Aspects of AdS/CFT: Black Solutions in Gauged Supergravity and Holographic Conductivities

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Zusammenfassung

Der Themenbereich dieser Dissertation gliedert sich in zwei Teilgebiete, die beide durch die AdS/CFT-Korrespondenz motiviert sind. Zum einen werden neue schwarze Gravitationslösungen im Bereich geeichter Supergravitation in vier und fünf Raumzeitdimensionen konstruiert. Zum anderen werden Methoden aus der Korrespondenz zwischen Eich- und Gravitationstheorien verwendet um holographisch den frequenzabhängigen Leitfähigkeitstensor für bestimmte Feldtheorien zu berechnen.

Den Ausgangspunkt zur Berechnung dieses Leitfähigkeitstensors bildet eine Schwarze-Loch-Lösung der vierdimensionalen Einstein-Yang-Mills-Wirkung mit $SU(2)$-Eichfeld. Die $SU(2)$-Symmetrie wird durch die Wahl des Eichfeldes explizit gebrochen und auf eine $U(1)$-Symmetrie in τ^3-Richtung reduziert. Es wurde bereits in anderen Arbeiten nachgewiesen, dass dieses System bei einer kritischen Temperatur einen Phasenübergang durchläuft, in dessen Folge die verbliebene $U(1)$-Symmetrie spontan gebrochen wird und ein schwarzes Loch mit Vektor-Haar entsteht. Dieses Ergebnis wurde dahingehend interpretiert, dass in der dualen Feldtheorie ein Phasenübergang zu einem p-Wellen-Supraleiter stattfindet.

Diese Doktorarbeit befasst sich mit den Strömen in τ^1- und τ^2-Richtung in der normalen, d.h. nicht-supraleitenden, Phase. Der frequenzabhängige Leitfähigkeitstensor wird mit Hilfe von Linearer-Response-Theorie basierend auf holographischen Methoden berechnet, die im Rahmen der AdS/CFT-Korrespondenz entwickelt wurden. Dabei konnte festgestellt werden, dass innerhalb eines bestimmten Frequenzbereichs ein negativer Eigenwert des hermiteschen Anteils des Leitfähigkeitstensors auftritt, was zu negativer Entropieproduktion zu führen scheint. Zum jetzigen Zeitpunkt konnte dieser Effekt noch nicht hinreichend geklärt werden.

Der zweite Teil dieser Dissertation hat die Suche nach schwarzen Lösungen mit AdS-Asymptotik innerhalb von $N = 2$ $U(1)$-geeichter Supergravitation in vier und fünf Dimensionen zum Inhalt. Hierbei werden nur Vektor-Multipletts berücksichtigt. Derartige Lösungen sind von Interesse, da schwarze Geometrien mit AdS-Asymptotik im Rahmen der Korrespondenz zwischen Gravitation und Eichtheorien Feldtheorien bei endlicher Temperatur und Ladungsdichte beschreiben.

Sowohl in vier als auch in fünf Dimensionen wird auf den Attraktor-Mechanismus für extremale schwarze Geometrien zurückgegriffen. Dieser garantiert, dass die relevanten Bewegungsgleichungen gewöhnliche Differentialgleichungen erster Ordnung sind. Diese

Flussgleichungen stellen eine Verallgemeinerung des asymptotisch flachen Falls dar auf den Fall mit AdS-Asymptotik. Im Folgenden werden unterschiedliche neue schwarze Lösungen konstruiert. In beiden Dimensionen konnten extremale schwarze Branen mit verschwindender Entropiedichte gefunden werden. Diese werden in Anlehnung an das Nernst-Gesetz der Thermodynamik als *Nernst-Branen* bezeichnet. Extremale schwarze Lösungen haben die Eigenschaft, dass ihre Temperatur gleich null ist. Viele bekannte extremale schwarze Lösungen wie Reissner-Nordström-Lösungen weisen jedoch einen nicht-verschwindende Entropie auf. Daher sind die Nernst-Lösungen von Interesse, die bei verschwindender Temperatur eine verschwindende Entropiedichte besitzen.

Weiterhin werden Konfigurationen untersucht, die keinen Nernst-Lösungen entsprechen. Hier konnten in vier Dimensionen sowohl singuläre Lösungen gefunden werden als auch eine numerische Lösung, die zwischen $AdS_2 \times \mathbb{R}^2$ und AdS_4 interpoliert. In fünf Dimensionen werden analytische Lösungen mit konstanten Skalaren sowie numerische Lösungen mit nicht-konstanten Skalaren konstruiert. Diese Lösungen verhalten sich asymptotisch wie AdS_5 und können am Horizont durch eine $BTZ \times \mathbb{R}^2$-Geometrie beschrieben werden.

Ferner wird der Zusammenhang zwischen den vier- und fünfdimensionalen Bewegungsgleichungen dargestellt.

Chapter 1

Introduction and Overview

"The most beautiful thing we can experience is the mysterious. It is the source of all true art and science."

Albert Einstein, "How I see the world"

With his words Einstein shows the idea of mankind being driven by an intrinsic urge for knowledge. A similar picture was drawn 150 years before by Wilhelm von Humboldt when he speaks of what he calls a man's "Bildungsbedürftigkeit" which may be translated as the "longing for education".

For Humboldt education implies a person's involvement with all the surrounding world, in the German original called "Aneignung von Welt". Surely from today's perspective exploring the laws of nature is closely related to this idea of education. People feel the need to know why apples fall down from trees or why there are myriads of stars in the sky.

Another consiredable philosopher, Immanuel Kant, sees man as an entity endowed with reason. From Kant's perspective it is possible to gain insight into the "things in themselves", which include the natural phenomena, by the use of reason. Of course, we all differ in our perceptions and processing of information but according to Kant this does not mean that cognition independently of an observer is impossible. For Kant this is true especially in mathematics and physics. Here his claim "sapere aude!" – "dare to know!" – seems to be fulfilled best. By the use of reason we are able to decide whether a theory describes well the physical phenomena or whether we should discard it. Of course, this process of acquiring knowledge goes hand in hand with experiences we make by perception, which from a physicist's point of view are experiments. Both traits, the thurst for knowledge and the use of reason, make a scientist.

Within the last 300 years a lot of progress has been made in understanding the laws of nature. Very often this process was accompanied by a change of paradigm. Some of the real revolutions in physics have been made because somebody went beyond the boundaries of existing knowledge – somebody "dared to know".

Some of the major changes in physics happened at the beginning of the twentieth century. At that time two theories involving completely new physics were found: Quantum mechanics and Einstein's theory of relativity.

Before it was believed that everything is completely determined by Newton's laws of mechanics, statistical mechanics and Maxwell's theory of electromagnetism. All these branches had in common the deterministic way of thinking. Pierre-Simon Laplace stated that an observer who knows the position and velocity of all masses at a given time could predict everything that will happen in the future. Such an observer who is able to determine all these quantities at a given moment is known as Laplace's demon. From this mechanistic perspective one can see two things which are typical for the leading paradigm in physics before the beginning of the twentieth century: The movement of any mass or particle is completely determined by the initial conditions and time as well as space are only background parameters that are the same for any observer.

Both these ideas were completely overthrown in the following. Einstein's theory of relativity states that space and time are different for different observers. There is no absolute knowledge about these quantities. Furthermore, mass and energy are equivalent and both backreact on space and time: Masses curve the space around them and influence in that way the movement of other masses or even light rays. The theory of relativity is very well tested experimentally, e.g. it was able to explain the mercury precession.

The second fully new theory was quantum mechanics. In contrast to relativity, which deals with very large structures and distances, quantum mechanics is a theory for microscopically small scales. Quantum mechanics was the end of the deterministic way of thinking that dominated hitherto. The uncertainty principle states that it is impossible to measure position and velocity of very small objects at the same time. The more exact one measures one of these two quantities the more uncertain becomes the other. Laplace's demon is in trouble!

Both of these theories – relativity and quantum mechanics – have made an enormous contribution to understand the laws of nature. But there is one real big problem: A full description of gravity must necessarily include quantum effects, but up to now there is no unified description of quantum mechanics and gravity within one theoretical framework. The whole twentieth century seems at large part devoted to the search for a "theory of everything". Such a theory would unify all known fundamental forces, the electromagnetic force, the weak and strong force as well as gravity. The principles of quantum mechanics have successfully been used in the formalism of quantum field theory with which three of the four fundamental forces can be described. The quantum field theory for the electromagnetic interaction is QED (quantum electrodynamics). This was unified with the weak interaction, which acts in processes like the nuclear beta decay, in the Glashow-Salam-Weinberg model. Also the strong force, which describes the interaction of quarks and gluons, can be formulated as a quantum field theory, namely QCD (quantum chromodynamics).

All our knowledge about the non-gravitational interaction of particles is governed by the standard model of particle physics. The standard model has been very successful and it is also very well tested experimentally. All the predicted particles have been found, the latest discovery being a boson that is very likely to be the Higgs boson in 2012.

But we are still lacking a unified theory for all four forces including gravity. One approach for such a theory is string theory. In string theory the particles are replaced by very tiny strings which can vibrate in different modes. The different modes represent different particles. One of those vibrational states is the graviton, the force carrier of the gravitational field. Gravity is inherently included in string theory and moreover, it is also a quantum theory. So is string theory "the theory of everything"? At present we cannot decide on that. String theory is a very complicated object and up to now there is no experimental verification of it. It is even hard to obtain an experimentally testable prediction from string theory. Nevertheless it is a consistent approach for the unification of all fundamental forces and therefore due to this very deep and fundamental aim it is worth working on it. Moreover, there is a large interplay between string theory and mathematics which makes string theory very interesting from a theoretical point of view.

But not only that. In the late 1990ies a duality was found between string theory on a space called AdS space and field theories living on the boundary of this space. This conjecture is called the AdS/CFT correspondence. It opened the possibility to apply string theory, even though in a more indirect way, to strongly coupled field theories. It is hoped that this approach might lead to new insights into condensed matter systems as for instance superconductors.

The work presented in this thesis takes the AdS/CFT correspondence as its starting point. We will especially perform conductivity calculations using AdS/CFT methods and construct black solutions with AdS asymptotics. The presented results have partly been published in [1, 2]. To be more precise, this thesis is organized as follows.

Chapters 2 and 3 are introductory chapters. Here the necessary tools which lay the foundation of our work are presented.

This includes basic facts about black holes and branes. Especially the attractor mechanism for extremal black holes is explained. We introduce this mechanism in the context of $N = 2$ supergravity. The attractor mechanism forces the values of the fields on the horizon to be independent of their asymptotic values and fixes them solely in terms of the charges of the black hole. Furthermore we will give a brief review on the scalar geometry of $N = 2$ $U(1)$ gauged supergravity in four and five dimensions, namely special geometry and very special (real) geometry.

In chapter 3 the AdS/CFT correspondence is introduced and an example is given of how to apply AdS/CFT to compute conductivities in the dual field theory using linear response. In addition, a short introduction to holographic superconductivity is presented. Just in the sense of Einstein's quotation from the beginning we will meet a little bit of art at some point on our way through these two chapters.

In chapter 4 we compute the frequency dependent conductivity tensor for a theory with $SU(2)$ flavor symmetry dual to a four-dimensional black hole solution in Einstein-Yang-Mills theory.

Theories with global $U(1)$ symmetry in the field theory have been used in AdS/CFT to construct toy models for s-wave superconductors holographically. In those approaches the phase transition to the superconducting state is modeled by the development of a black hole with scalar hair in the bulk. Hairy black holes are not allowed to exist in asymptotically flat space due to the no-hair conjecture (cf. e.g. [3]). In spaces which asymptote to AdS, though, it is well possible for a black hole to have hair (cf. e.g. [4]).

Of course, it is interesting to transfer AdS/CFT techniques to theories with non-Abelian symmetry group as such theories play a vital role in very different fields of physics. For instance, the original formulation of the AdS/CFT correspondence, which states the equivalence of IIB string theory on $AdS_5 \times S^5$ and $N = 4$ SYM living on the boundary of AdS_5 in the large N limit [5], involves a global $SU(4)$ symmetry on the field theory side. This is the R-symmetry which rotates to SUSY charges into each other.

Furthermore, black hole solutions with AdS asymptotics have been constructed within four-dimensional Einstein-Yang-Mills theory with $SU(2)$ gauge fields. This setup was used before in [6] to show that a black hole with vector hair occurs below a certain critical temperature. In [7] the frequency dependent conductivity for the current in τ^3 direction in flavor space was computed with AdS/CFT methods. In [7] the authors interpreted the delta peak at zero frequency as well as the appearing frequency gap as a sign of p-wave superconductivity.

In contrast to [7] we consider the frequency dependent conductivity tensor in the normal state. Our calculations involve currents in τ^1 and τ^2 direction in the normal state. We find that for a certain range of frequencies the Hermitean part of the conductivity develops a negative eigenvalue and therefore the entropy production rate turns negative. At the time of writing this thesis we could not find a satisfying explanation of this result. Some possibilities are discussed at the end of chapter 4.

Also in the following chapters we stay within the AdS/CFT context, but now we concern ourselves more with the gravitational part of the AdS/CFT correspondence. Our main field of interest will be to construct black solutions within the framework of $N = 2$ $U(1)$ gauged supergravity in four and five dimensions with AdS asymptotics.

Black geometries play an important role in research. In the 1970ies remarkable analogies between the classical black hole mechanics and the laws of thermodynamics were found. Because of this, it is possible to assign temperature and entropy to a black hole. For example, the entropy is proportional to the horizon area of the black hole. We address this point in chapter 2 in more detail. Some of the most interesting black holes are the extremal black holes. These have a lot of appealing features. E.g. they obey the attractor mechanism and display the striking feature of having zero temperature. We will comment on that more extensively in chapter 2.

Simultaneously, there is another critical way in which black holes play an essential role in string theory. Within the AdS/CFT framework black holes, which are states in the bulk, correspond to thermal ensembles in the dual field theory at the same temperature as the black hole. The dynamics of bulk fields in the black hole background therefore provides information on the interaction of the corresponding operators in the thermal ensemble of the dual field theory.

From this point of view it is particularly important to construct black hole solutions in spaces with AdS asymptotics. Charged black holes with AdS asymptotics were discussed for instance in [8]. These are the so-called AdS Reissner-Nordström black holes. The extremal Reissner-Nordström black holes have the property that their entropy is non-zero though the temperature vanishes [1]. This means that the extremal Reissner-Nordström black hole violates the Nernst law of thermodynamics which states that for zero temperature the entropy is typically also zero. From the AdS/CFT perspective then the dual field theory displays the same behavior, i.e. also on the field theory side the entropy is non-zero at vanishing temperature. This means that many different states are still accessible at $T = 0$ (cf. e.g. [8]), and the interpretation of that is not clear.

The natural question that arises is if there exist black solutions that fulfill the Nernst law. In chapters 5 and 6 we construct such solutions within $N = 2$ $U(1)$ gauged supergravity in four and five dimensions. It is important to mention that one of the features of the solution space of these theories is the existence of horizons with non-spherical topology, such as $\mathbb{R}^2$. These solutions are called black branes. We obtain extremal black brane solutions with zero entropy density in four and five dimensions. We call these solutions *"Nernst branes"* because they fulfill the aforementioned Nernst law.

In chapter 5 we consider $N = 2$ $U(1)$ gauged supergravity in four dimensions with only vector multiplets.

At first we study the equations of motion. The presence of the attractor mechanism halves the degrees of freedom and hence the flow equations are given as ordinary first-order differential equations. In practice, this means that we rewrite the action in terms of squares of first order flow equations. Here we find it convenient to introduce a combination of charges and fluxes whose phase is encoded in a parameter γ (see chapter 5 for the technical details). We also obtain a flow equation for this phase parameter γ. Then we proceed with exploring the solution space of the flow equations.

We first choose the simplest prepotential[2] encoding one complex vector multiplet scalar field, given by $F = -i X^0 X^1$, and obtain a class of solutions representing non-Nernst black branes. These turn out to be generalizations of the solution discussed in [9]. This serves as a consolidating check on the formalism. Furthermore, we show a full numerical solution that interpolates between a near-horizon $AdS_2 \times \mathbb{R}^2$ geometry and an asymptotic AdS_4 geometry for the prepotential $F = -\frac{(X^1)^3}{X^0}$. All these solutions have constant γ. We also find solutions with flowing γ but these turn out to be singular. It might be possible, though, that they describe the asymptotic region of a global solution once higher derivative corrections are taken into account.

We finally pursue the question of finding Nernst brane solutions. For this purpose, we consider the STU model which is based on the prepotential $F(X) = -\frac{X^1 X^2 X^3}{X^0}$. We only look at solutions for which the physical scalars have vanishing real part. We find a solution with constant γ and a fixed point at the zero of the radial coordinate. At this point, one of the scalars flows to zero. Moreover, at this point, the metric has a coordinate singularity and the area density in the constant time and constant radial coordinate hyperplane vanishes. The geometry near this fixed point has an infinitely long radial throat which

[1] This is also true for the charged black holes in flat space and not only in asymptotically AdS space. You find more information on Reissner-Nordström black holes in the introductory chapters 2 and 3.

[2] In chapter 2 it is shown how prepotentials arise within $N = 2$ supergravity.

suppresses fluctuations in the scalar fields such that their solutions become independent of their asymptotic values. We will take this infinite throat property to mean that the solution is extremal (cf. chapter 2), and the vanishing of the area density to mean that the solution has zero entropy density. However, these solutions asymptote to geometries that are not AdS_4 and, thus, it is not clear which role they might play in the gauge/gravity correspondence.

Because of these difficulties in finding Nernst geometries with AdS asymptotics in four dimensions, we then shift focus to gauged supergravity in five dimensions with the hope of finding Nernst solutions in asymptotically AdS_5 space.

Chapter 6 is devoted to the construction of five-dimensional black solutions with flat horizon in $N = 2$ $U(1)$ gauged supergravity. As in four dimensions we begin by rewriting the action in terms of squares of first-order flow equations. We also check that it is possible to derive the first order equations from a superpotential. In addition the relation between the five-dimensional first order equations and the four-dimensional ones obtained in chapter 5 is discussed.

The solutions we construct include Nernst solutions in asymptotic AdS_5 backgrounds as well as non-Nernst black brane solutions that describe extremal BTZ$\times\mathbb{R}^2$-solutions.

We find exact solutions with constant scalar fields that may have magnetic fields and rotation. These solutions do not carry electric charge. We construct extremal BTZ$\times\mathbb{R}^2$ solutions that are supported by magnetic fields, as well as rotating Nernst geometries in asymptotic AdS_5 backgrounds. Then we obtain numerical solutions with non-vanishing scalar fields, with and without rotation. These have BTZ$\times\mathbb{R}^2$ near horizon geometry and are asymptotically AdS_5. They constitute generalizations of a solution given in [10] to the case with several running scalar fields and rotation. We can immediately compute the entropy density of the BTZ$\times\mathbb{R}^2$ black brane by using the Cardy formula of the dual CFT. A salient aspect of the first-order rewriting that gives rise to these black branes is the fact that the angular momentum, the electric quantum numbers and the magnetic fields are organized into quantities which are invariant under the spectral flow of the theory (cf. [11] for the ungauged case).

In appendix E we turn to a different first-order rewriting in five dimensions. This is motivated by the search for solutions with electric fields. This rewriting is based on the one performed in [12] for static black hole solutions, which we adapt to the case of stationary black branes in the presence of magnetic fields. The resulting first-order flow equations allow for the non-extremal black brane solutions constructed in [13], as well as for the extremal electric solutions obtained in [14, 10]. So far we could not obtain any dyonic solution.

Chapter 2

Black Holes and Attractors

"Now there's a look in your eyes like black holes in the sky..."

Pink Floyd, "Shine on you crazy diamond"

As the quotation shows the terminus "black hole" has meanwhile become part of the popular culture. Indeed it is not surprising that these physical objects generate interest also outside the scientific community. Losely speaking, a black hole is an object whose gravitational forces are so strong that not even light can escape from it. Everything that passes a special surface, the event horizon, is doomed to fall into a spacetime singularity. On the way to the singularity the gravitational tidal forces will become larger and larger so that finally everything is torn into pieces – no wonder that this dramatic and frightening picture has made its way out to the public and did not stay hidden behind the "horizon" of science.

However, a lot of work has been done in this fruitful field of research from experimental physics even to pure mathematics. It is known that most galaxies have a black hole in their centre whose mass is typically about 0.1 per cent of the mass of the galaxy. Recently it was discovered that in the centre of the galaxy NGC 1277 there is a black hole that contains 59 per cent of the galaxy's mass which is very unusual [15]. Apart from the experimentalists' search for observational data concerning real existing black holes, black holes provide an interesting field for theorists to work on. We will now give some background for the theoretical and mathematical description of black holes.

The first black hole solution to the vacuum Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$ was found by the German physicist Karl Schwarzschild [16]. We will use this solution as a toy model to present important concepts of black holes. All this can be found in textbooks on general relativity like e.g. [3, 17, 18]. The line element for a Schwarzschild black hole with mass M is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.1}$$

with time coordinate t, radial coordinate r and $d\Omega^2$ the area element of the unit two-sphere. At $r = 2M$ the metric becomes singular, since g_{tt} vanishes and g_{rr} diverges. This surface where g_{tt} vanishes is called the event horizon. So what happens to an observer moving towards the event horizon? The answer is: Nothing special!

It turns out that the singularity is merely a coordinate singularity which can be removed by choosing appropriate coordinates. This means that an observer passing the event horizon will not die from very large tidal forces – at least not yet. However, this does not mean that the event horizon is not useful for our description of black holes. The horizon is indeed a surface which is special, and it can also be defined independently of the chosen coordinate system. For $r > 2M$ surfaces of constant r are timelike while for $r < 2M$ they are spacelike. For $r = 2M$ it is a null surface. This is an important feature to characterize the horizon independently of the chosen coordinates: A horizon is a null surface. Moreover, the horizon does not change in time – it is static. More generally, for a surface to be an event horizon it is even enough that it is stationary, i.e. invariant under the timelike Killing vector $\frac{\partial}{\partial t}$. The latter case includes also Kerr black holes which are rotating. To summarize, a horizon is a stationary null surface (cf. e.g. [19, 20] which are nice reviews of many of the concepts to be discussed here). We use the conventions of [19].

The line element (2.1) has another singularity at $r = 0$. In contrast to the coordinate singularity at $r = 2M$, the one at $r = 0$ cannot be removed by changing coordinates. It is really a singularity of the spacetime. For r tending to zero the curvature of the spacetime keeps on growing and the tidal forces diverge. Fortunately, this really bad singularity is hidden behind the event horizon. What happens inside the black hole has no effect on the world outside. This can be seen when one considers a light ray propagating radially outwards from the black hole horizon. An observer far away from the black hole would state that it takes an infinite amount of time for this light ray to reach him[1]. But this means that no information from the inside of the black hole can reach an observer outside the black hole. Such an observer does not need to worry about the spacetime singularity. This statement is governed by the cosmic censorship conjecture: There are no naked singularities, i.e. spacetime singularities without an event horizon hiding them in our universe (see e.g. [22] for a discussion).

After having defined the horizon of a black hole, let us introduce one further quantity, the surface gravity. In Newtonian mechanics this is just the gravitational force a unit test mass feels at the surface of an object. But since Newtonian mechanics is not the appropriate theory to describe black holes, a different definition of the surface gravity is needed. Consider an observer at infinity and a unit test mass closer to the black hole. The observer at infinity will measure a certain force that is needed to keep the test mass at rest. The surface gravity is then the limit of this force when the test mass approaches the horizon.

For the Schwarzschild black hole the surface gravity is simply given by (cf. e.g. [20])

$$\kappa = \frac{1}{4M} \,. \tag{2.2}$$

A mathematically rigorous definition of the surface gravity for the case of an arbitrary stationary black hole can be found e.g. in [17].

[1] Calculations can be found e.g. in [21].

Schwarzschild's black hole solution is of course not the only one. It is the simplest possible black hole solution since it depends only on one parameter, namely the mass of the black hole. But there are other black hole solutions that are charged and / or rotating. Charged black holes are described by the Reissner-Nordström metric. We will focus on them a bit later. Rotating black holes are the Kerr black holes we have already mentioned. They give rise to interesting research topics even in mathematics (see e.g. [23] for a mathematical introduction to this topic). For instance, S.T. Yau et. al. examined the stability of the Kerr black hole against perturbations of spin $s \in \{0, \frac{1}{2}, 1, 2\}$. There are rigorous mathematical proofs concerning scalar perturbations which are based on methods from functional analysis and differential equations. Yau et. al. were able to prove that solutions to the scalar wave equation in the Kerr geometry decay rapidly enough which shows stability of the Kerr black hole. An overview of their results also concerning higher spin perturbations can be found in [24] and the references therein.

As was already mentioned in the introduction, another remarkable feature of black holes is that the laws of black hole mechanics are very similar to the laws of thermodynamics. These ideas were presented for the first time in [25].

- Zeroth law: A body in thermodynamical equilibrium has constant temperature. The analog for black holes is that the surface gravity is constant at the horizon of a stationary black hole.

- First law: This is energy conservation. In thermodynamics the first law reads

$$dE = TdS + \mu dQ + \Omega dJ\,. \tag{2.3}$$

  For black holes we have

$$dM = \frac{\kappa}{8\pi}dA + \mu dQ + \Omega dJ\,. \tag{2.4}$$

  Here Q is the charge, μ the chemical potential, Ω the angular velocity and J the angular momentum. One can see that the surface gravity κ plays the role of temperature which is not surprising in view of the zeroth law. The area of the black hole horizon is in correspondence with the entropy which is also stated in the second law.

- Second law: In a thermodynamic process the total entropy S can never decrease, $\Delta S \geq 0$. Similarly, the area A of a black hole can never decrease, $\Delta A \geq 0$. This means it is impossible that a black hole splits into two smaller ones. By contrast the reversed process is not forbidden by the laws of black hole mechanics.

Are these merely analogies or is there a deeper connection between thermodynamics and black holes? To answer this question let us consider a system with large entropy like the desk of a hard working physicist with a lot of papers lying on it. What happens when we throw this desk into a black hole? Then the entropy of our universe would decrease in contradiction to classical thermodynamics. From considerations like this Bekenstein concluded that black holes have to have entropy to ensure that the entropy in the universe cannot decrease [26]. Having energy and entropy a black hole also has a temperature (cf. e.g. [20]) by virtue of

$$\frac{1}{T} = \frac{\partial S}{\partial E} \,. \tag{2.5}$$

Stephen Hawking showed in [27] that the temperature of a black hole is related to its surface gravity precisely by

$$T = \frac{\kappa}{2\pi} \,. \tag{2.6}$$

From all this the famous formula for the Bekenstein-Hawking entropy of a black hole follows:

$$S = \frac{A}{4} \,. \tag{2.7}$$

Now that we have learned about horizons and the laws of black hole mechanics we want to discuss the four dimensional Reissner-Nordström black hole in asymptotically flat space in more detail. This will provide us with tools we need for the later consideration of more complicated cases. We will follow [21] and [20]. The Reissner-Nordström black hole is a solution to the equations of motion derived from the Langrangian density

$$\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{2.8}$$

with the electromagnetic field strength given by

$$F = P\sin\theta d\theta \wedge d\phi + \frac{Q}{r^2}dt \wedge dr \,. \tag{2.9}$$

Here Q and P are the electric and magnetic charge, respectively. The line element for the Reissner-Nordström black hole reads

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{P^2+Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{P^2+Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2 \,. \tag{2.10}$$

One can see immediately that in the uncharged case, $P = Q = 0$, this reduces to the Schwarzschild solution. Just like the Schwarzschild metric also the Reissner-Nordström metric contains a spacetime singularity at $r = 0$. In contrast to the Schwarzschild case now we have in general two coordinate singularities since g_{tt} contains a quadratic term in r. This yields two horizons, an inner and an outer one with radii given by

$$r_\pm = M \pm \sqrt{M^2 - (P^2+Q^2)} \,. \tag{2.11}$$

The event horizon is then located at the larger root, i.e. at $r = r_+$. We only get real solutions for $M^2 \geq P^2 + Q^2$, otherwise there would be solutions with an imaginary part different from zero. The latter correspond to naked singularities which are considered unphysical due to the cosmic censorship conjecture. So we obtain that the mass of the Reissner-Nordström black hole is bounded from below. For $M^2 = P^2 + Q^2$ there is only one real solution and the two horizons coincide. Such black holes are called *extremal*.

Extremal black holes have a number of interesting properties. The temperature and entropy of the Reissner-Nordström black hole are given by (cf. e.g. [20])

$$\begin{aligned} T &= \frac{\sqrt{M^2-(P^2+Q^2)}}{2\pi(2M(M+\sqrt{M^2-(P^2+Q^2)})-(P^2+Q^2))}\,, \\ S &= \pi r_+^2 = \pi(M+\sqrt{M^2-(P^2+Q^2)})^2\,. \end{aligned} \tag{2.12}$$

One can see immediately that the temperature of an extremal Reissner-Nordström black hole is zero. This is also true in more general cases than Reissner-Nordström. Extremal black holes have vanishing temperature and because of this they are thermodynamically stable at the classical level. Further, the entropy of the Reissner-Nordström black hole is non-zero at the horizon and its value is determined only in terms of the black hole charges.

Furthermore the horizon of an extremal black hole lies at the end of an infinitely long throat. To see this we need to have a closer look at the near horizon geometry of the Reissner-Nordström black hole. With the coordinate change

$$\rho = r - M \tag{2.13}$$

the extremal Reissner-Nordström black hole takes the form

$$ds^2 = -\left(1+\frac{M}{\rho}\right)^{-2} dt^2 + \left(1+\frac{M}{\rho}\right)^2 (d\rho^2 + \rho^2 d\Omega^2)\,. \tag{2.14}$$

These coordinates are called isotropic (cf. e.g. [20, 28]). Now the tt-component of the metric has a double zero at $\rho = 0$ and therefore in isotropic coordinates the horizon is located there. Near the horizon, i.e. for $\rho \to 0$, the line element can be written as

$$ds^2 = -\frac{\rho^2}{M^2}dt^2 + \frac{M^2}{\rho^2}d\rho^2 + M^2 d\Omega^2\,. \tag{2.15}$$

One can see that in this particular limit the spacetime is a direct product of a two-sphere of radius M and a two-dimensional spacetime with metric element

$$d\tilde{s}^2 = -\frac{\rho^2}{M^2}dt^2 + \frac{M^2}{\rho^2}d\rho^2\,. \tag{2.16}$$

This is the line element of the two-dimensional Anti-de-Sitter space AdS_2. Anti-de-Sitter spaces play an important role in string theory and we will give further information on them in section 3.2. But let us have a closer look at the line element (2.16). The radial distance d of an observer at $\rho_0 > 0$ to the horizon is (up to overall constant prefactors) determined by (cf. e.g. [28])

$$d \sim \lim_{\epsilon\to 0}\int_{\epsilon}^{\rho_0} \frac{1}{\rho}d\rho = \lim_{\epsilon\to 0}(\log(\rho_0) - \log(\epsilon)) = \infty. \tag{2.17}$$

In this calculation we assumed that is it possible to use the near-horizon approximation (2.16). We see, that the horizon is infinitely far away from the observer.

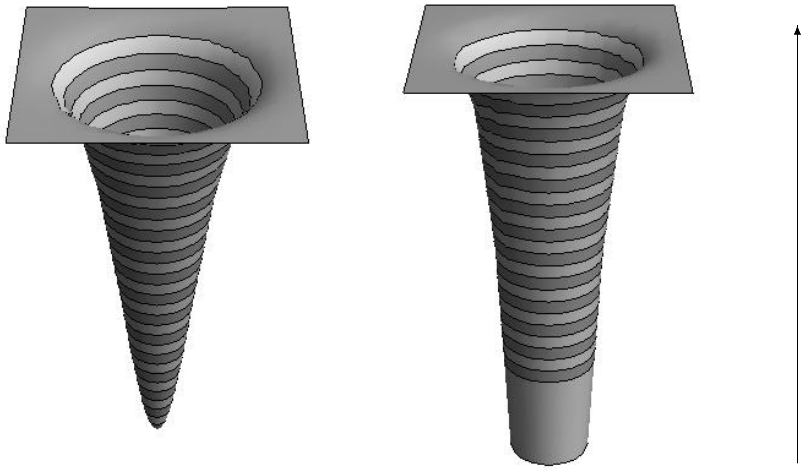

Figure 2.1: Non-extremal and extremal black hole. The extremal one on the right displays an infinitely long throat.

But this means that the horizon lies at the end of an infinite throat (cf. e.g. [20]). An analogous calculation shows that the situation is different for the non-extremal Reissner-Nordström black hole. Here the event horizon is at finite distance from an observer at finite ρ_0 (cf. [21]). You find an illustration of the two different situations in figure 2.1. This property of having a throat geometry will be very important in chapter 5 where we will use this property to distinguish whether a black solution is extremal or not.

In chapter 5 and 6 we will also look at horizon geometries including an AdS_2 factor but with horizon topology $AdS_2 \times \mathbb{R}^2$ instead of $AdS_2 \times S^2$. Such black solutions with non-spherical but flat horizon topology are called *black branes*. Such objects cannot exist in asymptotically flat space as is known from statements on the uniqueness black hole constructions (cf. e.g. [29, 30, 31]). In a space with asymptotic AdS behavior it is well possible for a black object to have non-spherical horizon topology. Therefore black branes are especially important in the context of the AdS/CFT correspondence.

2.1 The attractor mechanism

For our purpose of studying black brane solutions another interesting feature of extremal black configurations is important, namely the attractor mechanism. Early work on this subject can be found for instance in [32, 33, 34, 35, 36, 37, 32, 38]. An overview is given e.g. in [21]. In general, to study the attractor mechanism black solutions in supergravity are considered. In chapters 5 and 6 we will look at the four- and five-dimensional case of $N = 2$ supergravity, so we will restrict ourselves to these particular dimensions. For the introduction of the attractor mechanism we will consider the four-dimensional case for simplicity. The chosen supergravity theory always contains a set of scalar fields, and the general idea of the attractor mechanism is that at the horizon of the black hole the scalar fields included in the theory flow towards a value that is independent of the initial conditions far away from the horizon. This value is completely determined by the charges of the black hole. In the following we will present the basic features of the attractor mechanism. Here we stick closely to [21, 39]. The scalar fields of the supergravity under consideration typically have an effect on the couplings of the vector fields in the theory. The bosonic part of an action describing such a theory in four dimensions is given by (cf. e.g. [21])

$$\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{4}\mathcal{I}_{IJ}(\phi)F^I_{\mu\nu}F^{J\,\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}(\phi)\frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}F^I_{\mu\nu}F^J_{\rho\sigma}\right) . \quad (2.18)$$

Here $I, J = 0, \ldots, n$ denotes the number of vector fields and $i, j = 1, \ldots, n_S$ the number of scalars. These numbers are specified when choosing a specific model. The matrix $\mathcal{I}$ is negative definite and describes the gauge kinetic couplings and it depends on the scalar fields ϕ^i. The matrix $\mathcal{R}$ is a generalization of the θ-angle terms (cf. e.g. [21]). In a more general case one can also look at gauged supergravity. Then there is an additional scalar potential in the Lagrangian as we will see in chapters 5 and 6. For the purpose of explaining the attractor mechanism we can set the scalar potential to zero.

The scalar fields can be viewed as coordinates on a manifold with metric g_{ij}. In four dimensions and $N = 2$ supergravity this manifold has some interesting features: It is a special Kähler manifold. We will address this point later and give a precise definition and some properties of special Kähler spaces.

But for the moment the metric can be kept arbitrary and we will turn to the more specific case of $N = 2$ supergravity later.

In [21] an ansatz is made for the metric of a static spherically symmetric and charged black hole solution, namely

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}dr^2 + r^2 d\Omega^2 \quad (2.19)$$

with $d\Omega^2$ the line element of a two-sphere. Since all quantities U and ϕ^i are assumed to depend only on the radial variable r the action (2.18) can be reduced to an effective 1-dimensional action [21]

$$\mathcal{L}_{1d} = (U')^2 + \frac{1}{2}g_{ij}(\phi^i)'(\phi^j)' + e^{2U}V_{BH} . \quad (2.20)$$

Here V_{BH} is the so-called black hole potential which is given by [40]

$$V_{BH} = -\frac{1}{2}Q^T\mathcal{M}Q \quad (2.21)$$

with the matrix

$$\mathcal{M} = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} \quad (2.22)$$

and Q the charge vector

$$Q = \begin{pmatrix} p^I \\ q_I \end{pmatrix} \tag{2.23}$$

obtained from the vector field strengths and their duals by

$$p^I = \frac{1}{4\pi} \int_{S^2} F^I \,, \quad q_I = \frac{1}{4\pi} \int_{S^2} G_I \,. \tag{2.24}$$

The sphere S^2 is taken to be at infinity and the dual field strength G_I is related to F^I through [21]

$$G_I = -\frac{\delta \mathcal{L}}{\delta F^I} = \mathcal{R}_{IJ} F^J - \mathcal{I}_{IJ} \star F^J \,. \tag{2.25}$$

The equations of motion for the metric function U and the scalar fields are

$$\begin{aligned} U'' &= e^{2U} V_{BH} \,, \\ (\phi^i)'' + \Gamma^{\,i}_{jk} (\phi^j)'(\phi^k)' &= e^{2U} g^{ij} \partial_j V_{BH} \,. \end{aligned} \tag{2.26}$$

Here Γ^i_{jk} are the Christoffel symbols. To make sure that solutions to the equations of motion coming from the 1-dimensional effective action solve also the original 4-dimensional equations of motion one has to implement the Hamiltonian constraint (cf. e.g. [21])

$$\mathcal{H} = 0 \iff (U')^2 + \frac{1}{2} g_{ij} (\phi^i)'(\phi^j)' - e^{2U} V_{BH} = 0 \,. \tag{2.27}$$

At this point we are able to establish the attractor mechanism. The equations (2.26) describe a dynamical system in the radial variable r, and thus one can ask if they give rise to an attractor. To define an attractor in a rigorous mathematical way, one has to convert the given dynamical system to a system of first-order differential equations, which is always possible. This means one writes the higher order differential equations in the form $\frac{d}{dr}\vec{x}(r) = \vec{f}(\vec{x}(r), r)$ (cf. e.g. [41]). Here $\vec{x}$ is the vector $(U, U', \phi^1, (\phi^1)', \dots)^T$. The dots stand for the other functions that occur in the given system of second-order differential equations. $\vec{f}$ describes how the system of first-order differential equations explicitly looks like. It may also depend explicitly on r.

With the notion of the flux Φ_r which maps the initial conditions $\vec{x}_0$ to $\vec{x}(r)$ one can define an attractor in the following way.

Definition. An attractor is a compact subset A of the domain of definition of the function $\vec{f}$ such that

1. A is invariant under the flux Φ_r, i.e. $\Phi_r(A) \subset A$ and

2. there exists an open neighbourhood U of A that is invariant under Φ_r and has the additional property that for any neighbourhood V of A with $V \subset U$ there is a value r_V such that $\Phi_r(U) \subset V$ for any $r > r_V$.

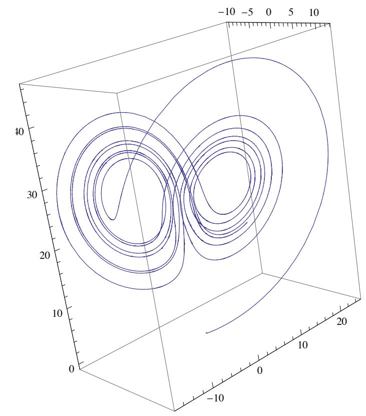

Figure 2.2: The Lorenz attractor

Furthermore, it is often postulated that the attractor set A is connected [41]. The second property explains the "attractivity" of the set A. Points close enough to A are mapped by Φ_r to points that are even closer to A. This means that while the system evolves the points in U finally reach the attractor A and can never escape from it again. The largest such set U with the property mentioned in the second part of the definition is called the basin of attraction. For initial conditions within this set the solution eventually flows to the attractor set.

For general dynamical systems attractors can exhibit a very complicated structure, as e.g. the Lorenz attractor does. It was found by the meteorologist Edward Lorenz when he studied the convection and heat distribution on a rectangular area (cf. e.g.[42]). This attractor is an example for a so-called strange attractor. It is a fractal, i.e. a geometrical object with non-integer dimension and properties like self-similarity (cf. e.g. [42]). You find a picture of the Lorenz attractor in figure 2.2.

On the other hand, attractors can also be very simple objects, as is the case for black holes. Here the attractor is only a single point in the domain of definition for the radial variable r. It is a fixed point of the dynamical system. The expression "fixed point" is to be understood in the following way. We could consider the sequence of points $\{\vec{x}(r_n)\}_{n\in\mathbb{N}}$ for appropriately chosen values[2] r_n. Then one can look at the map G that takes $\vec{x}(r_n)$ to $\vec{x}(r_{n+1})$. Reaching a point attractor means that the map G has a fixed point, i.e. $G(\vec{x}^*) = \vec{x}^*$ and the attractor is only the single point $\vec{x}^*$. Pointlike attractors occur e.g. in the presence of a potential that displays a minimum. A very simple example is a damped spring pendulum. The potential is quadratic in the amplitude and has a minimum at the

[2]There are several techniques of how to choose this sequence, but since this is not that important for understanding the main principles it will be omitted here.

equilibrium length l_0 of the spring. For any initial condition the amplitude finally tends to the equilibrium length l_0 and the velocity of the pendulum mass goes to zero. In this case the vector $(x, x')^T = (l_0, 0)^T$ is the attractor point.

Since in our case (2.26) there is also a potential we can ask if there is a similar mechanism at work. This means actually that we have to look for minima of the black hole potential. If we assume that the black hole potential has a minimum at ϕ^{i*} then we can conclude from the second equation in (2.26) that $\phi^i(r) = \phi^{i*} = const$ is a solution to the equations of motion for the ϕ^i since all derivatives vanish in this case. this means $\begin{pmatrix} \phi^i(r) \\ (\phi^i)'(r) \end{pmatrix} = \begin{pmatrix} \phi^{i*} \\ 0 \end{pmatrix}$ is a fixed point of the dynamical system. Since we chose the ϕ^{i*} such that the potential is minimized we can conclude that the fixed point is an attractor just like in the case of the spring pendulum mentioned before.

Apart from the dependence on the scalar fields ϕ^i the only expressions that enter the black hole potential are the electric and magnetic charges. This implies that the values ϕ^{i*} depend only on the charges of the black hole. This is also a way to characterize the horizon of an extremal black hole: The horizon lies at that radial point where the critical values ϕ^{i*} are reached that minimize the black hole potential (cf. e.g. [21]). To summarize, the scalar fields are driven to constant values at the horizon that are determined completely by the charges of the black hole and that do not depend on the initial conditions for the ϕ^i (here it is assumed that the basin of attraction corresponds to the maximal domain of definition for the ϕ^i, the appearance of different basins of attraction was e.g. discussed in [43, 44]).

The question that naturally arises at this stage is how do we know about the existence of minima of the black hole potential? An answer is given in [45]. To understand the main argument in [45] we need to relate the black hole potential to the central charge Z of the supergravity theory under consideration. We will not discuss the most general case but instead only give the argument for $N = 2$ theories in four dimensions. In this case the central charge of the underlying supersymmetry algebra is the related to the black hole potential via (cf. e.g. [21])

$$V_{BH} = |Z|^2 + 4g^{i\bar{j}}\partial_i|Z|\bar{\partial}_{\bar{j}}|Z| \tag{2.28}$$

with $g_{i\bar{j}}$ the Kähler metric on the moduli space (see also section 2.3). In this case the more general Lagrangian density (2.20) can be written in the form [21]

$$\mathcal{L} = (U')^2 + g_{i\bar{j}}(z^i)'(\bar{z}^{\bar{j}})' + e^{2U}(|Z|^2 + 4g^{i\bar{j}}\partial_i|Z|\bar{\partial}_{\bar{j}}|Z|)\,. \tag{2.29}$$

From now on the scalars will be denoted by z^i instead of ϕ^i to depict the restriction to 4-dimensional $N = 2$ supergravity. In [21] it is proven that a critical point of $|Z|$, i.e. a point r^* with $\partial_i|Z(r^*)| = 0$, is also a critical point of the black hole potential. Moreover, it can be shown that values of the radial variable for which the central charge is minimized also yield minima of the black hole potential. This means that the existence of minima of the central charge implies the existence of minima of the black hole potential. In [45] conditions for the existence of minima of the central charge are discussed. In theorem 2.5.1.

of [45] it is stated that a non-zero minimum of $|Z|^2$ (which implies a minimum of $|Z|$ since the map $|Z|^2 \mapsto |Z|$ is strictly increasing) exists under certain conditions on the charges of the black hole. This non-zero minimum of the central charge exists if the electric and magnetic charges are integral and if there exists a number $C \in \mathbb{C}$ such that [45]

$$\begin{aligned} \bar{C}X^I - C\bar{X}^I &= ip^I \,, \\ \bar{C}F_I - C\bar{F}_I &= iq_I \,. \end{aligned} \tag{2.30}$$

Here X^I are coordinates on the so-called big moduli space and for the cases which are important for us F_I are dericatives of a function called the prepotential. At this point we will not go into the details. We will provide more information on the X^I and F_I in section 2.3. For the moment the most important point in this discussion is that for a given set of black hole charges with "good enough" properties we know that there is an attractor of the flow equations located at the black hole horizon. Further, the attractor values of the scalar fields and of the black hole potential are completely fixed by the black hole charges.

Another important feature of extremal black hole solutions is that they admit a description in terms of first-order flow equations. In four-dimensional ungauged supergravity these equations were first constructed and solved for black holes in [32, 33]. As mentioned before, solutions to the equations of motion for the Lagrangian density (2.20) have to obey the Hamiltonian constraint (2.27)

$$(U')^2 + g_{i\bar{\jmath}}(z^i)'(\bar{z}^{\bar{\jmath}})' = e^{2U}(|Z|^2 + 4g^{i\bar{\jmath}}\partial_i|Z|\bar{\partial}_{\bar{\jmath}}|Z|) \,. \tag{2.31}$$

A very simple ansatz to fulfill the Hamiltonian constraint is to equate the appearing squares in an appropriate way:

$$\begin{aligned} U' &= \pm e^U|Z| \,, \\ (z^i)' &= \pm 2e^U g^{i\bar{\jmath}}\bar{\partial}_{\bar{\jmath}}|Z| \,. \end{aligned} \tag{2.32}$$

It can be shown that also the second-order equations of motion are satisfied if one chooses the same sign in both equations (cf. [21]). In order to reproduce the extremal Reissner-Nordström black hole in flat space one has to choose the minus sign in (2.32). The extremal Reissner-Nordström black hole has the interesting property of being supersymmetric. We will comment on that later in this section. Thus, taking the minus sign in (2.32) leads to supersymmetric solutions and results in the first-order flow equations

$$\begin{aligned} U' &= -e^U|Z| \,, \\ (z^i)' &= -2e^U g^{i\bar{\jmath}}\bar{\partial}_{\bar{\jmath}}|Z| \,. \end{aligned} \tag{2.33}$$

Moreover, the Lagrangian density (2.29) can be rewritten as a sum of these first-order flow equations up to a total derivative:

$$\mathcal{L} = (U' + e^U|Z|)^2 + g_{i\bar{\jmath}}((z^i)' + 2e^U g^{i\bar{k}}\bar{\partial}_{\bar{k}}|Z|)((\bar{z}^{\bar{\jmath}})' + 2e^U g^{\bar{\jmath}l}\partial_l|Z|) + TD \,. \tag{2.34}$$

The variation of the Lagrangian density written in the form (2.34) vanishes if the first-order differential equations (2.33) are satisfied. To summarize, the solutions to the first-order equations fulfill the Hamiltonian constraint and therefore also the second-order equations. Of course, the converse need not be true: There might be solutions to the second-order equations that do not obey the first-order equations. Inspite of this there is a great advantage of having first-order flow equations instead of second-order ones. Generally speaking first-order differential equations are easier to solve than second-order equations since we only have to specify half the number of boundary conditions. In combination with the attractor mechanism the appearance of first-order equations can be interpreted as follows. As the horizon values for the scalars and their derivatives are completely determined by the black hole charges the number of boundary conditions one needs to specify a solution is already reduced. Therefore first-order equations provide enough information to fix a solution completely.

As you can see in (2.33) the first-order equations in this example can be written in terms of derivatives of the central charge. In more general cases it is possible to express the first-order equations in terms of a so-called *superpotential* instead of the central charge. Also in this case the flow equations are determined by derivatives of the superpotential. We will see an example of a superpotential in chapter 6.

So far we have seen how the attractor mechanism works and also how to rewrite the equations of motion as first-order differential equations.

Since the attractor mechanism "halves" the number of free parameters (i.e. the number of boundary conditions) we might ask if there is a connection to BPS states. BPS states in supersymmetry are states that are invariant under a certain number (usually half or a quarter) of supersymmetry transformations. We will have a closer look at BPS states in the next section.

2.2 Black holes as BPS states

BPS states are very important in supersymmetry. Given a theory with $N \geq 1$ supersymmetries there are N spinorial charges Q^A_α with $A = 1, \dots N$ and $\alpha = 1, 2$. Due to the theorem of Haag, Lopuzanski and Sohnius (cf. [46]) the most general SUSY algebra is then (cf. e.g. [47], [48])

$$\begin{aligned} \{Q^A_\alpha, (Q^B_\beta)^\dagger\} &= 2\sigma^\mu_{\alpha\beta} P_\mu \delta^{AB} \,, \\ \{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta} Z^{AB} \,. \end{aligned} \tag{2.35}$$

Here P_μ is the generator of translations, $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\sigma^\mu = (-\mathbb{1}_2, \sigma_i)$ with σ_i the Pauli matrices. Z^{AB} is the central charge matrix. We want to consider massive representations in the case $N = 2$. Here we are left with only one central charge Z^{12}. In the rest frame we have $P_\mu = (-M, 0, 0, 0)$ and the algebra (2.35) simplifies to (with the notation conventions used in [47])

$$\begin{aligned}\{Q^A_\alpha, (Q^B_\beta)^\dagger\} &= 2M\delta_{\alpha\beta}\delta^{AB}\,,\\ \{Q^A_\alpha, Q^B_\beta\} &= |Z^{12}|\epsilon_{\alpha\beta}\epsilon^{AB}\,.\end{aligned} \tag{2.36}$$

Via $U(2)$ transformations applied to the Q^A_α the central charge matrix can be brought into the form

$$Z^{12} = \begin{pmatrix} 0 & 2|Z| \\ -2|Z| & 0 \end{pmatrix} \tag{2.37}$$

with $2|Z| = |Z^{12}|$. One can then define creation and annihilation operators by taking appropriate linear combinations of the SUSY charges Q^A_α (cf. e.g. [48]):

$$\begin{aligned}a_\alpha &= \frac{1}{\sqrt{2}}\left(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right)\,,\\ b_\alpha &= \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right)\,.\end{aligned} \tag{2.38}$$

These operators fulfill the anticommutation relations

$$\begin{aligned}\{a_\alpha, a^\dagger_\beta\} &= 2(M - |Z|)\delta_{\alpha\beta}\,,\\ \{b_\alpha, b^\dagger_\beta\} &= 2(M + |Z|)\delta_{\alpha\beta}\,.\end{aligned} \tag{2.39}$$

The left hand side of the first equation is a positive semi-definite operator, so also the right hand side has to be positive semi-definite. This leads to the so-called BPS bound

$$M \geq |Z|\,. \tag{2.40}$$

If $M > |Z|$ we have $2N = 4$ creation operators and one can create $2^{2N} = 16$ states starting from a state with minimal helicity. When the BPS bound is saturated, we have to set the operators a_α, $\alpha \in \{1, 2\}$ to zero and so one can only create only $2^{2(N-1)} = 4$ different states starting from a given helicity. This procedure can easily be generalized to theories with more than two SUSY charges. Then the central charge matrix is block-diagonal with each block of the form (2.37) and one defines creation and annihilation operators in analogy to (2.38). For $N > 2$ there are more possibilities of how many creation operators have to be set to zero in the BPS case. More precisely, it is possible that $k \leq \frac{N}{2}$ of the creation operators have to be set to zero (cf. e.g. [48]). It is important to mention that BPS states are the states with minimal mass for a given central charge, i.e. for a given set of quantum numbers.

So far we have discussed BPS states in supersymmetry, but since we want to interpret black holes as BPS states, we need a notion of BPS states in a gravity theory and the theory at hand is (not surprisingly) supergravity. Supergravity is the local version of supersymmetry, i.e. all supersymmetry transformations that were so far described by constant (spinorial) parameters ϵ are now described in terms of parameters $\epsilon(x)$ that depend on the spacetime coordinates. To understand this better we first have to take a step backwards and look how SUSY transformations can be realized on fields. Here it is useful to enlarge the space on which the SUSY fields live by anticommuting Grassmannian coordinates θ and $\bar{\theta}$. This space with coordinates $(x^\mu, \theta, \bar{\theta})$ (with x^μ the usual spacetime coordinates) is called superspace and fields depending on these coordinates are therefore named superfields. The idea is now to realize the SUSY generators Q_α and their conjugates as differential operators on superspace. This means that $i\epsilon^\alpha Q_\alpha$ is supposed to generate a translation in the Grassmannian θ-direction by a constant infinitesimal spinor ϵ and in addition a translation in spacetime [48].The explicit form of the differential operator Q_α is (cf. [49])

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\beta}\bar{\theta}^\beta\partial_\mu\,. \tag{2.41}$$

Then the transformation law for a superfield F is [49]

$$\delta_\epsilon F(x^\mu, \theta, \bar{\theta}) = (\epsilon Q + \bar{\epsilon}Q^\dagger)F\,. \tag{2.42}$$

The operator $\epsilon Q + \bar{\epsilon}Q^\dagger$ generates a coordinate shift by the infinitesimal parameter ϵ:

$$(x^\mu, \theta, \bar{\theta}) \mapsto (x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon})\,. \tag{2.43}$$

In supergravity the transformation parameters ϵ may depend on the spacetime coordinates, i.e. now one wants to take into account motions in superspace of the form [49]

$$(x^\mu, \theta, \bar{\theta}) \mapsto (x^\mu + i\theta\sigma^\mu\bar{\epsilon}(x) - i\epsilon(x)\sigma^\mu\bar{\theta}, \theta + \epsilon(x), \bar{\theta} + \bar{\epsilon}(x))\,. \tag{2.44}$$

One can see here that the infinitesimal shift in the coordinates depends on the spacetime position x^μ and so a coordinate transformation is generated that might be not only a translation. Since the theory dealing with coordinate transformations (i.e. diffeomorphisms) is general relativity, one can see immediately that for having supersymmetry realized locally one has to include gravity. Indeed, the gauge field for those coordinate transformations is the graviton. Its superpartner, the gravitino, is the gauge field for the local SUSY transformations (cf. e.g. [47]).

Also in supergravity BPS states are states which are invariant under a subset of SUSY transformations, but these are now local transformations. Invariance means here the following. We can look at a classical solution to the equations of motion and consider the variation of all fields under local supersymmetry. The solution is invariant under local SUSY if the variation of all fields vanish when evaluating it in the classical background solution. Schematically this can be expressed as

$$\delta(\epsilon(x))\text{boson} = \epsilon(x)\text{fermion} = 0\,, \quad \delta(\epsilon(x))\text{fermion} = \epsilon(x)\text{boson} = 0\,. \tag{2.45}$$

We will not write down the explicit expressions for the field variations since this requires lots of technical details. Instead we will restrict ourselves to a more qualitative description and refer to [50] for details. On a classical solution all fermionic fields are zero anyway so that the first condition is automatically satisfied. Setting the variation of all fermionic fields to zero leads to conditions on the infinitesimal parameter $\epsilon(x)$. For example, when taking the simplest background solution, namely flat Minkowski space, $\epsilon(x)$ has to be constant [50]. In general, setting the variations of the fermions to zero, leads to a set of first-order differential equations in the bosonic fields like the metric, scalars and vector fields and also in the transformation parameter $\epsilon(x)$. For example, the first-order equations (2.33) can be obtained from setting the variation of the gravitino and the gaugino to zero (cf. e.g. [21]). In the Minkowski example one gets for the variation of the graviton and the gravitino the following equations [50]

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}(x)\gamma^a\psi_\mu = 0\,, \quad \delta\psi_\mu = D_\mu\epsilon(x) = 0\,. \tag{2.46}$$

Here e^a_μ is the vielbein and ψ_μ denotes the gravitino. In classical four-dimensional Minkowski space the gravitino field vanishes, so the first equation is automatically satisfied. The second one sets restrictions on the transformation parameter $\epsilon(x)$. In Minkowski space the solutions to the second equation in (2.46) are four independent constant Majorana spinors ϵ_α, $\alpha = 1, \dots, 4$ (cf. e.g. [50]). In the general case a solution for the transformation parameters is built of some number k of independent spinors $\epsilon_\alpha(x)$ with $\alpha = 1, \dots, k$. These independent spinors are called Killing spinors and each of them stands for a preserved supersymmetry. They are the analogon of Killing vectors in general relativity. Just as the latter describe isometries of the spacetime, Killing spinors decribe the preserved supersymmetries. When k of the original N supersymmetries are preserved, the solution is called $\frac{k}{N}$-BPS.

It was discussed in [45] under what conditions on the central charge BPS states exist. There are three distinct cases:

1. $|Z|^2$ has a non-zero minimum. Then BPS states are expected to exist. As already mentioned the conditions on the black hole charges for the existence of a nonvanishing minimum are also examined in [45].

2. $|Z|^2$ does not have a stationary point on the interior of its domain of definition. It might vanish at the boundary. Then the supergravity approximation is not valid and it is not possible to decide whether BPS states exist or not.

3. For some choice of charges the minimal value of $|Z|^2$ can be zero. Then there are no BPS states in the theory.

An example of a BPS state in supergravity is the extremal Reissner-Nordström black hole in flat space. Here $|Z| = \sqrt{P^2 + Q^2}$ (cf. e.g. [51]) and the extremal black hole saturates the BPS bound (cf. the discussion below (2.11)). This is only true in asymptotically flat space. In section 3.5 we will consider the Reissner-Nordström black hole in asymptotically AdS space. In this case the mass of the extremal black hole is strictly larger than the BPS mass bound, i.e. the extremal Reissner-Nordström black hole with AdS asymptotics is not supersymmetric. The BPS solution on the other hand displays a naked singularity (cf. [8]).

Now that we have seen how the attractor mechanism works and how supersymmetry enters the discussion of black holes we want to come back to what we mentioned at the beginning of this section, namely the structure of the scalar manifold.

2.3 Special geometry

Before we will study the geometry of the scalar manifold in $N = 2$, $D = 4$ supergravity we review some features of the vector fields in the action (2.18) and their duals (2.25). First of all, generically there are three main types of multiplets in this theory. There is the gravity multiplet containing the graviton and two gravitini as well as a vector field called graviphoton. Then there are n vector multiplets, each equipped with a vector field, a gaugino and a complex scalar. In addition, there is a number of hypermultiplets which also contain scalars. For simplicity we do not consider hypermultiplets here. Since we have n vector multiplets each containing a vector field and in addition the graviphoton there are $n + 1$ vector fields.

The equations of motion and the Bianchi identities for the vector fields are (cf. [21])

$$dF^I = 0\,, \quad dG_I = 0\,. \tag{2.47}$$

From this one can see immediately that a transformation

$$\begin{pmatrix} F \\ G \end{pmatrix} \mapsto \begin{pmatrix} F' \\ G' \end{pmatrix} = S \begin{pmatrix} F \\ G \end{pmatrix} \tag{2.48}$$

with a matrix $S \in GL(2(n+1), \mathbb{R})$ leaves the equations of motion and Bianchi identities invariant because of the linearity of the derivative. When the transformation is performed also the Lagrangian changes and one has to take care that expression for the transformed dual field strength G'_I is consistent with its definition according to (2.25). This requirement restricts the matrix S to be an element of the symplectic group $Sp(2(n+1), \mathbb{R}) \subset GL(2(n+1), \mathbb{R})$, i.e. S has to obey the equation

$$S^T \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix} S = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix} . \tag{2.49}$$

Writing $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in blocks of size $(n+1) \times (n+1)$ yields the following conditions on A, B, C, D:

$$A^T C - C^T A = 0 = B^T D - D^T B\,, \quad A^T D - C^T B = \mathbb{1}_{n+1}\,. \tag{2.50}$$

Details of this calculations can be found e.g. in [52]. There are some comments in order at this point. Firstly, due to additional matter couplings in the Lagrangian there might be more restrictions on the transformation matrix such that it has to be an element only of a subgroup of the group of symplectic matrices. This subgroup (which of course can be identical with the full group of symplectic matrices) is called the U-duality group. Secondly, the stress-energy tensor is not changed under U-duality rotations and so neither the Einstein equations nor the metric are modified. This means that one can start with a solution to the equations for the Maxwell fields, i.e. a black hole solution with a fixed set of electric and magnetic charges, and perform a U-duality rotation to generate new solutions with different charges from the known one without changing gravitational properties (cf. [21]). Thirdly, it is important to mention that U-duality is in general not a symmetry of the Lagrangian, i.e. the Lagrangian will usually change its form when performing a U-duality transformation. This statement is encoded in the consistency requirement on G'_I when deriving it from the Lagrangian which was already mentioned above.

But what happens now to the scalar fields when one performs a duality rotation among the vector fields? The coupling matrices $\mathcal{I}$ and $\mathcal{R}$ in the Lagrangian (2.18) are also changed under a duality transformation and since they depend on the scalar fields the scalars get modified, too. This can be understood as follows. The scalars can be interpreted as coordinates on some manifold $\mathcal{M}_{scalar}$ and a duality rotation of the vector fields leads to a transformation of the scalars by a diffeomorphism of the scalar manifold. This means that the diffeomorphisms on the scalar manifold are somehow related to the duality transformations of the vector fields [52]. Of course, this sets restrictions on the manifolds that come into consideration for $\mathcal{M}_{scalar}$.

In the case $N = 2$, $D = 4$ the scalar manifold can be written as a Cartesian product of manifold for the scalars sitting in the hypermultiplets and a different one for those sitting in the vector multiplets. So we have

$$\mathcal{M}_{scalar} = \mathcal{M}_v \times \mathcal{M}_h \tag{2.51}$$

with $\mathcal{M}_{v/h}$ the manifold for the scalars in the vector multiplets or the hypermultiplets respectively. Because of this factorized structure of the scalar manifold and since the scalars in the hypermultiplets are not important for our discussion, we will only explain the properties of $\mathcal{M}_v$ in more detail.

As we already mentioned at some point, the manifold for the complex scalars in the vector multiplets in our case is a Kähler manifold. But not only this, it also has some additional properties which make the the manifold even more special - indeed its geometry is called "special geometry".

But let us first define a Kähler manifold and then look at what provides it with special geometry. The definitions for Kähler manifolds are taken from [50] and [53]. But before we come to that, we have to make some preparing definitions.

A complex manifold has a generic line element of the form

$$ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha \bar{z}^{\bar{\beta}} + g_{\alpha\beta}dz^\alpha dz^\beta + g_{\bar{\alpha}\bar{\beta}}d\bar{z}^{\bar{\alpha}}d\bar{z}^{\bar{\beta}}\,. \tag{2.52}$$

A metric on a complex manifold is called Hermitian if there exist coordinates in which

$$g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0 \tag{2.53}$$

so that the line element takes the form

$$ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha \bar{z}^{\bar{\beta}} . \tag{2.54}$$

The fundamental 2-form for a Hermitian metric is defined as

$$\mathcal{K} = -2ig_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^{\bar{\beta}} . \tag{2.55}$$

Now we can define a Kähler manifold.

Definition. A Kähler manifold is a complex manifold with Hermitian metric whose fundamental 2-form is closed, i.e. $d\mathcal{K} = 0$. In this case the fundamental 2-form is called the Kähler form.

That the Kähler form is closed is equivalent to the condition that

$$\partial_\gamma g_{\alpha\bar{\beta}} - \partial_\alpha g_{\gamma\bar{\beta}} = 0 \tag{2.56}$$

which means that in every coordinate patch the metric can be expressed through the second derivatives of a function K called the Kähler potential:

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z, \bar{z}) . \tag{2.57}$$

The Kähler potential is not determined uniquely since the potential

$$K'(z, \bar{z}) = K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}) \tag{2.58}$$

yields the same metric as the original potential K. In fact, in the overlapping regime of two coordinate patches, the different Kähler potentials are related by a transformation as in (2.58) which are thence called Kähler transformations. Often Kähler manifolds are defined in a different way which we will present shortly. For this purpose we need two additional definitions.

Definition. A homomorphism of vector bundles $J : T\mathcal{M} \to T\mathcal{M}$ is called an almost complex structure on the manifold $\mathcal{M}$, if $J^2 = -\mathrm{Id}_{T\mathcal{M}}$.

The prefix "almost" indicates that the manifold $\mathcal{M}$ need not be equipped with a complex structure although the tangent spaces $T_x\mathcal{M}$ exhibit a complex structure.

Definition. A Riemannian manifold $\mathcal{M}$ with metric g and orthogonal almost complex structure J, i.e. with

$$g(JX, JY) = g(X, Y) \tag{2.59}$$

for all vector fields X, Y, is called an almost Hermitian manifold.

Now the following theorem can be stated.

Theorem. A manifold is a Kähler manifold if and only if it is an almost Hermitian manifold whose almost complex structure J is parallel, i.e. for which $\nabla^g J = 0$. Here ∇^g is the Levi-Civita-connection.

Now that we know what a Kähler manifold is we can proceed and turn to special geometry (see e.g. [54, 55, 56] for mathematical approaches to special geometry). There are two types of Kähler manifolds which are called special manifolds. On the one hand, there are the so-called "rigid" or "affine" special Kähler manifolds. They arise in the context of rigid $N = 2$ supersymmetry. Their local counterpart is called "projective" special geometry or often only special geometry. In the following we will give definitions for special geometry either in the rigid and in the local case. For this purpose we will rely on [57]. There the authors display different definitions for special geometry and show their equivalence.

Generally speaking, in special geometry the scalars are affected by the symplectic transformations for the vector field strengths. To see how this works let us have a look at rigid special geometry first.

Definition. An n-dimensional Kähler manifold is a rigid special Kähler manifold if the following conditions are satisfied:

1. On every coordinate chart there are n independent holomorphic functions $X^I(z), I = 1, \ldots, n$ and a holomorphic function $F(X)$ called the prepotential such that the Kähler potential takes the form
$$K(z, \bar{z}) = i \left(X^I \frac{\partial}{\partial \bar{X}^I} \bar{F}(\bar{X}) - \bar{X}^I \frac{\partial}{\partial X^I} F(X) \right) . \tag{2.60}$$

2. On the intersection of two charts $U_i \cap U_j \neq \emptyset$ there are transition functions of the following form:
$$\begin{pmatrix} X \\ \partial F \end{pmatrix}_{(i)} = e^{ic_{ij}} M_{ij} \begin{pmatrix} X \\ \partial F \end{pmatrix}_{(j)} + b_{ij} \tag{2.61}$$
with $c_{ij} \in \mathbb{R}$, $M_{ij} \in Sp(2n, \mathbb{R})$ and $b_{ij} \in \mathbb{C}^{2n}$.

3. The transition functions satisfy the cocycle condition, i.e. on the intersection of three charts $U_i \cap U_j \cap U_k \neq \emptyset$
$$e^{c_{ij}}\, e^{c_{jk}}\, e^{c_{ki}} = 1 \,, \quad M_{ij}\, M_{jk}\, M_{ki} = \mathbb{1}_{2n} \,. \tag{2.62}$$

Equivalently, a rigid special Kähler geometry can be defined as follows.

Definition. A manifold $\mathcal{M}$ is a rigid special Kähler manifold if

1. there exists a $U(1) \times ISp(2n, \mathbb{R})$[3] vector bundle over $\mathcal{M}$ with constant transition functions as in (2.61) and a holomorphic section V such that the Kähler form can be

[3]The group $ISp(2n, \mathbb{R})$ is defined as $ISp(2n, \mathbb{R}) = Sp(2n, \mathbb{R}) \ltimes T(2n, \mathbb{R})$, i.e. it is the semi-direct product of the symplectic group $Sp(2n, \mathbb{R})$ and the translations on $\mathbb{R}^{2n}$. $ISp(2n, \mathbb{R})$ is called the inhomogeneous symplectic group (cf. [58]).

written as

$$\mathcal{K} = -\frac{1}{2\pi}\partial\bar{\partial}\langle V, V\rangle \tag{2.63}$$

with the symplectic inner product defined as

$$\langle V, V\rangle := V^T \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} V\,. \tag{2.64}$$

and

2. if in addition

$$\langle \partial_\alpha V, \partial_\beta V\rangle = 0 \tag{2.65}$$

where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$.

From the second definition we see the connection to the symplectic transformations we discussed before. In [57] also a third equivalent definition is given, but we restrict ourselves to the two definitions displayed above since the main ingredients we need for further applications can already be seen from them. These main features are the prepotential and the underlying symplectic structure.

Now that we explained the structure of rigid special geometry we come to the more general case of local special geometry. From here on we will use the term "special geometry" to describe *local* special geometry. In the local case the Kähler manifold is required to be a Kähler-Hodge manifold. This means that its Kähler form has to be of integer cohomology. Such manifolds are also called "Kähler manifolds of restricted type" or simply "Hodge manifolds". This additional condition on the Kähler manifold comes from the fermions which impose a quantization condition on the Kähler form (cf. [57]).

At this point we are ready to define special Kähler manifolds. We will again show two equivalent definitions.

Definition. An n-dimensional Kähler-Hodge manifold is called special Kähler if it has the following properties:

1. On every chart there exist projective coordinate functions $X^I(z), I = 0, \dots, n$ and a holomorphic function $F(X)$ which is homogeneous of degree two, such that the Kähler potential can be written as

$$K(z,\bar{z}) = -\log\left(i\bar{X}^I \frac{\partial}{\partial X^I}F(X) - iX^I\frac{\partial}{\partial \bar{X}^I}\bar{F}(\bar{X})\right)\,. \tag{2.66}$$

F is the prepotential.

2. On the intersection of two charts $U_i \cap U_j \neq \emptyset$ there are transition functions of the following form:

$$\begin{pmatrix} X \\ \partial F \end{pmatrix}_{(i)} = e^{f_{ij}(z)} M_{ij} \begin{pmatrix} X \\ \partial F \end{pmatrix}_{(j)} \tag{2.67}$$

with holomorphic functions f_{ij} and $M_{ij} \in Sp(2(n+1), \mathbb{R})$.

3. The transition functions satisfy the cocycle condition, i.e. on the intersection of three charts $U_i \cap U_j \cap U_k \neq \emptyset$

$$e^{f_{ij}(z)}\, e^{f_{jk}(z)}\, e^{f_{ki}(z)} = 1\,, \quad M_{ij}\, M_{jk}\, M_{ki} = \mathbb{1}_{2(n+1)}\,. \tag{2.68}$$

Comparing this definition with the one for rigid special geometry one can see the following differences. Local special geometry shows a projective structure: There are n physical scalars in the theory but we can describe everything equivalently in terms of the $(n+1)$ functions X^I. We call these functions coordinates on the big moduli space, and they are related to the physical scalars z^i by $z^i = \frac{X^i}{X^0}$, $i \in \{1, \ldots, n\}$. Here the projective nature of the X^I is manifest. Multiplying them with an arbitrary holomorphic function $e^{f(z)}$ does not change the physics as the scalars z^i are not touched by such a transformation. Another important point to mention is that the prepotential is now more restricted than in the case of rigid special geometry. Not only has it to be holomorphic but also it is required that the prepotential is homogeneous of degree two, i.e. $F(\lambda X) = \lambda^2 F(X)$. This implies

$$F_I = F_{IJ} X^J\,, \quad F_{IJK} X^K = 0 \tag{2.69}$$

with $F_I = \frac{\partial F(X)}{\partial X^I}$, $F_{IJ} = \frac{\partial^2 F(X)}{\partial X^I \partial X^J}$ etc.

The problem with having a definition based on a prepotential is that this function is in general not invariant under symplectic transformations [57]. Therefore one would like to define special geometry independently of a prepotential. There are even supergravity actions for which there does not exist a prepotential. An example is given in [59]. This is not a contradiction to the definition of special manifolds given above. In [57]the authors state that the geometry of the scalars can always be described in terms of a prepotential. But this does not exclude the possibility that there are supergravity actions which cannot be obtained using a prepotential. It is still possible that even when one starts with a theory with prepotential the symplectically transformed theory cannot be written down in terms of a prepotential.

For the applications we developed in $N = 2$, $D = 4$ we always work with models based on a prepotential, so we do not need to bother about these kind of subtleties. However, let us display an alternative definition of special geometry (cf. [57]):

Definition. An n-dimensional Kähler-Hodge manifold $\mathcal{M}$ is special Kähler if it has the following two properties:

1. There exists a holomorphic $Sp(2(n+1), \mathbb{R})$ vector bundle $\mathcal{H}$ over $\mathcal{M}$ and a holomorphic section $v(z)$ of $L \otimes \mathcal{H}$, such that the Kähler form is given by

$$\mathcal{K} = -\frac{i}{2\pi} \partial\bar{\partial} \log(i\langle \bar{v}, v\rangle)\,. \tag{2.70}$$

   L denotes a holomorphic line bundle over $\mathcal{M}$ with transition functions given by $U(1)$ transformations. The first Chern class of L has to be equal to the cohomology class of the Kähler form.

2. The section $v(z)$ satisfies
$$\langle v, \partial_\alpha v\rangle = 0\,. \tag{2.71}$$

It is possible to formulate similar expressions as in the above definition using a different section $V = e^{K/2}v$ with $V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}$. With

$$U_\alpha := D_\alpha V = \partial_\alpha V + \frac{1}{2}(\partial_\alpha K)V\,, \quad D_{\bar\alpha} V = \partial_{\bar\alpha} V - \frac{1}{2}(\partial_{\bar\alpha} K)V \tag{2.72}$$

and analogously for the complex conjugates one can derive the following constraints [57]:

$$\begin{aligned} \langle V, \bar V\rangle &= i\,, \\ D_{\bar\alpha} V &= 0\,, \\ \langle V, U_\alpha\rangle &= 0\,, \\ \langle U_\alpha, U_\beta\rangle &= 0\,. \end{aligned} \tag{2.73}$$

The first one of these equations is equivalent to

$$i\left(\bar X^I F_I - \bar F_I X^I\right) = 1\,. \tag{2.74}$$

We will use this equation later in the section on $N = 2$, $D = 4$ gauged supergravity to put a constraint on the scalar kinetic term in the big moduli space. This constraint will be very important because of the projective nature of special Kähler manifolds. We can either describe everything in terms of the n physical scalars $z^i = \frac{X^i}{X^0}$ or in terms of the $(n+1)$ functions X^I. But then we need a constraint to get rid of the additional degree of freedom. Further identities of special geometry which we will use in chapter 5 are presented in appendix A.

Since later we will not only work in four dimensions but also in five, let us have a look at the scalar geometry in five dimensions.

2.4 The scalar geometry for $N = 2$ supergravity in five dimensions

Also in five dimensions the scalar manifold is equipped with a typical structure. In contrast to the four-dimensional case the manifolds for five-dimensional supergravity are real manifolds instead of complex ones. These manifolds are called *very special (real) manifolds*. When one performs a dimensional reduction to four dimensions one is lead to special Kähler manifolds. Of course, this does not imply that every special Kähler manifold can be obtained by dimensional reduction from a very special manifold. The map that takes a very special manifold to a special one is called the **r**-map (cf. [60]).

The geometry of the scalar manifold in five-dimensional supergravity with Abelian vector multiplets was worked out in [61]. We will present the essential results and display some identities for very special geometry which we use in chapter 6.

First of all let us have a look at the field content we are dealing with in five dimensions. In five-dimensional supergravity one has the gravity multiplet and three kinds of matter multiplets, namely the vector, tensor and hypermultiplets (cf. e.g. [62]). Each of the latter contain scalars but we will focus only on the ones in the vector multiplet since only these will be necessary for the applications in chapter 6.

The gravity multiplet contains (just as in four dimensions) the graviton, two gravitini and the graviphoton. Each vector mulitplet contains a vector field, one $SU(2)$ doublet of spin-$\frac{1}{2}$ fermions and one real scalar field [62]. One can see here that the number n_S of scalars $\{\phi^i\}_{i=1\dots,n_S}$ is related to the number n of vector fields by

$$n_S = n - 1 \tag{2.75}$$

since there is one more vector field than scalars, namely the graviphoton.

In [61] it is shown that the $(n-1)$-dimensional scalar manifold in this case can be embedded into an n-dimensional Riemannian space in a particular way. More precisely, the scalar manifold is a hypersurface with vanishing second fundamental form and the embedding is given by the cubic polynomial

$$\frac{1}{6}C_{ABC}X^A X^B X^C = v\,. \tag{2.76}$$

Here v is a constant, the X^A, $A=1,\dots,n$ are real coordinates on the n-dimensional Riemannian manifold and C_{ABC} are constants that have to be chosen in a way such that C_{ABC} is symmetric in its indices. Often the constant v is taken to be equal to one. But for some applications it is necessary to choose a different value for v as you can see in section 6.2 and appendix D where we used $v=\frac{1}{2}$.

The physical scalars ϕ^i can now be interpreted as coordinates on the $(n-1)$-dimensional hypersurface determined by (2.76). Later we will not work with the physical scalars but with the X^A instead – again we will work in big moduli space. For the purpose of using X^A instead of ϕ^i we have to relate both kinds of scalar fields. To do so we need to take into account the form of the bosonic part of the supergravity action with Chern-Simons term in five dimensions. For Abelian vector fields the Lagrangian density for this action is given by (cf. e.g. [63, 64])

$$\mathcal{L} = \sqrt{-g}\left(R - g_{ij}\partial_M\phi^i\partial^M\phi^j - \frac{1}{2}G_{AB}F^A_{MN}F^{B\,MN}\right) - \frac{1}{24v}C_{ABC}F^A_{KL}F^B_{MN}A^C_P\epsilon^{KLMNP} \tag{2.77}$$

Here $M,N,K,L,P\in\{t,r,x,y,z\}$, $A,B,C\in\{1,\dots,n\}$ and $i,j\in\{1,\dots,n-1\}$. F^A_{MN} are the field strengths associated to the vector fields.

Now we want to relate the various terms in the action to very special geometry. The identities displayed below are taken from [63].

If one defines

$$X_A = \frac{1}{6}C_{ABC}X^B X^C\,, \tag{2.78}$$

$$G_{AB} = \frac{1}{v}\left(-\frac{1}{2}C_{ABC}X^C + \frac{9}{2}\frac{X_A X_B}{v}\right)\,, \tag{2.79}$$

then one gets the following relations:

$$X_A X^A = v \tag{2.80}$$

and with G^{AB} the inverse of G_{AB}

$$X_A = \frac{2v}{3} G_{AB} X^B \,, \quad X^A = \frac{3}{2v} G^{AB} X_B \,. \tag{2.81}$$

Here G_{AB} is the matrix in the supergravity action appearing in the vector kinetic terms. Here one can see the interplay of the big moduli scalars X^A and the vector fields.

$v = const$ implies $\partial_i v = 0$ and therefore

$$(\partial_i X_A) X^A = 0 \,. \tag{2.82}$$

From the equations (2.78) and (2.79) it follows that

$$G_{AB} \partial_i X^B = -\frac{3}{2v} \partial X_A \,. \tag{2.83}$$

The metric on the scalar manifold is given in terms of the X^A through

$$g_{ij} = G_{AB} \partial_i X^A \partial_j X^B \tag{2.84}$$

so that the scalar kinetic term becomes

$$g_{ij} \partial_M \phi^i \partial^M \phi^j = G_{AB} \partial_M X^A \partial^M X^B \,. \tag{2.85}$$

In [63] it is also shown that

$$g^{ij} \partial_i X^A \partial_j X^B = G^{AB} - \frac{2}{3} X^A X^B \,. \tag{2.86}$$

These are the tools one needs to deal with very special geometry.

2.5 Gauged supergravity in four and five dimensions

In chapters 5 and 6 we will not work with pure supergravity but with a more complicated version called *gauged supergravity* instead. Gauged supergravities allow for a broader class of black solutions and therefore these theories of gravity are of particular interest.

The symmetries of a generic supergravity theory are organized by its global symmetry group G. For theories with maximal or half-maximal number of supersymmetries the symmetry groups are uniquely determined in the various dimensions. A list of them can be found e.g. in [65]. Maximal supergravities in various dimensions have for instance been studied in [66, 67, 68, 69, 70]. In the case of $N = 2$ supergravity the situation is different. Here the symmetry group is less constrained and there are several possibilities depending on the specific model under consideration. Gauging the supergravity means that a subgroup G_0 of the symmetry group G is chosen and promoted to a local symmetry. This makes it difficult to explain how the gauging works for $N = 2$ since we do not have a fixed symmetry group G to work with. The examples we are interested in will be $U(1)$ gauged supergravities. Here a $U(1)^k$ group serves as gauge group with $k \leq n+1$ and $n+1$ the number of Abelian vector fields[4]. This is always possible due to the Abelian gauge symmetry $U(1)^{(n+1)}$ of the vector fields. In the general case the subgroup G_0 can be chosen via the embedding tensor formalism as is explained in [65]. This works as follows. One selects a subset of elements in the Lie algebra $\mathfrak{g}$ of G that generate a subalgebra. These generators are related to the generators of G by

$$\tilde{t}_I = \Theta_I^\alpha t_\alpha \,. \tag{2.87}$$

Here Θ_M^α is a constant tensor called the embedding tensor and t_α are the generators of the symmetry group G. One can see here that the embedding tensor is a tool to construct a subalgebra of the Lie Algebra $\mathfrak{g}$ which is the Lie algebra of the subgroup G_0 that will be promoted to a local symmetry. The embedding tensor has to fulfill a set of constraints to ensure local gauge invariance (cf. e.g. [65]). With the help of the generators $\tilde{t}_I$ one can perform the gauging in the standard way by defining a covariant derivative (cf. [65])

$$D_\mu = \partial_\mu - g A_\mu^I \tilde{t}_I \,. \tag{2.88}$$

Here g is the coupling constant and A_μ^I the vector fields that participate in the gauging. In the cases we consider in the chapters 5 and 6 the covariant derivative would only apply to the scalar fields in the hypermultiplets which we set to zero anyway. Therefore we do not need to deal with gauge covariant derivatives. In the general non-Abelian cases the field strengths of the vector fields have to be modified [65]. But for $U(1)$ gauged supergravity the field strengths do not change because of the commutativity of the group multiplication.

Now there are two things one has to take care of when constructing the Lagrangian for gauged supergravity: One has to make sure that it is gauge invariant and also that it is supersymmetric. To ensure both required properties one adds at first terms to the Lagrangian to make it gauge invariant and then applies the Noether procedure to restore supersymmetry [65]. This results in a scalar potential that appears at the order g^2. For the Lagrangians we will later use this scalar potential is the only modification we see compared to the ungauged case. The field strengths are left unchanged because we are looking at $U(1)$'s and also the covariant derivative is "invisible" since we do not consider hypermultiplet scalars.

In the scalar potential additional parameters appear which are called *fluxes*. Although in our calculations we will treat them just as a given set of parameters in four respectively five dimensions, one could ask where these parameters come from. Indeed there is a description of them in higher dimensions. In ten-dimensional type II superstring theory the fluxes are defined as integrals of field strength over appropriate cycles. For instance, the field strength of the NS B-field

$$H = dB \tag{2.89}$$

[4]Here we stick to our notation for the four-dimensional case.

in a Calabi-Yau compactification gives rise to electric and magnetic fluxes through [71]

$$\frac{1}{(2\pi)^2\alpha'}\int_{A_K} H = h^K\,, \quad \frac{1}{(2\pi)^2\alpha'}\int_{B^K} H = h_K\,. \tag{2.90}$$

Here (A_K, B^K) is a basis of 3-cycles and h^K, h_K are the magnetic and electric fluxes. Turning on fluxes corresponds to gauging some shift symmetries in the hypermultiplet sector. In [72] it was shown that the scalar potential of gauged supergravity agrees with the one obtained from compactifying type IIB supergravity on a Calabi-Yau manifold with fluxes. The author considered electric RR and NS fluxes and he showed that it is possible to express the Killing vectors for certain shift symmetries of the hypermultiplet scalars in terms of the electric fluxes. The covariant derivative defined in (2.88) can be expressed in terms of the Killing vectors according to (cf. [71])

$$D_\mu = \partial_\mu - k_I A^I_\mu \tag{2.91}$$

with the Killing vector k_I. From this example one can see the general mechanism. The fluxes coming from higher-dimensional field strengths are related to isometries in the hypermultiplet sector. These isometries are the shift symmetries we have already mentioned. Depending on which fluxes are switched on different shift symmetries are selected. Since the isometries of the hypermultiplet sector (like shift symmetries) are part of the symmetry group of the theory, one can use them to construct the subgroup of the symmetry group that will be gauged. The fluxes will then enter in the scalar potential of gauged supergravity.

In chapters 5 and 6 we will use Lagrangians from $N = 2$ $U(1)$ gauged supergravity to construct black brane solutions. These solutions are interesting since they allow for asymptotic AdS solutions which is important for applications within the AdS/CFT correspondence. The next chapter will be devoted to introducing this correspondence.

Chapter 3

The AdS/CFT Correspondence

Circle Limit IV, Maurits C. Escher

Several artworks of the Dutch artist Maurits C. Escher[1] deal with the artistical arrangement of a hyperbolic plane which can be identified with a disk. One of these is the xylograph "Circle Limit IV".

As you can see, the angels and devils become smaller and smaller the closer they get to the boundary. The size of the figures is a measure for their distance from the centre of the hyperbolic plane. From the geometrical point of view this means that the boundary is infinitely far away from the centre and no angel or devil could ever reach it.

Now you will certainly ask what this could have to do with physics. In this section we will review the AdS/CFT correspondence and as you will see shortly, the "AdS" part of that correspondence indeed involves hyperbolic spaces.

The AdS/CFT correspondence was proposed in 1997 by Maldacena [5].His famous conjecture states that IIB string theories on Anti-de Sitter spacetimes are dual to conformal field theories living on the boundary of these Anti-de Sitter spaces [5]. This means that there are two different theories that describe the same physics. It is just like in Escher's xylograph: Do you see the angels or the devils? Both describe the same arrangement of black and white.

To understand the AdS/CFT conjecture we need to introduce its basic ingredients. These are the Anti-de Sitter spaces we have already mentioned as well as the conformal field theory under consideration, namely $N = 4$ super-Yang-Mills theory. So let us start with the field theoretic side of AdS/CFT.

3.1 The field theory side of the duality

We will describe the main aspects of $N = 4$ super-Yang-Mills theory in four dimensions. Here we will mostly rely on [73]. To make it more precise, to understand AdS/CFT we need to deal with $N = 4$ $SU(N)$ super-Yang-Mills theory. As we have seen before theories with $N = 4$ supersymmetry contain four spinorial supercharges with commutation relations given in (2.35). These supercharges can be rotated into each other by $SU(4)$ transformations. Any theory dual to an $N = 4$ supersymmetric theory should better also display this kind of symmetry, and later we will see that a $SU(4)$ symmetry appears indeed in the gravitational dual to $N = 4$ SYM. The field content of $N = 4$ SYM consists only of a gauge multiplet. This contains four left Weyl fermions λ^a_α, $a = 1, \ldots, 4$, six real scalars X^i, $i = 1, \ldots, 6$ and a gauge field A_μ. Since we are considering a $SU(N)$ theory, A_μ is a $SU(N)$ gauge connection. For AdS/CFT it will be of particular interest to examine the large N behavior of the $SU(N)$ super-Yang-Mills theory.

The Lagrangian of $N = 4$ SYM takes the form (cf. [74])

$$\begin{aligned}\mathcal{L} \;=\; \mathrm{tr}\Big\{ &-\frac{1}{2g_{YM}^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_{a=1}^{4} i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_{i=1}^{6} D_\mu X^i D^\mu X^i \\ &+ \sum_{a,b,i} g_{YM} C_i^{ab}\lambda_a[X^i,\lambda_b] + \sum_{a,b,i} g_{YM}\bar{C}_{i\,ab}\bar{\lambda}^a[X^i,\bar{\lambda}^b] + \frac{g_{YM}^2}{2}\sum_{i,j}[X^i,X^j]^2 \Big\}\,. \end{aligned} \tag{3.1}$$

Here C_i^{ab} and $C_{i\,ab}$ are the structure constants of $SU(4)$. We will now review the symmetries of this Lagrangian. Depending on whether $\langle X^i\rangle = 0$ for all i or not there are two different phases for $N = 4$ SYM. If $\langle X^i\rangle = 0$ for all i the theory is said to be in the *superconformal phase*, otherwise it is in the *Coulomb phase*. In the superconformal phase none of the symmetries is broken. So let us specify what these symmetries are.

First of all the SYM Lagrangian is invariant under translations and Lorentz transformations. In addition, $N = 4$ SYM has conformal symmetry. A transformation is conformal if it preserves the angle between two vectors. So every isometry (like translations and Lorentz transformations) is of course a conformal transformation but there are further conformal transformations which are no isometries. Conformal transformations of this type are the dilations

$$x^\mu \mapsto \lambda x^\mu \,, \lambda \in \mathbb{R} \tag{3.2}$$

and the so-called special conformal transformations which can be written in the form (cf. e.g. [75])

$$x^\mu \mapsto \frac{x^\mu + \alpha^\mu x^2}{1 + 2\langle \vec{\alpha}, \vec{x} \rangle + \alpha^2 x^2} \,, \vec{\alpha} \in \mathbb{R}^4 \,. \tag{3.3}$$

The conformal transformations and the supersymmetry transformations can be combined to form the symmetry group of $N = 4$ SYM. This symmetry group is called the *supergroup* $SU(2,2|4)$. It has the following constituents (cf. [73]):

- The conformal transformations form the conformal group $SO(2,4) \sim SU(2,2)$. Here "$\sim$" means that $SU(2,2)$ is the universal cover of $SO(2,4)$. This difference will not be important for our purposes, so we will treat both groups as if they were identical.

  The generators of the Lie algebra of $SO(2,4)$ are the translations, Lorentz transformations, dilations and special conformal transformations.

- The $SU(4)$ symmetry which rotates the supercharges Q^a_α into each other. This is called the R-symmetry.

- The supersymmetries generated by the supercharges Q^a_α and their conjugates.

- Furthermore, there are additional supersymmetries $S_{\alpha a}$ and their conjugates. These are called *conformal supersymmetries*. Their existence is due to the fact that the supercharges Q^a_α do not commute with the special conformal transformations and so their commutator has to be an additional symmetry. These are the conformal supersymmetries.

These generators form a Lie algebra and the group associated to this Lie algebra is called $SU(2,2|4)$.

Now we can proceed looking at the gravitational part of the AdS/CFT conjecture.

3.2 The gravitational side of the duality

At first we will define Anti-de Sitter spaces of arbitrary dimension and give some explanation of what is called the boundary of this space. We will also review the symmetries of AdS spaces and show how these spaces enter in the AdS/CFT correspondence, namely in a low energy limit of some closed string theories.

AdS spacetimes arise as solutions to the Einstein-Hilbert action with cosmological constant (cf. e.g. [75]):

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{|g|} (R - \Lambda) \tag{3.4}$$

Here Minkowskian signature is used. κ is related to the D-dimensional Newton's constant G_D through $\kappa^2 = 8\pi G_D$. R denotes the Ricci scalar and $\Lambda = \frac{(D-1)(D-2)}{L^2}$ is the cosmological constant. The meaning of L will become clear momentarily. For AdS spaces we have $\Lambda < 0$. AdS spaces are spacetimes with constant negative scalar curvature. The $D = d+1$ dimensional solution can be embedded as a submanifold into $\mathbb{R}^{2,d}$. This is a space with indefinite inner product given by

$$(x, y) = \langle x, \mathrm{diag}(-1, -1, 1, \dots, 1, 1)\, y \rangle \tag{3.5}$$

and $\langle\, ,\, \rangle$ the standard scalar product on $\mathbb{R}^{d+2}$. For $y = (y_0, y_1, \dots, y_d, y_{d+1}) \in \mathbb{R}^{2,d}$ the "length squared" is determined by the indefinite inner product, $y^2 = (y, y)$, and we can describe AdS_{d+1} as the quadric (cf. e.g. [73, 75])

$$y^2 = -R^2 = const. \tag{3.6}$$

for a real number R. This is a $(d+1)$-dimensional surface with negative curvature in $\mathbb{R}^{2,d}$.

From this embedding we can see a fact that will be important in understanding Maldacena's conjecture. It is well known from Mathematics that the group of rotations in $\mathbb{R}^{2,d}$ is $SO(2, d)$. Since rotations preserve the inner product, it is clear that the quadric (3.6) is invariant under the transformation $\vec{y} \mapsto \Lambda\vec{y}$ for $\Lambda \in SO(2, d)$. But this means that every $SO(2, d)$ transformation is an isometry of AdS_{d+1}. One can see that in particular AdS_5 has a $SO(2, 4)$ rotational symmetry. We met $SO(2, 4)$ already as the conformal group in four dimensional space when we discussed the symmetries of $N = 4$ SYM, so here is the first indication of a possible duality.

But let us come back to the geometry of AdS_{d+1}. The following discussion is based on [76]. The surface (3.6) is a connected hyperboloid in the $(d+2)$-dimensional ambient space $\mathbb{R}^{2,d}$ (see figure 3.1). One can then introduce coordinates (t, ρ, Ω_i) on the hyperboloid such that

$$\begin{aligned} y_0 &= R\cosh(\rho)\cos(\tau)\,, \\ y_1 &= R\cosh(\rho)\sin(\tau)\,, \\ y_i &= R\sinh(\rho)\,\Omega_i\,, i \in \{2, \dots, d+2\}\ , \sum_i \Omega_i^2 = 1\,. \end{aligned} \tag{3.7}$$

When taking $\rho \in \mathbb{R}_0^+$ and $\tau \in [0, 2\pi)$ the entire hyperboloid is covered once. The "time" coordinate τ is periodic and therefore AdS_{d+1} has closed timelike curves. One can avoid this problem by going to the universal cover instead since in this space the circle in the τ direction is unwrapped. Often also the universal cover of AdS_{d+1} is itself called AdS_{d+1} and in our applications in the later chapters we also work with the universal cover.

There are two further important aspects of AdS_{d+1}. Firstly, one can change to imaginary time. The Euclidean version of AdS_{d+1} is then given by the hyperbola

$$-(y^0)^2 + \sum_{i=1}^{d+1} (y^i)^2 = -R^2\,. \tag{3.8}$$

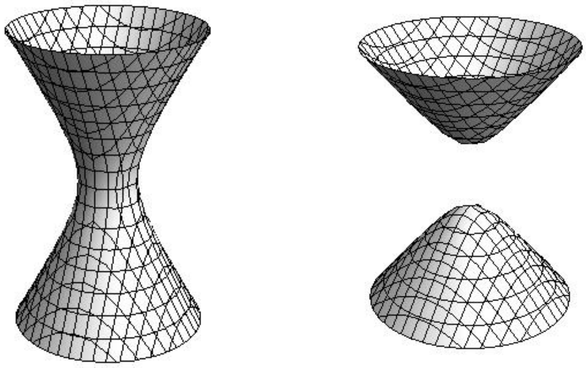

Figure 3.1: AdS space in Minkowski signature (on the left) and Euclidean signature.

As you can see in figure 3.1 this is now a disconnected hyperboloid. AdS_{d+1} is considered to be only one of the two connected leaves of the hyperboloid when one wants to work in a connected space.

Secondly, one can identify AdS_{d+1} with the upper halfplane

$$H_{n+1} = \{(z_0, \vec{z})|z_0 \in \mathbb{R}^+, \vec{z} \in \mathbb{R}^n\}. \tag{3.9}$$

This can be done by the coordinate transformation [73]

$$y_0 + y_1 = \frac{1}{z_0}, \quad z_i = z_0 y_i, i = 2, \ldots, d+1. \tag{3.10}$$

The metric on H_{d+1} is then given by

$$ds^2 = \frac{1}{z_0^2}(dz_0^2 + d\vec{z}^2). \tag{3.11}$$

By adding a point at infinity the upper halfplane H_{d+1} can be compactified. This procedure gives the unit ball B_{d+1} as the compactification of Euclidean AdS_{d+1}. Often both descriptions are used equivalently and no distinction is made between AdS_{d+1} and its compactification.

Now that we got to know AdS_{d+1} we are interested in the structure of its boundary. To examine the boundary of AdS_{d+1} one can introduce "light cone coordinates" [75] [2]

$$u = y_0 + iy_1, \quad v = y_0 - iy_1. \tag{3.12}$$

[2]We use a different sign convention than [75].

In these coordinates AdS_{d+1} is given by

$$-uv + \vec{y}^2 = -R^2 \tag{3.13}$$

with $\vec{y} = (y_2, \ldots, y_{d+1})$. The "boundary" of AdS_{d+1} are the points lying at infinity. For the quadric (3.13) these points are given by

$$uv - \vec{y}^2 = 0\,. \tag{3.14}$$

This can be seen in the following way. Going to infinity means that we consider points of the form $\lambda(u, v, \vec{y})$ with large $|\lambda|$. For such points in AdS_{d+1} (3.13) takes the form

$$-\lambda^2 uv + \lambda^2 \vec{y}^2 = -R^2\,. \tag{3.15}$$

Dividing this equation by $\lambda^2 \neq 0$ and taking the limit $\lambda \to \infty$ yields (3.14). One can see immediately that for a point $(u, v, \vec{y})$ satisfying (3.14) also $\lambda(u, v, \vec{y})$ fulfills (3.14) for any $\lambda \in \mathbb{R}$. This means that the boundary of AdS_{d+1} is the set of equivalence classes of points satisfying (3.14).

This set of equivalence classes itself is furnished with a projective structure. Therefore the boundary of AdS_{d+1} is automatically compactified. For the Euclidean version it has the topology of the sphere S^d (cf. e.g. [73]) whereas in the Minkowskian case the boundary is the conformal compactification of the d-dimensional Minkowski space (cf. e.g. [76]). But let us come back to the Euclidean case and examine the properties of the boundary of AdS_{d+1} a bit in more detail. One can consider the points with $v = 0$ as the points at infinity. We are looking now at the boundary itself, i.e. "infinity" corresponds now to the points that constitute the projective structure of the boundary and not of the entire AdS space.

For boundary points with $v \neq 0$ we can assume $v = 1$ because of the scaling invariance and then rewrite (3.14) as

$$u = \vec{y}^2\,. \tag{3.16}$$

Since this equation defines a d-dimensional surface, we can see that the boundary of AdS_{d+1} has indeed dimension d as desired. These arguments are presented in more detail in [75]. There it is also shown that the group of rotations of AdS_{d+1} acts as the conformal group on the boundary of AdS_{d+1}. In [75] this is discussed for the Euclidean case. Then the group of rotations is $SO(1, d+1)$ and it acts on the boundary of AdS_{d+1} as

$$\Lambda[y] := [\Lambda y]\,. \tag{3.17}$$

Here $\Lambda \in SO(1, d+1)$ and

$$[y] = [(u, v, \vec{y})] = \{(u', v'\vec{y}')|\exists\lambda \in \mathbb{R} \text{ with } (u', v', \vec{y}') = \lambda(u, v, \vec{y})\}\,, \tag{3.18}$$

i.e. $[y]$ is the equivalence class of the boundary point $y = (u, v, \vec{y})$. In [75] it is stated that a transformation $\Lambda \in SO(1, d+1)$ infinitesimally close to the identity can be written as the composition of infinitesimal conformal transformations. This shows that the group

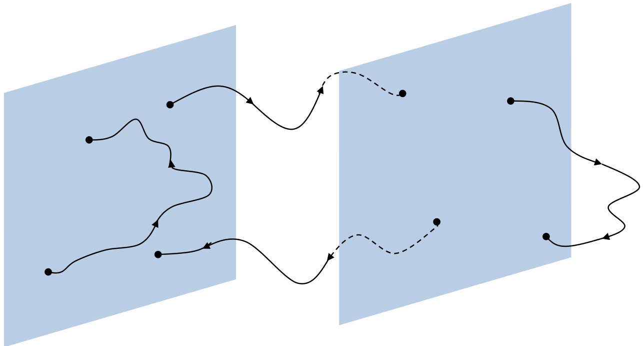

Figure 3.2: Open strings attached to branes.

of rotations in Euclidean space acts as the conformal group on the boundary of AdS_{d+1}. This result can be transferred to the Minkowski case and we can conclude that the group $SO(2,d)$ acts as the conformal group on the boundary of Minkowskian AdS_{d+1}. From this point of view it makes sense to say that the conformal field theory $N = 4$ SYM lives on the boundary of AdS_5 because both are equipped with $SO(2,4)$ conformal symmetry.

But how do AdS spaces arise from string theory? They appear in the low energy limit of Dp-brane solutions to $10D$ string theory! Generally speaking a Dp-brane is just an object on which open strings can end (see figure 3.2). The "D" stands for "Dirichlet" since the coordinates of the strings attached to the brane fulfill Dirichlet boundary conditions in the directions transverse to the brane. A *Dp-brane* is a p-dimensional brane (cf. e.g. [77]). Dp-brane metrics are solutions in type II string theory. For the basic understanding of the AdS/CFT correspondence solutions with N coincident $D3$ branes are of particular interest, so we will restrict ourselves to the discussion of this kind of branes. Such a solution is then given in terms of the metric (cf. e.g. [73])

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} \eta_{ij} dx^i dx^j + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2)\,. \tag{3.19}$$

Here η_{ij}, $i,j \in \{0,1,2,3\}$ denotes the metric on four dimensional Minkowski space and $d\Omega_5^2$ is the usual line element on the unit sphere in $\mathbb{R}^6$. The *radius* L is related to the number of coincident branes by

$$L^4 = 4\pi g_S N (\alpha')^2\,. \tag{3.20}$$

Furthermore, $D3$-brane solutions in string theory have constant axion and dilaton. Both are scalar fields in string theory. The dilaton determines the string coupling constant by $g_S = e^{\phi}$ with ϕ the expectation value of the dilaton field (cf. e.g. [73]). The axion is a

Goldstone boson originating from spontaneously broken Peccei-Quinn symmetry (cf. e.g. [78])[3].

However, in supergravity Dp-brane solutions are known to be $\frac{1}{2}$-BPS states, i.e. they preserve half of the supersymmetries (cf. [73]).

But let us now look at the asymptotic behavior of the $D3$-brane metric (3.19) for $r \to 0$ and $r \to \infty$. One can see immediately that for $r \to \infty$ the metric becomes flat. The so-called near-horizon limit $r \to 0$ can be examined as follows (cf. e.g. [73]). Defining the new variable $u = \frac{L^2}{r}$ and taking $u \to \infty$ one can see that asymptotically the line element (3.19) becomes

$$ds^2 = L^2 \left(\frac{1}{u^2} \eta_{ij} dx^i dx^j + \frac{1}{u^2} du^2 + d\Omega_5^2 \right) . \tag{3.21}$$

This of the form $AdS_5 \times S^5$ with AdS radius and radius of the sphere each given by L. One can see that the horizon region is infinitely far away from any observer, since it lies at the end of an infinite throat as it was also the case for the horizon of an extremal black hole[4].

Now that we have learned about $N = 4$ $SU(N)$ SYM and AdS spaces we are ready to state the AdS/CFT correspondence.

3.3 The AdS/CFT correspondence

The AdS/CFT correspondence states a duality between type IIB string theory on $AdS_5 \times S^5$ and the conformal field theory $N = 4$ SYM in four dimensions which is said to live on the boundary of AdS_5. This can be motivated as follows. In the presentation of the basic ideas we stick closely to [81]. As of now the number N of coincident branes respectively the dimension of the fundamental representation of the $SU(N)$ gauge group is considered to be large. The principal idea of AdS/CFT is then to increase $g_S N$ from very small values to very large ones and look at low energy excitations. Then one compares the decoupled theories one gets for small and for large $g_S N$.

For small values of $g_S N$ we can neglect gravitational effects of the branes and so we can treat them just if they were in flat space. One can see from (3.20) that for $g_S N \to 0$ L goes to zero as well and therefore the metric (3.19) becomes flat. The only effect of the $D3$-branes is that we have to determine appropriate boundary conditions for the strings ending on them. We will discuss the case $g_S N$ small in more detail below.

In contrast, for large values of $g_S N$ gravitational effects are important and we need to deal with the complicated geometry of the $D3$-branes that curve the spacetime in a non-trivial way.

As we have seen before the near horizon geometry is described by $AdS_5 \times S^5$. For understanding AdS/CFT we need to consider on the one hand low energy excitations of

[3] This symmetry was postulated to solve the strong CP problem [79]. The axion is seen as a possible dark matter candidate. The role of axions in string theory is discussed for instance in [80].

[4] The calculation here works analogous to (2.17).

closed strings near the horizon and on the other hand such modes that propagate in the asymptotic flat region far away from the horizon.

An observer in asymptotically flat space perceives a very low energy for modes with finite energy coming from the horizon region since these modes are redshifted. The closer such modes are brought to the horizon the harder it becomes for them to escape from the throat. But this means that an observer at asymptotic infinity just cannot see those modes at all, they are trapped in the throat (cf. e.g. [76]). Low energy modes in the asymptotically flat region correspond to modes with large wavelength compared to the size of the horizon. These excitations are not affected by the throat geometry, they do not see the mouth of the throat. More precicely, one can compute the absorption cross section for low energy excitations. This quantity goes to zero when the energy is lowered (cf. e.g. [76]). Such computations have been done in [82, 83].

All in all we have two systems of low energy modes in type IIB string theory that decouple from each other. One is a system of low energy closed strings propagating in flat ten dimensional Minkowski space. The second system are closed strings in the curved near horizon region with geometry $AdS_5 \times S^5$.

So what happens for small values of $g_S N$? Since $G_{10} \sim g_S^2(\alpha')^4$ (cf. [81]), G_{10} goes to zero in the case $g_S N << 1$, which implies $g_S << 1$ for N large. But G_{10} governs on the one hand the interactions of closed strings propagating in spacetime and on the other hand the interactions of spacetime fields and fields on the branes. So when the coupling goes to zero we get a system of decoupled closed strings in $10D$ Minkowski space and secondly the fields on the branes. Here we consider the N D3-branes as surfaces on which open strings can end and do not take into account any gravitational effects from the branes. Low energy excitations of Dp-branes are always governed by supersymmetric non-Abelian gauge theories (cf. e.g. [77]), and the gauge theory describing the low energy modes of strings ending on $D3$-branes is $N = 4$ SU(N) Yang-Mills theory. Also in the case $g_S N << 1$ we get two decoupled systems. Again one are low energy closed strings in flat ten dimensional space and the second one is the SYM theory describing the dynamics on the branes.

Comparing the situation for small values of $g_S N$ and for large ones, we can see that we always end up with two decoupled systems one of which are closed strings in flat space. The idea of AdS/CFT is now that the respective second system describes the same physics. A priori in $SU(N)$ SYM it is not necessary to restrict to small values of $g_S N$. Similarly, type IIB string on $AdS_5 \times S^5$ theory is also a good theory for small values of $g_S N$ and not only for large ones. So if both theories do not need a particular range of $g_S N$, why should we not assume that they are both valid for all values of $g_S N$ and that the two decoupled systems one gets in both cases are equivalent?

The AdS/CFT correspondence conjectures exactly this. $N = 4$ $SU(N)$ Yang-Mills and type IIB superstring theory on $AdS_5 \times S^5$ are dual to each other. A theory of gravity in a very non-trivial geometry and a conformal supersymmetric field theory describe the same physics!

At this point it is necessary to mention that at present there has been no rigorous proof of the AdS/CFT conjecture. Today AdS/CFT has the status of a conjecture. Nonetheless, all calculations done within this framework indicate that the AdS/CFT correspondence holds.

What makes AdS/CFT also appealing is that the idea of holography is present in it. As already mentioned the field theory is often said to live on the boundary of AdS. When the field theory on the boundary and the gravitational theory in the bulk describe the same physics, all relevant information is already encoded in the boundary which has one dimension less than the bulk. This is analogous to a hologram in which all three-dimensional data is incorporated in a two-dimensional surface.

It is possible to state the AdS/CFT correspondence in different versions (cf. e.g. [73]). The strongest one postulates an equivalence of $N = 4$ SYM theory in four dimensions and the full quantum type IIB string theory on $AdS_5 \times S^5$ for every value of $g_S N$ with the identifications $g_S = g_{YM}^2$ and as mentioned earlier $L^4 = 4\pi g_S N(\alpha')^2$. N is also related to the 5-form flux through the sphere S^5 (cf. e.g. [73]). The problem is that at present there is no tractable string theory quantization on a curved manifold like $AdS_5 \times S^5$ so that we need to turn to versions of AdS/CFT that are accessible in an easier way.

One possibility is to look at the t'Hooft limit. Here the t'Hooft coupling

$$\lambda = g_{YM}^2 N = g_S N \tag{3.22}$$

is kept fixed while letting $N \to \infty$. The t'Hooft coupling controls the field theory amplitudes in the large N limit. This can be seen in the following way (cf. e.g. [81]). Looking at the open strings propagating on the N branes, splitting and rejoining again, we can describe their interactions in a diagrammatic way. The amplitude for a string having its ends on the i-th and j-th brane and apart from that doing nothing at all is simply a constant without any dependence on N. The situation is different when the string splits into two strings and rejoins again. Now there are two interaction vertices either of which produces a factor of g_{YM} in the amplitude (see picture a) in fig. 3.3). Moreover, we need to sum over all possibilities for the brane on which the string could split. This results in a factor of N for the N possible branes. In summa, we get for the amplitude

$$A_1 = c_1 g_{YM}^2 N\,, \tag{3.23}$$

with c_1 a constant. The same method applies when we are considering a string splitting n times. For each splitting we get an additional factor of $g_{YM}^2 N$, and two interaction points as well as one boundary are added to the diagram (see picture b) in fig. 3.3). To summarize, the full amplitude for these kinds of diagrams is given through

$$A = \sum_{n=0}^{\infty} c_n (g_{YM}^2 N)^n = \sum_{n=0}^{\infty} c_n \lambda^n\,. \tag{3.24}$$

Up to here we only looked at interactions where new boundaries are created, but it is also possible to add strips to the diagrams that decrease the number of boundaries (see picture c) in fig. 3.3). Such diagrams cannot be drawn on a sheet of paper or a blackboard since they are not planar in contrast to the ones we considered before. The non-planar diagrams

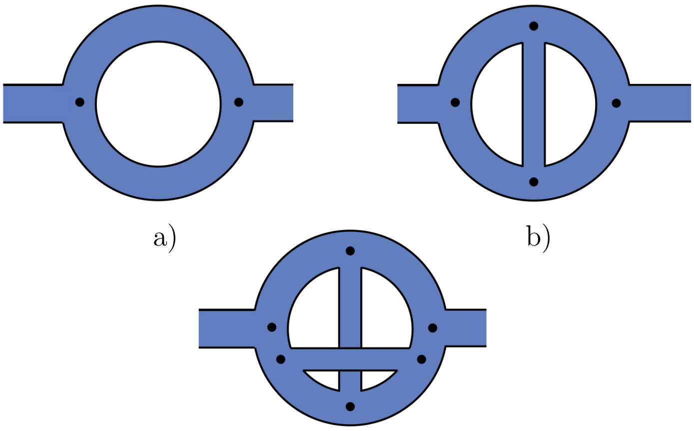

Figure 3.3: Planar and non-planar diagrams for string interactions.

contribute to the full amplitude by a factor of $\frac{g_{YM}^2}{N} = \frac{\lambda}{N^2}$ for each strip that decreases the number of boundaries.

The full amplitude is then given by

$$\mathcal{A} = \sum_{n=0}^{\infty} c_n \lambda^n + f_2(\lambda)\frac{1}{N^2} + f_4(\lambda)\frac{1}{N^4} + \ldots \,. \tag{3.25}$$

At the moment it is not necessary to determine the functions $f_i(\lambda)$ explicitly. From (3.25) one can see immediately that for $N \to \infty$ while λ is fixed only the summands with positive power of N contribute while the others disappear in the large N limit. Further, the amplitude is determined in terms of the t'Hooft coupling rather than the Yang-Mills coupling.

So the t'Hooft limit is well-defined on the field theoretic side. On the gravity side (3.25) corresponds to a string loop expansion. Here $g_S = \frac{\lambda}{N} \to 0$ for fixed λ and $N \to \infty$, so we reach a weak coupling regime. Therefore we may treat the type IIB theory perturbatively in g_S.

Another important limit is the Maldacena limit. Here we let in addition to $N \to \infty$ also $\lambda \to \infty$. This corresponds to strong coupling on the field theory side. On the other hand, on the AdS side we have (cf. (3.20))

$$\frac{L^2}{\alpha'} = \sqrt{4\pi\lambda}\,. \tag{3.26}$$

From this one can see that $\lambda \to \infty$ implies $\alpha' \to 0$, and we end up with classical type IIB supergravity. From here we can see that AdS/CFT provides a tool for treating strongly coupled field theories. Instead of performing calculations in a hardly tractable strongly coupled regime in SYM we could do the computation in supergravity!

Not only that AdS/CFT supports a duality between theories for open and closed strings but also between strongly and weakly coupled theories. Before we come to a more rigorous formulation of AdS/CFT, let us look at a further hint of the duality, namely the symmetries of both theories (cf. e.g.[73, 75]).

We have already reviewed the symmetry group of $N = 4$ SU(N) SYM and found it to be the supergroup $SU(2, 2|4)$ which includes the conformal group $SU(2, 2) \sim SO(2, 4)$ and the R-symmetry group $SU(4) \sim SO(6)$ as subgroups.

Both groups appear also on the gravity side. As mentioned before, the elements of $SO(2, 4)$ describe isometries of AdS_5. Furthermore, the isometries of the sphere S^5 are simply the elements of $SO(6)$ as those are rotations in $\mathbb{R}^6$.

3.4 The AdS/CFT dictionary

Having described the AdS/CFT correspondence in a more "intuitive" way we can now proceed giving a more rigorous concept of AdS/CFT. For this purpose we need to relate gauge invariant operators in $N = 4$ SYM on the one hand and fields in IIB theory on the other hand. This correspondence is given through the field-operator map. We follow mainly [73],[84] and [85]. The ideas were first presented in [84, 86].

Since the $N = 4$ SYM is interpreted to live on the boundary of AdS_5 we actually want to define boundary values for the bulk fields. We will show the concept for the case of a scalar field. On $AdS_5 \times S^5$ all scalar fields can be decomposed into Kaluza-Klein towers on S^5, i.e. they can be expanded in spherical harmonics Y_Δ (cf. also the article by J. Erdmenger in [87]):

$$\varphi(\tilde{x}, \vec{y}) = \sum_{\Delta=0}^{\infty} \varphi_\Delta(\tilde{x}) Y_\Delta(\vec{y}) , \tag{3.27}$$

with $\vec{y}$ coordinates on the 5-sphere and $\tilde{x} = (r, \vec{x})$ coordinates on AdS_5 equipped with the metric

$$ds^2 = \frac{L^2}{r^2}(dr^2 + d\vec{x}^2) . \tag{3.28}$$

Note that the line element can be taken with Euclidean or Minkowskian signature. Here the boundary is located at $r = 0$.

The dimensional reduction to AdS_5 yields a massive wave equation for φ_Δ:

$$(\Box_5 + L^2 m_\Delta^2)\varphi_\Delta(\tilde{x}) = 0 . \tag{3.29}$$

The equation of motion for φ_Δ and also the relation between m_Δ and Δ depends on what kind of field one considers. For a scalar field the relation is given by [5]

$$L^2 m_\Delta^2 = \Delta(\Delta - 4) . \tag{3.30}$$

[5] For arbitrary dimensions AdS_{d+1} m_Δ is given through $L^2 m_\Delta^2 = \Delta(\Delta - d)$ (cf. e.g. [85]).

The wave equation (3.29) has two linearly independent solutions with different asymptotic behavior for $r \to 0$:

$$\varphi_\Delta(r, \vec{x}) \sim \begin{cases} \left(\frac{r}{L}\right)^{\Delta} & \text{(normalizable)} \\ \left(\frac{r}{L}\right)^{4-\Delta} & \text{(non} - \text{normalizable)} \end{cases} \tag{3.31}$$

One comment is in order here. It is not always a priori clear which mode is the normalizable one. Usually, the asymptotic behavior of the normalizable mode is determined by the larger root of (3.30), but in [88] it is argued that for a certain mass range also the smaller root can lead to a normalizable mode.

Also the case $\Delta = 2$ is special since then the two solutions in (3.31) coincide. A second independent solution is obtained by adding a logarithmic term [89]. Here we will not address these more specific cases.

We can now proceed and define boundary values using the non-normalizable modes:

$$\varphi_\Delta^{(0)}(\vec{x}) := \lim_{r\to 0} \varphi_\Delta(r, \vec{x}) \left(\frac{r}{L}\right)^{\Delta-4} . \tag{3.32}$$

Correlation functions in $N = 4$ SYM and fields on $AdS_5 \times S^5$ can now be related as follows.

On the gravity side we have the type IIB action $S[\varphi_\Delta]$ for the bulk fields φ_Δ. This action can be the type IIB supergravity action or might also include α' corrections. Now let $Z[\varphi_\Delta^{(0)}]$ denote the (string theory or supergravity) partition function for φ_Δ equipped with the correct boundary conditions. In the case the supergravity approximation is valid and φ_Δ is a solution to the equations of motion, we can write the partition function in terms of the on-shell supergravity action (cf. [84])

$$Z[\varphi_\Delta^{(0)}] = \exp(-S_{on-shell}[\varphi_\Delta]) . \tag{3.33}$$

For Minkowski signature this reads

$$Z[\varphi_\Delta^{(0)}] = \exp(iS_{on-shell}[\varphi_\Delta]) . \tag{3.34}$$

From now on we will drop the subscript "on-shell" but you should always keep in mind that the action has to be evaluated on a solution.

One can apply these formulae also outside the supergravity regime, but then one has to include e.g. string theory corrections (cf. [84]).

The boundary values appear as sources for the field theory operators $\mathcal{O}_\Delta$ via the identification

$$Z[\varphi_\Delta^{(0)}] = \left\langle \exp\left(\int_{\partial AdS_5} d^4x\, \varphi_\Delta^{(0)} \mathcal{O}_\Delta\right)\right\rangle . \tag{3.35}$$

The expression in brackets is to be understood as the path integral over the canonical fields X^i, λ_a and A_μ of $N = 4$ SYM weighted by the action. The operators one considers are in general polynomials in the canonical fields or their derivatives. It can be shown that Δ is the conformal dimension of the operator $\mathcal{O}_\Delta$ [84].

There is an analogous formula to (3.35) for Minkowskian signature (cf. e.g. [85]),

$$Z[\varphi_\Delta^{(0)}] = \left\langle \exp\left(i \int_{\partial AdS_5} d^4x \, \varphi_\Delta^{(0)} \mathcal{O}_\Delta \right) \right\rangle . \tag{3.36}$$

The equations (3.33) (respectively (3.34)) and (3.35) (respectiveley (3.36)) lay the foundation of the AdS/CFT correspondence. We can compute field theory correlation functions using the bulk partition function.

For example, the expectation value of an operator $\mathcal{O}_\Delta$ can be computed (using Minkowski signature) by

$$\langle \mathcal{O}_\Delta \rangle = -i \frac{\delta \log Z[\varphi_\Delta^{(0)}]}{\delta \varphi_\Delta^{(0)}} = \frac{\delta S[\varphi_\Delta^{(0)}]}{\delta \varphi_\Delta^{(0)}} . \tag{3.37}$$

The last equality is valid if (3.34) applies (cf. e.g. [85]). Usually the action will also include boundary terms. These can come from the renormalization on the one hand but on the other hand there may also be finite local counter terms in the action (cf. [90]). These finite counter terms can always be chosen such that the expectation value takes the particular simple form we display below. It can be shown that for a scalar field with asymptotic behavior for $r \to 0$ according to (3.31)

$$\varphi_\Delta = \left(\frac{r}{L}\right)^{4-\Delta} \varphi_\Delta^{(0)} + \left(\frac{r}{L}\right)^{\Delta} \varphi_\Delta^{(1)} + \dots \tag{3.38}$$

the expectation value of the dual operator $\mathcal{O}_\Delta$ is given by

$$\langle \mathcal{O}_\Delta \rangle = \frac{2\Delta - 4}{L} \varphi_\Delta^{(1)} . \tag{3.39}$$

Indeed, the derivation of (3.39) makes use of boundary terms in the action. We refer to [85] for the details of the calculation. It is often said that the non-normalizable mode gives the source of the dual operator and the normalizable mode its expectation value. The latter is now clear in light of (3.39). This result can be generalized for other kinds of fields than scalars. In AdS/CFT the expectation value of the dual operator is always given in terms of the coefficient of the subdominant independent solution in the expansion of the field near the boundary. The coefficient of the dominant term however gives the source term for the operator. We will see in subsection 3.6 how this works in a particular example.

3.5 Generalizations and applications of AdS/CFT

So far we have established the AdS/CFT conjecture for $AdS_5 \times S^5$ and the field theoretic dual $N = 4$ SYM. But this is not the only possibility for having field theories with gravitational duals. One straightforward generalization of the presented ideas is to change from $AdS_5 \times S^5$ to different dimensions for the AdS space. In M-theory $AdS_4 \times S^7$ and $AdS_7 \times S^4$ are of interest (cf. e.g.[75]). All the formulas from the previous section generalize to different dimensions in a straightforward way. For this reason we will from now on work in arbitrary dimension of the anti-de Sitter space. It is also possible that the sphere is replaced by a different kind of space. Such a case was considered e.g. in [91] where it was argued that there exists a field theory dual to string theory on the space $AdS_5 \times X_5$ with $X_5 = (SU(2) \times SU(2))/U(1)$.

Moreover, one could ask whether there exist gravity duals for field theories with much less symmetry than $N = 4$ SYM. For example, it would be very nice to have a description of QCD which is strongly coupled in terms of a weakly coupled theory of gravity.

One way of breaking some symmetry on the field theoretic side is to introduce temperature. A temperature gives rise to an energy scale in a natural way. This does not a priori mean that a theory at finite temperature cannot be conformal. We will see that there are theories in which all non-zero temperatures are equivalent and therefore the theory is still conformal. To break conformal invariance one has to introduce a second energy scale, e.g. a chemical potential. What then really introduces a scale in the theory is the ratio of the temperature and the chemical potential. We will come back to that shortly.

So how could one introduce in a first step a temperature on the gravity side? Right – a good idea might be to look at a black hole in anti-de Sitter space!

As already mentioned AdS_{d+1} is a solution to the Einstein gravity action with negative cosmological constant, which we recapitulate here for convenience:

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x\sqrt{-g} \left(R + \frac{d(d-1)}{L^2} \right) . \tag{3.40}$$

The following discussion is mainly based on [85]. A black hole solution to the equations of motion derived from this action is the AdS Schwarzschild solution with line element (in Minkowski signature)

$$ds^2 = \frac{L^2}{r^2} \left(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^i dx^i \right) \tag{3.41}$$

with

$$f(r) = 1 - \left(\frac{r}{r_+} \right)^d . \tag{3.42}$$

At $r = r_+$ the tt-component of the metric vanishes and hence there is a horizon at $r = r_+$. For $r \to 0$ instead we rediscover the metric of AdS_{d+1}. For $r \to 0$ we see that $f(r) \to 1$ and so asymptotically the line element looks like

$$ds^2 = L^2 \left(-\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + \frac{dx^i dx^i}{r^2} \right) \tag{3.43}$$

which is AdS_{d+1}.

The temperature of the AdS Schwarzschild black hole is

$$T = \frac{d}{4\pi r_+} . \tag{3.44}$$

One method to obtain the temperature of a black hole that is different from the method involving the surface gravity we mentioned in chapter 2 is to Wick rotate to Euclidean time and to set some periodicity conditions for the Euclidean imaginary time. We will present this method briefly.

For the AdS black hole one gets in the case $t \mapsto -i\tau$

$$ds^2 = \frac{L^2}{r^2}\left(f(r)d\tau^2 + \frac{dr^2}{f(r)} + dx^i dx^i\right) . \tag{3.45}$$

For the solution to be regular at the horizon $r = r_+$ one has to require periodicity of the Euclidean time according to[6]

$$\tau \sim \tau + \frac{4\pi}{|f'(r_+)|} = \tau + \frac{4\pi r_+}{d} . \tag{3.46}$$

The temperature of the black hole is then given by the inverse of the periodicity, i.e. we get for the AdS Schwarzschild black hole exactly formula (3.44). To show that the black hole temperature is also the temperature of the dual field theory one can argue as follows like it was presented e.g. in [85]. The bulk metric can be pulled back to the boundary of AdS_{d+1} and gives there rise to a boundary metric $g^{(0)}_{\mu\nu}$ subject to

$$g_{\mu\nu}(r) = \frac{L^2}{r^2} g^{(0)}_{\mu\nu} + \dots \tag{3.47}$$

for r going to zero [7]. Comparing (3.45) with (3.47) one can see that

$$ds^{2,(0)} = d\tau^2 + dx^i dx^i \tag{3.48}$$

with τ identified periodically as in (3.46). This metric can be taken as the background metric for the field theory living on the boundary. From quantum field theory we know that a field theory at finite temperature corresponds to formally having a Euclidean time with a periodicity of the inverse temperature (cf. e.g. [92]). But this leads to the same temperature as the Hawking temperature of the black hole in the bulk. The only parameter appearing in the expression for the temperature is the horizon radius r_+. By the coordinate transformation $(t, r, x^i) \mapsto r_+(t, r, x^i)$ the radius r_+ can be removed from the metric. But then r_+ also does not enter in the temperature which reads now $T = \frac{d}{4\pi} \neq 0$. This means that all non-zero temperatures are equivalent and the dual field theory is still conformal.

---

[6] This can be seen in the following way. Close to the horizon one can approximate $f(r) \approx f'(r_+)(r - r_+)$ as $f(r_+) = 0$. By introducing the new coordinates $\rho^2 = \frac{4}{|f'(r_+)|}(r - r_+)$ and $\phi = \frac{|f'(r_+)|}{2}\tau$ we can rewrite (in the near-horizon approximation) $f(r)d\tau^2 + \frac{dr^2}{f(r)} = d\rho^2 + \rho^2 d\phi^2$. To avoid a conical singularity $\phi \sim \phi + 2\pi$. This leads to the periodicity of τ in(3.46).

[7] Earlier we showed that the boundary of AdS_5 has a conformal structure and the same is of course true for higher dimensional anti-de Sitter spaces. Therefore the metric at the boundary is only defined up to an overall rescaling but this does not affect the main argument.

Here a natural question arises: How can we implement a theory that distinguishes between different temperatures? To do so we have to include a Maxwell field in the gravitational action,

$$S = \int d^{d+1}x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}\right] \tag{3.49}$$

where $F = dA$ is the field strength. In chapter 4 we will also include non-Abelian gauge fields, but let us first show the principle mechanisms in the Abelian case. Again we follow the discussion in [85].

We require that all relevant quantities depend only on the radial variable r. As was done in the AdS Schwarzschild case for the metric, we assume that the vector potential can be pulled back to the boundary, i.e.

$$A_\mu(r) = A_\mu^{(0)} + \dots \quad \text{for}\, r \to 0\,. \tag{3.50}$$

To introduce a scale in the field theory one can add e.g. a chemical potential $\mu = A_t^{(0)}$. As we have already seen, to break conformal invariance it is not enough to introduce a temperature in the theory. We need an additional scale with which the temperature can be compared. This scale will be given by the chemical potential.

Having included a chemical potential one has to look for solutions to the Einstein and Maxwell equations with a vector potential of the form

$$A = A_t(r)dt\,. \tag{3.51}$$

Another possibility to introduce an energy scale is to add a magnetic field $B = F_{xy}^{(0)}$ (see e.g. [93, 94, 95, 96, 97]), but we will restrict ourselves to the case of adding a chemical potential since we will do exactly this in the computations in chapter 4.

A solution to Einstein-Maxwell theory with a chemical potential is given by the AdS Reissner-Nordström black hole with line element

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^i dx^i\right) \tag{3.52}$$

where the function f looks now more complicated than in the AdS Schwarzschild case:

$$f(r) = 1 - \left(1 + \frac{r_+^2\mu^2}{\gamma^2}\right)\left(\frac{r}{r_+}\right)^d + \frac{r_+^2\mu^2}{\gamma^2}\left(\frac{r}{r_+}\right)^{2(d-1)} \tag{3.53}$$

with

$$\gamma^2 = \frac{(d-1)g^2L^2}{(d-2)\kappa^2}\,. \tag{3.54}$$

The horizon is located at r_+. The metric is in this form given in [85]. A more general discussion of AdS Reissner-Nordström black holes is presented in [8]. With the conventions used in [85] the t-component of the gauge potential is

$$A_t(r) = \mu\left[1 - \left(\frac{r}{r_+}\right)^{d-2}\right]\,. \tag{3.55}$$

From this one can see that

$$\lim_{r\to 0} A_t(r) = \mu \tag{3.56}$$

and comparing this with the lines below (3.50) shows that μ is the chemical potential. It is required that A_t vanishes at the horizon [98] because of the bifurcation surface of the Killing vector $\frac{\partial}{\partial t}$ located at the horizon.

Going to Euclidean time the temperature of the AdS Reissner-Nordström black hole can be obtained in the same manner as for AdS Schwarzschild and one gets

$$T = \frac{1}{4\pi r_+}\left(d - \frac{(d-2)r_+^2\mu^2}{\gamma^2}\right). \tag{3.57}$$

Rescaling the coordinates in the same way as in the AdS Schwarzschild case, one finds that r_+ appears in the temperature only in the combination $\mu^2 r_+^2$. This means that the temperature will always depend on the chemical potential and so we have different temperatures for different chemical potentials. We achieved what we wanted – the chemical potential μ together with the temperature sets a scale in our theory.

Moreover, it is also possible to go to zero temperature in a continuous way, i.e. to the extremal Reissner-Nordström black hole, which also could not be done for AdS Schwarzschild.

3.6 Linear response

Up to now we have seen how to include finite temperature and a chemical potential on the field theory side and we have found gravity duals for those. So what happens when we add a small perturbation to our solutions? As long as the perturbations are small we can work with linear response theory. Here the basic object one wants to compute is the retarded Green's function. It gives the linear dependence of the expectation values of operators on their sources, i.e. a definition for the retarded Green's function in frequency space is (cf. e.g. [85])

$$\langle \mathcal{O}_A \rangle(\omega, \vec{k}) = G^R_{\mathcal{O}_A\mathcal{O}_B}\varphi_B^{(0)}(\omega, \vec{k}). \tag{3.58}$$

Here φ_B denotes a bulk field with boundary value $\varphi_B^{(0)}$ as before.

Our discussion of Green's functions and transport coefficients in this section is again based on [85].

Perturbing the boundary values causes a perturbation in the whole bulk and it is common to make the following separation ansatz for the bulk perturbation

$$\varphi_A(r) \mapsto \varphi_A(r) + \tilde{\varphi}_A(r)e^{i(\vec{k}\cdot\vec{x}-\omega t)}. \tag{3.59}$$

Plugging $\tilde{\varphi}_A$ into the linearized bulk equations of motion yields the equation of motion for the perturbation. Again we need to impose correct boundary conditions on the perturbation. We will display these conditions for the scalar field case. The condition at the AdS boundary ($r \to 0$) is the same as in the unperturbed case,

$$\tilde{\varphi}_A(r) = \left(\frac{r}{L}\right)^{(d-\Delta)}\tilde{\varphi}_A^{(0)} + \ldots . \tag{3.60}$$

The boundary condition at the horizon is a bit more complicated. We do not want modes to come out of the event horizon of the black hole. In order to obtain the retarded Green's function instead of the advanced one we require the perturbation to be ingoing at the horizon. This means that for $r \to r_+$ (cf. [85])

$$\tilde{\varphi}_A(r) = \begin{cases} e^{-i\frac{4\pi\omega}{T}\log(r-r_+)}(c_A + \dots)\,, & T \neq 0\,, \\ e^{i\frac{\omega L_2^2}{r-r_+}}(\tilde{c}_A + \dots)\,, & T = 0\,. \end{cases} \tag{3.61}$$

The c_A, $\tilde{c}_A$ and the dots denote functions of r that are subleading compared to the exponential function. L_2 is the radius of the AdS_2 part that arises in the near-horizon geometry in the extremal case (cf. e.g. [85]). We have seen this near-horizon behavior already in chapter 2 even for the asymptotically flat case.

Now we can read off the Green's function

$$G^R_{\mathcal{O}_A\mathcal{O}_B} = \frac{\delta\langle\mathcal{O}_A\rangle}{\delta\tilde{\varphi}_B^{(0)}} = \frac{2\Delta_A - d}{L}\frac{\delta\tilde{\varphi}_A^{(1)}}{\delta\tilde{\varphi}_B^{(0)}}\,. \tag{3.62}$$

The first equality follows from (3.58) and the second from the generalization to d dimensions of (3.39). The first equality is of course not only true for scalar fields but applies to every kind of perturbation one could think of.

As an example we will show how to compute the electrical conductivity in a 2+1 dimensional field theory. Some real material like Graphene can be described by such a theory at low energies and therefore it would be interesting to compare the AdS/CFT results with the "real world" measurements in Graphene [85].

The gravitational background we consider is given by the four-dimensional extremal AdS Reissner-Nordström black hole we discussed before.

According to Ohm's law the current (here in x-direction) and the electric field are related by

$$\langle J_x\rangle = \sigma E_x\,. \tag{3.63}$$

We will only look at the zero momentum case $\vec{k} = 0$. We need to add a perturbation in the bulk which causes the expectation value of the current in the boundary field theory. At zero momentum the electric field strength at the boundary is given in terms of the vector potential via

$$E_x = -\partial_t A_x^{(0)}\,. \tag{3.64}$$

So a natural choice for perturbing the background is to add a perturbation $A_x(r,t) = \tilde{A}_x(r)e^{-i\omega t}$ in the Maxwell vector potential.

At this point it is important to mention that at the boundary we have an electric field but no dynamical photon. This means that in the field theory the $U(1)$ symmetry is a global symmetry. In the bulk instead the $U(1)$ symmetry is gauged. This is also true in cases with a different symmetry group than $U(1)$: A global symmetry of the field theory coresponds to a gauged symmetry in the bulk (cf. e.g. [85]). In chapter 4 we will look at a global $SU(2)$ symmetry on the field theory side. We call this flavor symmetry in analogy to the global flavor symmetry in QCD.

But let us proceed with the calculation of the conductivity. Equation (3.59) together with setting the momentum to zero yields

$$E_x = -\partial_t \left(\tilde{A}_x^{(0)} e^{-i\omega t} \right) = i\omega \tilde{A}_x^{(0)} e^{-i\omega t} = i\omega A_x^{(0)} \,. \tag{3.65}$$

Using Ohm's law again we can determine the relation between the Green's function and the conductivity. As a first step we have

$$\langle J_x \rangle = \sigma E_x = \sigma \, i\omega A_x^{(0)} \,. \tag{3.66}$$

This is a relation between a source term and an expectation value, so we know from (3.58) that the proportionality is given by the retarded Green's function,

$$\langle J_x \rangle = G^R_{J_x J_x} A_x^{(0)} \,. \tag{3.67}$$

From the last two equations it follows that

$$\sigma = -\frac{i}{\omega} G^R_{J_x J_x} \,. \tag{3.68}$$

Now we need to determine the asymptotic behavior of the perturbation close to the boundary at $r = 0$. To do so one computes the linearized equation of motion for A_x. It can be shown that close to the boundary the solution looks like (cf. e.g. [85])

$$A_x(r) = A_x^{(0)} + \frac{r}{L} A_x^{(1)} + \ldots \,. \tag{3.69}$$

The constant term to leading order in r yields the source for J_x. We will see shortly how the expectation value of J_x is related to the next subleading term. From (3.69) we can see that at the boundary

$$\begin{aligned} A_x^{(0)} &= \lim_{r\to 0} A_x(r) \,, \\ A_x^{(1)} &= \lim_{r\to 0} (L A_x'(r)) \,. \end{aligned} \tag{3.70}$$

Here $'$ denotes differentiation with respect to r.

Now we can use (3.37) in order to determine the expectation value of J_x in AdS/CFT:

$$\langle J_x \rangle = \frac{\delta S}{\delta A_x^{(0)}} = \lim_{r\to 0} \left(\frac{f(r)}{g^2} A_x'(r) \right) = \frac{1}{g^2 L} A_x^{(0)} \,. \tag{3.71}$$

Here S is the Einstein-Maxwell action (3.49) in four dimensions and the function f was defined in (3.53). The second equality follows from plugging the background solution into the Einstein-Maxwell action and subsequently varying with respect to $A_x^{(0)}$. More details can be found in e.g. [85].

From (3.62) we can compute the Green's function

$$G^R_{J_xJ_x} = \frac{\delta\langle J_x\rangle}{\delta A_x^{(0)}} = \frac{1}{g^2L}\frac{\delta A_x^{(1)}}{\delta A_x^{(0)}} \tag{3.72}$$

which yields by virtue of (3.68) the conductivity

$$\sigma(\omega) = \frac{-i}{\omega g^2 L}\frac{\delta A_x^{(1)}}{\delta A_x^{(0)}}\,. \tag{3.73}$$

In general, the conductivity has to be computed numerically, and we will give explicit examples later. It can be shown that for AdS_4 Schwarzschild (i.e. $\mu = 0$) the conductivity takes the constant value (cf. e.g [99])

$$\sigma = \frac{1}{g^2L}\,. \tag{3.74}$$

The fact that the conductivity in that case is constant was first observed in [100].

Transport coefficients have been calculated with AdS/CFT methods in different setups. One very interesting framework is the construction of possible gravitational duals to superconductors. We will review this in the next section.

3.7 Basic aspects of holographic superconductivity

"...Aber warum weigern Sie sich nicht, Licht anzudrehen, wenn Sie von Elektrizität nichts verstehen?"

"...But why do you not refuse to turn on the light if you do not know anything about electricity?"

"Newton" to the inspector in Friedrich Dürrenmatts "The Physicists"

Surely the quotation above from Dürrenmatts satiric drama "The Physicists" has to be interpreted within the context of the threat of a nuclear war in the 1960ies where a person who does not know anything about the physical background might be able to fire a nuclear bomb.

However, it makes one rethink whether we use physical phenomena in technical applications without having a profound knowledge about them. But when can we claim at all to understand something properly?

It is a similar situation when we consider superconductors. On the one hand it is possible to use superconductors technically, e.g. to generate huge magnetic fields. This is done for instance at the LHC. On the other hand there are many phenomena in superconductivity which are not understood very well like high temperature superconductors.

A powerful theory for describing superconductivity is the BCS theory. In this model the charge carriers are pairs of electrons (so-called Cooper pairs) bound by the exchange of phonons, i.e. excitations of the atomic lattice. The spins of electrons in a Cooper pair have opposite directions which results in a total spin of zero for the Cooper pair (cf. e.g. [101]). In most superconductors the Cooper pairs have also zero angular momentum and are therefore called s-wave superconductors.

However, there are superconducting materials which cannot be described by BCS theory. For example, the spins of two electrons forming a Cooper pair can couple into a triplet state. Then the angular momentum L has to be an odd number due to the requirement of antisymmetry of the wave function. The possibility with the lowest energy is $L = 1$. These kinds of superconductors are then called p-wave superconductors (cf. e.g. [102]).

One approach to gain insight into the mechanism of superconductivity apart from BSC theory is to construct gravity duals for the field theory describing superconductivity. Within this framework gravitational toy models have been constructed which were interpreted as the duals of s- and p-wave superconductors.

We will at first present the basic ideas of the holographic description of superconductivity in the case of s-wave superconductors. In chapter 4 we will address in depth p-wave superconductors. A holographic approach to s-wave superconductors is e.g. presented in [103, 104, 99]. This section is based on these papers.

For the holographic description of an s-wave superconductor one considers a field theory in 2+1 dimensions and thence a four-dimensional gravitational dual. This is motivated by the fact that many unconventional superconductors like the cuprates are layered. Thus an effective 2+1 dimensional description is adequate (cf. [103]).

The basic idea is to add an additional field to the four-dimensional Lagrangian which in the case of an s-wave superconductor will be a charged scalar field. The operator dual to the scalar field develops a non-zero vacuum expectation value below a critical temperature T_c. This is interpreted as the onset of superconductivity in the field theory. On the gravity side the scalar field corresponds to a "hairy" black hole solution.

In four dimensions the no-hair conjecture states that a stationary black hole in asymptotically flat space is completely determined by its mass, charge and angular momentum (cf. e.g. [3]).

Of course this need not be true for a black hole in asymptotically AdS_4. Here it is well possible for a black hole to have a "hair", which in the case considered in [103, 104] is a non-trivial profile for the scalar field.

In [104] the following ansatz for the Lagrangian is used:

$$\mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{2}{L^2} |\psi|^2 - |\nabla\psi - iqA\psi|^2 \,. \tag{3.75}$$

This is the usual Einstein-Maxwell Lagrangian with an additional scalar field which is complex and carries electric charge. That particular choice can be motivated by Landau-Ginzburg theory for superconductivity. Here a scalar field is introduced to describe the Cooper pairs. This field carries electric charge due to the two electrons forming a Cooper pair and therefore it has to be complex (cf. e.g. [92]). In [104] the ansatz

$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{g(r)} + r^2(dx^2 + dy^2) \tag{3.76}$$

for the line element of the hairy black hole is used together with

$$A = \phi(r)dt\,, \quad \psi = \psi(r)\,. \tag{3.77}$$

Note that for this ansatz the radial coordinate ranges over $[r_+, \infty[$, i.e. the boundary is in this coordinates located at $r = \infty$. From the equations of motion it can be derived that close to the boundary ϕ and ψ behave as

$$\begin{aligned} \phi(r) &= \mu - \frac{\rho}{r} + \dots\,, \\ \psi(r) &= \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + \dots\,. \end{aligned} \tag{3.78}$$

There are two different choices of boundary conditions for ψ. Setting one of the $\psi^{(i)}$ to zero, the other one determines the expectation value of the dual operator $\mathcal{O}_i$ (cf. [104]). There is no source term for the operator $\mathcal{O}_i$ and so the symmetry is now spontaneously broken. In [104] the authors show that above a certain critical temperature T_c the solution derived from the Lagrangian (3.75) is the Reissner-Nordström black hole with the scalar field $\psi = 0$ whereas for lower values of the temperature one of the operators $\mathcal{O}_i$,$i \in \{1, 2\}$ develops a non-zero expectation value which means that the scalar field ψ is non-trivial (see also fig. 3.4[8]). Furthermore, at T close to T_c one has $\langle \mathcal{O}_i \rangle \sim T_c^i(1 - \frac{T}{T_c})^{\frac{1}{2}}$ for $i \in \{1, 2\}$ which is typical for a second-order phase transition [104].

At the end of the day, one wants to compute the frequency dependent conductivity to see whether superconductivity appears below T_c. For this purpose one needs to implement the method described in the previous section. Also for studying the superconducting case one includes a perturbation A_x with time dependence proportional to $e^{-i\omega t}$. The equation of motion for A_x is again obtained by plugging the perturbation into the linearized equations of motion. Often one works in a limit in which the scalar and gauge field do not back react on the metric. This limit is called the probe limit. In [104] the equations are worked out even with back-reaction but the probe limit was used successfully in the earlier paper [103]. The asymptotic behavior of the perturbation can be obtained from its equation of motion and comes out to be (for $L = 1$)

$$A_x(r) = A_x^{(0)} + \frac{A_x^{(1)}}{r} + \dots \quad \text{for}\, r \to \infty\,. \tag{3.79}$$

[8]We would like to thank Christopher Herzog for the permission to use the graphics from his homepage.

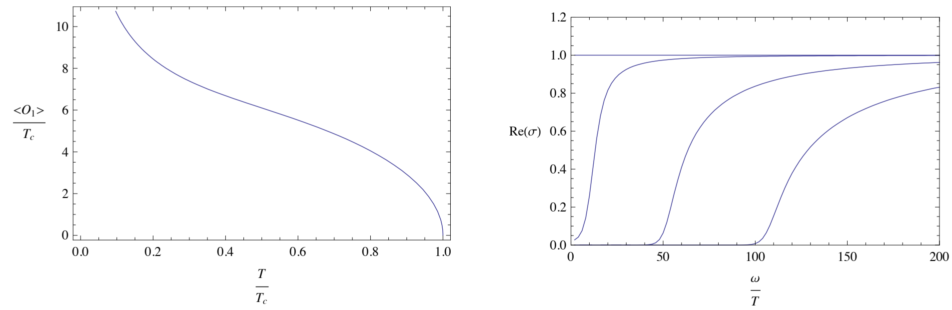

Figure 3.4: Development of the condensate of the operator $\mathcal{O}_1$ (on the left) and the formation of the frequency gap for this operator.

This is the same as (3.69) when taking into account the different radial variables. The conductivity can now be computed exactly in the same manner as for the non-superconducting case. The principle method is the same, but of course the differential equations one actually has to solve are different ones. Let us recall that

$$\sigma = \frac{-i}{\omega g L^2}\frac{\delta A_x^{(1)}}{\delta A_x^{(0)}} . \tag{3.80}$$

Here g is the coupling constant for the Maxwell field which in (3.75) was absorbed in the field strength.

Some of the results of [104] and [103] are the following. The imaginary part of the conductivity has a pole at $\omega = 0$ which leads to a delta function contribution in the real part of the conductivity due to the Kramers-Kronig relations. This delta function appears also in the normal state when going away from the probe limit. In the latter case the delta function is interpreted as a result of translational invariance instead of superconductivity and the coefficient in front of it differs from the superconducting state. In the superconducting phase the real part of the conductivity drops for frequencies less than a gap frequency ω_g (see fig. 3.4). Both of these results are consistent with results for conventional superconductors. A certain amount of energy is needed to break a Cooper pair and create two independent electrons. Then ω_g matches this energy. In [104] the authors found that the frequency gap matches the BCS result only in the probe limit indicating that there are different mechanisms at work.

Also the delta function contribution in the real part of the conductivity appears in conventional superconductors. One way to obtain the conductivity is to calculate it using the Drude model and then taking the limit of the relaxation time going to infinity. This procedure also leads to a delta peak in the real part of the conductivity (cf. e.g. [105]).

Having introduced the AdS/CFT correspondence and the holographic approach to field theoretic problems like the description of s-wave superconductivity, we will see in the next chapter how these techniques can be applied in a specific setup involving a non-Abelian symmetry group. Also in the later chapters 5 and 6 we will stay within the context of AdS/CFT. There we will concern ourselves with the attractor mechanism established at the beginning of chapter 2 and construct some black solutions including such with AdS asymptotics.

Chapter 4

Holographic Flavor Conductivity

In this chapter we investigate some features of the frequency dependent conductivity tensor in gauge theories dual to four-dimensional Einstein-Yang-Mills theory with $SU(2)$ gauge group. This theory was used before in [7]in order to describe a p-wave superconductor holographically.

We find that the Hermitean part of the conductivity tensor in the normal state of the dual field theory has a negative eigenvalue for a certain range of frequencies. This leads to a negative entropy production rate.

At the time of writing this thesis we could not give a satisfying explanation of this result. We will discuss this issue in 4.3.

We have already seen in 3.7 how gravity duals for s-wave superconductors have been constructed. One of the main differences between s-wave and p-wave superconductors is that in the first case the order parameter that indicates the phase transition to superconductivity is a complex scalar. For a p-wave superconductor instead it is a vector (see [106] for a summary of p-wave superconductors).

In the construction of a gravitational dual of a p-wave superconductor one is therefore led to search for black holes with vector hair rather than scalar hair. Such hairy black holes have been constructed within Einstein-Yang-Mills theory, especially for the case of an $SU(2)$ gauge group. Results on Yang-Mills black holes in asymptotic AdS space even with more general gauge groups are reviewed in [4].

In [6, 7] it was shown how a symmetry breaking condensate of a non-Abelian gauge field arises in four-dimensional Einstein-Yang-Mills theory with gauge group $SU(2)$. In [7] the $SU(2)$ symmetry is explicitly broken by the background gauge field $A = \Phi(r)\tau^3 dt$. Below a critical temperature the background becomes unstable and the true ground state is given by the charged black hole with vector hair, for instance in τ^1 direction. This vector hair breaks the $U(1)_3$ gauge group spontaneously. By looking at fluctuations of the gauge field around this configuration [7] calculated the conductivity related to the $U(1)_3$ subgroup and found a gap at low temperature and low frequency as well as a delta function peak at frequency $\omega = 0$. This was interpreted as a sign of superconductivity in the dual field theory.

We consider the other components of the conductivity tensor in the normal state of this theory, in particular those involving currents in the τ^1 and τ^2 directions. Even in the normal state we find that for small frequencies one of the eigenvalues of the Hermitean part of the conductivity tensor turns negative.

4.1 Setup

In view of the gauge/gravity correspondence, we consider the four-dimensional Einstein-Yang-Mills theory with a cosmological constant

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{4} (F^a_{\mu\nu})^2 + \frac{6}{L^2} \right] , \tag{4.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu , \quad \mu, \nu \in \{t, r, x, y\} , \tag{4.2}$$

is the field strength of an $SU(2)$ gauge group. We are interested in describing a thermal state of the dual field theory, and, thus, our starting point is a black hole solution of (4.1). A charge can be introduced into the thermal ensemble by adding a charge for the black hole. We are only considering the limit of large gL for simplicity. It was argued in [6] that in this limit the backreaction of the gauge fields on the metric can be neglected. Thus, we can consider the AdS_4-Schwarzschild black hole metric

$$ds^2 = \frac{1}{L^2} \left(-f(r)dt^2 + r^2(dx^2 + dy^2) \right) + L^2 \frac{1}{f(r)} dr^2 \tag{4.3}$$

with (setting the horizon to $r_H = 1$)

$$f(r) = r^2 - \frac{1}{r} . \tag{4.4}$$

The temperature of the black hole is then given by (cf. [7])

$$T = \frac{3}{4\pi L^2} . \tag{4.5}$$

In addition, the background contains a gauge field

$$A = \Phi(r)\tau^3 dt + w(r)\tau^1 dx , \tag{4.6}$$

where in general (cf. [7])

$$\Phi(r) = \mu - \frac{\rho L \kappa^2}{r} + \mathcal{O}(1/r^2) \tag{4.7}$$

with μ the chemical potential and ρ the charge density of the dual field theory. Moreover, $w(r)$ in (4.6) describes the condensate that develops when the temperature drops below a critical value [7]. The background gauge field in the normal state leaves the $U(1)_3$ subgroup of $SU(2)$ unbroken, which is furthermore broken spontaneously, once the vector hair $w(r)$ forms. Since [7] was the starting point for our considerations we will review their calculations here for completeness. So let us for the time being keep the condensate w although later we will perform our calculations only in the normal state where $w = 0$.

In [7] the components of the gauge field are rescaled according to

$$\tilde{\Phi} = gL^2\Phi \,, \quad \tilde{w} = gL^2 w \,. \tag{4.8}$$

It is also convenient to introduce a rescaled chemical potential via

$$\tilde{\mu} = gL^2\mu \,. \tag{4.9}$$

The relevant Yang-Mills equations for the rescaled functions are then

$$\begin{aligned} \tilde{\Phi}'' + \frac{2}{r}\tilde{\Phi} - \frac{1}{r(r^3-1)}\tilde{w}\tilde{\Phi} &= 0 \,, \\ \tilde{w}'' + \frac{1+2r^3}{r(r^3-1)}\tilde{w}' + \frac{r^2}{(r^3-1)^2}\tilde{\Phi}^2\tilde{w} &= 0 \,. \end{aligned} \tag{4.10}$$

The goal is now to solve this coupled system of differential equations together with appropriate boundary conditions for $\tilde{\Phi}$ and $\tilde{w}$ both at the horizon and at asymptotic infinity (i.e. at the boundary of the asymptotically AdS_4 spacetime).

It is demanded that at asymptotic infinity the fields $\tilde{\Phi}$ and $\tilde{w}$ behave like

$$\begin{aligned} \tilde{\Phi}(r) &= \tilde{p}_0 + \frac{\tilde{p}_1}{r} + \dots \,, \\ \tilde{w}(r) &= \frac{\tilde{W}_1}{r} + \dots \,. \end{aligned} \tag{4.11}$$

Actually, it follows from the Yang-Mills equations that at infinity

$$\tilde{w}(r) = \tilde{W}_0 + \frac{\tilde{W}_1}{r} \,. \tag{4.12}$$

But since an additional constant term $\tilde{W}_0$ in the expansion of $\tilde{w}$ would result in a source term for the corresponding current such a term is forbidden when one wants to model spontaneous symmetry breaking. The near-horizon expansion is

$$\begin{aligned} \tilde{\Phi}(r) &= \tilde{\Phi}_1(r-1) + \dots \,, \\ \tilde{w}(r) &= \tilde{w}_0 + \tilde{w}_2(r-1)^2 + \dots \,. \end{aligned} \tag{4.13}$$

Analogously as for asymptotic infinity, also this form of $\tilde{\Phi}$ and $\tilde{w}$ close to the horizon can be obtained from solving the equations (4.10) in this regime. The coefficient $\tilde{w}_2$ is not independent of $\tilde{w}_0$. To summarize, the method is the following. The general form of the conditions at the boundary and close to the horizon is obtained by solving the Yang-Mills equations in both regimes. These boundary conditions are then supplemented by the additional requirement of spontaneous symmetry breaking, i.e. that there is no constant term $\tilde{W}_0$. Then one needs to search for a solution that fulfills both kinds of boundary conditions.

The coefficients $\tilde{p}_0$ and $\tilde{p}_1$ are related to the chemical potential and charge density via (cf. [7])

$$\begin{aligned} \tilde{p}_0 &= gL^2\mu = \tilde{\mu}\,, \\ \tilde{p}_1 &= -g\kappa^2 L^3 \rho\,. \end{aligned} \tag{4.14}$$

The order parameter is $\tilde{W}_1$ since it is related to the expectation value of the $SU(2)$ currents j_i^a, $i \in \{x, y\}$[1] through (cf. [7])

$$\langle j_i^a \rangle \sim \tilde{W}_1 \delta_i^1 \delta_1^a\,. \tag{4.15}$$

An additional constant term in the expansion of $\tilde{w}$ in (4.11) would correspond to a source term for the current j_i^1. Since there is no source but only a non-zero expectation value for the current the symmetry is spontaneously broken.

A solution to the described boundary value problem can be obtained using the so-called shooting technique. With this method one can transform a boundary value problem into an initial value problem. The latter can easily be solved by Mathematica's NDSolve. The first step to start the shooting algorithm is to transform the equations (4.10) into a system of first order differential equations. The boundary conditions at infinity are then translated into the problem of finding the zeros of an objective function. We do not want to go into the technical details here. An instruction how to implement the shooting technique and some examples can be found in [107]. Basically for the case we consider one includes an additional constant term $\tilde{W}_0$ in the expansion for $\tilde{w}$ (4.11). Then one integrates (4.10) numerically from the horizon to the boundary using the initial conditions in terms of $\tilde{\Phi}_1$ and $\tilde{w}_0$. For $\tilde{w}_0 \neq 0$ this will lead to a non-trivial numerical solution for the condensate $\tilde{w}$. Of this numerical solution it is required that it obeys (4.11), i.e. that the additional constant $\tilde{W}_0$ vanishes. So what one actually does is to compute $\tilde{W}_0(\tilde{\Phi}_1, \tilde{w}_0)$ and to look for the zeros of this function. Then one takes the smallest zero (it is argued in [7] that the larger zeros correspond to thermodynamically disfavoured solutions). The values of $\tilde{\Phi}_1$ and the corresponding values of $\tilde{w}_0$ such that there is no constant term in the expansion of the condensate are displayed in figure 4.1. This is basically the same as figure 2 in [7].

Figure 4.1 can be interpreted as follows. The normal state with $\tilde{w} = 0$ is a solution to the equations of motion (4.10) independently of the chosen value of $\tilde{\Phi}_1$. But for values of $\tilde{\Phi}_1$ below a certain critical value there exists also a solution to (4.10) with non-zero $\tilde{w}$ which has lower free energy. In [7] this was interpreted as a phase transition from the normal state to a superconducting one. The critical temperature of the phase transition can be obtained in the following way.

[1] In our notational convention Greek indices run over the bulk indices $\{t, r, x, y\}$ and latin indices over field theory indices $\{t, x, y\}$.

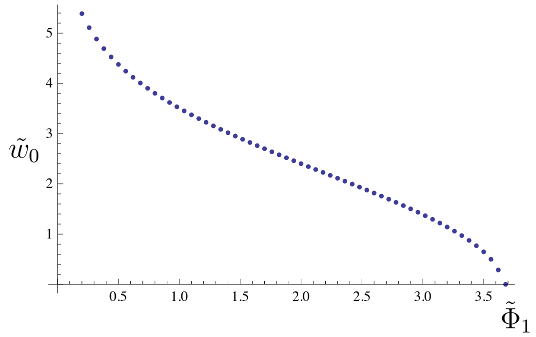

Figure 4.1: Forming of the symmetry breaking condensate below a critical value for $\tilde{\Phi}_1$.

In the normal state the gauge field takes the form

$$\Phi(r) = \mu\left(1 - \frac{1}{r}\right)\;, \tag{4.16}$$

The latter is the form $\Phi(r)$ takes in AdS_4 Reissner-Nordström (see equation (3.55) which is also appropriate for considering AdS_4 Schwarzschild in the probe limit. From the comparison of (4.7) and (4.16) we read off that in the normal state

$$\mu = \rho L \kappa^2\,. \tag{4.17}$$

Comparing this with (4.14) together with (4.9) leads to the rescaled gauge field in the normal state

$$\tilde{\Phi}(r) = \tilde{\mu}\left(1 - \frac{1}{r}\right)\;, \tag{4.18}$$

i.e. the coefficient $\tilde{\Phi}_1$ in the near-horizon expansion (4.13) is in the normal state given by

$$\tilde{\Phi}_1 = \tilde{\mu}\,. \tag{4.19}$$

This can be translated into a temperature $\tilde{T}$ in units of $\sqrt{\tilde{\mu}}$ via

$$\tilde{T} = \frac{T}{\sqrt{\tilde{\mu}}} = \frac{T}{\sqrt{\tilde{\Phi}_1}} = \frac{3}{4\pi L^2\sqrt{\tilde{\Phi}_1}} \tag{4.20}$$

where in the last step we used (4.5). The second equality is only valid in the normal state. From figure 4.1 one reads off the value $\tilde{\Phi}_{1,c} \approx 3.7$ at which the phase transition occurs. This implies a critical temperature of $\frac{3}{4\pi L^2\sqrt{\tilde{\Phi}_{1,c}}}$. In figure 4.1 in the superconducting state (i.e. for non-trivial $\tilde{w}$) the rescaled chemical potential $\tilde{\mu}$ grows as $\tilde{\Phi}_1$ is lowered. This means that in the superconducting state smaller values of $\tilde{\Phi}_1$ correspond to smaller values of $\tilde{T}$.

As already mentioned the development of the condensate $\tilde{w}$ was interpreted in [7] as the phase transition to a superconducting state. This was analyzed in [7] by considering the conductivity for the current j^3.

In contrast, we are interested in the conductivities describing the linear response of j^1 and j^2 to electric fields E^1 and E^2 in the normal state of the theory.

In order to obtain the conductivities holographically, we have to look at fluctuations a^a_μ of the gauge fields around the background. In general, in spacetimes which are asymptotic to AdS_4, the gauge fields with zero momentum can be expanded as (cf. (3.79) in the introduction)

$$A^a_\mu = e^{-i\omega t}\left[A^{a,(0)}_\mu + \frac{A^{a,(1)}_\mu}{r} + \mathcal{O}(r^{-2})\right] . \tag{4.21}$$

Here $A^{a,(0)}_\mu$ is the boundary value of the gauge field. We choose a gauge such that $A_r = 0$.

We can define the conductivities according to Ohm's law

$$j^a_i = \sigma^{ab}_{ij} E^b_j\,, \quad i,j \in \{x,y\}\,, \tag{4.22}$$

where

$$E^a_i = -F^{a,(0)}_{ti} = -\partial_t A^{a,(0)}_\mu - g\epsilon^{abc} A^{b,(0)}_t A^{c,(0)}_i\,. \tag{4.23}$$

There is no term $\partial_i A^{(0)}_t$ in (4.23) since A_t depends only on r. With this notion of the conductivity tensor we are interested in σ^{ab} with $a, b = 1, 2$. In earlier literature the focus was on σ^{33}, which was calculated for the normal state in [108] and for the superconducting state in [7].

One comment is in order here. In the non-Abelian case, Ohm's law would rather be given by (cf. formula (2) of [109])

$$j^a_i = \sigma^{ab}_{ij}\left(E^b_j - (D\mu)^b_j\right) , \tag{4.24}$$

where

$$(D\mu)^b_j = \partial_j \mu^b - g\epsilon^{bcd}\mu^c A^{d,(0)}_j . \tag{4.25}$$

This is the non-Abelian generalization of the fact that the gradient of the chemical potential causes a diffusion current. In the Abelian case the current is driven not only by the electric field but by a field $\tilde{\mathcal{E}} = E + \frac{1}{e}\nabla\mu$ (cf. e.g. [110]). Of course, in the Abelian case ∇ denotes the ordinary derivative and not a covariant one.

However, care has to be taken with the expression (4.24). The chemical potential in formula (2) of [109] is really a dynamical field whereas all quantities coming from the background, that are imprinted on the field theory from the outside, are included in the expression E^b_μ of [109][2]. This is different from the case we consider. What we call the

[2]We thank Yaron Oz for discussing this point with us.

chemical potential, i.e. $A_t^{3,(0)}$ is part of the external field and thus already included in E_i^a (cf. (4.23)). No other components of A_t are sourced by the external sources we are going to consider.

To determine the conductivities the perturbations a_i^a of the gauge field have to be expanded according to (4.21). Then the currents caused by linear response to the gauge fields are given by the following relation between the $a_i^{a,(0)}$ and $a_i^{a,(1)}$ in the expansion for a_i^a,

$$j_i^a = \frac{2}{\kappa^2} a_i^{a,(1)} = -K_{ij}^{R,ab} a_j^{b,(0)} \ , \tag{4.26}$$

where

$$K_{ij}^{R,ab} = -\frac{2}{\kappa^2} \frac{\delta a_i^{a,(1)}}{\delta a_j^{b,(0)}} \ . \tag{4.27}$$

We used the symbol $K_{ij}^{R,ab}$ instead of $G_{ij}^{R,ab}$ in order to indicate that (4.27) indeed defines the response function which in general is different from the retarded Green's function. This fact was stressed for instance in [111] and it is well known in the field of condensed matter physics. The discrepancy between the retarded Green's function and the response arises whenever the current itself depends on the gauge field, cf. e.g. the discussion in chapter 7 of [112].

For concreteness, we only consider the components σ_{yy}^{ab}, $a, b = 1, 2$, of the conductivity tensor in the normal state. The relevant coupled equations of motion for a_y^1 and a_y^2 were already given in sec. 5.1 of [7]. For $\tilde{w} = 0$ they read

$$\begin{aligned}
\left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)}\partial_r + \frac{r^2(\omega^2 L^4 + \tilde{\Phi}^2)}{(r^3-1)^2}\right] a_y^1 - \frac{2i\omega L^2 r^2 \tilde{\Phi}}{(r^3-1)^2} a_y^2 \;&=\; 0 \ , \\
\left[\partial_r^2 + \frac{2r^3+1}{r(r^3-1)}\partial_r + \frac{r^2(\omega^2 L^4 + \tilde{\Phi}^2)}{(r^3-1)^2}\right] a_y^2 + \frac{2i\omega L^2 r^2 \tilde{\Phi}}{(r^3-1)^2} a_y^1 \;&=\; 0 \ .
\end{aligned} \tag{4.28}$$

Once the asymptotic form of the solutions is known one can calculate the response functions according to (4.26). In order to obtain the conductivities, we need to translate this into the form (4.22). For Abelian gauge fields this is straightforward because of the linear relation between the gauge potential and the field strength as we have seen in section 3.7.

For a non-Abelian theory things are in general more complicated due to the nonlinear structure of the field strength (4.23). However, for the case at hand, i.e. with a_y^1, a_y^2 and A_t^3 being the only non-vanishing components of the gauge fields, there is still a linear relation between the field strengths and the fluctuations of the gauge fields, i.e.

$$\begin{aligned}
E_y^1 \;&=\; \left.(i\omega a_y^1 + g\Phi a_y^2)\right|_{r\to\infty} = i\omega a_y^{1,(0)} + \frac{\tilde{\mu}}{L^2} a_y^{2,(0)} \ , \\
E_y^2 \;&=\; \left.(i\omega a_y^2 - g\Phi a_y^1)\right|_{r\to\infty} = i\omega a_y^{2,(0)} - \frac{\tilde{\mu}}{L^2} a_y^{1,(0)} \ .
\end{aligned} \tag{4.29}$$

For $\omega^2 L^4 \neq \tilde{\mu}^2$ this can be inverted to give

$$\begin{pmatrix} a_y^{1,(0)} \\ a_y^{2,(0)} \end{pmatrix} = \frac{1}{\omega^2 - \frac{\tilde{\mu}^2}{L^4}} \begin{pmatrix} -i\omega & \frac{\tilde{\mu}}{L^2} \\ -\frac{\tilde{\mu}}{L^2} & -i\omega \end{pmatrix} \begin{pmatrix} E_y^1 \\ E_y^2 \end{pmatrix} . \tag{4.30}$$

Combining this with (4.26), we obtain

$$\begin{aligned} \begin{pmatrix} j_y^1 \\ j_y^2 \end{pmatrix} &= \frac{1}{\omega^2 - \frac{\tilde{\mu}^2}{L^4}} \begin{pmatrix} K_{yy}^{R,11} & K_{yy}^{R,12} \\ K_{yy}^{R,21} & K_{yy}^{R,22} \end{pmatrix} \begin{pmatrix} i\omega & -\frac{\tilde{\mu}}{L^2} \\ \frac{\tilde{\mu}}{L^2} & i\omega \end{pmatrix} \begin{pmatrix} E_y^1 \\ E_y^2 \end{pmatrix} \\ &\equiv \begin{pmatrix} \sigma_{yy}^{11} & \sigma_{yy}^{12} \\ \sigma_{yy}^{21} & \sigma_{yy}^{22} \end{pmatrix} \begin{pmatrix} E_y^1 \\ E_y^2 \end{pmatrix} , \end{aligned} \tag{4.31}$$

where in the last line we defined the conductivity.

It is often more convenient to work with a rescaled conductivity as in [7]. This is obtained by dividing σ by the value that the conductivity takes for AdS_4-Schwarzschild, i.e.

$$\sigma_\infty \equiv \frac{2L^2}{\kappa^2} . \tag{4.32}$$

Then we define

$$\tilde{\sigma}_{yy}^{ab} = \frac{1}{\sigma_\infty} \sigma_{yy}^{ab} \quad , \quad a, b = 1, 2 , \tag{4.33}$$

which can be expressed in terms of $\tilde{\mu} = gL^2\mu$ and

$$\tilde{K}_{yy}^{R,ab} = \frac{\kappa^2}{2} K_{yy}^{R,ab} \tag{4.34}$$

as

$$\tilde{\sigma}_{yy}^{ab} = \frac{1}{\omega^2 L^4 - \tilde{\mu}^2} \begin{pmatrix} \tilde{K}_{yy}^{R,11} & \tilde{K}_{yy}^{R,12} \\ \tilde{K}_{yy}^{R,21} & \tilde{K}_{yy}^{R,22} \end{pmatrix} \begin{pmatrix} i\omega L^2 & -\tilde{\mu} \\ \tilde{\mu} & i\omega L^2 \end{pmatrix} . \tag{4.35}$$

Given that the equations of motion (4.28) only depend on the combination ωL^2 it is obvious that $\tilde{\sigma}_{yy}^{ab}$ only depends on this dimensionless quantity. Using (4.5) one sees that ωL^2 is proportional to ω/T. In the following section we will compute the conductivities $\tilde{\sigma}_{yy}^{ab}$ as functions of ωL^2. However, in order to avoid cluttering of the formulas, we will set $L = 1$. Nevertheless, ω should always be understood as the dimensionless quantity ωL^2.

4.2 Conductivities

In this section we calculate the conductivities $\tilde{\sigma}_{yy}^{ab}$, $a, b = 1, 2$, both analytically and numerically in the normal state of the field theory.

4.2.1 Analytical solution

In this section we follow the calculation of [113] by expanding the (ingoing) solution for small $\tilde{\mu}$ and ω,

$$\begin{aligned} a_y^1 &= a_y^{1(0)} \left(\frac{r-1}{r+1}\right)^{-\frac{i\omega}{3}} \Big[1 + \omega\, a_y^{1,\omega}(r) + \tilde{\mu}\, a_y^{1,\tilde{\mu}}(r) + \omega\tilde{\mu}\, a_y^{1,\omega\tilde{\mu}}(r) + \omega^2\, a_y^{1,\omega^2}(r) \\ &\qquad + \tilde{\mu}^2\, a_y^{1,\tilde{\mu}^2}(r) + \ldots\Big] \end{aligned} \tag{4.36}$$

and similarly for a_y^2. Plugging this into (4.28) and solving the resulting equations order by order in the ω and $\tilde{\mu}$ expansion (demanding that all functions $a_y^{1,\#}(r)$ are regular at the horizon, $r = r_H = 1$, and that they vanish at the boundary, $r \to \infty$, so that $a_y^1 \to a_y^{1(0)}$ and $a_y^2 \to a_y^{2(0)}$ as $r \to \infty$) leads to

$$\begin{aligned} a_y^1 &\overset{r\to\infty}{\longrightarrow} a_y^{1(0)} \left(1 + \frac{2i\omega}{3r}\right) \Bigg[1 + \frac{i\omega}{3r} - \frac{2i\pi}{3\sqrt{3}}\frac{\omega\tilde{\mu}}{r}\frac{a_y^{2(0)}}{a_y^{1(0)}} + \frac{\sqrt{3}\pi - 3\ln 3}{6}\frac{\tilde{\mu}^2}{r} \\ &\qquad + c_1 \frac{\omega^2\tilde{\mu}}{r}\frac{a_y^{2(0)}}{a_y^{1(0)}} + c_2\frac{\omega^2\tilde{\mu}^2}{r} + \ldots\Bigg] , \\ a_y^2 &\overset{r\to\infty}{\longrightarrow} a_y^{2(0)} \left(1 + \frac{2i\omega}{3r}\right) \Bigg[1 + \frac{i\omega}{3r} + \frac{2i\pi}{3\sqrt{3}}\frac{\omega\tilde{\mu}}{r}\frac{a_y^{1(0)}}{a_y^{2(0)}} + \frac{\sqrt{3}\pi - 3\ln 3}{6}\frac{\tilde{\mu}^2}{r} \\ &\qquad - c_1 \frac{\omega^2\tilde{\mu}}{r}\frac{a_y^{1(0)}}{a_y^{2(0)}} + c_2\frac{\omega^2\tilde{\mu}^2}{r} + \ldots\Bigg] , \end{aligned} \tag{4.37}$$

with c_1 and c_2 two real constants that we could not determine analytically, but could read off from the numerical results of the next section. We left out a term of the order $\mathcal{O}(\omega\tilde{\mu}^2)$, as it is always subleading with respect to the term proportional to ω (note that it would also come with a factor of i, being linear in ω). Moreover, there is no ω^2-term (or higher order in ω without any $\tilde{\mu}$-factor). This can be understood because in the limit $\tilde{\mu} = 0$ the equations can be solved explicitly and the expansion of the solution only has a term linear in ω at order $1/r$, cf. formula (38) of [7].

From (4.26) and (4.37) we read off

$$\begin{aligned} \tilde{K}_{yy}^{R,11}(k=0,\omega) &= -i\omega - \frac{\sqrt{3}\pi - 3\ln 3}{6}\tilde{\mu}^2 + c_2\omega^2\tilde{\mu}^2 \ldots = \tilde{K}_{yy}^{R,22}(k=0,\omega) , \\ \tilde{K}_{yy}^{R,12}(k=0,\omega) &= \frac{2i\pi}{3\sqrt{3}}\omega\tilde{\mu} + c_1\omega^2\tilde{\mu} + \ldots = -\tilde{K}_{yy}^{R,21}(k=0,\omega) . \end{aligned} \tag{4.38}$$

The relations

$$\tilde{K}_{yy}^{R,22} = \tilde{K}_{yy}^{R,11} , \quad \tilde{K}_{yy}^{R,21} = -\tilde{K}_{yy}^{R,12} , \qquad \text{(normal phase)} , \tag{4.39}$$

are consistent with formula (4.60) of [113]. Plugging (4.38) into (4.31), we obtain to leading order in ω and μ

$$\tilde{\sigma}^{11}_{yy} = \frac{\omega^2}{\omega^2-\tilde{\mu}^2} + i\frac{0.8512\tilde{\mu}^2\omega}{\omega^2-\tilde{\mu}^2} + \ldots = \tilde{\sigma}^{22}_{yy} \,, \tag{4.40}$$

$$\tilde{\sigma}^{12}_{yy} = \frac{0.3576\tilde{\mu}^3 - 1.2092\tilde{\mu}\omega^2}{\omega^2-\tilde{\mu}^2} + i\frac{\tilde{\mu}\omega}{\omega^2-\tilde{\mu}^2} + \ldots = -\tilde{\sigma}^{21}_{yy} \,, \tag{4.41}$$

where we also used

$$\frac{\sqrt{3}\pi - 3\ln 3}{6} \approx 0.3576 \,. \tag{4.42}$$

4.2.2 Numerics

Numerically, we do not have to restrict to small values of ω and $\tilde{\mu}$. In order to be able to compare with the analytical results of the last section, let us nevertheless start by solving (4.28) numerically for small values of ω and for $\tilde{\mu} = 0.01$. Let us first outline how to calculate the retarded Green's functions numerically. We follow closely [7]. The near horizon expansion is

$$\begin{aligned} a^1_y &= (r-1)^{-i\omega/4\pi T}\left[\beta_1 + \alpha_1(r-1) + \ldots\right] \,, \\ a^2_y &= (r-1)^{-i\omega/4\pi T}\left[\beta_2 + \alpha_2(r-1) + \ldots\right] \,, \end{aligned} \tag{4.43}$$

where β_1 and β_2 are two independent constants which correspond to the two free constants of integration for (4.28) which are left after specifying the solution to be ingoing at the horizon.[3] The other coefficients in the expansion (4.43) are determined by β_1 and β_2. For α_1 and α_2 we find

$$\alpha_1 = -\frac{2\beta_2\omega\tilde{\Phi}_1}{9i+6\omega} \quad , \quad \alpha_2 = \frac{2\beta_1\omega\tilde{\Phi}_1}{9i+6\omega} \,, \tag{4.44}$$

where $\tilde{\Phi}_1$ is the coefficient in the near horizon expansion. Let us recall for convenience that close to the horizon the expansion of $\tilde{\Phi}$ is given by (cf. (4.13))

$$\tilde{\Phi} = \tilde{\Phi}_1(r-1) + \ldots \,. \tag{4.45}$$

Now notice the following. The equations (4.28) consist of two different types of terms. The ones which do not mix a^1_y with a^2_y are even under a change $\tilde{\Phi} \mapsto -\tilde{\Phi}$ and the terms mixing a^1_y with a^2_y are odd under $\tilde{\Phi} \mapsto -\tilde{\Phi}$. As the equations (4.28) are linear, we can

[3] In contrast to the situation in sec. 4.2 of [7], there are no constraints from the linearized Yang-Mills equations and there is no residual gauge invariance, which reduced the number of independent solutions in that case.

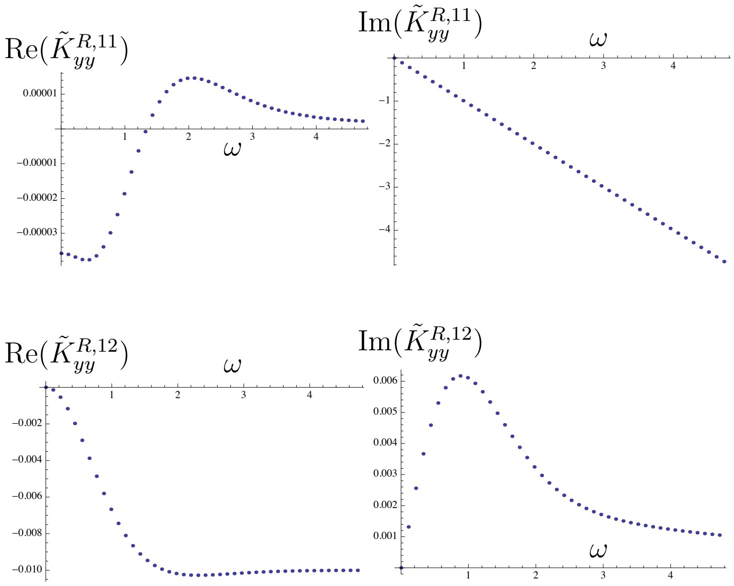

Figure 4.2: $\mathrm{Re}(\tilde{K}^{R,11}_{yy})$, $\mathrm{Im}(\tilde{K}^{R,11}_{yy})$, $\mathrm{Re}(\tilde{K}^{R,12}_{yy})$ and $\mathrm{Im}(\tilde{K}^{R,12}_{yy})$ as a function of ω for $\tilde{\mu} = 0.01$.

expand the asymptotic solutions as

$$\begin{aligned} a^1_y &= a^{1,(0)}_y + \frac{a^{1,(1)}_y}{r} + \ldots = \beta_1 a^{1,(0),e} + \beta_2 a^{1,(0),o} + \frac{\beta_1 a^{1,(1),e} + \beta_2 a^{1,(1),o}}{r} + \ldots \ , \\ a^2_y &= a^{2,(0)}_y + \frac{a^{2,(1)}_y}{r} + \ldots = \beta_2 a^{2,(0),e} + \beta_1 a^{2,(0),o} + \frac{\beta_2 a^{2,(1),e} + \beta_1 a^{2,(1),o}}{r} + \ldots \ , \end{aligned} \tag{4.46}$$

where $a^{1,(0),e}$, $a^{1,(0),o}$, $a^{1,(1),e}$, $a^{1,(1),o}$, $a^{2,(0),e}$, $a^{2,(0),o}$, $a^{2,(1),e}$, $a^{2,(1),o}$ are constants (depending on ω and $\tilde{\mu}$) which are either even (e) or odd (o) under $\tilde{\Phi} \mapsto -\tilde{\Phi}$, indicated by their superscript.

In order to calculate the response functions, we make use of formula (4.26). We notice that

$$\begin{pmatrix} a^{1,(0)}_y \\ a^{2,(0)}_y \end{pmatrix} = \begin{pmatrix} a^{1,(0),e} & a^{1,(0),o} \\ a^{2,(0),o} & a^{2,(0),e} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \tag{4.47}$$

which can be inverted to give

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a^{1,(0),e} & a^{1,(0),o} \\ a^{2,(0),o} & a^{2,(0),e} \end{pmatrix}^{-1} \begin{pmatrix} a^{1,(0)}_y \\ a^{2,(0)}_y \end{pmatrix}. \tag{4.48}$$

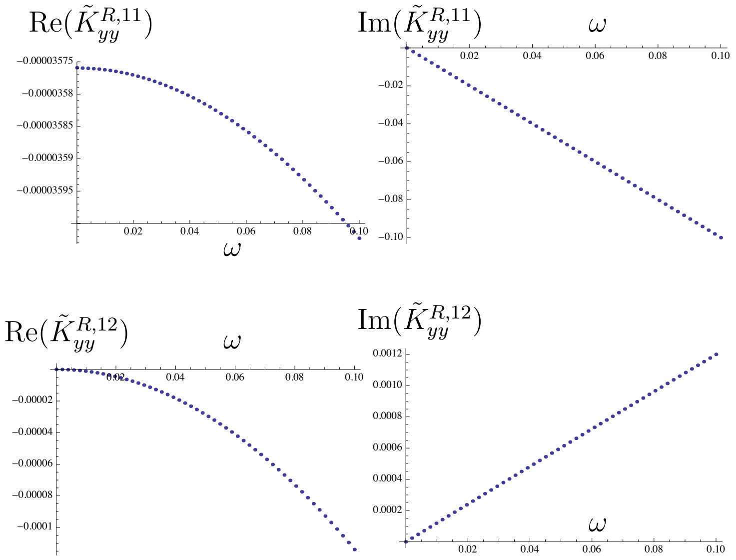

Figure 4.3: $\mathrm{Re}(\tilde{K}_{yy}^{R,11})$, $\mathrm{Im}(\tilde{K}_{yy}^{R,11})$, $\mathrm{Re}(\tilde{K}_{yy}^{R,12})$ and $\mathrm{Im}(\tilde{K}_{yy}^{R,12})$ as a function of ω for $\tilde{\mu} = 0.01$.

This then leads to

$$\begin{pmatrix} a_y^{1,(1)} \\ a_y^{2,(1)} \end{pmatrix} = \begin{pmatrix} a^{1,(1),e} & a^{1,(1),o} \\ a^{2,(1),o} & a^{2,(1),e} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \tag{4.49}$$

$$= \begin{pmatrix} a^{1,(1),e} & a^{1,(1),o} \\ a^{2,(1),o} & a^{2,(1),e} \end{pmatrix} \begin{pmatrix} a^{1,(0),e} & a^{1,(0),o} \\ a^{2,(0),o} & a^{2,(0),e} \end{pmatrix}^{-1} \begin{pmatrix} a_y^{1,(0)} \\ a_y^{2,(0)} \end{pmatrix} \tag{4.50}$$

from which one can read off the retarded response functions

$$\begin{pmatrix} K_{yy}^{R,11} & K_{yy}^{R,12} \\ K_{yy}^{R,21} & K_{yy}^{R,22} \end{pmatrix} = -\frac{2}{\kappa^2} \begin{pmatrix} a^{1,(1),e} & a^{1,(1),o} \\ a^{2,(1),o} & a^{2,(1),e} \end{pmatrix} \begin{pmatrix} a^{1,(0),e} & a^{1,(0),o} \\ a^{2,(0),o} & a^{2,(0),e} \end{pmatrix}^{-1} . \tag{4.51}$$

Notice that the coefficients $a^{1,(0),e}$ etc. are independent of β_1 and β_2 and, thus, can be calculated for arbitrary (non-vanishing) choices. The coefficients $a^{1,(0),e}$ etc. can easily be calculated numerically, by solving (4.28) for a background $\tilde{\Phi}$ and for $-\tilde{\Phi}$ and then taking sums or differences of the results in an obvious way.

The result for $\tilde{K}_{yy}^{R,ab}$ is displayed in fig. 4.2 and a zoom into the region of small ω values is given in fig. 4.3 (this will be important to compare with the analytic result of the previous section). We only show $\tilde{K}_{yy}^{R,11}$ and $\tilde{K}_{yy}^{R,12}$, as these are the only independent quantites in the normal phase, due to (4.39).

The result agrees nicely with (4.38) for $c_1 \approx -1.15$ and $c_2 \approx -0.28$ and using (4.42). The value of $\tilde{K}_{yy}^{R,11}$ in the limit of $\omega = 0$ depends on $\tilde{\mu}$ and one can easily obtain this dependence for larger values of $\tilde{\mu}$ numerically, cf. fig. 4.4. Obviously it diverges at the phase transition to the superconducting phase which occurs at $\tilde{\mu} = \Phi_{1,c}$. This is understandable

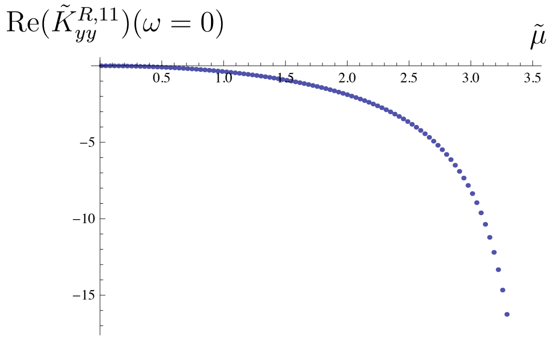

Figure 4.4: $\mathrm{Re}(\tilde{K}^{R,11}_{yy})(k=0,\omega=0)$ as a function of $\tilde{\mu}$.

from the analysis of the quasinormal mode spectrum performed in sec. 5.1 of [7]. At the phase transition a Goldstone mode appears in the spectrum of modes involving a^1_y. This leads to a pole at $\omega=0$ in $\tilde{K}^{R,11}_{yy}$.

Using (4.31), we can calculate the conductivity and find the result displayed in fig. 4.5. Again we note that

$$\tilde{\sigma}^{22}_{yy} = \tilde{\sigma}^{11}_{yy}\ , \quad \tilde{\sigma}^{21}_{yy} = -\tilde{\sigma}^{12}_{yy}\,. \tag{4.52}$$

It is even more instructive to look at the eigenvalues of the Hermitean part

$$\Sigma^{ab}_h = \frac{1}{2}(\tilde{\sigma}^{ab}_{yy} + (\tilde{\sigma}^{ba}_{yy})^*) \tag{4.53}$$

and the anti-Hermitean part

$$\Sigma^{ab}_{ah} = \frac{1}{2i}(\tilde{\sigma}^{ab}_{yy} - (\tilde{\sigma}^{ba}_{yy})^*)\ . \tag{4.54}$$

They are plotted in fig. 4.6.

Some comments are in order. Obviously the matrix Σ^{ab} is *not* positive semi-definite. Its eigenvalues go to zero, though, in the DC limit $\omega\to 0$. Moreover, the conductivity shows a characteristic pole at $\omega=\tilde{\mu}$ which arises due to the denominator in equation (4.31). Finally note that all conductivities go to their values in AdS-Schwarzschild (without chemical potential) for large values of ω, as expected.

Let us make a few more comments here. The fact that the matrix Σ develops a negative eigenvalue for $\omega<\tilde{\mu}$ indicates that the entropy production rate becomes *negative* for that frequency range, potentially leading to an instability. This can be checked explicitly. The entropy production rate is generally given by

$$\begin{aligned}\dot{s} = \mathrm{Re}\Big(\vec{j}^a\cdot(\vec{E}^a)^*\Big) \;&=\; -\mathrm{Re}\Big[(\tilde{K}^{R,11}_{yy}a^{1,(0)}_y + \tilde{K}^{R,12}_{yy}a^{2,(0)}_y)(i\omega a^{1,(0)}_y + \tilde{\mu}a^{2,(0)}_y)^* \\ &\quad +(\tilde{K}^{R,21}_{yy}a^{1,(0)}_y + \tilde{K}^{R,22}_{yy}a^{2,(0)}_y)(i\omega a^{2,(0)}_y - \tilde{\mu}a^{1,(0)}_y)^*\Big],\end{aligned} \tag{4.55}$$

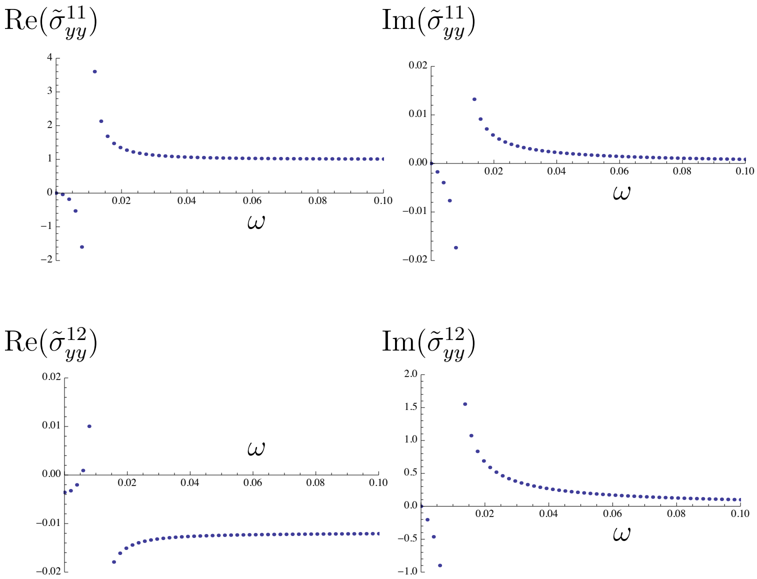

Figure 4.5: $\mathrm{Re}(\tilde{\sigma}^{11}_{yy})$, $\mathrm{Im}(\tilde{\sigma}^{11}_{yy})$, $\mathrm{Re}(\tilde{\sigma}^{12}_{yy})$ and $\mathrm{Im}(\tilde{\sigma}^{12}_{yy})$ as a function of ω for $\tilde{\mu} = 0.01$.

where s is the entropy density. It turns out that the eigenvector of Σ corresponding to the negative eigenvalue is given by $(i, 1)$, independently of ω. In other words, there is a relation between the two electrical fields, $E^1_y = iE^2_y$. Comparing this with (4.29), we see that this can be achieved if

$$a_y^{1,(0)} = i a_y^{2,(0)} \ , \tag{4.56}$$

i.e. the two gauge fields differ by $\pi/2$ in phase. From (4.28) and (4.43) we see that choosing the initial conditions $\beta_1 = i\beta_2$ at the horizon will lead, for the normal state of the dual field theory, to a solution which fulfills $a^1_y = ia^2_y$ throughout (and, thus, to a solution with (4.56)). In that case both equations of (4.28) reduce to the same equation for, say, a^1_y. Now, using (4.56) in (4.55), we obtain

$$\dot{s} = 2(\omega - \tilde{\mu})|a_y^{2,(0)}|^2 \Big(\mathrm{Re}\, \tilde{K}^{R,12}_{yy} - \mathrm{Im}\, \tilde{K}^{R,11}_{yy}\Big) \ . \tag{4.57}$$

The factor $(\omega - \tilde{\mu})$ comes from the fact that for the gauge fields (4.56) the field strengths are given by

$$E^1_y = iE^2_y = -a_y^{2,(0)}(\omega - \tilde{\mu}) \ . \tag{4.58}$$

Obviously, they vanish at $\omega = \tilde{\mu}$, where $\tilde{\sigma}$ has its pole. Thus, the entropy production rate stays finite there (it actually vanishes). Fig. 4.7 shows the entropy production rate for $\beta_1 = i\beta_2 = i$ and for $\tilde{\mu} = 0.01$. The change of sign at $\omega = \tilde{\mu}$ arises because the electrical field strengths change their sign, whereas the term in brackets in (4.57) does not. Rather

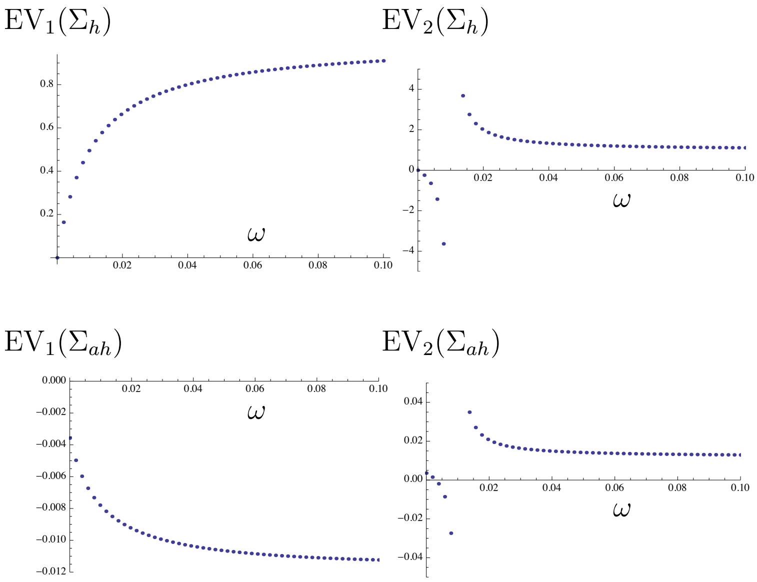

Figure 4.6: Eigenvalues of the matrices Σ_h^{ab} and Σ_{ah}^{ab}, given in (4.53) and (4.54), as a function of ω for $\tilde{\mu} = 0.01$.

it is positive, going to zero for $\omega \to 0$. Obviously, the problem with the negative entropy production rate arises because the currents j_y^1 and j_y^2 stay finite even though the field strengths E_y^1 and E_y^2 vanish and change sign at $\omega = \tilde{\mu}$. The negative entropy production rate might indicate an instability of the background, sourced by the electrical field. We will discuss this in more detail in the next section.

4.3 Discussion of our results

First of all it is important to notice that the theory seems well behaved from the field theory point of view. This becomes clear when we look at the response function $\tilde{K}_{yy}^{R,ab}$ instead of the conductivity. From the discussion in section 2 of [7] it follows that the anti-Hermitean part of the response function has to be negative semi-definite. In figure 4.8 the eigenvalues of the anti-Hermitean part of $\tilde{K}_{yy}^{R,ab}$ are plotted for the range in ω in which the negative eigenvalue of the Hermitean part of the conductivity appears. The eigenvalues of the anti-Hermitean part of the response function are indeed negative implying that $\tilde{K}_{yy}^{R,ab}$ is negative semi-definite.

One possibility to see whether there is an instability on the gravity side is to take into account backreaction. The full backreacted equations of motion to linear order can be found in [114]. Their equations (3.12a) to (3.12c) show that the perturbations a_y^1 and a_y^2 we consider do not source metric perturbations in the normal state and therefore it is not very likely that including backreaction at the linearized level could cure the problem.

At the moment the most promising idea is to look at the full Einstein-Yang-Mills

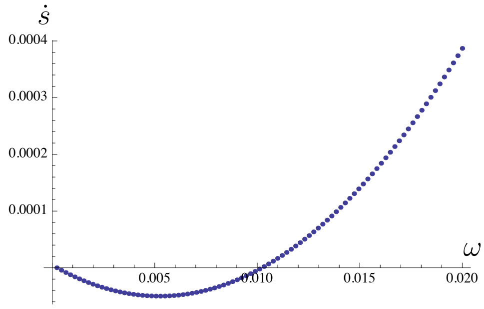

Figure 4.7: Entropy production rate $\dot{s}$, eq.(4.57), as a function of ω for $\tilde{\mu} = 0.01$.

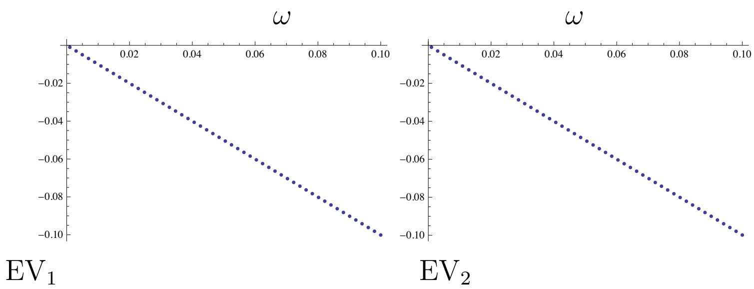

Figure 4.8: Eigenvalues of the anti-Hermitean part of the response function $\tilde{K}^{R,ab}_{yy}$ as a function of ω for $\tilde{\mu} = 0.01$.

equation instead of the linearized ones. The reason for that is the following.

For $\dot{s} = \mathrm{Re}\left(\vec{j}^a \cdot (\vec{E}^a)^*\right)$ we have according to (4.29) and (4.26)

$$\begin{aligned} E^1_y &= i\omega a^{1,(0)}_y + \frac{\tilde{\mu}}{L^2} a^{2,(0)}_y \,, \\ E^2_y &= i\omega a^{2,(0)}_y - \frac{\tilde{\mu}}{L^2} a^{1,(0)}_y \end{aligned} \tag{4.59}$$

as well as

$$j^a_i \sim a^{a,(1)}_i \,. \tag{4.60}$$

But this means that the entropy production rate $\dot{s}$ depends quadratically on the perturbations a^1_y and a^2_y. Such an effect cannot be seen directly at the linearized level in perturbation theory. We therefore hope that by considering the full Einstein-Yang-Mills equations we might see directly that the horizon area decreases in the presence of non-Abelian sources. This would clearly indicate an instability on the gravity side.

Chapter 5

Nernst branes in four-dimensional gauged supergravity

In this chapter we study static black brane solutions in the context of $N = 2$ $U(1)$ gauged supergravity in four dimensions with only vector multiplets. Using the formalism of first-order flow equations, we construct novel extremal black brane solutions including examples of *Nernst branes*, i.e. extremal black brane solutions with vanishing entropy density. This chapter is based on [1].

In this chapter we are particularly interested is black solutions which asymptote to AdS_4. As we know from our discussion of the AdS/CFT correspondence in chapter 3 black solutions in an asymptotically AdS space represent gravity duals for field theories at finite temperature. Therefore exploring the rich variety of black geometries might lead to various field theoretic applications.

We are especially interested in extremal black brane solutions. Firstly, as we have mentioned in chapter 2, in flat space it is impossible to have a near-horizon geometry that is not a sphere. Therefore black branes, i.e. objects with flat near-horizon geometry like $\mathbb{R}^2$ naturally arise in asymptotically non-flat backgrounds such as AdS_4. Secondly, the great thing about extremal black objects is that the attractor mechanism is at work as we have explained in chapter 2. Furthermore, extremality comes accompanied with vanishing temperature (cf. chapter 2). This is important when one wants to construct solutions that obey the Nernst law.

We will construct extremal black branes with vanishing entropy density. As those fulfill the Nernst law known from thermodynamics we call them *Nernst branes*. Prior, smooth Nernst configurations in AdS have been found in [115, 116, 117].

As already mentioned extremal black geometries exhibit the attractor mechanism. In chapter 2 we presented the attractor mechanism for the ungauged case. In the next two chapters we consider the more complicated case of gauged supergravity instead. This means that we will also include fluxes in the theory (cf. chapter 2). At first, we will focus on the four-dimensional case. In the presence of fluxes, extremal supersymmetric configurations in four dimensions were first discovered by [118] and subsequently discussed in [119, 120]. The first-order formalism in the presence of fluxes was first presented in [119]. More results on generalizing the attractor mechanism to gauged supergravity are presented in [121, 122, 118, 123, 119, 120, 124].

In this setup of $N = 2$ $U(1)$ gauged supergravity we find new black solutions. Among them are not only the aforementioned Nernst branes but also non-Nernst configurations. We find e.g. a solution that generalizes a solution presented in [9] as well as a numerical solution that interpolates between $AdS_2 \times \mathbb{R}^2$ and AdS_4.

But let us begin with the derivation of the first-oder flow equations.

5.1 Flow equations for extremal black branes in four dimensions

First-order flow equations for supersymmetric black holes and black branes were recently obtained in [119] by a rewriting of the action, where they were given in terms of physical scalar fields $z^i = Y^i/Y^0$ $(i = 1, \ldots, n)$. Here, we will re-derive them by working in big moduli space, so that the resulting first-order flow equations will now be expressed in terms of the Y^I. The formulation in big moduli space becomes particularly useful when discussing the coupling to higher-derivative curvature terms [125].

5.1.1 First-order flow equations in big moduli space

Following [118, 119, 120] we make the ansatz for the black brane line element,

$$ds^2 = -\mathrm{e}^{2U}\,dt^2 + \mathrm{e}^{-2U}\left(dr^2 + \mathrm{e}^{2\psi}(dx^2 + dy^2)\right) \ , \tag{5.1}$$

where $U = U(r)\,,\ \psi = \psi(r)$. The black brane will be supported by scalar fields that only depend on r.

The Lagrangian we will consider is given in (A.25). It is written in terms of fields X^I of big moduli space. The rewriting of this Lagrangian as a sum of squares of first-order flow equations will, however, not be in terms of the X^I, but rather in terms of rescaled variables Y^I defined by

$$Y^I = \mathrm{e}^A\,\tilde{X}^I = \mathrm{e}^A\,\bar{\varphi}\,X^I \ . \tag{5.2}$$

Here $A = A(r)$ denotes a real factor that will be determined to be given by

$$A = \psi - U \ , \tag{5.3}$$

while $\bar{\varphi}$ denotes a phase with a $U(1)$-weight that is opposite to the one of X^I. Thus, the $\tilde{X}^I = \bar{\varphi}X^I$ denote homogeneous coordinates that are $U(1)$ invariant and satisfy (A.3) (with X^I replaced by $\tilde{X}^I$) as well as

$$N_{IJ}\,\mathcal{D}_r X^I\,\mathcal{D}_r\bar{X}^J = N_{IJ}\,\tilde{X}'^I\,\bar{\tilde{X}}'^J \ , \tag{5.4}$$

where $\tilde{X}'^I = \partial_r \tilde{X}^I$. Observe that in view of (A.3),

$$e^{2A} = -N_{IJ}\, Y^I\, \bar{Y}^J \;, \tag{5.5}$$

and that

$$e^{2A}\, A' = -\tfrac{1}{2}\, N_{IJ}\, \left(Y'^I\, \bar{Y}^J + Y^I\, \bar{Y}'^J\right) \;, \tag{5.6}$$

where we used the second homogeneity equation of (A.1).

We will first discuss electrically charged extremal black branes in the presence of electric fluxes h_I only, so that for the time being the flux potential (A.30) reads

$$V(\tilde{X}, \bar{\tilde{X}}) = \left(N^{IJ} - 2\, \tilde{X}^I\, \bar{\tilde{X}}^J\right)\, h_I\, h_J \;. \tag{5.7}$$

Subsequently, we will extend the first-order rewriting to the case of dyonic charges as well as dyonic fluxes.

We take $F^I_{tr} = E^I(r)$ as well as $X^I = X^I(r)$. Inserting the line element (5.1) into the action (A.25) yields the one-dimensional Lagrangian

$$\mathcal{L}_{1d} = \sqrt{-g}\, L - Q_I\, E^I \;, \tag{5.8}$$

where L is given in (A.25) and the Q_I denote the electric charges. Extremizing with respect to E^I yields

$$-\, e^{2\psi - 2U}\, \mathrm{Im}\mathcal{N}_{IJ}\, E^J = Q_I \;, \tag{5.9}$$

and hence

$$E^I = -e^{2U-2\psi}\, \left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{IJ}\, Q_J \;. \tag{5.10}$$

The associated one-dimensional action reads,

$$\begin{aligned} -\, S_{1d} \;\; = \;\; & \int dr\, e^{2\psi}\, \Big\{ U'^2 - \psi'^2 + N_{IJ}\, \tilde{X}'^I\, \bar{\tilde{X}}'^J - \tfrac{1}{2}\, e^{2U-4\psi}\, Q_I\, \left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{IJ}\, Q_J \\ & \qquad + g^2\, e^{-2U} V(\tilde{X}, \bar{\tilde{X}}) \Big\} \\ & + \int dr \frac{d}{dr}\left[e^{2\psi}\, (2\psi' - U')\right] \;, \end{aligned} \tag{5.11}$$

in accordance with [119] for the case of black branes. Next, we rewrite (5.11) in terms of the rescaled variables Y^I. We also find it convenient to introduce the combination

$$q_I = e^{U - 2\psi + i\gamma}\, \left(Q_I - i\, g\, e^{2(\psi - U)} h_I\right) \;, \tag{5.12}$$

where γ denotes a phase which can depend on r. Using

$$\tilde{X}'^I = e^{-A}\, \left(Y'^I - A'\, Y^I\right) \;, \tag{5.13}$$

as well as (5.6), we obtain the intermediate result

$$
\begin{aligned}
- S_{1d} &= \int dr\, \mathrm{e}^{2\psi} \Big\{ U'^2 - \psi'^2 \\
&\quad + \mathrm{e}^{-2A}\, N_{IJ}\, \left(Y'^I - \mathrm{e}^A\, N^{IK}\, \bar{q}_K\right) \left(\bar{Y}'^J - \mathrm{e}^A\, N^{JL}\, q_L\right) + \left(A' + \mathrm{Re}\left[\tilde{X}^I\, q_I\right]\right)^2 \\
&\quad - \tfrac{1}{2}\, \mathrm{e}^{2U-4\psi}\, Q_I \left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{IJ} Q_J - q_I\, N^{IJ}\, \bar{q}_J - \left(\mathrm{Re}\left[\tilde{X}^I\, q_I\right]\right)^2 + g^2\, \mathrm{e}^{-2U} V(\tilde{X}, \bar{\tilde{X}}) \Big\} \\
&\quad + 2 \int dr\, \mathrm{e}^{2\psi}\; \mathrm{Re}\left[\tilde{X}'^I\, q_I\right] + \int dr \frac{d}{dr}\left[\mathrm{e}^{2\psi}\,(2\psi' - U')\right] \; .
\end{aligned} \tag{5.14}
$$

Next, using the identity,

$$
- \tfrac{1}{2}\, \left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{IJ} = N^{IJ} + \tilde{X}^I\, \bar{\tilde{X}}^J + \tilde{X}^J\, \bar{\tilde{X}}^I \; , \tag{5.15}
$$

as well as the explicit form of the potential (5.7), we obtain

$$
\begin{aligned}
- S_{1d} &= \int dr\, \mathrm{e}^{2\psi} \Big\{ U'^2 - \psi'^2 \\
&\quad + \mathrm{e}^{-2A}\, N_{IJ}\, \left(Y'^I - \mathrm{e}^A\, N^{IK}\, \bar{q}_K\right) \left(\bar{Y}'^J - \mathrm{e}^A\, N^{JL}\, q_L\right) + \left(A' + \mathrm{Re}\left[\tilde{X}^I\, q_I\right]\right)^2 \\
&\quad + 2\, \mathrm{e}^{2U-4\psi}\, Q_I \tilde{X}^I\, Q_J \bar{\tilde{X}}^J - \left(\mathrm{Re}\left[\tilde{X}^I\, q_I\right]\right)^2 - 2\, g^2\, \mathrm{e}^{-2U}\, h_I\, \tilde{X}^I\, h_J\, \bar{\tilde{X}}^J \Big\} \\
&\quad + 2 \int dr\, \mathrm{e}^{2\psi}\; \mathrm{Re}\left[\tilde{X}'^I\, q_I\right] + \int dr \frac{d}{dr}\left[\mathrm{e}^{2\psi}\,(2\psi' - U')\right] \; .
\end{aligned} \tag{5.16}
$$

Inserting the expression (5.12) into the fourth line of (5.16) yields

$$
\begin{aligned}
2 \int dr\, \mathrm{e}^{2\psi}\; \mathrm{Re}\left[\tilde{X}'^I\, q_I\right] &= 2 \int dr\, \frac{d}{dr}\left[\mathrm{e}^U\; \mathrm{Re}\left(\mathrm{e}^{i\gamma}\, \tilde{X}^I\, Q_I\right) + g\, \mathrm{e}^{2\psi - U}\; \mathrm{Im}\left(\mathrm{e}^{i\gamma}\, \tilde{X}^I\, h_I\right)\right] \\
&\quad - 2 \int dr\, \mathrm{e}^U\, U'\; \mathrm{Re}\left[\mathrm{e}^{i\gamma}\, \tilde{X}^I\, Q_I\right] \\
&\quad - 2\, g \int dr\, \mathrm{e}^{2\psi - U}\; (2\psi' - U')\; \mathrm{Im}\left[\mathrm{e}^{i\gamma}\, \tilde{X}^I\, h_I\right] \\
&\quad + 2 \int dr\, \gamma' \left[\mathrm{e}^U\, \mathrm{Im}\left(\mathrm{e}^{i\gamma}\, \tilde{X}^I\, Q_I\right) - g\, \mathrm{e}^{2\psi - U}\, \mathrm{Re}\left(\mathrm{e}^{i\gamma}\, \tilde{X}^I\, h_I\right)\right] \; .
\end{aligned} \tag{5.17}
$$

Combining the terms proportional to ψ'^2 and to ψ' into a perfect square, and the terms proportional to U'^2 and to U' into a perfect square, leads to

$$
- S_{1d} = S_{\mathrm{BPS}} + S_{\mathrm{TD}} \; , \tag{5.18}
$$

where

$$
\begin{aligned}
S_{\mathrm{BPS}} &= \int dr\, \mathrm{e}^{2\psi} \left\{ \left[U' - \mathrm{e}^{U-2\psi}\, \mathrm{Re}\left(\mathrm{e}^{i\gamma}\, \tilde{X}^I\, Q_I\right) + g\, \mathrm{e}^{-U}\; \mathrm{Im}\left(\mathrm{e}^{i\gamma}\, \tilde{X}^I\, h_I\right)\right]^2 \right. \\
&\quad - \left(\psi' + 2\, g\, \mathrm{e}^{-U}\; \mathrm{Im}\left[\mathrm{e}^{i\gamma}\, \tilde{X}^I\, h_I\right]\right)^2 \\
&\quad \left. + \mathrm{e}^{-2A}\, N_{IJ}\, \left(Y'^I - \mathrm{e}^A\, N^{IK}\, \bar{q}_K\right) \left(\bar{Y}'^J - \mathrm{e}^A\, N^{JL}\, q_L\right) + \left(A' + \mathrm{Re}\left[\tilde{X}^I\, q_I\right]\right)^2 + \Delta \right\} \; ,
\end{aligned} \tag{5.19}
$$

and

$$\begin{aligned}\Delta &= 2\left[\mathrm{e}^{U-2\psi}\,\mathrm{Im}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,Q_I\right) - g\,\mathrm{e}^{-U}\,\mathrm{Re}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,h_I\right)\right] \\ &\quad \left[\gamma' + \mathrm{e}^{U-2\psi}\,\mathrm{Im}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,Q_I\right) + g\,\mathrm{e}^{-U}\,\mathrm{Re}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,h_I\right)\right] . \end{aligned} \tag{5.20}$$

Finally,

$$S_{\mathrm{TD}} = \int dr\,\frac{d}{dr}\left[\mathrm{e}^{2\psi}\,(2\psi' - U') + 2\,\mathrm{e}^{U}\,\mathrm{Re}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,Q_I\right) + 2\,g\,\mathrm{e}^{2\psi-U}\,\mathrm{Im}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,h_I\right)\right] . \tag{5.21}$$

Setting the squares in S_{BPS} to zero gives

$$\begin{aligned} U' &= \mathrm{e}^{U-2\psi}\,\mathrm{Re}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,Q_I\right) - g\,\mathrm{e}^{-U}\,\mathrm{Im}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,h_I\right) , \\ \psi' &= -2\,g\,\mathrm{e}^{-U}\,\mathrm{Im}\left[\mathrm{e}^{i\gamma}\,\tilde{X}^I\,h_I\right] , \\ A' &= -\,\mathrm{Re}\left[\tilde{X}^I\,q_I\right] , \\ Y'^I &= \mathrm{e}^{A}\,N^{IK}\,\bar{q}_K , \end{aligned} \tag{5.22}$$

while demanding the variation of Δ to be zero yields

$$\mathrm{e}^{U-2\psi}\,\mathrm{Im}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,Q_I\right) - g\,\mathrm{e}^{-U}\,\mathrm{Re}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,h_I\right) = 0 \tag{5.23}$$

as well as

$$\gamma' = -\mathrm{e}^{U-2\psi}\,\mathrm{Im}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,Q_I\right) - g\,\mathrm{e}^{-U}\,\mathrm{Re}\left(\mathrm{e}^{i\gamma}\,\tilde{X}^I\,h_I\right) . \tag{5.24}$$

Note that the first-order flow equations for the Y^I and for A are consistent with one another: the latter is a consequence of the former by virtue of (5.6). The flow equations given above are obtained by varying (5.19) with respect to the various fields and setting the individual terms to zero. Therefore, the solutions to these flow equations describe a subclass, but certainly not the entire class of solutions to the equations of motion stemming from the one-dimensional action (5.11).

Comparing the flow equations (5.22) with the ones obtained in the supersymmetric context in [119] shows that the flow equations derived above are the ones for supersymmetric black branes, and that the phase γ is to be identified with the phase α of [119] via $\gamma' = -\left(\alpha' + \mathcal{A}_r\right)$, with $\mathcal{A}_r$ given in (A.20).

Next, we study the dyonic case, with charges (Q_I, P^I) and fluxes (h_I, h^I) turned on. The above results can be easily extended by first writing the term $Q(\mathrm{Im}\mathcal{N})^{-1}Q$ in the action (5.11) as

$$\begin{aligned} V_{\mathrm{BH}} &= -\tfrac{1}{2}\,Q_I\,\left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{IJ}\,Q_J \\ &= \left(N^{IJ} + 2\,\tilde{X}^I\,\bar{\tilde{X}}^J\right)Q_I\,Q_J \\ &= g^{i\bar{j}}\,\mathcal{D}_i Z\,\bar{\mathcal{D}}_{\bar{j}}\bar{Z} + |Z|^2 , \end{aligned} \tag{5.25}$$

where we used (A.29) to write $V_{\rm BH}$ in terms of $Z = -Q_I\, X^I$. Turning on magnetic charges amounts to extending Z to [38]

$$Z = P^I\, F_I - Q_I\, X^I = \left(P^I\, F_{IJ} - Q_J\right) X^J = -\hat{Q}_I\, X^I\ , \tag{5.26}$$

where

$$\hat{Q}_I = Q_I - F_{IJ}\, P^J\ . \tag{5.27}$$

Similarly, the flux potential with dyonic fluxes can be obtained from the one with purely electric fluxes by the replacement of h_I by

$$\hat{h}_I = h_I - F_{IJ}\, h^J\ , \tag{5.28}$$

cf. (A.30).

Thus, formally the action looks identical to before, and we can adapt the computation given above to the case of dyonic charges and fluxes by replacing Q_I and h_I with $\hat{Q}_I$ and $\hat{h}_I$. Performing these replacements in (5.12) as well yields

$$q_I = \mathrm{e}^{U-2\psi+i\gamma}\left(\hat{Q}_I - i\, g\, \mathrm{e}^{2(\psi-U)}\hat{h}_I\right)\ . \tag{5.29}$$

The above procedure results in

$$-\, S_{1d} = S_{\rm BPS} + S_{\rm TD} + S_{\rm sympl}\ , \tag{5.30}$$

where $S_{\rm BPS}$ and $S_{\rm TD}$ are given as in (5.19) and (5.21), respectively, with Q_I and h_I replaced by $\hat{Q}_I$ and $\hat{h}_I$, and with q_I now given by (5.29). The third contribution, $S_{\rm sympl}$, is given by

$$S_{\rm sympl} = g\int dr\left(Q_I\, h^I - P^I\, h_I\right)\ . \tag{5.31}$$

Observe that this term is constant, independent of the fields, and hence it does not contribute to the variation of the fields. Imposing the constraint $\mathrm{S}_{1d} = 0$ (which is the Hamiltonian constraint, to be discussed below) on a solution yields the condition

$$Q_I\, h^I - P^I\, h_I = 0\ , \tag{5.32}$$

in agreement with [119] for the case of black branes. The condition (5.32) can also be written as

$$\mathrm{Im}\left(\hat{Q}_I\, N^{IJ}\, \bar{\hat{h}}_J\right) = 0\ . \tag{5.33}$$

The flow equations are now given by

$$\begin{aligned}
U' &= \mathrm{e}^{U-2\psi}\ \mathrm{Re}\left[\mathrm{e}^{i\gamma}\, \tilde{X}^I\, \hat{Q}_I\right] - g\, \mathrm{e}^{-U}\ \mathrm{Im}\left[\mathrm{e}^{i\gamma}\, \tilde{X}^I\, \hat{h}_I\right]\ ,\\
\psi' &= -2\, g\, \mathrm{e}^{-U}\ \mathrm{Im}\left[\mathrm{e}^{i\gamma}\, \tilde{X}^I\, \hat{h}_I\right]\ ,\\
A' &= -\,\mathrm{Re}\left[\tilde{X}^I\, q_I\right]\ ,\\
Y'^I &= \mathrm{e}^A\, N^{IJ}\, \bar{q}_J\ ,\\
\gamma' &= \mathrm{e}^{U-2\psi}\mathrm{Im}\left(\mathrm{e}^{i\gamma}\, \tilde{Z}\right) + g\, \mathrm{e}^{-U}\mathrm{Re}\left(\mathrm{e}^{i\gamma}\, \tilde{W}\right)\ ,
\end{aligned} \tag{5.34}$$

where $\tilde{Z}$ and $\tilde{W}$ denote Z and W with X replaced by $\tilde{X}$, as in (A.31). Inspection of the flow equations (5.34) yields

$$A' = (\psi - U)' \;, \tag{5.35}$$

and hence we obtain (5.3) (without loss of generality).

Observe that, as before, the flow equations for A and Y^I are consistent with one another. The latter can be recast into

$$\begin{pmatrix} (Y^I - \bar{Y}^I)' \\ (F_I - \bar{F}_I)' \end{pmatrix} = -2i\, \mathrm{e}^{-\psi}\, \mathrm{Im} \begin{pmatrix} \mathrm{e}^{i\gamma}\, N^{IK}\, \hat{Q}_K \\ \mathrm{e}^{i\gamma}\, \bar{F}_{IK}\, N^{KJ}\, \hat{Q}_J \end{pmatrix} + 2i\, g\, \mathrm{e}^{\psi-2U}\, \mathrm{Re} \begin{pmatrix} \mathrm{e}^{i\gamma}\, N^{IK}\, \hat{h}_K \\ \mathrm{e}^{i\gamma}\, \bar{F}_{IK}\, N^{KJ}\, \hat{h}_J \end{pmatrix} \tag{5.36}$$

where here $F_I = \partial F(Y)/\partial Y^I$. Each of the vectors appearing in this expression transforms as a symplectic vector under $\mathrm{Sp}(2(\mathrm{n}+1))$transformations, i.e. as

$$\begin{pmatrix} Y^I \\ F_I \end{pmatrix} \mapsto \begin{pmatrix} A^I{}_J & B^{IJ} \\ C_{IJ} & D_I{}^J \end{pmatrix} \begin{pmatrix} Y^J \\ F_J \end{pmatrix} \;, \tag{5.37}$$

where $A^T\, D - C^T\, B = \mathbb{1}_{n+1}$. For instance, under symplectic transformations,

$$N^{IJ}\, \bar{\hat{h}}_J \mapsto \mathcal{S}^I{}_K\, N^{KL}\, \bar{\hat{h}}_L \quad , \quad F_{IJ} \mapsto \big(D_I{}^L\, F_{LK} + C_{IK}\big)\, [\mathcal{S}^{-1}]^K{}_J \;, \tag{5.38}$$

where $\mathcal{S}^I{}_J = A^I{}_J + B^{IK}\, F_{KJ}$ [126]. Using this, it can be easily checked that the last vector in (5.36) transforms as in (5.37).

The constraint (5.23) becomes

$$\mathrm{e}^{U-2\psi}\, \mathrm{Im}\left(\mathrm{e}^{i\gamma}\, Z(Y)\right) - g\, \mathrm{e}^{-U}\, \mathrm{Re}\left(\mathrm{e}^{i\gamma}\, W(Y)\right) = 0 \;, \tag{5.39}$$

where

$$\begin{aligned} Z(Y) &= P^I\, F_I(Y) - Q_I\, Y^I \;, \\ W(Y) &= h^I\, F_I(Y) - h_I\, Y^I \;. \end{aligned} \tag{5.40}$$

Using the flow equations (5.34), we find that the constraint (5.39) is equivalent to the condition

$$q_I\, Y^I = \bar{q}_I\, \bar{Y}^I \;. \tag{5.41}$$

The phase γ is not an independent degree of freedom. It can be expressed in terms of $Z(Y)$ and $W(Y)$ as [119]

$$\mathrm{e}^{-2i\gamma} = \frac{Z(Y) - ig\, \mathrm{e}^{2(\psi-U)}\, W(Y)}{\bar{Z}(\bar{Y}) + ig\, \mathrm{e}^{2(\psi-U)}\, \bar{W}(\bar{Y})} \;. \tag{5.42}$$

When $\gamma = k\pi$ $(k \in \mathbb{Z})$, equation (5.42) implies

$$Z(Y) = \bar{Z}(\bar{Y}) \quad , \quad W(Y) = -\bar{W}(\bar{Y}) \;. \tag{5.43}$$

Differentiating (5.39) with respect to r yields a flow equation for γ' that has to be consistent with the last flow equation in (5.34). We proceed to check consistency of these two flow equations. Multiplying (5.39) with $\exp(2\psi - U)$ and differentiating the resulting expression with respect to r yields

$$\mathrm{Im}\Big[i\, q_I\, Y^I \left(\gamma' - 2g\, \mathrm{e}^{-\psi}\, \mathrm{e}^{i\gamma}\, W(Y) \right) \Big] = 0 \,, \tag{5.44}$$

where we used the flow equations for Y^I and for $(\psi - U)$. Using (5.41), this results in

$$\gamma' = 2g\, \mathrm{e}^{-\psi}\ \mathrm{Re}\Big[\mathrm{e}^{i\gamma}\, W(Y) \Big] \,, \tag{5.45}$$

which, upon using (5.39), equals the flow equation for γ given in (5.34). Thus, we conclude that upon imposing the constraint (5.39), the flow equation for γ' given in (5.34) is automatically satisfied.

Summarizing, we obtain the following independent flow equations. Since e^{2A} is expressed in terms of Y^I, it is not an independent quantity, cf. (5.5). As $U = \psi - A$ on a solution, U is also not an independent quantity. The independent flow equations are thus

$$\begin{aligned} \psi' &= 2\, g\, \mathrm{e}^{-\psi}\ \mathrm{Im}\left[\mathrm{e}^{i\gamma}\, W(Y) \right] \,, \\ Y'^I &= \mathrm{e}^{\psi - U}\, N^{IK}\, \bar{q}_K \,, \end{aligned} \tag{5.46}$$

with q_K given in (5.29) and with γ given in (5.42) (observe that the latter is, in general, r-dependent). In addition, a solution to these flow equations has to satisfy the reality condition (5.41) as well as the symplectic constraint (5.32). The flow equations for the scalar fields Y^I can equivalently be written in the form (5.36). Observe that in view of (5.5) and (5.41), the number of independent variables in the set (U, ψ, Y^I) is the same as in the set $(U, \psi, z^i = Y^i/Y^0)$, which was used in [119]. Indeed, the flow equations (5.46) are equivalent to the ones presented there and, thus, they describe supersymmetric brane solutions.

Observe that the right hand side of the flow equations (5.46) may be expressed in terms of the Y^I only by redefining the radial variable into $\partial/\partial\tau = \mathrm{e}^{\psi}\, \partial/\partial r$, in which case they become

$$\begin{aligned} \frac{\partial \psi}{\partial \tau} &= 2\, g\ \mathrm{Im}\left[\mathrm{e}^{i\gamma}\, W(Y) \right] \,, \\ \frac{\partial Y^I}{\partial \tau} &= \mathrm{e}^{-i\gamma}\, N^{IK} \left(\bar{\hat{Q}}_K + i\, g\, \mathrm{e}^{2A} \bar{\hat{h}}_K \right) \,, \end{aligned} \tag{5.47}$$

with A expressed in terms of the Y^I according to (5.5).

Let us now discuss the Hamiltonian constraint mentioned above. For a Lagrangian density $\sqrt{-g}\,(\,\tfrac{1}{2}R + \mathcal{L}_M\,)$ it is given by the variation of the action w.r.t. g^{00} as

$$\tfrac{1}{2} R_{00} + \frac{\delta \mathcal{L}_M}{\delta g^{00}} - \tfrac{1}{2}\, g_{00} \left(\tfrac{1}{2} R + \mathcal{L}_M \right) = 0 \,. \tag{5.48}$$

Using the matter Lagrangian (A.25) as well as the metric ansatz (5.1), and replacing the gauge fields by their charges, as in (5.10), gives

$$\mathrm{e}^{2\psi}\left\{U'^2 - \psi'^2 + N_{IJ}\,\tilde{X}^{I\prime}\,\bar{\tilde{X}}^{J\prime} + g^2\,\mathrm{e}^{-2U}\,V_{\rm tot}(\tilde{X},\bar{\tilde{X}})\right\} + \left[\mathrm{e}^{2\psi}\,(2\psi' - 2\,U')\right]' = 0\ , \tag{5.49}$$

where $V_{\rm tot}$ denotes the combined potential

$$\begin{aligned} V_{\rm tot}(\tilde{X},\bar{\tilde{X}}) &= g^2\,V(\tilde{X},\bar{\tilde{X}}) + \mathrm{e}^{-4A}\,V_{\rm BH}(\tilde{X},\bar{\tilde{X}}) \\ &= g^2\left[N^{IJ}\,\partial_I\tilde{W}\partial_{\bar{J}}\bar{\tilde{W}} - 2|\tilde{W}|^2\right] + \mathrm{e}^{-4A}\left[N^{IJ}\,\partial_I\tilde{Z}\partial_{\bar{J}}\bar{\tilde{Z}} + 2|\tilde{Z}|^2\right]\ . \end{aligned} \tag{5.50}$$

The Hamiltonian constraint (5.49) can now be rewritten (up to a total derivative term) as

$$L_{\rm BPS} + L_{\rm sympl} = 0\ , \tag{5.51}$$

where $L_{\rm BPS}$ and $L_{\rm sympl}$ denote the integrands of (5.19) (with the charges and fluxes replaced by their hatted counterparts) and (5.31), respectively. Since $L_{\rm BPS} = 0$ on a solution to the flow equations, it follows that the Hamiltonian constraint reduces to the symplectic constraint (5.32). The total derivative term vanishes by virtue of the flow equation for $A = \psi - U$.

In the following, we briefly check that (5.36) reproduces the standard flow equations for supersymmetric domain wall solutions in AdS_4. Setting $Q_I = P^I = 0$, and choosing the phase $\gamma = \pi/2$, we obtain $\psi = 2U$, $A = U$ as well as

$$\begin{pmatrix} (Y^I - \bar{Y}^I)' \\ (F_I - \bar{F}_I)' \end{pmatrix} = i\,g\begin{pmatrix} h^I \\ h_I \end{pmatrix}\ , \tag{5.52}$$

which describes the supersymmetric domain wall solution of [127]. Observe that the choice $\gamma = 0$ leads to a solution of the type (5.52) with the imaginary part of $(Y^I, F_I)'$ replaced by the real part. The flow equations (5.52) can be easily integrated and yield

$$\begin{pmatrix} Y^I - \bar{Y}^I \\ F_I - \bar{F}_I \end{pmatrix} = i\,g\begin{pmatrix} H^I \\ H_I \end{pmatrix} = i\,g\begin{pmatrix} \alpha^I + h^I\,r \\ \beta_I + h_I\,r \end{pmatrix}\ , \tag{5.53}$$

where (α_I, β^I) denote integration constants. This solution satisfies $W(Y) = \bar{W}(\bar{Y})$, which is precisely the condition (5.39). In addition, making use of (5.5), we obtain $\mathrm{e}^{2U} = g\left(H^I\,F_I(Y) - H_I\,Y^I\right)$.

5.1.2 Generalizations of the first-order rewriting

In the preceding section the first-order rewriting was done for a general prepotential. In minimal gauged supergravity other first-order rewritings are possible. For a detailed discussion see appendix B. Moreover, in the first-order rewriting performed above, what was used operationally was the fact that i) for an arbitrary charge and flux configuration the one-dimensional bulk Lagrangian can be written as a sum of squares and ii) imposing the symplectic constraint (5.32), the vanishing of the squares gives a vanishing bulk Lagrangian, which in turn is equal to the Hamiltonian density. This ensures that both the equations of motion as well as the Hamiltonian constraint are satisfied. Given this procedure, we may identify invariances of the action under transformations of the charges and/or fluxes, since any such transformation will automatically produce a new, possibly physically distinct, rewriting of the action.

In the ungauged case in four dimensions this has been already explored in terms of an $\mathcal{S}$-matrix that operates on the charges, while keeping the action invariant [39][1]. In the gauged case, the presence of both charges and fluxes allows for a wider class of transformations, in which both sets of quantum numbers are transformed. Thus finding transformations on both charges and fluxes that leave the action invariant allows for more general rewritings. In [1] such transformations for the charges and fluxes are constructed. Also some explicit examples are given. Here we will not go into the details and refer the reader to [1].

Furthermore, in [1] a class of non-extremal black brane solutions that are based on first-order flow equations is constructed. This is done by turning on a deformation parameter in the line element describing extremal black branes. In the context of $N = 2$ $U(1)$ gauged supergravity in five dimensions, it was shown in [12] that there exists a class of non-extremal charged black hole solutions that are based on first-order flow equations, even though their extremal limit is singular. There the non-extremal parameter is not only encoded in the line element, but also in the definition of the physical charges. In [1] only that case is studied where the deformation parameter enters in the line element. A different procedure for constructing non-extremal black hole solutions has recently been given in [128] in the context of ungauged supergravity in four dimensions. With the rewriting obtained in [1] it is possible to reproduce e.g. the non-extremal versions of some of our solutions, such as the solution that will be presented in (5.95). This includes the solution discussed in [9]. Moreover, the non-extremal deformation of the extremal domain wall solution (5.53) and the non-extremal version of the interpolating solution near AdS_4 and near $AdS_2 \times \mathbb{R}^2$ to be discussed in section 5.2.1 can be reproduced. Again we refer the reader to [1] for further information.

5.1.3 $AdS_2 \times \mathbb{R}^2$ backgrounds

In the following, we will consider space-times of the type $AdS_2 \times \mathbb{R}^2$, i.e. line elements (5.1) with constant A. In view of the relation (5.5), we thus demand that the Y^I are constant in this geometry. Observe that the latter differs from the case of black holes in ungauged supergravity. There, the appropriate Y^I variable is not given in terms of (5.3), but rather in terms of $A = -U$, and the associated flow equations are solved in terms of $Y^I = Y^I(r)$ rather than in terms of constant Y^I.

For constant Y^I, their flow equation yields $q_I = 0$, which implies $\hat{Q}_I = i\, g\, \mathrm{e}^{2A}\, \hat{h}_I$. Inserting this into the flow equation for U gives

$$\mathrm{e}^A (\mathrm{e}^U)' = -2g\, \mathrm{Im}\left[\mathrm{e}^{i\gamma}\, Y^I\, \hat{h}_I\right] \ , \tag{5.54}$$

[1] See also [63] for related transformations in five dimensions.

which equals $(\mathrm{e}^{\psi})'$. Hence we obtain $\psi' = U'$, which is consistent with $A = \psi - U =$ constant. Combining the flow equations (5.45) and (5.54) gives

$$\mathrm{e}^{A}\,\left(\mathrm{e}^{U-i\gamma}\right)' = 2\,i\,g\,Y^I\,\hat{h}_I = \text{constant}\;, \tag{5.55}$$

which yields

$$\mathrm{e}^{A}\,\mathrm{e}^{U-i\gamma} = 2\,i\,g\,Y^I\,\hat{h}_I\,r + c\;,\qquad c\in\mathbb{C}\;. \tag{5.56}$$

It follows that

$$\begin{aligned}\mathrm{e}^{A}\,\mathrm{e}^{U} &= \mathrm{Re}\left[2\,i\,g\,\mathrm{e}^{i\gamma}\,Y^I\,\hat{h}_I\,r + c\,\mathrm{e}^{i\gamma}\right]\\ &= -2\,g\,\mathrm{Im}\left[\mathrm{e}^{i\gamma}\,Y^I\,\hat{h}_I\right]r + \mathrm{Re}\left[c\,\mathrm{e}^{i\gamma}\right]\;.\end{aligned} \tag{5.57}$$

Now recall that γ is given by (5.42), which takes a constant value, since the Y^I are constant. Hence $\mathrm{Re}\,[c\,\mathrm{e}^{i\gamma}]$ is constant, and it can be removed by a redefinition of r, resulting in

$$\mathrm{e}^{A+U} = -2\,g\,\mathrm{Im}\left[\mathrm{e}^{i\gamma}\,Y^I\,\hat{h}_I\right]r\;. \tag{5.58}$$

Observe that $\mathrm{Im}\left[\mathrm{e}^{i\gamma}\,Y^I\,\hat{h}_I\right] \neq 0$ to ensure that the space-time geometry contains an AdS_2 factor.

Contracting $\hat{Q}_I = i\,g\,\mathrm{e}^{2A}\,\hat{h}_I$ with Y^I yields the value for e^{2A} as

$$\mathrm{e}^{2A} = -i\,\frac{Y^I\,\hat{Q}_I}{g\,Y^J\,\hat{h}_J} = -i\,\frac{Z(Y)}{g\,W(Y)} = i\,\frac{\bar{Z}(\bar{Y})}{g\,\bar{W}(\bar{Y})}\;, \tag{5.59}$$

and hence

$$\hat{Q}_I = \frac{Z(Y)}{W(Y)}\,\hat{h}_I\;. \tag{5.60}$$

The values of the Y^I are, in principle, obtained by solving (5.60), or equivalently,

$$\begin{aligned}Q_I - \tfrac{1}{2}(F_{IJ}+\bar{F}_{IJ})P^J &= \tfrac{1}{2}g\,\mathrm{e}^{2A}\,N_{IJ}h^J\;,\\ -\tfrac{1}{2}N_{IJ}\,P^J &= g\,\mathrm{e}^{2A}\left(h_I - \tfrac{1}{2}(F_{IJ}+\bar{F}_{IJ})h^J\right)\;.\end{aligned} \tag{5.61}$$

There may, however, be flat directions in which case some of the Y^I remain unspecified, and an example thereof is given in section 5.2. The reality of e^{2A} forces the phases of $Z(Y)$ and of $W(Y)$ to differ by $\pi/2$ [119].

This relation (5.60) has an immediate consequence, similar to the one for black holes derived in [119]. Namely, using (5.60) and (5.59) in (5.33) leads to

$$0 = \mathrm{Im}\left(\frac{Z(Y)}{W(Y)}\,\hat{h}_I N^{IJ}\bar{\hat{h}}_J\right) = \mathrm{Im}\left(i\,g\,\mathrm{e}^{2A}\,\hat{h}_I N^{IJ}\bar{\hat{h}}_J\right) = g\,\mathrm{e}^{2A}\,\hat{h}_I N^{IJ}\bar{\hat{h}}_J\;. \tag{5.62}$$

Given that $g\,\mathrm{e}^{2A}\neq 0$, we infer that for any $AdS_2\times\mathbb{R}^2$ geometry

$$\hat{h}_I N^{IJ}\bar{\hat{h}}_J = 0\;. \tag{5.63}$$

We will use this fact below in section 5.2.1.

5.2 Exact solutions

In the following, we consider exact (dyonic) solutions of the flow equations in specific models. In general, solutions fall into two classes, namely solutions with constant γ and solutions with non-constant γ along the flow.

5.2.1 Solutions with constant γ

Let us discuss dyonic solutions in the presence of fluxes. For concreteness, we choose $\gamma = 0$ in the following. The flow equations (5.34) for U, ψ and A read,

$$\begin{aligned}
(\mathrm{e}^U)' &= \mathrm{e}^{-3A}\,\mathrm{Re}\left[Y^I\,\hat{Q}_I\right] - g\,\mathrm{e}^{-A}\,\mathrm{Im}\left[Y^I\,\hat{h}_I\right]\ , \\
(\mathrm{e}^{\psi})' &= -2\,g\,\mathrm{Im}\left[Y^I\,\hat{h}_I\right]\ , \\
(\mathrm{e}^{A})' &= -\,\mathrm{Re}\left[Y^I\,q_I\right]\ ,
\end{aligned} \tag{5.64}$$

while the flow equations for the Y^I are

$$\begin{pmatrix}(Y^I-\bar{Y}^I)' \\ (F_I-\bar{F}_I)'\end{pmatrix} = 2i\,\mathrm{e}^{-\psi}\left[-\,\mathrm{Im}\begin{pmatrix}N^{IK}\,\hat{Q}_K \\ \bar{F}_{IK}\,N^{KJ}\,\hat{Q}_J\end{pmatrix} + g\,\mathrm{e}^{2A}\,\mathrm{Re}\begin{pmatrix}N^{IK}\,\hat{h}_K \\ \bar{F}_{IK}\,N^{KJ}\,\hat{h}_J\end{pmatrix}\right]. \tag{5.65}$$

In the presence of both fluxes and charges, the flow equations (5.65) cannot be easily integrated. A simplification occurs whenever $\mathrm{Re}\,F_{IJ} = 0$. This is, for instance, the case in the model $F = -iX^0\,X^1$, to which we now turn.

The $F(X) = -iX^0\,X^1$ model

For this choice of prepotential, we have $F(X) = (X^0)^2\,\mathcal{F}(z)$, where $\mathcal{F}(z) = -iz$ with $z = X^1/X^0 = Y^1/Y^0$. The associated Kähler potential reads $K = -\ln(z+\bar{z})$, and z is related to the dilaton through $\mathrm{Re}\,z = \mathrm{e}^{-2\phi}$. We also have

$$N_{IJ} = -2\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \quad , \quad N^{IJ} = -\tfrac{1}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\ , \tag{5.66}$$

as well as

$$\mathcal{N}_{00} = -iz \quad , \quad \mathcal{N}_{11} = -i/z \quad , \quad \mathcal{N}_{01} = 0\ . \tag{5.67}$$

The flow equations (5.65) become

$$\begin{pmatrix}(Y^0-\bar{Y}^0)' \\ (Y^1-\bar{Y}^1)' \\ -i(Y^1+\bar{Y}^1)' \\ -i(Y^0+\bar{Y}^0)'\end{pmatrix} = i\,\mathrm{e}^{-\psi}\,\mathrm{Im}\begin{pmatrix}\hat{Q}_1 \\ \hat{Q}_0 \\ i\,\hat{Q}_0 \\ i\,\hat{Q}_1\end{pmatrix} - i\,g\,\mathrm{e}^{-\psi+2A}\,\mathrm{Re}\begin{pmatrix}\hat{h}_1 \\ \hat{h}_0 \\ i\,\hat{h}_0 \\ i\,\hat{h}_1\end{pmatrix}\ . \tag{5.68}$$

This yields

$$\begin{pmatrix} (Y^0 - \bar{Y}^0)' \\ (Y^1 - \bar{Y}^1)' \\ -i(Y^1 + \bar{Y}^1)' \\ -i(Y^0 + \bar{Y}^0)' \end{pmatrix} = i\,\mathrm{e}^{-\psi}\,\mathrm{Re}\begin{pmatrix} P^0 - g\,\mathrm{e}^{2A}\,h_1 \\ P^1 - g\,\mathrm{e}^{2A}\,h_0 \\ Q_0 + g\,\mathrm{e}^{2A}\,h^1 \\ Q_1 + g\,\mathrm{e}^{2A}\,h^0 \end{pmatrix} . \tag{5.69}$$

In order to gain some intuition for finding a solution with all charges and fluxes turned on, let us first consider a simpler example. We retain only one of the four charge/flux combinations appearing in (5.69), namely the one with $Q_1 \neq 0, h^0 \neq 0$. Observe that any of these four combinations satisfies the symplectic constraint (5.32). The associated flow equations (5.69) are

$$\begin{pmatrix} (Y^0 - \bar{Y}^0)' \\ (Y^1 - \bar{Y}^1)' \\ -i(Y^1 + \bar{Y}^1)' \\ -i(Y^0 + \bar{Y}^0)' \end{pmatrix} = i\,\mathrm{e}^{-\psi}\begin{pmatrix} 0 \\ 0 \\ 0 \\ Q_1 + g\,\mathrm{e}^{2A}\,h^0 \end{pmatrix} , \tag{5.70}$$

which yields $Y^1 =$ constant and

$$(Y^0)' = -\tfrac{1}{2}\,\mathrm{e}^{-\psi}\,\left(Q_1 + g\,\mathrm{e}^{2A}\,h^0\right) . \tag{5.71}$$

The flow equations for ψ and A read,

$$\begin{aligned} (\mathrm{e}^{\psi})' &= -2g\,\mathrm{Re}\left(Y^1\right)h^0 , \\ \left(\mathrm{e}^{2A}\right)' &= -2\mathrm{e}^{-\psi}\,\mathrm{Re}\left(Y^1\right)\left(Q_1 + g\,\mathrm{e}^{2A}\,h^0\right) . \end{aligned} \tag{5.72}$$

For the equation for A we used $q_0 = 0$ and $q_1 = \mathrm{e}^{-A-\psi}\left(Q_1 + g\,\mathrm{e}^{2A}\,h^0\right)$. The flow equation for ψ can be readily integrated and yields

$$\mathrm{e}^{\psi} = -2g\,\mathrm{Re}\left(Y^1\right)\,h^0\,(r + c) , \tag{5.73}$$

where c denotes an integration constant. Inserting this into the flow equation for A gives

$$\left(\mathrm{e}^{2A}\right)' = \frac{Q_1 + g\,\mathrm{e}^{2A}\,h^0}{g\,h^0\,(r + c)} , \tag{5.74}$$

which can be integrated to

$$\mathrm{e}^{2A} = \frac{\mathrm{e}^{\beta}(r + c) - Q_1}{g\,h^0} , \tag{5.75}$$

where β denotes another integration constant. Plugging this into the flow equation for Y^0, we can easily integrate the latter,

$$Y^0 = \frac{\mathrm{e}^{\beta}}{4gh^0\,(\mathrm{Re}\,Y^1)}\,(r + \delta) , \tag{5.76}$$

where δ denotes a third integration constant. Using (5.5), we infer

$$\delta = c - e^{-\beta} Q_1 \, . \tag{5.77}$$

Moreover, for U we obtain

$$e^{2U} = e^{2\psi - 2A} = \frac{4g^3\,(h^0)^3\,\left(\mathrm{Re}\,Y^1\right)^2\,(r+c)^2}{e^{\beta}(r+c) - Q_1} \, . \tag{5.78}$$

We take the horizon to be at $r + c = 0$, i.e. we set $c = 0$ in the following. Summarizing, we thus obtain

$$\begin{aligned}
Y^1 &= \text{constant} \, , \\
Y^0 &= \frac{e^{\beta}\, r - Q_1}{4\, g h^0\, (\mathrm{Re}\, Y^1)} \, , \\
\mathrm{Re}\, z &= \frac{4\, g\, h^0\, (\mathrm{Re}\, Y^1)^2}{e^{\beta}\, r - Q_1} \, , \\
e^{2A} &= \frac{e^{\beta}\, r - Q_1}{g\, h^0} = \frac{4\, (\mathrm{Re}\, Y^1)^2}{\mathrm{Re}\, z} \, , \\
e^{2U} &= \frac{4g^3\,(h^0)^3\,\left(\mathrm{Re}\,Y^1\right)^2\, r^2}{e^{\beta}\, r - Q_1} \, , \\
e^{2\psi} &= 4g^2\,\left(\mathrm{Re}\,Y^1\right)^2\,(h^0)^2\, r^2 \, .
\end{aligned} \tag{5.79}$$

We require $h^0 > 0$ and $Q_1 < 0$ to ensure positivity of $\mathrm{Re}\, z, e^{2A}$ and e^{2U}. This choice is also necessary in order to avoid a singularity of $\mathrm{Re}\, z$ and e^{2U} at $r = e^{-\beta} Q_1$. The resulting brane solution has non-vanishing entropy density, i.e. $e^{2A(r=0)} \neq 0$. The imaginary part of z is left unspecified by the flow equations. However, demanding the constraint (5.41) imposes Y^1 to be real, so that $\mathrm{Im}\, z = 0$. Note that $\mathrm{Re}\, Y^1$ corresponds to a flat direction (see also [129]). The above describes the extremal limit of the solution discussed in sec. 7 of [9] in the context of AdS/CMT (see also [130, 131, 132]).[2]

We now want to solve the equations (5.69) when all charges and fluxes are non-zero. Since the equations are quite difficult to solve directly, we will make an ansatz for e^{ψ} and e^{2A} and then solve for the Y^I. In the example above we saw that e^{ψ} and e^{2A} were linear functions of r, so we choose the following ansatz for these functions,

$$\begin{aligned}
e^{\psi(r)} &= a\, r \, , \\
e^{2A(r)} &= b\, r + c \, .
\end{aligned} \tag{5.80}$$

When plugging this ansatz into the flow equations (5.69) we get

$$\begin{aligned}
(Y^0(r))' &= -\frac{1}{2ar}\left(\bar{\hat{Q}}_1 + igc\bar{\hat{h}}_1\right) - \frac{gb}{2a} i\bar{\hat{h}}_1 \, , \\
(Y^1(r))' &= -\frac{1}{2ar}\left(\bar{\hat{Q}}_0 + igc\bar{\hat{h}}_0\right) - \frac{gb}{2a} i\bar{\hat{h}}_0 \, .
\end{aligned} \tag{5.81}$$

[2] In order to compare the solutions, one would have to set $\gamma = \delta = 1$ in (7.1) of [9], take their solution to the extremal limit $m^2 = \frac{q^2}{8}$ and relate the radial coordinates according to $r^{(\mathrm{us})} = \frac{1}{2l}(r^{(\mathrm{them})})^2 - \frac{1}{2}\frac{q}{\sqrt{8}}$.

These equations can easily be integrated to give

$$
\begin{aligned}
Y^0(r) &= -\frac{1}{2a}\left(\bar{\hat{Q}}_1 + igc\bar{\hat{h}}_1\right)\ln r - \frac{gb}{2a} i\bar{\hat{h}}_1 r + C^0\,, \\
Y^1(r) &= -\frac{1}{2a}\left(\bar{\hat{Q}}_0 + igc\bar{\hat{h}}_0\right)\ln r - \frac{gb}{2a} i\bar{\hat{h}}_0 r + C^1\,.
\end{aligned} \tag{5.82}
$$

In order for (5.80) and (5.82) to constitute a solution, several conditions on the parameters a, b, c, the charges and the fluxes have to be fulfilled. On the one hand, one has to impose the constraints

$$
\operatorname{Im}\left(\hat{Q}_I Y^I\right) = 0 \quad , \quad \operatorname{Re}\left(\hat{h}_I Y^I\right) = 0\,, \tag{5.83}
$$

following from (5.34) and (5.39) for $\gamma = 0$. On the other hand, further constraints on the parameters arise from (5.5), (5.33) and the equation for ψ in (5.64). Note that the equation for U does not give additional information, as U is determined, once the Y^I and ψ are given.

First, the constraints (5.83) imply

$$
\begin{aligned}
\operatorname{Re}\left(\hat{Q}_I N^{IJ}\bar{\hat{h}}_J\right) &= 0\,, \\
\operatorname{Im}\left(\hat{Q}_I C^I\right) &= 0\,, \\
\operatorname{Re}\left(\hat{h}_I C^I\right) &= 0\,.
\end{aligned} \tag{5.84}
$$

Next, let us have a look at the equation for ψ. Using our ansatz for e^{ψ}, it reads

$$
a = -2\,g\,\operatorname{Im}\left(Y^I\,\hat{h}_I\right)\,. \tag{5.85}
$$

With the form of the Y^I given in (5.82) and demanding that the right hand side of (5.85) is a constant, we obtain the constraint

$$
\operatorname{Re}\left(\bar{\hat{h}}_1\hat{h}_0\right) = 0\,. \tag{5.86}
$$

This directly leads to a vanishing of the term linear in r. Together with the symplectic constraint (5.33) it also implies that the logarithmic term vanishes. Moreover, we read off

$$
a = -2g\,\operatorname{Im}\left(C^I\hat{h}_I\right)\,. \tag{5.87}
$$

Next, we analyze the constraints coming from $\mathrm{e}^{2A} = -N_{IJ}Y^I\bar{Y}^J$. With our ansatz (5.80) and the form of N_{IJ} given in (5.66), one obtains

$$
b\,r + c = 4\operatorname{Re}\left(Y^0\bar{Y}^1\right)\,. \tag{5.88}
$$

This fixes the constant c to be

$$
c = 4\operatorname{Re}(C^0\bar{C}^1)\,. \tag{5.89}
$$

Using (5.87) the term linear in r on the right hand side of (5.88) is just identically $b\,r$, i.e. there is no constraint on b (except that it has to be positive in order to guarantee the positivity of e^{2A} for all r). All other terms on the right hand side vanish if one demands

$$\begin{aligned} \hat{Q}_I N^{IJ} \bar{\hat{Q}}_J &= 0\,, &(5.90)\\ \mathrm{Re}\left[\left(\hat{Q}_I - igc\hat{h}_I\right) C^I\right] &= 0\,, &(5.91)\end{aligned}$$

in addition to (5.86) and the symplectic constraint (5.33).

We now show that (5.91) is fulfilled because the even stronger constraint

$$\hat{Q}_I - igc\hat{h}_I = 0 \qquad (5.92)$$

holds. To do so, we note that (5.91) together with (5.84) and (5.87) imply

$$\begin{aligned} \hat{Q}_0 C^0 + \hat{Q}_1 C^1 &= -\frac{ac}{2}\,,\\ \hat{h}_0 C^0 + \hat{h}_1 C^1 &= -i\frac{a}{2g}\,. \end{aligned} \qquad (5.93)$$

To get a solution for C^0 and C^1 one of the two following conditions has to be valid:

i) If and only if $\hat{Q}_0\hat{h}_1 - \hat{h}_0\hat{Q}_1 = \rho \neq 0$ there is a unique solution for C^0 and C^1.

ii) If the two lines in (5.93) are multiples of each other then there exists a whole family of solutions.

We will now show that case i) can be ruled out. To do so we combine the determinant condition of case i) with the first constraint in (5.84), i.e. we look at the system of linear equations for $\hat{Q}_0$ and $\hat{Q}_1$

$$\begin{aligned} \hat{Q}_0\hat{h}_1 - \hat{Q}_1\hat{h}_0 &= \rho\,,\\ \hat{Q}_0\bar{\hat{h}}_1 + \hat{Q}_1\bar{\hat{h}}_0 &= 0\,. \end{aligned} \qquad (5.94)$$

Obviously, the two lines cannot be multiples of each other for $\rho \neq 0$. Thus, in order to find a solution at all, the determinant $\hat{h}_1\bar{\hat{h}}_0 + \bar{\hat{h}}_1\hat{h}_0$ has to be non-vanishing. This is, however, in conflict with (5.86). Thus, the two lines in (5.93) have to be multiples of each other, implying (5.92). Note that this implies the absence of the logarithmic terms in the solutions for Y^I.

To summarize we have found the following solution:

$$\begin{aligned} Y^0(r) &= -\frac{gb}{2a} i\bar{\hat{h}}_1 r + C^0\,,\\ Y^1(r) &= -\frac{gb}{2a} i\bar{\hat{h}}_0 r + C^1\,,\\ \mathrm{e}^{\psi(r)} &= ar\,,\\ \mathrm{e}^{2A(r)} &= br + c\,. \end{aligned} \qquad (5.95)$$

In addition, the parameters have to fulfill the conditions

$$\mathrm{Re}(\hat{h}_0\bar{\hat{h}}_1) = 0\,, \quad \hat{Q}_I = igc\hat{h}_I\,, \quad \hat{h}_I C^I = -i\frac{a}{2g}\,, \quad \mathrm{Re}(C^0\bar{C}^1) = \frac{c}{4}\,, \tag{5.96}$$

while the parameter b can be any non-negative number. For $b = 0$ this solution falls into the class discussed in section 5.1.3.

Let us finally also mention that one can show that the $F(X) = -iX^0\,X^1$ model does not allow for Nernst brane solutions (i.e. solutions with vanishing entropy) of the first-order equations. Making an ansatz $\mathrm{e}^U \sim r^\alpha, \mathrm{e}^\psi \sim r^\beta$ and $\mathrm{e}^A \sim r^{\beta-\alpha}$ for the near-horizon geometry, one can show that the only solutions with both non-vanishing charges and fluxes have $\alpha = \beta = 1$ and are captured by the solution discussed in this section after setting $b = 0$.

The $F = -(X^1)^3/X^0$ model: Interpolating solution between AdS_4 and $AdS_2 \times \mathbb{R}^2$

Next, we would like to construct a (supersymmetric) solution that interpolates between an AdS_4 vacuum at spatial infinity, and an $AdS_2 \times \mathbb{R}^2$ background with constant Y^I at $r = 0$. Thus, asymptotically, we require the solution to be of the domain wall type (see (5.52)) with $\gamma = \pi/2$. Hence the Y^I satisfy (5.53) with $\alpha^I = \beta_I = 0$, so that we may write $Y^I = y^I_\infty\, r$ with $y^I_\infty = g\,N^{IJ}\,\bar{\hat{h}}_J =$ constant. The latter can be established as follows. Using $(Y-\bar{Y})^I = ig\,h^I\,r$ and $F_I - \bar{F}_I = ig\,h_I\,r$, we obtain

$$iN_{IJ}\,Y^J = F_I - \bar{F}_{IJ}\,Y^J = ig\,\bar{\hat{h}}_I\,r\;, \tag{5.97}$$

so that

$$\tfrac{1}{2}(Y+\bar{Y})^I = g\left(N^{IJ}\,\bar{\hat{h}}_J - \tfrac{1}{2}i\,h^I\right) r\;. \tag{5.98}$$

It follows that asymptotically,

$$Y^I = g\,N^{IJ}\,\bar{\hat{h}}_J\,r\,, \tag{5.99}$$

and hence, in an AdS_4 background,

$$\mathrm{e}^{2A} = \mathrm{e}^{2U} = -g^2\,\hat{h}_I\,N^{IJ}\,\bar{\hat{h}}_J\,r^2\;. \tag{5.100}$$

Since this expression only depends on the Y^I through the combinations N_{IJ} and F_{IJ}, which are homogeneous of degree zero, the r-dependence scales out of these quantities, and thus $\hat{h}_I\,N^{IJ}\,\bar{\hat{h}}_J$ is a constant. For e^{2A} to be positive, we need to require $a^2 \equiv -\hat{h}_I\,N^{IJ}\,\bar{\hat{h}}_J > 0$.

At $r = 0$, on the other hand, we demand $\gamma = \gamma_0$ as well as $Y^I =$ constant, with A given by (5.59). Thus, we want to construct a solution with a varying $\gamma(r)$ that interpolates between the values $\pi/2$ and γ_0. From (5.63) we know that $\hat{h}_I\,N^{IJ}\,\bar{\hat{h}}_J = 0$ at the horizon. The Y^I appearing in this expression are evaluated at the horizon. In general, their values will differ from the asymptotic values, so that the flow will interpolate between an asymptotic AdS_4 background satisfying $\hat{h}_I\,N^{IJ}\,\bar{\hat{h}}_J < 0$ and an $AdS_2 \times \mathbb{R}^2$ background satisfying

$\hat{h}_I\, N^{IJ}\, \bar{\hat{h}}_J = 0$. This, however, will not be possible whenever F_{IJ} is independent of the Y^I, such as in the $F = -iX^0X^1$ model, as already observed in [119]. Thus, in the example below, we will consider the $F = -\,(X^1)^3/X^0$ model instead.

The interpolating solution has to have the following properties. At spatial infinity, where $\mathrm{Im}\, W(Y) = 0$, γ is driven away from its value $\pi/2$ by the term $\mathrm{Re}\, Z(Y)$ in the flow equation of γ,

$$\gamma' = \left[\mathrm{e}^{2U-3\psi}\ \mathrm{Re}\, Z(Y)\right]_\infty = \left[\mathrm{e}^{-4U}\ \mathrm{Re}\, Z(Y)\right]_\infty = \frac{\mathrm{Re}\, Z(y_\infty)}{a^4\, r^3}\ , \tag{5.101}$$

and hence

$$\gamma(r) \approx \frac{\pi}{2} - \frac{\mathrm{Re}\, Z(y_\infty)}{2a^4\, r^2}\ . \tag{5.102}$$

Our example below has $\mathrm{Re}\, Z(y_\infty) = 0$ though and, thus, the phase γ will turn out to be constant. Near $r = 0$, on the other hand, the deviation from the horizon values can be determined as follows. Denoting the deviation by $\delta Y^I = \beta^I\, r^p$, we obtain $\delta(\mathrm{e}^{2A}) = c\, r^p$ with

$$c = -N_{IJ}(\beta^I\, \bar{Y}^J + Y^I\, \bar{\beta}^J)\ , \tag{5.103}$$

where in this expression the Y^I and N_{IJ} are calculated at the horizon. Using this, we compute the deviation of $\bar{q}_I$ from its horizon value $\bar{q}_I = 0$,

$$\delta\bar{q}_I = \mathrm{e}^{-A-\psi-\mathrm{i}\gamma_0}\left(-\bar{F}_{\bar{I}\bar{J}\bar{L}}(p^J + \mathrm{i}gh^J)\bar{\beta}^L + \mathrm{i}g\bar{\hat{h}}_I\, c\right)\, r^p\ , \tag{5.104}$$

where all the quantities that do not involve β^I are evaluated on the horizon. The flow equations for the Y^I then yield

$$p\, N_{IK}\, \beta^K = \frac{\mathrm{e}^{-i\gamma_0}}{\Delta}\left(-\bar{F}_{\bar{I}\bar{J}\bar{L}}(p^J + \mathrm{i}gh^J)\bar{\beta}^L + \mathrm{i}g\bar{\hat{h}}_I\, c\right)\ , \tag{5.105}$$

where

$$\Delta = -2g\, \mathrm{Im}\left(\mathrm{e}^{i\gamma_0}\, \hat{h}_I\, Y^I\right)\ . \tag{5.106}$$

Contracting (5.105) with $\bar{Y}^I$ yields $p = 1$. Inserting this value into (5.105) yields a set of equations that determines the values of the β^I.

Next, using the flow equation for γ, we compute the deviation from the horizon value γ_0, which we denote by $\delta\gamma = \Sigma\, r$. We obtain

$$\Sigma = -\frac{g}{\Delta}\, \mathrm{Re}\left(\mathrm{e}^{i\gamma_0}\, \beta^I\, \hat{h}_I\right)\ . \tag{5.107}$$

The example below will have a vanishing Σ, consistent with a constant γ. Finally, using the flow equation for ψ we compute the change of ψ,

$$\delta\psi = -\frac{g}{\Delta}\, \mathrm{Im}\left(\mathrm{e}^{i\gamma_0}\, \beta^I\, \hat{h}_I\right)\, r\ . \tag{5.108}$$

We now turn to a concrete example of an interpolating solution. As already mentioned, this will be done in the context of the $F(X) = -\frac{(X^1)^3}{X^0}$ model. To obtain an interpolating solution we first need to specify the form of the solution at both ends.

Let us first have a look at the near horizon $AdS_2 \times \mathbb{R}^2$ region. According to our discussion in sec. 5.1.3, we have to solve (5.63) and $\hat{Q}_I = ig\mathrm{e}^{2A}\hat{h}_I$ under the assumption that the Y^I are constant. It turns out that these constraints can be solved, for instance, by choosing $Q_0, P^1, h_1, h^0 \neq 0$ and all other parameters vanishing. Of course, due to the symplectic constraint (5.32), the four non-vanishing parameters are not all independent, but have to fulfill

$$P^1 = \frac{Q_0 h^0}{h_1}\,. \tag{5.109}$$

For $F(Y) = -\frac{(Y^1)^3}{Y^0}$, and introducing $z = z_1 + iz_2 = \frac{Y^1}{Y^0}$, we have

$$N_{IJ} = \begin{pmatrix} -4\,\mathrm{Im}\,(z^3) & 6\,\mathrm{Im}\,(z^2) \\ 6\,\mathrm{Im}\,(z^2) & -12\,\mathrm{Im}\,(z) \end{pmatrix}\,. \tag{5.110}$$

Using

$$\mathrm{e}^{2A} = 8|Y^0|^2 z_2^3\,, \tag{5.111}$$

which follows from (5.5), shows that one can fulfill $\hat{Q}_I = ig\mathrm{e}^{2A}\hat{h}_I$ and (5.63) by fixing

$$\begin{aligned} z_1 &= 0\,, \\ z_2 &= \sqrt{\frac{(3+2\sqrt{3})h_1}{3h^0}}\,, \\ |Y^0| &= \frac{3^{\frac{1}{4}}\sqrt{Q_0}}{2\sqrt{2}z_2^2\sqrt{h_1}}\,. \end{aligned} \tag{5.112}$$

Interestingly we find that the axion, which is in this model given by z_1, has to vanish and all parameters Q_0, P^1, h_1, h^0 have to have the same sign.

In order to describe the asymptotic AdS_4 region, we note that asymptotically the charges Q_0 and P^1 can be neglected and we can read off the asymptotic form of the solution from (5.53) with vanishing integration constants α^I and β_I. More precisely, we have to allow that the asymptotic form of the interpolating solution differs from (5.53) by an overall factor. This is because, apriori, we only know that asymptotically $(\psi - 2U)' = 0$, cf. (5.34) for vanishing charges. Without the need to match the asymptotic region to a near horizon $AdS_2 \times \mathbb{R}^2$ region, we could just absorb the constant $\psi - 2U$ in a rescaling of the coordinates x and y. However, when we start from the hear horizon solution and integrate out to infinity, we can not expect to end up with this choice of convention. Allowing for a non-vanishing constant $\psi - 2U = C \neq 0$, the flow equations (5.52) for Y^I and, thus, also the solutions (5.53), would obtain an overall factor e^C. Concretely, we obtain

$$\begin{aligned} Y^0_{AdS_4}(r) &= i\mathrm{e}^C \frac{g}{2} h^0 r\,, \\ Y^1_{AdS_4}(r) &= -\mathrm{e}^C \frac{g\sqrt{h^0 h_1}}{2\sqrt{3}} r\,, \end{aligned} \tag{5.113}$$

with a constant e^C to be determined by numerics below. Note, however, that the dilaton $\mathrm{e}^{-2\phi} = z_2 = \sqrt{\frac{h_1}{3h^0}}$ is independent of this factor and it comes out to be real if h_1 and h^0 have the same sign, consistently with what we found from the near horizon region.

We would now like to discuss the interpolating solution, which we obtain by specifying boundary conditions at the horizon and then integrating numerically from the horizon to infinity using Mathematica's NDSolve. Given that in the AdS_4 region Y^0 is purely imaginary and Y^1 is purely real, we choose the same reality properties at the horizon, i.e. we fulfill (5.112) by

$$\begin{aligned} Y^0_h &= i\frac{3^{\frac{5}{4}} h^0 \sqrt{Q_0}}{2\sqrt{2}(3+2\sqrt{3})h_1^{\frac{3}{2}}}\,, \\ Y^1_h &= iY^0_h z_2 = -\frac{3^{\frac{3}{4}}\sqrt{h^0 Q_0}}{2\sqrt{2(3+2\sqrt{3})h_1}}\,. \end{aligned} \tag{5.114}$$

The subscript h means that these values pertain to the horizon. Now we can determine the phase γ_0 at the horizon using (5.42). We get $\mathrm{e}^{-2i\gamma_0} = -1$ independently of h_1, h^0, Q_0 and P^1. We choose $\gamma_0 = \pi/2$ as in the asymptotic AdS_4 region and we will see that this leads to a constant value for γ throughout.

Next we have to determine how the solution deviates near the horizon from $AdS_2 \times \mathbb{R}^2$. We do this following our discussion at the beginning of this section, cf. (5.103) - (5.108). Given the reality properties of Y^I in both limiting regions, we choose β^0 purely imaginary and β^1 purely real. Solving (5.105) for the β's results in

$$\operatorname{Im}\beta^0 = \operatorname{Re}\beta^1 \frac{h^0\big((\sqrt{3}-2)\sqrt{3+2\sqrt{3}}h_1\sqrt{\frac{h_1}{h^0}} + (5\sqrt{3}-9)Q_0\big)}{h_1^2}\,. \tag{5.115}$$

Given this, the deviation of all the fields close to the horizon can be determined (resulting in initial conditions at, say $r_i = 10^{-6}$) and a numerical solution for the flow equations (5.34) can be found, integrating from the horizon outwards. To do the numerics we chose

$$g = 1 \quad , \quad \operatorname{Re}\beta^1 = 0.01 \quad , \quad Q_0 = 1 \quad , \quad h_1 = 1 \quad , \quad h^0 = 10\,. \tag{5.116}$$

One can see that asymptotically $\operatorname{Im} Y^0$ and $\operatorname{Re} Y^1$ are linear functions of r (cf. fig. 5.1), whereas the real part of Y^0 and imaginary part of Y^1 are zero within the numerical tolerances. The constant e^C in (5.113) can be determined to be roughly $\mathrm{e}^C \approx 422$. Futhermore, for $r \to \infty$, the functions e^A, e^U and e^ψ behave as expected for the AdS_4 background, i.e. $\mathrm{e}^A = \mathcal{O}(r)$, $\mathrm{e}^U = \mathcal{O}(r)$ and $\mathrm{e}^\psi = \mathcal{O}(r^2)$ (cf. fig. 5.2 and 5.3).[3] Another important feature is that $\gamma(r) = \frac{\pi}{2}$ for any r, cf. fig. 5.3.

[3]Note the difference in the plotted r-range in fig. 5.3 compared to figs. 5.1 and 5.2. Plotting $\operatorname{Im} Y^0$, $\operatorname{Re} Y^1$, e^A and e^U for larger values would just confirm the linearity in r.

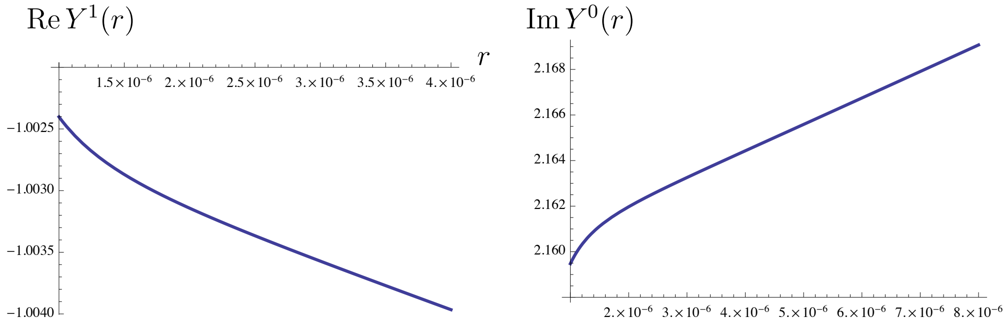

Figure 5.1: Interpolating solutions for Re Y^1 and Im Y^0.

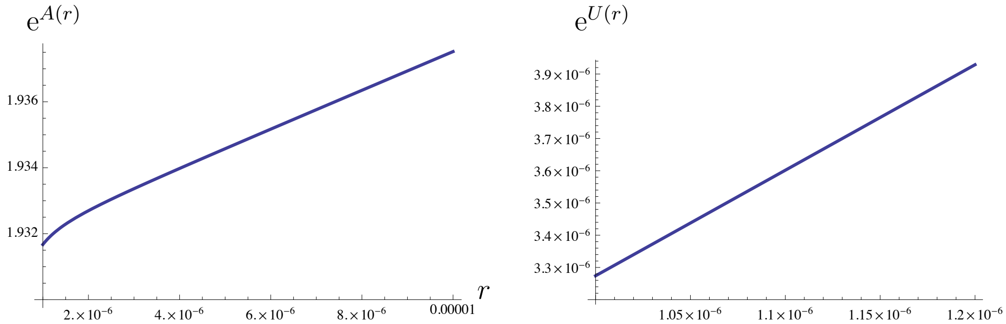

Figure 5.2: Interpolating solutions for e^A and e^U.

5.2.2 Solutions with non-constant γ

Next, we turn to solutions with non-constant γ along the flow. We focus on two models.

The $F = -i\,X^0\,X^1$ model

We first consider the model $F = -i\,X^0\,X^1$ and restrict ourselves to a case involving only the electric charge Q_0 and the electric flux h_0. Since N^{IJ} is off-diagonal, the flow equation for Y^0 reads $(Y^0)' = 0$. This gives $Y^0 = C^0$, with C^0 a non-vanishing c-number, which we take to be real, since the phase of C^0 can be absorbed into γ. Then, the relation (5.5) yields $\exp(2A) = 4C^0\,\mathrm{Re}\left(\bar{Y}^1\right)$, from which we infer that Re $Y^1 = \exp(2A)/(4C^0)$. In the following we take $Q_0, h_0 > 0$ and $C^0 < 0$, for concreteness.

The constraint equation (5.39) yields the following relation between γ and A,

$$\tan[\gamma] = \frac{g\,h_0}{Q_0}\,e^{2A}\,. \tag{5.117}$$

Using the τ-variable introduced in (5.47), the flow equation for A can be written as

$$\begin{aligned} \dot{A} &= -\,\mathrm{Re}\left[(Q_0\,e^{-2A} - i\,g\,h_0)\,C^0\,e^{i\gamma}\right] \\ &= -\,C^0\,\left(Q_0\,e^{-2A}\cos[\gamma] + g\,h_0\,\sin[\gamma]\right)\,, \end{aligned} \tag{5.118}$$

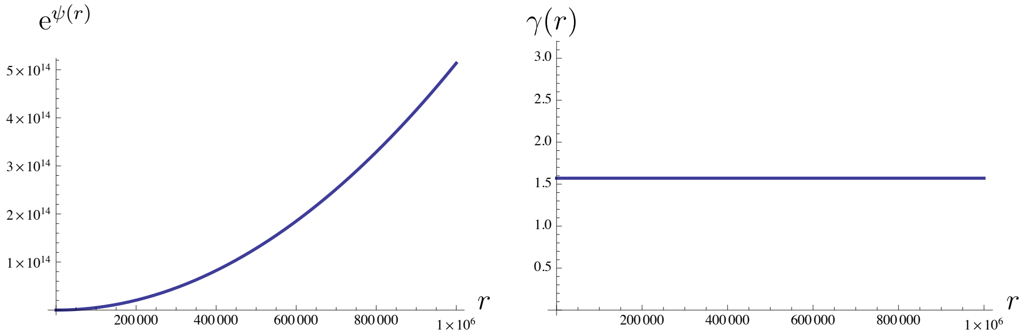

Figure 5.3: Interpolating solutions for e^{ψ} and γ.

where $\dot{A} = \partial A/\partial\tau$. Using (5.117) and (assuming $\gamma \in [-\pi/2, \pi/2]$)

$$\sin[\gamma] = \frac{\tan[\gamma]}{\sqrt{1+\tan^2[\gamma]}} \tag{5.119}$$

in (5.118) gives

$$\frac{\partial}{\partial\tau} e^{2A} = -2\,C^0 \sqrt{Q_0^2 + g^2\,h_0^2\,e^{4A}}\,. \tag{5.120}$$

This is solved by

$$e^{2A} = \alpha \left(e^{\delta\,\tau} - \frac{Q_0^2}{4\,\alpha^2\,g^2\,h_0^2}\, e^{-\delta\,\tau} \right) , \tag{5.121}$$

where $\delta = -2\,g\,h_0\,C^0 > 0$ and α denotes an integration constant. Setting $\alpha = Q_0/(2gh_0)$ this becomes

$$e^{2A} = \frac{Q_0}{gh_0} \sinh(\,\delta\,\tau\,)\,, \tag{5.122}$$

and demanding $\exp(2A)$ to be positive restricts the range of τ to lie between 0 and ∞.

The flow equation for ψ is given by

$$\dot{\psi} = \delta\,\sin[\gamma] = \delta\,\tanh(\,\delta\,\tau\,)\,, \tag{5.123}$$

where we used (5.119), (5.117) and (5.122). This can easily be solved by

$$e^{\psi-\psi_0} = \cosh(\,\delta\,\tau\,)\,, \tag{5.124}$$

where ψ_0 denotes an integration constant which we set to zero. Using $dr = \exp(\psi)\,d\tau$, we establish

$$r - r_0 = \frac{\sinh(\,\delta\,\tau\,)}{\delta} > 0\,. \tag{5.125}$$

As $\tau \to 0$ we have $\exp(2A) \to 0$ and $r - r_0 \to 0$.

The dilaton is given by

$$\mathrm{Re}\,z = \mathrm{Re}\left(\frac{Y^1}{Y^0}\right) = \frac{e^{2A}}{4\,(C^0)^2}\,. \tag{5.126}$$

We notice here that the dilaton $e^{\phi} = (\sqrt{\mathrm{Re}\, z})^{-1}$ blows up at $r \to r_0$. The same happens with the curvature invariants like the Ricci scalar. As the curvature invariants vanish asymptotically for large r (we checked this up to second order in the Riemann tensor), one could still hope that our solution describes the asymptotic region of a global solution, once higher curvature corrections are taken into account.

The STU model

Now we consider the STU-model, which is based on $F(Y) = -Y^1Y^2Y^3/Y^0$. We denote the $z^i = Y^i/Y^0$ (with $i = 1, 2, 3$) by $z^1 = S$, $z^2 = T$, $z^3 = U$. For concreteness, we will only consider solutions that are supported by an electric charge Q_0 and an electric flux h_0. In addition, we will restrict ourselves to axion-free solutions, that is solutions with vanishing $\mathrm{Re}\, S$, $\mathrm{Re}\, T$, $\mathrm{Re}\, U$. The solutions will thus be supported by $S_2 = \mathrm{Im}\, S$, $T_2 = \mathrm{Im}\, T$, $U_2 = \mathrm{Im}\, U$, so that

$$
\begin{aligned}
N_{IJ} &= \begin{pmatrix} 4S_2T_2U_2 & 0 & 0 & 0 \\ 0 & 0 & -2U_2 & -2T_2 \\ 0 & -2U_2 & 0 & -2S_2 \\ 0 & -2T_2 & -2S_2 & 0 \end{pmatrix}, \\
N^{IJ} &= \frac{1}{4S_2T_2U_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & S_2^2 & -S_2T_2 & -S_2U_2 \\ 0 & -S_2T_2 & T_2^2 & -T_2U_2 \\ 0 & -S_2U_2 & -T_2U_2 & U_2^2 \end{pmatrix}.
\end{aligned} \tag{5.127}
$$

The flow equations for the Y^i imply that they are constant. To ensure that S, T and U are axion-free, we take the constant Y^i to be purely imaginary, i.e. $Y^i = i\, C^i$, where the C^i denote real constants. Using (5.5) in the form

$$
\mathrm{e}^{2A} = 8\, |Y^0|^2\, S_2\, T_2\, U_2\,, \tag{5.128}
$$

the flow equation (5.47) for Y^0 gives

$$
\dot{Y}^0 = 2\, |Y^0|^2\, (\, Q_0\, \mathrm{e}^{-2A} + i\, g\, h_0\,)\, \mathrm{e}^{-i\gamma}\,. \tag{5.129}
$$

Using the constraint equation (5.39) in the form

$$
\mathrm{Im}\, \Big[(Q_0\, \mathrm{e}^{-2A} + i\, g\, h_0\,)\, \mathrm{e}^{-i\gamma}\, \bar{Y}^0 \Big] = 0 \tag{5.130}
$$

as well as the flow equation for A,

$$
\dot{A} = -\, \mathrm{Re}\, \Big[\big(Q_0\, \mathrm{e}^{-2A} + i\, g\, h_0\big)\, \mathrm{e}^{-i\gamma}\, \bar{Y}^0 \Big], \tag{5.131}
$$

we can rewrite (5.129) as

$$
\dot{Y}^0 = -\, 2\, Y^0\, \dot{A}\,. \tag{5.132}
$$

This immediately gives

$$Y^0 \;=\; C^0\, \mathrm{e}^{-2A}\,, \tag{5.133}$$

where C^0 denotes an integration constant which we take to be real, for simplicity. Then, it follows from (5.128) that

$$8\,\frac{C^1\, C^2\, C^3}{C^0} \;=\; 1\,. \tag{5.134}$$

For concreteness we take $C^I < 0$ ($I = 0, \ldots, 3$) in the following (ensuring the positivity of S_2, T_2 and U_2), as well as $h_0, Q_0 > 0$.

The constraint equation (5.130) yields the following relation between γ and A,

$$\tan\left[\,\gamma\,\right] \;=\; \frac{g\,h_0}{Q_0}\,\mathrm{e}^{2A}\,. \tag{5.135}$$

The flow equation (5.131) can be written as

$$\tfrac{1}{2}\,\frac{\partial}{\partial\tau}\mathrm{e}^{2A} \;=\; -\,C^0\,\left(Q_0\,\mathrm{e}^{-2A}\,\cos\left[\,\gamma\,\right] \,+\, g\,h_0\,\sin\left[\,\gamma\,\right]\right) \tag{5.136}$$

which, using (5.135), leads to

$$\tfrac{1}{4}\,\frac{\partial}{\partial\tau}\mathrm{e}^{4A} = -C^0\,\sqrt{Q_0^2 \,+\, g^2\,h_0^2\,\mathrm{e}^{4A}}\,. \tag{5.137}$$

This can be solved to give

$$\mathrm{e}^{4A} \;=\; \frac{4\,g^4\,h_0^4\,\left(C^0\right)^2\,(\tau\,+\,c\,)^2\,-\,Q_0^2}{g^2\,h_0^2}\;, \tag{5.138}$$

where c denotes a further integration constant. Hence, the prefactor of the planar part of the metric is

$$\mathrm{e}^{2A} \;=\; \frac{\sqrt{4\,g^4\,h_0^4\,\left(C^0\right)^2\,(\tau\,+\,c\,)^2\,-\,Q_0^2}}{g\,h_0}\,. \tag{5.139}$$

This is well behaved provided that $(\tau + c)^2 \geq Q_0^2/(4g^4\,h_0^4\,(C^0)^2)$.

The flow equation for ψ,

$$\dot{\psi} \;=\; -2\,g\,h_0\,C^0\,\mathrm{e}^{-2A}\,\sin\left[\,\gamma\,\right] \tag{5.140}$$

is solved by (using (5.119), (5.135) and (5.138))

$$\mathrm{e}^{\psi} \;=\; \frac{1}{\tau\,+\,c}\;. \tag{5.141}$$

The radial coordinate is then related to the τ variable by

$$r\left[\,\tau\,\right] \;=\; \log\left[\,\tau\,+\,c\,\right], \tag{5.142}$$

where we set an additional integration constant to zero.

The physical scalars are given by

$$S_2 = \frac{C^1}{C^0}\,\mathrm{e}^{2A} \quad , \quad T_2 = \frac{C^2}{C^0}\,\mathrm{e}^{2A} \quad , \quad U_2 = \frac{C^3}{C^0}\,\mathrm{e}^{2A} \; . \tag{5.143}$$

They are positive as long as e^{2A} is. However, as in the previous example, they vanish at the lower end of the radial coordinate (indicating that string loop and α' corrections should become important). Again also the curvature blows up there and one can at best consider this solution as an asymptotic approximation to a full solution which might exist after including higher derivative terms. As we said in the introduction, it would be worthwhile to further pursue the search for everywhere well behaved solutions with non-constant γ as they might be radically different from the ungauged case.

5.3 Nernst brane solutions in the STU model

In the following we construct Nernst brane solutions (i.e. solutions with vanishing entropy density) in a particular model, namely the STU-model already discussed in sec. 5.2.2. As there, we denote the $z^i = Y^i/Y^0$ (with $i = 1, 2, 3$) by $z^1 = S$, $z^2 = T$, $z^3 = U$. For concreteness, we will only consider solutions that are supported by the electric charge Q_0 and the electric fluxes h_1, h_2, h_3. In addition, we will restrict ourselves to axion-free solutions, that is solutions with vanishing $\mathrm{Re}\, S$, $\mathrm{Re}\, T$, $\mathrm{Re}\, U$. The solutions will thus be supported by $S_2 = \mathrm{Im}\, S$, $T_2 = \mathrm{Im}\, T$, $U_2 = \mathrm{Im}\, U$, and the corresponding matrices N_{IJ} and N^{IJ} are given in eq. (5.127). We thus have that e^{2A} is determined by (5.128). In the following, we take Y^0 to be real, so that the Y^i will be purely imaginary.

Let us consider the flow equation (5.47) for the Y^I. We set $\gamma = 0$. Instead of working with a τ coordinate defined by $d\tau = \mathrm{e}^{-\psi}\, dr$, we find it convenient to work with $d\tau = -\mathrm{e}^{-\psi}\, dr$. We obtain

$$\begin{aligned} \dot{Y}^0 &= -\frac{Q_0}{4 S_2 T_2 U_2} \, , \\ \dot{Y}^i &= 2\, i\, g\, Y^i \Big[2\, Y^i\, h_i \, - \, Y^j\, h_j \Big] \, , \end{aligned} \tag{5.144}$$

where $i, j = 1, 2, 3$ and $\dot{Y}^I = \partial Y^I / \partial \tau$. Here, i is not being summed over, while j is. The flow equations for Y^i are solved by

$$Y^i = -\frac{i}{2\; g\, h_i\, \tau} \; . \tag{5.145}$$

Next, using that for an axion-free solution $S_2 = -\,i\, Y^1/Y^0$, $T_2 = -\,i\, Y^2/Y^0$ and $U_2 = -\,i\, Y^3/Y^0$, and inserting (5.145) into the flow equation for Y^0, we obtain

$$\dot{Y}^0 = 2 g^3\, Q_0\, h_1 h_2 h_3\, \left(Y^0 \right)^3 \tau^3 \; . \tag{5.146}$$

This equation can be easily solved to give

$$Y^0 = -\frac{1}{\sqrt{-g^3\, Q_0\, h_1\, h_2\, h_3\, (\,\tau^4 + C^0\,)}}\,, \tag{5.147}$$

where C^0 denotes an integration constant. We also take $Q_0 < 0$ and $h_1, h_2, h_3 > 0$. This ensures that the physical scalars

$$\begin{aligned} S_2 &= \frac{1}{2\,g\,h_1\,\tau}\,\sqrt{-g^3\;Q_0\,h_1\,h_2\,h_3\,(\,\tau^4 + C^0)}\,, \\ T_2 &= \frac{1}{2\,g\,h_2\,\tau}\,\sqrt{-g^3\,Q_0\,h_1\,h_2\,h_3\,(\,\tau^4 + C^0)}\,, \\ U_2 &= \frac{1}{2\,g\,h_3\,\tau}\,\sqrt{-g^3\,Q_0\,h_1\,h_2\,h_3\,(\,\tau^4 + C^0)} \end{aligned} \tag{5.148}$$

take positive values.

Next, we consider the flow equation for ψ following from (5.34). Using (5.145), we obtain

$$\dot{\psi} = 2\,g\;\mathrm{Im}\left[Y^i\,h_i\right] = -\frac{3}{\tau}\,, \tag{5.149}$$

which upon integration yields

$$\mathrm{e}^{\psi-\psi_0} = \frac{1}{\tau^3}\,, \tag{5.150}$$

where ψ_0 denotes an integration constant, which we set to $\psi_0 = 0$. We can thus take τ to range between 0 and ∞. The relation $d\tau = -\mathrm{e}^{-\psi}\,dr$ then results in

$$r\,[\,\tau\,] = \frac{1}{2\,\tau^2} + C_r\,, \tag{5.151}$$

where C_r denotes a real constant, which we take to be zero in the following. This sets the range of r to vary from ∞ to 0 as τ varies from 0 to ∞.

Using (5.128), we obtain

$$\mathrm{e}^{2A} = \frac{\sqrt{-Q_0}}{\sqrt{g^3\,h_1\,h_2\,h_3}}\,\frac{\sqrt{\tau^4 + C^0}}{\tau^3}\,, \tag{5.152}$$

and hence

$$\mathrm{e}^{-2U} = \frac{\sqrt{-Q_0}}{\sqrt{g^3\,h_1\,h_2\,h_3}}\,\sqrt{\tau^4 + C^0}\,\tau^3\,. \tag{5.153}$$

We take C_0 to be non-vanishing and positive to ensure that the curvature scalar remains finite througout and that the above describes a smooth solution of Einstein's equations.

Asymptotically ($\tau \to 0$) we obtain

$$\begin{aligned} \mathrm{e}^{2A} &\overset{r\to\infty}{\longrightarrow} \alpha\left(1 + \frac{1}{8\,C^0\,r^2}\right)(2\,r)^{3/2}\,, \\ \mathrm{e}^{-2U} &\overset{r\to\infty}{\longrightarrow} \alpha\left(1 + \frac{1}{8\,C^0\,r^2}\right)\left(\frac{1}{2\,r}\right)^{3/2}\,, \\ \mathrm{e}^{2U} &\overset{r\to\infty}{\longrightarrow} \alpha^{-1}\left(1 - \frac{1}{8\,C^0\,r^2}\right)(2\,r)^{3/2}\,, \end{aligned} \tag{5.154}$$

where $\alpha = \sqrt{-Q_0\, C^0/(g^3\, h_1\, h_2\, h_3)}$. Observe that both $\exp(2A)$ and $\exp(2U)$ have an asymptotic $r^{3/2}$ fall-off, which is rather unusual. Finally, observe that the scalar fields S_2, T_2 and U_2 grow as $(C^0\, r)^{1/2}$ asymptotically.

The near-horizon solution (which corresponds to $\tau \to \infty$) can be obtained by setting $C^0 = 0$ in the above. This yields

$$\begin{aligned} e^{2A} &\overset{r\to 0}{\longrightarrow} \frac{\sqrt{-Q_0}}{\sqrt{g^3\, h_1\, h_2\, h_3}}\, (2\, r)^{1/2}\,, \\ e^{-2U} &\overset{r\to 0}{\longrightarrow} \frac{\sqrt{-Q_0}}{\sqrt{g^3\, h_1\, h_2\, h_3}}\, \frac{1}{(2\, r)^{5/2}}\,, \end{aligned} \tag{5.155}$$

while S_2, T_2 and U_2 approach the horizon as $r^{-1/2}$. The near-horizon geometry has an infinitely long radial throat as well as a vanishing area density, which indicates that the solution describes an extremal black brane with vanishing entropy density. It describes a supersymmetric Nernst brane solution that is valid in supergravity, and hence in the supergravity approximation to string theory when switching off string theoretic α'-corrections. When embedding this solution into type IIA string theory, where S, T and U correspond to Kähler moduli, the fact that asymptotically and at the horizon S_2, T_2 and U_2 blow up indicates that this solution should be viewed as a good solution only in ten dimensions. In the heterotic description, where S_2 is related to the inverse of the heterotic string coupling g_S, the behaviour $S_2 = \infty$ (which holds asymptotically and at the horizon) is consistent with working in the classical limit $g_S \to 0$.[4] One problem that comes across when studying these Nernst solutions is that they suffer from divergent tidal forces. These may get cured by quantum or stringy effects [133].

We observe that the Nernst brane solution constructed above differs substantially from the one constructed in [122]. This is related to the fact that here we deal with a flux potential in gauged supergravity, and not with a cosmological constant as in [122].

The solution given above is supported by electric charges and fluxes, only. Additional supersymmetric Nernst solutions can be generated through the technique of symplectic transformations.

[4] In fact, by analyzing this solution in the STU model viewed as arising from heterotic compactification on $K3 \times T^2$, one can explicitly verify that (for appropriately chosen flux parameters) the Nernst solution is a smooth and consistent solution of the 10D supergravity action arising as the low-energy limit of the heterotic string. In particular, the 10D string coupling is given by $g^2_{10\,s} = V_{K3}\, \frac{h_1}{h_3}$ and, thus, can be made small for appropriate choices of h_1 and h_3, even for values of the $K3$ volume V_{K3} which are large enough to allow the neglect of α'-corrections.

Chapter 6

Extremal black brane solutions in gauged supergravity in $D = 5$

In this chapter we study stationary black brane solutions in the context of $N = 2$, $U(1)$ gauged supergravity in five dimensions. Using the formalism of first-order flow equations, we construct examples of extremal black brane solutions including Nernst branes, i.e. extremal black brane solutions with vanishing entropy density, as well as black branes with cylindrical horizon topology. The work presented here is based on [2].

Since we could not determine Nernst solutions in four dimensions with AdS_4 asymptotics we turn our attention to the case of five dimensions trying to find a Nernst solution with the desired asymptotic behavior.

Just as in four dimensions the attractor mechanism for extremal black objects guarantees the existence of first-order flow equations. Various types of extremal five-dimensional black solutions with flat horizons have already been discussed e.g. in [8, 134, 135, 136, 137, 117, 138, 139, 140, 14, 10, 141].

In the following we proceed by deriving the first-order flow equations. We find that the solution space of these equations allows for Nernst solutions with constant scalars and AdS_5 asymptotics. Moreover, we find solutions with constant scalars that may have magnetic fields or rotation, but no electric charges. Some of them are extremal $BTZ \times \mathbb{R}^2$ solutions. For these solutions we can compute the entropy density using the Cardy formula of the dual CFT. The angular momentum, electric quantum numbers and magnetic fields are combined in such a way that the resulting quantitiy is invariant under the spectral flow of the theory, exactly as in the ungauged case [11]. We also construct solutions with running scalar fields numerically. These interpolate between a $BTZ \times \mathbb{R}^2$ near horizon geometry and AdS_5. We will also display the general relation between the four-dimensional first-order equations derived in the last chapter and the five dimensional ones.

Because of the difficulties in finding solutions with electric charge we perform a different first-order rewriting in appendix E. Here we derive a different set of first-order equations based on [12]. In this setup we are able to construct solutions with electric charge. We can also reproduce solutions found in [13, 14, 10].

But let us start with the derivation of the first of the two sets of first-order flow equations.

6.1 First-order equations for stationary solutions in five dimensions

In the following, we derive first-order flow equations for extremal stationary black brane solutions in $N = 2, U(1)$ gauged supergravity in five dimensions with n Abelian vector multiplets. We work in big moduli space. We follow the exposition given in [142] for the ungauged case and adapt it to the gauged case.

6.1.1 Flow equations in big moduli space

Following [117], we make the ansatz for the black brane line element,

$$ds^2 = -e^{2U(r)}dt^2 + e^{2V(r)}dr^2 + e^{2B(r)}(dx^2 + dy^2) + e^{2W(r)}(dz + C(r)dt)^2 \,, \tag{6.1}$$

while for the Abelian gauge fields A^A_M $(A = 1, \ldots, n)$ we take

$$A^A_M dx^M = A^A_t \, dt + P^A \, x \, dy + A^A_z \, dz = (e^A + A^A_z \, C(r)) \, dt + P^A \, x \, dy + A^A_z \, dz \,. \tag{6.2}$$

Here the P^A are constants and A^A_t, A^A_z depend only on r. The associated field strength components read

$$\begin{aligned} F^A_{rt} &= (A^A_t)' = (e^A)' + A^A_z \, C' + (A^A_z)' \, C \,, \\ F^A_{xy} &= P^A \,, \\ F^A_{rz} &= (A^A_z)' \,, \end{aligned} \tag{6.3}$$

where $'$ denotes differentiation with respect to r, and $(e^A)'$ corresponds to the four-dimensional electric field upon dimensional reduction. The solutions we seek will be supported by real scalar fields $X^A(r)$ and by electric fluxes h_A. The ansatz (6.1) and (6.2) is the most general ansatz with translational invariance in the coordinates t, x, y and z and with rotational invariance in the x, y-plane, cf. [141].

The bosonic part of the five-dimensional action describing $N = 2, U(1)$ gauged supergravity is given by [143, 144]

$$\begin{aligned} S \;=\; \int dx^5 \Big[\sqrt{-g} \Big(& R - G_{AB} \, \partial_M X^A \partial^M X^B - \frac{1}{2} G_{AB} F^A_{MN} F^{B\,MN} \\ & -g^2 (G^{AB} h_A h_B - 2(h_A X^A)^2) \Big) \\ & - \frac{1}{24} C_{ABC} F^A_{KL} F^B_{MN} A^C_P \, \epsilon^{KLMNP} \Big] \,, \end{aligned} \tag{6.4}$$

where the scalar fields X^A satisfy the constraint $\frac{1}{6}C_{ABC}X^AX^BX^C = 1$. The target space metric G_{AB} is given by

$$G_{AB} = -\frac{1}{2}C_{ABC}X^C + \frac{9}{2}X_AX_B \ , \tag{6.5}$$

where

$$X_A = \frac{2}{3}G_{AB}\,X^B = \frac{1}{6}C_{ABC}X^BX^C \ . \tag{6.6}$$

Inserting the solution ansatz into this action, we find that the Ricci scalar contributes

$$\begin{aligned}\sqrt{-g}\,R \;\; &= \;\; e^{2B+W+U-V}\left(2B'^2 + 2U'W' + 4B'W' + 4B'U' + \frac{1}{2}e^{2W-2U}C'^2\right) \\ &\quad - \left[2e^{2B+W+U-V}(2B' + W' + U')\right]' \ ,\end{aligned} \tag{6.7}$$

while the gauge field kinetic terms contribute

$$\begin{aligned}\sqrt{-g}\left(-\frac{1}{2}G_{AB}F^A_{MN}F^{B\,MN}\right) &= e^{2B+W+U-V}\Big(- G_{AB}P^AP^Be^{2V-4B} + G_{AB}F^A_{rt}F^B_{rt}e^{-2U} \\ &\quad -G_{AB}(A^A_z)'(A^B_z)'(e^{-2W} - e^{-2U}C^2) - 2G_{AB}F^A_{rt}(A^B_z)'e^{-2U}C\Big) \ ,\end{aligned} \tag{6.8}$$

with F^A_{rt} given in (6.3). The Chern-Simons term, on the other hand, can be rewritten as

$$\int dx^5 \left(-\frac{1}{24}C_{ABC}F^A_{KL}F^B_{MN}A^C_P\,\epsilon^{KLMNP}\right) = \int dx^5 \left(-C_{ABC}F^A_{rt}F^B_{xy}A^C_z + TD\right) \ , \tag{6.9}$$

where TD denotes a total derivative term. Inserting these expressions into (6.4) yields the one-dimensional Lagrangian $\mathcal{L}$,

$$\begin{aligned}\mathcal{L} \;\; = \;\; & e^{2B+W+U-V}\Big(2B'^2 + 2U'W' + 4B'W' + 4B'U' + \frac{1}{2}e^{2W-2U}C'^2 - G_{AB}(X^A)'(X^B)' \\ & - G_{AB}P^AP^Be^{2V-4B} + G_{AB}(e^A)'(e^B)'e^{-2U} + G_{AB}A^A_zA^B_zC'^2e^{-2U} \\ & + 2G_{AB}(e^A)'A^B_zC'e^{-2U} - G_{AB}(A^A_z)'(A^B_z)'e^{-2W} \\ & - g^2e^{2V}(G^{AB}h_Ah_B - 2(h_AX^A)^2)\Big) \\ & - C_{ABC}(e^A)'P^BA^C_z - \; C_{ABC}A^A_zP^BA^C_zC' - \; C_{ABC}(A^A_z)'P^BA^C_zC \ ,\end{aligned} \tag{6.10}$$

where we dropped total derivative terms.

Now we express the electric field $(e^A)'$ in terms of electric charges Q_A by performing the Legendre transformation $\mathcal{L}_L = \mathcal{L} - Q_A(e^A)'$, and obtain

$$(e^A)' = \frac{1}{2}e^{-2B-W+U+V}G^{AB}\,\hat{q}_B - A^A_zC' \ , \tag{6.11}$$

where

$$\hat{q}_A = Q_A + C_{ABC}P^B A_z^C \,. \tag{6.12}$$

Substituting this relation in (6.10) gives

$$\begin{aligned}\mathcal{L} \;=\; & e^{2B+W+U-V}\Big(2B'^2 + 2U'W' + 4B'W' + 4B'U' + \frac{1}{2}e^{2W-2U}C'^2 - G_{AB}(X^A)'(X^B)' \\ & - G_{AB}P^AP^Be^{2V-4B} - \frac{1}{4}G^{AB}\hat{q}_A\hat{q}_B\, e^{-4B-2W+2V} \\ & - G_{AB}(A_z^A)'(A_z^B)'e^{-2W} - g^2e^{2V}(G^{AB}h_Ah_B - 2(h_AX^A)^2)\Big) \\ & - C_{ABC}(A_z^A)'P^BA_z^CC + Q_AA_z^AC' \,. \end{aligned} \tag{6.13}$$

Furthermore, using

$$(C_{ABC}A_z^AP^BA_z^CC)' = 2C_{ABC}(A_z^A)'P^BA_z^CC + C_{ABC}A_z^AP^BA_z^CC' \,, \tag{6.14}$$

we obtain

$$- C_{ABC}(A_z^A)'P^BA_z^CC = \frac{1}{2}C_{ABC}A_z^AP^BA_z^CC' + TD \,, \tag{6.15}$$

where TD denotes again a total derivative, which we drop in the following.

Next, we express C' in terms of a constant quantity J which, in the compact case, corresponds to angular momentum. We do this by performing the Legendre transformation $\mathcal{L}_L = \mathcal{L} - JC'$, and obtain

$$C' = e^{-2B-3W+U+V}\,\hat{J} \,, \tag{6.16}$$

where

$$\hat{J} = J - Q_AA_z^A - \frac{1}{2}C_{ABC}A_z^AA_z^BP^C \,. \tag{6.17}$$

This results in

$$\begin{aligned}\mathcal{L} \;=\; & e^{2B+W+U-V}\Big(2B'^2 + 2U'W' + 4B'W' + 4B'U' - G_{AB}(X^A)'(X^B)' \\ & -G_{AB}(A_z^A)'(A_z^B)'e^{-2W} - G_{AB}P^AP^Be^{2V-4B} - \frac{1}{4}G^{AB}\hat{q}_A\hat{q}_B\, e^{-4B-2W+2V} \\ & -\frac{1}{2}e^{-4B-4W+2V}\hat{J}^2 - g^2e^{2V}(G^{AB}h_Ah_B - 2(h_AX^A)^2)\Big) \,. \end{aligned} \tag{6.18}$$

Now we rewrite the one-dimensional Lagrangian (6.18) as a sum of squares of first-order flow equations. To this end, we use the relation

$$\Big(e^{U-W}(\hat{q}_AA_z^A - \frac{1}{2}C_{ABC}A_z^AP^BA_z^C)\Big)' = e^{U-W}\Big((U'-W')(-\hat{J}) + \hat{q}_A(A_z^A)'\Big) + \big(Je^{U-W}\big)' \,, \tag{6.19}$$

and obtain

$$\begin{aligned}
\mathcal{L} \;=\; & e^{2B+W+U-V}\Bigg[-e^{-2W}G_{AB}\Big((A_z^A)'+\frac{1}{2}G^{AC}\hat{q}_Ce^{-2B+V}\Big)\Big((A_z^B)'+\frac{1}{2}G^{BD}\hat{q}_De^{-2B+V}\Big) \\
& \qquad -\frac{1}{2}\left(\hat{J}e^{-2B-2W+V}-(U'-W')\right)^2 \\
& \qquad -\left(B'-\frac{1}{2}(U'+W')+\tfrac{3}{2}X_AP^A\,e^{V-2B}\right)^2 \\
& \qquad +\frac{1}{3}\left(3\left(B'+\tfrac{1}{2}(U'+W')\right)-2gX^Ah_Ae^V+\frac{3}{2}X_A\,P^A\,e^{V-2B}\right)^2 \\
& -G_{AB}\left(X'^A-e^V\left[\frac{2}{3}X^C(gh_C+G_{CD}P^De^{-2B})X^A-G^{AC}(gh_C+G_{CD}P^De^{-2B})\right]\right) \\
& \qquad \left(X'^B-e^V\left[\frac{2}{3}X^E(gh_E+G_{EF}P^Fe^{-2B})X^B-G^{BE}(gh_E+G_{EF}P^Fe^{-2B})\right]\right)\Bigg] \\
& +2\left(e^{2B+W+U}\left(gX^Ah_A-\frac{3}{2}X_AP^A\,e^{-2B}\right)\right)' \\
& +\left(e^{U-W}(\hat{q}_AA_z^A-\frac{1}{2}C_{ABC}A_z^AP^BA_z^C)\right)'-\left(Je^{U-W}\right)' \\
& +2ge^{W+U+V}\,h_AP^A\,.
\end{aligned} \tag{6.20}$$

This concludes the rewriting of the effective one-dimensional Lagrangian.

Setting the squares in (6.20) to zero yields the first-order flow equations

$$\begin{aligned}
(A_z^A)' &= -\frac{1}{2}G^{AC}\hat{q}_C\,e^{V-2B}\,, \\
(X^A)' &= \frac{2}{3}X^C(gh_Ce^V+G_{CD}P^De^{V-2B})X^A-G^{AC}(gh_Ce^V+G_{CD}P^De^{V-2B})\,, \\
U'-W' &= \hat{J}\,e^{-2B-2W+V}\,, \\
0 &= B'-\frac{1}{2}(U'+W')+\frac{3}{2}X_AP^A\,e^{V-2B}\,, \\
0 &= 3\left(B'+\frac{1}{2}(U'+W')\right)-2gX^Ah_Ae^V+\frac{3}{2}X_A\,P^A\,e^{V-2B}\,.
\end{aligned} \tag{6.21}$$

These flow equations are supplemented by (6.11) and (6.16), and solutions to these equations are subjected to the constraint

$$h_AP^A=0\,, \tag{6.22}$$

which follows from the last line of (6.20). Note that the flow equations (6.21) show an interesting decoupling: The scalar fields X^A and the metric coefficient e^B are completely determined by the magnetic fields and the fluxes, whereas the electric charges only enter in the equations for the metric functions e^U and e^W and the A_z^A-components of the gauge fields. This will be helpful in the search for solutions, cf. sec. 6.3.

Subtracting the fourth from the fifth equation in (6.21) gives

$$B' + U' + W' = g\, h_A X^A\, e^V . \tag{6.23}$$

When B is constant, this yields a flow equation for $U + W$ that, when compared with the fourth equation of (6.21), yields the condition

$$g\, X^A h_A = 3 X_A P^A\, e^{-2B} . \tag{6.24}$$

Also observe that (6.3), (6.11) and the first equation of (6.21) implies

$$F^A_{rt} = \frac{1}{2}(1 - \hat{C})\, e^{-2B-W+U+V} G^{AB} \hat{q}_B , \tag{6.25}$$

where

$$\hat{C} \equiv C\, e^{-(U-W)} , \tag{6.26}$$

while the third equation, together with (6.16), gives

$$C' = (U' - W')\; e^{U-W} = \left(e^{U-W}\right)' , \tag{6.27}$$

and hence

$$\hat{C} = 1 + \lambda\, e^{-(U-W)} , \tag{6.28}$$

with λ a real integration constant.[1] Inserting this into (6.25) gives

$$F^A_{rt} = -\frac{\lambda}{2}\, e^{-2B+V} G^{AB} \hat{q}_B . \tag{6.29}$$

However, the electric field is actually given by

$$(F^A)^{tr} = F^A_{tr} g^{tt} g^{rr} + F^A_{zr} g^{tz} g^{rr} = \frac{1}{2} e^{-2B-W-U-V} G^{AB}\, \hat{q}_B , \tag{6.30}$$

where we used the form of the inverse metric, the third equation of (6.3), together with the first equation of (6.21), and (6.29). Comparing this with (6.11), we see that this can also be expressed as

$$(F^A)^{tr} = e^{-2U-2V}((e^A)' + A^A_z C') . \tag{6.31}$$

Obviously, the electric field is independent of the integration constant λ and is non-vanishing whenever some of the charges $\hat{q}_A$ are non-vanishing.

In contrast, the five-dimensional magnetic field component $(F^A)^{rz}$ does depend on λ according to

$$(F^A)^{rz} = F^A_{rz} g^{rr} g^{zz} + F^A_{rt} g^{rr} g^{tz} = \frac{1}{2}\lambda e^{-2B-W-U-V} G^{AB}\, \hat{q}_B . \tag{6.32}$$

[1] Given the relation $C = e^{U-W} + \lambda$ between three of the metric functions, the solution set of the first-order equations (6.21) is naturally more restricted than the one obtained by looking at the second-order equations of motion. In particular, the charged magnetic brane solution of [138, 141] is not a solution of (6.21).

Notice, however, that both the electric field and the rz-component of the magnetic field are determined by the charges $\hat{q}_A$. As a consequence, the combination $G_{AB}F^A_{MN}F^{BMN}$ vanishes (independently of λ) for vanishing P^A on any solution of (6.21), i.e.

$$G_{AB}F^A_{rt}F^{Brt} = -G_{AB}F^A_{rz}F^{Brz} \ . \tag{6.33}$$

On the other hand, inserting (6.28) into the line element (6.1) results in

$$ds^2 = e^{2W}\,\lambda\left(\lambda + 2\,e^{U-W}\right)dt^2 + 2e^{2W}\,C\,dt\,dz + e^{2W}dz^2 + e^{2V}\,dr^2 + e^{2B}(dx^2+dy^2)\,. \tag{6.34}$$

Thus, we see that the sign and the magnitude of the integration constant λ determine the nature of the warped line element. In particular, a vanishing λ will give a null-warped metric, i.e. $g_{tt} = 0$.

Let us now briefly display the flow equations for static, purely magnetic solutions. They are obtained by setting $Q_A = A^A_z = J = 0$, which results in $\hat{q}_A = \hat{J} = C' = 0$, so that the non-vanishing flow equations are

$$\begin{aligned}
(X^A)' &= \frac{2}{3}X^C(gh_Ce^V + G_{CD}P^De^{V-2B})X^A - G^{AC}(gh_Ce^V + G_{CD}P^De^{V-2B})\,,\\
U' - W' &= 0\,,\\
0 &= B' - \frac{1}{2}(U'+W') + \frac{3}{2}X_AP^A\,e^{V-2B}\,,\\
0 &= 3\left(B' + \frac{1}{2}(U'+W')\right) - 2gX^Ah_Ae^V + \frac{3}{2}X_A\,P^A\,e^{V-2B}\,.
\end{aligned} \tag{6.35}$$

These flow equations need again to be supplemented by the constraint $h_AP^A = 0$. Magnetic supersymmetric $AdS_3\times\mathbb{R}^2$ solutions to these equations were studied in [136, 139, 140, 10].

Finally, we would like to show that the flow equations (6.21) follow from a superpotential. To do so, it is convenient to introduce the scalars

$$\phi_1 = B - \frac{1}{2}(U+W) \quad , \quad \phi_1 = B + \frac{1}{2}(U+W) \quad , \quad \phi_3 = U - W\ . \tag{6.36}$$

Using them and introducing the physical scalars φ^i, the one-dimensional Lagrangian (6.18) takes the form

$$\begin{aligned}
\mathcal{L} \;=\; & -e^{2\phi_2-V}(\phi_1')^2 + 3e^{2\phi_2-V}(\phi_2')^2 - \frac{1}{2}e^{2\phi_2-V}(\phi_3')^2\\
& -e^{2\phi_2-V}G_{ij}(\varphi^i)'(\varphi^j)' - e^{\phi_1+\phi_2+\phi_3-V}G_{AB}(A^A_z)'(A^B_z)'\\
& -e^{-2\phi_1+V}G_{AB}P^AP^B - \frac{1}{4}e^{-\phi_1-\phi_2+\phi_3+V}G^{AB}\hat{q}_A\hat{q}_B\\
& -\frac{1}{2}e^{-2\phi_2+2\phi_3+V}\hat{J}^2 - g^2e^{2\phi_2+V}(G^{AB}h_Ah_B - 2(h_AX^A)^2)\,,
\end{aligned} \tag{6.37}$$

where we used (cf. section 2.4 with g_{ij} replaced by G_{ij})

$$G_{ij} = G_{AB}\partial_iX^A\partial_jX^B\ . \tag{6.38}$$

The advantage of working with the scalars (6.36) is that the sigma-model metric is then block diagonal with

$$g_{\phi_1\phi_1} = e^{2\phi_2 - V} \quad , \quad g_{\phi_2\phi_2} = -3e^{2\phi_2 - V} \quad , \quad g_{\phi_3\phi_3} = \frac{1}{2}e^{2\phi_2 - V} \ ,$$
$$g_{ij} = e^{2\phi_2 - V} G_{ij} \quad , \quad g_{AB} = e^{\phi_1 + \phi_2 + \phi_3 - V} G_{AB} \ . \tag{6.39}$$

It is now a straightforward exercise to show that the potential $\mathcal{V}$ of the one-dimensional Lagrangian (i.e. the last two lines of (6.37)) can be expressed as

$$\mathcal{V} = -g^{\phi_1\phi_1}\left(\frac{\partial \mathcal{Z}}{\partial \phi_1}\right)^2 - g^{\phi_2\phi_2}\left(\frac{\partial \mathcal{Z}}{\partial \phi_2}\right)^2 - g^{\phi_3\phi_3}\left(\frac{\partial \mathcal{Z}}{\partial \phi_3}\right)^2 - g^{ij}\frac{\partial \mathcal{Z}}{\partial \varphi^i}\frac{\partial \mathcal{Z}}{\partial \varphi^j} - g^{AB}\frac{\partial \mathcal{Z}}{\partial A_z^A}\frac{\partial \mathcal{Z}}{\partial A_z^B} \ , \tag{6.40}$$

with the superpotential

$$\mathcal{Z} = \frac{1}{2}e^{\phi_3}\hat{J} + \frac{3}{2}e^{\phi_2 - \phi_1}P^A X_A - e^{2\phi_2} g h_A X^A \ . \tag{6.41}$$

In doing so, one has to make use of the constraint $h_A P^A = 0$, of (6.38) and the identities of section 2.4

$$G^{ij}\partial_i X^A \partial_j X^B = G^{AB} - \frac{2}{3}X^A X^B \quad , \quad \partial_i X_A = -\frac{2}{3}G_{AB}\partial_i X^B \quad , \quad X_A = \frac{2}{3}G_{AB}X^B \ . \tag{6.42}$$

Using the superpotential (6.41), it is straightforward to check that the first-order flow equations (in the physical moduli space) can be expressed as[2]

$$\phi_1' = g^{\phi_1\phi_1}\frac{\partial \mathcal{Z}}{\partial \phi_1} \ , \ \phi_2' = g^{\phi_2\phi_2}\frac{\partial \mathcal{Z}}{\partial \phi_2} \ , \ \phi_3' = g^{\phi_3\phi_3}\frac{\partial \mathcal{Z}}{\partial \phi_3} \ , \ (A_z^A)' = g^{AB}\frac{\partial \mathcal{Z}}{\partial A_z^B} \ , \ (\varphi^i)' = g^{ij}\frac{\partial \mathcal{Z}}{\partial \varphi^j} \ , \tag{6.43}$$

In order to derive the flow equation for φ^i, one has to multiply the flow equation for X^A by $G_{AB}\partial_j X^B$ and use (6.38), (6.42) and

$$X_A \partial_i X^A = 0 \ . \tag{6.44}$$

6.1.2 Hamiltonian constraint

Next, we discuss the Hamiltonian constraint and show that it equals the constraint $h_A P^A = 0$ that we encountered in the rewriting of the Lagrangian in terms of first-order flow equations.

The Einstein equations take the form

$$\begin{aligned} R_{MN} \ &= \ \frac{1}{3}g_{MN}\, g^2\left(G^{AB}h_A h_B - 2(h_A X^A)^2\right) - \frac{1}{6}g_{MN}G_{AB}F^A_{KL}F^{B\,KL} \\ &\quad + G_{AB}(X^A)'(X^B)'\delta_{Mr}\delta_{Nr} + G_{AB}F^A_{MK}F^B_{NL}g^{KL} \ . \end{aligned} \tag{6.45}$$

[2]Note that (6.40) would hold also for any combination of signs in $\mathcal{Z} = \frac{1}{2}e^{\phi_3}\hat{J} \pm \frac{3}{2}e^{\phi_2-\phi_1}P^A X_A \pm e^{2\phi_2}h_A X^A$, but (6.43) requires the signs given in (6.41).

There are only five independent equations, namely the ones corresponding to the tt-, rr-, xx-,zz- and tz-component of the Ricci tensor, which we have displayed in appendix C. To obtain the Hamiltonian constraint, we consider the tt-component of Einstein's equations. We use the rr-, xx-,zz- and tz-equations to obtain expressions for the second derivatives U'', C'', B'' and W'', which we then insert into the expression for the tt-component. This yields the following equation, which now only contains first derivatives,

$$\begin{aligned} 0 &= 2B'^2 + 2U'W' + 4B'W' + 4B'U' + \frac{1}{2}e^{2W-2U}C'^2 - G_{AB}(X^A)'(X^B)' + G_{AB}F^A_{rt}F^B_{rt}e^{-2U} \\ &\quad + G_{AB}P^AP^Be^{2V-4B} - G_{AB}(A^A_z)'(A^B_z)'(e^{-2W} - e^{-2U}C^2) - 2G_{AB}F^A_{rt}(A^B_z)'e^{-2U}C \\ &\quad + g^2e^{2V}\left(G^{AB}h_Ah_B - 2(h_AX^A)^2\right) . \end{aligned} \tag{6.46}$$

This equation turns out to be equivalent to the rr-component of Einstein's equations,

$$R_{rr} - \frac{1}{2}g_{rr}R - \frac{1}{2}g_{rr}\mathcal{L}_M + \frac{\delta\mathcal{L}_M}{\delta g^{rr}} = 0 , \tag{6.47}$$

where $\mathcal{L}_M$ denotes the matter Lagrangian.

Next, using (6.11), (6.16), (6.27) and the flow equation for $(A^A_z)'$ in (6.46), we obtain the intermediate result

$$\begin{aligned} 0 &= 2B'^2 + 2U'W' + 4B'W' + 4B'U' + \frac{1}{2}(U'-W')^2 - G_{AB}(X^A)'(X^B)' \\ &\quad + G_{AB}P^AP^Be^{2V-4B} + g^2e^{2V}\left(G^{AB}h_Ah_B - 2(h_AX^A)^2\right) . \end{aligned} \tag{6.48}$$

Then, using the first-order flow equation (6.21) for $(X^A)'$, we get

$$\begin{aligned} 0 &= 2B'^2 + 2U'W' + 4B'W' + 4B'U' + \frac{1}{2}(U'-W')^2 - \frac{4}{3}g^2e^{2V}(h_AX^A)^2 \\ &\quad + \frac{2}{3}(G_{AB}X^AP^B)^2e^{2V-4B} + \frac{4}{3}ge^{2V-2B}(h_AX^A)(G_{AB}X^AP^B) - 2ge^{2V-2B}h_AP^A . \end{aligned} \tag{6.49}$$

In the next step we use (6.23) as well as the fourth flow equation of (6.21) to obtain

$$\begin{aligned} 0 &= 2B'^2 + 2U'W' + 4B'W' + 4B'U' + \frac{1}{2}(U'-W')^2 - \frac{4}{3}(B'+U'+W')^2 \\ &\quad + \frac{2}{3}\left(-B' + \frac{1}{2}(U'+W')\right)^2 + \frac{4}{3}(B'+U'+W')\left(-B' + \frac{1}{2}(U'+W')\right) \\ &\quad - 2ge^{2V-2B}h_AP^A . \end{aligned} \tag{6.50}$$

Then, one checks that all the terms containing B',U' and W' cancel out, so that the on-shell Hamiltonian constraint (6.50) reduces to (6.22).

6.2 Reducing to four dimensions

The five-dimensional stationary solutions to the flow equations (6.21) may be related to a subset of the four-dimensional static solutions discussed in [119] and chapter 5 by performing a reduction on the z-direction. We briefly describe this below. A detailed check of the matching of the five- and four-dimensional flow equations is performed in appendix D.

The five-dimensional solutions are supported by electric fluxes $h_A^{5\rm d}$, electric charges $Q_A^{5\rm d}$, magnetic fields $P_{5\rm d}^A$, and rotation J.[3] The relevant subset of four-dimensional solutions is supported by electric fluxes $h_A^{4\rm d}$, electric charges $Q_I = (Q_0, Q_A)$ and magnetic fields $P_{4\rm d}^A$.

The five-dimensional $N = 2$, $U(1)$ gauged supergravity action (6.4) is based on real scalar fields $X_{5\rm d}$ which satisfy the constraint $\frac{1}{6}C_{ABC}X_{5\rm d}^A X_{5\rm d}^B X_{5\rm D}^C = 1$ for some constants C_{ABC}, while the four-dimensional $N = 2$, $U(1)$ gauged supergravity action considered in [119] and chapter 5 is based on complex scalar fields $X_{4\rm D}^I$ with a cubic prepotential function

$$F(X_{4\rm d}) = -\frac{1}{6}\frac{C_{ABC}X_{4\rm d}^A X_{4\rm d}^B X_{4\rm d}^C}{X_{4\rm d}^0} \; . \tag{6.51}$$

The four-dimensional physical scalar fields are $z^A = X_{4\rm d}^A / X_{4\rm d}^0$, which we decompose as $z^A = C^A + i\hat{X}^A$.

Now we relate the real four-dimensional fields $(C^A, \hat{X}^A)$ to the fields appearing in the five-dimensional flow equations. To do so, we find it convenient to use a different normalization for the scalar constraint equation, namely

$$\frac{1}{6}C_{ABC}X_{5\rm d}^A X_{5\rm d}^B X_{5\rm D}^C = v \; . \tag{6.52}$$

We will show in the appendix that the matching between the four-dimensional and the five-dimensional flow equations requires to choose $v = \frac{1}{2}$, a value which was already obtained in [64] when matching the gauge kinetic terms in four and five dimensions. Choosing the normalization (6.52) amounts to replacing C_{ABC} by C_{ABC}/v, a change that affects the normalization of the Chern-Simons term in the five-dimensional action (6.4), as well as the quantities $\hat{q}_A$ and $\hat{J}$ given in (6.12) and (6.17), respectively. On the other hand, if we stick to the definition $X_A^{5\rm d}$, we get for $X_A^{5\rm d}$ and G_{AB} the expressions (2.81) and (2.79) which we repeat here for convenience:

$$X_A^{5\rm d} = \frac{2v}{3}G_{AB}X_{5\rm d}^B \; , \tag{6.53}$$

with G_{AB} given by

$$G_{AB}(X_{5\rm d}) = \frac{1}{v}\left(-\frac{1}{2}C_{ABC}X_{5\rm d}^C + \frac{9}{2v}X_A^{5\rm d}X_B^{5\rm d}\right) \; . \tag{6.54}$$

[3] J generates translations in the z-direction. When the coordinate z is compact, J has the interpretation of angular momentum.

Using this normalization, we obtain the following dictionary between the four-dimensional quantities that appeared in chapter 5 and the five-dimensional quantities that enter in (6.21),

$$\begin{aligned}
\hat{X}^A &= e^W X^A_{5\mathrm{d}} \,, \\
C^A &= A^A_z \,, \\
h^{4\mathrm{d}}_A &= -h^{5\mathrm{d}}_A \,, \\
P^A_{4\mathrm{d}} &= -P^A_{5\mathrm{d}} \,, \\
Q^{4\mathrm{d}}_A &= -\frac{1}{2} Q^{5\mathrm{d}}_A \,, \\
Q^{4\mathrm{d}}_0 &= \frac{1}{2} J \,.
\end{aligned} \tag{6.55}$$

The five- and four-dimensional line elements are related by

$$ds_5^2 = e^{2\phi} ds_4^2 + e^{-4\phi}(dz + C dt)^2 \,, \tag{6.56}$$

where

$$ds_4^2 = -e^{2U_4} dt^2 + e^{-2U_4} dr^2 + e^{-2U_4+2\psi}(dx^2 + dy^2) \tag{6.57}$$

and

$$2\phi = -W \,. \tag{6.58}$$

This yields

$$\begin{aligned}
U &= U_4 + \phi \,, \\
V &= -U_4 + \phi \,, \\
B &= \psi - U_4 + \phi \,.
\end{aligned} \tag{6.59}$$

6.3 Solutions

In the following, we construct solutions to the flow equations (6.21). First we consider exact solutions with constant scalars X^A. Subsequently we numerically construct solutions with running scalars X^A.

6.3.1 Solutions with constant scalar fields X^A

We pick $V = 0$ in the following. We will consider two distinct cases. In the first case, all the magnetic fields P^A are taken to be non-vanishing. In the second case, we set all the P^A to zero. Other cases where only some of the P^A are turned on are also possible, and their analysis should go along similar lines.

Taking $P^A \neq 0$, $\hat{q}_A = 0$, $\hat{J} \neq 0$

Here we consider the case when all the P^A are turned on. Demanding X^A = constant yields

$$X^C(gh_C + G_{CD}P^D e^{-2B})X_A = gh_A + G_{AB}P^B e^{-2B} \; . \tag{6.60}$$

Observe that G_{AB} is constant, and so is B. We set $B = 0$ in the following, which can always be achieved by rescaling x and y. Combining (6.60) with (6.24), we express the magnetic fields P^A in terms of h_A and X^A as

$$P^A = -g\, G^{AB} \left(h_B - \frac{3}{2} \left(h_C\, X^C \right) X_B \right) \; . \tag{6.61}$$

This relation generically fixes the scalars $X^A = X^A(h_B, P^B)$ in terms of the fluxes and magnetic fields, as we will see in the explicit examples of sec. 6.3.2. Contracting (6.61) with h_A and using the constraint (6.22) we obtain

$$G^{AB}\, h_A h_B = \left(h_A\, X^A \right)^2 \tag{6.62}$$

as well as

$$G_{AB} P^A P^B = \frac{1}{2} g^2 \left(h_A\, X^A \right)^2 \; . \tag{6.63}$$

Observe that (6.62) together with $h_A\, X^A = 0$ would imply $h_A = 0$. Thus, in the following, we take $h_A\, X^A \neq 0$.

We obtain from (6.23),

$$U + W = g h_A X^A \left(r - r_0 \right) \; , \tag{6.64}$$

where r_0 denotes an integration constant. Inserting this into the third equation of (6.21) gives

$$\left(e^{-(U-W)} \right)' = -\hat{J}\, e^{g h_A X^A (r_0 - r)} \; . \tag{6.65}$$

Next we set $\hat{q}_A = 0$, so that the A^A_z take constant values. These are determined by

$$Q_A + C_{ABC}\, P^B\, A^C_z = 0 \; . \tag{6.66}$$

Defining $C_{AB} = C_{ABC} P^C$, this is solved by

$$A^A_z = -C^{AB}\, Q_B \; , \tag{6.67}$$

where $C^{AB} C_{BC} = \delta^A{}_C$. Here, we assumed that C_{AB} is invertible, which generically is the case when all the P^A are turned on.

For constant A^A_z, $\hat{J}$ is also constant, and we can solve (6.65). Taking $h_A X^A \neq 0$, we get

$$e^{-(U-W)} = \frac{\hat{J}\, e^{g h_A X^A (r_0 - r)}}{g h_A X^A} + b \; , \tag{6.68}$$

where b denotes an integration constant. Combining this result with (6.64) gives

$$e^{2W} = \frac{\hat{J}}{gh_A X^A} + b\, e^{gh_A X^A\,(r-r_0)} \tag{6.69}$$

as well as

$$e^{-2U} = \frac{\hat{J}\, e^{2gh_A X^A (r_0-r)}}{gh_A X^A} + b\, e^{gh_A X^A\,(r_0-r)} \ . \tag{6.70}$$

To bring these expressions into a more palatable form, we introduce a new radial variable

$$\tau = \alpha\, e^{gh_A X^A\,(r-r_0)} \tag{6.71}$$

with $\tau \geq 0$ and

$$\alpha = gh_A X^A > 0 \ . \tag{6.72}$$

(If $gh_A X^A < 0$ we have $\tau \leq 0$.) Then (assuming $\hat{J} \geq 0$)

$$\begin{aligned} e^{2W} &= \alpha^{-1}\left[\hat{J} + b\,\tau\right] \ , \\ e^{-2U} &= \alpha\left[\frac{\hat{J}}{\tau^2} + \frac{b}{\tau}\right] \ , \\ C &= \frac{\tau}{\hat{J} + b\,\tau} + \lambda \ , \end{aligned} \tag{6.73}$$

and the associated line element reads

$$\begin{aligned} ds^2 &= -\alpha^{-1}\frac{\tau^2}{\hat{J} + b\,\tau}\,dt^2 + \frac{\alpha^{-2}}{\tau^2}\,d\tau^2 \\ &\quad +\alpha^{-1}\left(\hat{J} + b\,\tau\right)\left(dz + \left[\frac{\tau}{\hat{J} + b\,\tau} + \lambda\right]dt\right)^2 \\ &\quad + \left(dx^2 + dy^2\right) \ , \end{aligned} \tag{6.74}$$

Now we notice that for

$$\lambda = -\frac{1}{b} \quad \text{and} \quad b = 4\alpha^{-3} > 0 \tag{6.75}$$

and assuming z to be compact, this is nothing but the metric of the extremal BTZ black hole in AdS_3 times $\mathbb{R}^2$, so that the space time is asymptotically $AdS_3 \times \mathbb{R}^2$. This can be made manifest by the coordinate redefinitions

$$\tau = \rho^2 - \frac{\hat{J}}{b} \ , \quad z = \frac{l}{b}\,\phi \ , \tag{6.76}$$

where $l^2 = \alpha\, b = 4/(gh_A X^A)^2$. Introducing

$$j = \frac{2\hat{J}}{bl} \ , \tag{6.77}$$

the line element becomes

$$ds^2 = -\left(\frac{\rho}{l} - \frac{j}{2\rho}\right)^2 dt^2 + \left(\frac{\rho}{l} - \frac{j}{2\rho}\right)^{-2} d\rho^2 + \rho^2 \left(d\phi - \frac{j}{2\rho^2}\, dt\right)^2 + \left(dx^2 + dy^2\right) . \quad (6.78)$$

This describes an extremal BTZ black hole with angular momentum j and mass $M = j/l$ [145], where l denotes the radius of AdS_3. The horizon is at $\rho_+^2 = jl/2 = \hat{J}/b$, which corresponds to $\tau = 0$. The entropy of the BTZ black hole (and hence the entropy density of the extremal BTZ $\times \mathbb{R}^2$ solution (6.78)) is

$$\mathcal{S}_{\rm BTZ} = \frac{2\pi\, \rho_+}{4} = \pi \frac{\sqrt{\hat{J}}}{4}\, \alpha^{3/2} \; . \quad (6.79)$$

Observe that α is determined in terms of the fluxes h_A and the P^A through (6.61), and so it is independent of J and Q_A.

In deriving the above solution, we have assumed that all the P^A are turned on so as to ensure the invertibility of the matrix C_{AB}. In this generic case, the constant values of the scalar fields X^A and A_z^A are entirely determined in terms of the h_A, P^A and Q_A. When switching off some of the P^A, some of the A_z^A may be allowed to have arbitrary constant values, but these are expected not to contribute to the entropy density.

The BTZ$\times\mathbb{R}^2$-solution given above can be found in any $N = 2$, $U(1)$ gauged supergravity model. This can also be inferred as follows. Setting $B' = (X^A)' = (A_z^A)' = \hat{q}_A = 0$ in the Lagrangian (6.18) and using the relations (6.62) and (6.63) yields a one-dimensional Lagrangian that descends from a three-dimensional Lagrangian describing Einstein gravity in the presence of an anti-de Sitter cosmological constant $\Lambda = -1/l^2$ determined by the flux potential ($4/l^2 = (g h_A X^A)^2$). As is well-known, the associated three-dimensional equations of motion allow for extremal BTZ black hole solutions with rotation. As shown in [136], the near-horizon geometry of the BTZ$\times\mathbb{R}^2$-solution (which is supported by the magnetic fields (6.61)) preserves half of the supersymmetry.

The entropy of the BTZ black hole solution depends on $\hat{J}$, which takes the form

$$J + \frac{1}{2} C^{AB} Q_A Q_B \; . \quad (6.80)$$

This combination is invariant under the transformation

$$J \to \hat{J} \quad , \quad Q_A \to \hat{q}_A \; , \quad (6.81)$$

with $\hat{J}$ and $\hat{q}_A$ given in (6.17) and (6.12), respectively. In the absence of fluxes, this transformation is called spectral flow transformation and can be understood as follows from the supergravity perspective [11]. The rewriting of the five-dimensional Lagrangian in terms of first-order flow equations makes use of the combinations $\hat{q}_A$ and $\hat{J}$. These combinations have their origin in the presence of the gauge Chern-Simons term. When the A_z^A are constant, the shifts $Q_A \to \hat{q}_A$ and $J \to \hat{J}$ take the form of shifts induced by a large gauge transformation of A_z^A, i.e. $A^A \to A^A + k^A$, where k^A denotes a closed one-form.

These transformations constitute a symmetry of string theory, and this implies that the entropy of a black hole should be invariant under spectral flow. It must therefore depend on the combination (6.80). In the presence of fluxes, we find that the BTZ $\times\mathbb{R}^2$-solution (6.78) respects the spectral flow transformation (6.81).

The three-dimensional extremal BTZ black hole geometry, resulting from dimensionally reducing the black brane solution (6.78) on $\mathbb{R}^2$, is a state in the two-dimensional CFT dual to the asymptotic AdS_3, with left-moving central charge $c = \frac{3l}{2G_3}$ and $L_0 - \frac{c}{24} = \frac{\rho_+^2}{4\,G_3\,l}$, as well as $\tilde{L}_0 - \frac{\tilde{c}}{24} = 0$, cf. [146]. Hence, the large charge leading term in the entropy of the black hole is given by the Ramanujan-Hardy-Cardy formula for the dual CFT,

$$\mathcal{S}_{\rm BTZ} \;=\; 2\,\pi\,\sqrt{\frac{c}{6}\,\left(L_0 \,-\, \frac{c}{24}\right)}\;. \tag{6.82}$$

This is exactly equal to the Bekenstein-Hawking entropy (6.79) computed above (in units of $G_3 = 1$), and can be regarded as a microscopic computation of the bulk black brane entropy from the holographic dual CFT.

Taking $P^A = 0, Q_A = 0, \hat{J} \neq 0$

Now we consider the case when all the P^A vanish. Then, (6.60) reduces to

$$\left(h_C X^C\right) X_A = h_A \;, \tag{6.83}$$

which determines the constants X^A in terms of the fluxes h_A. Contracting (6.83) with $G^{AB} h_B$ yields

$$G^{AB}\,h_A h_B = \frac{2}{3}\left(h_A\,X^A\right)^2 \;. \tag{6.84}$$

Thus we take $h_A\,X^A \neq 0$ in the following, since otherwise $h_A = 0$.

Combining the fourth equation of (6.21) with (6.23) results in

$$B' = \frac{1}{3} g h_A X^A \;, \tag{6.85}$$

which can be readily integrated to give

$$e^B = e^{\beta}\,e^{\frac{1}{3} g h_A X^A\,r} \;, \tag{6.86}$$

where β denotes an integration constant which we set to zero. The combination $U + W$ is given by

$$U + W = 2B + u \;, \tag{6.87}$$

where u denotes an integration constant which we also set to zero.

The flow equation for $U - W$ reads

$$\left(e^{-(U-W)}\right)' = -\hat{J}\,e^{-4B} = -\hat{J}\,e^{-\frac{4}{3} g h_A X^A\,r} \;. \tag{6.88}$$

Next, let us consider the flow equation for A_z^A,

$$\left(A_z^A\right)' = -\tfrac{1}{2} G^{AC}\, Q_C\, e^{-2B} \; . \tag{6.89}$$

A non-vanishing Q_A yields a running scalar field $A_z^A \sim G^{AB} Q_B\, e^{-\frac{2}{3} g h_A X^A r}$. In the chosen coordinates, the line element can only have a throat at $|r| = \infty$. At either of these points, either the area element e^B or A_z^A blows up. If we demand that both e^B and A_z^A stay finite at the horizon, we are thus led to take A_z^A to be constant, which can be obtained by setting $Q_A = 0$. Therefore, we set $Q_A = 0$ in the following. This implies that $\hat{J}$ is constant, which we take to be non-vanishing.

Taking $h_A X^A \neq 0$, (6.88) is solved by

$$e^{-(U-W)} = \frac{3}{4} \frac{\hat{J}}{g h_A X^A}\, e^{-\frac{4}{3} g h_A X^A r} + \gamma \; , \tag{6.90}$$

where γ denotes an integration constant. Using (6.87), this results in

$$\begin{aligned} e^{2W} &= \frac{3}{4} \frac{\hat{J}}{g h_A X^A}\, e^{-\frac{2}{3} g h_A X^A r} + \gamma\, e^{\frac{2}{3} g h_A X^A r} \; , \\ e^{-2U} &= \frac{3}{4} \frac{\hat{J}}{g h_A X^A}\, e^{-2 g h_A X^A r} + \gamma\; e^{-\frac{2}{3} g h_A X^A r} \; . \end{aligned} \tag{6.91}$$

Redefining the radial coordinate,

$$\tau = e^{\frac{1}{3} g h_A X^A r} \quad , \quad \tau \geq 0 \; , \tag{6.92}$$

yields

$$\begin{aligned} e^B &= \tau \; , \\ e^{2W} &= \frac{3}{4} \frac{\hat{J}}{g h_A X^A}\, \tau^{-2} + \gamma\, \tau^2 \; , \\ e^{-2U} &= \frac{3}{4} \frac{\hat{J}}{g h_A X^A}\, \tau^{-6} + \gamma\, \tau^{-2} \; , \\ C &= \frac{\tau^4}{\frac{3}{4} \frac{\hat{J}}{g h_A X^A} + \gamma\, \tau^4} + \lambda \; . \end{aligned} \tag{6.93}$$

In the chosen coordinates, the line element reads

$$ds^2 = -e^{2U(\tau)} dt^2 + (\tfrac{1}{3} g h_A X^A)^{-2} \left(\frac{d\tau}{\tau}\right)^2 + e^{2B(\tau)}(dx^2 + dy^2) + e^{2W(\tau)}(dz + C(\tau) dt)^2 \, . \tag{6.94}$$

It exhibits a throat as $\tau \to 0$. In the following we set $\lambda = 0$, and we take $\hat{J}/(g h_A X^A) > 0$.

In the throat region, the terms proportional to γ do not contribute (recall that we are taking $\hat{J}$ to be non-vanishing) and the line element becomes

$$ds^2 = -\tau^6 dt^2 + (\tfrac{1}{3} g h_A X^A)^{-2} \left(\frac{d\tau}{\tau}\right)^2 + \tau^2(dx^2 + dy^2) + \tau^{-2}(dz + \tau^4 dt)^2 \,, \tag{6.95}$$

where we rescaled the coordinates by various constant factors. Then, performing the coordinate transformation

$$\tilde{\tau} = \tau^3 \quad , \quad \tilde{t} = 3t \,, \tag{6.96}$$

and setting $(\frac{1}{3} g h_A X^A)^2 = 1$ for convenience, the line element becomes

$$\begin{aligned} ds^2 &= \frac{1}{9}\left(-\tilde{\tau}^2 d\tilde{t}^2 + \left(\frac{d\tilde{\tau}}{\tilde{\tau}}\right)^2\right) + \tilde{\tau}^{2/3}(dx^2 + dy^2) + \tilde{\tau}^{-2/3}\left(dz + \tfrac{1}{3}\tilde{\tau}^{4/3} d\tilde{t}\right)^2 \\ &= \frac{2}{3}\,\tilde{\tau}^{2/3} d\tilde{t} dz + \tilde{\tau}^{-2/3} dz^2 + \frac{1}{9}\left(\frac{d\tilde{\tau}}{\tilde{\tau}}\right)^2 + \tilde{\tau}^{2/3}(dx^2 + dy^2) \,, \end{aligned} \tag{6.97}$$

which describes a null-warped throat. The entropy density vanishes, $\mathcal{S} \sim e^{2B+W}|_{\tau=0} = 0$.

In the limit $\tau \to \infty$, on the other hand, there are two distinct cases. When $\gamma \neq 0$ (we take $\gamma > 0$),

$$\begin{aligned} e^B &= \tau \,, \\ e^{2W} &\approx \gamma \tau^2 \,, \\ e^{-2U} &\approx \gamma \tau^{-2} \,, \\ C &\approx \gamma^{-1} \,, \end{aligned} \tag{6.98}$$

and the line element becomes

$$ds^2 = -\tau^2 dt^2 + (\tfrac{1}{3} g h_A X^A)^{-2} \left(\frac{d\tau}{\tau}\right)^2 + \tau^2(dx^2 + dy^2) + \tau^2(dz + dt)^2 \,, \tag{6.99}$$

where we rescaled the coordinates. Observe that this describes a patch of AdS_5.

The other case corresponds to setting $\gamma = 0$, in which case the behavior at $\tau \to \infty$ is determined by

$$\begin{aligned} e^B &= \tau \,, \\ e^{2W} &= \frac{3}{4}\,\frac{\hat{J}}{g h_A X^A}\,\tau^{-2} \,, \\ e^{-2U} &= \frac{3}{4}\,\frac{\hat{J}}{g h_A X^A}\,\tau^{-6} \,, \\ C &= \frac{4}{3}\,\frac{g h_A X^A}{\hat{J}}\,\tau^4 \,, \end{aligned} \tag{6.100}$$

and the associated line element is again of the form (6.95) and (6.97).

Thus, we conclude that the solution (6.94) with $\gamma \neq 0$ describes a solution that interpolates between AdS_5 and a null-warped Nernst throat at the horizon with vanishing entropy, in which all the scalar fields are kept constant. This is a purely gravitational stationary solution that is supported by electric fluxes h_A. It is an example of a Nernst brane (i.e. a solution with vanishing entropy density), and is the five-dimensional counterpart of the four-dimensional Nernst solution constructed in chapter 5. As already mentioned in section 5.3 in Nernst geometries the tidal forces diverge.

6.3.2 Solutions with non-constant scalar fields X^A

Here we present numerical solutions that are supported by non-constant scalar fields X^A and that interpolate between a near horizon solution of the type discussed above in sec. 6.3.1 and an asymptotic AdS_5-region with metric

$$ds^2 = -e^{2r}dt^2 + dr^2 + e^{2r}(dx^2 + dy^2 + dz^2) \ , \tag{6.101}$$

i.e. the metric functions $U(r), B(r)$ and $W(r)$ in (6.1) all asymptote to the linear function r and C becomes 0 (or constant, since a constant C can be removed by a redefinition of the z-variable). To be concrete, we work within the STU-model. Within this model, a solution with a single running scalar and with $\hat{J} = 0$ was already given in sec. 2.3 of [10]. In order to facilitate the comparison with their results, we will work with physical scalars in this section, i.e. we solve the constraint $X^1X^2X^3 = 1$ via

$$X^1 = e^{-\frac{1}{\sqrt{6}}\phi^1 - \frac{1}{\sqrt{2}}\phi^2} \ , \quad X^2 = e^{-\frac{1}{\sqrt{6}}\phi^1 + \frac{1}{\sqrt{2}}\phi^2} \ , \quad X^3 = e^{\frac{2}{\sqrt{6}}\phi^1} \ . \tag{6.102}$$

In an asymptotically AdS_5-spacetime the two scalars ϕ^i have a leading order expansion

$$\phi^i(r) = a_i r e^{-2r} + b_i e^{-2r} + \mathcal{O}(e^{-4r}) \ , \quad i = 1, 2 \ , \tag{6.103}$$

i.e. they both correspond to dimension 2 operators of the dual field theory and a_i and b_i correspond to the sources and 1-point functions, respectively.

For all the following numerical solutions, we choose

$$g = 1 \ , h_1 = h_2 = h_3 = 1 \ . \tag{6.104}$$

Solution with a single running scalar

Let us first consider the case with a single running scalar field and with vanishing $\hat{J}$. For concreteness, we choose the magnetic fields

$$P^1 = P^2 = 4^{1/3} \ , \quad P^3 = -2 \cdot 4^{1/3} \ , \tag{6.105}$$

so as to satisfy the constraint (6.22). One could choose a different overall normalization for the magnetic fields P^A by rescaling the x and y coordinates. This, on the other hand, would also imply a rescaling of e^B and, thus, we would not have $B = 0$ anymore, as was assumed in sec. 6.3.1. Solving (6.61) for these values of P^A leads to

$$X^1_{(0)} = X^2_{(0)} = 4^{-1/3} \ , \quad X^3_{(0)} = 4^{2/3} \ , \tag{6.106}$$

where we added the subscript (0) in order to distinguish the constant near horizon values from the full, non-constant solution to be discussed momentarily. These values lead to

$$\phi^1_{(0)} \approx 1.132 \ , \quad \phi^2_{(0)} = 0 \ , \quad \alpha = 3 \cdot 2^{1/3} \approx 3.780 \ , \tag{6.107}$$

with α defined in (6.72). Finally, using (6.71), (6.73) and (6.75), we obtain

$$U_{(0)} = -\frac{1}{2} \ln\left(4\alpha^{-3}\right) + \frac{\alpha}{2} r \quad , \quad W_{(0)} = \frac{1}{2} \ln\left(4\alpha^{-3}\right) + \frac{\alpha}{2} r \quad , \quad C_{(0)} = 0. \tag{6.108}$$

In order to obtain an interpolating solution with an asymptotic AdS_5-region, we slightly perturb around this solution. In particuar, we make the following ansatz for small r

$$\begin{aligned} &X^1 = X^1_{(0)} + c_1 e^{c_2 r} \ , \quad X^2 = X^2_{(0)} + c_3 e^{c_4 r} \ , \\ &B = \delta B \ , \quad U = U_{(0)} + \delta U \ , \quad W = W_{(0)} + \delta W \end{aligned} \tag{6.109}$$

with $c_2, c_4 > 0$. Plugging this into the flow equations (6.21), we obtain the following conditions:

$$\begin{aligned} &c_1 = c_3 \ , \quad c_2 = c_4 = 2^{-2/3}\left(\sqrt{33} - 1\right) \approx 2.989 \ , \\ &\delta U = \delta W = -\frac{1}{2}\frac{c_3(2^{2/3}c_2 + 14)}{2} e^{c_2 r} \ , \quad \delta B = 2^{-2/3} c_3 (3 \cdot 2^{1/3} + c_2) e^{c_2 r} \ . \end{aligned} \tag{6.110}$$

The constant c_3 is undetermined and sets the value of the source and 1-point function of ϕ^1, cf. (6.103). We will see this explicitly in the example in sec. 6.3.2 below. Using (6.110), we can find the initial conditions needed to solve (6.21) numerically. In practice it is most convenient to solve (6.21) in the τ variable (6.71), as the horizon is at $\tau = 0$. This allows to set the initial conditions for instance at $\tau = 10^{-13}$ and then integrate outwards. Doing so and choosing $c_3 = 1$, we obtain the result depicted in figure 6.1 (note that the plot makes use of the r-variable, i.e. the primes denote derivatives with respect to r as before). Moreover, $C \equiv 0$. Even though the functions U and W always have the same derivative, they differ by a shift, as can be seen in the right part of figure 6.1. This is due to the fact that we chose $b = 4\alpha^{-3}$ according to (6.75), instead of $b = 1$, with b being introduced in (6.68). This solution is very similar to the one discussed in sec. 2.3 of [10] and it is obvious that the metric asymptotically becomes of the form (6.101).

Solution with two running scalars, $\hat{J} = 0$

We now turn to the case of two non-trivial scalars, first still with vanishing $\hat{J}$ and then with $\hat{J} \neq 0$ in the next subsection. In all cases we choose

$$P^1 = \left(\frac{4}{3}\right)^{1/3} \quad , \quad P^2 = 2 \cdot \left(\frac{4}{3}\right)^{1/3} \quad , \quad P^3 = -3 \cdot \left(\frac{4}{3}\right)^{1/3} \quad . \tag{6.111}$$

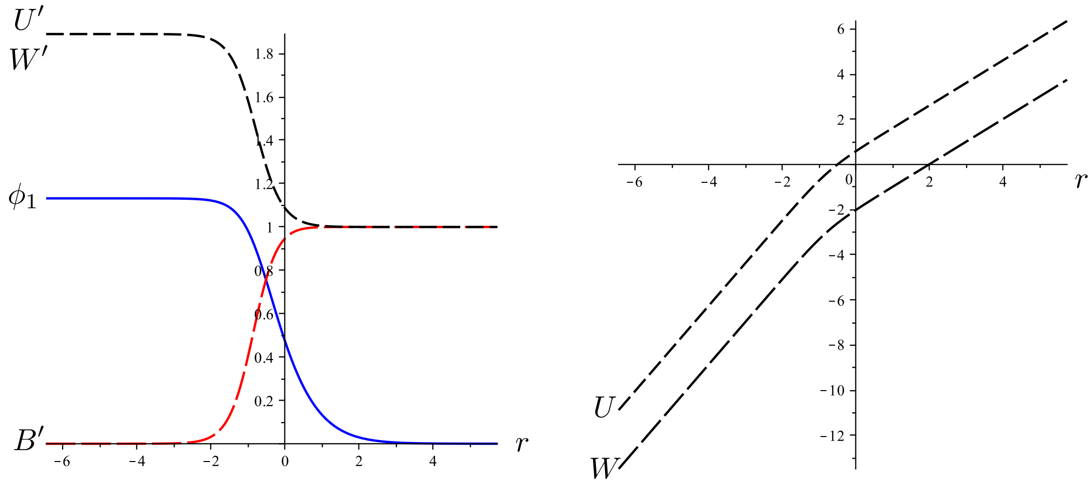

Figure 6.1: The case of one scalar with vanishing $\hat{J}$.

Again, the overall normalization of the P^A is imposed on us by demanding $B = 0$ in the near-horizon region. This time solving (6.61) leads to

$$X^1_{(0)} = \frac{1}{4}\left(\frac{4}{3}\right)^{2/3} \quad , \quad X^2_{(0)} = \left(\frac{4}{3}\right)^{2/3} \quad , \quad X^3_{(0)} = \frac{3}{2}6^{1/3} \; , \tag{6.112}$$

which implies

$$\phi^1_{(0)} \approx 1.228 \; , \quad \phi^2_{(0)} \approx 0.980 \; , \quad \alpha = \frac{7}{6}\cdot 4^{2/3}\cdot 3^{1/3} \approx 4.240 \; . \tag{6.113}$$

The functions $U_{(0)}$, $W_{(0)}$ and $C_{(0)}$ are again given by (6.108), now using the value of α given in (6.113).

Perturbing around the near-horizon solution utilizes the same ansatz as in (6.109). This time, we obtain the conditions

$$\begin{aligned} c_1 &= -\frac{c_3}{8}\frac{41\cdot 6^{1/3}+33c_2}{29\cdot 6^{1/3}-3c_2} \quad , \quad c_2 = c_4 \approx 3.694 \; , \\ \delta U = \delta W &= \frac{1}{16}\frac{c_3(21\cdot 6^{2/3}c_2^2+588c_2-178\cdot 6^{1/3})}{c_2(-3c_2+29\cdot 6^{1/3})}e^{c_2 r} \; , \\ \delta B &= \frac{6^{1/3}}{16}\frac{c_3(-27c_2^2-42\cdot 6^{1/3}c_2+49\cdot 6^{2/3})}{-3c_2+29\cdot 6^{1/3}}e^{c_2 r} \; . \end{aligned} \tag{6.114}$$

Again, the parameter c_3 determines the sources and 1-point functions of the scalars ϕ^1 and ϕ^2.

Using (6.114) in (6.109) we obtain the initial conditions to solve the flow equations numerically. We do not find any solution for $c_3 = 1$, but inverting the sign, i.e. choosing $c_3 = -1$, leads to the solution depicted in figure 6.2, which in addition has $C \equiv 0$ and which is asymptotically AdS_5.

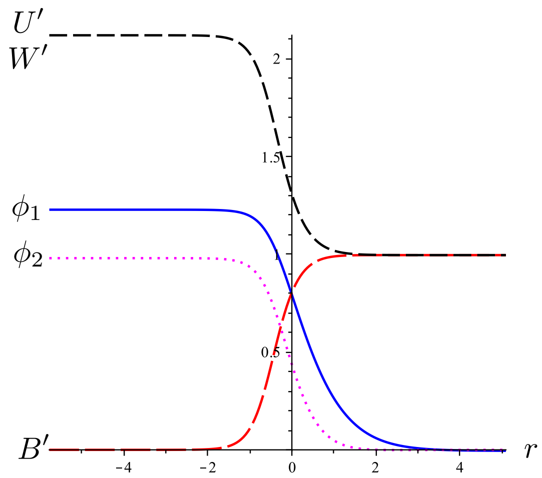

Figure 6.2: The case of two scalars with vanishing $\hat{J}$.

Solutions with two running scalars, $\hat{J} \neq 0$

Finally, let us look at the more general case, where we have two running scalars and a constant non-vanishing $\hat{J}$. We choose the same values for the magnetic fields as in the last example, i.e. (6.111). Given that (6.61) does not depend on $\hat{J}$ at all, it is not surprising that this leads to the same values for the $X^A_{(0)}$ as in (6.112) (and, thus, also (6.113) does not change). The main change arises for U, W and C, as their flow equations explicitly depend on $\hat{J}$. They take on the near-horizon form

$$U_{(0)} = -\frac{1}{2}\ln\left(\frac{\hat{J}}{\alpha} + \frac{4e^{\alpha r}}{\alpha^3}\right) + \alpha r \quad , \quad W_{(0)} = \frac{1}{2}\ln\left(\frac{\hat{J}}{\alpha} + \frac{4e^{\alpha r}}{\alpha^3}\right) ,$$
$$C_{(0)} = \frac{\alpha^3}{4}\left(\frac{1}{\hat{J}\frac{\alpha^2}{4}e^{-\alpha r} + 1} - 1\right) . \tag{6.115}$$

Notice, in particular, the different behavior of $U_{(0)}$ and $W_{(0)}$ very close to the horizon, i.e. for $r \to -\infty$. Whereas the slope of $U_{(0)}$ and $W_{(0)}$ was $\alpha/2$ in the case of vanishing $\hat{J}$, cf. (6.108), now it is α for $U_{(0)}$ and zero for $W_{(0)}$ in the case of non-vanishing $\hat{J}$. We will clearly see this in the numerical solutions.

Again, we perturb around the near-horizon solution by (6.109). We again infer that $c_2 = c_4$ and that c_1 and c_3 are related as in (6.114). Moreover, $\delta U, \delta W$ and δB are all proportional to $e^{c_2 r}$. Without going into the details, we present the resulting numerical solutions for different values of $\hat{J}$ in figure 6.3. All these plots were produced using $c_3 = -0.1$ and $b = 4\alpha^{-3}$. One can nicely see that the main difference to the case of vanishing $\hat{J}$ appears in the U and W sector. The different slope of U and W close to the horizon, mentioned in the last paragraph, is apparent. It is also obvious that for small $\hat{J}$, U and W

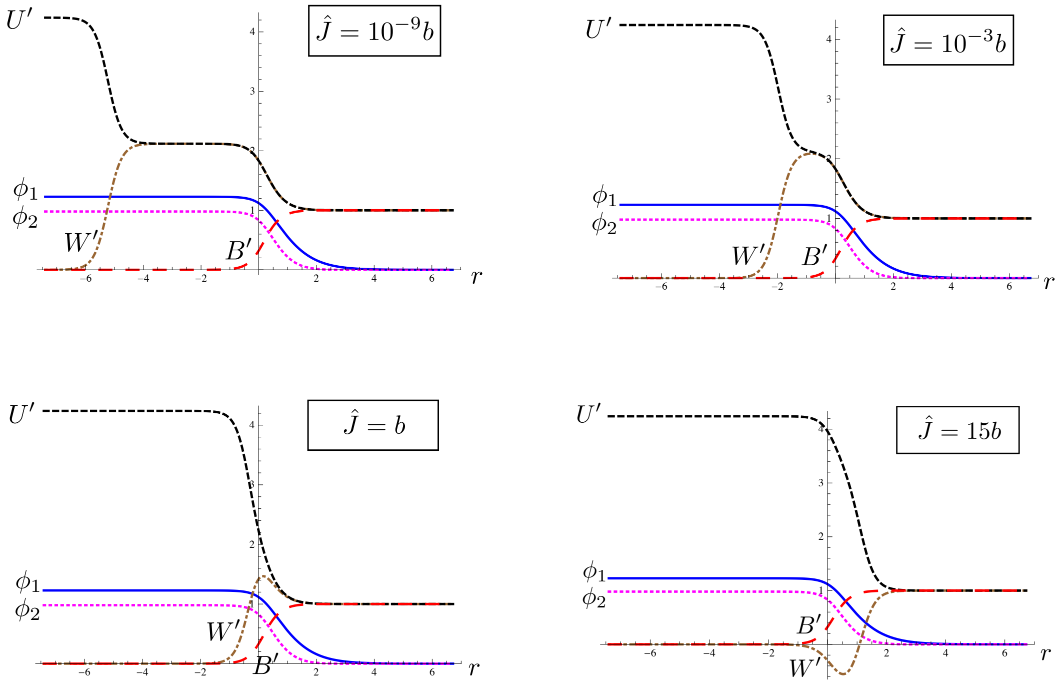

Figure 6.3: The case of two scalars witht non-vanishing $\hat{J}$.

first behave as in the case with vanishing $\hat{J}$ when approaching the horizon from infinity. I.e. they start out showing the same slope of $\alpha/2$ until the $\hat{J}$-term starts dominating very close to the horizon, where the slope of U doubles and W becomes constant. With increasing $\hat{J}$ the intermediate region, where U and W have the same slope of $\alpha/2$, becomes smaller and smaller and finally disappears altogether.

The function $C = e^{U-W} - 1/b$ is shown (for $\hat{J} = b$) in figure 6.4. Obviously, asymptotically it becomes constant and, thus, the asymptotic region is indeed given by AdS_5.

Finally, in figure 6.5, we plot ϕ^1 and ϕ^2, multiplied with e^{2r}, for two different values of c_3. As expected from (6.103), the graphs show a linear behavior with non-vanishing sources and 1-point functions for the two operators dual to the scalars. Obviously, these sources and 1-point functions depend on the value of c_3.

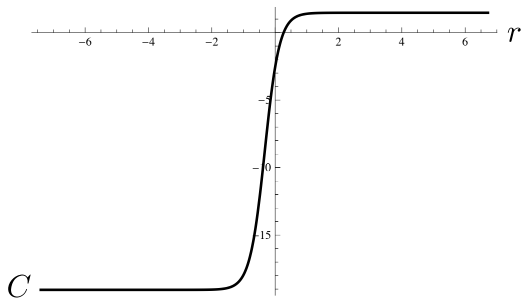

Figure 6.4: The function C for $\hat{J} = b$.

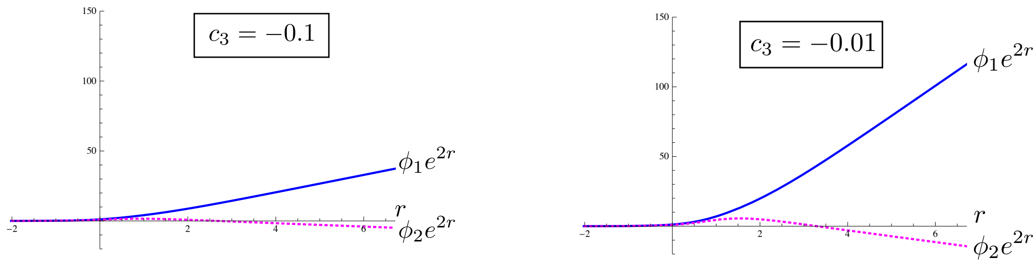

Figure 6.5: The two scalars multiplied with e^{2r}.

Chapter 7

Conclusions and Outlook

In the last chapters we have met some interesting results within the wide subject of the AdS/CFT correspondence.

We have seen how to apply AdS/CFT techniques to calculate the frequency dependent conductivity tensor for field theories dual to a black hole in Einstein-Yang-Mills theory with $SU(2)$ gauge group. Further, we have constructed several new black solutions in $N = 2$ $U(1)$ gauged supergravity in four and five dimensions. The larger part of these solutions behave asymptotically like AdS which makes them interesting within the AdS/CFT context. In addition we found extremal black branes with zero entropy density – the Nernst branes.

Nonetheless we are left with some yet unsolved problems. It will be very interesting to see what causes the negative entropy production rate we found in chapter 4 for the normal state of the field theory. As already mentioned at the end of chapter 4, the next task will be to see whether we can find an instability on the gravity side looking at the full Einstein-Yang-Mills equations.

Also our work on supergravity solutions in four and five dimension exhibits some "loose ends". Since all our four-dimensional Nernst solutions were axion-free it would be nice to find one with axions excited. Moreover, it would be interesting to see whether the singular solutions with flowing γ could be cured by taking into account higher derivative corrections or whether there exist non-singular solutions with non-constant γ.

In five dimensions we met problems when adding electric charge. At present we could not find a dyonic solution and we had the impression that having electric charges and having magnetic fields seemed to be somehow complementary to each other. We saw these difficulties even at the beginning when we performed the first-order rewriting since the first-order rewriting in chapter 6 leads to flow equations for the scalars X^A which only contain magnetic fields and fluxes but no electric charges. The latter only influence the equations of motion for the X^A in an indirect way. However, as we have seen in appendix E it is possible to find different rewritings. It would be interesting to see if there is another rewriting that contains electric charges directly in the flow equations for the X^A or if otherwise there is an explanation why we could not find a rewriting of that type. This would also lead to a deeper understanding and this is what one is actually searching for.

We can only learn something new when getting involved with the puzzles and problems that come across. What would science be without open questions – or to say it with Einstein's words from the beginning, without the mysterious?

Appendix A

Special geometry

We already stated in section 2.3 that the Lagrangian describing the couplings of $N = 2$ vector multiplets to $N = 2$ supergravity is encoded in a holomorphic function $F(X)$, called the prepotential, that depends on $n + 1$ complex scalar fields X^I ($I = 0, \ldots, n$). Here, n counts the number of physical scalar fields. We will repeat here some of the identities displayed in section 2.3. Moreover, we will provide the foundations for our calculations in chapter 5.

For the prepotential $F(X)$ it is known that the coupling to supergravity requires it to be homogeneous of degree two, i.e. $F(\lambda X) = \lambda^2 F(X)$, from which one derives the homogeneity properties

$$\begin{aligned} F_I &= F_{IJ}\, X^J \ , \\ F_{IJK}\, X^K &= 0 \ , \end{aligned} \tag{A.1}$$

where $F_I = \partial F(X)/\partial X^I$, $F_{IJ} = \partial^2 F/\partial X^I \partial X^J$, etc. The X^I are coordinates on the big moduli space, while the physical scalar fields $z^i = X^i/X^0$ ($i = 1, \ldots, n$) parametrize an n-dimensional complex hypersurface, which is defined by the condition that the symplectic vector $(X^I, F_I(X))$ satisfies the constraint

$$i\left(\bar{X}^I\, F_I - \bar{F}_I\, X^I\right) = 1 \ . \tag{A.2}$$

This can be written as

$$-\, N_{IJ}\, X^I\, \bar{X}^J = 1 \ , \tag{A.3}$$

where

$$N_{IJ} = -i\left(F_{IJ} - \bar{F}_{IJ}\right) \ . \tag{A.4}$$

The constraint (A.3) is solved by setting

$$X^I = \mathrm{e}^{K(z,\bar{z})/2}\, X^I(z) \ , \tag{A.5}$$

where $K(z, \bar{z})$ is the Kähler potential,

$$\mathrm{e}^{-K(z,\bar{z})} = |X^0(z)|^2 \left[-N_{IJ}\, Z^I\, \bar{Z}^J\right] \tag{A.6}$$

with $Z^I(z) = (Z^0, Z^i) = (1, z^i)$. Writing

$$F(X) = \left(X^0\right)^2 \mathcal{F}(z) \,, \tag{A.7}$$

which is possible in view of the homogeneity of $F(X)$, we obtain

$$F_0 = X^0 \left(2\mathcal{F}(z) - z^i \mathcal{F}_i\right) \,, \tag{A.8}$$

where $\mathcal{F}_i = \partial\mathcal{F}/\partial z^i$. In addition, we compute

$$\begin{aligned} F_{00} &= 2\mathcal{F} - 2z^i \mathcal{F}_i + z^i z^j \mathcal{F}_{ij} \,, \\ F_{0j} &= \mathcal{F}_j - z^i \mathcal{F}_{ij} \,, \\ F_{ij} &= \mathcal{F}_{ij} \,, \end{aligned} \tag{A.9}$$

to obtain

$$- N_{IJ} Z^I \bar{Z}^J = i \left[2 \left(\mathcal{F} - \bar{\mathcal{F}}\right) - \left(z^i - \bar{z}^i\right) \left(\mathcal{F}_i + \bar{\mathcal{F}}_i\right)\right] \,, \tag{A.10}$$

and hence

$$\mathrm{e}^{-K(z,\bar{z})} = i \, |X^0(z)|^2 \left[2 \left(\mathcal{F} - \bar{\mathcal{F}}\right) - \left(z^i - \bar{z}^i\right) \left(\mathcal{F}_i + \bar{\mathcal{F}}_i\right)\right] \,. \tag{A.11}$$

As already mentioned in chapter 2 the $X^I(z)$ are defined projectively, i.e. modulo multiplication by an arbitrary holomorphic function,

$$X^I(z) \to \mathrm{e}^{-f(z)} \, X^I(z) \,. \tag{A.12}$$

This transformation induces the Kähler transformation

$$K \to K + f + \bar{f} \tag{A.13}$$

on the Kähler potential, while on the symplectic vector $(X^I, F_I(X))$ it acts as a phase transformation, i.e.

$$(X^I, F_I(X)) \to \mathrm{e}^{-\frac{1}{2}(f - \bar{f})} \, (X^I, F_I(X)) \,. \tag{A.14}$$

The resulting geometry for the space of physical scalar fields z^i is a special Kähler geometry, with Kähler metric

$$g_{i\bar{j}} = \frac{\partial^2 K(z,\bar{z})}{\partial z^i \, \partial \bar{z}^j} \tag{A.15}$$

based on a Kähler potential of the special form (A.11).

Let us relate the Kähler metric (A.15) to the metric (A.4) on the big moduli space. Differentiating e^{-K} yields

$$\begin{aligned} \partial_k \mathrm{e}^{-K} &= -\partial_k K \, \mathrm{e}^{-K} = i \, |X^0(z)|^2 \left[\mathcal{F}_k - \bar{\mathcal{F}}_k - (z^i - \bar{z}^i)\mathcal{F}_{ik}\right] + \partial_k \ln X^0(z) \, \mathrm{e}^{-K} \,, \\ \partial_k \partial_{\bar{l}} \mathrm{e}^{-K} &= \left[-\partial_k \partial_{\bar{l}} K + \partial_k K \, \partial_{\bar{l}} K\right] \mathrm{e}^{-K} = \left[-g_{k\bar{l}} + \partial_k K \, \partial_{\bar{l}} K\right] \mathrm{e}^{-K} \\ &= i \, |X^0(z)|^2 \left(\mathcal{F}_{kl} - \bar{\mathcal{F}}_{kl}\right) - \Big[\partial_k \ln X^0(z) \, \partial_{\bar{l}} \ln \bar{X}^0(\bar{z}) \\ &\qquad -\partial_k \ln X^0(z) \, \partial_{\bar{l}} K - \partial_k K \, \partial_{\bar{l}} \ln \bar{X}^0(\bar{z})\Big] \mathrm{e}^{-K} \,. \end{aligned} \tag{A.16}$$

Using (A.9) we have

$$N_{ij} = -i\left(\mathcal{F}_{ij} - \bar{\mathcal{F}}_{ij}\right) \ , \tag{A.17}$$

and hence we infer from (A.16) that

$$g_{i\bar{j}} = N_{ij}\,|X^0|^2 + \frac{1}{|X^0(z)|^2}\mathcal{D}_i X^0(z)\mathcal{D}_{\bar{j}}\bar{X}^0(\bar{z}) \ , \tag{A.18}$$

where

$$\mathcal{D}_i X^0(z) = \partial_i X^0(z) + \partial_i K\, X^0(z) \tag{A.19}$$

denotes the covariant derivative of $X^0(z)$ under the transformation (A.12), i.e. $\mathcal{D}_i\left(\mathrm{e}^{-f} X^0(z)\right) = \mathrm{e}^{-f}\,\mathcal{D}_i X^0(z)$.

Next, let us consider the combination $N_{IJ}\,\mathcal{D}_\mu X^I\,\mathcal{D}^\mu \bar{X}^J$, where the space-time covariant derivative $\mathcal{D}_\mu$ reads

$$\mathcal{D}_\mu X^I = \partial_\mu X^I + i\mathcal{A}_\mu X^I = \partial_\mu X^I + \frac{1}{2}\left(\partial_i K\,\partial_\mu z^i - \partial_{\bar{i}} K \partial_\mu \bar{z}^i\right) X^I \ , \tag{A.20}$$

which is a covariant derivative for $U(1)$ transformations (A.14). The combination $N_{IJ}\,\mathcal{D}_\mu X^I\,\mathcal{D}^\mu \bar{X}^J$ is thus invariant under $U(1)$-transformations. Observe that

$$\mathcal{D}_\mu X^0 = \mathrm{e}^{K/2}\,\mathcal{D}_i X^{(0)}(z)\,\partial_\mu z^i \ . \tag{A.21}$$

Using (A.3) we obtain

$$\begin{aligned} N_{IJ}\,\mathcal{D}_\mu X^I\,\mathcal{D}^\mu \bar{X}^J \ &= \ |X^0|^2\,N_{ij}\,\partial_\mu z^i\,\partial^\mu \bar{z}^j - \frac{1}{|X^0|^2}\,\mathcal{D}_\mu X^0\,\mathcal{D}^\mu \bar{X}^0 \\ &\quad + \frac{X^0}{\bar{X}^0}\,N_{iJ}\bar{X}^J\,\partial_\mu z^i\,\mathcal{D}^\mu \bar{X}^0 + \frac{\bar{X}^0}{X^0}\,N_{Ij}X^I\,\partial_\mu \bar{z}^j\,\mathcal{D}^\mu X^0 \ . \end{aligned} \tag{A.22}$$

Next, using

$$X^0\,\bar{X}^J\,N_{kJ} = \frac{1}{X^0(z)}\,\mathcal{D}_k X^0(z) \ , \tag{A.23}$$

as well as (A.18) and (A.21) we establish

$$N_{IJ}\,\mathcal{D}_\mu X^I\,\mathcal{D}^\mu \bar{X}^J = g_{i\bar{j}}\,\partial_\mu z^i\,\partial^\mu \bar{z}^j \ , \tag{A.24}$$

which relates the kinetic term for the physical fields z^i to the kinetic term for the fields X^I on the big moduli space. Observe that both sides of (A.24) are invariant under Kähler transformations (A.13).

Using the relation (A.24), we express the bosonic Lagrangian (describing the coupling of n vector multiplets to $N = 2$ $U(1)$ gauged supergravity [147, 148]) in terms of the fields X^I of big moduli space,

$$L = \tfrac{1}{2}R - N_{IJ}\,\mathcal{D}_\mu X^I\,\mathcal{D}^\mu \bar{X}^J + \tfrac{1}{4}\mathrm{Im}\mathcal{N}_{IJ}\,F^I_{\mu\nu}\,F^{\mu\nu J} - \tfrac{1}{4}\mathrm{Re}\mathcal{N}_{IJ}\,F^I_{\mu\nu}\,\tilde{F}^{\mu\nu J} - g^2 V(X,\bar{X}) \ , \tag{A.25}$$

where

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + i\,\frac{N_{IK}\,X^K\,N_{JL}\,X^L}{X^M\,N_{MN}\,X^N}\ , \tag{A.26}$$

which satisfies the relation

$$\mathcal{N}_{IJ}\,X^J = F_I\ . \tag{A.27}$$

The flux potential reads

$$V = g^{i\bar{\jmath}}\,\mathcal{D}_i W\,\bar{\mathcal{D}}_{\bar{\jmath}}\bar{W} - 3|W|^2 \quad , \quad W = h^I\,F_I - h_I\,X^I\ , \tag{A.28}$$

where $\mathcal{D}_i X^I = \partial_i X^I + \frac{1}{2}\partial_i K\,X^I$. Here, (h^I, h_I) denote the magnetic/electric fluxes. AdS_4 with cosmological constant $\Lambda = -3g^2$ corresponds to a constant W with $|W| = 1$. Switching off the flux potential corresponds to setting $g = 0$.

Using the identity (see (23) of [125])

$$N^{IJ} = g^{i\bar{\jmath}}\,\mathcal{D}_i X^I\,\bar{\mathcal{D}}_{\bar{\jmath}}\bar{X}^J - X^I\,\bar{X}^J\ , \tag{A.29}$$

the flux potential can be expressed as

$$\begin{aligned} V(X,\bar{X}) \;&=\; \left[g^{i\bar{\jmath}}\,\mathcal{D}_i X^I\,\bar{\mathcal{D}}_{\bar{\jmath}}\bar{X}^J - 3X^I\bar{X}^J\right]\left(h^K\,F_{KI} - h_I\right)\left(h^K\,\bar{F}_{KJ} - h_J\right) \\ &= \left[N^{IJ} - 2X^I\bar{X}^J\right]\,\left(h^K\,F_{KI} - h_I\right)\left(h^K\,\bar{F}_{KJ} - h_J\right) \\ &= N^{IJ}\,\partial_I\tilde{W}\partial_{\bar{J}}\bar{\tilde{W}} - 2\,|\tilde{W}|^2\ , \end{aligned} \tag{A.30}$$

where in the last line $\tilde{W}$ is expressed in terms of $U(1)$-invariant fields $\tilde{X}^I$,

$$\tilde{W} = h^I\,F_I(\tilde{X}) - h_I\,\tilde{X}^I = \left(h^I\,F_{IJ} - h_J\right)\,\tilde{X}^J\ . \tag{A.31}$$

Appendix B

First-order rewriting in minimal gauged supergravity in $D = 4$

The $N = 2$ Lagrangian in minimal gauged supergravity is given in terms of the prepotential $F = -i\,(X^0)^2$. Only a single scalar field, X^0 is turned on, whose absolute value is set to a constant, $|X^0| = \frac{1}{4}$ by the constraint (A.3), and the consequent bulk 1-D Lagrangian density for the metric ansatz (2.1) is given by

$$\mathcal{L}_{1D} = \mathrm{e}^{2A+2U}\,[\,-A'^2 - 2\,A'\,U' + \frac{1}{4}\,\mathrm{e}^{-2U-4A}\,(\,|\hat{Q}_0|^2 - 3\,g^2\,|\hat{h}_0|^2\mathrm{e}^{4A}\,)\,]\,. \tag{B.1}$$

We now perform a first-order rewriting for this Lagrangian inspired by the corresponding rewriting for black holes in minimal gauged supergravity, as done in [149]. Denoting $\phi^1 = A$, $\phi^2 = U$, $W = \mathrm{e}^{U}\,(\,|\hat{Q}_0|^2 + g^2\,|\hat{h}_0|^2\,\mathrm{e}^{4A} + \gamma\,\mathrm{e}^{A}\,)$,[1] and the matrix, $m = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, we can write the above Lagrangian density as

$$\mathcal{L}_{1D} = -\,\mathrm{e}^{2A+2U}\,[\,\phi^{a\prime}\,m_{ab}\,\phi^{b\prime} + \frac{1}{4}\,\mathrm{e}^{-4U-4A}\,m^{ab}\,W_a\,W_b\,]\,. \tag{B.2}$$

Here, $W_1 = W_A$ is the derivative of W w.r.t A and $W_2 = W_U$ is the derivative of W w.r.t U, given explicitly as

$$\begin{aligned} W_A &= \frac{\mathrm{e}^{2U}}{2\,W}\,(\,4\,g^2\,|\hat{h}_0|^2\,\mathrm{e}^{4A} + \gamma\,\mathrm{e}^{A}\,)\,, \\ W_U &= W\,. \end{aligned} \tag{B.3}$$

The first-order rewriting is then simply,

$$\mathcal{L}_{1D} = -\,\mathrm{e}^{2A+2U}\,[m_{ab}\,(\,\phi^{a\prime} - \frac{1}{2}\,\mathrm{e}^{-2U-2A}\,m^{ac}\,W_c\,)\,(\,\phi^{b\prime} - \frac{1}{2}\,\mathrm{e}^{-2U-2A}\,m^{bd}\,W_d\,)\,] - \phi^{a\prime}\,W_a\,. \tag{B.4}$$

[1] Here γ is an arbitrary real number.

Using the chain-rule of derivatives on $W = W(\phi^a)$, the last term in the above equation becomes $\phi^{a\prime} W_a = W'$. Hence the first-order rewritten Lagrangian can finally be written as

$$\mathcal{L}_{1D} = -\,\mathrm{e}^{2A+2U}\,[m_{ab}\,(\,\phi^{a\prime} - \frac{1}{2}\,\mathrm{e}^{-2U-2A}\,m^{ac}\,W_c\,)\,(\,\phi^{b\prime} - \frac{1}{2}\,\mathrm{e}^{-2U-2A}\,m^{bd}\,W_d\,)\,] - W'\,. \tag{B.5}$$

The first-order equations for the metric functions can be written as

$$A' = \frac{1}{2}\,\mathrm{e}^{-2U-2A}\,W_U\,, \tag{B.6}$$

$$U' = \frac{1}{2}\,\mathrm{e}^{-2U-2A}\,(\,W_A - W_U\,)\,. \tag{B.7}$$

The Hamiltonian density can be written as $\mathcal{H} = \mathcal{L}_{1D} + [\,\mathrm{e}^{2(A+U)}\,(\,2\,A'\,)\,]$. On-shell, using the first-order equations, this can be written as $\mathcal{H} = \mathcal{L}_{1D} + [\,\mathrm{e}^{2(A+U)}\,(\,W'\,)\,]$. Substituting for $\mathcal{L}_{1D}$ from (B.6), we see that the on-shell Hamiltonian density becomes

$$\mathcal{H} = -\,\mathrm{e}^{2A+2U}\,[m_{ab}\,(\,\phi^{a\prime} - \frac{1}{2}\,\mathrm{e}^{-2U-2A}\,m^{ac}\,W_c\,)\,(\,\phi^{b\prime} - \frac{1}{2}\,\mathrm{e}^{-2U-2A}\,m^{bd}\,W_d\,)\,]\,. \tag{B.8}$$

This is trivially zero on-shell as the perfect squares vanish due to the first-order equationss. Hence the Hamiltonian constraint from General Relativity is satisfied for field configurations obeying these first-order equations. This completes the first-order rewriting for the Lagrangian in minimally gauged supergravity. All smooth black brane solutions to these first-order first-order equationss are necessarily non-supersymmetric. The first non-supersymmetric black brane solutions in minimally gauged supergravity were written down in [8] where it was also shown that the supersymmetric solutions are singular.

There is a different rewriting of this Lagrangian which gives an $AdS_2 \times \mathbb{R}^2$ background, given below as

$$\begin{aligned}\mathcal{L}_{1D} = & -[\,(\,(\mathrm{e}^{A})'\,)^2\,\mathrm{e}^{2U} + 2\,\mathrm{e}^{A+U}\,(\,(\mathrm{e}^{A})'\,)\,(\,\mathrm{e}^{U'} - \frac{1}{\sqrt{v_1}}\,) \\ & + 2\,(\,(\mathrm{e}^{A})'\,)\,\mathrm{e}^{A+U}\,\frac{1}{\sqrt{v_1}} - \frac{1}{4}\,\mathrm{e}^{-2A}\,(\,|\hat{Q}_0|^2 - 3\,g^2\,|\hat{h}_0|^2\mathrm{e}^{4A}\,)\,]\,.\end{aligned} \tag{B.9}$$

The first-order first-order equationss following from the rewriting above are

$$(\,\mathrm{e}^{A}\,)' = 0\,, \tag{B.10}$$

$$(\,\mathrm{e}^{U}\,)' = \frac{r}{\sqrt{v_1}}\,, \tag{B.11}$$

with $v_1 = 2\frac{\mathrm{e}^{4A}}{|\hat{Q}_0|^2}$. The Hamiltonian density vanishes on-shell provided $\mathrm{e}^{4A} = \frac{|\hat{Q}_0|^2}{3\,g^2\,|\hat{h}_0|^2}$. The $AdS_2 \times \mathbb{R}^2$ background which is the near-horizon black brane geometry in the presence of fluxes, is a solution to these first-order equationss.

Appendix C

Einstein equations in five dimensions

When evaluated on the solution ansatz (6.1), the independent Einstein equations take the following form:
tt-component:

$$
\begin{aligned}
&\frac{1}{2}e^{-2(U+V)}\Big(- e^{2(U+W)}C'^2 - 2e^{2(U+W)}C(2B'C' - C'(U'+V'-3W') + C'') \\
&\qquad +2e^{4U}(2B'U' + U'^2 - U'V' + U'W' + U'') \\
&\qquad -e^{2W}C^2\big(e^{2W}C'^2 + 2e^{2U}(2B'W' + U'W' - V'W' + W'^2 + W'')\big)\Big) \\
&= \ (-e^{2U} + e^{2W}C^2)\left(\frac{1}{3}g^2\left(G^{AB}h_Ah_B - 2(h_AX^A)^2\right) - \frac{1}{6}G_{AB}F^A_{KL}F^{B\,KL}\right) + G_{AB}F^A_{rt}F^B_{rt}e^{-2V}
\end{aligned}
\tag{C.1}
$$

rr-component:

$$
\begin{aligned}
&-2B'^2 + \frac{1}{2}e^{2W-2U}C'^2 - U'^2 + 2B'V' + U'V' + V'W' - W'^2 - 2B'' - U'' - W'' \\
&= \ G_{AB}(X^A)'(X^B)' + e^{2V}\left(\frac{1}{3}g^2\left(G^{AB}h_Ah_B - 2(h_AX^A)^2\right) - \frac{1}{6}G_{AB}F^A_{KL}F^{B\,KL}\right) \\
&\quad +G_{AB}\left(F^A_{rt}F^B_{rt}(-e^{-2U}) + (A^A_z)'(A^B_z)'(e^{-2W} - e^{-2U}C^2) + 2F^A_{rt}(A^B_z)'e^{-2U}C\right)
\end{aligned}
\tag{C.2}
$$

xx-component:

$$
\begin{aligned}
&-e^{2B-2V}\left(2B'^2 + B'U' - B'V' + B'W' + B''\right) = \\
&e^{2B}\left(\frac{1}{3}g^2\left(G^{AB}h_Ah_B - 2(h_AX^A)^2\right) - \frac{1}{6}G_{AB}F^A_{KL}F^{B\,KL}\right) + G_{AB}P^AP^Be^{-2B}
\end{aligned}
\tag{C.3}
$$

zz-component:

$$\begin{aligned} &\frac{1}{2}e^{2W-2U-2V}\left(-e^{2W}C'^2 - 2e^{2U}(2B'W' + U'W' - W'V' + W'^2 + W'')\right) \\ &= \quad e^{2W}\left(\frac{1}{3}g^2\left(G^{AB}h_Ah_B - 2(h_AX^A)^2\right) - \frac{1}{6}G_{AB}F^A_{KL}F^{B\,KL}\right) + G_{AB}(A^A_z)'(A^B_z)'e^{-2V} \end{aligned} \tag{C.4}$$

tz-component:

$$\begin{aligned} &\frac{1}{2}e^{2W-2U-2V}\Bigg(-2e^{2U}(B'C' + 2CB'W') + e^{2U}(C'U' + C'V' - 3C'W' - C'') - e^{2W}CC'^2 \\ &\qquad +2e^{2U}C(U'W' - V'W' + W'^2 + W'')\Bigg) \\ &= \quad e^{2W}C\left(\frac{1}{3}g^2\left(G^{AB}h_Ah_B - 2(h_AX^A)^2\right) - \frac{1}{6}G_{AB}F^A_{KL}F^{B\,KL}\right) + 2G_{AB}F^A_{rt}(A^B_z)'e^{-2V} \end{aligned} \tag{C.5}$$

Appendix D

Relating five- and four-dimensional flow equations

We relate the four-dimensional flow equations for black branes (5.34) in big moduli space to the five-dimensional flow equations (6.21). We set $g = 1$ throughout. For literature on the $4D - 5D$ connection see e.g. [150, 151].

The four-dimensional $N = 2$, $U(1)$ gauged supergravity theory is based on complex scalar fields X^I, $I = 0, \ldots, n$ encoded in the cubic prepotential (with $A = 1, \ldots, n$)

$$F(X) = -\frac{1}{6}\frac{C_{ABC}X^A X^B X^C}{X^0} = (X^0)^2 \mathcal{F}(z) \,, \tag{D.1}$$

where $z^A = X^A/X^0$ denote the physical scalar fields and

$$\mathcal{F}(z) = -\frac{1}{6}C_{ABC}z^A z^B z^C \,. \tag{D.2}$$

Differentiating with respect to z^A yields

$$\begin{aligned} \mathcal{F}_A &= -\frac{1}{2}C_{ABC}z^B z^C \,, \\ \mathcal{F}_{AB} &= -C_{ABC}z^C \,, \end{aligned} \tag{D.3}$$

where $\mathcal{F}_A = \partial\mathcal{F}/\partial z^A$, etc. The Kähler potential $K(z, \bar{z})$ is determined in terms of $\mathcal{F}$ by

$$\begin{aligned} e^{-K} &= i\left(2(\mathcal{F} - \bar{\mathcal{F}}) - (z^A - \bar{z}^A)(\mathcal{F}_A + \bar{\mathcal{F}}_A)\right) \\ &= \frac{i}{6}C_{ABC}(z^A - \bar{z}^A)(z^B - \bar{z}^B)(z^C - \bar{z}^C) \,. \end{aligned} \tag{D.4}$$

The Kähler metric $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K(z, \bar{z})$ can be expressed as

$$g_{A\bar{B}} = K_A K_{\bar{B}} - ie^K \left(\mathcal{F}_{AB} - \bar{\mathcal{F}}_{AB}\right) \,, \tag{D.5}$$

where $K_A = \partial K/\partial z^A$ and

$$K_A = -K_{\bar{A}} = -e^K \tfrac{i}{2} C_{ABC}(z^B - \bar{z}^B)(z^C - \bar{z}^C) \,. \tag{D.6}$$

In the following we pick the gauge $X^0(z) = 1, X^A(z) = z^A$ (with $X^I(z) \equiv X^I e^{-K/2}$), so that the complex scalar fields X^I and the z^A are related by $X^0 = e^{K/2}$ and $X^A = e^{K/2}\, z^A$.

Using the dictionary (6.55) that relates the quantities appearing in the four- and five-dimensional flow equations, in particular $z^A - \bar{z}^A = 2ie^W\, X^A_{\rm 5d}$, we obtain

$$e^{-K} = 8v\, e^{3W} \tag{D.7}$$

as well as

$$\begin{aligned} K_{\bar{A}} &= -\frac{3i}{2v}\, e^{-W}\, X_A^{\rm 5d}\,, \\ g_{A\bar{B}} &= 4ve^{K+W} G_{AB} = \frac{1}{2} e^{-2W} G_{AB}\,, \end{aligned} \tag{D.8}$$

where G_{AB} denotes the target space metric in five dimensions, cf. (6.54), and the $X_A^{\rm 5d}$ were defined in (6.53). The factor of v in (D.7) arises due to the normalization in (6.52). Using these expressions, we establish

$$g^{A\bar{B}}\, K_{\bar{B}} = -(z^A - \bar{z}^A)\,. \tag{D.9}$$

In the big moduli space, the four-dimensional flow equations were expressed in terms of rescaled complex scalar fields Y^I given by $Y^0 = |Y^0|e^{i\alpha}$ and $Y^A = Y^0\, z^A$, where $|Y^0| = e^{K/2+\psi-U_4}$. On a solution to the four-dimensional flow equations we can relate the phase α to the phase γ that enters in the four-dimensional flow equations. We obtain $\alpha = -\gamma$, which we establish as follows. Writing $e^{2i\alpha} = Y^0/\bar{Y}^0$ we get

$$\alpha' = -\frac{i}{2} e^{-2i\alpha} \left(\frac{(Y^0)'}{\bar{Y}^0} - (\bar{Y}^0)' \frac{Y^0}{(\bar{Y}^0)^2} \right) = -\frac{i}{2\bar{Y}^0} \left(e^{-2i\alpha} (Y^0)' - (\bar{Y}^0)' \right). \tag{D.10}$$

The flow equation for Y^0 reads (cf. (5.34))

$$\begin{aligned} (Y^0)' &= e^{\psi - U_4}\, N^{0J}\, \bar{q}_J \\ &= e^{\psi - U_4 + K} \Big[g^{A\bar{B}}\, K_A\, \bar{q}_{\bar{J}}\, (\partial_{\bar{B}} + K_{\bar{B}})\, \bar{X}^{\bar{J}}(\bar{z}) - \bar{q}_{\bar{J}}\, \bar{X}^{\bar{J}}(\bar{z}) \Big] \,, \end{aligned} \tag{D.11}$$

where we used

$$N^{IJ} = e^K \Big[g^{A\bar{B}}\, (\partial_A + \partial_A K) X^I(z)\, (\partial_{\bar{B}} + \partial_{\bar{B}} K) \bar{X}^J(\bar{z}) - X^I(z) \bar{X}^J(\bar{z}) \Big] \,, \tag{D.12}$$

cf. for instance [125]. In (D.11) the q_I denote the four-dimensional quantities

$$q_I = e^{U_4 - 2\psi + i\gamma} \left(\hat{Q}_I - i e^{2(\psi - U_4)}\, \hat{h}_I \right)\,. \tag{D.13}$$

The quantities $\hat{Q}_I$ and $\hat{h}_I$ are combinations of the four-dimensional charges and fluxes given by (cf. (5.27),(5.28))

$$\begin{aligned} \hat{Q}_I &= Q_I - F_{IJ}\, P^J\,, \\ \hat{h}_I &= h_I - F_{IJ}\, h^J\,. \end{aligned} \tag{D.14}$$

For later use, we also introduce the quantities

$$\begin{aligned} Z(Y) &= -\hat{Q}_I\, Y^I \ , \\ W(Y) &= -\hat{h}_I\, Y^I \ . \end{aligned} \tag{D.15}$$

Inserting the flow equation (D.11) in (D.10) yields

$$\begin{aligned} \alpha' = & -\frac{i}{2} e^{i\alpha} e^{K/2} \Big[\Big(g^{A\bar{B}} K_A K_{\bar{B}} - 1 \Big) \Big(e^{-2i\alpha} (\bar{q}_0 + \bar{q}_{\bar{C}} \bar{z}^{\bar{C}}) - (q_0 + q_C z^C) \Big) \\ & \qquad + g^{A\bar{B}} (e^{-2i\alpha} K_A \bar{q}_{\bar{B}} - q_A K_{\bar{B}}) \Big] \ , \end{aligned} \tag{D.16}$$

where we used the relation

$$\bar{q}_J \left(\partial_{\bar{B}} + K_{\bar{B}} \right) \bar{X}^J(z) = \bar{q}_{\bar{B}} + K_{\bar{B}} \left(\bar{q}_0 + \bar{q}_A \bar{z}^A \right) \tag{D.17}$$

as well as $|Y^0| = e^{K/2+\psi-U_4}$. Next, using that on a four-dimensional solution we have $q_I Y^I = \bar{q}_I \bar{Y}^{\bar{I}}$, we obtain

$$e^{-2i\alpha} \left(\bar{q}_0 + \bar{q}_{\bar{C}} \bar{z}^{\bar{C}} \right) = q_0 + q_C z^C \ . \tag{D.18}$$

Inserting this in (D.16) results in

$$\alpha' = -\frac{i}{2} e^{i\alpha} e^{K/2} g^{A\bar{B}} \left(e^{-2i\alpha} K_A \bar{q}_{\bar{B}} - q_A K_{\bar{B}} \right) \ . \tag{D.19}$$

Using (D.9), we obtain

$$\begin{aligned} g^{A\bar{B}} \left(e^{-2i\alpha}\, K_A\, \bar{q}_{\bar{B}} - q_A\, K_{\bar{B}} \right) &= \left(z^A - \bar{z}^A \right) \left(e^{-2i\alpha}\, \bar{q}_{\bar{A}} + q_A \right) \\ &= e^{-2i\alpha} z^A \bar{q}_{\bar{A}} - \bar{z}^A q_A - e^{-2i\alpha} \bar{z}^A \bar{q}_{\bar{A}} + q_A z^A \\ &= e^{-2i\alpha} (\bar{q}_0 + z^A \bar{q}_{\bar{A}}) - (q_0 + \bar{z}^A q_A) \ , \end{aligned} \tag{D.20}$$

where we used (D.18) in the last equality. Now we notice that

$$q_I \bar{z}^I = q_0 + \bar{z}^A q_A = -e^{i(\alpha+\gamma)} e^{-K/2+2U_4-3\psi} \left(\bar{Z}(\bar{Y}) - i e^{2(\psi-U_4)} \bar{W}(\bar{Y}) \right) \tag{D.21}$$

and

$$e^{-2i\alpha} \bar{q}_{\bar{I}} z^I = e^{-2i\alpha} (\bar{q}_0 + z^A \bar{q}_{\bar{A}}) = -e^{-i(3\alpha+\gamma)} e^{-K/2+2U_4-3\psi} \left(Z(Y) + i e^{2(\psi-U_4)} W(Y) \right) , \tag{D.22}$$

so that

$$\alpha' = -e^{2U_4-3\psi}\, \mathrm{Im} \left(e^{-i(2\alpha+\gamma)} Z(Y) \right) - e^{-\psi}\, \mathrm{Re} \left(e^{-i(2\alpha+\gamma)} W(Y) \right) \ . \tag{D.23}$$

This is precisely the flow equation for γ, provided $\alpha = -\gamma$.

Using this result, we now relate the flow equations for the z^A to the five-dimensional flow equations for $X^A_{5\mathrm{d}}$ and A^A_z. Using the four-dimensional flow equations for Y^0 and Y^A we obtain

$$\begin{aligned}(z^A)' &= \frac{1}{Y^0}\left((Y^A)' - z^A (Y^0)'\right)\\ &= \frac{e^{\psi - U_4}}{Y^0}\left(N^{AJ} - z^A N^{0J}\right)\bar{q}_J\\ &= e^{-\frac{K}{2}+i\gamma}\left(N^{AJ} - z^A N^{0J}\right)\bar{q}_J\,. \end{aligned} \tag{D.24}$$

Then, using (D.12), one derives

$$N^{AJ} - z^A N^{0J} = e^K g^{A\bar{B}}\left(\partial_{\bar{B}} + K_{\bar{B}}\right)\bar{X}^J(z)\,, \tag{D.25}$$

which implies

$$(z^A)' = e^{\frac{K}{2}+i\gamma} g^{A\bar{B}}\bar{q}_J\left(\partial_{\bar{B}} + K_{\bar{B}}\right)\bar{X}^J(z)\,. \tag{D.26}$$

Now we specialize to four-dimensional solutions that are supported by electric charges Q_I, magnetic charges P^A and electric fluxes h_A. Decomposing $z^A = C^A + i\hat{X}^A$ and using the expression (D.13) gives

$$\begin{aligned}\bar{q}_0 + \bar{q}_A\bar{z}^A = e^{U_4 - 2\psi - i\gamma}\Big[&Q_0 + \frac{1}{2}C_{ABC}P^A C^B C^C + Q_A C^A - \frac{1}{2}C_{ABC}P^A\hat{X}^B\hat{X}^C\\ &+ e^{2(\psi-U_4)}h_A\hat{X}^A + i\left(-Q_A\hat{X}^A - C_{ABC}P^A C^B\hat{X}^C + e^{2(\psi-U_4)}h_A C^A\right)\Big]\,. \end{aligned} \tag{D.27}$$

Then, using (D.17) and (D.8) leads to (we recall (6.58))

$$\begin{aligned}(z^A)' = 2e^{\frac{K}{2}+U_4-2\psi-4\phi}G^{AB}\Big[&-12ie^{K-4\phi}X_B^{5\mathrm{d}}\Big(Q_0 + \frac{1}{2}C_{CDE}P^C C^D C^E\\ &+ Q_E C^E - \frac{1}{2}C_{CDE}P^C X^D_{5\mathrm{d}}X^E_{5\mathrm{d}}e^{-4\phi} + e^{2(\psi-U_4)}h_E X^E_{5\mathrm{d}}\,e^{-2\phi}\Big)\\ &+ Q_B + C_{BEF}C^E P^F\\ &+12e^{K-4\phi}X_B^{5\mathrm{d}}\Big(-Q_E X^E_{5\mathrm{d}}e^{-2\phi} - C_{CDE}P^C C^D X^E_{5\mathrm{d}}e^{-2\phi} + e^{2(\psi-U_4)}h_E C^E\Big)\\ &+ i\left(e^{2(\psi-U_4)}h_B - C_{BEF}P^E X^F_{5\mathrm{d}}e^{-2\phi}\right)\Big]\,. \end{aligned} \tag{D.28}$$

Thus, we obtain for the real part,

$$\begin{aligned}(C^A)' = 2e^{\frac{K}{2}+U_4-2\psi-4\phi}G^{AB}\Big[&Q_B + C_{BEF}C^E P^F\\ &+12e^{K-4\phi}X_B^{5\mathrm{d}}\Big(-Q_E\hat{X}^E - C_{CDE}P^C C^D\hat{X}^E + e^{2(\psi-U_4)}h_E C^E\Big)\Big]\,. \end{aligned} \tag{D.29}$$

Next we show that the second line of this equation vanishes by virtue of the four-dimensional flow constraint

$$\mathrm{Im}\left(e^{i\gamma}Z(Y)\right) - e^{2(\psi-U_4)}\mathrm{Re}\left(e^{i\gamma}W(Y)\right) = 0\,. \tag{D.30}$$

We have

$$Z(Y) = Y^0\left(-\frac{1}{2}C_{ABC}P^A z^B z^C - Q_A z^A - Q_0\right) \tag{D.31}$$

and

$$W(Y) = -Y^0 h_A z^A\,. \tag{D.32}$$

This leads to

$$\mathrm{Im}\left(e^{i\gamma}Z(Y)\right) = |Y^0|\left(-C_{ABC}P^A C^B \hat{X}^C - Q_A\hat{X}^A\right) \tag{D.33}$$

as well as

$$\mathrm{Re}\left(e^{i\gamma}\,W(Y)\right) = -|Y^0|h_A C^A\,. \tag{D.34}$$

This gives

$$\begin{aligned} &|Y^0|\left(-C_{ABC}P^A C^B\hat{X}^C - Q_A\hat{X}^A + e^{2(\psi-U_4)}h_A C^A\right) = \\ &\qquad = \mathrm{Im}\left(e^{i\gamma}Z(Y)\right) - e^{2(\psi-U_4)}\mathrm{Re}\left(e^{i\gamma}W(Y)\right)\,, \end{aligned} \tag{D.35}$$

which vanishes due to (D.30), so that (D.29) becomes

$$(C^A)' = \frac{1}{\sqrt{2}v}e^{V-2B}G^{AB}\left(Q_B + C_{BEF}C^E P^F\right)\,, \tag{D.36}$$

where we used the relations (6.59) and (D.7).

For the imaginary part of $(z^A)'$ we get

$$\begin{aligned} \left(e^{-2\phi}X^A_{\mathrm{5d}}\right)' \;=\; 2e^{\frac{K}{2}+U_4-2\psi-4\phi}\Bigg[&-e^{2\phi}X^A_{\mathrm{5d}}\Bigg(Q_0 + \frac{1}{2}C_{BCD}P^B C^C C^D + Q_B C^B \\ &- \frac{1}{2}C_{BCD}P^B X^C_{\mathrm{5d}}X^D_{\mathrm{5d}}e^{-4\phi} + e^{2(\psi-U_4)}h_B X^B_{\mathrm{5d}}e^{-2\phi}\Bigg) \\ &+ G^{AB}\left(e^{2(\psi-U_4)}h_B - C_{BEF}P^E X^F_{\mathrm{5d}}e^{-2\phi}\right)\Bigg]\,. \end{aligned} \tag{D.37}$$

Using (6.54) we obtain

$$-\,C_{ABC}X^C_{\mathrm{5d}} = 2vG_{AB} - \frac{9}{v}X^{\mathrm{5d}}_A X^{\mathrm{5d}}_B\,. \tag{D.38}$$

Contracting this expression once with $P^A X^B_{\mathrm{5d}}$ and once with P^A, we rewrite the two expressions containing $C_{BCD}P^B X^C_{\mathrm{5d}}X^D_{\mathrm{5d}}$ and $C_{BEF}P^E X^F_{\mathrm{5d}}$ in (D.37). Using (D.7) as well we obtain

$$
\begin{aligned}
\left(e^{-2\phi}X^A_{5\mathrm{d}}\right)' &= \frac{1}{\sqrt{2v}}e^{U_4-2\psi-\phi}\Big[-e^{2\phi}X^A_{5\mathrm{d}}\left(Q_0+\frac{1}{2}C_{BCD}P^BC^CC^D+Q_BC^B\right)\\
&\qquad +2vP^Ae^{-2\phi}-3X^A_{5\mathrm{d}}(X^{5\mathrm{d}}_BP^B)e^{-2\phi}\\
&\qquad +e^{2(\psi-U_4)}G^{AB}\left(h_B-\frac{3}{2v}X^{5\mathrm{d}}_B(h_CX^C_{5\mathrm{d}})\right)\Big]\\
&= \left(e^{-2\phi}\right)'X^A_{5\mathrm{d}}+e^{-2\phi}(X^A_{5\mathrm{d}})'\,.
\end{aligned}
\tag{D.39}
$$

Using $X^{5\mathrm{d}}_A(X^A_{5\mathrm{d}})'=0$ (and (6.58), (6.59)) we infer

$$
\begin{aligned}
\left(e^{-2\phi}\right)' &= \frac{1}{\sqrt{2v}}e^{U_4-2\psi-\phi}\Big[-e^{2\phi}\left(Q_0+\frac{1}{2}C_{BCD}P^BC^CC^D+Q_BC^B\right)\\
&\qquad -e^{-2\phi}X^{5\mathrm{d}}_BP^B-\frac{1}{3}e^{2(\psi-U_4)}h_CX^C_{5\mathrm{d}}\Big]\\
&= -\frac{1}{\sqrt{2v}}\Big[e^{-2B-2W+V-2\phi}\left(Q_0+\frac{1}{2}C_{BCD}P^BC^CC^D+Q_BC^B\right)\\
&\qquad +e^{V-2B-2\phi}X^{5\mathrm{d}}_BP^B+\frac{1}{3}e^{-U}h_CX^C_{5\mathrm{d}}\Big]
\end{aligned}
\tag{D.40}
$$

as well as

$$
\begin{aligned}
(X^A_{5\mathrm{d}})' &= \frac{1}{\sqrt{2v}}e^{U_4-2\psi+\phi}\Big[2e^{-2\phi}\left(vP^A-X^A_{5\mathrm{d}}(X^{5\mathrm{d}}_BP^B)\right)\\
&\qquad +e^{2(\psi-U_4)}G^{AB}\left(h_B-\frac{1}{v}X^{5\mathrm{d}}_B(h_CX^C_{5\mathrm{d}})\right)\Big]\,.
\end{aligned}
\tag{D.41}
$$

The former should match the flow equation for e^W. To check this, we note that the five-dimensional flow equations (6.21) imply

$$
U'+W'=\frac{2}{3}h_AX^A_{5\mathrm{d}}e^V+\frac{1}{v}X^{5\mathrm{d}}_AP^Ae^{V-2B}\,.
\tag{D.42}
$$

Subtracting this from the third equation of (6.21) gives

$$
W'=\frac{1}{3}h_AX^A_{5\mathrm{d}}e^V+\frac{1}{2v}X^{5\mathrm{d}}_AP^Ae^{V-2B}-\frac{1}{2}\hat{J}e^{-2B-2W+V}\,,
\tag{D.43}
$$

so that

$$
\left(e^W\right)'=e^{-2\phi}W'=\frac{1}{3}h_AX^A_{5\mathrm{d}}e^{V-2\phi}+\frac{1}{2v}X^{5\mathrm{d}}_AP^Ae^{V-2B-2\phi}-\frac{1}{2}\hat{J}e^{-2B-2W+V-2\phi}\,.
\tag{D.44}
$$

This matches (D.40) if we set $v=1/2$ and perform the identifications (6.55). Under these identifications, the flow equations (D.36) and (D.41) precisely match those for A^A_z and $X^A_{5\mathrm{d}}$ appearing in (6.21). Similarly, the flow equations for the four-dimensional warp factors U_4 and ψ match those of the five-dimensional warp factors U and B using (6.59).

Appendix E

A different first-order rewriting in five dimensions

We present a different first-order rewriting that allows for solutions with electric fields. This rewriting is the one performed in [12] for static black hole solutions, which we adapt to the case of stationary black branes in the presence of magnetic fields.

We consider the metric (6.1) and the gauge field ansatz (6.2) with $A^A_z = 0$, so that $\hat{q}_A = Q_A$ and $\hat{J} = J$. The starting point of the analysis is therefore Lagrangian (6.18), with $A^A_z = 0$, $\hat{q}_A = Q_A$ and $\hat{J} = J$.

We perform the following g-split of U and V,

$$\begin{aligned} U &= U_0 + \frac{1}{2}\log f \,, \\ V &= V_0 - \frac{1}{2}\log f \,, \\ f &= f_0(r) + g^2 f_2(r) = -\mu\, r^2 + g^2 e^{2U_2(r)} \,, \\ V_0 &= 2B(r) + W(r) + U_0(r) + \log(r) \,. \end{aligned} \tag{E.1}$$

In addition, we perform the rescaling

$$P^A = g\, p^A \quad , \quad J = g\, j \,, \tag{E.2}$$

and we organize the terms in the Lagrangian into powers of g. This yields $\mathcal{L} = \mathcal{L}_0 + g^2\, \mathcal{L}_2$. First, we analyze $\mathcal{L}_0$,

$$\begin{aligned} \mathcal{L}_0 \;=\; e^{2B+W+U_0-V_0} f_0 \Big(& 2B'^2 + 2U'W' + 4B'W' + 4B'U' - G_{AB}(X^A)'(X^B)' \\ & - \frac{1}{4} G^{AB} Q_A Q_B\, e^{-4B-2W+2V_0} (f_0)^{-1} \Big) \,. \end{aligned} \tag{E.3}$$

We perform a first-order rewriting of $\mathcal{L}_0$ by introducing parameters $\tilde{q}_A$ and γ_A that are related to the electric charges q_A by

$$\frac{1}{4} G^{AB} Q_A Q_B = -\mu\, G^{AB}\, \tilde{q}_A \gamma_B \,. \tag{E.4}$$

We obtain

$$\begin{aligned}
\mathcal{L}_0 &= -\mu\, r\, e^{U_0}\tilde{q}_A G^{AB}\left(6X_B + e^{U_0}(r^2\tilde{q}_B - \gamma_B)\right) \\
&\quad + e^{2B+W+U_0-V_0} f_0 \Big[\frac{1}{2}(2B' + U_0')(2B' + 4W' + 3U_0') \\
&\qquad - \frac{9}{4} G^{AB}\big(X_A' - U_0' X_A + \frac{2}{3} e^{-2B-W+V_0}\tilde{q}_A\big)\big(X_B' - U_0' X_B + \frac{2}{3} e^{-2B-W+V_0}\tilde{q}_B\big)\Big] \\
&\quad -2\left(e^{U_0}\tilde{q}_A X^A f_0\right)' - \mu(2W' + 4B')\,.
\end{aligned} \tag{E.5}$$

This yields the first-order flow equations

$$\begin{aligned}
X_A' &= U_0' X_A - \frac{2}{3} e^{-2B-W+V_0}\tilde{q}_A\,, \\
B' &= -\frac{1}{2} U_0'\,, \\
2B' &= -4W' - 3U_0'\,.
\end{aligned} \tag{E.6}$$

Using (E.1) as well as $X^A (X_A)' = 0$, these equations yield

$$\begin{aligned}
W' &= B' = -\frac{1}{2} U_0' \\
\left(e^{-U_0}\right)' &= -\frac{2}{3} r\, X^A \tilde{q}_A\,, \\
\left(e^{-U_0} X_A\right)' &= -\frac{2}{3}\, r\, \tilde{q}_A\,.
\end{aligned} \tag{E.7}$$

Integrating the latter gives

$$e^{-U_0} X_A = \frac{1}{3} H_A \quad , \quad H_A = \tilde{\gamma}_A - r^2\, \tilde{q}_A\,, \tag{E.8}$$

where $\tilde{\gamma}_A$ denote integration constants. Contracting this with X^A results in

$$e^{-U_0} = \frac{1}{3} H_A X^A\,, \tag{E.9}$$

which satisfies the second equation of (E.7) by virtue of $X_A (X^A)' = 0$.

$\mathcal{L}_0$ contains, in addition, the first line, which is not the square (or the sum of squares) of a first-order flow equation. Its variation with respect to U_0 gives

$$\tilde{q}_A\, G^{AB}\left(3X_B + e^{U_0}\,\left(r^2\,\tilde{q}_B - \gamma_B\right)\right) = 0\,. \tag{E.10}$$

Comparing with (E.8) yields

$$\tilde{q}_A\, G^{AB}\,(\tilde{\gamma}_B - \gamma_B) = 0\,. \tag{E.11}$$

Since G^{AB} is positive definite, we conclude that this can only be fulfilled for arbitrary values of $\tilde{q}_A$ if $\tilde{\gamma}_B = \gamma_B$. On the other hand, varying the first line of $\mathcal{L}_0$ with respect to X^A and using (E.8) gives

$$-\,2\mu\, r\, e^{U_0}\,\tilde{q}_A\left(2\,\delta X^A + G^{AC}\,\delta G_{CD}\, X^D\right)\,, \tag{E.12}$$

which vanishes by virtue of $\delta G_{CD}\, X^D = 3\,\delta X_C = -2\, G_{CD}\,\delta X^D$. Thus, we conclude that the set of variational equations derived from $\mathcal{L}_0$ is consistent.

Now we turn to $\mathcal{L}_2$,

$$
\begin{aligned}
\mathcal{L}_2 \;=\; & e^{2B+W+U_0-V_0+2U_2}\Bigg[-\frac{1}{2}\left(j\,e^{-2B-2W+V_0-U_2}+(W'-(U_0'+U_2'))\right)^2 \\
& \qquad -\left(B'-\frac{1}{2}(U_0'+U_2'+W')+\frac{3}{2}X_Ap^A\,e^{-2B+V_0-U_2}\right)^2 \\
& \qquad +\frac{1}{3}\left[3\left(B'+\frac{1}{2}(U_0'+U_2'+W')\right)-2X^Ah_Ae^{V_0-U_2}+\frac{3}{2}X_A\,p^A\,e^{-2B+V_0-U_2}\right]^2 \\
& -G_{AB}\left(X'^A-\left[\frac{2}{3}X^C(h_Ce^{V_0-U_2}+G_{CD}p^De^{-2B+V_0-U_2})X^A-G^{AC}(h_Ce^{V_0-U_2}+G_{CD}p^De^{-2B+V_0-U_2})\right]\right) \\
& \left(X'^B-\left[\frac{2}{3}X^E(h_Ee^{V_0-U_2}+G_{EF}p^Fe^{-2B+V_0-U_2})X^B-G^{BE}(h_Ee^{V_0-U_2}+G_{EF}p^Fe^{-2B+V_0-U_2})\right]\right)\Bigg] \\
& +2\left(e^{2B+W+U_0+U_2}\left(X^Ah_A-\frac{3}{2}X_Ap^A\,e^{-2B}\right)\right)' \\
& -\left(j\,e^{-W+U_0+U_2}\right)'+2e^{W+U_0+V_0}\,h_Ap^A\,.
\end{aligned}
\tag{E.13}
$$

This yields the first-order flow equations

$$
\begin{aligned}
(X^A)' &= \frac{2}{3}X^C(h_Ce^{V_0-U_2}+G_{CD}p^De^{-2B+V_0-U_2})X^A-G^{AC}(h_Ce^{V_0-U_2}+G_{CD}p^De^{-2B+V_0-U_2})\,, \\
W' &= U_0'+U_2'-j\,e^{-2B-2W+V_0-U_2}\,, \\
0 &= B'-\frac{1}{2}(U_0'+U_2'+W')+\frac{3}{2}X_Ap^A\,e^{-2B+V_0-U_2}\,, \\
0 &= 3\left(B'+\tfrac{1}{2}(U_0'+U_2'+W')\right)-2X^Ah_Ae^{V_0-U_2}+\frac{3}{2}X_A\,p^A\,e^{-2B+V_0-U_2}\,,
\end{aligned}
\tag{E.14}
$$

as well as the constraint

$$h_Ap^A=0\,. \tag{E.15}$$

We have thus derived two sets of first-order flow equations (one derived from $\mathcal{L}_0$ and the other derived from $\mathcal{L}_2$) that need to be mutually consistent. Consistency of these two sets implies certain relations which we now derive.

Adding the third and fourth equation of (E.14) gives

$$B'=\frac{1}{3}X^Ah_Ae^{V_0-U_2}-X_A\,p^A\,e^{-2B+V_0-U_2}\,. \tag{E.16}$$

Combining this with the first equation of (E.7) yields

$$U_0'=-\frac{2}{3}X^Ah_Ae^{V_0-U_2}+2X_A\,p^A\,e^{-2B+V_0-U_2}\,. \tag{E.17}$$

Comparing with the second equation of (E.7) yields

$$\frac{1}{3}e^{-2B-W}X^A\tilde{q}_A = -\frac{1}{3}X^A h_A e^{-U_2} + X_A\, p^A\, e^{-2B-U_2}\,. \tag{E.18}$$

Contracting the first equation of (E.14) with G_{AB} results in

$$X'_A = -(\frac{2}{3}X^C h_C e^{V_0-U_2} + X_C p^C e^{-2B+V_0-U_2})X_A + \frac{2}{3}(h_A e^{V_0-U_2} + G_{AD}p^D e^{-2B+V_0-U_2})\,, \tag{E.19}$$

Using (E.17) gives

$$X'_A = \left(U'_0 - 3X_C p^C e^{-2B+V_0-U_2}\right) X_A + \frac{2}{3}(h_A e^{V_0-U_2} + G_{AD}p^D e^{-2B+V_0-U_2})\,. \tag{E.20}$$

Using $W' = -\frac{1}{2}U'_0$ (from the first equation of (E.7)) in the second equation of (E.14) gives

$$U'_2 = -\frac{3}{2}\,U'_0 + j\, e^{-2B-2W+V_0-U_2}\,, \tag{E.21}$$

which expresses U_2 in terms of U_0.

The second equation of (E.14) can be rewritten as

$$U'_0 + U'_2 + W' = 2W' + j\, e^{-2B-2W+V_0-U_2} \tag{E.22}$$

which, when inserted into the third equation of (E.14), gives

$$-\, j\, e^{-2W} + 3X_A p^A = 0\,. \tag{E.23}$$

Using $-2W = U_0$ (ignoring an additive constant) as well as (E.8), this results in

$$j = H_A\, p^A\ . \tag{E.24}$$

Since the left hand side is constant, we conclude that

$$\tilde{q}_A p^A = 0 \quad , \quad j = \gamma_A p^A\ . \tag{E.25}$$

Finally, comparing the first equation of (E.6) with (E.20)

$$-\,3X_C p^C e^{-2B-U_2} X_A + \frac{2}{3}(h_A e^{-U_2} + G_{AD}p^D e^{-2B-U_2}) + \frac{2}{3}\tilde{q}_A e^{-2B-W} = 0\ . \tag{E.26}$$

Note that the contraction of (E.26) with X^A gives back (E.18). On the other hand, contracting (E.26) with p^A and using (E.15) and (E.25) gives

$$3\left(X_A p^A\right)^2 = \frac{2}{3}p^A G_{AB}\, p^B\ . \tag{E.27}$$

Observe that both sides are positive definite. This relation is, for instance, satisfied for the STU-model $X^1X^2X^3 = 1$. Inserting the relation

$$G_{AB} = -\frac{1}{2}C_{ABC}X^C + \frac{9}{2}X_AX_B \tag{E.28}$$

into (E.27) we obtain

$$X^A C_{ABC}p^B p^C = 0\ . \tag{E.29}$$

Thus, we conclude that the two sets (E.7) and (E.14) are mutually consistent, provided that (E.26) is satisfied and the constraints (E.15) and (E.25) hold.

In the following, we solve the first-order flow equations for the case when $p^A = 0$. Then $j = 0$, so that from (E.21) we obtain $U_2' = -\frac{3}{2}\,U_0'$, which also equals $W' + 2B'$ by virtue of the first equation of (E.7). Thus $U_2 = W + 2B$, up to an additive constant. Then, (E.26) is satisfied provided we set $\tilde{q}_A = -h_A$. Summarizing, when $p^A = 0$, we obtain

$$\begin{aligned}
\frac{1}{4}G^{AB}Q_AQ_B &= \mu\, G^{AB}\, h_A\gamma_B\ ,\\
e^{-U_0}X_A &= \frac{1}{3}\,H_A \quad , \quad H_A = \gamma_A + r^2\, h_A\ ,\\
e^{-U_0} &= \frac{1}{3}\,H_AX^A\ ,\\
B &= W = -\frac{1}{2}U_0\ ,\\
U_2 &= -\frac{3}{2}U_0\ ,\\
V_0 &= -\frac{1}{2}U_0 + \log r\ ,\\
f &= -\mu\, r^2 + g^2 e^{-3U_0}\ ,\\
(e^A)' &= \frac{1}{2}\,e^{2U_0}\, r\, G^{AB}Q_B\ .
\end{aligned} \tag{E.30}$$

This is the black brane analog of the black hole solutions discussed in [12].

For this class of solutions, we now check the Hamiltonian constraint

$$R_{tt} + \frac{\delta\mathcal{L}_M}{\delta g^{tt}} - \frac{1}{2}g_{tt}\,(R + \mathcal{L}_M) = 0\ , \tag{E.31}$$

where $\mathcal{L}_M$ denotes the matter Lagrangian. Using

$$\sqrt{-g}\, g^{tt}\left(R_{tt} - \frac{1}{2}g_{tt}R\right) = -3\, e^{3B+U-V}\left(B'^2 + B'\,U'\right) + 3\left(e^{3B+U-V}\, B'\right)'\ , \tag{E.32}$$

as well as

$$\frac{\delta\mathcal{L}_M}{\delta g^{tt}} - \frac{1}{2}g_{tt}\,\mathcal{L}_M = \frac{1}{2}\,g_{tt}\,\left[e^{-6B}\,G^{AB}Q_AQ_B + e^{-2V}\,G_{AB}X'^AX'^B + g^2\left(G^{AB} - 2X^AX^B\right)h_Ah_B\right]\ , \tag{E.33}$$

where we replaced the electric fields by their charges, we obtain for (E.31),

$$\mathcal{L}_0 + \mathcal{L}_2 - 6\left(e^{3B+U-V}\,B'\right)' = 0\ . \tag{E.34}$$

Imposing the first-order flow equations, this reduces to

$$\begin{aligned}&\left(e^{U_0} f_0\,\tilde{q}_A\,X^A + 3\mu\,B\right)' + g^2\left(e^{3B+U_0+U_2}\,\tilde{q}_A\,X^A\right)' + 3\left(e^{3B+U_0-V_0}\,f\;B'\right)'\\&+\mu\,r\,e^{U_0}\,\tilde{q}_A\,X^A = 0\ .\end{aligned} \tag{E.35}$$

Then, using the second equation of (E.7) and

$$3\,r\,B'' = -U_0'\,e^{U_0}\,r^2\,\tilde{q}_A\,X^A - e^{U_0}\,r\,\tilde{q}_A\,X^A - e^{U_0}\,r^2\,\tilde{q}_A\,X'^A\ , \tag{E.36}$$

we find that (E.35) is satisfied on a solution to the first-order flow equations. Thus, the Hamiltonian constraint (E.31) does not lead to any further constraint.

The class of static solutions (E.30) was obtained long time ago in [13] by solving the equations of motion. Their mass density is determined by μ. Let us consider a black solution, with the horizon located at $f(r_H) = 0$ with $e^{2U_0}(r_H) \neq 0$. Its temperature is then given by

$$T_H = \left[\frac{e^{U_0-V_0}}{4\pi}|f'|\right]_{r=r_H} = \left[\frac{e^{\frac{3}{2}U_0}}{4\pi\,r}|f'|\right]_{r=r_H}\ , \tag{E.37}$$

where we used (E.30) to express V_0 in terms of U_0. The horizon condition $f(r_H) = 0$ gives

$$\mu\,r_H^2 = g^2\,e^{-3U_0(r_H)}\ , \tag{E.38}$$

and hence $r_H \neq 0$. For the solution to be extremal, its temperature has to vanish. Imposing $f'(r_H) = 0$ results in

$$\mu\,r_H + \frac{3}{2}g^2\,e^{-3U_0(r_H)}\,U_0'(r_H) = 0\ . \tag{E.39}$$

Combining both equations gives

$$1 + \frac{3}{2}\,r_H\,U_0'(r_H) = 0\ . \tag{E.40}$$

As an example, consider the STU-model $X^1X^2X^3 = 1$ and take $\gamma_A = c\,h_A$ with $c > 0$, so that $H_A = h_A\,H$ with $H = c + r^2$. Then

$$\begin{aligned}&e^{-3U_0} = H_1H_2H_3 = h_1h_2h_3\,H^3\quad ,\quad X^1 = \frac{(H_2H_3)^{1/3}}{H_1^{2/3}} = \frac{(h_2h_3)^{1/3}}{h_1^{2/3}}\ ,\\&X^2 = \frac{(H_1H_3)^{1/3}}{H_2^{2/3}} = \frac{(h_1h_3)^{1/3}}{h_2^{2/3}}\quad ,\quad X^3 = \frac{(H_1H_2)^{1/3}}{H_3^{2/3}} = \frac{(h_1h_2)^{1/3}}{h_3^{2/3}}\ .\end{aligned} \tag{E.41}$$

The scalars X^A are thus constant. We set $g = 1$ and $\alpha^3 = h_1h_2h_3 > 0$. The horizon condition (E.38) yields

$$\mu\,H(r_H) - \alpha^3 H^3(r_H) = \mu\,c\ , \tag{E.42}$$

while the condition (E.39) gives

$$r_H \left(\mu - 3\alpha^3\, H^2(r_H)\right) = 0 \ , \tag{E.43}$$

which implies (we take $r_H \neq 0$),

$$\mu = 3\alpha^3\, H^2(r_H) \ . \tag{E.44}$$

Inserting (E.44) into (E.42) gives

$$2\alpha^3 H^3(r_H) = \mu\, c \ . \tag{E.45}$$

Combining (E.45) with the first equation of (E.30), which takes the form

$$\mu\, c = \frac{1}{4} \frac{Q_A\, G^{AB}\, Q_B}{h_A\, G^{AB}\, h_B} \ , \tag{E.46}$$

yields the entropy density as

$$\mathcal{S} = \frac{1}{4} e^{3B(r_H)} = \frac{1}{4}\left(\alpha^3 H^3(r_H)\right)^{1/2} = \frac{1}{4}\left(\frac{\mu\, c}{2}\right)^{1/2} = \frac{1}{8}\left(\frac{1}{2}\, \frac{Q_A\, G^{AB}\, Q_B}{h_A\, G^{AB}\, h_B}\right)^{1/2} \ . \tag{E.47}$$

Extremal electric solutions of this type have been considered recently in [14, 10].

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