A NOTE ON THE MOTION OF AN OCEAN BACTERIUM IN A LINEAR SHEAR FLOW

KALVIS M. JANSONS

Abstract. This paper treats a simple model, which can be exactly solved, motivated by the back-and-forth motion of ocean bacteria. In particular, the probability is determined that a bacterium moving randomly along a fluid line through the origin in a linear shear flow hits the origin before time t.

§1. *Introduction*. Many ocean bacteria do not use the much studied 'run-and-tumble' motion, but rather a 'back-and-forth' motion [**3**]. In this study, we consider a simplified system, motivated by the back-and-forth motion of ocean bacteria in a linear shear flow, for which an exact solution is possible. This solution is of some interest in its own right, and would also be useful in testing numerical code for more realistic models.

Luchsinger *et. al.* [**3**] compare the effectiveness of the run-and-tumble motion and the back-and-forth motion in simple linear-flow models of the turbulent marine environment. They conclude that the back-and-forth motion is better at keeping the bacterium in a nutrient rich patch than the run-and-tumble motion would be.

In this study however, we consider the motion of a bacterium in a linear shear flow on a length scale that is much larger than the individual back-and-forth steps and investigate the distance of the bacterium as a function of time from a point on the line of the bacterium's motion, but we ignore rotational diffusion of the bacterium. In effect, we study the effectiveness of this motion as a search strategy.

Other related work considers the energetics of bacterial motion, and in particular the influence of body size [**4**].

§2. *The model*. Throughout, we assume a flow field $v(x, y) = (0, \gamma x)$ in Cartesian coordinates, where γ is the shear rate. It does not appear that the exact result given below extends easily to a general linear flow. We assume a frame in which the point 'food source' is at the origin, so in this frame it is fixed for all time. We further suppose that the back-and-forth motion has sufficiently small steps that we can replace it by a Brownian motion on a line that moves with the flow and passes through the origin.

Both rigid and flexible bodies tumble in a linear shear flow (see [**1**, **2**, **5**]), but elongated bodies, for example a rod-like body, move approximately like a fluid line element for a significant part of the cycle, and we shall consider the motion only in that phase of the motion. For an axisymmetric ellipsoid, with aspect ratio r, the period of the tumbling motion is

$$T = 2\pi(r + r^{-1})/\gamma, \tag{1}$$

and we therefore need to assume that $t \ll T$, where T is the tumbling time for the bacterium (see [**1**]). For a more detailed discussion of the various timescales for the motion of a flexible body in a shear flow, which in their case is a polymer molecule, see [**5**].

Our main goal is to find the probability that such a bacterium, which is moving along a correctly oriented line, has of reaching the origin by time t.

Countless important effects have been ignored, such as drift (or a bias to the motion), rotational diffusion of the bacterium and the finite step size, but the inclusion of any of these would rob us of the exact solution.

Since the bacterium diffuses along a line through the origin that is advected by the shear flow, the angle θ_t of this line to the x axis is changed only by the advection, and we find that

$$\tan\theta_t = \gamma t + \tan\theta_0. \tag{2}$$

If the bacterium hits the origin, then the process is stopped; otherwise, the position (X, Y) of the bacterium satisfies the Itô equation

$$dX_t = \cos(\theta_t)\sigma dW_t, \qquad dY_t = \sin(\theta_t)\sigma dW_t + \gamma X_t dt, \tag{3}$$

where W is standard Brownian motion and σ is a constant. The diffusivity along the line through the origin is $D \equiv \frac{1}{2}\sigma^2$.

The probability $\Omega_t(x, y)$ of hitting the origin by time t for a bacterium, starting at $(X_0, Y_0) = (x, y)$, can also be expressed in the form of a partial differential equation

$$\frac{\partial\Omega}{\partial t} = D\frac{\partial^2\Omega}{\partial r^2} + \gamma x\frac{\partial\Omega}{\partial y}, \tag{4}$$

where $r \equiv (x^2 + y^2)^{1/2}$; the initial condition is

$$\Omega_0(x, y) = \begin{cases} 1, & \text{if } (x, y) = (0, 0), \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

and the boundary conditions are $\Omega_t(0, 0) = 1$ and Ω_t is bounded for all $t \geqslant 0$. However, it is a little easier to use stochastic calculus to solve this problem.

2.1. *The exact solution.* First note that

$$(dX_t)^2 = \sigma^2\cos^2(\theta_t)dt = \gamma^{-1}\sigma^2 d\theta_t. \tag{6}$$

Thus

$$X_t \stackrel{d}{=} \gamma^{-1/2}\sigma B(\theta_t), \tag{7}$$

where B is a standard Brownian motion, and '$\stackrel{d}{=}$' means 'equal in law'. So X is a pure Brownian motion in terms of a 'time' θ_t. (The situation is more complex in the case of general linear flow.)

Changing to plane polar coordinates, the position of the bacterium (R, θ) is given by

$$R_t \stackrel{d}{=} \gamma^{-1/2}\sigma B(\theta_t)\sec\theta_t. \tag{8}$$

Using (2), the distance from the bacterium to the origin can be written as

$$R_t \stackrel{d}{=} \gamma^{-1/2}\sigma\left(1+(\gamma t+\tan\theta_0)^2\right)^{1/2} B(\arctan(\gamma t+\tan\theta_0)). \qquad (9)$$

Provided that $X_0 > 0$, the event of the bacterium hitting the origin is equivalent to $X_t = 0$. So, applying the reflection principle to (7) and using (2), we find that

$$\Omega_t(x,y) = \operatorname{erfc}\left[x\left(4D\gamma^{-1}(\arctan(\gamma t+y/x)-\arctan(y/x))\right)^{-1/2}\right] \qquad (10)$$

for $x > 0$. The solution in $x < 0$ follows from the symmetry $\Omega_t(-x,-y) = \Omega_t(x,y)$, and at $x = 0$ by the obvious limit.

2.2. *The long-time limit of the exact solution.* Also of interest is the probability of ever hitting the origin, namely,

$$\omega(x,y) \equiv \lim_{\gamma t\to\infty} \Omega_t(x,y) = \operatorname{erfc}\left[x\left(4D\gamma^{-1}\left(\frac{\pi}{2}-\arctan(y/x)\right)\right)^{-1/2}\right], \qquad (11)$$

with $D\gamma^{-1}$ fixed.

Notice that $\lim_{x\downarrow 0}\omega(x,y) = 1$ for all y, which is to be expected as the line $x = 0$ is unaffected by the flow, and that $\omega(x,y) < 1$ elsewhere.

§3. *Conclusion.* The analytical result for the probability of a bacterium hitting the origin before time t when moving randomly along a fluid line through the origin is likely to be useful in improving estimates of stages of more complex models of ocean bacterial motion. Also, it should help to determine the benefits of the back-and-forth motion as opposed to the run-and-tumble motion, even though we have considered a somewhat simplified model.

It seems unlikely that exact solutions exist for much more complex, or more realistic, models. It might be possible, however, to include some of the neglected effects, for example, rotational diffusion or a bias to the motion, using asymptotic methods. Even the extension to a general linear flow appears to be algebraically much more complex.

Also, such exact solutions provide useful checks on more realistic numerical models.

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Dr. K. M. Jansons,
Department of Mathematics,
University College London,
Gower Street,
London WC1E 6BT,
U.K.
E-mail: swimming@kalvis.com

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