the main issue under consideration in this chapter.

Secondary advantages of JRuby include its inherently cross platform nature (as it is based on the Java Virtual Machine it will run on a wide variety of platforms) which allows the creation of things like graphical user interfaces which will across different systems without modification, see figure 5.4. Ruby has a clear, concise syntax, but unlike Python is very flexible, which makes creating a framework which often appears fairly like normal written English possible. Finally it has access to both Ruby and Java libraries, so for tasks ranging from scientific numeric computing to `pdf` generation there are already multiple existing libraries which could be used.

The Ambl is extremely extensible and just about everything can be customised, but if you use the defaults it is very easy and concise. Ambl and its examples currently stand at around 6000 lines of Ruby code, not including additional templates and other supporting files[22]. The equivalent in languages like Java would probably be several tens of thousands of lines of code, although much of the functionality would be impossible to implement in anything like its current form in most languages. In appendix C technical details, including how to obtain and run Ambl for yourself, are provided.

5.5 A simple example

In this section we look at building a simple agent-based modelling using our prototype framework. This simple model contains a set of agents which have an energy level. This energy level essentially undergoes a (possibly biased) random walk. The agents have a utility which depends on their energy level. For the world of these agents we have a couple of global properties (minimum utility and total utility)

[21] `http://www.scala-lang.org/`

[22] As estimated via `find ./ -name '*.rb' -print0 | xargs -0 cat| wc -l`

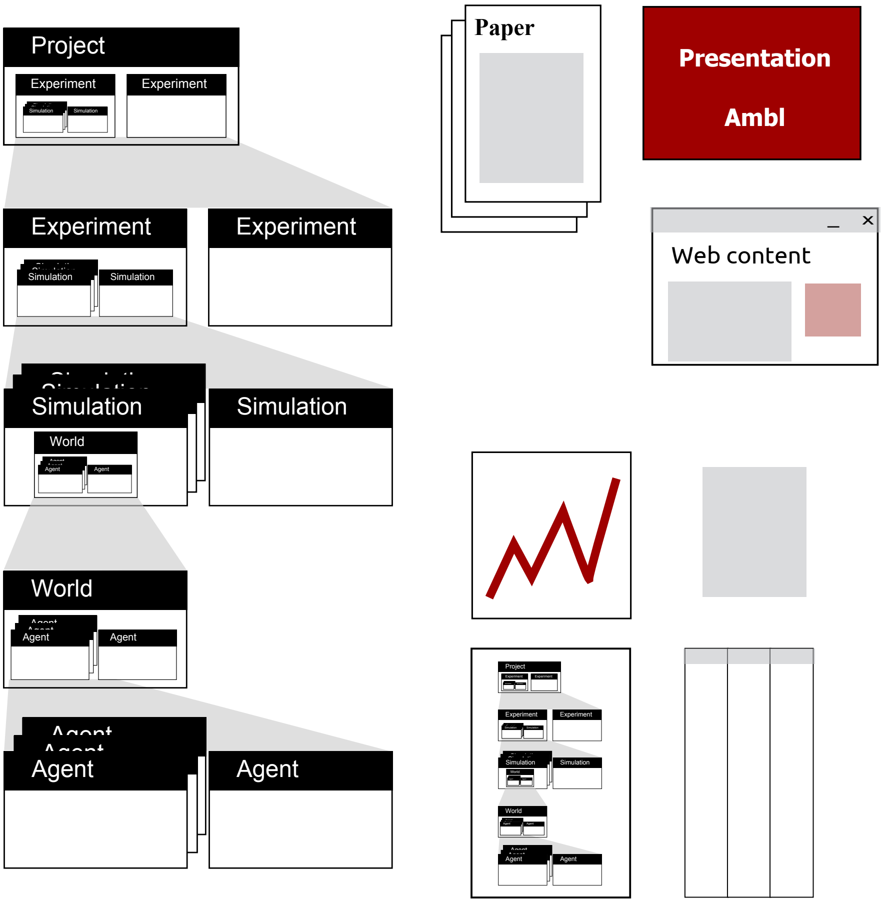

Figure 5.3: An overview of Ambl and the outputs which it can generate.

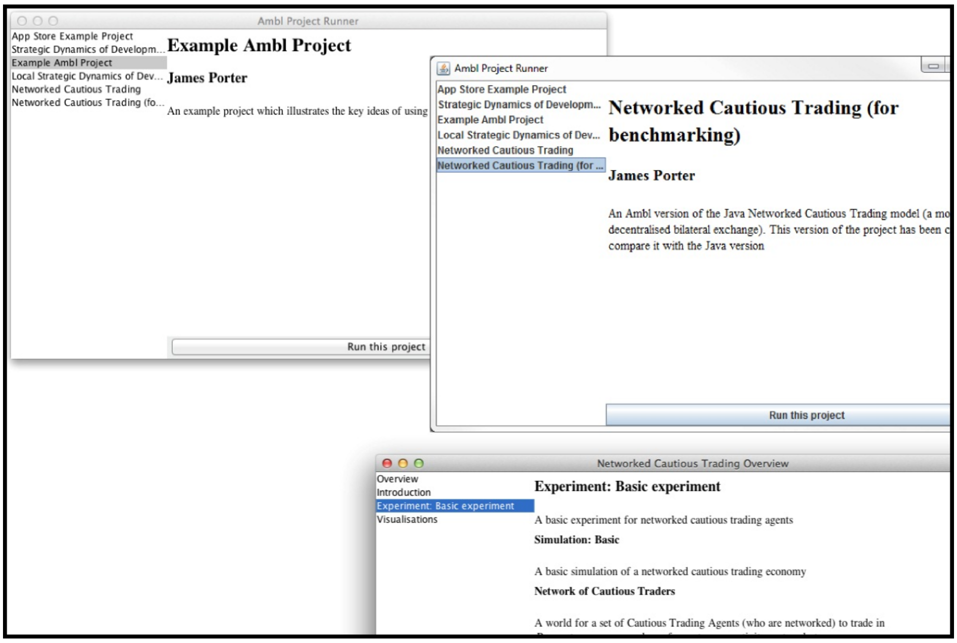

Figure 5.4: Graphical user interfaces for providing an overview of an Ambl project and for running specific Ambl projects; both work across multiple platforms with no required modifications. (Composite screenshot from Windows 7 and Mac OS 10.7.)

which we are interested in. We carry out simulations collecting these values as they vary as determined by the individual random walks. The experiment we will carry out will be for an unbiased random walk. Finally this experiment will be the only one contained in a project with some additional information. The remainder of this section will break down the above into components and explain how they each fit into Ambl.

5.5.1 An Agent

Below we define the agent. Basically there are three main things to think about for an agent: the things which are part of the definition (description, list of parameters and so on), the things that an individual agent can do (the methods) and the properties of an individual agent (the attributes, these are preceded with an `@` symbol in Ruby, and hence Ambl).

On line 2 we add a description of agent which can then be used for documenting and outputting to various formats. On line 7 we declare a variable `x` which we can then refer to as `@x` (Ambl takes care of setting it up). The next line defines our agent's parameters in a similar way. Lines 9 and 10 document certain methods (to be defined below as being of special significance). On lines 12-15 we define our first method which returns the utility level of the agent; lines 17-19 and 21-23 define methods which increase or decrease the energy level (essentially take steps in a random walk). By using the kind of internal domain specific language that is found in lines 2-10 we have saved ourselves a lot of effort and ensured that our model of an agent is much more concise and readable than it would otherwise be.

Listing 5.1: Example Agent

```
1  class ExampleProjectAgent < Ambl::Agent
2    add_description "A simple example agent with two parameters.
3  It has some energy level, which is determined by a random walk of
4  step size :delta. The utility of the agent is the energy level
5  multiplied by :alpha"
6
7    has_variables :x
8    has_parameters :alpha, :delta
9    has_observables :utility
10   has_methods :deplete_energy, :increase_energy
11
12   def utility
13     #can also use parameters(:alpha)
14     @x ** @alpha
15   end
16
17   def deplete_energy
18     @x -= @delta
19   end
20
21   def increase_energy
22     @x += @delta
23   end
24 end
```

5.5.2 A World

Below we define the world that our agents 'live' in. In Ambl the world will typically be the most complex component of a model. In line 2 as for the agent we add a description. Line 5 adds a collection to the world, giving it a name, declaring a class that the objects in the collection will belong to and declaring a set-up method (where we initialise the collection). Line 6 and 8 do similar things to the agent example, declaring a parameter and some 'observable' properties we may be interested in. Line 7 declares an update. In Ambl we declare a set (possibly with only one member) of updates which are carried out in each time period (Ambl, as is standard, assumes a succession of time periods).

In lines 11-17 we build our collection of agents. Line 12 illustrates a trivial use of the Ambl options system. In Ambl class `@options` is a special attribute which stores the options that have been passed to it. By using the `fix` method we are adding additional options, in this case it is trivial as we are adding the same values for every agent, in practice we will most likely want to vary options[23] in some fashion. In line 15 the Agents are created and added to the collection (Ambl automatically generates the collection for us).

In lines 19-27 we have our update: it is relatively straightforward: for each agent it randomly increases or depletes its energy. Lines 29-31 and 33-35 report on global 'observables'; they use Ruby's concise functional programming style syntax.

If you are not already familiar with Ruby some of the above may not have been clear. The good news is that the remainder of the components are in much more of a domain specific style; the agent and world must, because they vary so much from model to model, be somewhat lower level and hence depend largely directly on the language's syntax.

Listing 5.2: Example World

```
1  class ExampleProjectWorld < Ambl::World
2    add_description "A simple example world containing many ExampleAgents
       .
3  A parameter :beta controls the bias of the random walk of the agents"
4
5    has_collection :example_agents, :object_class => ExampleProjectAgent,
         :set_up => :build_collection
6    has_parameter :beta
7    has_updates :update_energy
8    has_observables :total_utility, :minimum_utility
9
10
11   def build_collection
12     options = @options.fix(:x => 1.0, :alpha => 2.0, :delta => 0.9)
13
14     1.upto(100) do |id|
15       @example_agents << ExampleProjectAgent.new(options, id)
16     end
17   end
18
19   def update_energy
20     @example_agents.each do |agent|
```

[23]Ambl includes specific support for having options which are agent dependent, rather than having to call this repeatedly.

```
      if rand > @beta
        agent.increase_energy
      else
        agent.deplete_energy
      end
    end
  end

  def total_utility
    @example_agents.inject(0.0) {|total, agent| total + agent.utility }
  end

  def minimum_utility
    @example_agents.inject(@example_agents[0].utility){|min, agent| min
        = agent.utility if min > agent.utility}
  end
end
```

5.5.3 A Simulation

The below simulation almost doesn't need explanation; everything is pretty close to plain English. On line 3 is declares its World and line 4 specifies a parameter (as before). Line 5 specifies a result which should be collected. Ambl has a sophisticated results system, but most of the time you shouldn't have to worry about it. You specify an observable in the world to collection, optionally specify the resolution (in what proportion of periods should it be recorded), and it will just collect it and report it to the Experiment where it can be processed, outputted to a file for further analysis and so on. Line 6 specifies something which should be collected at the end of the simulation and line 7 specifies for how many periods to run it.

Listing 5.3: Example Simulation

```
1 class ExampleProjectSimulation < Ambl::Simulation
2   add_description "A simulation of the world with agents performing a
      random walk (possibly with bias :beta)"
3   has_world ExampleProjectWorld
4   has_parameters :n
5   collect_result :total_utility
6   collect_final :minimum_utility
7   run_for 25
8 end
```

5.5.4 An Experiment

The experiment in this case is particularly simple as we only have one simulation, to which we specify a `beta` value of 0.5, making it an unbiased random walk. Line 9 could be omitted as by default if only one simulation is supplied it is run. There are more sophisticated run modes which will do things like carry out repetitions or compare simulations.

Listing 5.4: Example Experiment

```
1  class ExampleProjectExperiment < Ambl::Experiment
2    add_description "An example experiment which runs in the default way.
       "
3
4    #Include simulations; by default all would then be run,
5    #but can more precisely specify way they are run
6    has_simulation :one, ExampleProjectSimulation, :beta => 0.5
7
8    #The below is the default so could have been omitted
9    run_simulation :one
10 end
```

5.5.5 A Project

The below listing concludes our simple example project. As with the Experiment and Simulation above it should be relatively straightforward to follow. We add a title, author and description. We add some sections and some text. The interesting lines are 6-8 where we declare several outputs and line 13 where we declare the experiment. The position of that line is significant: it determines the order in which sections will appear in the outputs and is somewhat in keeping with the idea of Literate Programming, see Knuth [1984], where we mix description and code.

Listing 5.5: Example Agent

```
1  class BasicProject < Ambl::Project
2    add_title "Example Ambl Project"
3    add_author "James Porter"
4    add_description "An example project which illustrates the key ideas
       of using the Ambl framework."
5
6    has_output "basic project", HTMLProcessor
7    has_output "basic-project-draft", LatexProcessor
8    has_output "basic project", PresentationProcessor
9
10   add_section "Introduction"
```

```
  add_text "This is a simple project with only one set of agents."

  has_experiment :simple, ExampleProjectExperiment

  add_section "Conclusion"
  add_text "Hopefully these examples have been helpful. Good luck with your own project(s)."
end
```

5.6 A fuller example

In this section we carry out an example trans-disciplinary investigation using Agent-Based Modelling. As the example we focus on an investigation of commercial strategies that could be used by a developer when creating 'Apps' for an online distributions store such as the Android Market[24] or Apple App Store[25]. This is an ideal example for a number of reasons:

- The global, competitive market means that many simplifying economic assumptions we may wish to make are in fact fairly realistic.
- The large number of Apps (in the hundreds of thousands) and downloads (in the tens of billions) means that we have a lot of actual data on which to base our model (and the potential to gain more with collaboration from developers).
- This kind of environment can be considered paradigmatic of many current and future online retail methods.
- They are popular and well understood markets (many, if not most, readers will own smart-phones and have personal experience using such markets).
- I have personally developed applications for the Android Market and thus can play both sides of this trans-disciplinary investigation.

Using an agent-based model we can create a framework in which a developer can simulate many different approaches and calibrate for their particular current situation, audience and chosen application area. While a more traditional mathematical model could no doubt be constructed (and would probably be of interest) the agent-based approach offers more flexibility and a greater ease of dealing with some of the complex interrelated features of a App Market.

[24] See `http://market.android.com`

[25] See `http://www.apple.com`

5.6.1 The problem

The problem we examine is the choice between a developer creating one higher quality application or more than one application, which will then presumably be lower in quality. The difficulty in this decision comes from the way in which application sales are driven. While typically an application is available to every buyer in the world, they do not know ahead of time the quality of the application. In assessing this they may depend on a number of channels, such as:

1. the reviews from previous buyers,
2. their ownership of another application from that developer
3. and promotion by the App Store.

The first and third will depend on the quality of the application. The second will depend on their already owning another application from the developer. While one could develop this model much further, we use the development of a basic case in order to illustrate the use of the framework.

5.6.2 Modelling the agents and world

First we need to consider at what level to model our scenario. In this model we will just consider one seller in isolation, who produces one or more applications each of whom have a set of buyers. There are more efficient ways of modelling this than the approach we adopt below, but this approach allows for the future addition more sophisticated interrelationships, which could be driven by empirical data gained from actual App Stores.

Below is our first type of agent, the seller. The seller will produce one or more application and the the question we want to examine is how revenue is determined by the choice of the number of applications to develop. The model we built is highly simplified but many of the components that are included, or could be included, are in practice measurable either in aggregate, or in some cases at the individual level via the use of software analytical tools by the developers.

Lines 1-5 function as in our previous example project (detailed above). We declare a new `AppStoreSeller` object and set up several parameters. We also flag a couple of methods as being observable (for documentation purposes) along with additional methods. The `has_many` in line 5 is new and essential says that under the name `apps` instances of this object should have a collection of `AppStoreApp` or applications. This will be initialised for us and is made available as `@apps`.

We then have several methods which create a new application (lines 7-11), develop the number of applications specified by our parameter (lines 13-16) and update buyers of each application (lines 22-26). The method in lines 18-20 return the current revenue for the seller (the sum of the revenues for each application). The code takes advantage of a number of Ruby features, such as the functional programming style `inject`[26] to keep things clear and concise.

Listing 5.6: An App Store Seller

```
1  class AppStoreSeller < Ambl::Agent
2    has_parameters :skill, :capital, :number_of_apps
3    has_observables :revenue
4    has_methods :create_app, :develop, :update_buyers_of_apps_at_rate
5    has_many :apps, AppStoreApp
6
7    def create_app(effort)
8      a = create_with_id(AppStoreApp, {:effort => effort}, @apps.length)
9      a.seller = self
10     @apps << a
11   end
12
13   def develop
14     effort = @capital / @number_of_apps.to_f
15     @number_of_apps.times { create_app(effort) }
16   end
17
18   def revenue
19     @apps.inject(0){|sum, app| sum + app.revenue}
20   end
21
22   def update_buyers_of_apps_at_rate(rate)
23     @apps.each do |app|
24       app.update_buyers_at_rate(rate)
25     end
26   end
27 end
```

Below is the code for an App Store application (or App). It is the most complex part of the model (currently) but is still quite concise as it takes advantage of many of the features of Ambl. Lines 5-11 declare the various Ambl properties. It has a certain quality that is determined by the effort put into producing it. Each buyer gives is a score and the rate of new buyers depends on these scores. It has an

[26]In other languages this often goes by the name `reduce` or `fold`; both would in fact be valid Ruby but `inject` is more standard.

observable property called revenue which we have already summed above to get the total revenue per seller. It has a function `setup` which is declared as to be used to set up the object instance. It also has both one seller and many buyers.

As is mentioned in the comment on lines 16-17 the current implementation of setting the quality and score is basic and we should probably customise it for particular App Stores based on their scoring system. Lines 18-39 define the functions to set up the App, to get a score (this scoring is done by buyers, but it easiest to define this here) and to give an overall quality score for the app (which will determine future sales). Lines 49-55 define how we add buyers. This is done at a rate determined by a value passed by the world (the `rate`), by the quality score and with a noise term `rand`.

Listing 5.7: An App Store App

```
1  class AppStoreApp < Ambl::Agent
2    add_description "An application (or \"app\"); it has a quality determined by effort (in the simple case directly). It can report its revenue
3  (total sales) and maintains a list of buyers."
4
5    has_variables :quality, :scores,:total_score
6    has_parameters :effort
7    has_observables :revenue
8    has_methods :add_buyer, :update_buyers
9    has_setup :setup
10   has_one :seller, :AppStoreSeller
11   has_many :buyers, :AppStoreBuyer
12
13   #Assume quality is on unit interval
14   #This is a simple function which has desirable properties
15   #i.e. diminishing marginal returns to effort, in unit interval, doesn't attain 1 (no 'perfect' apps)
16   #Probably want to consider something more sophisticated
17   #Should be based on user scores/other metrics
18   def setup
19     @quality = (3 / 4.0) * @effort ** 2.0
20     @scores = []
21     @total_score = 0
22   end
23
24   #Determined probabilistically via quality
25   def get_score
26     score = rand < @quality? 0 : 1
27     @scores << score
```

```
      @total_score += score
      score
    end

    #a more sophisticated notion of quality taking into account the
        actual quality along with user feedback
    def quality_impression
      if @scores.length > 0
        @quality * @total_score / @scores.length.to_f
      else
        @quality
      end
    end

    def revenue
      @buyers.length
    end

    def add_buyer(buyer)
      @buyers << buyer
    end

    def update_buyers_at_rate(rate)
      (rand * rate * quality_impression).round.times do
        buyer = AppStoreBuyer.new(@options, @buyers.length)
        buyer.buy self
        add_buyer buyer
      end
    end
  end
```

Our final agent class if for the buyers. If you were attempting to optimise the performance of this model you would probably remove this class as it currently stands; but we leave it here as if we wanted to make the model more realistic we will almost certainly need to build on this. Currently the interesting part is the possibility that if a buyer likes an App (lines 19-21), they buy all the other Apps sold by the seller (lines 24-30). Of course we should probably replace this with a probability based on observed data[27].

Listing 5.8: An App Store Buyer

```
1 class AppStoreBuyer < Ambl::Agent
2   add_description "A buyer of the smart phone application. Has a list
```

[27]In my experience the assumption that someone who likes an App immediately buys all other Apps on sale from the same seller is heroically optimistic.

```
    of applications and has a score for each (the score is 0 for
    dislike, 1 for like; a realistic
model would follow scoring of App Store), currently if the buyer likes
  an app she buys all other apps from the seller, should
be based on empirical data."

  has_variables :apps
  has_methods :buy_other_apps_from_seller_of
  has_setup :setup_scores_hash
  has_many :apps, :AppStoreApp

  def setup_scores_hash
    @scores = {}
  end

  def buy(app)
    @apps << app
    @scores[app] = app.get_score

    if @scores[app] > 0
      buy_other_apps_from_seller_of(app)
    end
  end

  #Buy apps from the seller of the app which the buyer does not already
      have (i.e. @apps)
  def buy_other_apps_from_seller_of(app)
    (app.seller.apps - @apps).each do |a|
        buy a
        a.add_buyer self
    end
  end
end
```

The world in this case is relatively simple as we only have one seller and interactions between agents do not occur explicitly in the world (but are handled by agents). We declare a collection of sellers in line 7, even though in this case it is a simple collection of 1 and set it up in lines 12-18; getting each seller (though we only have one) to develop applications. We can work out the total revenue as in lines 20-22; this code will work when we have more than one seller. We only have one update per period, which is defined in lines 24-27 and as you can see just calls the update method of each seller; again this code will work when we have more than one seller.

Listing 5.9: App Store World

```ruby
class OneSellerAppStoreWorld < Ambl::World
  add_description "A world with one App seller. As buyers buy the App they
may increase the rate of sales of the App itself
and (if applicable) possibly buy other Apps from the same seller."

  has_parameter :rate
  has_collection :sellers, :object_class => AppStoreSeller, :set_up => :create_list_of_sellers
  has_observables :revenue
  has_updates :update_world

  #Currently just one seller
  def create_list_of_sellers
    options = @options.fix(:capital => 1.0, :skill => 1.0)
    @sellers = [AppStoreSeller.new(options, 0)]
    @sellers.each do |seller|
      seller.develop
    end
  end

  def revenue
    @sellers.inject(0){|sum, seller| sum += seller.revenue}
  end

  def update_world
    @sellers.each do |seller|
      seller.update_buyers_of_apps_at_rate(@rate)
    end
  end
end
```

5.6.3 Setting up an experiment

For this illustration we set up a set of simulation which vary the number of applications produced by a seller. We declare the World, a parameter to be supplied by the experiment, we collect the `revenue` observable every 5 periods and run this for 100 periods. All of those feature can be easily modified.

Listing 5.10: Example Agent

```ruby
class AppStoreSimulation < Ambl::Simulation
  add_description "A simulation of a seller with a fixed strategy"
  has_world OneSellerAppStoreWorld
```

```
  has_parameters :number_of_apps
  collect_result :revenue, :resolution => 5
  run_for 100
end
```

Below is our experiment which runs the above simulation with different `number_of_apps` parameters (running from 1 to 5); it repeats each simulation 10 times and will automatically average the results.

Listing 5.11: Example Agent

```
class AppStoreExperiment < Ambl::Experiment
  add_description "A set of simulations of a single seller on an App Store
comparing the revenues obtained by different fixed strategies (number of applications developed)."
  has_simulation :comparison, AppStoreSimulation, :rate => 100, :capital => 1.0

  vary_parameter_for_simulation :comparison, :vary_parameter => :number_of_apps, :over_values => [1,2,3,4,5]
  repeat 10
end
```

We omit the project as in its current form it is very similar to the previous example. A full project for this would include multiple experiments with different versions of the App Store model. When we run this model (either via the command line, our software tools or the graphical user interface in figure 5.4) we obtain the final revenue figures presented in figure 5.5. When you run an Ambl project the results are saved in a standard `.csv` file and default visualisations are created for each result collected. In this experiment the results do depend heavily on the way multiple applications are evaluated, so it would be of critical importance to replace details of the model with empirically calibrated values before drawing conclusions.

5.7 Evaluation

There are two major areas to evaluate for a methodology for agent-based modelling. The first areas is that of the method and we might ask questions such as how easy is it to use, does it accomplish it goals and so on. The second concerns performance. While a framework might meet all its stated objectives if is extremely slow then it is unlikely to be successful. Related questions concern the difficulty in optimising a model and how it interoperates with other frameworks. Ambl is currently missing

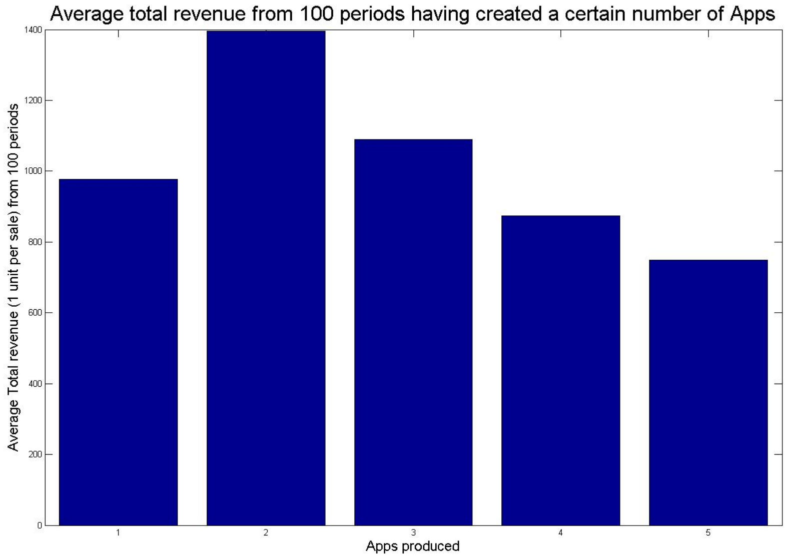

Figure 5.5: An example final revenue result for the App Store model over 100 periods. With these parameters producing 2 applications maximises revenue.

several features which should probably be included in such a general framework, we leave discussion of theses until the next section.

5.7.1 Method

In section 5.3 we covered (briefly) how Ambl implements each of the key ideas we covered and in the final section we discuss some future additional possibilities but here we focus on evaluating some of the unique (or unusual) features of Ambl.

A key aspect of Ambl is in the automated generation of outputs with no additional work on the part of the user. Figure **??** shows one example of a generate output from an example Ambl project. It is `pdf` generated from LaTeX source which has been compiled unmodified from that generated by Ambl. Ambl takes a project with its experiments, simulations, worlds and agents and automatically summarises it. It runs the simulations and picks appropriate visualisations for each kind of experiment, so for example if the experiment compares different parameter values it plots all results on the same plot so the can be compared; if it repeats a simulation Ambl averages over your results. It outputs these results, then converts them to a format suitable for compilation via `pdflatex`. While this will not produce a final article it quickly produces something that can be shared, amended and expanded upon. A future version of this framework could quite reasonably be expected to produce outputs something along the lines of first draft of an article, though this would require a few more features, particularly improvement in visualisation (see the final section).

A simple example of the largely unmodified[28] output of an example included with Ambl is provided in Appendix D. While this is far from a polished report or paper it provides a basic $\LaTeX$ outline, summaries of the key parts of the agent-based model and the experiment undertaken; along with actual results from those experiments; all with no additional work.

Visualisation of results is the key challenge in advancing this kind of workflow. Trying to create something that can produce results without much human intervention is hard. All of the standard tools which I have tried appear to be in some way clumsy, ugly or limited. Ambl uses Ruby-Vis[29] which is a Ruby implementation of Proto-Vis, a visualisation toolkit favoured by many visualisation professionals[30] as it can produce very pretty results and is extremely flexible. In figure 5.6 example visualisations created using this framework are shown to demonstrate something of the range of possible visualisations. It is currently relatively straightforward for experienced programmers to to adapt these examples for their own use to create state of the art visualisations; however, it is far from accessible to inexperienced programmers, let alone non-programmers, and may take quite a bit of time; which is why I remain unsatisfied with the visualisation aspects of Ambl. However, as Ambl saves results in a standard format users can use whatever tools they are comfortable with (I actually used Matlab for most of the figures in this thesis). Ambl also saves visualisations in a user-editable vector format (`.svg`) so it is possible to prettify visualisations using standard graphics tools, which is in fact the approach suggested in the visualisation text Yau [2011].

5.7.2 Performance

In chapters 3 and 4 we utilised a carefully optimised model written in Java for numerical realisations of the economies. In this section we take the model from 4 and recreate it in our framework in order to compare performance. There are two major ways we will compare performance; firstly, in terms of raw speed of execution and secondly in terms of how clear the code is, how long it would take to write and so on.

The original Java application is written in the typical way. It does not

[28]Some changes were necessary for it to be included within this thesis document.

[29]`http://rubyvis.rubyforge.org/`

[30]See for example `http://flowingdata.com/index.php?s=protovis`.

Figure 5.6: A collection of sample visualisations which use the Ruby-Vis library.

use any agent-based modelling libraries but outputs results to a standard `.csv` file for processing and visualisation. The code for the model has been repeatedly improved. It is clear and relatively concise (for Java, a language with a reputation for verboseness). However the Ambl files, are in my opinion at least, much clearer. A particular problem for the Java project was in the creating of lots of files to run various simulations; these tended to contain a lot of boiler plate code and repetition. While that problem could be partially solved by the use of an appropriate Java framework, it is highly unlikely that even then we would get as concise and clear simulation and experiment running files as those made possible by Ambl.

Figures 5.7 and 5.8 compare the performance of the Java and Ambl versions of the networked cautious trading model from chapter 4. This is not an entirely fair comparison as the Ambl version has a built in benchmarking mode which turns off result collection so that you are just testing the main algorithm; the Java version is building up arrays of results (though not outputting them). The Ambl performance is slightly better on average; however if result collection is enabled this will no longer be the case. The result collection mechanism in Ambl could be further optimised for performance; though as this would make the simulation running code considerably more complex this should probably only be done once simulation running has been finalised. Also the Java version will significantly outperform the Ambl version once we have more closely converged agents; at this point the numerical random generation of proposals becomes the important issue. A further caveat on performance is that JRuby will soon be updated to a version with potentially large performance improvements[31] so I'm reluctant to draw any strong conclusions here; however, the performance of Ambl seems to be satisfactory.

[31]Updated to version 1.7 which takes advantages of new features in the JVM.

The architecture of Ambl allows the lower level components (the agents and world) to be replaced by code written in a lower level language. It should be straightforward to write a JRuby adapter[32] for a suitably written and compiled Java version of a model; which would give nearly all the performance benefits of a pure Java model, but with all of the documentation and output benefits of Ambl.

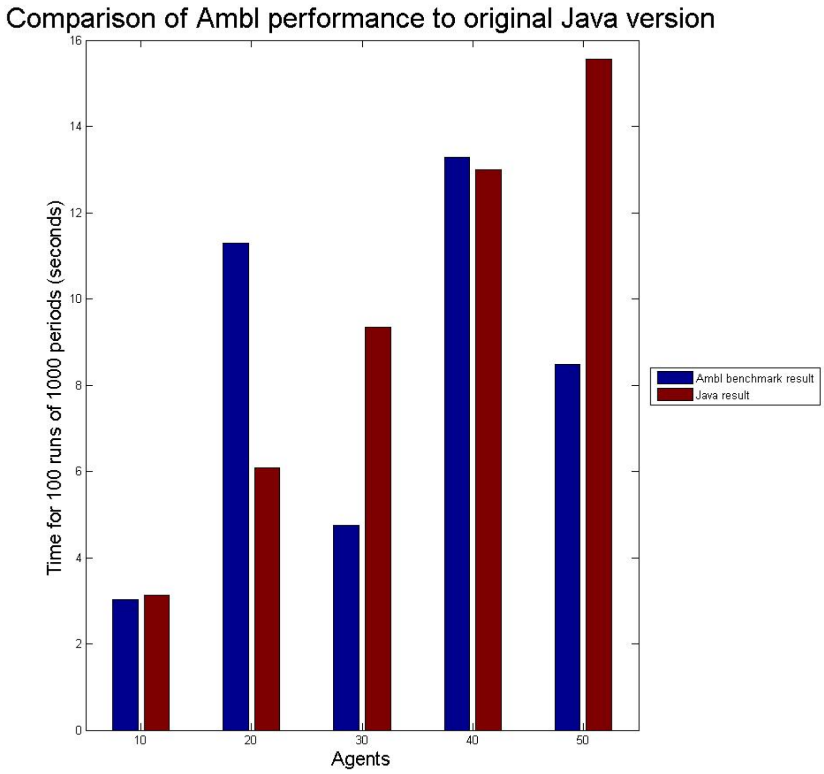

Figure 5.7: Total run time in seconds for 100 runs of 1000 periods, comparing the Java version to the Ambl version as we vary the number of agents in the networked economy. This is not an entirely fair test as the Ambl version has a built in benchmarking mode which turns off result collection so that you are just testing the main algorithm; the Java version is building up arrays of results (though not outputting them).

[32]The idea of an adapter is to take a piece of code and create a wrapper round it (an adapter) so that it can fit in with other code which expects certain functionality. The authoritative reference is Gamma et al. [1994]. Here we would actually be replacing Ambl code at the agent and world level with a compiled Java replacement. A simple adapter World could be set up to call the Java code appropriately but also to use the original Ambl code for documentation and so on.

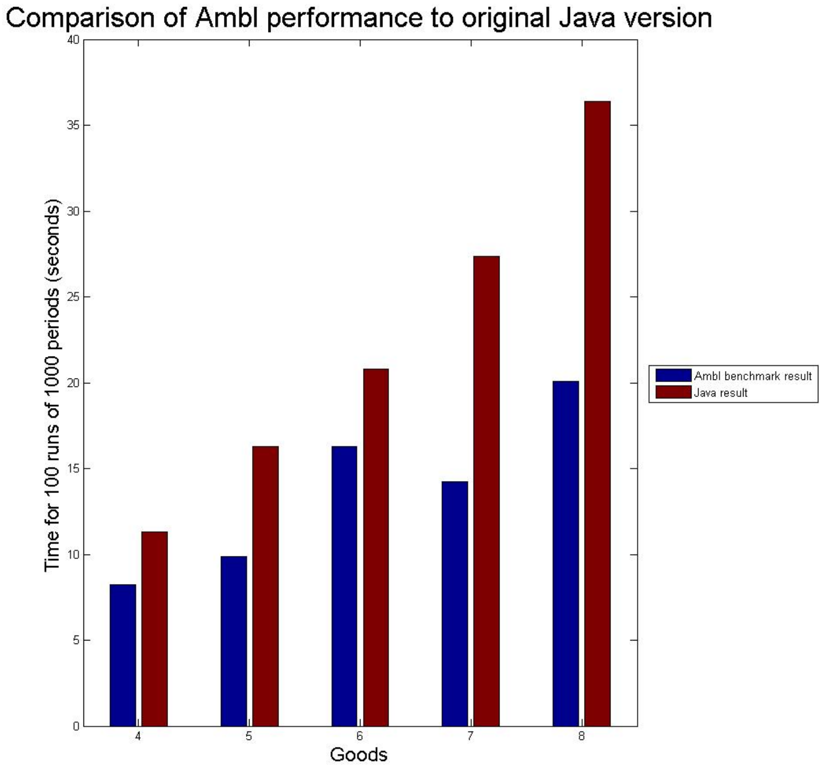

Figure 5.8: n
Total run time in seconds for 100 runs of 1000 periods, comparing the Java version to the Ambl version as we vary the number of goods in the networked economy. This is not an entirely fair test as the Ambl version has a built in benchmarking mode which turns off result collection so that you are just testing the main algorithm; the Java version is building up arrays of results (though not outputting them).

5.8 Future work

Hopefully this chapter has convinced you both of the feasibility and usefulness for agent-based modelling of version control, testing and thinking about agent-based models in a more holistic project-based way. Each of these ideas to some extent is included or at least could be included by most general purpose agent-based modelling systems. Our argument is that they should be, perhaps to a greater extent. Explicit support for many of these ideas could make them more accessible to newcomers and generally encourage their use. Proper testing of models should be an essential part of agent-based modelling and there are a lot of existing tools from software development more generally which could be used for this purpose.

There are various directions one could take to build on Abml in order to to make this into a 'full framework' (as I've argued it is already useful for many purposes). There is a clear need for improved documentation, both in the form of a reference and tutorials for newcomers; though one can say the same about most agent-based modelling frameworks, or indeed most software frameworks in general. The appendices B and C along with the source code, for those with the requisite skill in Ruby, are starting points.

In terms of modelling currently there is only explicitly support for time in the form of periods; one can add continuous time modelling via the Gillespie algorithm, but it would be helpful to have explicit support for this kind of thing. Many agent-based modelling platforms offer some kind of "event driven" notion of processes; Ruby as a language can quite naturally support such ideas, but again some kind of explicit support might be useful, particularly if it allowed for automatic documentation generation along with explanatory diagrams.

Visualisation is, as remarked above, a key challenge. Ambl seeks to automate this as much as possible (at least for the initial version). One would ideally have a system where visualisations are automatically generated but where a tool exists to tweak or replace them as desired. There is a recently started open source project which aims to make these kinds of things, among others, much more accessible[33].

In terms of outputs a natural generalisation of the current system might be to think of a project and output in terms somewhat inspired by aspect orientated programming: we might have different aspects of our project (report, presentation, visualisations, models or custom combinations of these) which are woven together in various formats; so we could produce a LaTeX report or a web based model overview. Including more sophisticated model diagramming, perhaps something like UML, see

[33] `http://sciruby.com/`

Bersini [2012], could be useful (though it is easier to use this kind of thing when working in statically typed languages). One might also want to think about how to include actual write ups and how to work effectively with collaborators on projects using Ambl.

But Ambl is, in its current form, very much ready for actual agent-based modelling work. The agent-based models created for chapter 6 were created using Ambl (and are included as examples with Ambl). The only addition required was some code to calculate the average values of a property across periods for each agent; everything else was essentially 'pure' Ambl. In fact this highlights one of the key advantages of an internal domain specific language: when additional functionality is required for some special purpose it is typically easy to add.

Chapter 6

Strategic Segregation

6.1 Introduction

A pure equilibrium approach is unlikely to suffice for those major economic and social issues dealing with questions of economic or social change with non-trivial micro-macro interrelationships. Models which are either intrinsically dynamic or which can be straightforwardly extended to dynamic models are necessary but, at the microeconomic level in particular, relatively rare. In this chapter we consider a model where agents make choices about their effort, but where the outcome depends (stochastically) on the actions of others. It is thus, at one level, a strategic model. But it can form the basis of more sophisticated dynamic models, of models with simple local learning and (potentially) of policy discussions.

In our model agents are given the choice of whether to acquire human capital (to "get educated"), in which case with some probability they may escape poverty, achieve success and so on, or to remain at the status quo, in which case they have a guaranteed utility which is higher than that is they put in the effort but fail to succeed. The probability of success if they do put in effort depends non-trivially on the decisions of others in their community.

This kind of model can incorporate a wide range of social scenarios from the university attainment rates from the classroom of a developed country to urban migration from a village in a developing country; in each case an individual's probability of success depends on the action of their peers. Two major factors will be in play here: competition and cooperation, the model we present shortly can be used to model purely competitive cases, purely complementary cases and our benchmark case where both effects occur.

6.2 Related Work

I previously introduced the Schelling Segregation model (Schelling [1971]) in section 2.7 as a classic example of agent-based modelling[1], but it is additionally (and more specifically) a fundamental model of dynamic segregation. The typical area to which it is applied is residential segregation, but of course it can be applied more widely. Some of the major, more recent, efforts in modelling polarisation and segregation are explored in section 2.10. In the remainder of this section we look at some of the work more specifically relevant to our approach.

Roland Benabou looks at a model of residential segregation in Benabou [1993]. His model links residential choice, educational investment and production with distinct communities. Homogeneous agents choose between becoming high-skill or low skill workers or remaining outside formal labour force. It is a general equilibrium model, with the distribution of agents' abilities and tastes being determined endogenously. The key elements of the model are: a continuum of identical agents, a city: agents choose to live in one district or remain outside production (with constant returns to scale) and education as a local public good. An equilibrium solution for a fully integrated community (i.e. one district) is found; a simple dynamic is formulated to consider how agents might switch between skill sets. Under particular conditions [2] segregation occurs (concentration of high skill workers). Results are generalised for any number of districts.

Bowles et al. [2009] explores group inequality resulting from segregation and its resultant effects on the cost of human capital investment. In their model there are two social groups, parents invest in the human capital of their children without discounting (i.e. they fully internalise the resultant utility of their children). Occupational status is split into two categories, attaining the higher requires greater investment. The model considers a continuum of agents.

Output in period t is given by $f(h_t, l_t)$, where h_t (proportion of high skill workers) is less than s_t (proportion who have higher skills) and f satisfies constant returns to scale and diminishing marginal returns. $\tilde{h}$ is value of h for which marginal products are equal, so $h_t = \min\{s_t, \tilde{h}\}$.

There are two groups with population proportions β,$1-\beta$ and skill shares s_t^1,s_t^2. For every individual a proportion η of peers is from the group he belongs to (this is the *correlation ratio*) and he has a ability drawn from $G(a)$ with support $[0, \infty)$.

[1] In Schelling [2006] a retrospective account of how this primitive agent based model arose from manually running the model on paper is provided.

[2] Detailed on page 636, proposition 2.

Segregation is $\sigma_t^i = \eta s_t^i + (1-\eta)s_t$. Cost of human capital is function $c(a, \sigma)$ which assuming no discrimination and no discounting there will be a threshold level of ability above which individuals in each group will acquire the human capital.

This framework results in a dynamic specified by

$$s_t^i = G(\tilde{a}(\delta(s_t), \sigma_{t-1}^i))$$

where $\delta(s_t)$ is anticipated future wage differential. Depending on initial level of inequality and level of segregation we get different (persistent) outcomes.

In Mookherjee et al. [2010] looks at both the model from Bowles et al. [2009] that we have just examined and a model of the authors. In their model roughly speaking there is a 1-dimensional continuum geography which locally determine agents' behaviour. A location determines cost of investing in skills; this is a combination of global and local factors. There are global wages for low and high skills (determined by the fraction at each skill level). At each location there will be a resultant skill proportion (the proportion of agents at that point with skills). The resulting equilibrium can be symmetric (they use the term unsegregated) or asymmetric (where the geography is split into alternating regions of investing and non-investing). In a somewhat similar approach in Ray and Robson [2010] the idea of status is examined using this kind of model.

In Hanaki et al. [2011] a model of car parking is studied, as an example of situation where agents may learn to be 'lucky'. The basic idea is that there is a group of workers, a one-way street where you can choose a point a which to take the next available parking space (it is assumed you cannot look ahead); so their strategy is some $k \in K$, where K is the set of spaces[3]. If no parking spaces are free you end up in car park paying a fixed cost, whereas if you find a space you pay a cost proportional to the distance from the workplace.

Initially this model is analysed from a Nash equilibrium perspective. There may be both symmetric Nash equilibria (the details depend on the model, but a one might be where all agents take the first available space) or most interestingly asymmetric equilibria where agents may be in 'lucky' groups or 'unlucky' groups; that is there are some that because they expect a better result are in fact more likely to obtain it; there may in fact be a totally asymmetric equilibrium where agents $1, \ldots, N$ have strategies to park in spaces that are some permutation of $(1, \ldots, N)$; that is they each end up with a particular space, and if they defect there is some probability of ending up in the car park such that their expected utility is worse

[3]There may be more or less spaces than agents.

that settling for their own space, however bad it might be.

The model is then formulated as a dynamic one with agent learning. In each period there is a fixed probability of an agent entering or leaving the city and there is a set of period $t \in \{1, 2, \ldots\}$. Depending on the model parameters regions where symmetric and asymmetric outcomes are stable can be identified. This kind of model is attractive as it should be applicable to a wide range of important situations where you can't go back on a choice, such as a job offer or dating. Interestingly it suggests that instead of supposing internal and heterogeneous risk appetites, the response to risk may be learned.

6.3 Our model

In our model agents are given the choice of whether to put in effort or to remain at the status quo. It is a strategic model, as the probability of success depends on the choices of other agents. In this section we precisely define the general form of the model then consider three major types of the model, carrying out an equilibrium characterisation for each. In this section we focus on a continuum version of the model with homogeneous agents as it makes the concepts involved clearer. We develop a discrete version of the model with heterogeneous agents in a later section.

Our economy E consists of a continuum of individuals. Each agent has a base utility u_l, if they put in effort there is cost e and if they succeed (as determined by a economy wide probability of success) they attain utility u_h. That is for an agent i:

$$u_i = \begin{cases} u_h - e & \text{if } i \text{ attains high status} \\ u_l - e & \text{if } i \text{ aspires to high status but fails to attain it} \\ u_l & \text{if } i \text{ doesn't aspire to high status} \end{cases}$$

We assume that $u_h - a_h > u_l$ (and $u_l > u_l - a_h$). Let the probability of success for an agent be a function $q(m_h)$, where $m_h \in [0, 1]$ is the measure of agents who do put in the effort e.

6.3.1 Complementarity model

First we consider the case where effort by one's peers is complementary to one's own effort, that is q is increasing in m_h. Consider the linear form of q_m (or course we could consider more complex increasing functions but qualitatively the same conclusions would hold, at least generically with respect to the Nash equilibrium characterisation). Let $q_m = \rho m_h$, where $\rho \in (0, 1]$. Then if the agent puts in effort e

$$E(u) = \rho m_h u_h + (1 - \rho m_h)u_l - e.$$

otherwise

$$E(u) = u_l$$

As $u_h > u_l$, for suitable values of ρ we have the situation illustrated in figure 6.1

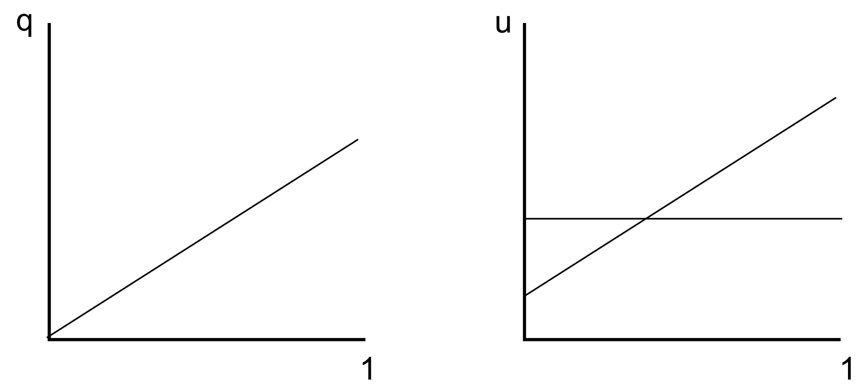

Figure 6.1: The left hand graph just shows the probability of success given effort as it increases with the fraction of agents m_h putting in effort e. The right hand figure shows the expected utility depending on the choice of the agent, given the measure of agents putting in the effort. The constant utility is for no effort, the linearly increasing line if effort is expended. When these lines do not intersect the outcome will be trivial.

There are up to three qualitatively distinct Nash equilibria: $m_h = 0$, $m_h = 1$ and an unstable Nash Equilibria where $m_h = \frac{e+u_l}{\rho(u_h-u_l)+u_l}$. The third is clearly unstable, as any additional non-zero fraction of agents expending effort e_h will make the choice of effort e_h optimal for all agents. The second equilibrium is a strict Pareto improvement on the first, though if ρ is small relative to $\frac{u_l}{u_h}$ this equilibrium may not exist.

6.3.2 Competitive model

Now consider the case where effort by one's peers is competitive with one's own effort, that is q is decreasing in m_h. Consider the linear form of q_m (once more it should be noted that while we could consider more complex decreasing functions qualitatively the same conclusions would hold, at least generically with respect to the Nash equilibrium characterisation). Let $q_m = \beta - \rho m_h$, where $\beta, \rho \in (0, 1]$ and $\beta u_h - e > u_l$. Then if the agent puts in effort e:

$$E(u) = (\beta - \rho m_h)u_h + (1 - \beta + \rho m_h)u_l - e.$$

otherwise

$$E(u) = u_l$$

As $u_h > u_l$, for suitable values of β, ρ we have the situation illustrated in figure 6.2

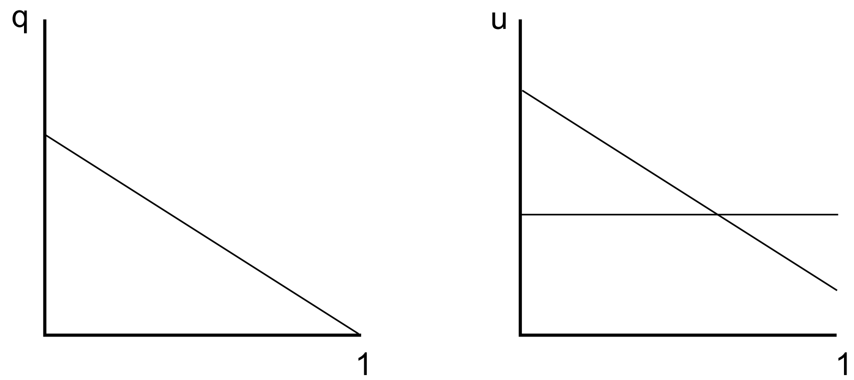

Figure 6.2: The left hand graph just shows the probability of success given effort as it decreases with the fraction of agents m_h putting in effort e. The right hand figure shows the expected utility depending on the choice of the agent, given the measure of agents putting in the effort. The constant utility is for no effort (maintaining status quo), the linearly decreasing line if effort is expended. When these lines do not intersect the outcome will be trivial.

In this case there is one Nash equilibrium where our two alternative actions intersect, that is $m_h = \frac{\beta(u_h - u_l) - e}{\rho(u_h - u_l)}$. Where they do not intersect there is a trivial equilibrium.

6.3.3 Benchmark model

We turn now to the basic form of what will be our benchmark model. We want to incorporate both competition and complementarity in our model and our benchmark case is that of an inverted u-shaped probability of success curve. The idea here is that if no other, or only a small number of other agents are putting in effort then it makes it highly unlikely that any agent will succeed. As more agents put in effort the probability of success increases. However, once a certain threshold is crossed the probability decreases. This idea could include a wide variety of situations, particularly with respect to eduction of a group and its effect on attainment within a wider society which is competitive.

The most interesting of the qualitative variations of such a model is illustrated in figure 6.3 (as when there are fewer intersections we are either in a trivial case or in one of the previous two cases). You can see that in a local sense we include the equilibria of the previous models: there is the Pareto suboptimal Nash equilibrium with $m_h = 0$, a Pareto optimal equilibrium with positive $m_h < 1$. And again there is an unstable Nash equilibrium.

In order to be more precise let us consider the purest form of this model in figure 6.4. Here we have a linearly increasing probability of success with increasing participation until some threshold after which the probability linearly decreases. We analyse this form, though the results obtained could be applied to any functional form of probability of success which is initially strictly increasing, then after a threshold is strictly decreasing. Here we have some threshold proportion of agents n such that:

$$q(m_h) = \begin{cases} \underline{\rho} m_h & \text{if } m_h < n \\ \beta - \overline{\rho} m_h & \text{if } m_h \geq n \end{cases}$$

where $\rho \in$ and $\beta = n(\underline{\rho} + \overline{\rho})$, that is for consistency we require $n\underline{\rho} = \beta - \overline{\rho}$. For this scenario there are up to three Nash equilibria. There is a stable Nash equilibrium where $m_h = 0$ as at this point the expected utility of making effort e_h is $u_l - e_h < u_l$. There may be an additional stable Nash equilibrium if $\beta - \overline{\rho} < u_l$. In this case

$$m_h = \frac{\beta(u_h - u_l) - e_h}{\overline{\rho}(u_h - u_l)}$$

is that Nash equilibrium. There is an unstable Nash equilibrium, as in the complementarity case, of

$$m_h = \frac{e_h + u_l}{\underline{\rho}(u_h - u_l) + u_l}.$$

6.4 Finite set of agents

Assuming a continuum of agents makes the intuition clearer and is in keeping with related work such as Bowles et al. [2009]. However, as the population of agents under consideration may be a school class, a village or some other such population, the model's applicability to discrete, finite populations is an important matter. In this section we analyse the finite analogue of the model presented in the previous

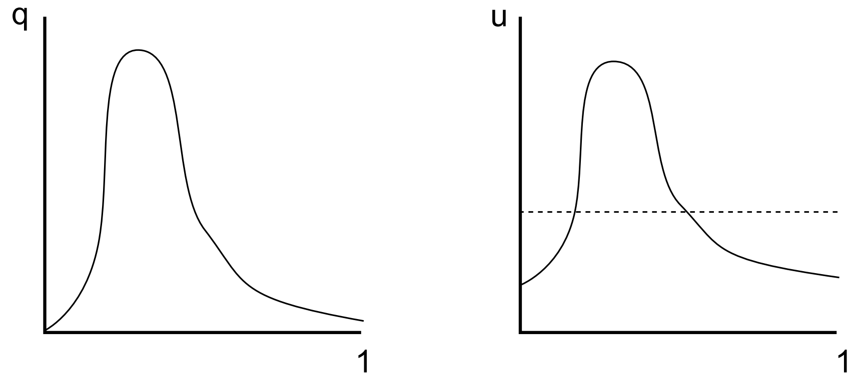

Figure 6.3: The left hand graph just shows the probability of success given effort as it varies with the fraction of agents m_h putting in effort e. The right hand figure shows the expected utility depending on the choice of the agent, given the measure of agents putting in the effort. The line of constant utility is for no effort, the other shows how expected utility varies with the proportion putting in effort.

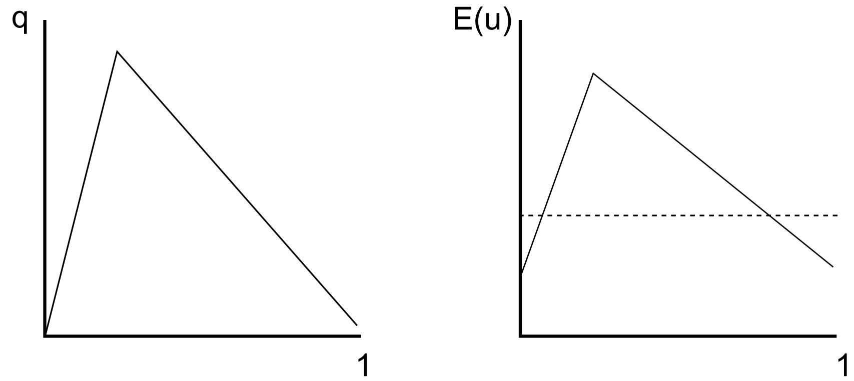

Figure 6.4: This figure is the linear special case of figure 6.3. It allows for clearer analysis but, at least in terms of equilibria, the results should be qualitatively the same.

section.

We assume that we have agents $i \in \{1, \ldots, I\}$ and as before the probability of success q depends on the fraction of agents who put in effort e_h. Letting this fraction be r_h we explore the finite versions of our model. In the below analysis we assume we have a sufficient number of agents for the model to be meaningful; so we have strictly positive fractions of agents in each partition (as required for identifying Nash equilibria).

6.4.1 Complementarity and Competitive scenarios

The complementarity scenario is now straightforward. Generically we cannot have a fraction of agents r_h putting in effort h such that $r_h = \frac{e+u_l}{\rho(u_h-u_l)+u_l}$; so the unstable Nash equilibrium will not exist. This leaves us with up to two Nash equilibria which will be on the boundaries, that is $r_h = 0$ and $r_h = 1$. The later exists if $\rho u_h \geq u_l$.

The competitive model is more interesting for a finite set of agents. Assuming $u_h\beta > u_l$ and $u_l > u_h(\beta - \rho)$ then generically there exists a least proportion $\tilde{r}_h$ of agents strictly larger than $\frac{\beta(u_h-u_l)-e_h}{\rho(u_h-u_l)}$.

Proposition 5. *The proportion $\tilde{r}_h$ is the unique Nash Equilibrium for the competitive model, when $u_h\beta > u_l$ and $u_l > u_h(\beta - \rho)$.*

Proof. At $\tilde{r}_h$ there are two classes of agents, those currently putting in effort e_h and those not. Consider unilateral deviations for each class. For those putting in effort e_h their expected utility is $(\beta - \rho\tilde{r}_h)u_h > (\beta - \rho m_h)u_h = u_l$. That is their current expected utility is greater than if they deviate. Now consider an agent with effort $e = 0$. Their current (expected) utility is u_l; if they deviate it will be $(\beta - \rho(I\tilde{r}_h + 1)/I)u_h$ but by definition $\tilde{r}_h$ is the least proportion of agents larger than $\frac{\beta(u_h-u_l)-e_h}{\rho(u_h-u_l)}$ so the expected utility upon deviation will be less than u_l. Therefore $\tilde{r}_h$ is a Nash equilibrium.

For any $r_h > \tilde{r_h}$ the expected utility upon deviation for those agents who do put in effort e_h is strictly larger than their current utility (so it cannot be a Nash equilibrium) and for $r_h < \tilde{r_h}$ the same holds for agents who are not putting in effort e_h. Therefore $\tilde{r_h}$ is the unique Nash equilibrium when $u_h\beta > u_l$ and $u_l > u_h(\beta - \rho)$. □

6.4.2 Benchmark scenario

We turn now to our benchmark model for a finite set of agents. For this we will have up to two Nash equilibria. Again, generically, the unstable Nash equilibrium will not exist (as it would require a specific proportion of agents to be possible). As with the complementarity case we will have a Nash equilibrium at $r_h = 0$. There may be a further Nash equilibrium when I is large enough and $u_h\beta - n\rho_2 > u_l$ and $u_l > u_h(\beta - \rho_2)$. Generically there exists a least proportion $\tilde{r}_h$ of agents strictly larger than $\frac{\beta(u_h-u_l)-e_h}{\rho(u_h-u_l)}$, this is a second Nash equilibrium.

Proposition 6. *For the finite version of the benchmark model with large enough I there are up to two Nash equilibria. One where $r_h = 0$ and a second at $\tilde{r}_h$.*

Proof. The $r_h = 0$ case is straightforward as we only need to consider deviations of any agent from no effort to e_h. But for I large enough the expected utility upon deviation $\rho_1 u_h + (1-\rho_1)u_l - e_h < u_l$.

Now consider $\tilde{r}_h$. The key point to note is that for Nash equilibrium characterisation we consider only unilateral deviations. For I large enough we are considering a situation that is locally similar to that in proposition 5; so in terms of unilateral deviations (and hence Nash equilibrium) the same result holds. □

6.5 The long run behaviour of the economy

In both our benchmark case and in the complementarity case multiple stable Nash Equilibria arise. Using more sophisticated dynamic models it may be possible to say more; indeed we pick up this thread in section 6.7. However using the criterion of stochastic stability, see Young [1993, 2008], we can analyse the long run behaviour of the model. Instead of calculating the Nash Equilibria we ask in which states the economy spends most of its time when agents play best responses to a sample of the history of the system and there is a small probability of making a mistake.

We consider a version of our economy with N agents. In each period a single agent is drawn and with probability $1-\epsilon$ plays their best response to a sample of the history of the economy. With probability ϵ they experiment and play a strategy at random; in our case this means that with probability $\frac{\epsilon}{2}$ they play their best response and with $\frac{\epsilon}{2}$ a strategy which is not.

Let us consider a sample size and history size of N (so we sample the entire history)[4]. Let the the number of agents in that history h_t at period t who have chosen e_h be n_h. The agent chooses the best response $BR(n_h)$ and this is added to the history. So we either have a transition from n_h to $n_h - 1$, n_h or $n_h + 1$. At

[4]Keeping a sample history size of N is in keeping with the idea of an agent responding to the behaviour of the population as a whole. One could consider a more complex model where all N players simultaneously make a choice but such a model would be both subject to switching behaviour and, aside perhaps from very small values of N, intractable. If one wishes to pursue a more complex dynamic approach then one is probably better using numerical techniques as we do in section 6.7.

the boundaries of 0 and N only two of these will be possible. This is illustrated in figure 6.5.

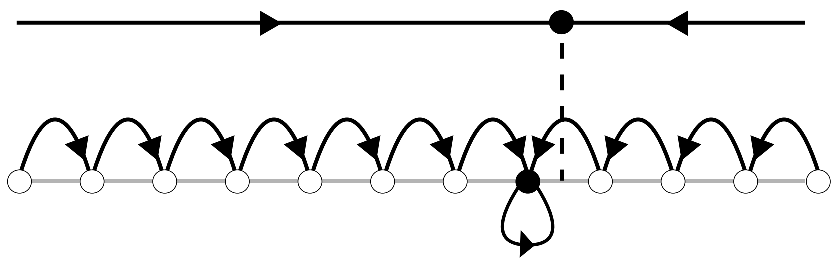

Figure 6.5: This figure shows the Markov chain for an economy of size N, with history and sample size N and (above it) how this relates to the best response dynamics for the system with a continuum of agents. In each period an agent plays their best response to the last N strategies. For this scenario there is only one Nash equilibrium and the dynamics are entirely deterministic as the economy moves towards this Nash equilibrium. We could include experimentation but this would not change the basic behaviour.

Shortly we consider two cases: versions of the complementarity case and the benchmark case from before. We do not analyse the competitive case in this way as it has only a single, stable Nash equilibrium so any such process should result in long run behaviour which spends essentially all of its time at this equilibrium.

6.5.1 Complementarity case

Consider the complementarity case with base utilities for high and low achievement of u_h and u_l respectively, fix parameters ρ and e_h. Index agents by $1, \ldots, N$ and let $\tilde{n}$ be the largest index such that $u_h(\rho\frac{\tilde{n}}{N}) + u_l(1-\rho\frac{\tilde{n}}{N}) - e_h < u_l$.

Proposition 7. *For large enough N, if $\frac{e+u_l}{\rho(u_h-u_l)+u_l} > \frac{1}{2}$ then N is stochastically stable, whereas if $\frac{e+u_l}{\rho(u_h-u_l)+u_l} < \frac{1}{2}$ then 0 is stochastically stable.*

Proof. We have set of states $\{0, 1, \ldots, \tilde{n}, \ldots, N\}$, that is the number of agents who put in effort e_h in our history of length N. We have a finite irreducible aperiodic[5] Markov chain so it must be ergodic, that is there will be a long run probability μ_i of being in state i which is independent of our starting state.

In order to determine stochastic stable states we do not need to precisely calculate this long run probability, instead we characterise it to orders of ϵ. The

[5]Recall that we can have transitions from a state to itself, though we have omitted these from the figure for clarity.

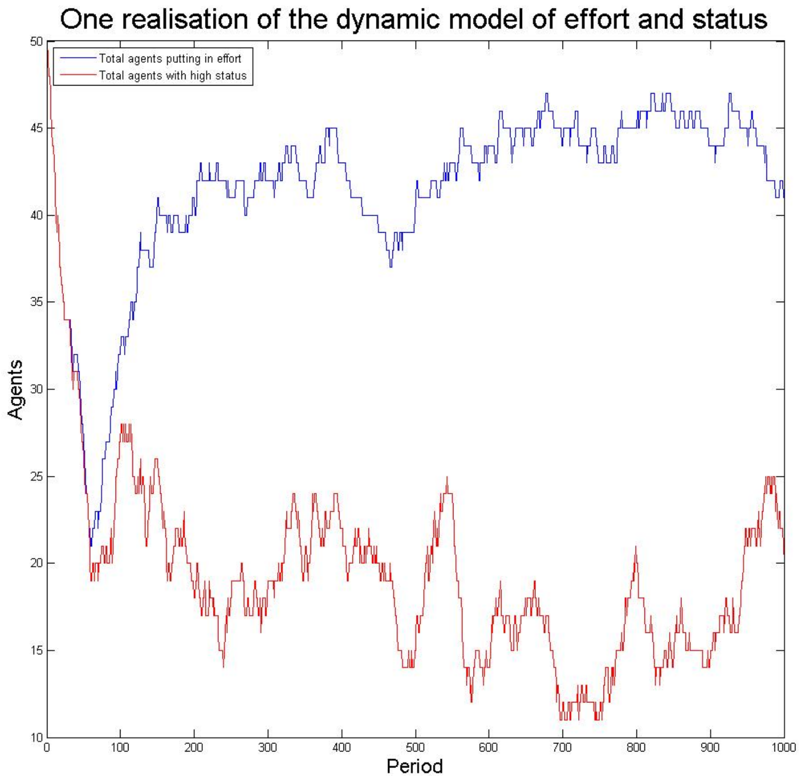

Figure 6.6: This figure shows an example realisation of our dynamic process which we consider from the angle of stochastic stability. Here we have set ϵ at a high value (0.1) so there is quite a lot of noise but the state, after an initial transition from our starting state, seems to remain close to a Nash equilibrium of the model.

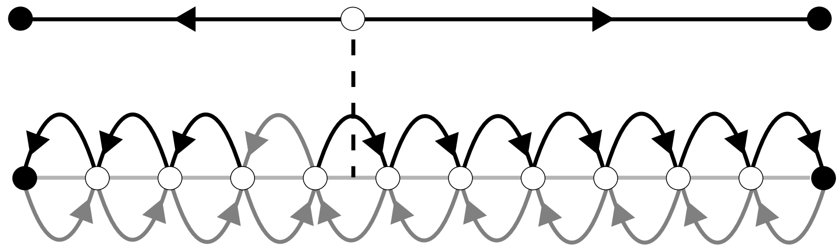

Figure 6.7: Here we consider the complementarity case. We do not label transitions from a state to itself, but label the dominant transitions in black. Transitions only possible via experimentation (or mistakes) are labelled in grey. Analogously to the continuum system (included above the finite system) we have two boundary Nash equilibria and similar dynamics. The long run behaviour (with experimentation or mistakes) will however depend on the details of the system and is characterised in proposition 7.

Markov chain is one dimensional so we know that the probability of being in a state i is proportional to the product of probabilities on edges of the directed tree of transitions towards state i. So if we let R_n be the probability of a right-transitions from state n and L_n be the probability of a left transition from a state n. Then the long run probability of being in state i is proportional to

$$\prod_{n<i} R_n \prod_{n>i} L_n.$$

So consider the probabilities of being in each of the Nash equilibria, which for this case are 0 and N, to orders of ϵ. So $\mu_0 \propto \prod_{n>0} L_n$. That is $\mu_0 \propto \epsilon^{N-\tilde{n}+1}$ whereas $\mu_N \propto \prod_{n<N} R_n$, that is $\mu_N \propto \epsilon^{\tilde{n}}$. The stochastically stable state(s) is the Nash equilibrium which in the long run the process spends almost all of its time at. So for this case if $\tilde{n} < \frac{N-1}{2}$ then N is stochastically stable, whereas if $\tilde{n} > \frac{N-1}{2}$ then 0 is stochastically stable.

If we are considering large values for N then can neglect the -1 constant term, so essentially we must consider the value of $\frac{e+u_l}{\rho(u_h-u_l)+u_l}$. From the above it follows that if $\frac{e+u_l}{\rho(u_h-u_l)+u_l} > \frac{1}{2}$ then for large enough N, N is stochastically stable. Similarly if $\frac{e+u_l}{\rho(u_h-u_l)+u_l} < \frac{1}{2}$ then 0 is stochastically stable for N large enough. $\square$

6.5.2 Benchmark case

Consider the benchmark case with base utilities for high and low achievement of u_h and u_l respectively, fix parameters $\underline{\rho}$, $\overline{\rho}$ and e_h. Index agents by $1, \ldots, N$ and let n_1

be the largest index such that

$$\frac{n_1}{N} < \frac{e_h + u_l}{\underline{\rho}(u_h - u_l) + u_l}.$$

And let n_2 be largest index such that

$$\frac{n_2}{N} < \frac{e_h + u_l}{\underline{\rho}(u_h - u_l) + u_l}.$$

In this system states 0 and n_2 correspond to the Nash equilibria.

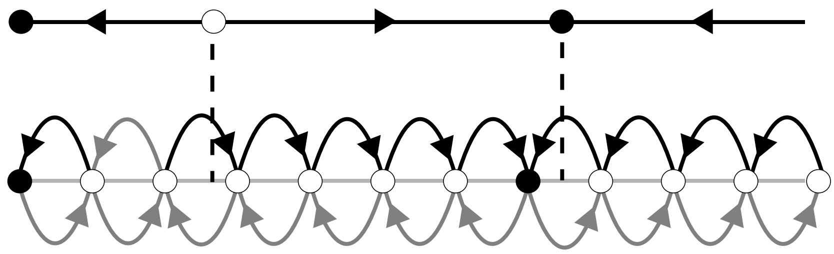

Figure 6.8: Here we consider our benchmark case. Again we do not label transitions from a state to itself and label the dominant transitions in black. Transitions only possible via experimentation (or mistakes) are labelled in grey. Analogously to the continuum system (included above the finite system) we have two Nash equilibria and similar dynamics (when there are a large enough number of agents). The long run behaviour will depend on the details of the system and is characterised in proposition 8.

Proposition 8. *Let m_1, m_2 be the unstable and stable interior Nash equilibria of the benchmark model for a continuum of agents. For N large enough if $m_1 < m_2$ then n_2 is stochastically stable. And if $m_1 > m_2$ then 0 is stochastically stable.*

Proof. Similarly to proposition 7 we have set of states of our Markov chain $\{0, 1, \dots N\}$, that is the number of agents who put in effort e_h in our history of length N. We have a finite irreducible aperiodic Markov chain so it must be ergodic, that is there will be a long run probability μ_i of being in state i which is independent of our starting state.

Once again in order to determine stochastic stable states we do not need to precisely calculate this long run probability, instead we characterise it to orders of ϵ. The Markov chain is one dimensional so we know that the probability of being in a state i is proportional to the product of probabilities on edges of the directed tree of transitions towards state i. As in the proof of proposition 7 let R_n be the probability of a right-transitions from state n and L_n be the probability of a left transition from a state n.

We consider the relative probabilities of each Nash equilibrium up to orders of ϵ. So $\mu_0 \propto \prod_{n>0} L_n$. That is

$$\mu_0 \propto \epsilon^{n_2 - n_1 + 1}.$$

For the other Nash equilibrium, $\mu_{n_2} \propto \prod_{n<n_2} R_n \prod_{n>n_2} L_n$, that is $\mu_N \propto \epsilon^{n_1}$. The stochastically stable state(s) is the Nash equilibrium which in the long run the process spends almost all of its time at. So for this case if $n_1 < \frac{n_2-1}{2}$ then n_2 is stochastically stable, whereas if $n_1 > \frac{n_2-1}{2}$ then 0 is stochastically stable.

If we are considering large values for N (and thus n_1, n_2) then we can neglect the constant term, so essentially we must consider the relative values of n_1, n_2 and N or analogously the interior Nash equilibria (of the continuous system). Let

$$m_1 = \frac{e_h + u_l}{\underline{\rho}(u_h - u_l) + u_l}$$

the unstable Nash equilibrium of the benchmark model and

$$m_2 = \frac{\beta(u_h - u_l) - e_h}{2\overline{\rho}(u_h - u_l)}$$

the stable interior Nash equilibrium of the benchmark model. So for N large enough if $m_1 < m_2$ then n_2 is stochastically stable. And if $m_1 > m_2$ then 0 is stochastically stable. $\square$

6.6 Agent heterogeneity

So far we have assumed homogeneous agents, however it is fairly natural to consider heterogeneity with respect to cost of effort, which we do in this section. There are perhaps two more obvious ways of framing heterogeneity. We could think of an agent i as having a particular intrinsic skill level s_i in which their cost of effort decreases. Alternatively, and perhaps more satisfactorily we can think of variation in the cost of human capital acquisition, say some c_i, which determines the cost of effort. Depending on the issue under consideration either may be the more natural framing, but as we can consider cost of effort as either $\frac{e_h}{s_i}$ of $e_h c_i$ the are essentially equivalent.

6.6.1 Continuum of agents

If we have a cost of effort e we introduce a new cost of effort for individual i of $e_i = \frac{e}{s_i}$ where s_i is the skill of that individual. To simplify matters we consider a continuous distribution of skill with full support over some range in skill[6]. The result of a change in skill is to shift the expected utility upon effort up or down. While this may be problematic out-of-equilibrium, in equilibrium we can assume that agents are in a sense ordered, as the following result shows.

Proposition 9. *For the model with a continuum of agents, in equilibrium there exists two disjoint subsets (one possibly empty) of total measure* 1*, such that the skill of any agent s_1 in the first set S_1 is greater than the skill of any agent s_2 in the other set S_2 and where agents in S_1 put in effort e and agents in S_2 do not.*

Proof. Let S_1 and S_2 be the sets of agents putting in effort and not putting in effort respectively. If we have a boundary equilibrium the result holds trivially (one of S_1 or S_2 is empty) . Assume we have an interior equilibrium and let $s_3 \in S_1$ be an agent with skill strictly greater than some agent $s_4 \in S_2$. In equilibrium both agents must not benefit from unilateral deviation, so when p is probability of an agent succeeding we require $pu_h + (1-p)u_l - e_h s_3 \geq u_l$ and $u_l \geq pu_h + (1-p)u_l - e_h s_4$. That is $s_4 \leq s_3$, a contradiction. Therefore we must have the ordering result. □

6.6.2 Sub-economies

Figure 6.9 illustrate how we can deal with more general cases within this framework. For the preceding results we are only really interested in those parameter regimes where reserve utility u_l line is crossed (perhaps multiple times) by the expected utility given a proportion of agents putting in effort curve. If there is a particular proportion of agents, such that for any such agent i with a skill level such that $q(m_h)u_h + (1-q(m_h))u_l - e_h c_i > u_l$ for all m_h or $q(m_h)u_h + (1-q(m_h))u_l - e_h c_i < u_l$ for all m_h (or the analogous finite situation) then we can simply remove this portion from consideration as their decision is independent of others. Of course if we were considering policies which might reduce or increase the cost of effort for these agents then they once more can be explicitly considered within the model.

6.6.3 Finite set of agents

Assume we have a set of I agents indexed by i with the cost of human capital acquisition c_i of each agent drawn from the distribution of human capital acquisition

[6]We will look at this again when considering discrete agents.

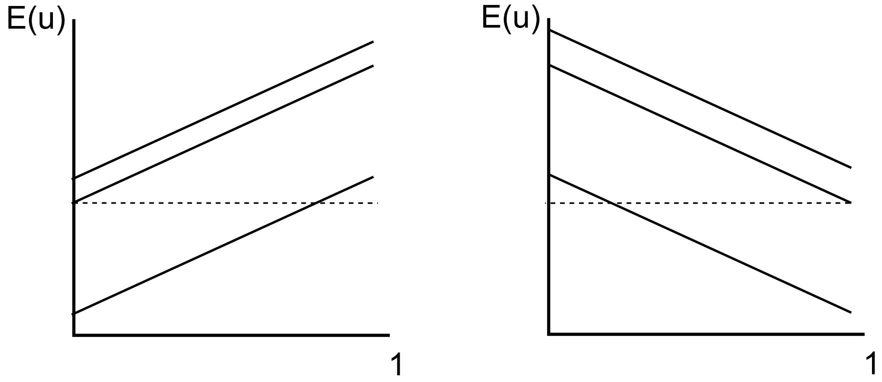

Figure 6.9: When considering behaviour there may be a proportion of agents whose behaviour is unaffected by the proportion of agents putting in effort. We can remove this proportion from consideration as we can take it to be fixed.

in the continuous form of the model. In propositions 5 and 6 we identified the Nash equilibria for the finite model. While introducing heterogeneity may mean that we do not obtain such a clear characterisation we can still obtain an analogous result to proposition 9 as long as the variation in cost is large relative to the variation driven by the economy wide probability of success.

Without loss of generality re-index agents such that $c_i < c_j$ for $1 \leq i < j \leq I$ and let

$$\Delta q_{\max} = \max_{i \in 1 \ldots I-1, j=i+1} \{|q(m_j) - q(m_i)|\}$$

and let

$$\Delta c_{\min} = \min_{i \neq j} \{|c_i - c_j|\}$$

that is let $\Delta q_{\max}$ be the largest change in probability of success from a change of one agent's effort and let $\Delta c_{\min}$ be the smallest variation in cost between any two agents. As agents are ordered we only need to consider consecutive pairs of i, j. Now let the *distinguishable cost condition* be

$$(u_h - u_l)\Delta q_{\max} < e_h \Delta c_{\min}.$$

Proposition 10. *For a finite set of agents, in equilibrium there exists two disjoint subsets (one possibly empty) of total proportion* 1*, such that the cost of any agent* c_1 *in the first set* C_1 *is less than the cost of any agent* c_2 *in the other set* C_2 *and where agents in* C_1 *put in effort* e_h *and agents in* C_2 *do not, if the distinguishable* *cost condition holds.*

Proof. The distinguishable cost condition means that for agents with costs $c_1 < c_2$ if $e_1 = 0$ then $e_2 = 0$ as $c_1 - c_2 > \Delta c_{\min}$, so by the distinguishable cost condition the change in q is sufficiently small to implies that our two sets are ordered in the sense of this proposition. $\square$

The distinguishable cost condition is sufficient, but not necessary. In particular forms of the model a weaker condition could replace it.

6.6.4 The marginal agent: equilibrium characterisation for heterogeneous agents

The important thing to consider for a particular measure of agents putting in the effort is the marginal utility *over agents*. Figure 6.10 illustrates this.

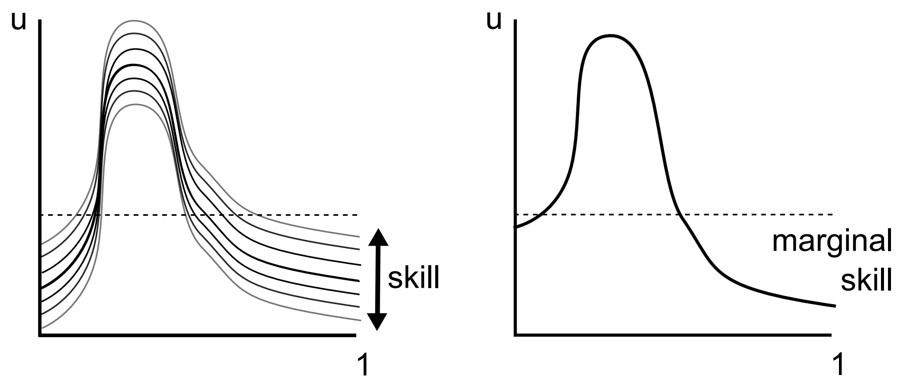

Figure 6.10: The left hand figure illustrates how the expected utility varies with cost of human capital acquisition (it shifts up or down). When considering the macro behaviour the relevant consideration is the marginal agent's expected utility. We are considering the expected utility of the lowest skilled agent who is putting in effort or the highest skilled agent who is not.

In moving from the expected utility curve of the agent with the lowest cost of effort to the curve for the marginal agent the effect is that of adding a decreasing function to this curve. This is straightforward in both continuum and finite cases. It is clear that the curve for the marginal agent will have the same qualitative form as that for any particular agent; so our original results hold.

However things get more complex for our complementarity and benchmark models. Restricting our attention to the models with a continuum of agents and equilibrium existence we can make a number of observations. Firstly, for the benchmark model, the equilibrium characterisation for the decreasing portion of the expected utility curves is as before (as each of the individual curves are decreasing, the marginal agent's curve must also be decreasing). Essentially we only need to focus on the increasing portion of the agents' expected utility curves (that is to focus on the competitive model).

Here we can, if skill is sufficiently stratified with respect to the expected utility curve, we can get additional pairs of unstable and stable Nash equilibria as 'gaps' in skill create differentiated layers of skills. This is illustrated in figure 6.11. While the precise form of any such model will be very dependent on the distribution of skill; when skill is sufficiently evenly distributed and the probability of success of certain forms this kind of differentiation will not occur, as the following example (Example 6.6.4) illustrates.

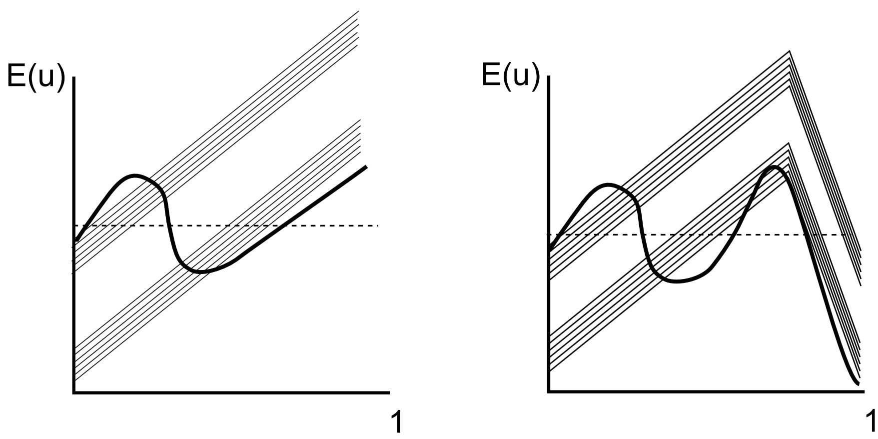

Figure 6.11: Here we can see how layers of equilibria may occur. The bold line is the expected utility of the marginal agent and the other lines are representing regions of the skill interval with greater probability (or in the finite case actual skill levels).

Example Consider the complementarity case, with cost of human capital acquisition distributed uniformly on an interval $[c_1, c_2]$ such that $u_l - e_h c_1 < \rho u_h + (1 - \rho)u_l - e_h c_2$; this entails that the lowest skilled agent's expected utility upon putting in effort (when the rest of the agents are putting in effort) is strictly larger than the highest skilled agent's expected utility upon putting in effort when a measure 0 of the rest of the economy is also putting in effort.

The marginal agent will have expected utility upon effort e_h of

$$u_h \rho m_h + (1 - \rho m_h)u_l - (c_2 - c_1)m_h e_h.$$

The key term is the $-(c_2 - c_1)m_h e_h$; while is decreasing in m_h, which could be problematic if it decreases faster than the $u_h\rho m_h + (1 - \rho m_h)u_l$ increases (and it decreases sufficiently such that the expected utility having gone above u_l once more goes below it). The first derivative with respect to m_h of expected marginal utility is

$$\rho(u_h - u_l) - (c_2 - c_1)e_h$$

so we want to know[7] if $\rho(u_h - u_l) > (c_2 - c_1)e_h$. But this follows immediately from our example definition.

6.7 Endogenous Segregation or Polarisation

In this section we consider a variation on our dynamic model, whereby there is only one type of agent, but we have local learning. We use this as a way of thinking about endogenous segregation. In contrast to say the model of segregation in Schelling [1971] or more recent work mentioned in the introduction to this chapter, the segregation, if it arises, would be fully endogenous (rather than as a result of membership of pre-existing communities).

We extend our benchmark model to a lattice of agents, each of which is in one of two states (low and high). Each knows what it and its neighbours did in the last m periods, where m is a memory length. In particular it can form an estimate of its probability of success based on assumed similarity to it and its neighbours experiences in previous periods. A basis for thinking about similarity based learning can be found in Gilboa et al. [2010] but in our model we take a very simple form; the average of the all the experiences. In our model we either put in effort or not; so only those periods where an agent and its neighbours put in effort are counted and of these we establish an individual estimate of the probability of success. From this estimate and knowledge of the cost of effort e, the reserve utility u_l and high status utility u_h the agent can decide on the basis of maximising expected utility whether to put in effort or not.

The aggregate outcome is determined as before: depending on the fraction putting in effort there is a global probability of success which is applied to each agent. There is no advantage conferred for being in a high status position in the current period, though of course such an addition would be a natural extension of the model. At an aggregate level this will look much as before as an example run in figure 6.12 makes clear. There is more noise, because of the more rapid updating

[7] While in this example this is relatively straightforward, this is the key question for all cases, that is considering the marginal agent's expected utility's crossing of the u_l line.

(simultaneous actions) in this model, but the overall picture is similar with the economy being roughly at the competitive equilibrium.

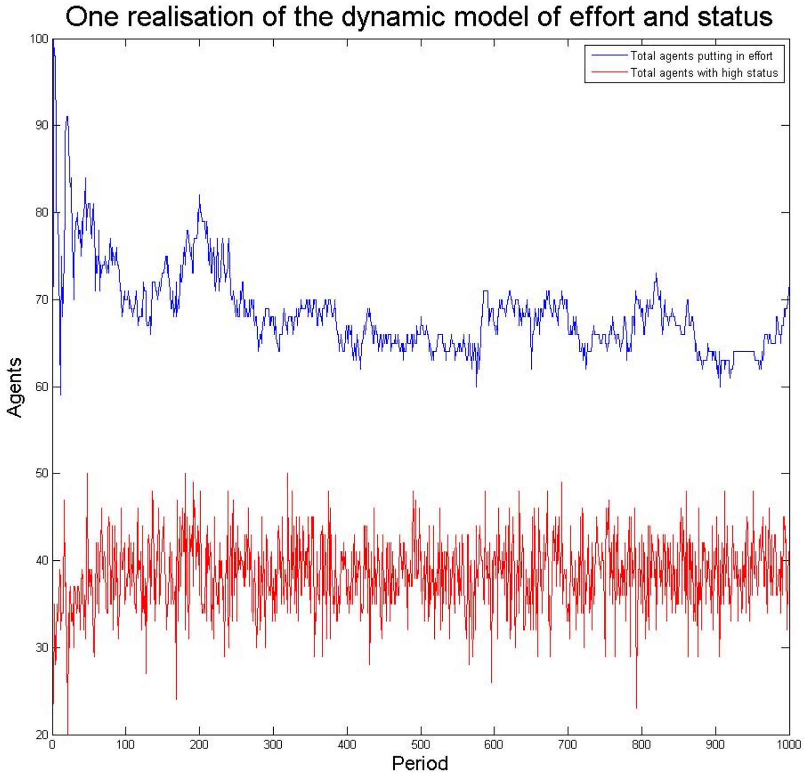

Figure 6.12: In this figure we see the total number of agents putting in effort (in blue) and total number achieving high status (in red). Agents base their actions on their own past success and that of their neighbours. We see something that at an aggregate level looks quite like a Nash equilibrium. At an individual level things are more interesting as can be seen in figure 6.13.

What we are interested in asking is whether agents can learn to put in effort and whether they might form segregated regions of high and low effort (and thus high and low status) and from looking at the average across many periods we see that this is the case. In figure 6.13 we see on the left the average effort put in and on the right the average resulting state (where low status is given weighting 0 and high status weighting 1 and the mean calculated for each individual). Similar patterns appear when we vary the number of agents and when we consider different numbers of periods to average over.

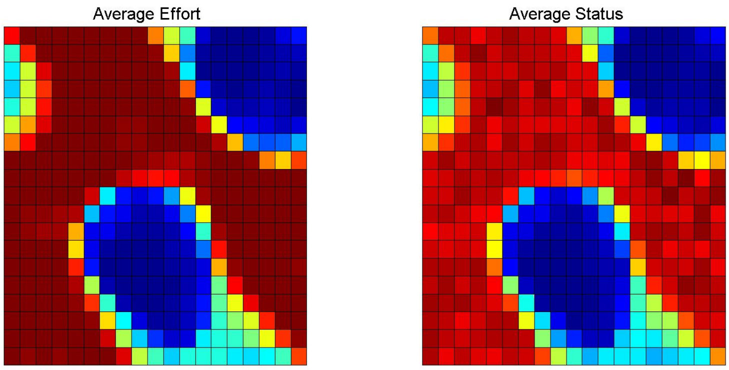

Figure 6.13: In this figure we see the average effort and average status of agents over 1000 periods. We have a lattice with toroidal boundaries; so the coloured regions represent a kind of endogenous segregation between high effort/high status and low effort/low status regions.

6.8 Discussion

In this chapter we have developed a set of strategic models of development which non-trivially relate individual effort and global outcomes. We have developed versions which capture the idea of competition in development, cooperation and our benchmark case where both effects occur. For these models the Nash equilibria were characterised. A version of the model was developed for finite sets of agents and again Nash equilibria characterised. Using the criterion of stochastic stability we analysed the long run behaviour of a dynamic version of this economy. We introduced agent heterogeneity both for the continuum model and for the finite model. We showed how our solutions may still hold when some agents have particularly high or low skill.

In the final section we looked at a spatial version of the model. This is a version of the model on a lattice and we show how we may obtain a kind of endogenous segregation; where certain regions learn to aim high and other to aim low, despite no agent actually having any advantage in terms of the cost of effort or probability of success. This model is of relevance to a wide variety of situations where agents learn from their peers/neighbours. Any discrimination would exacerbate the effects of segregation, beyond that observed in our model which assumes that outcomes are only related (stochastically) to one's effort.

6.9 Future Work

Economic opportunities for individuals (which may be rational for those individuals to pursue) may lead to outcomes which make some, or perhaps all, individuals worse off. This is a repeated theme in game theory and is one area the models presented above are of relevance to.

In this section we sketch how versions of our model can be of use in addressing a specific set of concerns in development economics of forced inclusion: it allows us to formally capture a process by which a group of individuals acting in their own interest can end up in a situation that is worse for everyone because of reduction in the availability of some non-market good. This, of course applies more generally to other areas of polarisation or segregation.

We can think of public goods, such as a community way of life or other community resource which either require a certain number of individuals to maintain or which degrade as fewer individuals engage with them. A concrete example is the possibility of migration from a rural village to urban environment in search of employment. There is a cost to migration and uncertain reward (which may be affected by other individuals from the village: a very small number of other individuals may not be helpful, but as the number increases it may make finding employment easier; though beyond a point this may have diminishing and eventually negative returns to the individual). As individuals migrate away from the village the quality of communal life or utility provided by another key public good may decrease.

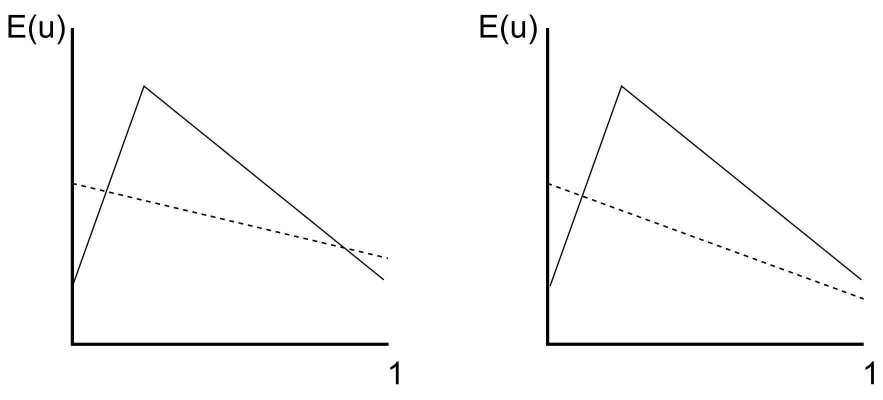

Figure 6.14: Here we consider two examples of forced inclusion for our benchmark case. As we can see, when the interior Nash equilibrium exists it now results in a lower utility level for all agents than the alternative $m_h = 0$ equilibrium.

Formally we need to make one further alteration to the model. Previously if an agent put in effort e_h but failed to attain the higher outcome the resulting utility would be $u_l - e$, now it should be a variable $\tilde{u}_l - e_h$; but by altering the cost e_h to include $u_l - \tilde{u}_l$ we are left with a vertically shifted curve for the expected utility for those agents who do put in effort e_h.

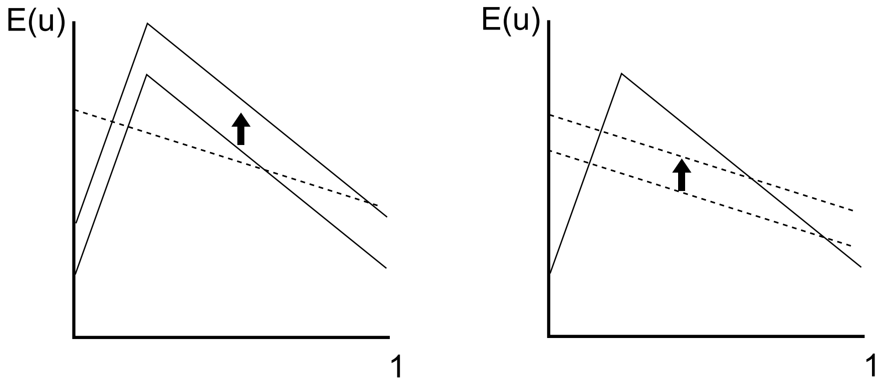

Figure 6.15: Where forced inclusion effects are significant there may be unintended consequences to policy changes. On the left we see the effect of a reduction in the cost of putting in effort; the non-trivial Nash equilibrium outcome now has lower utility for all agents. On the right is the effect of a general improvement to the public good; this may discourage effort.

In figure 6.16 we consider the idea of a threshold effect. This is particularly relevant when considering say the quality of communal life in a village or when thinking about maintaining other public goods.

The corresponding more optimistic scenario is that of "positive inclusion": while many individuals may fail to actually attain the improved status, job or so on which they might prefer, there may be spillover effects which benefit their community. notes for example that while failing to complete secondary education may mean that the child is unable to escape the village to become a teacher, the education may in fact lead to improved productivity in their life as part of the community. Banerjee and Duflo [2011] for example stresses the empirical observation that there are significant increases in average lifetime earnings to any additional years of education.

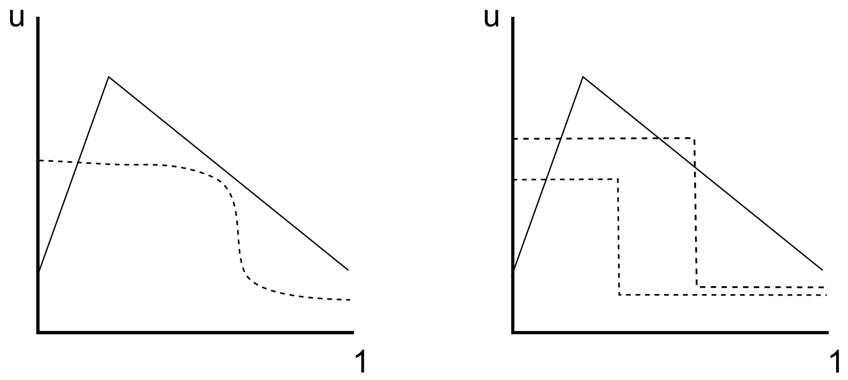

Figure 6.16: In this figure we consider the possibility of a threshold effect, whereby u_l remains constant (or approximately constant) until a threshold of agents putting in effort is crossed. The result will either be qualitatively as with decreasing utility but there may be more dramatic changes or the introduction of an additional Nash equilibrium.

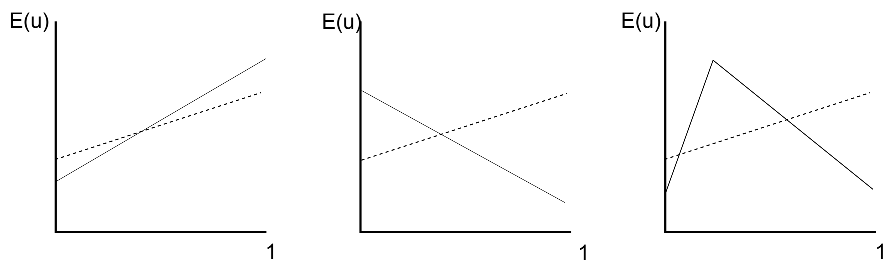

Figure 6.17: Positive inclusion for each of our three main models. In the first there is essentially no effect (as the boundary complementarity equilibria still hold). In the second, competitive case, and the third, benchmark case, the positive externalities of effort improve the interior Nash equilibrium.

Chapter 7

Conclusions and Future Work

This thesis has developed several models and approaches that contribute towards out-of-equilibrium economic modelling. In chapters 3 and 4 we developed models of exchange which relaxed many of the typical economic assumptions such as foresight and strong assumptions on knowledge. We showed that under certain conditions both the random matching and network-matched models will asymptotically result in Pareto optimal allocations. Numerically we went further and conclude that for our model, which with respect to informational aspects is pessimistic, coordination on this outcome is unexpectedly fast. We further showed that changes in wealth, considered from an a posteriori, realisation dependent, perspective can depend substantially on the network structure. Extending this basic model we see how endogenous network formation reveals agents preferences for moderately-connected trading partners and how agents might use a 'worthless good' for exchange when that is widely available and where endowments and utility functions are highly heterogeneous.

Following this theoretical work on out-of-equilibrium modelling we turned to models in a more practical sense. We developed a prototype of an agent-based modelling framework which shows how several key ideas from contemporary software engineering can be incorporated into agent-based modelling work. We built an internal domain specific language for agent-based modelling, including additional ideas such as version control, testing and a more holistic view of an agent-based modelling project than is typically taken. We demonstrated that such an approach has many benefits and offers a more efficient way of building agent-based models than many existing approaches. We identified key lessons of general relevance and suggest ways the framework could be developed further.

Finally in chapter 6 we looked at a set of 'complex' models where aggregate outcomes are determined from individual decisions in a non-trivial way. We investigated three classes of models: cooperative, complementary and our benchmark case which incorporated both cooperative and complementary effects. We identified Nash equilibria solutions to these models viewed strategically. We developed a finite population version of these models. Using the criterion of stochastic stability we characterised the long run behaviour of the resulting economies. Finally we drew together several stands of the thesis in developing a model of development with local learning and aggregate outcomes. We use the framework from chapter 5 to build this model and show how a kind of endogenous segregation may occur.

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Appendix A

Decentralised Exchange Implementation

A.1 Numerical Implementation

The numerical model was implemented using the Java programming language. The implementation explicitly models individual agents via an Agent class. Each instance of the class stores the agent's current bundle of goods, the parameters of its utility function and its current marginal rates of substitution. Each agent can make or consider offers, carry out trades and reset itself for another realisation of trading.

The following subsections outline the details of the models and implementation. The key source files are contained in appendix A.2 below.

A.1.1 Cautious Trading

Two files contain the key parts of the implementation of Cautious Trading: the *Agent* and *CautiousEconomy* classes. The former implements agents with Cobb-Douglas utility functions, random initial endowments and specifies the mechanics of trade proposals and trades. The later creates a collection of these agents and carries out simulated runs of the economy.

A.1.2 Scarf Example

A modified version of the *Agent* and *CautiousEconomy* classes was created to study the behaviour of an economy which in many settings may not converge. The implementation is broadly similar to the original, the main changes being to the endowments and utility functions.

A.1.3 Experimentation

The *CautiousEconomy* has been augmented with the possibility of experimentation. Essentially the *ExperimentingEconomy* class adds experimentation to *CautiousEconomy* via a scaling parameter to proposed trades. To be more precise an initial level of allowable experimentation is selected and the allowable level decreases linearly until it ceases. The probability of experimentation is fixed at an initial level and this too decreases over time.

A.2 Source Code

This section contains the key source code files; many more were actually used to model the cautious economy. The code is arranged into four distinct levels: agent, economy, experiment and simulation. The first two play obvious roles, the experiment code provides general code to investigate the cautious economy and the simulation code runs experiments and does some processing of results. Figures in this report were then produced using Matlab.

A.2.1 Agent Code

The below code is for the basic form of the Agent class.

lstinputlisting../../Developer/eclipse-workspace/CautiousExchange/src/com/porter/cautious

A.2.2 Cautious Economy Code

The below code presents the basic abstract economy class, which all economies subclass.

Listing A.1: Cautious Economy source code

```
package com.porter.cautious.model;
import java.io.*;
import java.util.ArrayList;
import com.porter.util.*;

/** The class Economy consists of a collection
 * of independent Agents, who trade
 * via Cautious Trading.
 */
public abstract class Economy{

  public ArrayList<Agent> agents;

```

```
  public int size;
  public int nGoods;
  public int trades, period, round;

  /** Reset the economy i.e. give each agent a random
   * allocation and utility function.
   */
  public void reset(){
    for (Agent a: agents) {
      a.reset();
    }
    resetCounters();
  }

  /**
     * Restore original state of economy
     */
    public void restart(){
    for (Agent a: agents) {
      a.restart();
    }
    resetCounters();
  }

    protected void resetCounters(){
        trades = 0;
        period = 0;
        round = 0;
    }

  /**
     * Return the total utility of all Agents in the Economy.
     */
  public double totalUtility(){
    double total = 0;
    for (int i = 0; i < this.agents.size(); i++) {
      total += agents.get(i).currentUtility;
    }
    return total;
  }
  /**
    * Attempt one exchange per member of the economy
  */
  public abstract void round();

  /**
```

```
   * Carry out multiple rounds of trading
   * @param n Number of rounds to run
   */
  public void runRounds(int n){
    for (int i = 0; i < n; i++) {
      round();
    }
  }
//
//    public void outputTotalUtility(FileWriter writer){
//        try{
//            writer.write("Total Utility: " + totalUtility());
//        }
//        catch(IOException e){
//            e.printStackTrace();
//        }
//    }

  /**
   * @param periods The number of periods
   * @param repetitions The number of realisations to average over
   * @throws IOException
   * */
  public double[] averageUtility(int periods, int repetitions){
    double results[] = new double[periods];
    Processing.initialiseArrayToZero(results);

    for (int r = 0; r < repetitions; r++) {
      for (int i = 0; i < periods; i++) {
        round();
        results[i]+=totalUtility();
      }
      restart();
    }
    for (int i = 0; i < results.length; i++) {
      results[i] /= repetitions;
    }
    return results;
  }

  /**
   * @param periods The number of periods
   * @param repetitions The number of realisations to average over
   * @throws IOException
   * */
  public double[][] manyUtility(int rounds, int repetitions){
```

```
    double results[][] = new double[rounds][repetitions];
    Processing.initialiseArrayToZero(results);

    for (int r = 0; r < repetitions; r++) {
      for (int i = 0; i < rounds; i++) {
        round();
        results[i][r] = totalUtility();
      }
      reset();
    }
    return results;
  }

    public double[][] manyUtilityFixedIC(int rounds, int repetitions){
        double results[][] = new double[rounds][repetitions];
        Processing.initialiseArrayToZero(results);

        for (int r = 0; r < repetitions; r++) {
      for (int i = 0; i < rounds; i++) {
                round();
        results[i][r] = totalUtility();
      }
      restart();
        }
    return results;
  }

    /**
     * Measure the rate of success of Cautious Trading.
     * @param rounds
     * @param intervals
     * @return An array of the numbers of trades taking place in a
         series of intervals
     */
  public int[] measureRateOfSuccess(int rounds, int intervals){
    int results[] = new int[rounds/intervals];
    results[0] = 0;
    int tradesSoFar;

    for (int p = 0; p < rounds/intervals; p++) {
      tradesSoFar = trades;
      for (int i = 0; i < intervals; i++) {
        round();
      }
      results[p] = trades - tradesSoFar;
    }
```

```
    return results;
  }

  public double[][] getsAverageMRS(){
    double[][] mrs = new double[nGoods][nGoods];
    Processing.initialiseArrayToZero(mrs);

    for (Agent a : agents) {
      Processing.add2dArrayInPlace(mrs, a.getMRS());
    }
    return mrs;
  }
}
```

A.2.3 Experimenting Economy Code

The below code shows how the above economy has been expanded to include the idea of experimentation. We were able to utilise much of the functionality of the CautiousEconomy superclass. The key changes are to the round method and to the counters which are now of type double for efficiency purposes as we would otherwise need to cast integers to doubles to calculate experimentation scaling in each round.

Listing A.2: Experimenting Economy source code

```
package com.porter.cautious.model;

/** The class Economy consists of a collection
 * of independent Agents, who trade
 * via Cautious Trading.*/
public class ExperimentingEconomy extends CautiousEconomy {
  public double acceptable, propensity, decay;
  public double doubleEndDecay, doubleRoundCount;
  public double baselineExperimentation;
  public int endDecay;

  /**An Economy is of size no. of agents each
   * of whom deal with nGoods no. of Goods.
   * @param size Number of agents in economy
   * @param nGoods Number of goods in economy
   * @param acceptable_loss_proportion The proportion of average
       initial
   * absolute utility that is initially
   * acceptable to lose in a trade.
   * This declines until 0 at endDecay. Obviously there are schemes
       which are
```

```
 * more analytically satisfying, this one is a compromise between
     this and ease of computation.
 * @param propensity_to_experiment How often to experiment
 * @param endDecay The point at which experimentation stops
 * */
public ExperimentingEconomy(int size,int nGoods, double
    acceptable_loss_proportion, double propensity_to_experiment, int
    endDecay)
      throws IllegalArgumentException{

      super(size,nGoods);
      //Check values of parameters
      if ( 0.0 > acceptable_loss_proportion ||
          acceptable_loss_proportion > 1.0
  || 0.0 > propensity_to_experiment || propensity_to_experiment > 1.0
  || 0 > endDecay){
    throw new IllegalArgumentException("Values must be in range (0,1]
        for Acceptable, Propensity and positive integer for endDecay"
      );
      }

      this.propensity = propensity_to_experiment;
      this.endDecay = endDecay;
      this.doubleEndDecay = (float) endDecay;

      this.baselineExperimentation = calculateBaselineExperimentation
          (acceptable_loss_proportion);
}

/**
 * No experimentation version of Economy, should perform as Cautious
     Economy
 * @param size
 * @param nGoods
 * @param acceptable
 * @param proportion
 * @throws IllegalArgumentException
 */
public ExperimentingEconomy(int size,int nGoods) throws
    IllegalArgumentException{
  this(size, nGoods, 0.0,1.0, 0);
}

protected double calculateBaselineExperimentation(double
    acceptable_loss_proportion){
  return acceptable_loss_proportion * calculateAverageAbsoluteUtility
```

```java
        ();
}

protected double calculateAverageAbsoluteUtility(){
  double total = 0.0;
  for (int i = 0; i < size; i++) {
    total += Math.abs(agents.get(i).currentUtility);
  }
  return total /(double)size;
}

  @Override
public void round(){
  double allowable_experimentation = this.baselineExperimentation *
                      (1.0 - doubleRoundCount/doubleEndDecay);
      int r;

      for (int i = 0; i < size; i++) {
        r = gen.nextInt(size);

        //get another agent at random
        while(r == i){
              r = gen.nextInt(size);
          }

          if(this.round < endDecay && gen.nextDouble() < propensity *
              (1.0 - doubleRoundCount/doubleEndDecay)){
          if(agents.get(i).propose(agents.get(r),
              allowable_experimentation)){
            trades++;
          }
          }
        else{
          if(agents.get(i).propose(agents.get(r))){
              trades++;
            }
        }
        period++;
      }
      round++;
      doubleRoundCount++;
  }

@Override
protected void resetCounters(){
  super.resetCounters();
```

```
            doubleRoundCount = 0.0f;
        doubleEndDecay = 0.0f;
    }
}
```

The code for networked economies can be found in the source files; but it is omitted here as it functions more or less as the above code.

Appendix B

A technical overview of Ambl

The following appendix (appendix C) explains how to get Ambl on your own system; in this section we focus on giving a technical overview and filling in some of the details that were omitted from chapter 5 as that focused on methodological issues and examples.

B.1 The design of the Ambl implementation

In chapter 5 we related Ambl to existing work in agent-based modelling, here we focus on some of the more precise technical details. Ambl is a set of libraries for agents based modelling, together with a way of structuring projects and a set of classes that allow programming in an internal domain specific language style. The current implementation is a prototype, so lacks some of the features that should be in a 'full' framework; however it follows a fairly conventional Ruby style as regards the project/library structure.

Figure B.1 shows the overall structure. In its prototype form both library and project are included in one folder (these should really be split apart once more fully developed). There is a doc folder for documentation, a lib folder for libraries and a log folder for recording of logs. The model, output, project, result folders are where it diverges from more typical projects. The model and project folders are 'special'; if you define models and projects here Ambl 'knows' about them and will include them in the project running application. The output folder is where Ambl generates its various outputs (see section B.3 for more on these). The result folder is where results from each run of each simulation in each experiment in each project are automatically saved (see figure B.4).

The remainder of the major folders include a spec folder where the RSpec

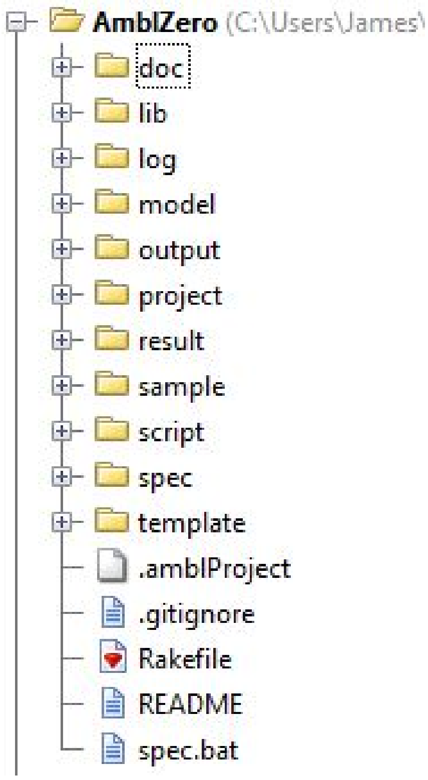

Figure B.1: The overall structure of the current Ambl project.

tests are contained (we remark further about testing in chapter 5). The script folder contains a number of scripts (command line programs) which perform tasks such running projects.

The template folder contains a set of templates which could be used to generate basic agents, worlds, simulations and experiments (if you supply the appropriate information). This is not fully implemented (and given the conciseness of the Ambl domain specific syntax) and not so important, but a script to do this can be found in the util folder of the lib folder. The files in the base folder include a .gitignore file for version control and a Rakefile (which uses the Ruby build tool Rake to perform various tasks such as running tests and generating documentation).

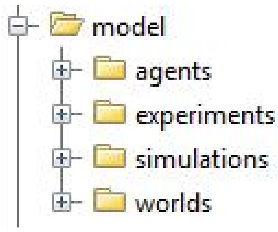

Figure B.2: Ambl model components should typically be saved in these folders.

Figure B.2 shows the model subfolders. Each performs the obvious role. An important point to make is that you can have many agents, worlds, simulations and experiments in each folder and they can interrelate in non-trivial ways. Ambl will load files from lower to higher levels of the agent, world, simulation, experiment, project hierarchy so lower level components can be included directly in higher level components files for convenience.

B.2 Ambl Libraries

Figure B.3 shows the main libraries folder and subfolders. The typical Ruby convention is to have one file named after the library which loads all of the other files and this is the case here (the ambl.rb file). Most of the core libraries are essential complete (though you may wish to augment them for specific purposes, something which Ruby makes very easy with its ability to 'reopen' classes).

Additional folders include a set of visualisations (in vis), various utility classes and helpers in util and a graphical user interface framework in swingfaster. We now look at key components of Ambl in more detail. This section attempts to summarise the key methods and abilities of the classes; chapter 5 goes through examples of usage of these classes.

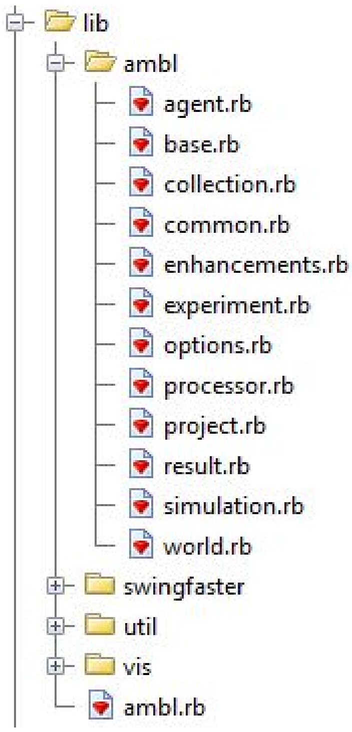

Figure B.3: The main library folder for Ambl.

B.2.1 Agents

The Agent and World classes share a some functionality, which is contained in the AmblBase parent class. A snippet from it is shown below as it illustrates the kind of techniques used frequently by these core Ambl classes. In addition to defining instance methods and attributes we also take advantage of Ruby's flexibility in defining class level methods and attributes. The syntax is a little strange but each of these methods below is defined at a class level (and when subclassed these attributes will be distinct for each subclass i.e. for each Agent or World class).

Listing B.1: Snippet from Base class

```
module Ambl
  class AmblBase
    class << self
      attr_accessor :name, :description, :observable_list, :method_list
          , :setup_method

      def add_name(n)
        @name = n
      end

```

```
10      def add_description(d)
11        @description = d
12      end
```

Line 1 sets up a module (use for namespacing). Line 3 is the interesting part: this functionality allows you to add behaviour at a class level, or technically to the 'eigenclass'. If you want to learn more about this kind of sophisticated technique see Perrotta [2010]. Using this kind of method we can allow for a natural domain specific way of describing a model. The agent class allows us to:

- Add a list of variables
- Add a list of parameters
- Give a list of methods for the Agent
- Give a list of 'observable' properties of the Agent
- Add a variable
- Add a parameter
- Declare a relationship with another Agent
- Declare a relationship to a set of other Agents

These concise declarations allow us to quickly 'sketch out' a model, filling in the details later. They also allow Ambl to 'understand' the model and to automatically set up various things for us and to output meaningful descriptions.

B.2.2 Worlds

The World class is similar in many respects to the Agent class. There are two important additional concepts: collections and updates. A collection is a set of agents and an update is something which happens in each period. It allows the specification of a list of updates via `has_updates` and the description of a collection via `has_collection`. Each update is a Ruby function so has the flexibility to accomplish various levels of sophistication from a simple update of a variable to calling a method for each agent and beyond. The collection is initialised via a specified function and will typically draw either on a built in Ruby collection or on one from the Ambl Collections module. The examples contained with Ambl should make the above clear.

B.2.3 Simulations

The Simulation class is relatively simple from the point of view of the Ambl modeller: it has a single world, can have a description and so on. However, the main elements are specifying which results to collect and in what way to run the simulation. Ambl offers support for the collection of results at every time period, every n time periods or at the end. It allows running for a fixed time period or while/until a condition is true.

When it is run it collects together the results which can then be processed by 'higher up' parts of Ambl. It also has an additional benchmark mode which can provide an estimate of how long a simulation will take to run.

B.2.4 Experiments

The Experiment class is fairly simple for the user. The idea here is that it defines an experiment which can involve running simulations, comparing simulations or varying a parameter in a simulation. Furthermore it can specify a certain number of repetitions of the simulation(s).

B.2.5 Projects

The Project class is quite complex; however, for the user it is very simple. You simply specify a description, author and title; along with perhaps multiple experiments and other content. Finally you should also specify what 'outputs' it has, for example LaTeXpaper or HTML (see will look in more detail at these below). From just this Ambl will run all the experiments, process the results, generate default visualisations and pull together all of these, generated summaries of model components and additional content into 'outputs'.

B.3 Ambl Outputs

We consider two types of output upon the running of Ambl projects: results and generated outputs.

B.3.1 Results

An important part of any agent-based investigation will be the recording and processing of results from realisations of the model. Ambl automatically saves your results, sorting them by simulation and by date as shown in figure B.4.

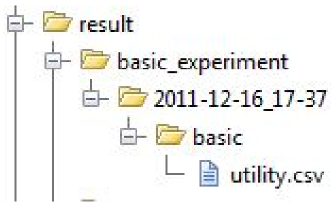

Figure B.4: An example of a result which has been automatically exported in a standard format (.csv) to a suitably titled folder.

B.3.2 Generated Outputs

Ambl as it currently stands includes four kinds of 'output' which can be automatically generated:

- LaTeXreport: a ready-to-compile LaTeXversion of your project, including visualisations converted to a suitable format, summaries of the components of your project and other content.
- HTML report: similar to the LaTeXreport but formatted for the web (presumably the future of scientific publishing) and ready to import to Microsoft Word and similar programs.
- Presentation format: a 3d-effect presentation format based on your project, using the `reveal.js`[1] library.
- Summary viewer: a simple graphical application which summarises your project.

But importantly it is easy to create your own outputs: simply create a new 'processor' (see the `processor.rb` file for the current output processors).

B.4 Future work

There are a multitude of additions which could be made to Ambl, but perhaps the most important issue, if Ambl is to be developed further, is to identify a core set of libraries and functionalities which form 'core Ambl' and then providing this restricted collection as Ambl, allowing users to easily customise and extend elements for their own projects.

[1] `http://lab.hakim.se/reveal-js/`

Appendix C

Getting Started with Ambl

A link to download the latest version of Ambl will be made available on my Warwick site[1]. Ambl is currently set up as a (J)Ruby project; to develop it further as a standalone agent-based modelling platform this will need to be transitioned into a globally available library (basically the contents of the lib directory and some of the scripts) and a individual project structure to use for actual Ambl work (basically this would include the current folder structure along with project specific libraries and models).

Ambl has been tested using JRuby 1.6.5[2] using the Ruby 1.9 mode [3] on both Windows 7 and Mac OS X Lion (and works without modification on both platforms). It should work fine on Linux (using the same JRuby version). It will mostly work with other versions of Ruby, though the graphical user interface elements will not be supported (as they explicitly require JRuby). It does use some of the new features of Ruby 1.9 such as `require_relative` and improved syntax for things like processing a list. However, with some small modifications and additions it could be altered to support JRuby in 1.8 mode and Ruby 1.8[4].

At the time of writing the JRuby project appears to be close to releasing JRuby version 1.7 which should both make explicitly invoking the 1.9 mode unnecessary and, by taking advantage of new features of the Java 7 virtual machine, significantly improve performance.

[1] `http://www.warwick.ac.uk/go/jamesporter`

[2] `http://jruby.org`

[3] Either supply the flag `--1.9` at the command line or configure your editor/IDE appropriately.

[4] If for some reason you have no choice over which version of Ruby to use.

C.1 Installation Guide

In order to install Ambl you should download the latest version via my site[5]. You should also install JRuby 1.6.5 from `http://jruby.org`. There are various download options for different operating systems including easy to use installers.

Ambl uses the rspec ruby library for testing which can be installed via the command: `jgem install rspec`. It also uses the Apache open source SVG toolkit Batik[6] which is already included with Ambl. This is used for image format conversion and isn't really a core part of Ambl; but if you want to use this you will need a working Java installation. Finally, you should extract Ambl to the folder you wish to work within.

C.2 Testing and Using Ambl

To confirm that Ambl is working on your system you should run the command `rake spec` from the directory you extracted it to. This will run all of the rspec tests. This tests the core parts of the framework and a full example project. You may also wish to run `jruby --1.9 scripts/run.rb` which will run all of the projects included with the Ambl download. If both of these work then Ambl is almost certainly fully working on your system.

To run a project you should use the supplied project runner (see figure C.1) which allows you to choose from all Ambl projects it finds and run them. This project runner runs each project on a new thread (so you could run multiple projects on multiple computer cores simultaneously) and something like this should probably form part of an 'Ambl Assistant' application to make Ambl easier to use for those unfamiliar with command line usage. Alternatively (for those who are happier using the command line) to run a particular project you simply supply the project class name as a command line parameter to the run script.

C.3 Creating a project

Ambl includes several example projects and it is probably best to look at these when getting started. In addition the previous appendix B gives an explicit overview of what is possible and chapter 5 presents several actual examples. For convenience it is worth pointing out that Ambl loads files from lower to higher levels of the agent, world, simulation, experiment, project hierarchy, so initially it may be easiest just to

[5] `http://www.warwick.ac.uk/go/jamesporter`
[6] `http://xmlgraphics.apache.org/batik/`

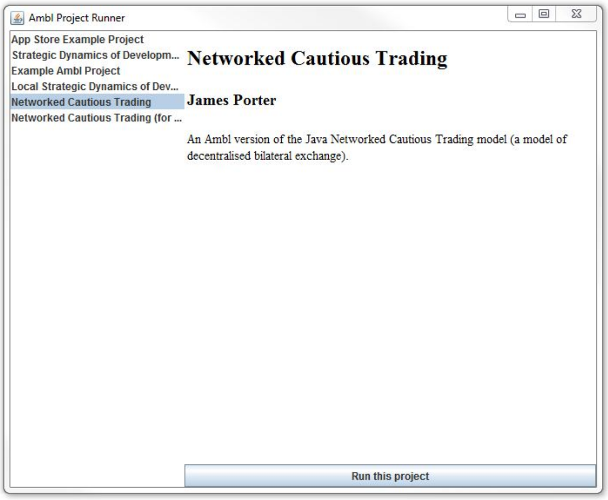

Figure C.1: The graphical user interface for running projects. The list of available projects appears on the left, a description and run button on the right.

work with a single project file containing (one or more) agents, worlds, simulations and experiments[7]. To see an example of this look at the `example_project.rb` file in the project folder. This uses most of the key features of Ambl so is a good place to start.

[7]Unlike many languages (J)Ruby doesn't demand that individual components be split into individual files.

Appendix D

An example automatically generated output from Ambl

Abstract

An example project which illustrates the key ideas of using the Ambl framework.

Introduction

This is a simple project which uses all types of experiment for a simple world/simulation. In learning Ambl it is a good place to start.

Experiment: Simple

An example experiment which runs in the default way.

Simulation: One

A simulation of the world with agents performing a random walk (possibly with bias :beta)

ExampleProjectWorld

A simple example world containing many ExampleAgents. A parameter :beta controls the bias of the random walk of the agents

> **Parameters:** beta
>
> **Observable values:** total utility, minimum utility

ExampleProjectAgent

A simple example agent with two parameters. It has some energy level, which is determined by a random walk of step size :delta. The utility of the agent is the energy level multiplied by :alpha

> **Parameters:** alpha, delta
> **Variables:** x
> **Methods:** deplete energy, increase energy
> **Observable values:** utility

Experiment: Comparison

An example experiment which compares two simulations.

Simulation: Low beta

A simulation of the world with agents performing a random walk (possibly with bias :beta)

ExampleProjectWorld

A simple example world containing many ExampleAgents. A parameter :beta controls the bias of the random walk of the agents

> **Parameters:** beta
> **Observable values:** total utility, minimum utility

ExampleProjectAgent

A simple example agent with two parameters. It has some energy level, which is determined by a random walk of step size :delta. The utility of the agent is the energy level multiplied by :alpha

> **Parameters:** alpha, delta
> **Variables:** x
> **Methods:** deplete energy, increase energy
> **Observable values:** utility

Simulation: High beta

A simulation of the world with agents performing a random walk (possibly with bias :beta)

ExampleProjectWorld

A simple example world containing many ExampleAgents. A parameter :beta controls the bias of the random walk of the agents

Parameters: beta

Observable values: total utility, minimum utility

ExampleProjectAgent

A simple example agent with two parameters. It has some energy level, which is determined by a random walk of step size :delta. The utility of the agent is the energy level multiplied by :alpha

Parameters: alpha, delta

Variables: x

Methods: deplete energy, increase energy

Observable values: utility

Experiment: Varying a parameter

An example experiment which repeats a particular simulation while varying a parameter

Simulation: Three

A simulation of the world with agents performing a random walk (possibly with bias :beta)

ExampleProjectWorld

A simple example world containing many ExampleAgents. A parameter :beta controls the bias of the random walk of the agents

Parameters: beta

Observable values: total utility, minimum utility

ExampleProjectAgent

A simple example agent with two parameters. It has some energy level, which is determined by a random walk of step size :delta. The utility of the agent is the energy level multiplied by :alpha

Parameters: alpha, delta

Variables: x
Methods: deplete energy, increase energy
Observable values: utility

Experiment: Simple repeated

An example experiment which runs some repetitions of a single experiment.

Simulation: One

A simulation of the world with agents performing a random walk (possibly with bias :beta)

ExampleProjectWorld

A simple example world containing many ExampleAgents. A parameter :beta controls the bias of the random walk of the agents
Parameters: beta
Observable values: total utility, minimum utility

ExampleProjectAgent

A simple example agent with two parameters. It has some energy level, which is determined by a random walk of step size :delta. The utility of the agent is the energy level multiplied by :alpha
Parameters: alpha, delta
Variables: x
Methods: deplete energy, increase energy
Observable values: utility

Experiment: Comparison repeated

An example experiment which compares two simulations (carrying out repetitions of each).

Simulation: Low beta

A simulation of the world with agents performing a random walk (possibly with bias :beta)

ExampleProjectWorld

A simple example world containing many ExampleAgents. A parameter :beta controls the bias of the random walk of the agents

> **Parameters:** beta
> **Observable values:** total utility, minimum utility

ExampleProjectAgent

A simple example agent with two parameters. It has some energy level, which is determined by a random walk of step size :delta. The utility of the agent is the energy level multiplied by :alpha

> **Parameters:** alpha, delta
> **Variables:** x
> **Methods:** deplete energy, increase energy
> **Observable values:** utility

Simulation: High beta

A simulation of the world with agents performing a random walk (possibly with bias :beta)

ExampleProjectWorld

A simple example world containing many ExampleAgents. A parameter :beta controls the bias of the random walk of the agents

> **Parameters:** beta
> **Observable values:** total utility, minimum utility

ExampleProjectAgent

A simple example agent with two parameters. It has some energy level, which is determined by a random walk of step size :delta. The utility of the agent is the energy level multiplied by :alpha

> **Parameters:** alpha, delta
> **Variables:** x
> **Methods:** deplete energy, increase energy
> **Observable values:** utility

Experiment: Varying a parameter repeated

An example experiment which repeats a particular simulation while varying a parameter and repeating for each

Simulation: Three

A simulation of the world with agents performing a random walk (possibly with bias :beta)

ExampleProjectWorld

A simple example world containing many ExampleAgents. A parameter :beta controls the bias of the random walk of the agents

Parameters: beta

Observable values: total utility, minimum utility

ExampleProjectAgent

A simple example agent with two parameters. It has some energy level, which is determined by a random walk of step size :delta. The utility of the agent is the energy level multiplied by :alpha

Parameters: alpha, delta

Variables: x

Methods: deplete energy, increase energy

Observable values: utility

Conclusion

Hopefully these examples have been helpful. Good luck with your own project(s).

Visualisations

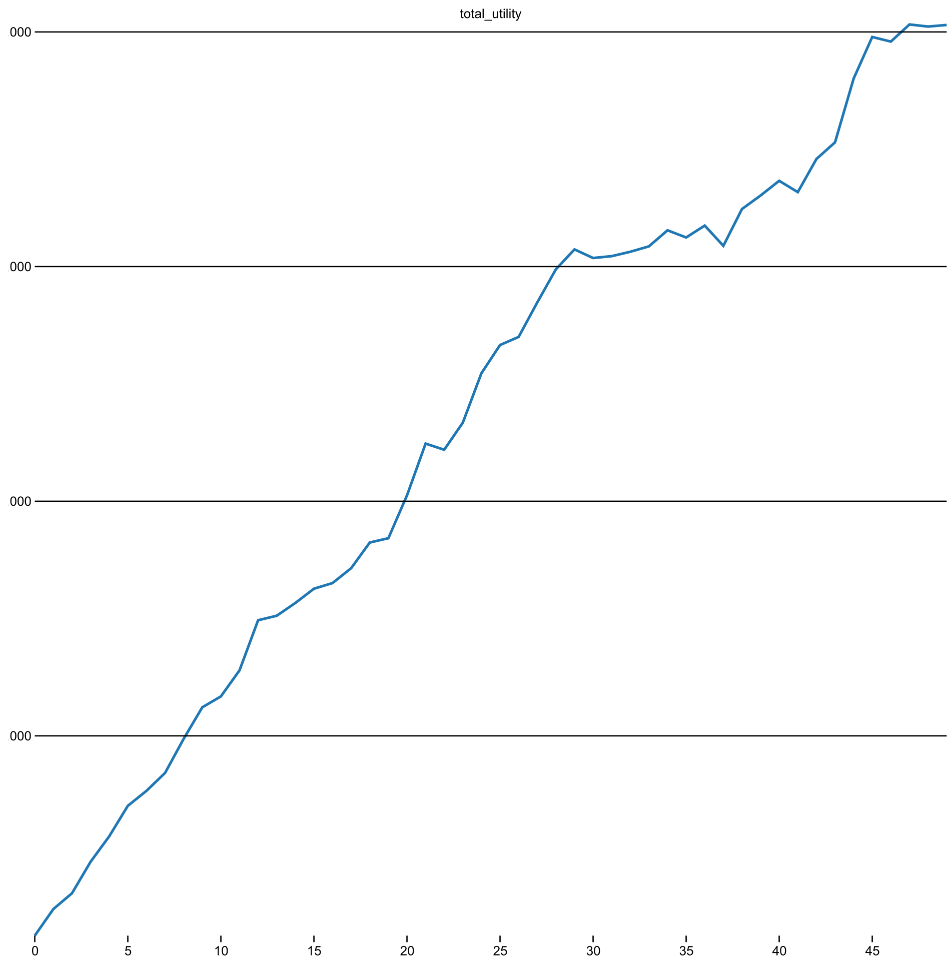

Figure D.1: Automatically generated figure for total utility0. You will probably want to customise.

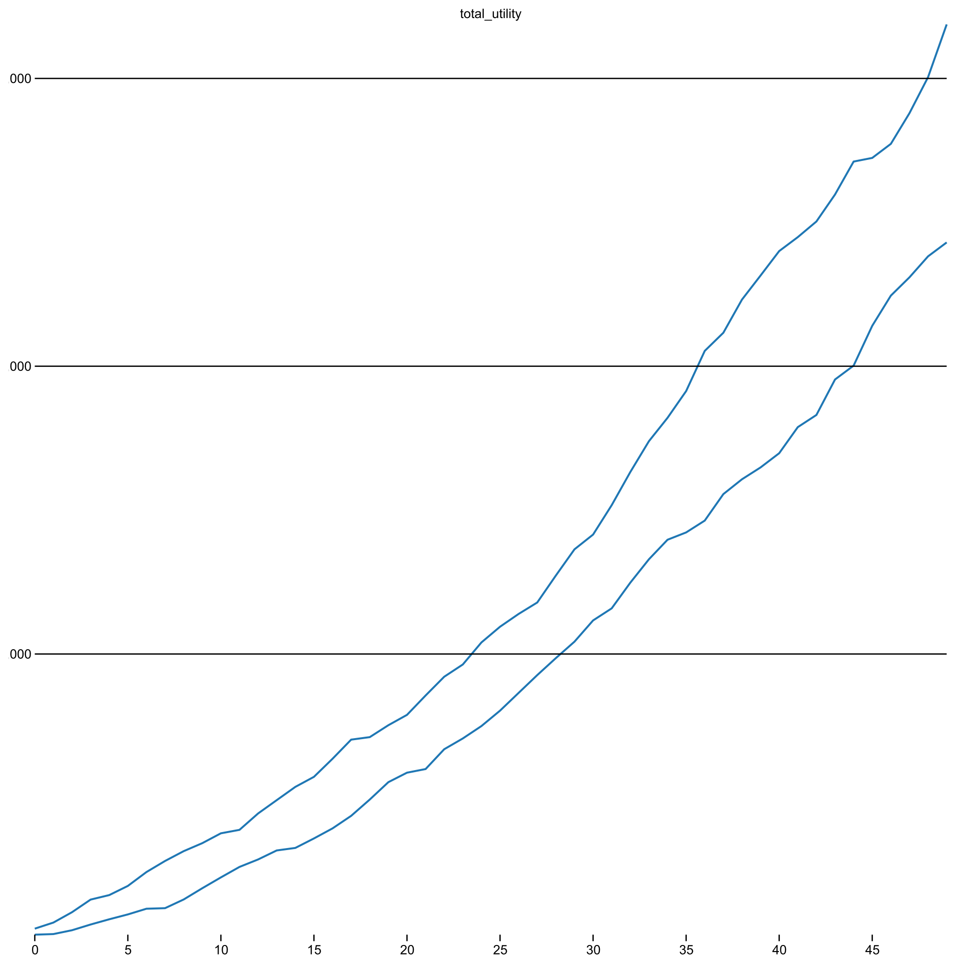

Figure D.2: Automatically generated figure for total utility11. You will probably want to customise.

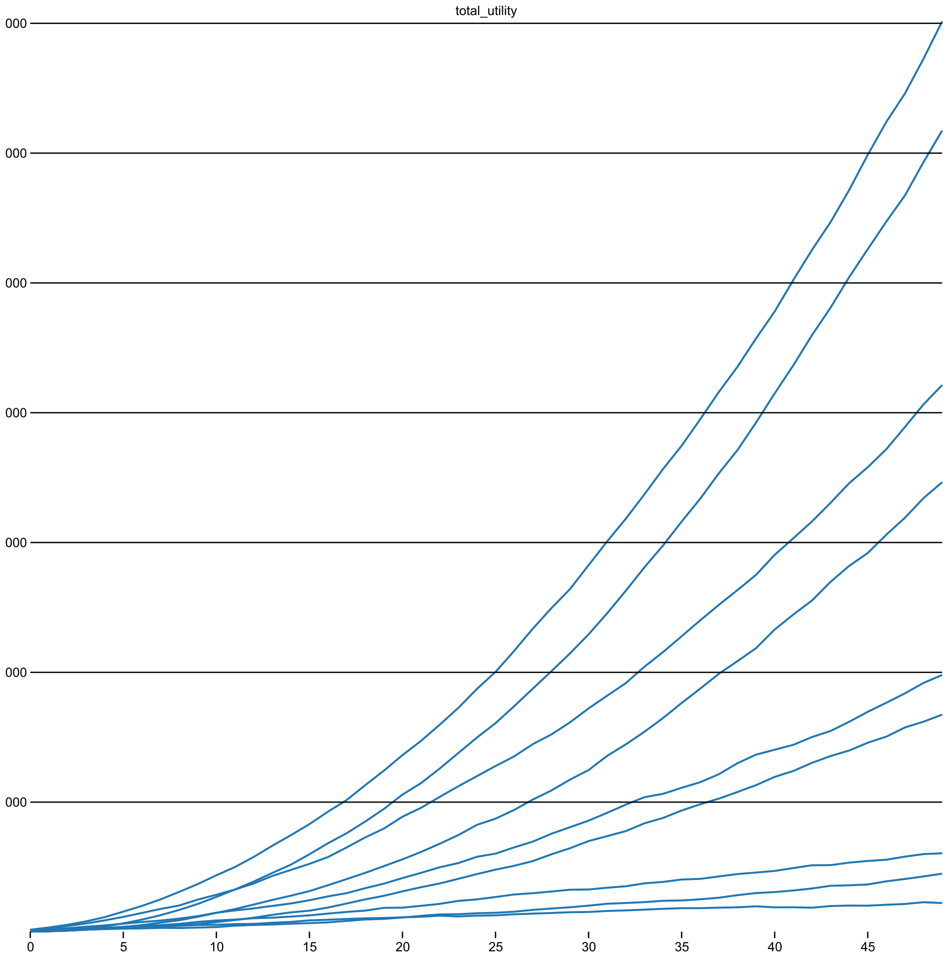

Figure D.3: Automatically generated figure for total utility22. You will probably want to customise.

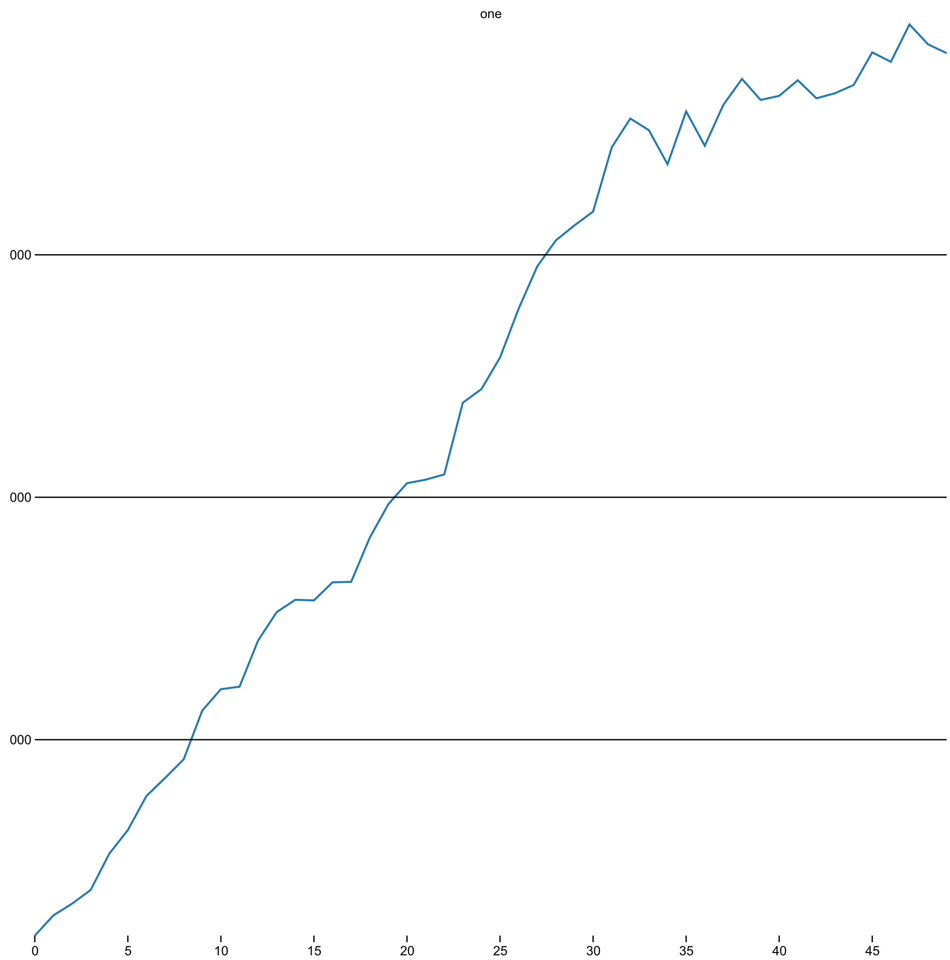

Figure D.4: Automatically generated figure for total utility33. You will probably want to customise.

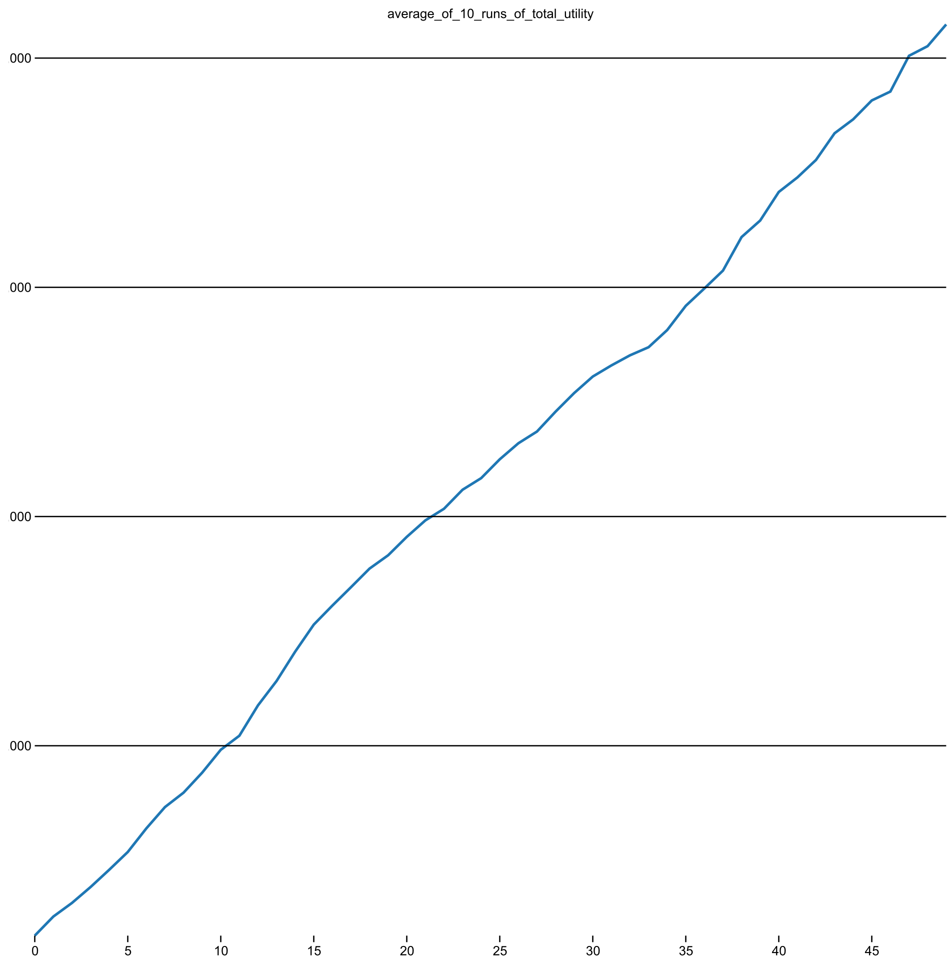

Figure D.5: Automatically generated figure for total utility34. You will probably want to customise.

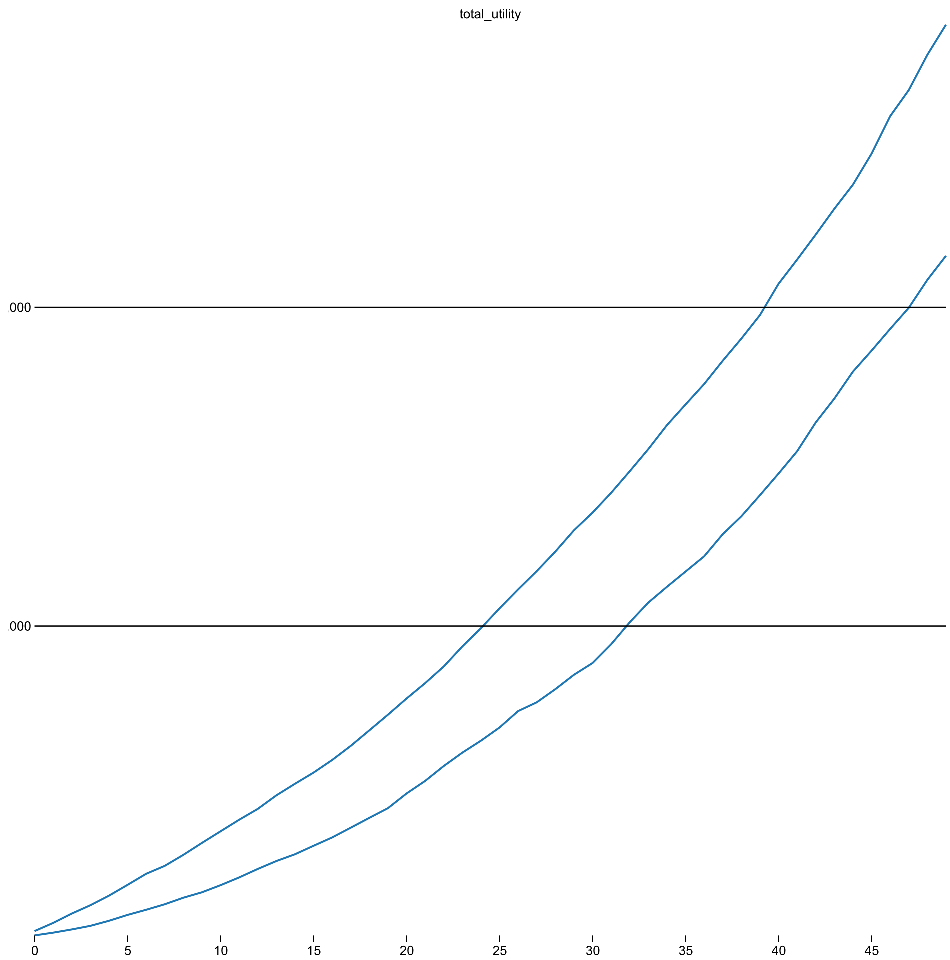

Figure D.6: Automatically generated figure for total utility45. You will probably want to customise.

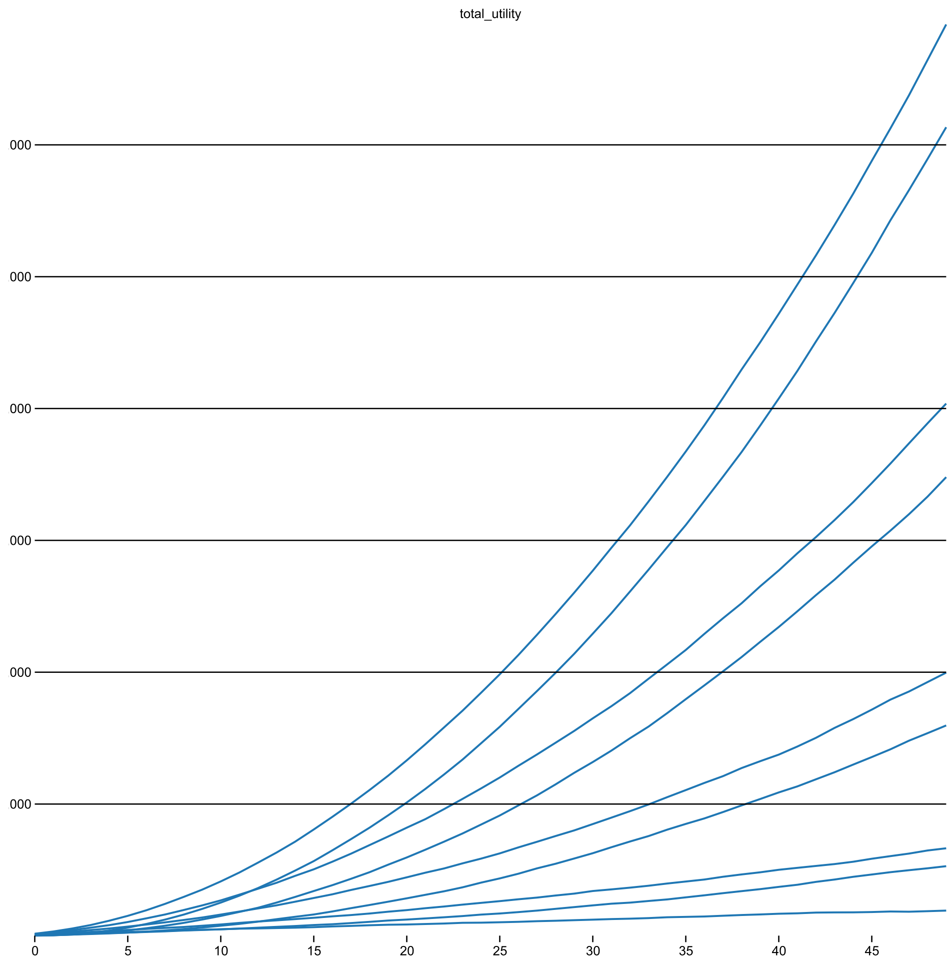

Figure D.7: Automatically generated figure for total utility56. You will probably want to customise.