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Models and String Topology

by

Ana Lucía García Pulido

Thesis

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for the degree of

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This thesis is dedicated to the memory of my grandmother Teresa.

Declarations

I declare that this work

- has not been published in any form.
- is my own except for the research undertaken in Chapter 4 that is joint work with my thesis advisor, Prof. John D. S. Jones. This is currently being prepared for publication as a joint paper, for which both authors contributed equally in both content and writing.
- has not been submitted for any degree at another university.

Ana Lucía García Pulido, December 2012.

Abstract

This thesis concerns the study of *string topology*, a relatively new branch of algebraic topology.

We begin with a survey of the background of string topology. In particular, this includes a summary of the papers [4] by Chas and Sulivan, [15] by Jones and [5] by Cohen and Jones that provide the background for the original work of this thesis.

We then proceed to give a new efficient technique to do systematic computations of the full structure of the string topology for a large family of manifolds. For this, we first use the results of Jones [15] and Cohen and Jones [5] to reduce the problem to calculating Hochschild homology and cohomology. Secondly, we use the concept of models to compute Hochschild homology and cohomology and obtain some further Hochschild structure. Thus, most of this work is devoted to developing this technique for calculating Hochschild homology and cohomology via models.

This research contributes to the area by providing the first general and systematic method of computing the full structure of string topology. In addition, we give multiple, transparent examples of our new theory.

Chapter 1

Introduction

The fundamental problem in topology is to distinguish between topological spaces, for which many tools have been developed in order to give a partial classification. In the history of algebraic topology there is a wide number of theories invented for this purpose - the fundamental group, higher homotopy theories, homology and cohomology - each with their own strengths and weaknesses.

One branch of recent development was initiated by Chas and Sullivan in [4]. Here they study the homology of the *free loop space* of a manifold, the set of all loops in that manifold. In this paper it is shown that the homology of this space has indeed a very rich structure that they call the *string topology*.

The free loop space of a topological space has been the object of study for a long time. In [15], Jones gave a very important relation between the existing algebraic theories of Hochschild homology and the cohomology of the free loop space of a simply connected space. More recently, in [5], Cohen and Jones gave a dual relation to this, relating Hochschild cohomology to the homology of the free loop space of a manifold. Furthermore, by using methods different to those above, the string homology of the projective spaces and the spheres were computed by Cohen, Jones and Yan in [6].

Within this thesis, we give a new efficient technique to do systematic computations of the full structure of the string topology for a large family of manifolds. This technique consists of two essential components. First we use the results of Jones [15] and Cohen and Jones [5] to reduce the problem of obtaining the string topology into calculating Hochschild homology and cohomology. Secondly, we use the concept of models (see below) to compute Hochschild homology and cohomology and obtain some further Hochschild structure. Thus, most of this paper is devoted to developing this technique for calculating Hochschild homology and cohomology via models. The work presented here is joint with Professor John D. S. Jones.

We now discuss these ideas in more detail. Let M be a closed, oriented d-dimensional manifold. The *free loop space* of M is the set of smooth maps $LM = \mathrm{Maps}(S^1, M)$. In [4], Chas and Sullivan prove that the shifted homology groups $H_{*+d}(LM)$ have a very interesting product, bracket (known respectively as the *string product* and *string bracket*) and an additional operator called the *Batalin-Vilkovisky operator*. This product and bracket make $H_{*+d}(LM)$ into a *Gerstenhaber algebra* and together with the Batalin-Vilkovisky operator give $H_{*+d}(LM)$ the structure of a *Batalin-Vilkovisky algebra*. In Chapter 2 we give full the definitions and some examples of Gerstenhaber algebras and Batalin-Vilkovisky algebras. We also describe Chas and Sullivan's construction of the string product, string bracket and Batalin-Vilkovisky operator on $H_{*+d}(LM)$.

In Chapter 3 we turn our attention to the algebraic preliminaries. We give the definition of Hochschild homology and cohomology of an algebra A and discuss some basic facts in these theories - the product and the Connes or B-operator in Hochschild homology and the Gerstenhaber algebra structure of Hochschild cohomology. We then give the more general definition of Hochschild homology of a differential graded algebra. We finish the chapter with the theorems of Jones and Cohen and Jones which relate the Hochschild theories with the homology and cohomology of LM together with the operators from string topology.

In Chapter 4 we introduce the idea of a model. For an algebra A, a model is a differential graded algebra P, that is free as a graded commutative algebra, together with a map of differential graded algebras (where A is considered with zero differential) such that it induces an isomorphism in cohomology. We discuss a model for the Hochschild homology (respectively cohomology) of a free graded commutative algebra. Then, inspired by the free case, in Theorem 4.2.3 (respectively Theorem 4.4.1) we construct a model for the Hochschild homology (respectively cohomology) of a differential graded algebra and the corresponding B-operator (respectively Gerstenhaber algebra structure).

The first results of this kind were due to Smith in [23], based on work of Tate (see [24]). However, the B-operator is not discussed and it is assumed that $K = \mathbb{Q}$ and that A is a graded complete intersection algebra. Our work began by seeing if these methods could be adapted to give full computations of the string topology.

Finally, we use our theory to highlight how these models simplify working with string topology. We also give explicit use of our theory by computing the cohomology of the free loop space and the full Batalin-Vilkovisky structure of the spheres, the projective spaces and the Grassmann manifold of two planes in $\mathbb{C}^4$.

Chapter 2

Topological background

In this chapter we discuss the definition of string topology and its operators - the string product, the string bracket and the Batalin-Vilkovisky operator. The definitions given here can also be found in [5] and [7].

Let V be a graded vector space. If $x \in V_p$, p is called the degree of x and we will denote it by $|x|$. A product $\bullet \colon V \times V \to V$ on V is called graded commutative if $x \bullet y = (-1)^{pq} y \bullet x$ for every $x \in V_q$ and $y \in V_q$.

A Gerstenhaber algebra V is a graded vector space with an associative and graded commutative product $x \bullet y$ and a bracket $[,]$ that satisfy the following properties.

1. The bracket has degree 1, that is

   $|[a, b]| = |a| + |b| + 1$

2. The bracket is a graded derivation of the product (Poisson identity), that is

   $[a, b \bullet c] = [a, b] \bullet c + (-1)^{|b|(|a|+1)} b \bullet [a, c]$

3. The bracket is a graded derivation of the bracket (Jacobi identity), that is

   $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+1)(|b|+1)} [b, [a, c]]$

4. The bracket is antisymmetric, that is

   $[a, b] = -(-1)^{(|a|+1)(|b|+1)} [b, a]$

where $a, b, c \in V$.

Note that conditions (1), (3) and (4) imply that $[,]$ is a graded Lie bracket of degree $+1$. Gerstenhaber algebras were introduced by Murray Gerstenhaber, see [11].

There is an additional structure that we can sometimes find in a Gerstenhaber algebra, namely a Batalin-Vilkovisky structure.

A Batalin-Vilkovisky algebra is a Gerstenhaber algebra with an additional operator $\Delta\colon V_n \to V_{n+1}$, the Batalin-Vilkovisky operator, such that

1. $\Delta \circ \Delta = 0$
2. $[a, b] = (-1)^{|a|}\Delta(a \bullet b) - (-1)^{|a|}(\Delta a) \bullet b - a \bullet (\Delta b)$ for every $a, b \in V$.

Notice that in a Batalin-Vilkovisky algebra, the Lie bracket measures how much the Batalin-Vilkovisky operator fails to be a derivation. A direct calculation expresses the Poisson identity in terms of Δ.

$$\begin{aligned}\Delta(a \bullet b \bullet c) &= \Delta(a \bullet b) \bullet c + (-1)^{|a|} a \bullet \Delta(b \bullet c) + (-1)^{|b|(|a|+1)} b \bullet \Delta(a \bullet c) \\ &- (\Delta a) \bullet b \bullet c - (-1)^{|a|} a \bullet (\Delta b) \bullet c - (-1)^{|a|+|b|} a \bullet b \bullet (\Delta c).\end{aligned}$$

Another direct calculation leads to the following formula.

$$\Delta([a, b]) = [\Delta a, b] - (-1)^{|a|-1}[a, \Delta b]. \tag{2.0.1}$$

Example 2.0.1. The free Gerstenhaber algebra on one generator x of odd degree is the free graded commutative algebra generated by x with vanishing bracket. That is,

$$x^2 = 0, \quad [x, x] = 0.$$

Indeed, in this case, using the antisymmetry of the bracket we get

$$\begin{aligned}[x,x] &= -(-1)^{(|x|+1)(|x|+1)}[x,x] \\ &= -[x,x].\end{aligned}$$

Thus $[x,x]=0$.

The free Batalin-Vilkovisky algebra on one generator x of odd degree is the free graded commutative algebra generated by x and $\Delta(x)$. From before we know that $[x,x]=0$ and by using Equation (2.0.1) we have

$$\begin{aligned}0 &= \Delta([x,x]) \\ &= [\Delta x, x] - (-1)^{|x|-1}[x, \Delta x] \\ &= [\Delta x, x] - [x, \Delta x] \\ &= [\Delta x, x] + (-1)^{(|x|+1)(|\Delta x|+1)}[\Delta x, x] \\ &= 2[\Delta x, x] \\ &= 2(\Delta((\Delta x)x) - (\Delta x)^2).\end{aligned}$$

From this it follows that

$$\Delta((\Delta x)x) = (\Delta x)^2$$

and

$$[\Delta x, \Delta x] = \Delta((\Delta x)^2) = 0.$$

Thus the bracket vanishes and Δ is given by the formula

$$\Delta((\Delta x)x) = (\Delta x)^2.$$

If x has even degree then $[x,x]$ has odd degree so its square is zero. As an algebra the free Gerstenhaber algebra generated by x is the free graded commutative algebra generated by x and $[x, x]$. In fact, using the Poisson and Jacobi identity we see that the bracket is given by the following formulas

$$[x^n, x^m] = nm[x, x]x^{n+m-2},$$
$$[x^n, [x, x]] = 2n(n-1)([x, x])^2 x^{n-2}.$$

The free Batalin-Vilkovisky algebra in one generator of even degree, x, is equal to the free graded commutative algebra generated by x, $\Delta(x)$ and $\Delta(x^2)$. To show this, we will prove that the bracket and the Batalin-Vilkovisky operator are given by the following formulas.

$$\begin{aligned}
[x, x] &= \Delta(x^2) - 2x(\Delta x),\\
[x^n, x^m] &= nm(\Delta(x^2)x^{n+m-2} - 2(\Delta x)x^{n+m-1}),\\
[x^n, [x, x]] &= 0,\\
[\Delta x, x^n] &= 0,\\
\Delta(x^k) &= a_k\Delta(x^2)x^{k-2} + b_k(\Delta x)x^{k-1},\\
\Delta(x^k(\Delta x) &= a_k(\Delta x)\Delta(x^2)x^{k-2},\\
\Delta(x(\Delta x)) &= 0,\\
\Delta(x^2(\Delta x)) &= \Delta(x^2)(\Delta x),\\
\Delta(x^k(\Delta x^2)) &= (b_k - 2)(\Delta x)\Delta(x^2)x^{k-1},\\
\Delta(x^2(\Delta x^2)) &= -4(\Delta x)\Delta(x^2)x,\\
\Delta(x^1(\Delta x^2)) &= -(\Delta x)\Delta(x^2),\\
\Delta(x^n(\Delta x^k)) &= (a_k(b_{n+k-2} - 2(n+k-2)) + a_{n+k-1}b_k)(\Delta x)(\Delta x^2)x^{n+k-3},\\
[x^n, \Delta(x^2)] &= -2n(\Delta x)(\Delta x^2)x^{n-1},
\end{aligned}$$

where $n, m \geq 1$, $k > 2$ and a_n, b_n are defined for $n > 2$ by $a_n = a_{n-1} + n - 1$ and $b_n = b_{n-1} - (2n-3)$ with $a_3 = 3$ and $b_3 = -3$.

Now we show the above identities. By definition of Δ, the bracket $[x, x]$ is given in terms of x, $\Delta(x)$ and $\Delta(x^2)$ by

$$\begin{aligned} [x, x] &= (-1)^{|x|}\Delta(x^2) - (-1)^{|x|}(\Delta x)x - x(\Delta x) \\ &= \Delta(x^2) - 2x(\Delta x). \end{aligned}$$

We can use Equation (2.0.1) to compute $\Delta([x, x])$ but also we can compute it using the last identity and the definition of Δ. Combining these calculations we get

$$0 = \Delta([x, x]) = -2\Delta((\Delta x)x),$$

and so

$$\begin{aligned} [\Delta x, x] &= (-1)^{|\Delta x|}\Delta((\Delta x)x) - (-1)^{|\Delta x|}(\Delta(\Delta x))x - (\Delta x)^2 \\ &= -\Delta((\Delta x)x) \\ &= 0, \end{aligned}$$

since $(\Delta x)^2 = 0$ because Δx has odd degree. By induction and the Poisson identity we see that

$$[\Delta x, x^n] = 0.$$

This implies that

$$\Delta(x^n(\Delta x)) = \Delta(x^n)(\Delta x).$$

We next show that, for $n > 2$, there exist non zero constants a_n, b_n such that

$$\Delta(x^n) = a_n\Delta(x^2)x^{n-2} + b_n(\Delta x)x^{n-1}$$

and so

$$\Delta(x^n(\Delta x)) = a_n(\Delta x)\Delta(x^2)x^{n-2}.$$

Indeed,

$$\begin{aligned}[x, x^{n-1}] &= (n-1)[x, x]x^{n-2} \\ &= (n-1)(\Delta(x^2) - 2x(\Delta x))x^{n-2} \\ &= (n-1)\Delta(x^2)x^{n-2} - 2(n-1)(\Delta x)x^{n-1},\end{aligned}$$

where the first equality is proved above. On the other hand, we have that

$$\begin{aligned}[x, x^{n-1}] &= (-1)^{|x|}\Delta(x^n) - (-1)^{|x|}(\Delta x)x^{n-1} - x\Delta(x^{n-1}) \\ &= \Delta(x^n) - (\Delta x)x^{n-1} - x\Delta(x^{n-1})\end{aligned}$$

and putting together both equations we get

$$\Delta(x^n) = \Delta(x^{n-1})x + (n-1)\Delta(x^2)x^{n-2} - (2n-3)(\Delta x)x^{n-1}.$$

In particular, we can calculate $\Delta(x^3)$ either using this method or using the formula for Δ of a triple product to obtain

$$\Delta(x^3) = 3(\Delta x^2)x - 3(\Delta x)x^2,$$

which gives $a_3 = 3$ and $b_3 = -3$. By induction, if $\Delta(x^{n-1}) = a_{n-1}\Delta(x^2)x^{n-3} + b_{n-1}(\Delta x)x^{n-2}$, we obtain

$$\begin{aligned}\Delta(x^n) &= \Delta(x^{n-1})x + (n-1)\Delta(x^2)x^{n-2} - (2n-3)(\Delta x)x^{n-1} \\ &= a_{n-1}\Delta(x^2)x^{n-2} + b_{n-1}(\Delta x)x^{n-1} + (n-1)\Delta(x^2)x^{n-2} - (2n-3)(\Delta x)x^{n-1} \\ &= (a_{n-1} + n - 1)\Delta(x^2)x^{n-2} + (b_{n-1} - (2n-3))(\Delta x)x^{n-1}\end{aligned}$$

so the general formula holds with $a_n = a_{n-1} + n - 1$ and $b_n = b_{n-1} - (2n-3)$ for $n > 3$.

We will compute $\Delta(x^n(\Delta x^2))$ in terms of products of x, $\Delta(x)$ and $\Delta(x^2)$. By definition of Δ and using the formula proved before for $\Delta(x^n)$ when $n \geq 3$, we get

$$\begin{aligned}
[x^n, \Delta x^2] &= (-1)^{|x^n|}\Delta(x^n(\Delta x^2)) - (-1)^{|x^n|}(\Delta x^n)(\Delta x^2) \\
&= \Delta(x^n(\Delta x^2)) - (a_n(\Delta x^2)x^{n-2} + b_n(\Delta x)x^{n-1})(\Delta x^2) \\
&= \Delta(x^n(\Delta x^2)) - b_n(\Delta x)(\Delta x^2)x^{n-1},
\end{aligned}$$

as $(\Delta x^2)^2 = 0$ since Δx^2 has odd degree.

On the other hand, we can use the formulas

$$\begin{aligned}
\Delta x^2 &= [x, x] + 2x(\Delta x) \\
[x^n, [x, x]] &= 2n(n-1)([x, x])^2 x^{n-2} \\
[x^n, x] &= n(\Delta x^2)x^{n-1} - 2n(\Delta x)x^n \\
[x^n, \Delta x] &= 0 \\
(\Delta x)^2 &= 0
\end{aligned}$$

to get

$$\begin{aligned}
[x^n, \Delta x^2] &= [x^n, [x, x] + 2x(\Delta x)] \\
&= [x^n, [x, x]] + 2[x^n, x(\Delta x)] \\
&= 2n(n-1)([x, x])^2 x^{n-2} + 2[x^n, x](\Delta x) + (-1)^{|x|(|x^n|+1)}[x^n, \Delta x] \\
&= 2n(n-1)(\Delta(x^2) - 2x(\Delta x))^2 x^{n-2} + 2(n(\Delta x^2)x^{n-1} - 2n(\Delta x)x^n)(\Delta x) \\
&= 2n(n-1)(-2\Delta(x^2)x(\Delta x) - 2x(\Delta x)(\Delta x^2))^2 x^{n-2} + 2n(\Delta x^2)x^{n-1}(\Delta x) \\
&= 2n(n-1)(2x(\Delta x)\Delta(x^2) - 2x(\Delta x)(\Delta x^2))^2 x^{n-2} - 2n(\Delta x)(\Delta x^2)x^{n-1} \\
&= -2n(\Delta x)(\Delta x^2)x^{n-1}.
\end{aligned}$$

Putting these two equations together we get

$$\Delta(x^n(\Delta x^2)) = (b_n - 2)(\Delta x)(\Delta x^2)x^{n-1}.$$

when $n \geq 3$ and

$$\Delta(x^2(\Delta x^2)) = -4(\Delta x)(\Delta x^2)x$$
$$\Delta(x(\Delta x^2)) = -(\Delta x)(\Delta x^2).$$

So for $m \geq 3$

$$\begin{aligned}
\Delta(x^n\Delta(x^m)) &= \Delta(x^n[a_m\Delta(x^2)x^{m-2} + b_m(\Delta x)x^{m-1}]) \\
&= a_m\Delta(x^{n+m-2}\Delta(x^2)) + b_m\Delta(x^{n+m-1}(\Delta x)) \\
&= (a_m(b_{n+m-2} - 2(n+m-2)) + a_{n+m-1}b_m)(\Delta x)(\Delta x^2)x^{n+m-3}.
\end{aligned}$$

For a more detailed discussion of Batalin-Vilkovisky algebras see [12, Chapter 1].

2.1 String Topology

Let M be a closed, oriented manifold of dimension d and let $LM = C^\infty(S^1, M)$ denote the *free loop space of* M. In [4] Chas and Sullivan investigated the structure of $H_*(LM)$. This paper gave rise to the study of *string topology* due to the following important theorem.

Theorem 2.1.1 (Chas-Sullivan, 1999)**.** *The shifted homology groups* $\mathbb{H}_*(LM) = H_{*+d}(LM)$ *form a Batalin-Vilkovisky algebra.*

We call this structure the *String Homology of* M and we refer to its product and bracket as the *string product* and *string bracket*, respectively (see [4, Theorem 5.4]). This theorem provides very strong motivation to study Batalin-Vilkovisky algebras. In this chapter, we describe the operators of string topology as seen in the notes [7].

2.2 The String Product

In this section the definition of the umkehr (or intersection) map is discussed and then used to describe the string product. The definition of the string product described here is the homotopy theoretical definition due to Cohen and Jones, see [5].

For defining the umkehr map using the Pontryagin-Thom map one considers an embedding of compact manifolds $e\colon P \to M$, with dim $M = d$ and codim $P = k$, and an open tubular neighbourhood $\eta_e \subset M$ of $P \subset M$. Then we have a projection map

$$\pi_e\colon M \to \frac{M}{M - \eta_e}.$$

Notice that

$$\frac{M}{M - \eta_e} \cong \eta_e \cup \{\infty\}$$

and, by the tubular neighbourhood theorem, this is isomorphic to the Thom space P^{ν_e} of the normal bundle $\nu_e \to P$. Therefore, the Pontryagin-Thom map can be viewed as a map

$$\pi_e\colon M \to P^{\nu_e}.$$

We define the umkehr map $e_!$ to be the composition

$$e_!\colon H_q(M) \xrightarrow{(\pi_e)_*} H_q(P^{\nu_e}) \cong H_{q-k}(P),$$

where the last isomorphism is the Thom isomorphism.

Intuitively, a homology class $\theta \in H_q(M)$ which is represented by an embedded

q-dimensional manifold Q transverse to P induces a homology class

$$\alpha = i_*([Q \cap P]) \in H_{q-k}(P),$$

where $[Q \cap P]$ is the fundamental class of the $(q-k)$-dimensional manifold $Q \cap P$ and $i\colon Q \cap P \to P$ is the inclusion. In this case we would have that $e_!(\theta) = \alpha$.

A useful example is the umkehr map of the diagonal embedding $\Delta\colon M \to M \times M$. Since the diagonal map induces the cup product in cohomology, the following diagram is commutative.

$$\begin{tikzcd}
H_p(M) \otimes H_q(M) \arrow[r, "\mu"] \arrow[d, "\text{Poincaré Duality}"'] & H_{p+q-d}(M) \arrow[d, "\text{Poincaré Duality}"] \\
H^{d-p}(M) \otimes H^{d-q}(M) \arrow[r, "\text{cup}"] & H^{2d-p-q}(M)
\end{tikzcd}$$

Therefore, $\mu = \Delta_!\colon H_p(M) \otimes H_q(M) \to H_{p+q-d}(M)$.

We would like to mimic the above construction of μ to get a map

$$\mu\colon H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM)$$

for LM that will define the string product. First we will give a brief outline of the construction. Let

$$LM \times_M LM = \{(\alpha, \beta) \in LM \times LM | \alpha(1) = \beta(1)\}.$$

In this case, by using the diagonal embedding $\Delta : M \to M \times M$, we will prove that there is an embedding of codimension d

$$e\colon LM \times_M LM \to LM \times LM$$

and the existence of a natural tubular neighbourhood η_e of $LM \times_M LM$ in $LM \times LM$.

The normal bundle of $LM \times_M LM$ associated to the embedding e will be isomorphic to $ev^*(TM)$, where $ev\colon LM \to M$ is the evaluation map $ev(\alpha) = \alpha(1)$. This gives a Thom-Pontryagin map

$$\pi_e\colon LM \times LM \to (LM \times_M LM)^{ev^*(TM)}.$$

Then we define the umkehr map $e_! = (\pi_e)_* \cap u$, where u is the Thom class in $H^d((LM \times_M \times LM)^{ev^*(TM)})$ given by the orientation and $(\pi_e)_*$ is the map in homology induced by π_e. Finally, if $\gamma\colon LM \times_M LM \to LM$ denotes the usual composition of loops, the string product is the composition

$$\begin{aligned}\mu\colon H_p(LM) \otimes H_q(LM) &\to H_{p+q}(LM \times LM)\\ &\xrightarrow{e_!} H_{p+q-d}(LM \times_M LM) \xrightarrow{\gamma_*} H_{p+q-d}(LM).\end{aligned}$$

In [16, Proposition 2.4.1], Klingenberg shows that LM is a Hilbert manifold and therefore $LM \times LM$ is a Hilbert manifold. Note that $LM \times_M LM$ is a submanifold of codimension d of $LM \times LM$ and we have a pull-back square

$$\begin{tikzcd} LM \times_M LM \arrow[r, "e"'] \arrow[d, "ev"'] & LM \times LM \arrow[d, "ev\times ev"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd} \tag{2.2.1}$$

where $ev\colon LM \to M$ is the evaluation map $\alpha \mapsto \alpha(1)$ and Δ is the diagonal map.

Notice that, since the normal bundle ν_Δ of M in $M \times M$ is isomorphic to the tangent bundle TM and $ev \times ev\colon LM \times LM \to M \times M$ is a fibre bundle, then the normal bundle of $LM \times_M LM$ in $LM \times LM$ is isomorphic to

$$ev^*(\nu_\Delta) \cong ev^*(TM).$$

The existence of the above pull-back square (2.2.1) means that there is a natural tubular neighbourhood of $LM \times_M LM$ in $LM \times LM$: the inverse image of a tubular neighbourhood of the diagonal embedding $\eta_e = (ev \times ev)^{-1}(\eta_\Delta)$. Now η_e is homeomorphic to the (total space of the) normal bundle $ev^*(TM)$. This induces an isomorphism of the quotient space to the Thom space

$$\frac{LM \times LM}{(LM \times LM) - \eta_e} \cong (LM \times_M LM)^{ev^*(TM)}.$$

The projection

$$LM \times LM \to \frac{LM \times LM}{(LM \times LM) - \eta_e}$$

and the last homeomorphism define the Pontryagin-Thom map

$$\pi_e \colon LM \times LM \to (LM \times_M LM)^{ev^*(TM)}.$$

Then we can define the umkehr map to be the composition

$$e_! \colon H_*(LM \times LM) \xrightarrow{(\pi_e)_*} H_*((LM \times_M LM)^{ev^*(TM)}) \xrightarrow{\cap u} H_{*-d}(LM \times_M LM)$$

where $u \in H^d((LM \times_M LM)^{ev^*(TM)})$ is the Thom class given by the orientation.

Let $\gamma \colon LM \times_M LM \to LM$, be the standard multiplication of loops. The *string product* is the composition

$$\mu \colon \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \to \mathbb{H}_*(LM \times LM) \xrightarrow{e_!} \mathbb{H}_*(LM \times_M LM) \xrightarrow{\gamma_*} \mathbb{H}_*(LM).$$

Now the map $\alpha_t \colon LM \times LM \to LM$ given by

$$\alpha_t(\alpha, \beta)(s) = \begin{cases} \beta(2s - t) & \text{for } 0 \leq s \leq \frac{t}{2} \\ \alpha(2s - t) & \text{for } \frac{t}{2} \leq s \leq \frac{t+1}{2} \\ \beta(2s - t) & \text{for } \frac{t+1}{2} \leq s \leq 1 \end{cases} \tag{2.2.2}$$

gives a homotopy from $\gamma(\alpha, \beta)$ and $\gamma(\beta, \alpha)$; that is, the product is homotopy commutative. This fact and the naturality of the umkehr map imply that the string product is commutative (see [4, Theorem 3.3]).

2.3 The Batalin-Vilkovisky Algebra Structure

We will now introduce an operator Δ in $\mathbb{H}_*(LM)$ which arises naturally from the action of the circle on LM. We also discuss the definition of the string bracket in terms of Δ and give an alternate homotopy theoretical definition. These operators give $\mathbb{H}_*(LM)$ the structure of a Batalin-Vilkovisky algebra. Unless otherwise stated, from now on we will identify S^1 with $\mathbb{R}/\mathbb{Z}$ with the standard additive group structure.

There is an action $\rho\colon S^1 \times LM \to LM$ defined by $\rho(t,\alpha)(s) = \alpha(t+s)$. This defines an operator

$$\Delta\colon \mathbb{H}_*(LM) \xrightarrow{e_1 \times \bullet} H_1(S^1) \otimes \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(S^1 \times LM) \xrightarrow{\rho_*} \mathbb{H}_{*+1}(LM),$$

where $e_1 \in H_1(S^1)$ is the generator. Now, the commutativity of the diagram

$$\begin{tikzcd}
S^1 \times (S^1 \times LM) \arrow[r, "\mathrm{id} \times \rho"] \arrow[d, "\mu \times \mathrm{id}"'] & S^1 \times LM \arrow[r, "\rho"] & LM \\
S^1 \times LM \arrow[urr, "\rho"'] & &
\end{tikzcd}$$

implies that $\rho_*(\mu_* \times (id)_*) = \rho_*(id_* \times \rho_*)$, where $\mu\colon S^1 \times S^1 \to S^1$ is the map $\mu(r,s) = (r+s) \bmod 1$. Then $\Delta^2(\theta) = \rho_*(e_1^2 \otimes \theta)$ where e_1^2 denotes the Pontryagin product of e_1 with itself induced by μ. Now, by definition, the Pontryagin product induced by μ is the composition

$$H_1(S^1) \otimes H_1(S^1) \xrightarrow{\times} H_2(S^1 \times S^1) \xrightarrow{\mu_*} H_2(S^1)$$

and since $H_2(S^1) = 0$ then $e_1^2 = 0 \in H_2(S^1)$. Thus $\Delta^2(\theta) = 0$ since the Pontryagin product $e_1^2 = 0$.

From this we get an operation $\mathbb{H}_p(LM) \times \mathbb{H}_q(LM) \to \mathbb{H}_{p+q+1}(LM)$ given by

$$[\phi, \theta] = (-1)^p \Delta(\phi \bullet \theta) - (-1)^p \Delta(\phi) \bullet \theta - \phi \bullet \Delta(\theta).$$

The bracket $[,]$ is called the *string bracket* and Δ is the *Batalin-Vilkovisky* operator. In [4, Theorem 5.4], Chas and Sullivan proved the following theorem.

Theorem 2.3.1. *With this structure* $(\mathbb{H}_*(LM), \Delta, \bullet)$ *is a Batalin-Vilkovisky algebra.*

2.3.1 The String Bracket

Here we discuss a homotopy theoretical definition of the string bracket, making use of the actions of $\mathbb{Z}/2$ on S^1 (antipodal action) and on $LM \times LM$ (permutation action). This description can be found in [7].

Let $\mathcal{P} \subset S^1 \times LM \times LM$ be the space

$$\mathcal{P} = \{(t, \alpha, \beta) \colon \alpha(0) = \beta(t)\}.$$

There is a diffeomorphism

$$h \colon S^1 \times (LM \times_M LM) \to \mathcal{P}$$

given by $h(t, (\alpha, \beta)) = (t, \alpha, \beta_t)$, where $\beta_t(s) = \beta(s - t)$.

Now, there is a pullback square of fibrations

$$\begin{tikzcd} \mathcal{P} \arrow[r, "u"] \arrow[d, "\epsilon"'] & S^1 \times LM \times LM \arrow[d, "\epsilon"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd}$$

where $\epsilon \colon S^1 \times LM \times LM \to M \times M$ is defined by $(t, \alpha, \beta) \to (\alpha(0), \beta(t))$. By an analogous argument to before, this pullback square allows the definition of a

Pontryagin-Thom map

$$\pi_u \colon S^1 \times LM \times LM \to \mathcal{P}^{\epsilon^*(TM)}$$

and therefore of an umkehr map

$$u_! \colon H_*(S^1 \times LM \times LM) \to H_{*-d}(\mathcal{P}) \cong H_{*-d}(S^1 \times (LM \times_M LM)).$$

Notice that $\mathbb{Z}/2$ acts on S^1 by the antipodal action and, on $LM \times LM$ and on $M \times M$, by the permutation action. Moreover, $\Delta(M)$ is the fixed point set from the action of $\mathbb{Z}/2$ on $M \times M$. Thus the action on $S^1 \times LM \times LM$, together with the action on $M \times M$, induce a $\mathbb{Z}/2$-action on the above pullback diagram. Observe that the diffeomorphism $h \colon S^1 \times (LM \times_M LM) \to \mathcal{P}$ is equivariant, where $\mathbb{Z}/2$ acts antipodally on S^1 and permutes the two components of $LM \times_M LM$.

Therefore, the umkehr map $u_!$ is well defined on the homology of the orbits

$$u_! \colon H_*(S^1 \times_{\mathbb{Z}/2} (LM \times LM)) \to H_{*-d}(S^1 \times_{\mathbb{Z}/2} (LM \times_M LM))$$

as h is equivariant. Using this we get a homomorphism

$$\nu_* \colon H_*(LM \times LM) \to H_{*+1-d}(S^1 \times_{\mathbb{Z}/2} (LM \times_M LM))$$

defined by

$$\nu_*(\phi \otimes \theta) = u_!(e_1 \otimes (\phi \otimes \theta - (-1)^{(|\phi|+1)(|\theta|+1)} \theta \otimes \phi)).$$

Let $G \colon [0,1] \times (LM \times_M LM) \to LM$ be the homotopy given by 2.2.2 and recall that the standard multiplication of loops commutes up to homotopy via G. By identifying $[0,1]$ with the upper semicircle of S^1, G defines a map

$$G \colon S^1 \times_{\mathbb{Z}/2} (LM \times_M LM) \to LM.$$

Finally, we get an operation

$$G_* \circ \nu_* \colon H_*(LM \times LM) \to H_{*+1-d}(S^1 \times_{\mathbb{Z}/2} (LM \times_M LM)) \to H_{*+1-d}(LM).$$

We have the following theorem.

Theorem 2.3.2.

$$(G_* \circ \nu_*)(\phi \otimes \theta) = [\phi, \theta].$$

Proof. In [4, Definition 4.1] the operation $G_* \circ \nu_*$ is defined to be the string bracket. Taking Δ to be defined as above, in [4, Corollary 5.3] Chas and Sullivan proved that the following identity holds

$$(G_* \circ \nu_*)(\phi \otimes \theta) = (-1)^p \Delta(\phi \bullet \theta) - (-1)^p \Delta(\phi) \bullet \theta - \phi \bullet \Delta(\theta).$$

□

Chapter 3

Connections with Algebra

Hochschild homology and cohomology have become very important in the study of string topology as they provide us with new ways to compute string topology with its full structure. In the following sections we present the definitions of Hochschild homology and cohomology and give relations between these theories and string topology. The first three sections are dedicated to the study of the basic Hochschild theory, their content can be found in [18] and [26].

3.1 Hochschild Homology

Definition 3.1.1. Let K be a field, A a unital K-algebra and M an A-bimodule. The chain complex $(C_n(A,M), b)$ defined by

$$C_n(A,M) := M \otimes A^{\otimes n}$$

and

$$b(m, a_1, \ldots, a_n) := (ma_1, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n m, a_1, \ldots, a_{n-1})$$

is called the Hochschild complex. The homology of this complex is called the Hochschild Homology of A with coefficients in M. We denote it by $HH_*(A, M)$.

Throughout, we will write with no distinction both $(m, a_1, \ldots, a_n)$ and $m \otimes a_1 \otimes \ldots a_n$ for an element in $C_n(A, M)$ or in $HH_*(A, M)$.

An alternative way to compute Hochschild homology is in terms Tor when the algebra A is a projective as a K-module.

Proposition 3.1.2. *Let A be a unital algebra and $A^e = A \otimes A^{op}$ where A^{op} denotes the algebra with the opposite multiplication. If A is projective as a module over K, then for any A-bimodule M there is an isomorphism*

$$HH_n(A, M) = Tor_n^{A^e}(M, A).$$

For a proof of this proposition we refer the reader to [26, Corollary 9.1.5].

3.1.1 Properties and Computations of Hochschild Homology

We now describe some of the properties and computations in low degree of the Hochschild homology groups. We discuss the functoriality of $HH_n(-,-)$, compute $HH_0(A, M)$ and introduce the concept of Kähler differentials and their relation with $HH_1(A, M)$. This is a summary of some of the results from [18, Section 1.1].

Remark 3.1.3. Note that the construction of the Hochschild homology is functorial in M: given $f : M \to M'$ there exists a map

$$\begin{aligned} f_* : HH_*(A, M) &\to HH_*(A, M') \\ f_*(m, a_1, \ldots, a_n) &:= (f(m), a_1, \ldots, a_n). \end{aligned}$$

We also have functoriality in A: given $g : A \to A'$ and M' an A'-bimodule then g makes it into an A-bimodule and $g_*(m, a_1, \ldots, a_n) = (m, g(a_1), \ldots, g(a_n))$.

Remark 3.1.4. When $M = A$ we have that $HH_n(-,-)$ is a covariant functor, i.e. $f : A \to A'$ induces a homomorphism $f_* : HH_n(A, A) \to HH_n(A', A')$.

Also observe that

$$HH_0(A,M) = \frac{M}{\{am - ma : \ a \in A, \ m \in M\}}.$$

Definition 3.1.5. Let A be unital and commutative. The A-module $\Omega^1_{A|K}$, generated by the elements da with $a \in A$ subject to the relations

$$\begin{aligned} d(\lambda a + \mu b) &= \lambda da + \mu db \quad \lambda, \mu \in K \text{ and } a, b \in A \\ d(ab) &= (da)b + adb, \end{aligned}$$

is called the module of Kähler differentials .

Proposition 3.1.6. *If A is unital and commutative, then there is a canonical isomorphism $HH_1(A,A) \cong \Omega^1_{A|K}$. If M is a symmetric bi-module, i.e. $ma = am$ for every $m \in M$ and $a \in A$, then $HH_1(A,M) \cong M \otimes \Omega^1_{A|K}$.*

Proof. Since A is commutative, then $HH_1(A,A)$ is the quotient of $A \otimes A$ by the relation $ab \otimes c - a \otimes bc + ca \otimes b$. Then we have a map $HH_1(A,A) \to \Omega^1_{A|K}$ defined as $a \otimes b \mapsto adb$. It is easy to check that the map in the other direction which sends adb to the class of $a \otimes b$ is well defined because of the commutativity of A and that is also a module homomorphism. $\square$

3.1.2 Algebra Structure in HH_*

One of the interesting properties of Hochschild homology of a commutative algebra is the existence of a product $HH_p(A,A) \times HH_q(A,A) \to HH_{p+q}(A,A)$. Unless otherwise stated, the algebras considered here will be commutative.

Following [18, Chapter 4, Section 4.2] there is an action of the symmetric group S_n on $C_n(A,A)$ given by

$$\sigma \cdot (a_0, a_1, \ldots, a_n) = (a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}),$$

where $\sigma \in S_n$.

We define a (p,q)-shuffle to be a permutation $\sigma \in S_{p+q}$ such that

$$\sigma(1) < \cdots < \sigma(p) \quad \textit{and} \quad \sigma(p+1) < \cdots < \sigma(p+q).$$

Definition 3.1.7. Let A be a commutative algebra. The internal product

$$\times \colon HH_p(A,A) \otimes HH_q(A,A) \to HH_{p+q}(A,A)$$

is given by the following formula

$$\begin{aligned}(a_0, a_1, \dots, a_p) &\times (a'_0, a'_1, \dots, a'_q) \\ &= \left[\sum_{\sigma} \operatorname{sgn}(\sigma)(a_0a'_0, a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(p)}, a'_{\sigma^{-1}(p+1)}, \dots, a'_{\sigma^{-1}(p+q)}) \right]\end{aligned}$$

where the sum is over all (p,q)-shuffles and $[]$ denotes the class of an element in $HH_*(A,A)$.

Remark 3.1.8. In fact, for any unital K-algebras A, B there exists an external product

$$HH_p(A,M) \otimes HH_q(B,N) \to HH_{p+q}(A \otimes B, M \otimes N)$$

where M is an A-bimodule and N is a B-bimodule. When A is commutative, the multiplication is a homomorphism of K-algebras, and so it induces a map in Hochschild homology

$$\mu_* \colon HH_*(A \otimes A, A \otimes A) \to HH_*(A,A).$$

When $B = M = N = A$ we can compose the external product with μ_* to give rise to the internal product. For a complete account of these facts we refer the reader to [3, Chapter XI, Section 6], [18, Section 4.2] and [26, Section 9.4].

Corollary 3.1.9. *With the inner product given above, $HH_*(A,A)$ becomes an associative and graded commutative algebra. That is, the map*

$$\times : HH_p(A,A) \otimes HH_q(A,A) \to HH_{p+q}(A,A)$$

satisfies the identity $\alpha \times \beta = (-1)^{pq} \beta \times \alpha$ with $\alpha \in HH_p(A,A)$ and $\beta \in HH_q(A,A)$.

For a proof of this we refer the reader to [18, Section 4.2].

We now introduce the module of differential forms $\Omega^n_{A|K}$. For particular algebras A, we will use this module to completely describe the algebra $HH_*(A,A)$. Let $\Omega^0_{A|K} = A$ and define the module of differential forms as

$$\Omega^n_{A|K} = \Lambda^n_A \Omega^1_{A|K},$$

where Λ_A denotes the exterior product over A. Now since $\Omega^1_{A|K} \cong HH_1(A,A)$ we have a natural map of algebras

$$\psi \colon \Omega^n_{A|K} \to HH_n(A,A)$$

called the antisymmetrisation map. There are special cases when the antisymmetrisation map is an isomorphism, in particular, when A is a free algebra. The following theorem will be key in the following chapter.

Theorem 3.1.10 (Hochschild-Kostant-Rosenberg)**.** *For a smooth algebra A, the antisymmetrisation map is an isomorphism.*

For the definition of smooth see [18, Appendix E]. Most importantly for our applications, free commutative algebras are smooth. This result is originally due to Hochschild, Kostant and Rosenberg in [14]. For a proof of this theorem we refer the reader to [18, Section 3.4] and [26, Theorem 9.4.7].

We finish this section with standard theory that will be very useful in the future.

3.1.3 The Connes Operator

In later sections the Connes or B-operator will play an important role in the study of string topology.

Definition 3.1.11. Let $s\colon C_{n-1}(A, A) \to C_n(A, A)$ be defined by

$$s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n,$$

$t\colon C_n(A, A) \to C_n(A, A)$ defined by

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

and $N\colon C_n(A, A) \to C_n(A, A)$ defined by

$$N = \mathrm{id} + t + t^2 + \cdots + t^n.$$

Let $B\colon C_n(A, A) \to C_{n+1}(A, A)$ be the map

$$B = (1 - t)sN.$$

Define

$$b'(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n).$$

A direct calculation shows that the following identities are satisfied

$$b(1 - t) = (1 + t)b'$$

$$b's + sb' = \mathrm{id}$$

$$Nb = b'N.$$

Using these identities we have

$$\begin{aligned} bB + Bb &= b(1-t)sN + (1+t)sNb \\ &= (1+t)b'sN + (1+t)sb'N \\ &= (1+t)(b's + sb')N \\ &= 0. \end{aligned}$$

Therefore, B induces a map

$$B : HH_n(A, A) \to HH_{n+1}(A, A)$$

called the Connes or B-operator. A formula for B is given by

$$\begin{aligned} B(a_0, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{ni}(1, a_i, \ldots, a_n, a_0, \ldots a_{i-1}) \\ - (-1)^{n(i-1)}(a_i, 1, a_{i+1}, \ldots, a_n, a_0, \ldots a_{i-1}). \end{aligned}$$

We have the following commutative diagram:

$$\begin{tikzcd} \Omega^n_{A|K} \arrow[r, "\psi_n"] \arrow[d, "d"'] & HH_n(A, A) \arrow[d, "B"] \\ \Omega^{n+1}_{A|K} \arrow[r, "\psi_{n+1}"] & HH_{n+1}(A, A) \end{tikzcd} \tag{3.1.1}$$

where d denotes the exterior differential operator defined by

$$d(a_0 da_1 \cdots da_n) = da_0 da_1 \cdots da_n.$$

For the proof of this statement see [18, Proposition 2.3.3] and [26, Lemma 9.8.10].

Let A be a unital K-algebra and $\overline{A} = A/K$. We define the normalised

Hochschild complex as $\overline{C}_n(A,M) = M \otimes \overline{A}^{\otimes n}$ with boundary operator given by

$$b(m, a_1, \dots, a_n) = (ma_1, a_2, \dots, a_n) + \sum_{0<i<n} (-1)^i (m, a_1, \dots, a_i a_{i+1}, \dots, a_n) \\ + (-1)^n (a_n m, a_1, \dots, a_{n-1}).$$

This complex is quasi-isomorphic to the Hochschild complex. For a more detailed discussion of this please see [3, Chapter IX, Section 6]. In this case we have a normalised B operator which is defined as:

$$\overline{B}(a_0, \dots, a_n) = \sum_{i=0}^{n} (-1)^{ni} (1, a_i, \dots, a_n, a_0, \dots a_{i-1}).$$

3.2 Hochschild Cohomology

Let A be a K-algebra and M an A-bimodule. Define the Hochschild cochain complex

$$C^n(A, M) = \mathrm{Hom}(A^{\otimes n}, M).$$

The coboundary operator β is given by the following formula. Given a K-linear map $f : A^{\otimes n} \to M$

$$\beta(f)(a_1, \dots, a_{n+1}) = a_1 f(a_2, \dots, a_{n+1}) + \sum_{0<i<n+1} (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ + (-1)^{n+1} f(a_1, \dots, a_n) a_{n+1}.$$

Then the Hochschild cohomology of A with coefficients in M is defined by

$$HH^n(A, M) = H^n(C^n(A, M), \beta).$$

The following proposition describes Hochschild cohomology in terms of Ext.

Proposition 3.2.1. *Let A be a unital algebra and projective as a module over K.*

Recall that $A^e = A \otimes A^{\mathrm{opp}}$ where A^{opp} is the algebra A with the opposite multiplication. Then for any A-bimodule M there is an isomorphism

$$HH^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M).$$

For a proof of this proposition please see [26, Corollary 9.1.5].

3.2.1 Properties and Computations of Hochschild Cohomology

As in the case of the Hochschild homology, there is functoriality of $HH^n(-,-)$. In the following paragraphs we will also compute $HH^0(A, M)$ and $HH^1(A, M)$. These calculations can be found in [18, Section 1.5]. There, one may also find an expression for $HH^2(A, M)$, however we will not make use of such a computation here.

The functor $H^n(-,-)$ is contravariant in A: for a K-algebra homomorphism $f : A \to A'$ and an A'-bimodule M we have a natural A-module, denoted by f^*M, and a map

$$f^* : HH^n(A', M) \to HH^n(A, f^*M).$$

In the case of $n = 0$ we have that

$$HH^0(A, M) = \{m \in M | am = ma \text{ for any } a \in A\}.$$

When $n = 1$ a cocycle is a K-module homomorphism $D : A \to M$ satisfying the identity $D(ab) = a(Db) + (Da)b$. Such a homomorphism D is called a derivation from A to M and we denote by $\mathrm{Der}(A, M)$ the module of derivations from A to M. The homomorphism D is a coboundary if it has the form $L_m(a) = [m, a] = ma - am$, so if it is an *inner derivation.* Therefore

$$HH^1(A, M) = \frac{\mathrm{Der}(A, M)}{\text{Inner derivations}}.$$

3.2.2 The cup product in $HH^*(A, M)$

There is a multiplication in the Hochschild cochain complex which turns it into an associative and graded ring. For $f \in C^p(A, M)$ and $g \in C^q(A, M)$ their cup product $f \cup g \in C^{p+q}(A, M)$ is defined by

$$(f \cup g)(a_1 \otimes \ldots \otimes a_p \otimes b_1 \otimes \ldots \otimes b_q) = f(a_1 \otimes \ldots \otimes a_p)g(b_1 \otimes \ldots \otimes b_q).$$

We have that

$$\beta(f \cup g) = (\beta f) \cup g + (-1)^p f \cup (\beta g).$$

In consequence $\cup$ induces a product in cohomology

$$\cup \colon HH^p(A, M) \otimes HH^q(A, M) \to HH^{p+q}(A, M).$$

Although the cup product is not graded commutative at the chain level, in [11, Corollary 2, Section7], Gerstenhaber proved that it is graded commutative in cohomology. Since $HH^1(A, A) = \mathrm{Der}(A, A)$, when A is a commutative algebra, and the cup product is graded commutative, there is a natural map of algebras

$$\psi : \Lambda^n_A(\mathrm{Der}(A, A)) \to HH^n(A, A).$$

The following theorem is the cohomological counterpart of theorem 3.1.10 which will be key in the following chapter.

Theorem 3.2.2 (Hochschild-Kostant-Rosenberg)**.** *If A is a smooth algebra then the map*

$$\psi : \Lambda^n_A(\mathrm{Der}(A, A)) \to HH^n(A, A)$$

is an isomorphism of algebras.

For a proof of this, see [14, Theorem 5.2].

3.2.3 Lie Bracket on $HH^*(A, A)$

Given $f \in C^m(A, A)$ and $g \in C^n(A, A)$ we define $f \circ_i g \in C^{m+n-1}(A, A)$ by

$$f \circ_i g(a_1, \ldots, a_{m+n-1}) = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), a_{i+n}, \ldots, a_{m+n-1}).$$

Now we define $f \circ g$ by

$$f \circ g = \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f \circ_i g$$

and $[f, g]$ by

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f.$$

This bracket operation is the Gerstenhaber bracket and it is a Lie bracket of degree 1 as defined in [18, Chapter 1, E.1.5.2]. This was first proved by Gerstenhaber in [11]. Moreover, in the same paper, Gerstenhaber proved that the bracket together with the cup product make $HH^*(A, A)$ into a Gerstenhaber algebra.

3.3 Hochschild Homology and Cohomology for Differential Graded Algebras

In this section we introduce the concept of a differential graded algebra and the definitions of Hochschild homology and cohomology for these algebras. Throughout this manuscript we will use cohomological grading conventions. In particular, this will mean that the Hochschild boundary operator will increase degree by 1 whereas the Connes operator will decrease degree by 1.

Definition 3.3.1. A Differential Graded Algebra A, over the ground field K, is a graded algebra $A = \oplus_{n \in \mathbb{Z}} A_n$ with a unit together with a differential ∂ of degree $+1$ (i.e. $\partial^2 = 0$ and $\partial : A_n \to A_{n+1}$) which is a graded derivation for the product in A, that is,

$$\partial(ab) = \partial a \cdot b + (-1)^{|a|} a \cdot \partial b.$$

An element $a \in A_i$ is called homogeneous of degree $i = |a|$.

In our case, we will restrict to those differential graded algebras for which there exists $n \in \mathbb{Z}$ such that either $A_k = 0$ for every $k \geq n$ or $A_k = 0$ for every $k \leq n$.

A differential graded algebra (A, ∂) gives rise to a cochain complex

$$A_0 \to \cdots \to A_{n-1} \xrightarrow{\partial} A_n \to \cdots$$

whose cohomology is denoted $H^*(A, \partial)$.

3.3.1 Hochschild Homology for Differential Graded Algebras

In this subsection we discuss the definition of the Hochschild homology of a differential graded algebra. This material can be found in [26, Section 9.9.1] and [18, Section 5.3].

Let (A, ∂) be a differential graded algebra. The iterated tensor product of complexes $(A, \partial)^{\otimes n+1}$ is a complex with underlying module the tensor product $A^{\otimes n+1}$ and the differential on it is defined by

$$\partial(a_0, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{|a_0| + \cdots + |a_{i-1}|} (a_0, \ldots, a_{i-1}, \partial a_i, a_{i+1}, \ldots, a_n).$$

The (total) degree of an element $x = (a_0, \ldots, a_n) \in A^{\otimes n+1}$ is given by the formula $|x| = |a_0| + |a_1| + \ldots + |a_n| - n$.

If $0 \leq i < n$, define $d_i \colon A^{\otimes n+1} \to A^{\otimes n}$ by the formula

$$d_i(a_0, \ldots, a_n) = (a_0, \ldots, a_i a_{i+1}, \ldots, a_n)$$

and $d_n\colon A^{\otimes n+1} \to A^{\otimes n}$ by

$$d_n(a_0, \ldots, a_n) = (-1)^{|a_n|(|a_0|+|a_1|+\ldots+|a_{n-1}|)}(a_n a_0, a_1, \ldots, a_{n-1}).$$

Now we define $b\colon A^{\otimes n+1} \to A^{\otimes n}$ to be the operator

$$b = \sum_{i=0}^{n} (-1)^i d_i.$$

If we multiply ∂ by $(-1)^n$, we get a double complex

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
(A^{\otimes 3})_0 \arrow[r, "\partial"] \arrow[d, "b"'] & (A^{\otimes 3})_1 \arrow[r, "\partial"] \arrow[d, "b"'] & (A^{\otimes 3})_2 \arrow[r, "\partial"] \arrow[d, "b"'] & {} \\
(A^{\otimes 2})_0 \arrow[r, "-\partial"] \arrow[d, "b"'] & (A^{\otimes 2})_1 \arrow[r, "-\partial"] \arrow[d, "b"'] & (A^{\otimes 2})_2 \arrow[r, "-\partial"] \arrow[d, "b"'] & {} \\
A_0 \arrow[r, "\partial"] & A_1 \arrow[r, "\partial"] & A_2 \arrow[r, "\partial"] & {}
\end{tikzcd}$$

where $(A^{\otimes n+1})_p$ denotes the elements of the tensor product with inner degree equal to p, that is

$$(A^{\otimes n+1})_p = \bigoplus_{i_0+\cdots+i_n=p} A_{i_0} \otimes \ldots \otimes A_{i_n}.$$

Definition 3.3.2. The Hochschild complex of the differential graded algebra (A, ∂) is the total complex of the double complex

$$(C_{n,p}(A, \partial),\ b + (-1)^n \partial)$$

defined above, where $C_{n,p}(A, \partial) = (A^{\otimes n+1})_p$ and the module of total degree r is

$$\mathrm{Tot}_r(A, \partial) = \bigoplus_{p-n=r} C_{n,p}(A, \partial).$$

The Hochschild homology of (A, ∂) is the homology of the Hochschild complex and we denote it by $HH_*(A, \partial)$.

Although this definition is valid for any differential graded algebra (A, ∂), we will restrict our attention to simply connected differential graded algebras of finite type. That is, a positively graded differential graded algebra (A, ∂) of finite type, over K, with $H^0(A) = K$ and $H^1(A) = 0$.

3.3.2 Commutative Differential Graded Algebras

For a commutative differential graded algebra A there exists an analogue of the differential forms, of the antisymmetrisation map and of Theorem 3.1.10. Here we follow [18, Section 5.4].

Definition 3.3.3. Let A be a differential graded algebra. We call A a commutative differential graded algebra if the following identity holds

$$ab = (-1)^{|a||b|} ba$$

for a, b homogeneous elements.

Let A be a commutative differential graded algebra. Then the graded A-bimodule $A \otimes A$ forms a graded algebra with the product $(a \otimes b)(x \otimes y) = (-1)^{|b||x|} ax \otimes by$ for homogeneous elements $a, b, x, y \in A$.

In the case of a commutative differential graded algebra we also have analogous definitions to the non-graded case of the internal product in $HH_*(A, \partial)$ and a B-operator.

Definition 3.3.4. The graded module of differential forms $\Omega^1_{A|K}$ is defined as the graded A-bimodule generated by elements da, for $a \in A$ and with degree $|da| = |a|$

for homogeneous elements, subject to the relations

$$d(ab) = adb + (da)b = adb + (-1)^{|a||b|}bda$$
$$d(\lambda a + \mu b) = \lambda da + \mu db$$

for every $a, b \in A$ and $\lambda, \mu \in K$.

Definition 3.3.5. For a graded A-module M the graded exterior product of M with itself over A is given by the quotient $M\Lambda_A M = M \otimes_A M/\sim$ where the equivalence relation is generated by

$$m \otimes n \sim -(-1)^{|m||n|} n \otimes m$$

for homogeneous n, m.

In general, we define $\Lambda_A^n M = M^{\otimes_A n}/\approx$ where $\approx$ is the equivalence relation generated by $\sim$.

Definition 3.3.6. The graded exterior differential module of the commutative graded algebra (A, ∂) is defined as

$$\Omega^n_{A|K} = \Lambda^n_A \Omega^1_{A|K}.$$

Remark 3.3.7. For $x, y \in \Omega^n_{A|K}$ we have that $dxdy = -(-1)^{|x||y|}dydx$.

Observe that there is a natural extension δ of ∂ from A to $\Omega^n_{A|K}$ given by the formula

$$\delta(a_0 da_1 \dots da_n) = (-1)^n(\partial(a_0)da_1 \dots da_n + (-1)^{|a_0|}a_0 d(\partial a_1)da_2 \dots da_n + \dots$$
$$\dots + (-1)^{|a_0|+\dots+|a_{n-1}|}a_0 da_1 \dots d(\partial a_n)).$$

The extension of the differential of A to $\Omega^n_{A|K}$ gives rise to a complex that we will denote by $((\Omega^n_{A|K})_*, \delta)$, where an element $x = a_0 da_1 \dots da_n \in \Omega^n_{A|K}$ has (internal) degree $|x| = |a_0| + \dots + |a_n|$. We define $\Omega^*_{(A,\partial)}$ to be the total complex of the following double complex

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
(\Omega^2_{A|K})_0 \arrow[r, "\delta"'] \arrow[d, "0"'] & (\Omega^2_{A|K})_1 \arrow[r, "\delta"'] \arrow[d, "0"'] & (\Omega^2_{A|K})_2 \arrow[r, "\delta"'] \arrow[d, "0"'] & {} \\
(\Omega^1_{A|K})_0 \arrow[r, "-\delta"'] \arrow[d, "0"'] & (\Omega^1_{A|K})_1 \arrow[r, "-\delta"'] \arrow[d, "0"'] & (\Omega^1_{A|K})_2 \arrow[r, "-\delta"'] \arrow[d, "0"'] & {} \\
A_0 \arrow[r, "\delta"'] & A_1 \arrow[r, "\delta"'] & A_2 \arrow[r, "\delta"'] & {}
\end{tikzcd}$$

where the module of total degree n is

$$\Omega^n_{(A,\partial)} = \bigoplus_{p-q=n} (\Omega^q_{A|K})_p.$$

Notice that there exists a canonical map

$$\psi\colon \Omega^*_{(A,\partial)} \to (C_{*,*}(A,\partial), b+\partial)$$

given by

$$\psi_n(a_0 da_1 \dots da_n) = \sum_{\sigma \in S_n} \pm \operatorname{sgn}(\sigma)(a_0, a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}),$$

where $C_{*,*}(A,\partial)$ is defined in Definition 3.3.2. Here the sign convention $\pm$ means that when we commute two elements of the n-tuple of degree p and q then we multiply by $(-1)^{pq}$. This map defines a map in homology, the antisymmetrisation map for Hochschild homology of a differential graded algebra.

We also have a generalisation of the Hochschild-Kostant-Rosenberg isomorphism:

Theorem 3.3.8. *Let (A,∂) be a commutative differential graded algebra such that either*

1. $A = \Lambda V$ *for a free graded* K*-module* V*, or*

2. A *is smooth and* $\mathbb{Q} \subset K$*.*

Then the complex $C_{*,*}(A, \partial)$ *defined is quasi-isomorphic to* $\Omega^*_{(A,\partial)}$*. Thus*

$$HH_*(A, \partial) \cong H_*(\Omega^*_{(A,\partial)}).$$

For a definition of a smooth graded algebra and a proof of this theorem we refer the reader to [18, Section 5.4.5] and [18, Proposition 5.4.6], respectively.

Example 3.3.9. In the case of a free graded commutative algebra $A = K[V]$ the differential is zero and so the Hochschild complex associated with A coincides with the complex defined in the previous section. Notice that $\Omega^1_{A|K}$ is the $K[V]$-module generated by elements of the form $\sigma^{-1}v \in \Sigma^{-1}V$, where $\Sigma^{-1}V$ is the graded vector space in which $\Sigma^{-1}V_n = V_{n-1}$. It follows that $\Omega^*_{A|K} = K[V] \otimes K[\Sigma^{-1}V]$ and thus, using Theorem 3.1.10, there is an isomorphism of graded algebras $HH_*(K[V], 0) = K[V] \otimes K[\Sigma^{-1}V] = K[V \oplus \Sigma^{-1}V]$.

The Connes operator is defined on the generators as

$$B(v) = \sigma^{-1}v, \quad v \in V$$
$$B(\sigma^{-1}w) = 0, \quad w \in V$$

and extended as a derivation. Using a homogeneous basis $x_1, \ldots, x_n$ of V and the commutative diagram 3.1.1, we can see that $HH_*(K[V], 0)$ is generated by elements of the form

$$p(x_1, \ldots, x_n)Bx_{i_1}, \ldots Bx_{i_m}$$

where $p \in K[V]$.

3.3.3 Hochschild Cohomology for Differential Graded Algebras

Let (A, ∂) be a differential graded algebra. Given a K-linear map $f : A^{\otimes n} \to A$, we say that f has degree p if $f(a_1 \otimes \ldots \otimes a_n) \in A$ has degree $k+p$ for any homogeneous element $(a_1 \otimes \ldots \otimes a_n) \in A^{\otimes n}$ of degree k. Define $\mathrm{Hom}(A^{\otimes n}, A)$ to be the graded module generated by K-linear maps $f : A^{\otimes n} \to A$ of degree p, for any $p \in \mathbb{Z}$.

Let $C^{n,p}(A, \partial) = (\mathrm{Hom}(A^{\otimes n}, A))_p$ and define $\delta\colon C^{n,p}(A, \partial) \to C^{n,p+1}(A, \partial)$ by the formula

$$\delta(f)(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{|a_0|+\cdots+|a_{i-1}|} f(a_0 \otimes \cdots \otimes \partial(a_i) \otimes \cdots \otimes a_n)$$

for any $f \in \mathrm{Hom}(A^{\otimes n+1}, A)$. Notice that the Hochschild coboundary map

$$\beta\colon C^{n,\bullet}(A, \partial) \to C^{n+1,\bullet}(A, \partial)$$

together with δ make $C^{n,p}$ into the following double complex.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
(\mathrm{Hom}(A^{\otimes 2}, A))_0 \arrow[r, "\delta"] \arrow[u, "\beta"] & (\mathrm{Hom}(A^{\otimes 2}, A))_1 \arrow[r, "\delta"] \arrow[u, "\beta"] & (\mathrm{Hom}(A^{\otimes 2}, A)_2 \arrow[r] \arrow[u, "\beta"] & {} \\
(\mathrm{Hom}(A, A)_0 \arrow[r, "-\delta"] \arrow[u, "\beta"] & (\mathrm{Hom}(A, A))_1 \arrow[r, "-\delta"] \arrow[u, "\beta"] & (\mathrm{Hom}(A, A))_2 \arrow[r] \arrow[u, "\beta"] & {} \\
A_0 \arrow[r, "\delta"] \arrow[u, "\beta"] & A_1 \arrow[r, "\delta"] \arrow[u, "\beta"] & A_2 \arrow[r, "\delta"] \arrow[u, "\beta"] & {}
\end{tikzcd}$$

Definition 3.3.10. The Hochschild Cohomology of the differential graded algebra (A, ∂) is defined as the cohomology of the double complex

$$HH^*(A, \partial) = H^*(C^{*,*}(A, A), \beta + \delta).$$

Observe that the cup product and the Lie bracket are also defined for differential graded algebras using the same formulas used in previous sections.

Let (A, ∂) be a commutative differential graded algebra. A K-module homomorphism $f\colon A \to A$ is a graded derivation of degree p if f satisfies the identity $f(ab) = (f(a))b + (-1)^{|a|}a(f(b))$ and if for any element $a \in A_k$ we have $f(a) \in A_{k+p}$. We denote by $\mathrm{Der}(A)$ to the module of graded derivations.

For $f \in \mathrm{Der}(A)$, define D to be graded commutator

$$D = \partial \circ f + (-1)^{|f|} f \circ \partial.$$

There is a natural extension of D from $\mathrm{Der}(A)$ to $\Lambda^n_A(\mathrm{Der}(A))$ which, by abusing notation, we will also call D. This extension is given by the following formula

$$\begin{aligned} D(a \wedge f_1 \ldots \wedge f_n) = (-1)^n(\partial(a) \wedge f_1 \ldots \wedge f_n + (-1)^{|a|} a \wedge D(f_1) \wedge f_2 \ldots \wedge f_n + \ldots \\ \ldots + (-1)^{|a|+|f_1|+\ldots+|f_{n-1}|} a \wedge f_1 \ldots \wedge D(f_n)) \end{aligned}$$

for $a \in A$ and $f_i \in \mathrm{Der}(A)$. Here $\Lambda^0_A(\mathrm{Der}(A)) = A$.

Then we have the following double complex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
(\Lambda^2_A(\mathrm{Der}(A)))_0 \arrow[r, "D"] \arrow[u, "0"] & (\Lambda^2_A(\mathrm{Der}(A)))_1 \arrow[r, "D"] \arrow[u, "0"] & (\Lambda^2_A(\mathrm{Der}(A)))_2 \arrow[r] \arrow[u, "0"] & {} \\
(\Lambda^1_A(\mathrm{Der}(A)))_0 \arrow[r, "-D"] \arrow[u, "0"] & (\Lambda^1_A(\mathrm{Der}(A)))_1 \arrow[r, "-D"] \arrow[u, "0"] & (\Lambda^1_A(\mathrm{Der}(A)))_2 \arrow[r] \arrow[u, "0"] & {} \\
(\Lambda^0_A(\mathrm{Der}(A)))_0 \arrow[r, "D"] \arrow[u, "0"] & (\Lambda^0_A(\mathrm{Der}(A)))_1 \arrow[r, "D"] \arrow[u, "0"] & (\Lambda^0_A(\mathrm{Der}(A)))_2^{D} \arrow[r] \arrow[u, "0"] & {}
\end{tikzcd}$$

As in the non-graded case, there is a canonical map of graded algebras

$$\psi\colon (\Lambda^n_A(\mathrm{Der}(A)), D) \to (C^*(A, A), \beta + \delta)$$

using the cup product in $HH^*(A, \partial)$.

Theorem 3.3.11 (Hochschild-Kostant-Rosenberg)**.** *Let A be a commutative differential smooth graded algebra (A, ∂) then the map ψ is an isomorphism.*

Example 3.3.12. For a free graded commutative algebra $A = K[V]$ the Hochschild coboundary operator is equal to the coboundary map defined in Section 3.2. Now $HH^1(A, 0) = \mathrm{Der(A)}$ is equal to the $K[V]$-module generated by elements $(\sigma^{-1}v) \in (\Sigma^{-1}V)^*$. From this we get $\Lambda^*(\mathrm{Der}(A)) = K[V] \otimes K[(\Sigma^{-1}V)^*]$ and using Theorem 3.2.2 we get an isomorphism

$$HH^*(A, 0) = K[V] \otimes K[(\Sigma^{-1}V)^*] = K[V \oplus (\Sigma^{-1}V)^*].$$

Furthermore, using the fact that $(\Sigma^{-1}V)^* = \Sigma^{-1}V^*$ and the elementary properties of the Lie bracket in lower degrees we find that

$$\begin{aligned} [v, w] &= 0, \quad v,\ w \in V \\ [\sigma\phi, v] &= \phi(v), \quad v \in V\ \phi \in V^* \\ [\sigma\phi, \sigma\psi] &= 0, \quad \phi,\ \psi \in V^*. \end{aligned}$$

Using the Jacobi and Poisson identity completely determines the bracket in $HH^*(K[V], 0)$.

3.4 Hochschild Package and String Topology

Up to now we have denoted by $HH_*(A, \partial)$ (respectively, $HH^*(A, \partial)$) the Hochschild homology (respectively, cohomology) of a differential graded algebra (A, ∂). From now on we will use the notation $HH_*(A, A)$ (respectively, $HH^*(A, A)$) to remain consistent with the literature. Analogously we will write $C_*(A, A)$ and $C^*(A, A)$ for the Hochschild chains and cochains.

3.4.1 Hochschild Homology and $H^*(LM)$

In [15], Jones described the relation between Hochschild homology and the cohomology of the loop space of a simply connected topological space M. We summarise these results here.

Let M be a simply connected topological space and define

$$e_n\colon \Delta^n \times LM \to M^{n+1}$$

as

$$e_n((t_1, \dots, t_n), \gamma) = (\gamma(0), \gamma(t_1), \dots, \gamma(t_n)),$$

where Δ^n is the n-simplex

$$\Delta^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n | 0 \le t_1 \le \dots \le t_n \le 1\}.$$

This gives a map

$$\rho_n\colon (S^*(M))^{\otimes(n+1)} \xrightarrow{\times} S^*(M^{n+1}) \xrightarrow{e_n^*} S^*(\Delta^n \times LM) \xrightarrow{/\sigma_n} S^{*-n}(LM)$$

where the first map is the external product and the last map is the slant product with the fundamental n-simplex $\sigma_n \in S^n(\Delta^n)$. The construction of this map can also be found in [13].

Lemma 3.4.1. *The map ρ_n induces a map of chain complexes*

$$\rho_n\colon (C_n(S^*(M), S^*(M)), b + \delta) \to (S^{*-n}(LM), \delta)$$

where b is the Hochschild differential and δ is the coboundary operator on singular cochains.

Proof. Let $\Delta_i \colon M^n \to M^{n+1}$ be the map defined as

$$\Delta_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n)$$

for $1 \leq n$ and let $\delta_i \colon \Delta^n \to \Delta^{n+1}$ be the inclusion of the i-th face of Δ^{n+1}. Notice that the following diagram is commutative.

$$\begin{tikzcd}
\Delta^n \times LM \arrow[r, "e_n"] & M^{n+1} \\
\Delta^{n-1} \times LM \arrow[r, "e_{n-1}"] \arrow[u, "\delta_i \times 1"] & M^n \arrow[u, "\Delta_i"']
\end{tikzcd}$$

Therefore the diagram

$$\begin{tikzcd}
S^*(M^{n+1}) \arrow[r, "e_n^*"] \arrow[d, "\Delta_i^*"'] & S^*(\Delta^n \times LM) \arrow[d, "\delta_i^* \times 1"] \\
S^*(M^n) \arrow[r, "e_{n-1}^*"] & S^*(\Delta^{n-1} \times LM)
\end{tikzcd}$$

commutes.

Observe that the map

$$(S^*(M))^{\otimes n+1} \xrightarrow{\times} S^*(M^{n+1}) \xrightarrow{\Delta_i^*} S^*(M^n)$$

is given by

$$\Delta_i^*(x_1 \otimes \ldots \otimes x_{n+1}) = x_1 \otimes \ldots \otimes x_i x_{i+1} \otimes \ldots \otimes x_{n+1}$$

since the diagonal map induces the product in $S^*(M)$ and Δ_i is the product of identity maps and the diagonal map $M \to M \times M$.

A general formula for the coboundary of the slant product is

$$\delta(x/a) = \delta x/a + (-1)^{|x|} x/da.$$

Thus, if $x = x_1 \otimes \ldots \otimes x_{n+1} \in (S^*(M))^{\otimes n+1}$, then

$$\begin{aligned}
\delta(\rho_n(x)) &= \delta(e_n^*(x)/\sigma_n) \\
&= \delta(e_n^*(x))/\sigma_n + (-1)^{|x|} e_n^*(x)/d\sigma_n \\
&= e_n^*(\delta x)/\sigma_n + (-1)^{|x|} e_n^*(x)/d\sigma_n \\
&= \rho_n(\delta x) + (-1)^{|x|} e_n^*(x)/d\sigma_n.
\end{aligned}$$

Now we compute the second summand,

$$\begin{aligned}
e_n^*(x)/d\sigma_n &= \sum_{i=0}^{n} (-1)^i e_n^*(x)/(\delta_i)_*(\sigma_{n-1}) \\
&= \sum_{i=0}^{n} (-1)^i (\delta_i)^* e_n^*(x)/\sigma_{n-1} \\
&= \sum_{i=0}^{n} (-1)^i e_{n-1}^*((\Delta_i)^*(x))/\sigma_{n-1} \\
&= \rho_{n-1}(b(x)).
\end{aligned}$$

In conclusion, ρ_n is a map of chain complexes. $\square$

In fact, in [15], Jones proved the more general statement.

Theorem 3.4.2 (Jones). *Let M be a simply connected topological space. The map*

$$\rho_* : HH_*(S^*(M), S^*(M)) \to H^*(LM).$$

is an isomorphism of algebras. Moreover, via ρ, the B-operator defined on $HH_*(S^*(M), S^*(M))$ is the dual of the Batalin-Vilkovisky operator on $H_*(LM)$.

3.4.2 Hochschild Cohomology and String Homology

In [5], Cohen and Jones prove the following theorem.

Theorem 3.4.3 (Cohen, Jones)**.** *If M is a simply connected, oriented, closed smooth manifold, then there exists an isomorphism of graded algebras,*

$$f : \mathbb{H}_*(LM) \to HH^*(S^*(M), S^*(M)).$$

Moreover, there is a Batalin-Vilkovisky algebra structure in $HH^*(S^*(M), S^*(M))$ which is defined by Tradler in [25]. In [10], Felix and Thomas proved the following.

Theorem 3.4.4 (Felix, Thomas)**.** *Assume the hypothesis of Theorem 3.4.3 and in addition suppose that the coefficient field is of characteristic zero. Then the map f is an isomorphism of Batalin-Vilkovisky algebras.*

In the case of characteristic 2 and $M = S^2$, Menichi proved that f is not an isomorphism of Batalin-Vilkovisky algebras (see [20]).

Chapter 4

Models and String Topology

All topological spaces considered here are of the homotopy type of a countable CW-complex whose integral homology is of finite type. Also throughout this chapter, we will assume all the algebras A considered here to be finitely generated.

Definition 4.0.5. Let A be a graded commutative algebra. A model P for A is a differential graded commutative algebra, together with a map of differential graded algebras $\epsilon\colon P \to A$, where A is considered as a differential graded algebra with zero differential, such that

- P is free as a graded commutative algebra
- the induced homomorphism $H_*(P) \to A$ is an isomorphism.

We will explore some interesting examples.

Example 4.0.6. Let

$$A = \frac{K[x]}{(x^{n+1})}$$

where K is a characteristic zero field and $|x| = 2k$. Let $P = (K[x, y], \partial, \epsilon)$ where $|y| = 2k(n+1) - 1$, $\partial y = x^{n+1}$, $\partial x = 0$ and

$$\epsilon\colon K[x, y] \to K[x]$$

is the map of (differential) graded algebras defined by

$$x \mapsto [x]$$
$$y \mapsto 0.$$

Observe that $\mathrm{Ker}(\partial) = K[x]$ and $\mathrm{Im}(\partial) = (x^{n+1})$. It follows that the induced map $\epsilon_* \colon H^*(P) \to A$ is an isomorphism. By definition, P is a model for A.

Example 4.0.7. Let K be a characteristic zero field and

$$A = \frac{K[x,y]}{(x^3 - 2xy, x^2y - y^2)}$$

where $|x| = 2$ and $|y| = 4$. We will show that $P = (K[x,y,f,g], \partial)$ is a model for A, where $\partial x = \partial y = 0$, $\partial f = x^3 - 2xy$, $\partial g = x^2y - y^2$ with degrees $|x| = 2$, $|y| = 4$, $|f| = 5$ and $|g| = 7$. Given that f and g have odd degrees, their square is zero and so a general element $Q \in K[x,y,f,g]$ can be written in the form

$$Q = Q_1 + Q_2 f + Q_3 g + Q_4 f g$$

with $Q_i \in K[x,y]$. Using the fact that ∂ is $K[x,y]$-linear and the Leibniz rule we get the following expression

$$\partial Q = Q_2 \partial f + Q_3 \partial g + Q_4 (\partial f) g - Q_4 f (\partial g).$$

It follows that $Q \in \mathrm{Ker}\,\partial$ if and only if $Q_4 = 0$ and there exists $\tilde{Q} \in K[x,y]$ such that $Q_2 = \tilde{Q}\partial g$ and $Q_3 = -\tilde{Q}\partial f$. Therefore, if

$$Q = Q_1 + \tilde{Q}(\partial g) f + \tilde{Q}(\partial f) g$$

with $Q_1, \tilde{Q} \in K[x,y]$. Now, any element of $R \in \operatorname{Im}\partial$ is of the form

$$R = Q_2\partial f + Q_3\partial g + \tilde{Q}(\partial f)g - \tilde{Q}f(\partial g)$$

with $Q_2, Q_3, \tilde{Q} \in K[x,y]$.

Now consider the map of algebras

$$\epsilon\colon K[x,y,f,g] \to K[x,y]$$

which is defined on the generators by

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y \\ f &\mapsto 0 \\ g &\mapsto 0. \end{aligned}$$

From the description of $\operatorname{Ker}\partial$ and $\operatorname{Im}\partial$ we see that the map induced by ϵ in homology

$$H^*(P) \to A$$

is an isomorphism.

4.1 Free Graded Commutative Algebras

In this section we give models for the Hochschild homology and cohomology of a free differential graded algebra. In further sections, these models will be key in the construction of models for Hochschild homology and cohomology of a differential graded algebra.

In Example 3.3.9 we saw that, for a free graded commutative algebra $A = K[V]$, the Hochschild homology was given by $HH_*(K[V], K[V]) = K[V \oplus \Sigma^{-1}V]$.

Thus $(K[V\oplus\Sigma^{-1}V],0)$ is a model for the algebra $HH_*(A,A)$. Moreover, the Connes operator is defined on the generators by

$$B(v)=\sigma^{-1}v,\quad v\in V$$
$$B(\sigma^{-1}w)=0,\quad w\in V$$

and extended as a derivation.

Analogous to the case of homology, by Example 3.3.12, the Hochschild cohomology of a free graded commutative algebra $A=K[V]$ is equal to

$$HH^*(K[V],K[V])=K[V\oplus(\Sigma^{-1}V)^*].$$

In consequence, $(K[V\oplus(\Sigma^{-1}V)^*],0)$ is a model for the algebra $HH_*(A,A)$.

Using the fact that $(\Sigma^{-1}V)^*=\Sigma V^*$ and the elementary properties of the Gerstenhaber bracket in low degrees, we see that

$$[v,w]=0,\quad v,\ w\in V$$
$$[\sigma\phi,v]=\phi(v),\quad v\in V\ \phi\in V^*$$
$$[\sigma\phi,\sigma\psi]=0,\quad \phi,\ \psi\in V^*.$$

The Jacobi and Poisson identity completely determine the bracket in $HH^*(K[V],K[V])$.

4.2 Models and Hochschild Homology

Given a model $P=(K[V],\partial,\epsilon)$ for a graded commutative algebra A we will now construct a model $L(P)$ for $HH_*(A,A)$. As in Section 4.1, the underlying graded commutative algebra for this model is given by $K[V\oplus\Sigma^{-1}V]$. The following lemma will be very important for the construction of the differential of the model $L(P)$.

Lemma 4.2.1. *Let $P = (K[V], \partial)$ be a differential graded algebra. There exist unique derivations $\delta, \beta\colon K[V \oplus \Sigma^{-1}V] \to K[V \oplus \Sigma^{-1}V]$ of degree $+1$ and -1, respectively, such that*

1. $\delta^2 = \beta^2 = 0$,
2. $\delta\beta + \beta\delta = 0$,
3. $\delta v = \partial v$, *for every* $v \in V$
4. $\beta v = \sigma^{-1} v$ *for every* $v \in V$.

Proof. We will show the existence of δ and β. First set β to be as in 4. We then define β on $\Sigma^{-1}V$ by $\beta(\sigma^{-1}v) = 0$, for each $v \in V$, so that $\beta^2 = 0$ on V and $\Sigma^{-1}V$. We finally extend β to $K[V \oplus \Sigma^{-1}V]$ as a K-linear map which is also a (graded) derivation of the product in $K[V \oplus \Sigma^{-1}V]$. As a consequence, $\beta^2 = 0$ in $K[V \oplus \Sigma^{-1}V]$.

Now let δ be as in 3 and define δ on $\Sigma^{-1}V$ by $\delta(\sigma^{-1}v) = -\beta(\partial v)$ for every $v \in V$. We then extend δ to $K[V \oplus \Sigma^{-1}V]$ as a K-linear map and a derivation of the product.

To prove that 2 holds, we note that it is satisfied for elements $v \in V$ and $\sigma^{-1}v \in \Sigma^{-1}V$ by the definitions of δ and β and since $\beta^2 = 0$. Using the fact that δ and β are graded derivations, we see that it is satisfied for the whole of $K[V \oplus \Sigma^{-1}V]$.

To complete the proof of existence, we are left to prove that $\delta^2 = 0$. Observe that $\delta = \partial$ in $K[V]$ and since $\partial^2 = 0$, we have that $\delta^2 v = 0$ for every $v \in V$. Further, using the definition of β, the fact that $\delta^2 v = 0$ for every $v \in V$, and 2, we get the following relations

$$\delta^2(\sigma^{-1}v) = \beta(\delta^2 v) = 0.$$

As before, since δ is a (graded) derivation of the product, $\delta^2 = 0$ on $K[V \oplus \Sigma^{-1}V]$.

To show uniqueness consider δ, β as above and suppose that $\tilde{\delta}$ and $\tilde{\beta}$ also satisfy the hypothesis above.

Since β and $\tilde{\beta}$ are (linear) derivations and $\beta v = \sigma^{-1} v = \tilde{\beta} v$ for every $v \in V$, then $\beta = \tilde{\beta}$ in $K[V] \subset K[V \oplus \Sigma^{-1} V]$. Using the same argument, and that $\delta v = \partial v = \tilde{\delta} v$ for every $v \in V$, $\delta = \tilde{\delta}$ in $K[V] \subset K[V \oplus \Sigma^{-1} V]$.

By hypothesis, $\beta(v) = \sigma^{-1} v$ for $v \in V$. Now, using the fact that $\beta^2 = 0$ and that β is a derivation, for any $v_1, \ldots, v_n \in V$, we have that

$$\beta((\sigma^{-1} v_1) \ldots (\sigma^{-1} v_n)) = 0.$$

The same argument applies for $\tilde{\beta}$ and so $\beta = \tilde{\beta}$ in $K[\Sigma^{-1} V] \subset K[V \oplus \Sigma^{-1} V]$.

Since $\delta\beta + \beta\delta = 0$, then for any $v \in V$,

$$\begin{aligned} \delta(\sigma^{-1} v) &= \delta(\beta v) \\ &= -\beta(\delta v) \\ &= -\beta(\partial v). \end{aligned}$$

But $\partial v \in K[V]$ and we proved before that $\beta = \tilde{\beta}$ in $K[V]$ so

$$\begin{aligned} -\beta(\partial v) &= -\tilde{\beta}(\partial v) \\ &= \tilde{\delta}(\sigma^{-1} v) \end{aligned}$$

as $\tilde{\delta}\tilde{\beta} + \tilde{\delta}\tilde{\beta} = 0$.

Finally using the Leibniz rule for δ and $\tilde{\delta}$ (respectively, β and $\tilde{\beta}$) we see that $\delta = \tilde{\delta}$ (respectively $\beta = \tilde{\beta}$) in $K[V \oplus \Sigma^{-1} V]$. $\square$

We will also make use of the following standard lemma of double complexes that can be found in [18, Proposition 1.0.12].

Lemma 4.2.2. *Let $C_{**} \to C'_{**}$ be a map of double complexes which is a quasi-isomorphism when restricted to each column. Then the induced map on the total complexes is a quasi-isomorphism.*

We now prove the following theorem.

Theorem 4.2.3. *Let A be a graded commutative algebra and $P = (K[V], \partial, \epsilon)$ be a model for A. Then there is a map of differential graded algebras*

$$L(\epsilon)\colon (K[V \oplus \Sigma^{-1}V], \delta) \to (HH_*(A, A), 0)$$

such that

- *the map $L(\epsilon)$ makes $L(P) = ((K[V \oplus \Sigma^{-1}V], \delta), L(\epsilon))$ into a model for $HH_*(A, A)$ and*

- *the following diagram commutes*

$$\begin{tikzcd} H_n(K[V \oplus \Sigma^{-1}V], \delta) \arrow[r] \arrow[d, "\beta"'] & HH_n(A, A) \arrow[d, "B"] \\ H_{n-1}(K[V \oplus \Sigma^{-1}V], \delta) \arrow[r] & HH_{n-1}(A, A) \end{tikzcd} \tag{4.2.1}$$

where β is defined as in Lemma 4.2.1.

In [23], Smith proves the first result of this kind, based on work of Tate (see [24]). However, the B-operator is not discussed and it is assumed that $K = \mathbb{Q}$ and that A is a graded complete intersection algebra.

Proof. Throughout this proof we will be using the normalised Hochschild complex without making any distinction in notation. First recall that the chain level version of the antisymmetrisation map

$$\psi : K[V \oplus \Sigma^{-1}V] \to C_*(K[V], K[V])$$

arises from the extension as a map of algebras of the linear maps

$$V \hookrightarrow C_0(K[V], K[V]) = K[V]$$

and

$$\begin{array}{rcl} \Sigma^{-1}V & \hookrightarrow & C_1(K[V],K[V]) \\ \sigma^{-1}v & \mapsto & 1\otimes v = Bv \end{array}$$

where B represents the normalised Connes operator at the chain level.

In this case it is easy to check that $b\circ\psi = 0$. Indeed we have that b is zero in $C_1(K[V],K[V])$ because $K[V]$ is commutative and that b is zero in $C_0(K[V],K[V])$ by definition. These facts together with the fact that b is a derivation of the product in $C_*(K[V],K[V])$ show that $b\circ\psi = 0$. Thus the map

$$\psi : (K[V\oplus\Sigma^{-1}V],0) \to (C_*(K[V],K[V]),b)$$

is a map of differential graded algebras.

We now claim that ψ gives a map of chain complexes

$$\psi : (K[V\oplus\Sigma^{-1}V],\delta) \to (C_*(K[V],K[V]),b+\partial)$$

where, abusing notation, we denote by ∂ to the differential in $C_*(K[V],K[V])$ induced by the differential ∂ in $K[V]$ (see Subsection 3.3.2) and it satisfies the properties

$$\partial b + b\partial = 0, \quad \partial B + B\partial = 0.$$

The formula

$$\psi\circ\delta = (b+\partial)\circ\psi$$

can be verified directly on the generators $v\in V$ and $\sigma^{-1}v\in\Sigma^{-1}V$ of $K[V\oplus\Sigma^{-1}V]$ and from this follows that it is satisfied in the whole of $K[V\oplus\Sigma^{-1}V]$. By Example 3.3.9 ψ is a quasi-isomorphism when restricted to the columns, so the induced map of the total complexes is a quasi-isomorphism, by Lemma 4.2.2.

Similarly, the map of chain complexes

$$\tilde{\epsilon}\colon (C_*(K[V], K[V]), b + \partial) \to (C_*(A, A), b),$$

induced by the map in the model $\epsilon\colon (K[V], \partial) \to (A, 0)$, is a quasi-isomorphism restricted to the columns and therefore, by Lemma 4.2.2, the map of the total complexes is a quasi-isomorphism.

Finally, let $\eta\colon (K[V \oplus \Sigma^{-1}V], \delta) \to (C_*(A, A), b)$ be the composition $\eta = \tilde{\epsilon} \circ \psi$. It is clear that the induced homomorphism

$$\eta_*\colon H_*(K[V \oplus \Sigma^{-1}V], \delta) \to HH_*(A, A)$$

is an isomorphism of graded algebras.

We now proceed to prove the second part of the theorem. Although β is a derivation at the chain level, B is not. Fortunately, B is a derivation up to homotopy (see [18, Corollary 4.3.4]). Using this fact we prove the following lemma:

Lemma 4.2.4. *The following diagram commutes up to chain homotopy*

$$\begin{tikzcd}
K[V \oplus \Sigma^{-1}V]_{s,t} \arrow[r] \arrow[d, "\beta"'] & C_{s,t}(K[V], K[V]) \arrow[d, "B"] \\
K[V \oplus \Sigma^{-1}V]_{s-1,t} \arrow[r] & C_{s-1,t}(K[V], K[V])
\end{tikzcd}$$

Proof. By definition of ψ we have

$$\psi(p(\sigma^{-1}v_1)^{\alpha_1} \cdots (\sigma^{-1}v_n)^{\alpha_n}) = p(Bv_1)^{\alpha_1} \cdots (Bv_n)^{\alpha_n}$$

where $p \in K[V]$ and $v_i \in V$ for $i = 1, \dots, n$. In [18, Corollary 4.3.5] Loday proves that, at the chain level, $B(xBy) = BxBy$. From this it follows that

$$B(\psi(p(\sigma^{-1}v_1)^{\alpha_1} \cdots (\sigma^{-1}v_n)^{\alpha_n})) = (B(p))(Bv_1)^{\alpha_1} \cdots (Bv_n)^{\alpha_n}.$$

On the other hand, from the definition of β one gets

$$\begin{aligned}\psi \circ \beta(p(\sigma^{-1}v_1)^{\alpha_1}\cdots(\sigma^{-1}v_n)^{\alpha_n}) &= \psi(\beta(p)((\sigma^{-1}v_1))^{\alpha_1}\cdots((\sigma^{-1}v_n))^{\alpha_n} \\ &= \psi(\beta(p))(Bv_1)^{\alpha_1}\cdots(Bv_n)^{\alpha_n}.\end{aligned}$$

It is sufficient to prove that there is a chain homotopy that makes the following diagram commute

$$\begin{tikzcd} K[V] \arrow[r] \arrow[d, "\beta"] & K[V\oplus\Sigma^{-1}V] \arrow[r, "\psi"] & C_*(K[V],K[V]) \arrow[d, "B"] \\ K[V] \arrow[r] & K[V\oplus\Sigma^{-1}V] \arrow[r, "\psi"] & C_*(K[V],K[V]) \end{tikzcd}$$

Since B is a derivation up to homotopy, there exists a chain homotopy

$$h^{(2)}\colon (C_*(K[V],K[V]),b+\partial)^{\otimes 2} \to (C_*(K[V],K[V]),b+\partial)$$

between the maps

$$u\otimes v \mapsto B(u,v)$$

and

$$u\otimes v \mapsto (Bu)v + (-1)^{|u|}uBv.$$

This induces a chain homotopy

$$h^{(n)}\colon (C_*(K[V],K[V]),b+\partial)^{\otimes n} \to (C_*(K[V],K[V]),b+\partial)$$

between the maps

$$u_1\otimes\cdots\otimes u_n \mapsto B(u_1,\ldots,u_n)$$

and

$$u_1\otimes\cdots\otimes u_n \mapsto \sum_{i=1}^{n}(-1)^{|u_1|+\ldots+|u_{i-1}|}u_1\ldots u_{i-1}(Bu_i)u_{i+1}\ldots u_n.$$

Finally define

$$h : K[V \oplus \Sigma^{-1}V] \to C_*(K[V], K[V])$$

in terms of a homogeneous basis $x_1, \ldots, x_m$ of V as

$$h(x_1^{\alpha_1} \cdots x_m^{\alpha_m})\beta(x_{i_1})^{\gamma_{i_1}} \cdots \beta(x_{i_s})^{\gamma_{i_s}} = h^{(n)}(x_1^{\otimes\alpha_1} \otimes \cdots \otimes x_m^{\otimes\alpha_m})\beta(x_{i_1})^{\gamma_{i_1}} \cdots \beta(x_{i_s})^{\gamma_{i_s}}$$

where $n = \alpha_1 + \ldots + \alpha_m$. This is precisely a chain homotopy between $\psi\beta$ and $B\psi$. □

□

In the following theorem we arrange some of the results from [22] to obtain a model for $H^*(\Omega M)$. An alternative proof of this theorem can be found in [9].

Theorem 4.2.5. *Let M be a simply connected topological space and K a field of characteristic zero. If $P = (K[V], \partial)$ is a minimal model for M, then*

$$\Omega P := (K[\Sigma^{-1}V], 0)$$

is a model for $H^*(\Omega M)$.

Proof. Let X, Y, Z, topological spaces, X simply connected, $f\colon Z \to X$ a map and $g\colon Y \to X$ a fibre map. Define

$$Z \times_X Y = \{(z, y) \in Z \times Y | f(z) = g(y)\}.$$

In [22], Smith combines results from [8] to prove that there is an isomorphism of algebras

$$H^*(Z \times_X Y) \cong \mathrm{Tor}_{S^*(X)}(S^*(Z), S^*(Y)).$$

For a topological space $Z = \{*\} \subset M$ and PM the path space

$$PM = \{\gamma\colon [0,1] \to M | \quad \gamma(0) = *\},$$

we have a fibration $g\colon PM \to M$ defined by $g(\gamma) = \gamma(1)$. Together with the constant map $\{*\} \to M$, we get that

$$H^*(\{*\} \times_M PM) \cong \mathrm{Tor}_{S^*(M)}(K, S^*(PM)).$$

But we have a homeomorphism

$$\{*\} \times_M PM \cong \Omega M$$

and since PM is contractible we can conclude that

$$H^*(\Omega M) \cong \mathrm{Tor}_{S^*(M)}(K, K).$$

Note that this particular case, where $X = M$ is a topological space, Z is a point of M and $Y = PM$, was first proved by Adams in [1]. Thus

$$H^*(\Omega M) \cong HH_*(S^*(M), K).$$

We finish this theorem with the proof of the following lemma.

Lemma 4.2.6. *Let $P = (K[V], \partial)$ a minimal model for $H^*(M)$. Then*

$$H^*(\Omega M) \cong HH_*(K[V], K) \cong K[\Sigma^{-1}V].$$

Proof. Since $P = (K[V], \partial)$ and $(S^*(M), \delta)$ are quasi-isomorphic,

$$HH_*(K[V], K) \cong HH_*(S^*(M), K). \tag{4.2.2}$$

A direct computation shows that

$$HH_*(K[V], K) \cong K[\Sigma^{-1}V],$$

as $P = (K[V], \partial)$ is a minimal model for $H^*(M)$. $\square$

$\square$

Lemma 4.2.7. *Let $\epsilon\colon S^*M \to K$ be the augmentation map. The following diagram commutes*

$$\begin{tikzcd} H^*(LM) \arrow[r, "i^*"] & H^*(\Omega M) \\ HH_*(S^*M, S^*M) \arrow[u, "\cong"] \arrow[r, "\epsilon_*"] & HH_*(S^*M, K) \arrow[u, "\cong"'] \end{tikzcd}$$

where the left vertical isomorphism is described in Subsection 3.4.1 and the right, in Equation 4.2.2.

Proof. By construction of the vertical maps, the diagram above is commutative. $\square$

Lemma 4.2.8. *Let $P = (K[V], \partial)$ be a minimal model for M, $\gamma\colon P \to S^*M$ the quasi-isomorphism obtained from the model and $\epsilon\colon S^*M \to K$ and $\eta\colon P \to K$ augmentation maps. Then the following diagram commutes.*

$$\begin{tikzcd} HH_*(S^*M, S^*M) \arrow[r, "\epsilon^*"] & HH_*(S^*M, K) \\ HH_*(P, P) \arrow[u, "\gamma_*"] \arrow[r, "\eta_*"] & HH_*(P, K) \arrow[u, "\gamma_*"'] \end{tikzcd}$$

Proof. First notice that the following diagram commutes.

$$\begin{tikzcd} P \arrow[r, "\gamma"] \arrow[d, "\eta"'] & S^*M \arrow[d, "\epsilon"] \\ K \arrow[r, "\gamma"] & K \end{tikzcd}$$

Therefore the following diagram is commutative

$$\begin{tikzcd}
(S^*M)^{\otimes n+1} \arrow[r, "\epsilon^*"] & (S^*M)^{\otimes n} \otimes K \\
(K[V])^{\otimes n+1} \arrow[u, "\gamma_*"] \arrow[r, "\eta_*"] & (K[V])^{\otimes n} \otimes K \arrow[u, "\gamma_*"']
\end{tikzcd}$$

Finally, since the diagram commutes at the chain level, then the diagram commutes in homology. □

Lemma 4.2.9. *Let $P = (K[V], \partial)$ be a minimal model for M and $\eta\colon P \to K$ the augmentation map. Further, let $LP = (K[V \oplus \Sigma^{-1}V], \delta)$ be the model for LM given in Theorem 4.2.3 and $\Omega P = (K[\Sigma^{-1}V], 0)$ be the model for ΩM given in Theorem 4.2.5. Finally, let $p\colon K[V \oplus \Sigma^{-1}V] \longrightarrow K[\Sigma^{-1}V]$ be the projection onto the second factor. Then p is a map of chain complexes and the following diagram is commutative*

$$\begin{tikzcd}
HH_*(P,P) \arrow[r, "\eta^*"] & HH_*(P,K) \\
H^*(LP) \arrow[u] \arrow[r, "p^*"] & H^*(\Omega P) \arrow[u]
\end{tikzcd}$$

where the left vertical isomorphism is given by Theorem 4.2.3 and the right, by Theorem 4.2.5.

Proof. We first prove that p is a map of chain complexes. Let β be as in Theorem 4.2.3. Note that $p(\delta v) = 0$ because $\delta v = \partial v \in K[V]$. Since P is a minimal model, ∂v is a decomposable element and, using the fact that β is a derivation,

$$p(\sigma^{-1}(\partial v)) = p(\sigma^{-1}(\delta v)) \in I(V)$$

and so

$$p(\delta(\sigma^{-1}v)) = p(\sigma^{-1}(\delta v)) = 0.$$

In conclusion, p induces a map of chain complexes.

We now focus on the second part of the lemma. We will first analyse the composite map

$$C^*(K[V \oplus \Sigma^{-1}V], \delta) \longrightarrow C_*(K[V], K[V]) \longrightarrow C_*(K[V], K).$$

Recall that the map in the left is defined by the inclusions

$$V \to K[V] = C_0(K[V], K[V])$$

and

$$\begin{aligned} &\Sigma^{-1}V \to C_1(K[V], K[V]) = K[V] \otimes K[V] \\ &\sigma^{-1}w \mapsto 1 \otimes w \end{aligned}$$

and extended as a map of algebras. The map induced by η is defined by

$$\eta_*(r_0 \otimes r_1 \otimes \cdots \otimes r_n) = \eta(r_0) \otimes r_1 \otimes \cdots \otimes r_n$$

and so $v \mapsto 0$ and $1 \otimes w \mapsto 1 \otimes w$. It follows that the composite takes the generators $v \in V$ to 0 and $\sigma^{-1}w \in \Sigma^{-1}V$ to $1 \otimes w$ and extends into a map of algebras. In a similar fashion we analyse the second composite map

$$C^*(K[V \oplus \Sigma^{-1}V], \delta) \longrightarrow C^*(K[\Sigma^{-1}V], 0) \longrightarrow C_*(K[V], K).$$

Note that the map in the right is defined by the inclusion

$$\begin{aligned} &\Sigma^{-1}V \to C_1(K[V], K) = K \otimes K[V] \\ &\sigma^{-1}w \mapsto 1 \otimes w \end{aligned}$$

and extended as a map of algebras. Also, recall that the map in the left is the induced by the projection onto the second factor. Thus, the generators $v \in V$ and $\sigma^{-1}w \in \Sigma^{-1}V$ are send to 0 and $1 \otimes w$, respectively and the composite extends as a map of algebras.

We have just proved that the diagram is commutative. $\square$

The following theorem was first proved by Smith, in [22]. We give a different proof using theorem 4.2.3.

Theorem 4.2.10. *The map $H^*(LM) \xrightarrow{i^*} H^*(\Omega M)$ is surjective if and only if $H^*(M)$ is free.*

Proof. Let $P = (K[V], \partial)$ be a minimal model for M and LP, ΩP and $p\colon K[V \oplus \Sigma^{-1}V] \longrightarrow K[\Sigma^{-1}V]$ be as in Lemma 4.2.9. Suppose $\partial \not\equiv 0$. By combining the diagrams of Lemmas 4.2.7, 4.2.8 and 4.2.9, we have the following commutative diagram.

$$\begin{tikzcd} H^*(LM) \arrow[r, "i^*"] & H^*(\Omega M) \\ H^*(LP) \arrow[u, "\cong"] \arrow[r, "p^*"] & H^*(\Omega P) \arrow[u, "\cong"'] \end{tikzcd}$$

Thus i^* is not surjective if and only if p_* is not surjective.

Since $\partial \not\equiv 0$ there exist $v \in V$ of minimal degree such that $\partial v \neq 0$. Therefore, $\delta(\sigma^{-1}v) = \sigma^{-1}(\partial v) \neq 0$ and so $\sigma^{-1}v$ is not a cycle in LP but it is a cycle in ΩP. Notice that $\sigma^{-1}v$ is also an element in $\Sigma^{-1}V$ with lowest degree such that $\delta(\sigma^{-1}v) \neq 0$. As a consequence, by the Leibniz rule and the linearity of δ, we can deduce that for any homogeneous element $q \in K[V \oplus \Sigma^{-1}V]$ with $|q| < |\sigma^{-1}v| = |v| - 1$, then $\delta q = 0$.

Suppose that there exists a cycle α in LP such that $p(\alpha) = p(\sigma^{-1}v)$. Thus $\alpha = \sigma^{-1}v + w$ with $w \in I(V)$, the ideal generated by V. Since α is a cycle we have

$$0 = \delta(\alpha) = \delta(\sigma^{-1}v) + \delta(w)$$

and so $\delta(w) = -\delta(\sigma^{-1}v)$ is not a cycle in LP. Since $w \in I(V)$ is of the form

$$\sum x_i v_i$$

with $x_i \in LP$ and $v_i \in V$, then

$$\delta(w) = \sum x_i \delta(v_i) + \delta(x_i) v_i = \sum \delta(x_i) v_i = 0$$

because $|x_i|, |v_i| < |w| = |\alpha| < |v|$ and v was of minimal degree. This contradiction proves that i^* is surjective implies that $H^*(M)$ is free.

The converse is an easy exercise. □

4.3 Examples: Cohomology of the Loop Space and the B-operator

In this section we use Theorem 4.2.3 to describe the cohomology of the loop space of the sphere, projective spaces and Grassmann manifolds and to compute the respective B-operators. We will use the same method for each of manifold M.

- First we obtain the model for the cochains on the free loop space of M by using Theorem 4.2.3.

- Then we display the cohomology Leray-Serre spectral sequence for the fibration $\Omega M \hookrightarrow LM \to M$. (See [19, Chapter 5] for the construction of the cohomology Leray-Serre spectral sequence of a fibration.) Here the differential that we obtained from the model suggests differentials for this spectral sequence in the following fashion. We start with the E_2 term of this spectral sequence and we calculate the differential δ from the model of each of the elements x of bidegree (p, q). For such x we then set d_2 to be the leading terms of $\delta(x)$, that is, the elements of the right bidegree $((p+2, q-1))$ that appear as summands in $\delta(x)$. If there are no such elements of the right bidegree, then $d_2(x) = 0$. Once knowing the second differential we can calculate the elements of the third page $E_3^{*,*}$. Similarly, in order to calculate d_n we take the leading terms of $\delta - \sum_{i=2}^{n-1} d_i$. We note that, in our examples, this will give all the differentials since there is a very limited number of elements of each bidegree in a particular term of this spectral sequence. In most of the examples, there is only one non trivial differential.

- Once more we use the model to compute β, the derivation of degree -1 from Theorem 4.2.3, at the chain level. Using all of its properties this operator β is easy to compute and in homology these computations are equivalent to computing B. We will make no distinction in notation between β and B.

- From this, we get a description of $H^*(LM)$ with its cup product and the corresponding B-operator.

4.3.1 $H^*(LS^{2n+1})$ and the B-operator

In this case, the cohomology of the sphere is a free algebra on one generator x with degree $|x| = 2n + 1$, that is, the exterior algebra $A = K[x]$. As in Section 4.1 we obtain

$$H^*(LS^{2n+1}) = K[x, u]$$

where $|u| = 2n$. In this case the B-operator is given by $Bx = u$ and, since $B^2 = 0$, $Bu = 0$. These formulas completely determine the B-operator on $H^*(LS^{2n+1})$ as B is a derivation of the cup product.

4.3.2 $H^*(LS^{2n})$ and the B-operator

Let

$$A = H^*(S^{2n}) = \frac{K[x]}{(x^2)}$$

with $|x| = 2n$. From Example 4.0.6 $P = (K[x, y], \partial)$ is the model for A, where $|y| = 4n - 1$ and the differential ∂ is given by $\partial x = 0$ and $\partial y = x^2$ on the generators. By Theorem 4.2.3 a model for $H^*(LS^{2n})$ is given by

$$LP = (K[x, y, u, v], \delta)$$

where $|x| = 2n$, $|y| = 4n - 1$, $|u| = 2n - 1$, $|v| = 4n - 2$ and, by Theorem 4.2.1 (2), δ is defined on the generators by

$$\begin{aligned} \delta x &= 0 \\ \delta y &= x^2 \\ \delta u &= 0 \\ \delta v &= -2xu. \end{aligned}$$

Figure 4.1 shows the cohomology Leray-Serre spectral sequence associated to the path fibration $\Omega S^{2n} \hookrightarrow LS^{2n} \to S^{2n}$. There we compute the differentials using the model for $H^*(LS^{2n})$.

From Lemma 4.2.4 we see that, at the chain level, B is given by $Bx = u$, $By = v$, $Bu = B^2x = 0$ and $Bv = B^2y = 0$ since $B^2 = 0$. Let $z_i = xv^i$, $i = 0, 1, 2, \ldots$. Using the fact that B is a derivation of the product we have that $B(z_i) = uv^i$.

From Figure 4.1, notice that the algebra $H^*(LS^{2n})$ is the graded commutative algebra

$$K[z_i, B(z_i) : i \geq 0]$$

with degrees $|z_i| = 2n + i(4n - 2)$, $|B(z_i)| = 2n - 1 + i(4n - 2)$ and with the following

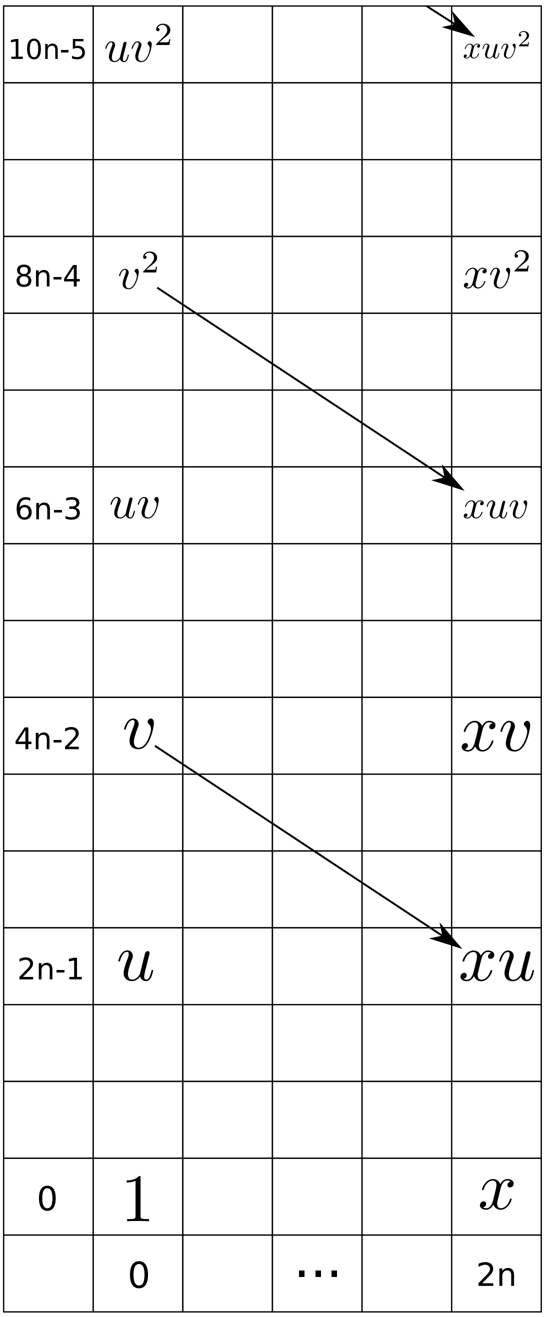

Figure 4.1: The cohomology Leray-Serre spectral sequence for $H^*(LS^{2n})$

relations:

$$z_i z_j = 0 \quad \text{for } i, j = 0, 1, 2, \ldots,$$
$$z_i B(z_j) = 0 \quad \text{for } i, j = 0, 1, 2, \ldots,$$
$$B(z_i)B(z_j) = 0 \quad \text{for } i \neq j.$$

That is, all the products in $H^*(LS^{2n})$ are trivial.

4.3.3 $H^*(L\mathbb{C}P^n)$ and the B-operator

Set

$$A = \frac{K[x]}{(x^{n+1})} = H^*(\mathbb{C}P^n)$$

with $|x| = 2$. In Example 4.0.6 it is shown that $P = (K[x, y], \partial)$, where ∂ is defined on the generators as $\partial x = 0$ and $\partial y = x^{n+1}$ and $|y| = 2n + 1$, is the model for A. Using Theorem 4.2.3 the model for $H^*(L\mathbb{C}P^n)$ is given by

$$LP = (K[x, y, u, v], \delta),$$

where δ is defined on the generators by

$$\delta x = 0$$
$$\delta y = x^{n+1}$$
$$\delta u = 0$$
$$\delta v = -(n + 1)x^n u$$

with degrees $|u| = 1$, $|v| = 2n$ and the degrees of x and y as above. Using the model for $H^*(L\mathbb{C}P^n)$ we compute the differentials of the cohomology Leray-Serre spectral sequence associated to the fibration $\Omega\mathbb{C}P^n \hookrightarrow L\mathbb{C}P^n \to \mathbb{C}P^n$, which are pictured in Figure 4.2.

8n+1	uv^4		xuv^4		x^2uv^4			x^nuv^4
8n	v^4		xv^4		x^2v^4			x^nv^4
6n+1	uv^3		xuv^3		x^2uv^3			x^nuv^3
6n	v^3		xv^3		x^2v^3			x^nv^3
4n+1	uv^2		xuv^2		x^2uv^2			x^nuv^2
4n	v^2		xv^2		x^2v^2			x^nv^2
2n+1	uv		xuv		x^2uv			x^nuv
2n	v		xv		x^2v			x^nv
1	u		xu		x^2u			x^nu
0	1		x		x^2			x^n
	0	1	2	3	4	...		2n

Figure 4.2: The cohomology Leray-Serre spectral sequence for $H^*(L\mathbb{C}P^n)$

Using the model for $H^*(L\mathbb{C}P^n)$ we see that, at the chain level, the B-operator is given by $Bx = u$, $By = v$, $Bu = B^2x = 0$ and $Bv = B^2y = 0$. Thus, if $z_i = xv^i$, then $Bz_i = uv^i$ for $i = 0, 1, \ldots$ as B is a derivation of the product.

From Figure 4.2 we see that $H^*(L\mathbb{C}P^n)$ is the graded commutative algebra

$$K[z_i, B(z_i) : i \geq 0]$$

where the generators satisfy the following relations

$$\begin{aligned} z_{i_1} z_{i_2} \cdots z_{i_{n+1}} &= 0 \\ Bz_i Bz_j &= 0 \\ z_i z_j &= z_0 z_{i+j} \\ z_{i_1} z_{i_2} \cdots z_{i_n} Bz_j &= 0, \end{aligned}$$

for any $i, j, i_1, \ldots, i_n, i_{n+1} \geq 0$.

4.3.4 $H^*(L(G_2(\mathbb{C}^4))$ and the B-operator

Let

$$A = H^*(G_2(\mathbb{C}^4)) = \frac{K[x, y]}{(x^3 - 2xy, x^2y - y^2)}$$

where $|x| = 2$ and $|y| = 4$. This expression for $H^*(G_2(\mathbb{C}^4))$ derives from the following. In the book [2, Proposition 23.2], Bott and Tu give a description of the cohomology ring of the Grassmann manifolds in terms of the Chern classes of the tautological bundle and its orthogonal complement. According to this description, the cohomology of $G_2(\mathbb{C}_4)$ is given by

$$H^*(G_2(\mathbb{C}^4)) = \frac{K[c_1, c_2, d_1, d_2]}{(1 + c_1 + c_2)(1 + d_1 + d_2) = 1},$$

where $|c_1| = |d_1| = 2$ and $|c_2| = |d_2| = 4$. By writting d_1 and d_2 in terms of c_1 and c_2 we get the expression above.

As in Example 4.0.7, a model for A is given by $P = (K[x, y, f, g], \partial)$, where ∂ is given on the generators by

$$\begin{aligned}
\partial x &= 0, \\
\partial y &= 0, \\
\partial f &= x^3 - 2xy, \\
\partial g &= x^2 y - y^2,
\end{aligned}$$

with degrees $|f| = 5$, $|g| = 7$. The model for $H^*(L(G_2(\mathbb{C}^4)))$ described in Theorem 4.2.3 is the commutative graded algebra

$$K[x, y, f, g, u, v, a, b]$$

with the degrees of x, y, f, g as above and $|u| = 1$, $|v| = 3$, $|a| = 4$ and $|b| = 6$. The differential of this model is defined on the generators by

$$\begin{aligned}
\delta x &= \delta y = \delta u = \delta v = 0, \\
\delta f &= x^3 - 2xy, \\
\delta g &= x^2 y - y^2, \\
\delta a &= -3x^2 u + 2yu + 2xv, \\
\delta b &= -2xu - x^2 v + 2yv.
\end{aligned}$$

Using this model we can calculate the differentials of the cohomology Leray-Serre spectral sequence for the path fibration $\Omega(G_2(\mathbb{C}^4)) \hookrightarrow L(G_2(\mathbb{C}^4)) \to G_2(\mathbb{C}^4)$. Standard definitions for spectral sequences tells us that $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ and that

$$E_{r+1}^{p,q} \cong \frac{\operatorname{Ker} d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}}{\operatorname{Im} d_r \colon E_r^{p-r,q+r-1} \to E_r^{p,q}},$$

see [19, Definition 1.1, Chapter 1]. To obtain E_2, E_4 and E_5 we first notice that

$$E_2^{pq} \cong H^p(G_2(\mathbb{C}^4)) \otimes H^q(\Omega(G_2(\mathbb{C}^4)))$$

and we calculate the second differential d_2 by taking the leading terms of the differential δ given by the model. This calculation leads us to the third term of this spectral sequence E_3, in which, we notice that $\operatorname{Im} d_3 = \{0\}$ since $E_3^{p,q} = \{0\}$ when p is odd. Then by definition $E_4^{p,q} \cong E_3^{p,q}$. In fact, this same argument applies to all the differentials d_{2k+1} of an odd term E_{2k+1} and so $d_{2k+1} = 0$ and $E_{2k+2}^{p,q} \cong E_{2k+1}^{p,q}$ for $k \geq 1$. We take d_4 to be the leading terms of $\delta - d_2 - d_3$, which determines $E_5^{p,q}$. We see that $d_6 = 0$ because the leading terms of $\partial - d_2 - d_3 - d_4 - d_5$ are zero. Analogously $d_8 = 0$. For $k \geq 9$ we have that $d_k = 0$ since $E_k^{p,q} = 0$ when $p \geq 9$. Thus, the spectral sequence collapses in the E_5-term.

The E_2, E_4 and E_5 terms of this spectral sequence are depicted in Figures 4.3, 4.4, 4.5, respectively.

At the chain level, the B-operator is given by

$$\begin{aligned}
Bx &= u,\\
By &= v,\\
Bu &= B^2x = 0,\\
Bv &= B^2y = 0,\\
Bf &= a,\\
Bg &= b,\\
Ba &= B^2f = 0,\\
Bb &= B^2g = 0,
\end{aligned}$$

Figure 4.3: The E_2-term of the cohomology Leray-Serre spectral sequence for $H^*(L(G_2(\mathbb{C}^4)))$

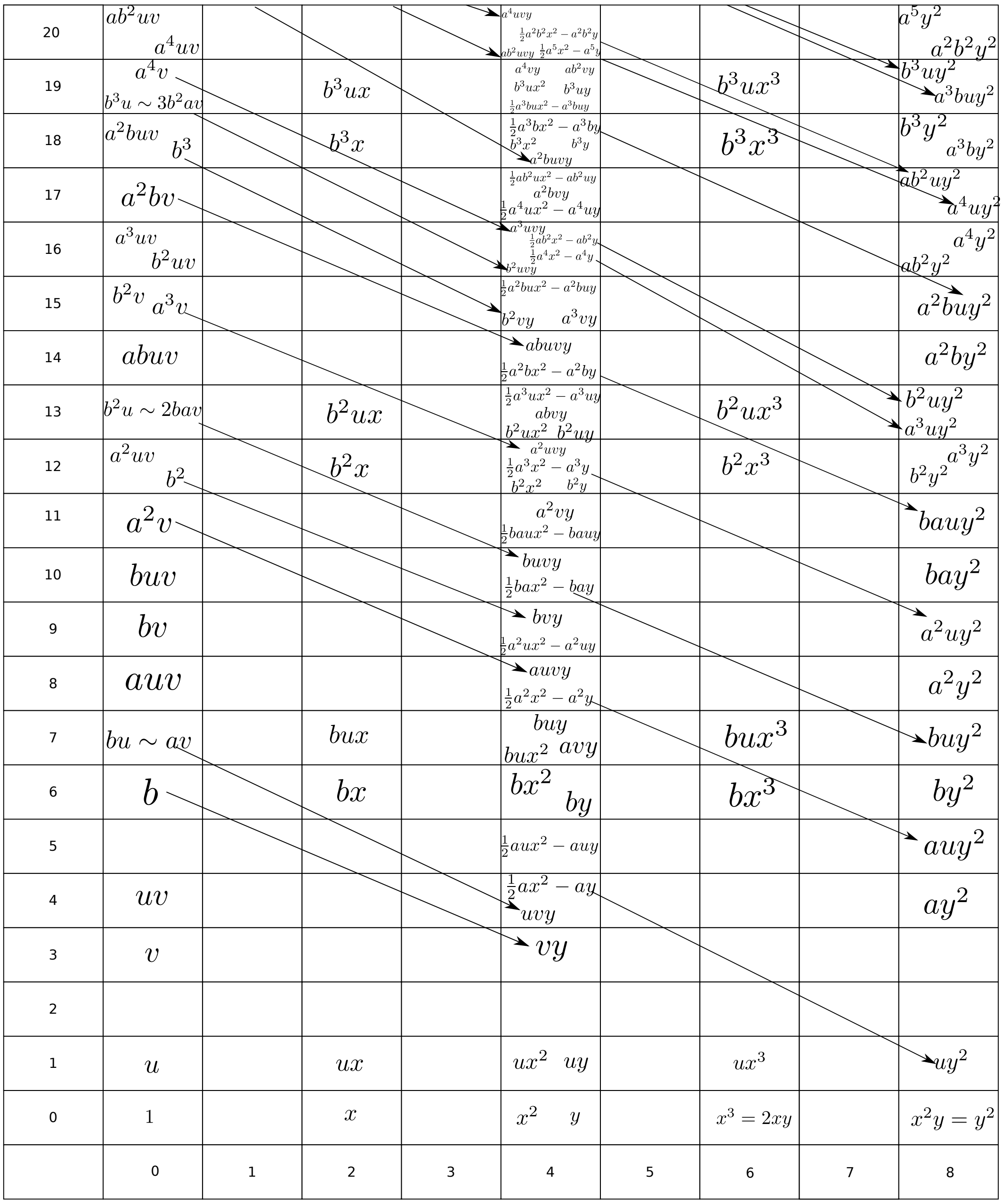

Figure 4.4: The E_4-term of the cohomology Leray-Serre spectral sequence for $H^*(L(G_2(\mathbb{C}^4)))$

20	ab^2uv a^4uv								a^5y^2 $a^2b^2y^2$
19	$b^3u - 3b^2av$		b^3ux		a^4vy ab^2vy b^3ux^2 b^3uy $\frac{1}{2}a^3bux^2 - a^3buy$		b^3ux^3		
18	a^2buv		b^3x		b^3x^2 b^3y		b^3x^3		b^3y^2 a^3by^2
17					$\frac{1}{2}ab^2ux^2 - ab^2uy$ a^2bvy $\frac{1}{2}a^4ux^2 - a^4uy$				
16	a^3uv b^2uv								a^4y^2 ab^2y^2
15	b^2v				$\frac{1}{2}a^2bux^2 - a^2buy$ a^3vy				
14	$abuv$								a^2by^2
13	$b^2u - 2bav$		b^2ux		$\frac{1}{2}a^3ux^2 - a^3uy$ $abvy$ b^2ux^2 b^2uy		b^2ux^3		
12	a^2uv		b^2x		b^2x^2 b^2y		b^2x^3		a^3y^2 b^2y^2
11					a^2vy $\frac{1}{2}baux^2 - bauy$				
10	buv								bay^2
9	bv				$\frac{1}{2}a^2ux^2 - a^2uy$				
8	auv								a^2y^2
7	$bu - av$		bux		buy bux^2 avy		bux^3		
6			bx		bx^2 by		bx^3		by^2
5					$\frac{1}{2}aux^2 - auy$				
4	uv								ay^2
3	v								
2									
1	u		ux		ux^2 uy		ux^3		
0	1		x		x^2 y		$x^3 = 2xy$		$x^2y = y^2$
	0	1	2	3	4	5	6	7	8

Figure 4.5: The $E_5 = E_\infty$-term of the cohomology Leray-Serre spectral sequence for $H^*(L(G_2(\mathbb{C}^4)))$

as $B^2 = 0$. Using that B is a derivation of the product, if we write

$$\begin{aligned}
\alpha_n &= b^n x, \\
\beta_n &= b^n y, \\
\delta_m &= b^m u - m b^{m-1} a v, \\
\epsilon_{n,m} &= b^n \left(\frac{1}{2} a^m u x^2 - a^m u y \right), \\
\lambda_{n,m} &= b^n a^m y^2,
\end{aligned}$$

for $m \geq 1$ and $n \geq 0$, then

$$\begin{aligned}
B(\alpha_n) &= b^n u, \\
B(\beta_n) &= b^n v =: \gamma_n, \\
B(\delta_m) &= 0, \\
B(\epsilon_{n,m}) &= -b^n a^m u v =: \theta_{n,m}, \\
B(\lambda_{n,m}) &= 2 b^n a^m v y =: 2\rho_{n,m},
\end{aligned}$$

for $m \geq 1$ and $n \geq 0$. From Figure 4.5 we see that $H^*(L(G_2(\mathbb{C}^4)))$ is the graded commutative algebra

$$K[\alpha_n, u, \beta_n, \gamma_n, \delta_m, \epsilon_{n,m}, \theta_{n,m}, \lambda_{n,m}, \rho_{n,m} : m \geq 1,\ n \geq 0]$$

with degrees

$$|\alpha_n| = 6n + 2,$$

$$|u| = 1,$$

$$|\beta_n| = 6n + 4,$$

$$|\gamma_n| = 6n + 3,$$

$$|\delta_m| = 6m + 1,$$

$$|\epsilon_{n,m}| = 6n + 4m + 5,$$

$$|\theta_{n,m}| = 6n + 4m + 4,$$

$$|\lambda_{n,m}| = 6n + 4m + 8,$$

$$|\rho_{n,m}| = 6n + 4m + 7$$

and satisfying the relations

$$\alpha_n\alpha_m = \alpha_0\alpha_{n+m},$$

$$\beta_n\beta_m = \beta_0\beta_{n+m},$$

$$\alpha_n\beta_m = \alpha_0\beta_{n+m} = \alpha_{n+m}\beta_0,$$

$$\alpha_0\alpha_0\alpha_n = 2\alpha_0\beta_n,$$

$$\alpha_0\alpha_0\beta_n = \beta_0\beta_n,$$

$$\delta_n\delta_m = \delta_1\delta_{n+m-1} = \frac{n+m}{n}u\delta_{n+m},$$

$$\beta_l\beta_m u = 0,$$

$$\alpha_n\delta_m = u\alpha_{n+m},$$

$$\beta_n\delta_m = u\beta_{n+m} - m\rho_{n+m-1,1},$$

$$\gamma_n\delta_m = u\gamma_{n+m},$$

and all other double products are trivial.

4.4 Models and Hochschild Cohomology

In this section we will prove the following theorem:

Theorem 4.4.1. *Let A be a graded commutative algebra and $P = (K[V], \partial, \epsilon)$ a model for A. For the bracket defined as in Section 4.1 there is a unique operator δ of $K[V \oplus (\Sigma^{-1}V)^*]$ such that $\delta^2 = 0$, δ is a derivation of the product, δ is a derivation of the bracket and*

$$\delta(v) = \partial(v), \quad v \in V.$$

Moreover, there is a natural isomorphism of algebras

$$G(\epsilon)\colon H^*(K[V \oplus (\Sigma^{-1}V)^*], \delta) \to HH^*(A, A),$$

that is, $(K[V \oplus (\Sigma^{-1}V)^*], \delta, G(\epsilon))$ is a model for $HH^*(A, A)$.

Proof. We will start the proof of the theorem with the following lemma.

Lemma 4.4.2. *Let ξ be a derivation of $K[V]$. Then there is a unique derivation D_ξ of $K[V \oplus (\Sigma^{-1}V)^*]$ such that*

- *If $u \in K[V] \subset K[V \oplus (\Sigma^{-1}V)^*]$, then $D_\xi(u) = \xi(u)$.*
- *If $u, v \in K[V \oplus (\Sigma^{-1}V)^*]$, then $D_\xi[u, v] = [D_\xi(u), v] + (-1)^{(|u|+1)(|v|+1)}[u, D_\xi(v)]$.*

The proof of this lemma is elementary and it is left to the reader. Set $\delta = D_\partial$ as in the lemma. Let $G(P)$ be the differential Gerstenhaber algebra with underlying free commutative algebra $K[V \oplus (\Sigma^{-1}V)^*]$ equipped with the differential δ, and the elementary bracket described in Section 4.1.

There is a map of algebras

$$\eta\colon K[V \oplus (\Sigma^{-1}V)^*] \to C^*(K[V], K[V]),$$

which is the extension, as a map of graded algebras, of the K-linear maps

$$V \hookrightarrow C^0(K[V], K[V]) = K[V]$$

and

$$(\Sigma^{-1}V)^* \hookrightarrow C^1(K[V], K[V]) = \mathrm{Hom}(K[V], K[V])$$

where an element $(\sigma^{-1}v)^*$ gets mapped to the unique derivation of $K[V]$ which agrees with v^* in V. This map is the antisymmetrisation map seen in Subsection 3.3.3.

We claim that $\beta \circ \eta = 0$, where β is the Hochschild coboundary map. Indeed $\beta \circ \eta = 0$ in $C^0(K[V], K[V])$ because $K[V]$ is graded commutative and $\beta \circ \eta((\sigma^{-1}v)^*) = 0$ since $\eta((\sigma^{-1}v)^*) \in C^1(K[V], K[V])$ is a graded derivation and $\mathrm{Ker}(\beta\colon C^1(K[V], K[V]) \to C^2(K[V], K[V])) = \mathrm{Der}(K[V])$. Given that β is a derivation of the cup product, we conclude that $\beta \circ \eta = 0$ (see Subsection 3.2.2). Thus the map η is compatible with the differentials and hence it gives a map of differential graded algebras

$$\eta\colon (K[V \oplus (\Sigma^{-1}V)^*], 0) \to (C^*(K[V], K[V]), \beta).$$

Furthermore, η induces a map of chain complexes

$$\eta\colon (K[V \oplus (\Sigma^{-1}V)^*], \delta) \to (C^*(K[V], K[V]), \beta + \partial),$$

where, abusing notation, ∂ is the differential in $C^*(K[V], K[V])$ induced by ∂. In fact, to see that $\eta \circ \delta = (\beta + \partial) \circ \eta$, first verify it directly on the generators $v \in V$ and $(\sigma^{-1}v)^* \in (\Sigma^{-1}V)^*$ and then notice that since η is a map of algebras and δ, β, ∂ are derivations this identity holds in the whole of $K[V \oplus (\Sigma^{-1}V)^*]$.

By the Hochschild-Kostant-Rosenberg theorem, η is an isomorphism of differential graded algebras.

Now, since $\epsilon : (K[V], \partial) \to (A, 0)$ is a quasi-isomorphism, by a result of Rickard (see [21]), there exists an isomorphism of differential graded algebras

$$\tilde{\epsilon}\colon HH^*(K[V], K[V]) \to HH^*(A, A).$$

Define $G(\epsilon)\colon (K[V \oplus (\Sigma^{-1}V)^*], \delta) \to HH^*(A, A)$ as the composition $G(\epsilon) = \tilde{\epsilon} \circ \eta$. By definition, $G(\epsilon)$ is an isomorphism of differential graded algebras. $\square$

Remark 4.4.3. We strongly believe that, by using either the constructions in [21] or the Kontsevich formality theorem [17], one can conclude that the ismorphism from Theorem 4.4.1 is also an isomorphism of Gerstenhaber algebras.

In our computations, we will make extensive use of the following theorem of Cohen, Jones and Yan [6].

Theorem 4.4.4 (Cohen-Jones-Yan)**.** *Let M be a closed, oriented, simply connected manifold. There is a second quadrant spectral sequence of algebras $\{E^r_{p,q}, d^r : p \leq 0, q \geq 0\}$ such that*

1. *$E^r_{*,*}$ is an algebra and the differential $d^r\colon E^r_{*,*} \to E^r_{*-r,*+r-1}$ is a derivation for each $r \geq 1$.*

2. *The spectral sequence converges to the loop homology $\mathbb{H}_*(LM)$ as algebras. That is, $E^\infty_{*,*}$ is the associated graded algebra to a natural filtration of the algebra $\mathbb{H}_*(LM)$.*

3. *For $m, n \geq 0$,*

$$E^2_{-m,n} \cong H^m(M; H_n(\Omega M)).$$

Here ΩM is the space of base point preserving loops in M. Furthermore the isomorphism $E^2_{-*,*} \cong H^*(M; H_*(\Omega M))$ is an isomorphism of algebras, where the algebra structure on $H^*(M; H_*(\Omega M))$ is given by the cup product on the cohomology of M with coefficients in the Pontryagin ring $H_*(\Omega M)$.

In fact we will make use of a dualised version of this spectral sequence. More precisely, we will consider the spectral sequence of Theorem 4.4.4 with the following adjustments:

- We regrade this spectral sequence into a fourth quadrant spectral sequence and so we make a slight change of notation; the spectral sequence becomes

$$\{E_r^{p,q}, d_r\ : p \geq 0, q \leq 0\} \quad \text{with differentials} \quad d_r\colon E_r^{*,*} \longrightarrow E_r^{*+r,*-r+1}.$$

- We also observe that, in this case, for $m, n \geq 0$

$$E_2^{m,-n} \cong H^m(M; H_n(\Omega M))$$

  and, from above, we have an isomorphism of algebras $E_2^{*,-*} \cong H^*(M, H_*(\Omega M))$.

- We will refer to this spectral sequence as the "Cohen-Jones-Yan spectral sequence".

By combining Theorems 4.4.1 and 4.4.4 we prove the following Corollary.

Corollary 4.4.5. *Let X be a simply connected manifold and $P = (K[V], \partial)$ a (minimal) model for $H^*(X)$. If N denotes*

$$\mathrm{Ker}(\delta : (\Sigma^{-1}V)^* \to K[V \oplus (\Sigma^{-1}V)^*]),$$

where δ is defined as in Theorem 4.4.1, then

$$K[N] \subset H_*(LX).$$

Proof. Notice that, for any $r = 1, 2, \ldots$, the elements of $(\Sigma^{-1}V)^*$ in the E_r-term of the Cohen-Jones-Yan spectral sequence appear in first column of that term. Therefore, if

$$x \in (\Sigma^{-1}V)^*,$$

then $x \notin \mathrm{Im}(\delta_r)$. So, if $x \in N$, then x survives to E_∞.

An analogous argument shows that, if $x, y \in (\Sigma^{-1}V)^*$, then xy is not a boundary in any of the E_r-terms of the Cohen-Jones-Yan spectral sequence and, using the Leibniz rule, $xy \in N$ whenever $x, y \in N$. This concludes the proof. □

4.5 Examples: String Homology of the Spheres, Projective Spaces and Grassmann Manifolds

In this section, using the model described in Theorem 4.4.1 we obtain the homology of the loop space of the spheres, projective spaces and of $G_2(\mathbb{C}^4)$ with its corresponding string product. We also compute the Batalin-Vilkovisky operator using the calculations from Section 4.3. We will use the same method for each manifold M.

- Using the isomorphism of graded algebras from Theorem 3.4.3 and then the model described in Theorem 4.4.1, we write the model for the (shifted) homology of the free loop space of M.

- Then we display the Cohen-Jones-Yan spectral sequence from Theorem 4.4.4. Here the differential that we obtained from the model suggests differentials for this spectral sequence in the following fashion. We start with the E^2 term of this spectral sequence and we calculate the differential δ from the model of each of the elements x of bidegree (p, q). For such x we then set d_2 to be the leading terms of $\delta(x)$, that is, the elements of the right bidegree $((p+2, q-1))$ that appear as summands in $\delta(x)$. If there are no such elements of the right bidegree, then $d^2(x) = 0$. Once knowing the second differential we can calculate the elements of the third page $E^3_{*,*}$. Similarly, in order to calculate d^n we take the leading terms of $\delta - \sum_{i=2}^{n-1} d^i$. We note that, in our examples, this will give all the differentials since there is a very limited number of elements of each bidegree in a particular term of this spectral sequence. In most of the examples, there is only one non trivial differential.

- From this, we obtain a description of $\mathbb{H}_*(LM)$ with the string product.
- Now we use the calculations from Section 4.3 and Poincaré duality on M to give an isomorphism (of vector spaces) between $\mathbb{H}_*(LM)$ and $H^*(LM)$.
- We use the isomorphism between $\mathbb{H}_*(LM)$ and $H^*(LM)$ and the calculations from 4.3 of the B-operator to obtain the Batalin-Vilkovisky operator $B^* = \Delta$ via the formula

$$s(B^*t) = Bs(t), \tag{4.5.1}$$

where $s \in H^*(LM)$ and $t \in \mathbb{H}_*(LM)$.

4.5.1 $\mathbb{H}_*(LS^{2n+1})$ and the String Product

This calculation is valid only in the case where $n > 0$, since Theorem 3.4.3 requires the manifold to be simply connected. In this case, the cohomology of the sphere is a free algebra on one generator x of degree $|x| = 2n + 1$, that is, A is the exterior algebra $K[x]$. Following an analogous idea to Section 4.3.1, since A is free we can use Section 4.1 to get

$$\mathbb{H}_*(LS^{2n+1}) = K[x, \nu]$$

with $|\nu| = -2n$.

Recall from Section 4.3 that $H^*(LS^{2n+1})$ is the free graded commutative algebra generated by x and u. Using Poincaré duality on S^{2n+1} we can construct an isomorphism of vector spaces

$$\mathbb{H}_*(LS^{2n+1}) \to H^{2n+1-*}(LS^{2n+1})$$

which is given by

$$x\nu^k \mapsto u^k$$
$$\nu^k \mapsto u^k x$$

for $k \geq 0$. Thus $B^*(1) = x\nu$ and $B^*(x\nu^k) = 0 = B^*(\nu^m)$ for $k \geq 0$ and $m \geq 1$.

As algebras with their respective products and disregarding the grading, $\mathbb{H}_*(LS^{2n+1}) = K[x, \nu]$ and $H^*(LS^{2n+1}) = K[x, u]$ are isomorphic.

4.5.2 $\mathbb{H}_*(LS^{2n})$ and the String Product

Let

$$A = H^*(S^{2n}) = \frac{K[x]}{(x^2)}$$

with $|x| = 2n$. From example 4.1 the model for A is $P = (K[x, y], \partial)$, where the differential ∂ is defined on the generators by $\partial x = 0$ and $\partial y = x^2$. By Theorem 4.4.1, a model for $\mathbb{H}_*(LS^{2n})$ is given by

$$GP = (K[x, y, a, b], \delta),$$

where $|x| = 2n$, $|y| = 4n - 1$, $|a| = 1 - 2n$, $|b| = 2 - 4n$ and δ is defined on the generators by

$$\delta x = 0$$
$$\delta y = x^2$$
$$\delta a = -2xb$$
$$\delta b = 0.$$

Figure 4.6 represents the Cohen-Jones-Yan spectral sequence from Theorem

4.4.4 when $M = S^{2n}$. From this we see that

$$\mathbb{H}_*(LS^{2n}) = \frac{K[x, b, \nu]}{(x^2,\ \nu x,\ 2bx)},$$

where $\nu = ax$ and the generators have degrees $|x| = 2n$, $|b| = 2 - 4n$, $|\nu| = 1$.

Now we will compute the Batalin-Vilkovisky operator $\Delta = B^*$ using Formula 4.5.1. In subsection 4.3.1 we saw that $H^*(LS^{2n})$ has generators z_i and $B(z_i)$, $i \geq 0$, with the relations described there. Using Poincaré duality on S^{2n} we have, for every k, an isomorphism of vector spaces

$$\mathbb{H}_k(LS^{2n}) \to H^{2n-k}(LS^{2n})$$

given by

$$\begin{aligned} x &\mapsto 1 \\ b^i\nu &\mapsto B(z_i) \\ b^i &\mapsto z_i. \end{aligned}$$

From this isomorphism and formula 4.5.1, we see that B^* is given by

$$\begin{aligned} B^*(x) &= 0 \\ B^*(b^i\nu) &= b^i \\ B^*(b^i) &= 0. \end{aligned}$$

Finally we observe that there is a substantial difference between the rings $\mathbb{H}_*(LS^{2n})$ and $H^*(LS^{2n})$. In the latter, all the products are trivial whereas the former contains the polynomial ring $K[b]$.

	0	1		...		2n
0	1					x
1-2n	a					ax
2-4n	b					bx
3-6n	ab					abx
4-8n	b^2					b^2x
5-10n	ab^2					ab^2x

Figure 4.6: The Cohen-Jones-Yan spectral sequence for $\mathbb{H}_*(LS^{2n})$

4.5.3 $\mathbb{H}_*(L\mathbb{C}P^n)$ and the String Product

As in Example 4.0.6, the model for

$$A = H^*(\mathbb{C}P^n) = \frac{K[x,y]}{(x^{n+1})}$$

is given by $P = (K[x,y], \partial)$, where $|x| = 2$, $|y| = 4n+1$, $\partial x = 0$ and $\partial y = x^{n+1}$. Using Theorem 4.4.1 we see that a model for $\mathbb{H}_*(L\mathbb{C}P^n)$ is given by

$$G(P) = (K[x,y,a,b], \delta),$$

with $|a| = -1$, $|b| = -2n$ and δ defined on the generators by

$$\begin{aligned}
\delta x &= 0,\\
\delta y &= x^{n+1},\\
\delta a &= -(n+1)x^n b,\\
\delta b &= 0.
\end{aligned}$$

In Figure 4.7 we find the Cohen-Jones-Yan spectral sequence (see Theorem 4.4.4) for $M = \mathbb{C}P^n$.

Let $\nu = ax$. Observe from Figure 4.7 that

$$\mathbb{H}_*(L(\mathbb{C}P^n)) = \frac{K[x,\nu,b]}{(x^{n+1},\ \nu x^n,\ (n+1)bx^n)},$$

where the generators have degree $|x| = 2$, $|\nu| = 1$ and $|b| = -2n$.

To compute the Batalin-Vilkovisky operator $\Delta = B^*$ on $\mathbb{H}_*(L\mathbb{C}P^n)$ we recall from Section 4.3.3 that $H^*(L\mathbb{C}P^n)$ has generators z_i and $B(z_i)$, $i \geq 0$, with the corresponding relations described there. Then, by Poincaré duality on $\mathbb{C}P^n$, we get

	0	1	2	3	4	...			2n
0	1		x		x^2				x^n
-1	a		ax		ax^2				ax^n
-2n	b		bx		bx^2				bx^n
-2n-1	ab		abx		abx^2				abx^n
-4n	b^2		b^2x		b^2x^2				b^2x^n
-4n-1	ab^2		ab^2x		ab^2x^2				ab^2x^n
-6n	b^3		b^3x		b^3x^2				b^3x^n
-6n-1	ab^3		ab^3x		ab^3x^2				ab^3x^n
-8n	b^4		b^4x		b^4x^2				b^4x^n
-8n-1	ab^4		ab^4x		ab^4x^2				ab^4x^n

Figure 4.7: The Cohen-Jones-Yan spectral sequence for $\mathbb{H}_*(L\mathbb{C}P^n)$

an isomorphism of vector spaces

$$\mathbb{H}_*(L\mathbb{C}P^n) \to H^{2n-*}(L\mathbb{C}P^n)$$

given by

$$\begin{aligned} x^k &\mapsto (z_0)^{n-k}, \\ x^l b^m &\mapsto (z_0)^{n-l-1}(z_m), \\ \nu x^l b^m &\mapsto (z_0)^{n-l-1} B(z_m), \end{aligned}$$

where $0 \le k \le n$, $0 \le l \le n-1$, $m \ge 0$. Using this isomorphism and Formula 4.5.1 to compute B^*, we get the following formulas

$$\begin{aligned} B^*(x^k) &= 0, \\ B^*(x^l b^m) &= 0, \\ B^*(\nu x^l b^m) &= (n-l-1)x^l b^m, \end{aligned}$$

for any $0 \le k \le n$, $0 \le l \le n-1$, $m \ge 0$.

Notice the great difference between the rings $\mathbb{H}_*(L(\mathbb{C}P^n))$ and $H^*(L(\mathbb{C}P^n))$. In the latter, all the products of length $n+1$ are trivial whereas this is not the case for the former. Indeed, the polynomial algebra $K[b]$ is contained in $\mathbb{H}_*(L(\mathbb{C}P^n))$.

Remark 4.5.1. The rings $\mathbb{H}_*(LS^{2n+1})$, $\mathbb{H}_*(LS^{2n})$ and $\mathbb{H}_*(L(\mathbb{C}P^n))$, computed in Sections 4.5.1, 4.5.2, 4.5.3 were first computed by Cohen, Jones and Yan in [6]. We observe that, up to regrading, both descriptions agree.

4.5.4 $\mathbb{H}_*(L(G_2(\mathbb{C}^4)))$ with the String Product

Let

$$A = H^*(G_2(\mathbb{C}^4)) = \frac{K[x,y]}{(x^3 - 2xy, y^2 - x^2 y)},$$

where $|x| = 2$ and $|y| = 4$. By Example 4.0.7, the model for A is given by $P = (K[x,y,f,g], \partial)$, where ∂ is defined on the generators by

$$\begin{aligned}
\partial x &= 0, \\
\partial y &= 0, \\
\partial f &= x^3 - 2xy, \\
\partial g &= x^2 y - y^2,
\end{aligned}$$

and with degrees $|f| = 5$, $|g| = 7$. Using Theorem 4.4.1, we obtain a model $G(P)$ for $\mathbb{H}_*(L(G_2(\mathbb{C}^4)))$ with underlying free commutative algebra

$$G(P) = K[x, y, f, g, \mu, \nu, \alpha, \beta]$$

with differential given by

$$\begin{aligned}
\delta x &= \delta y = \delta\alpha = \delta\beta = 0, \\
\delta f &= x^3 - 2xy, \\
\delta g &= x^2 y - y^2, \\
\delta\mu &= -3\alpha x^2 + 2\alpha y - 2\beta xy, \\
\delta\nu &= 2\alpha x - \beta x^2 + 2\beta y.
\end{aligned}$$

Figures 4.8, 4.9 and 4.10 represent the E^2, E^4 and $E^5 = E^\infty$-terms of the Cohen-Jones-Yan spectral sequence (see Theorem 4.4.4) when $M = G_2(\mathbb{C}^4)$. Notice that the spectral sequence collapses in the E^5-term.

Using these figures we obtain the algebra structure of $\mathbb{H}_*(L(G_2(\mathbb{C}^4)))$ with respect to the string product. Let $\gamma = \mu x$, $\epsilon = \mu y$, $\zeta = \nu y^2$, $\eta = \mu\nu y^2$ and $\theta = \alpha\nu x^2 - 2\alpha\nu y$. Then $\mathbb{H}_*(L(G_2(\mathbb{C}^4)))$ is equal to the differential graded commutative

	0	1	2	3	4	5	6	7	8
0	1		x		x^2, y		$x^3 = 2xy$		$y^2 = x^2y$
-1	μ		μx		μx^2, μy		μx^3		μy^2
-2									
-3	ν		νx		$\nu x^2 \sim 2\nu y$		νx^3		νy^2
-4	$\mu\nu$, α		$\mu\nu x$, αx		$\mu\nu x^2 \sim 2\mu\nu y$, αx^2, αy		$\mu\nu x^3$, αx^3		$\mu\nu y^2$, αy^2
-5	$\alpha\mu$		$\alpha\mu x$		$\alpha\mu x^2$, $\alpha\mu y$		$\alpha\mu x^3$		$\alpha\mu y^2$
-6	β		βx		βx^2, βy		βx^3		βy^2
-7	$\beta\mu$, $\alpha\nu$		$\beta\mu x$, $\alpha\nu x$		$\beta\mu x^2$, $\beta\mu y$, $\alpha\nu x^2 \sim 2\alpha\nu y$		$\beta\mu x^3$, $\alpha\nu x^3$		$\beta\mu y^2$, $\alpha\nu y^2$
-8	$\alpha\mu\nu$, α^2		$\alpha\mu\nu x$, $\alpha^2 x$		$\alpha\mu\nu x^2 \sim 2\alpha\mu\nu y$, α^2x^2, $\alpha^2 y$		$\alpha\mu\nu x^3$, $\alpha^2 x^3$		$\alpha\mu\nu y^2$, $\alpha^2 y^2$
-9	$\beta\nu$, $\alpha^2\mu$		$\beta\nu x$, $\alpha^2\mu x$		$\beta\nu x^2 \sim 2\beta\nu y$, $\alpha^2\mu x^2$, $\alpha^2\mu y$		$\beta\nu x^3$, $\alpha^2\mu x^3$		$\beta\nu y^2$, $\alpha^2\mu y^2$
-10	$\beta\mu\nu$, $\alpha\beta$		$\beta\mu\nu x$, $\alpha\beta x$		$\beta\mu\nu x^2 \sim 2\beta\mu\nu y$, $\alpha\beta x^2$, $\alpha\beta y$		$\beta\mu\nu x^3$, $\alpha\beta x^3$		$\beta\mu\nu y^2$, $\alpha\beta y^2$
-11	$\alpha\beta\mu$, $\alpha^2\nu$		$\alpha\beta\mu x$, $\alpha^2\nu x$		$\alpha\beta\mu x^2$, $\alpha\beta\mu y$, $\alpha^2\nu x^2 \sim 2\alpha^2\nu y$		$\alpha\beta\mu x^3$, $\alpha^2\nu x^3$		$\alpha\beta\mu y^2$, $\alpha^2\nu y^2$
-12	α^3, β^2, $\alpha^2\mu\nu$		$\alpha^3 x$, $\beta^2 x$, $\alpha^2\mu\nu x$		α^3x^2, α^3y, β^2x^2, β^2y, $\alpha^2\mu\nu x^2 \sim 2\alpha\mu\nu y$		α^3x^3, β^2x^3, $\alpha^2\mu\nu x^3$		α^3y^2, β^2y^2, $\alpha^2\mu\nu y^2$
-13	$\alpha^3\mu$, $\beta^2\mu$, $\alpha\beta\nu$		$\alpha^3\mu x$, $\beta^2\mu x$, $\alpha\beta\nu x$		$\beta^2\mu x^2$, $\beta^2\mu y$, $\alpha^3\mu x^2$, $\alpha^3\mu y$, $\alpha\beta\nu x^2 \sim 2\alpha\beta\nu y$		$\beta^2\mu x^3$, $\alpha^3\mu x^3$, $\alpha\beta\nu x^3$		$\beta^2\mu y^2$, $\alpha^3\mu y^2$, $\alpha\beta\nu y^2$
-14	$\alpha\beta\mu\nu$, $\alpha^2\beta$		$\alpha\beta\mu\nu x$, $\alpha^2\beta x$		$\alpha\beta\mu\nu x^2 \sim \alpha\beta\mu\nu y$, $\alpha^2\beta x^2$, $\alpha^2\beta y$		$\alpha\beta\mu\nu x^3$, $\alpha^2\beta x^3$		$\alpha\beta\mu\nu y^2$, $\alpha^2\beta y^2$
-15	$\beta^2\nu$, $\alpha^2\beta\mu$, $\alpha^3\nu$		$\beta^2\nu x$, $\alpha^2\beta\mu x$, $\alpha^3\nu x$		$\beta^2\nu x^2 \sim 2\beta^2\nu y$, $\alpha^2\beta\mu x^2$, $\alpha^2\beta\mu y$, $\alpha^3\nu x^2 \sim 2\alpha^3\nu y$		$\beta^2\nu x^3$, $\alpha^2\beta\mu x^3$, $\alpha^3\nu x^3$		$\beta^2\nu y^2$, $\alpha^2\beta\mu y^2$, $\alpha^3\nu y^2$
-16	$\alpha\beta^2$, $\alpha^3\mu\nu$, α^4, $\beta^2\mu\nu$		$\alpha\beta^2 x$, $\alpha^3\mu\nu x$, $\alpha^4 x$, $\beta^2\mu\nu x$		$\alpha\beta^2x^2$, $\alpha\beta^2y$, $\alpha^3\mu\nu x^2 \sim 2\alpha^3\mu\nu y$, $\beta^2\mu\nu x^2 \sim 2\beta^2\mu\nu y$, α^4x^2, α^4y		$\alpha\beta^2x^3$, $\alpha^3\mu\nu x^3$, α^4x^3, $\beta^2\mu\nu x^3$		$\alpha\beta^2y^2$, $\alpha^3\mu\nu y^2$, α^4y^2, $\beta^2\mu\nu y^2$
-17	$\alpha^4\mu$, $\alpha\beta^2\mu$, $\alpha^2\beta\nu$		$\alpha^4\mu x$, $\alpha\beta^2\mu x$, $\alpha^2\beta\nu x$		$\alpha^4\mu x^2$, $\alpha^4\mu y$, $\alpha\beta^2\mu x^2$, $\alpha\beta^2\mu y$, $\alpha^2\beta\nu x^2 \sim 2\alpha^2\beta\nu y$		$\alpha^4\mu x^3$, $\alpha\beta^2\mu x^3$, $\alpha^2\beta\nu x^3$		$\alpha^4\mu y^2$, $\alpha\beta^2\mu y^2$, $\alpha^2\beta\nu y^2$
-18	$\alpha^2\beta\mu\nu$, $\alpha^3\beta$, β^3		$\alpha^2\beta\mu\nu x$, $\alpha^3\beta x^2$, $\beta^3 x$		$\alpha^2\beta\mu\nu x^2 \sim 2\alpha^2\beta\mu\nu y$, β^3x^2, β^3y, $\alpha^3\beta x$, $\alpha^3\beta y$		$\alpha^2\beta\mu\nu x^3$, β^3x^3, $\alpha^3\beta x^3$		$\alpha^2\beta\mu\nu y^2$, β^3y^2, $\alpha^3\beta y^2$
-19	$\alpha^3\beta\mu$, $\alpha\beta^2\nu$, $\beta^3\mu$, $\alpha^4\nu$		$\alpha\beta^2\nu x$, $\alpha^3\beta\mu x$, $\alpha^4\nu x$, $\beta^3\mu x$		$\alpha\beta^2\nu x^2 \sim 2\alpha\beta^2\nu y$, $\alpha^4\nu x^2 \sim 2\alpha^4\nu y$, $\alpha^3\beta\mu x^2$, $\alpha^3\beta\mu y$, $\beta^3\mu x^2$, $\beta^3\mu y$		$\alpha\beta^2\nu x^3$, $\alpha^3\beta\mu x^3$, $\beta^3\mu x^3$, $\alpha^4\nu x^3$		$\alpha\beta^2\nu y^2$, $\alpha^3\beta\mu y^2$, $\beta^3\mu y^2$, $\alpha^4\nu y^2$
-20	$\alpha\beta^2\mu\nu$, $\alpha^2\beta^2$, α^5, $\alpha^4\mu\nu$		$\alpha\beta^2\mu\nu x$, $\alpha^2\beta^2 x$, $\alpha^5 x$, $\alpha^4\mu\nu x$		$\alpha\beta^2\mu\nu x^2 \sim 2\alpha\beta^2\mu\nu y$, $\alpha^2\beta^2x^2$, $\alpha^2\beta^2y$, α^5x^2, α^5y, $\alpha^4\mu\nu x^2 \sim 2\alpha^4\mu\nu y$		$\alpha\beta^2\mu\nu x^3$, $\alpha^2\beta^2x^3$, α^5x^3, $\alpha^4\mu\nu x^3$		$\alpha\beta^2\mu\nu y^2$, $\alpha^2\beta^2y^2$, α^5y^2, $\alpha^4\mu\nu y^2$

Figure 4.8: The E^2-term of the Cohen-Jones-Yan spectral sequence for $\mathbb{H}_*(L(G_2(\mathbb{C}^4)))$

	0	1	2	3	4	5	6	7	8
0	1		x		x^2 y		$x^3 = 2xy$		$y^2 = x^2y$
-1	μ		μx		μx^2 μy		μx^3		μy^2
-2									
-3					$\nu x^2 - 2\nu y$				νy^2
-4	α				$\mu\nu x^2 - 2\mu\nu y$ αy				$\mu\nu y^2$
-5	$\alpha\mu$				$\alpha\mu y$				
-6	β		βx		βx^2 βy		βx^3		βy^2
-7	$\beta\mu$		$\beta\mu x$		$\beta\mu x^2$ $\beta\mu y$ $\alpha\nu x^2 - 2\alpha\nu y$		$\beta\mu x^3$		$2\beta\mu y^2 - 2\alpha\nu y^2$
-8	α^2				$\alpha\mu\nu x^2 - 2\alpha\mu\nu y$ $\alpha^2 y$				$\alpha\mu\nu y^2$
-9	$\alpha^2\mu$				$\beta\nu x^2 - 2\beta\nu y$ $\alpha^2\mu y$				$\beta\nu y^2$
-10	$\alpha\beta$				$\beta\mu\nu x^2 - 2\beta\mu\nu y$ $\alpha\beta y$				$\beta\mu\nu y^2$
-11	$\alpha\beta\mu$				$\alpha\beta\mu y$ $\alpha^2\nu x^2 - 2\alpha^2\nu y$				$\alpha^2\nu y^2$
-12	α^3 β^2		$\beta^2 x$		$\alpha^3 y$ $\beta^2 x^2$ $\beta^2 y$ $\alpha^2\mu\nu x^2 - 2\alpha^2\mu\nu y$		$\beta^2 x^3$		$\beta^2 y^2$ $\alpha^2\mu\nu y^2$
-13	$\beta^2\mu$ $\alpha^3\mu$		$\beta^2\mu x$		$\beta^2\mu x^2$ $\beta^2\mu y$ $\alpha^3\mu y$ $\alpha\beta\nu x^2 - 2\alpha\beta\nu y$		$\beta^2\mu x^3$		$2\beta^2\mu y^2 - 2\alpha\beta\nu y^2$
-14	$\alpha^2\beta$				$\alpha\beta\mu\nu x^2 - 2\alpha\beta\mu\nu y$ $\alpha^2\beta y$				$\alpha\beta\mu\nu y^2$
-15	$\alpha^2\beta\mu$				$\beta^2\nu x^2 - 2\beta^2\nu y$ $\alpha^2\beta\mu y$ $\alpha^3\nu x^2 - 2\alpha^3\nu y$				$\beta^2\nu y^2$ $\alpha^3\nu y^2$
-16	$\alpha\beta^2$ α^4				$\alpha\beta^2 y$ $\alpha^4 y$ $\alpha^3\mu\nu x^2 - 2\alpha^3\mu\nu y$ $\beta^2\mu\nu x^2 - 2\beta^2\mu\nu y$				$\alpha^3\mu\nu y^2$ $\beta^2\mu\nu y^2$
-17	$\alpha\beta^2\mu$ $\alpha^4\mu$				$\alpha^4\mu y$ $\alpha\beta^2\mu y$ $\alpha^2\beta\nu x^2 - 2\alpha^2\beta\nu y$				$\alpha^2\beta\nu y^2$
-18	$\alpha^3\beta$ β^3		$\beta^3 x$		$\alpha^3\beta y$ $\beta^3 x^2$ $\beta^3 y$ $\alpha^2\beta\mu\nu x^2 - 2\alpha^2\beta\mu\nu y$		$\beta^3 x^3$		$\beta^3 y^2$ $\alpha^2\beta\mu\nu y^2$
-19	$\alpha^3\beta\mu$ $\beta^3\mu$		$\beta^3\mu x$		$\alpha\beta^2\nu x^2 - 2\alpha\beta^2\nu y$ $\alpha^4\nu x^2 - 2\alpha^4\nu y$ $\alpha^3\beta\mu y$ $\beta^3\mu x^2$ $\beta^3\mu y$		$\beta^3\mu x^3$		$\alpha^4\nu y^2$ $2\beta^3\mu y^2 - 2\alpha\beta^2\nu y^2$
-20	$\alpha^2\beta^2$ α^5				$\alpha^5 y$ $\alpha^2\beta^2 y$ $\alpha\beta^2\mu\nu x^2 - 2\alpha\beta^2\mu\nu y$ $\alpha^4\mu\nu x^2 - 2\alpha^4\mu\nu y$				$\alpha\beta^2\mu\nu y^2$ $\alpha^4\mu\nu y^2$

Figure 4.9: The E^4-term of the Cohen-Jones-Yan spectral sequence for $\mathbb{H}_*(L(G_2(\mathbb{C}^4)))$

	0	1	2	3	4	5	6	7	8
0	1		x		x^2 y		$x^3 = 2xy$		$y^2 = x^2y$
-1			μx		μx^2 μy		μx^3		μy^2
-2									
-3									νy^2
-4	α								$\mu\nu y^2$
-5					$\alpha\mu y$				
-6	β		βx		βx^2 βy		βx^3		
-7			$\beta\mu x$		$\beta\mu x^2$ $\beta\mu y$ $\alpha\nu x^2 - 2\alpha\nu y$		$\beta\mu x^3$		$2\beta\mu y^2 = 2\alpha\nu y^2$
-8	α^2								$\alpha\mu\nu y^2$
-9					$\alpha^2\mu y$				$\beta\nu y^2$
-10	$\alpha\beta$								$\beta\mu\nu y^2$
-11					$\alpha\beta\mu y$ $\alpha^2\nu x^2 - 2\alpha^2\nu y$				
-12	β^2 α^3		$\beta^2 x$		$\beta^2 x^2$ $\beta^2 y$		$\beta^2 x^3$		$\alpha^2\mu\nu y^2$
-13			$\beta^2\mu x$		$\beta^2\mu x^2$ $\beta^2\mu y$ $\alpha^3\mu y$ $\alpha\beta\nu x^2 - 2\alpha\beta\nu y$		$\beta^2\mu x^3$		$2\beta^2\mu y^2 = 2\alpha\beta\nu y^2$
-14	$\alpha^2\beta$								$\alpha\beta\mu\nu y^2$
-15					$\alpha^2\beta\mu y$ $\alpha^3\nu x^2 - 2\alpha^3\nu y$				$\beta^2\nu y^2$
-16	$\alpha\beta^2$ α^4								$\alpha^3\mu\nu y^2$ $\beta^2\mu\nu y^2$
-17					$\alpha^4\mu y$ $\alpha\beta^2\mu y$ $\alpha^2\beta\nu x^2 - 2\alpha^2\beta\nu y$				
-18	$\alpha^3\beta$ β^3		$\beta^3 x$		$\beta^3 x^2$ $\beta^3 y$		$\beta^3 x^3$		$\alpha^2\beta\mu\nu y^2$
-19			$\beta^3\mu x$		$\alpha\beta^2\nu x^2 - 2\alpha\beta^2\nu y$ $\alpha^4\nu x^2 - 2\alpha^4\nu y$ $\alpha^3\beta\mu y$ $\beta^3\mu x^2$ $\beta^3\mu y$		$\beta^3\mu x^3$		$2\beta^3\mu y^2 = 2\alpha\beta^2\nu y^2$
-20	$\alpha^2\beta^2$ α^5								$\alpha\beta^2\mu\nu y^2$ $\alpha^4\mu\nu y^2$

Figure 4.10: The $E^5 = E^\infty$-term of the Cohen-Jones-Yan spectral sequence for $\mathbb{H}_*(L(G_2(\mathbb{C}^4)))$

algebra

$$K[x, y, \alpha, \beta, \gamma, \epsilon, \zeta, \eta, \theta],$$

where the following products are trivial

$$\eta^2,\ \alpha x,\ \beta y^2,\ \zeta x,\ \eta x,\ \theta x,\ \alpha y,\ \zeta y,\ \eta y,\ \gamma\epsilon,\ \gamma\eta,\ \gamma\theta,\ \gamma\zeta,\ \alpha\gamma,\ \epsilon\zeta,\ \epsilon\eta,\ \zeta\eta,\ \zeta\theta,\ \eta\theta,$$

and subject to the relations

$$\begin{aligned}
x^3 &= 2xy,\\
y^2 &= x^2 y,\\
\gamma x^2 &= 2\gamma y = 2\epsilon x,\\
\gamma x^3 &= 2\epsilon y,\\
\theta y &= -\beta\epsilon y = -\alpha\zeta,\\
\epsilon\theta &= -\alpha\eta.
\end{aligned}$$

The degrees of the generators are

$$|x| = 2,\ |y| = 4,\ |\alpha| = -4,\ |\beta| = -6,\ |\gamma| = 1,\ |\epsilon| = 3,\ |\zeta| = 5,\ |\eta| = 4,\ |\theta| = -3.$$

Finally we will compute the Batalin-Vilkovisky operator $\Delta = B^*$ on $\mathbb{H}_*(L(G_2(\mathbb{C}^4)))$. In Subsection 4.3.4 we saw that $H^*(L(G_2(\mathbb{C}^4))$ is the quotient of the graded commutative algebra generated by

$$\alpha_n,\ \beta_n,\ \delta_m,\ \epsilon_{n,m},\ \lambda_{n,m},$$

and

$$u = B(\alpha_0),\ \gamma_n = B(\beta_n),\ \theta_{n,m} = -B(\epsilon_{n,m}),\ \rho_{n,m} = B\left(\frac{1}{2}\lambda_{n,m}\right)$$

subject to the relations described in the same subsection. We have an isomorphism of vector spaces

$$\mathbb{H}_*(L(G_2(\mathbb{C}^4)) \to H^{8-*}(L(G_2(\mathbb{C}^4))$$

given by

$$\begin{aligned}
y^2 &\mapsto 1 \\
\beta^m x^3 &\mapsto \alpha_m \\
\beta^m x^2 &\mapsto \alpha_m \alpha_0 \\
\beta^m x &\mapsto \alpha_m (\alpha_0)^2 \\
\beta^m y &\mapsto \beta_m \\
\alpha^k \beta^m &\mapsto \lambda_{k,n} \\
\beta^m &\mapsto \beta_m \beta_0 \\
\beta^m \gamma &\mapsto \alpha_m (\alpha_0)^2 B(\alpha_0) \\
\beta^m \gamma x &\mapsto \alpha_m (\alpha_0) B(\alpha_0) \\
\beta^m \gamma x^2 &\mapsto \alpha_m B(\alpha_0) \\
\alpha^k \beta^m \epsilon &\mapsto \epsilon_{m,k} \\
\beta^m \epsilon &\mapsto \beta_m B(\alpha_0) \\
\beta^k \epsilon y &\mapsto \delta_k \\
\mu y^2 &\mapsto B(\alpha_0) \\
\beta^m \zeta &\mapsto \gamma_n \\
\alpha^k \beta^m \eta &\mapsto \theta_{m,k} \\
\beta^m \eta &\mapsto B(\alpha_0) \gamma_m \\
\beta^m \alpha^n \theta &\mapsto \rho_{m,n+1}
\end{aligned}$$

for any $n, m \geq 0$, $k \geq 1$. A direct calculation shows that B^* is given by the following formulas

$$\begin{aligned}
B^*(y^2) &\mapsto 0\\
B^*(\beta^m x^3) &\mapsto 0\\
B^*(\beta^m x^2) &\mapsto 0\\
B^*(\beta^m x) &\mapsto 0\\
B^*(\beta^m y) &\mapsto 0\\
B^*(\alpha^k \beta^m) &\mapsto 0\\
B^*(\beta^m) &\mapsto 0\\
B^*(\beta^m \gamma) &\mapsto 0\\
B^*(\beta^m \gamma x) &\mapsto 3\beta^m x\\
B^*(\beta^m \gamma x^2) &\mapsto 2\beta^m x^2\\
B^*(\alpha^k \beta^m \epsilon) &\mapsto 0\\
B^*(\beta^m \epsilon) &\mapsto 0\\
B^*(\beta^k \epsilon y) &\mapsto \beta^m x^3\\
B^*(\mu y^2) &\mapsto x^3\\
B^*(\beta^m \zeta) &\mapsto \beta^m y\\
B^*(\alpha^k \beta^m \eta) &\mapsto -\alpha^k \beta^m \epsilon\\
B^*(\beta^m \eta) &\mapsto \beta^m \epsilon\\
B^*(\beta^m \alpha^n \theta) &\mapsto 2\alpha^{n+1} \beta^m
\end{aligned}$$

for any $n, m \geq 0$, $k \geq 1$.

Notice that the rings $\mathbb{H}_*(L(G_2(\mathbb{C}P^n)))$ and $H^*(L(G_2(\mathbb{C}P^n)))$ differ greatly. Indeed, the latter contains a polynomial algebra in two generators, whereas in the former every element has order at most 5.

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