Robust Disturbance Rejection by the Attractive Ellipsoid Method – Part II: Discrete-time Systems ⋆

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Abstract: This paper presents sufficient conditions for the robust stabilization of discrete-time polytopic systems subject to control constraints and unknown but bounded perturbations. The *attractive ellipsoid method* (AEM) is extended and applied to cope with this problem. To tackle the stabilization problem, new linear matrix inequality (LMI) conditions for robust state-feedback control are developed. These conditions ensure the convergence of state trajectories of the system to a minimal size ellipsoidal set despite the presence of non-vanishing disturbances. The developed LMI conditions for the AEM are extended to deal with the problem of gain-scheduled state-feedback control, where the scheduling parameters governing the time-variant dynamical system are unknown in advance but can be measured in real-time. A feature of the obtained conditions is that the state-space matrices and Lyapunov matrix are separated. The desired robust control laws are obtained by convex optimization. Numerical simulations are given to illustrate the feasibility of the proposed AEM for robust disturbance rejection.

Keywords: Robust control; disturbance rejection; attractive ellipsoid method; gain-scheduling; linear parameter-varying systems; Lyapunov methods.

1. INTRODUCTION

A prominent problem in robust control is the disturbance rejection with the ultimate goal to minimize the influence of uncertainty (e.g., parametric type, unmodeled dynamics, external persistent perturbations) on the performance of dynamical systems. Many techniques have been developed, for example, linear quadratic gaussian control (LQG) and $\mathcal{H}_\infty$ control (Petersen et al., 2000), sliding-mode control (Utkin et al., 2009), robust maximum principle (Boltyanski and Poznyak, 2012), active disturbance rejection control (Guo and Zhao, 2016). Among these techniques, the invariant ellipsoid method (Nazin et al., 2007; Polyak et al., 2008) for linear systems and the more recently developed technique, the *attractive ellipsoid method* (AEM) for nonlinear systems (Gonzalez-Garcia et al., 2009; Poznyak et al., 2014a), use the concept of asymptotically attractive (invariant) ellipsoids. The AEM, which is based on Lyapunov arguments, is a robust control design technique that minimizes the effect of non-vanishing and unmatched perturbations in nonlinear systems. In the presence of non-vanishing disturbances, it is not possible to keep the state of a dynamical system at the origin. The key idea of the AEM is to find a controller and a corresponding *asymptotically attractive ellipsoid* such that state trajectories of the system converge to a small (in a given sense) neighborhood of the origin. These ellipsoidal regions characterise the effect of the exogenous disturbances on state trajectories of the dynamical system.

⋆ A companion paper (García and Ampountolas, 2018) deals with the robust stabilization of continuous-time systems by the AEM.

In this paper, the AEM is extended to the problem of robust constrained stabilization of discrete-time polytopic systems subject to unknown but bounded perturbations. Newly developed conditions for the robust stabilization of continuous-time polytopic systems by the AEM are presented in García and Ampountolas (2018). To the best of our knowledge, in the AEM literature there does not exist LMI synthesis conditions for the robust stabilization state-feedback control problem of discrete-time systems by the AEM. Most of the corresponding literature deals with continuous-time systems (Lozada-Castillo et al. (2013), Poznyak et al. (2014b), Perez et al. (2015)). Therefore this work offers in the literature new LMI stability conditions for AEM-based approaches. The main feature of the obtained conditions is that the state-space matrices and Lyapunov matrices are decoupled allowing parameter-dependent Lyapunov functions to be employed for robust stabilization and performance. The new LMI conditions allow for the application of the AEM to discrete-time linear systems with and without structured uncertainty where the uncertainty is a bounded and convex polytope or ellipsoid. We extend the developed LMI conditions to the problem of gain-scheduled state-feedback control, where the scheduling parameters that govern the time-variant dynamical system are unknown in advance but can be measured in real-time (Shamma and Athans, 1991).

To ensure convergence of system state trajectories to a *minimal ellipsoidal set*, a semidefinite programming (SDP) problem is solved, which determines the minimum size of the corresponding attractive ellipsoid and the required robust state-feedback control. An important characteristic of the proposed approach is that a design parameter, which affects the feasibility of the obtained LMIs, is restricted in the interval $(0, 1]$. This significantly reduces the search space of solutions for the corresponding SDP problem. For continuous-time systems, this parameter is not restricted (García and Ampountolas, 2018). For the specification of this parameter, an Armijo-like rule which is based on successive stepsize reduction is proposed. Numerical examples are given to illustrate the feasibility of the proposed approach.

2. PRELIMINARIES

Notation. For matrices and vectors $(\cdot)^\mathsf{T}$ indicates transpose. For matrix elements $\star$ denotes the transposed symmetric element and $X \preceq 0$ indicates that X is negative semi-definite. For symmetric matrices, $X \prec 0$ indicates that X is negative definite. $\mathbb{S}^n$ denotes the space of square and symmetric real matrices of dimension n. For square matrices trace$(\cdot)$ denotes the trace of $(\cdot)$.

Problem Formulation. Consider a discrete-time linear time-invariant system

$$x_{k+1} = Ax_k + Bu_k + \omega_k, \qquad x_0 \text{ given}, \tag{1}$$

where the pair (A, B) is controllable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the control input, and $\omega_k \in \mathbb{R}^n$ is an unknown but bounded (at each discrete-time instant) perturbation,

$$\omega_k^\mathsf{T} W \omega_k \leq 1, \qquad \forall\, k \geq 0, \tag{2}$$

where the matrix $W \in \mathbb{S}^n \succ 0$ is given. No other constraints are imposed on the perturbation ω_k, however it is not considered to be random.

The objective of this paper is to design a robust state-feedback controller of the form $u_k = Kx_k$, where $K \in \mathbb{R}^{m\times n}$ is a gain matrix, for the system (1) to compensate the influence of external perturbations (2) on the system state such that the closed-loop system trajectories

$$x_{k+1} = (A + BK)\, x_k + \omega_k, \tag{3}$$

converge asymptotically to a minimal size ellipsoid, which includes the origin.

The following definition characterizes this minimal region.

Definition 1. (Ellipsoidal set). An ellipsoid $\mathcal{E}(P, \bar{x}) \subset \mathbb{R}^n$ with center $\bar{x}$ and shape matrix P is a set of the form,

$$\mathcal{E}(P, \bar{x}) := \left\{x \in \mathbb{R}^n : (x - \bar{x})^\mathsf{T} P^{-1} (x - \bar{x}) \leq 1\right\}, \tag{4}$$

where $P \in \mathbb{S}^n$ is a positive definite matrix.

If $\bar{x} = 0$ then the ellipsoid can be written as $\mathcal{E}(P) := \left\{x \in \mathbb{R}^n : x^\mathsf{T} P^{-1} x \leq 1\right\}$, $P = P^\mathsf{T} \succ 0$, and we further assume it is a *controlled invariant set* of (1).

Also, consider the following distance metric from a point x to a set $\mathcal{E}$, $\|x\|_{\mathcal{E}} := \inf_{y\in\mathcal{E}} \|x - y\|, \forall\, x \in \mathbb{R}^n$.

Definition 2. (Asymptotically Attractive Ellipsoid). The set $\mathcal{E}(P)$, is an asymptotically attractive ellipsoid for the system (1) if $\|x(k, x_0)\|_{\mathcal{E}(P)} \to 0$, as $k \to \infty$, for any $x_0 \in \mathbb{R}^n$.

For any initial condition x_0, convergence of state trajectories of system (1) to a minimal size ellipsoid is guaranteed by the asymptotic attractivity of the set $\mathcal{E}(P)$.

3. MAIN RESULTS

3.1 Attractive Ellipsoid Method for Discrete-time Systems

Consider the quadratic Lyapunov function

$$V(x_k) = x_k^\mathsf{T} P^{-1} x_k, \qquad P \succ 0.$$

Its total difference along the trajectories of system (3) is,

$$\begin{aligned} \Delta V(x_k) &= V(x_{k+1}) - V(x_k) \\ &= x_k^\mathsf{T} Q^\mathsf{T} P^{-1} Q x_k + x_k^\mathsf{T} Q^\mathsf{T} P^{-1} \omega_k \\ &\quad + \omega_k^\mathsf{T} P^{-1} Q x_k + \omega_k^\mathsf{T} P^{-1} \omega_k - x_k^\mathsf{T} P^{-1} x_k \end{aligned} \tag{5}$$

for all $(x_k, x_{k+1}, \omega_k) \neq 0$, where $Q := (A + BK)$ and $\omega_k^\mathsf{T} W \omega_k \leq 1$. Define now the augmended vector $z_k^\mathsf{T} = \left[x_k^\mathsf{T}\ \omega_k^\mathsf{T}\right]$, then $\Delta V(x_k) = z_k^\mathsf{T} \Omega_1 z_k < 0$ holds for all $z_k \neq 0$ with,

$$\Omega_1 := \begin{bmatrix} Q^\mathsf{T} P^{-1} Q - P^{-1} & Q^\mathsf{T} P^{-1} \\ P^{-1} Q & P^{-1} \end{bmatrix} \prec 0.$$

Adding and subtracting $\alpha x_k^\mathsf{T} P^{-1} x_k$ and $\alpha \omega_k^\mathsf{T} W \omega_k$ in (5), where $\alpha \in (0, 1]$, we obtain

$$\begin{aligned} \Delta V(x_k) &= z_k^\mathsf{T} \Omega_2 z_k - \alpha x_k^\mathsf{T} P^{-1} x_k + \alpha \omega_k^\mathsf{T} W \omega_k \\ &\leq z_k^\mathsf{T} \Omega_2 z_k + \alpha\left(1 - V(x_k)\right), \end{aligned}$$

with

$$\Omega_2 := \begin{bmatrix} Q^\mathsf{T} P^{-1} Q - P^{-1} + \alpha P^{-1} & Q^\mathsf{T} P^{-1} \\ \star & P^{-1} - \alpha W \end{bmatrix}.$$

If $\Omega_2 \prec 0$, this implies that $z_k^\mathsf{T} \Omega_2 z_k < 0$ for all $z_k \neq 0$ and the corresponding Lyapunov function $V(x_k)$ satisfies the inequality $z_k^\mathsf{T} \Omega_2 z_k + \alpha(1 - V(x_k)) < \alpha(1 - V(x_k))$. Then $\Delta V(x_k)$ is upper bounded as

$$\Delta V(x_k) < \alpha\left(1 - V(x_k)\right) \leq 0,$$

and $V(x_k) > 1$ guarantees that $\mathcal{E}(P)$ is an attractive ellipsoid of the closed-loop system (3).

The following LMI condition establishes that the ellipsoid $\mathcal{E}(P)$ is an attractive ellipsoid of the closed-loop system (3) with gain matrix $K = LG^{-1}$, where L and G (non-singular) are design matrices of appropriate dimension.

Theorem 3. If there exists a symmetric positive definite matrix $P \in \mathbb{S}^n$, matrices $L \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{n\times n}$ (non-singular), and a constant scalar $\alpha \in (0, 1]$, such that the following condition is satisfied,

$$\begin{bmatrix} P - G - G^\mathsf{T} & 0 & G^\mathsf{T} A^\mathsf{T} + L^\mathsf{T} B^\mathsf{T} & G^\mathsf{T} \\ \star & -\alpha W & I & 0 \\ \star & \star & -P & 0 \\ \star & \star & \star & -\dfrac{1}{\alpha} P \end{bmatrix} \prec 0 \tag{6}$$

then the ellipsoid $\mathcal{E}(P)$ is an attractive ellipsoid of the closed-loop system (3) with feedback gain matrix $K = LG^{-1}$ and perturbation matrix $W \in \mathbb{S}^n$.

Proof. Assume that condition (6) is feasible. Block $(1, 1)$ of (6) implies $P - G - G^\mathsf{T} \prec 0$, where P is positive definite and G is of full rank. Observe that $W \succ 0$ and $P \succ 0$ are guaranteed by the block $(2, 2)$ and $(3, 3)$ of (6), respectively. Then, we have the following property

$$(P - G)^\mathsf{T} P^{-1} (P - G) \succeq 0,$$

which is equivalent to, $-G^\mathsf{T} P^{-1} G \preceq P - G - G^\mathsf{T} \prec 0$. We introduce the change of variables $L := KG$ to obtain,

$$\begin{bmatrix} -G^\top P^{-1} G & 0 & G^\top A^\top + G^\top K^\top B^\top & G^\top \\ \star & -\alpha W & I & 0 \\ \star & \star & -P & 0 \\ \star & \star & \star & -\frac{1}{\alpha} P \end{bmatrix} \prec 0. \quad (7)$$

Let $\Gamma := \mathrm{diag}(G, I, I, I)$, decomposing (7) as $\Gamma^\top \Phi \Gamma \prec 0$,

$$\Phi := \begin{bmatrix} -P^{-1} & 0 & A^\top + K^\top B^\top & I \\ \star & -\alpha W & I & 0 \\ \star & \star & -P & 0 \\ \star & \star & \star & -\frac{1}{\alpha} P \end{bmatrix}, \quad (8)$$

implies that inequality (7) is valid if and only if $\Phi \prec 0$ is satisfied. Applying now the Schur complement to the block $\begin{bmatrix} -P & 0 \\ \star & -(1/\alpha)P \end{bmatrix}$ of (8), we obtain $\Omega_2 \prec 0$, which guarantees that $\mathcal{E}(P)$ is an attractive ellipsoid of the closed-loop system (3). The feedback control law may be retrieved as $u_k = Kx_k = LG^{-1}x_k$ for G non-singular. ■

Two important features of condition (6) are: (a) the state-space matrices and Lyapunov matrix are separated, and (b) the feedback gain K does not depend on the Lyapunov matrix P. The slack variable matrix G can be seen as an additional degree of freedom.

An optimal (minimum size) attractive ellipsoid can be obtained by using the trace criterion due to linearity of the trace function. The following SDP problem (with fixed α obtained from the line search subproblem, see Remark 4 below) solves the optimal robust stabilization problem and calculates the minimal size ellipsoid,

$$\min_{P,L,G} \quad \mathrm{trace}\,(P)$$
$$\text{subject to:} \quad (6).$$

The linear feedback gain $K = LG^{-1}$ minimizes the size of the attractive ellipsoid of the closed-loop system (3).

Remark 4. (*Line search subproblem*). Due to the presence of the decision variable α, condition (6) is not an LMI. However, for fixed α, this condition actually becomes an LMI. The idea here is to find the maximum α that minimizes the trace(P) subject to feasibility of (6) and $\Omega_2 \prec 0$. To find a suitable $\alpha \in (0, 1]$, a line-search subproblem can be solved starting from $\alpha = 1$, such that it keeps decreasing the value of α, by a certain factor, until the problem becomes infeasible, or stops when $\alpha \approx 0$. The Armijo-type rule is essentially a successive reduction rule, suitable for this line-search subproblem. Here, fixed scalars $\gamma > 0$ and β, with $\beta \in (0, 1)$ are chosen, and the Armijo rule is expressed as $\alpha^{(\ell)} = \beta^{\ell^*}\gamma$, where ℓ^* is the first nonnegative integer ℓ for which the problem becomes infeasible; ℓ also indicates the number of steps required for the Armijo rule to converge. In other words, the step sizes $\beta^\ell \gamma$, $\ell = 0, 1, \ldots$, are tried successively until (6) is infeasible or Ω_2 non-negative for $\ell = \ell^*$. Thus, we are not satisfied with just a cost improvement; the amount of the improvement has to be sufficiently large as per the tests described above. Alternatively the quantity trace(P) can be plotted against α and the maximum α that minimizes trace(P) would be selected. The reduction factor β is chosen close to 1 (e.g., 0.95), depending on the confidence we have on the quality of the initial step size γ. We can always take $\gamma = 1$ (i.e., $\alpha^{(0)} = 1$ for iteration $\ell = 0$).

3.2 Constrained control

Consider now the constrained control case, where the magnitude of the control signal $u_k = Kx_k$ inside an ellipsoid $\mathcal{E}(\Omega)$ is constrained as,

$$\|u_k\|_R^2 = x_k^\top K^\top R^{-1} K x_k < \mu^2, \quad (9)$$

for all x_k such that $x_k^\top \Omega^{-1} x_k \le 1$, where $\|\cdot\|_R^2$ is the 2-norm induced by the weighting matrix $R \in \mathbb{S}^m \succ 0$. Here μ is an appropriate positive constant scalar that restricts the inputs for given matrix R. Equivalently, $\left(1/\mu^2\right) x_k^\top K^\top R^{-1} K x_k < x_k^\top \Omega^{-1} x_k$, which implies $\left(1/\mu^2\right) K^\top R^{-1} K \prec \Omega^{-1}$ for all $x \neq 0$. Multiplying both sides of this inequality by Ω and applying Schur's lemma we obtain,

$$\begin{bmatrix} -\Omega & \Omega K^\top \\ K\Omega & -\mu^2 R \end{bmatrix} \prec 0.$$

Decomposing this inequality now as

$$\begin{bmatrix} -\Omega & 0 \\ 0 & -\mu^2 R \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} [\Omega \ 0] + \begin{bmatrix} \Omega \\ 0 \end{bmatrix} \left[0 \ K^\top\right] \prec 0.$$

will allow us to derive the main result of this section with the help of the following elimination lemma.

Lemma 5. Let us define a symmetric matrix Φ and U, M matrices with appropriate dimensions. Then the following conditions are equivalent:

(1) $\Phi \prec 0$ and $\Phi + UM^\top + MU^\top \prec 0$.
(2) The LMI problem

$$\begin{bmatrix} \Phi & M + UF \\ M^\top + F^\top U^\top & -F - F^\top \end{bmatrix} \prec 0,$$

is feasible with respect to F.

Proof. Proof is omitted; see Song and Yang (2011). ■

Using Lemma 5 with matrix assignments,

$$\Phi \leftarrow \begin{bmatrix} -\Omega & 0 \\ 0 & -\mu^2 R \end{bmatrix}, \quad U \leftarrow \begin{bmatrix} 0 \\ K \end{bmatrix}, \quad M \leftarrow \begin{bmatrix} \Omega \\ 0 \end{bmatrix}, \quad F \in \mathbb{S}^n,$$

the following LMI problem is feasible with respect to F,

$$\begin{bmatrix} -\Omega & 0 & \Omega \\ \star & -\mu^2 R & KF \\ \star & \star & -F - F^\top \end{bmatrix} \prec 0, \qquad \Omega \succ 0.$$

In particular, we can impose that $\Omega \succeq P$. If $\Omega := P$ the magnitude of the control signal inside the attractive ellipsoid will be bounded. Also, F can be selected as $F := G$ and $L := KG$, so

$$\begin{bmatrix} -\Omega & 0 & \Omega \\ \star & -\mu^2 R & L \\ \star & \star & -G - G^\top \end{bmatrix} \prec 0. \quad (10)$$

Finally, we obtain the following SDP problem (with fixed α, see Remark 4),

$$\begin{aligned} \min_{P,L,G,\Omega} \quad & \mathrm{trace}\,(P) \\ \text{subject to:} \quad & (6), (10). \end{aligned} \quad (11)$$

The obtained linear state-feedback gain $K = LG^{-1}$ from the solution of the SDP problem (11) minimizes the size of the attractive ellipsoid of the closed-loop system (3), while states inside Ω satisfy the control constraint (9).

3.3 Polytopic systems

Consider the class of discrete-time linear parameter varying (LPV) systems of the form

$$x_{k+1} = A(\theta)\,x_k + B(\theta)\,u_k + \omega_k, \tag{12}$$

where matrices $A(\theta)$ and $B(\theta)$ depend affinely on the unknown but *measurable* time-invariant vector of parameters θ. The vector θ takes values in the unit simplex Θ_N, $\theta \in \Theta_N \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, $N \geq 2$, where N is the number of vertices and Θ_N may be expressed as,

$$\Theta_N = \Big\{\theta \in \mathbb{R}^N : \sum_{i=1}^{N} \theta_i = 1, \theta_i \geq 0, \forall\, i = 1,\ldots,N\Big\}. \tag{13}$$

The affine assumption implies that matrices $A(\theta)$ and $B(\theta)$ are matrix polytopes and can be expressed as,

$$[A(\theta), B(\theta)] = \sum_{i=1}^{N} \theta_i\,(A_i, B_i), \tag{14}$$

where $\theta \in \Theta_N$ as above. Usually a large number of vertices N is required to arbitrarily approximate the original state space representation. Using a state feedback control $u = Kx$, the closed loop system is

$$x_{k+1} = [A(\theta) + B(\theta)\,K]\,x_k + \omega_k. \tag{15}$$

Assume that the ellipsoid $\mathcal{E}(P(\theta))$ is given by

$$\mathcal{E}(P(\theta)) := \left\{x \in \mathbb{R}^n : x^\top P^{-1}(\theta)\,x \leq 1\right\}, \tag{16}$$

with $P^{-1}(\theta) \in \mathbb{S}^n \succ 0$ as,

$$P^{-1}(\theta) := \left(\sum_{i=1}^{N} \theta_i P_i\right)^{-1},$$

is an invariant set of (15); $P_i \in \mathbb{S}^n$, $i = 1,\ldots,N$, are positive definite matrices. The resulting parameter-dependent Lyapunov function $V(x,\theta) = x^\top P^{-1}(\theta)\,x$. Robust stability requires stability to be checked for all $\theta \in \Theta_N$. This is equivalent of solving an infinite number of feasibility problems on $P(\theta) \succ 0$. Alternatively, quadratic stability, which implies robust stability, requires stability to be checked at the N vertices of A and B, provided that the matrix polytopes (14) can arbitrarily approximate the original state space. This is equivalent of solving N feasibility problems on $P \succ 0$. The resulting $V(x,\theta) = V(x) = x^\top Px$ does not depend directly on θ.

Theorem 6. The ellipsoid $\mathcal{E}(P)$ given by (16) is the attractive ellipsoid of the closed-loop system (15) with state-feedback gain $K = LG^{-1}$ if and only if there exist symmetric positive definite matrices $P_i \in \mathbb{S}^n$, $i = 1,\ldots,N$, and matrices $L \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{n\times n}$, and and constant scalar $\alpha \in (0,1]$, such that the following conditions are satisfied,

$$\begin{bmatrix} P_i - G - G^\top & 0 & G^\top A_i^\top + L^\top B_i^\top & G^\top \\ \star & -\alpha W & I & 0 \\ \star & \star & -P_i & 0 \\ \star & \star & \star & -\dfrac{1}{\alpha}P_i \end{bmatrix} \prec 0, \tag{17}$$

for all vertices, $i = 1,\ldots,N$.

Proof. Necessity is directly obtained from Theorem 3. Indeed, if we satisfy (6) for $P(\theta)$, $A(\theta)$ and $B(\theta)$, for all $\theta \in \Theta_N$ then this implies that the LMI must hold on the vertices. Now suppose that (17) holds for all A_i, B_i and P_i, $i = 1,\ldots,N$. Multiplying each inequality matrix by θ_i and summing for all $i = 1,\ldots,N$, by convexity of the sets, we conclude that (17) must hold for all $\theta \in \Theta_N$. ■

Minimization of the trace of $P(\theta)$ can be equivalently expressed as the minimization of a parameter $\eta > 0$ subject to,

$$\text{trace}(P_i) \leq \eta, \quad \text{for all } \ i = 1,\ldots,N. \tag{18}$$

We can now state the following SDP problem (given $\alpha \in (0,1]$) to address the robust stabilization problem of polytopic systems subject to bounded perturbations,

$$\begin{array}{cc} \min\limits_{P_i,L,G} & \eta \\ \text{subject to:} & (17),(18). \end{array} \tag{19}$$

3.4 Gain-scheduled dynamic feedback control

This section studies the problem of designing gain-scheduled feedback controllers for the time-varying linear system (12) in the special case where $B_i = B$, for all $i = 1, 2,\ldots,N$, $N \geq 2$, where N is the number of vertices.

Assume gain-scheduled state-feedback control of the form

$$u_k(x_k) = K(\theta)\,x_k, \qquad K(\theta) = \sum_{i=1}^{N} \theta_i K_i, \tag{20}$$

where K_i, $i = 1,\ldots,N$, are gain matrices, such that the system $[A(\theta) + BK(\theta)]\,x_k + \omega_k$ converges asymptotically to a minimal size ellipsoid, which includes the origin for all $\theta \in \theta_N \subseteq \mathbb{R}^N$ in (13).

Theorem 7. The ellipsoid $\mathcal{E}(P)$ given by (16) is the attractive ellipsoid of the closed-loop system (15) with gain-scheduled control (20) and feedback gains $K_i = L_iG^{-1}$, $i = 1,\ldots,N$, or

$$u_k = \left(\sum_{i=1}^{N} \theta_i L_i\right) G^{-1}x_k, \tag{21}$$

if and only if there exist symmetric positive definite matrices $P_i \in \mathbb{S}^n$, matrices $L_i \in \mathbb{R}^{m\times n}$, $i = 1,\ldots,N$, $G \in \mathbb{R}^{n\times n}$, and a constant scalar $\alpha \in (0,1]$, such that the following LMI conditions are satisfied

$$\begin{bmatrix} P_i - G - G^\top & 0 & G^\top A_i^\top + L_i^\top B^\top & G^\top \\ \star & -\alpha W & I & 0 \\ \star & \star & -P_i & 0 \\ \star & \star & \star & -\dfrac{1}{\alpha}P_i \end{bmatrix} \prec 0, \tag{22}$$

for all vertices, $i = 1,\ldots,N$.

Proof. Proof omitted since it is a direct application of Theorems 3 and 6. ■

Finally, we obtain the following SDP problem (given α)

$$\begin{array}{cc} \min\limits_{P_i,L_i,G} & \eta \\ \text{subject to:} & (18),(22). \end{array} \tag{23}$$

The obtained linear state-feedback gains $K_i = L_iG^{-1}$ from the solution of the SDP problem (23) minimize the size of the attractive ellipsoid of the closed-loop system $[A(\theta) + BK(\theta)]\,x_k + \omega_k$.

Remark 8. Note that the LMI conditions in Theorems 6 and 7 can be readily extended by LMI (10) to cope with constrained inputs as in Section 3.2. For the gain-scheduling problem, using L_i, $i = 1,\ldots,N$, and, $\Omega(\theta) :=$

$\sum_{i=1}^{N}\theta_i\Omega_i$, in LMI condition (10) results in the following problem (for given α),

$$\begin{array}{ll}\min\limits_{P_i,L_i,\Omega_i,G} & \eta \\ \text{subject to:} & (18)\,,(22)\,.\end{array} \tag{24}$$

Remark 9. The input matrix B can vary in a different matrix polytope given by, $B(\theta)=\sum_{j=1}^{N}\theta_j B_j, \theta_j\in\Theta_N$.

4. NUMERICAL EXAMPLES

This section presents two numerical examples both with three states and two controls to demonstrate the efficiency of the proposed AEM approach. Here we follow a slightly different notation compared to rest of the paper. The state vector is denoted as: $x_k := x(k) = [x_1(k)\ x_2(k)\ x_3(k)]^{\mathsf{T}}$, where $x_i(k)$ denotes the i-th element of vector x at discrete time instant k. Similarly, the control vector is denoted as: $u_k := u(k) = [u_1(k)\ u_2(k)]^{\mathsf{T}}$, where $u_i(k)$ denotes the i-th element of vector u at discrete time instant k.

4.1 Example 1

Consider the following discrete-time system

$$A=\begin{bmatrix}2&1&2\\6&-2&1\\1&0&1\end{bmatrix},B=\begin{bmatrix}-0.2&0\\1&0.1\\1&0.5\end{bmatrix},W=\begin{bmatrix}2.2&0&0\\0&0.04&0\\0&0&0.0004\end{bmatrix}\times 10^5,$$

with $R=I$. Let the exogenous disturbance input ω_k be

$$\omega_k = 0.2 + 0.5\sin(50k) + 0.4\sin(100k)\,(0.001, 0.01, 0.1)^{\mathsf{T}}\,.$$

This system is unstable with $u=0$. The control input (9) is bounded by $\mu^2=100$ with $R=I$. The linear feedback gain K and the ellipsoidal matrices P and Ω resulting from the solution of the SDP problem (11) are as follows,

$$K=\begin{bmatrix}-1.9780&3.2024&2.2909\\-6.2962&-7.5908&-11.8332\end{bmatrix},$$

$$P=\begin{bmatrix}0.6763&0.9373&-1.1563\\0.9373&1.3097&-1.6066\\-1.1563&-1.6066&2.0394\end{bmatrix},$$

$$\Omega=\begin{bmatrix}3.4095&4.7553&-5.8414\\4.7553&6.7163&-8.1787\\-5.8414&-8.1787&10.3236\end{bmatrix},$$

with $\alpha=0.33$, $\operatorname{trace}(P)=4.0254$ and $\Omega\succ P$. To illustrate the approach, consider the initial state $x_0=[1.78\ 2.45\ -3.1]^{\mathsf{T}}$. Fig. 1 illustrates the obtained results of the proposed AEM approach. Figs 1(a)–1(c) depict the state trajectories while Figs 1(d) and 1(e) the control signal trajectories. Fig. 1(f) displays the minimal size ellipsoid projected onto the subspace (u_1, u_2). As can be seen in Fig. 1(f) the control trajectories remain inside the attractive ellipsoid despite the presence of non-vanishing disturbances.

Fig. 2 displays the state trajectories while approaching the minimal size attractive ellipsoid P (red color) and the controlled invariant ellipsoid Ω (blue color); both ellipsoids projected onto the subspaces (x_1,x_2) and (x_2,x_3). As can be seen in Figs 2(a), 2(c), every state in state-space can be taken inside P (red ellipsoid) but only states inside Ω (blue ellipsoid) satisfy the control constraints (9); note that $\Omega\succ P$ holds in this example. Finally, Figs 2(b), 2(d) demonstrate that the state trajectories remain inside the attractive ellipsoid and converge to a small neighborhood of the origin despite the presence of non-vanishing disturbances. Therefore the proposed AEM approach is appropriate in selecting design parameters for the constrained stabilization and state-feedback control of discrete-time systems; and, for making the attractive ellipsoid of an appropriate small size.

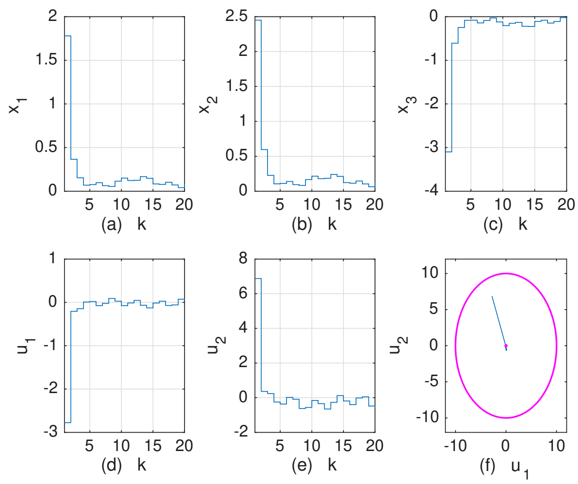

Fig. 1. Example 1: States of the system and control signal.

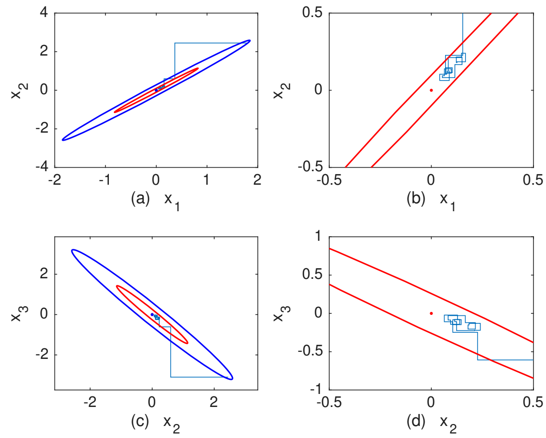

Fig. 2. Example 1: Projection onto the subspaces (x_1, x_2) and (x_2, x_3). The attractive ellipsoid P is indicated with red color; the controlled invariant ellipsoid Ω is indicated with blue color.

4.2 Example 2

Consider a discrete-time LPV system with the following affine parameter-dependent matrices,

$$A(\theta)=\begin{bmatrix}1&-1+\theta&0\\1-\theta&1&0.5\\\theta&-1&1-\theta\end{bmatrix},$$

$$B(\theta)=\begin{bmatrix}-1&-0.5\\0&1\\1&2\end{bmatrix},\quad W=\begin{bmatrix}2&0&0\\0&0.05&0\\0&0&0.0005\end{bmatrix}\times 10^5,$$

where θ is a time-invariant parameter, with $|\theta|\le 1$. The control input (9) is bounded by $\mu^2=100$, $R=I$ and $\Omega=P$. The matrix $A(\theta)$ is unstable for all θ. Let us consider the exogenous disturbance input ω_k be,

$$\omega_k = 0.5 + 0.2\sin(80k) + 0.2\sin(100k)\,(0.001, 0.01, 0.1)^{\mathsf{T}}\,.$$

To solve the stabilization problem, we consider gain-scheduled feedback control of the form (20) and express the LPV system into the polytopic form (12). The state matrix $A(\theta)$ can be expressed as in (14) with $N=2$ vertices,

$$A_1=\begin{bmatrix}1&-2&0\\2&1&0.5\\-1&-1&2\end{bmatrix},\qquad A_2=\begin{bmatrix}1&0&0\\0&1&0.5\\1&-1&0\end{bmatrix},$$

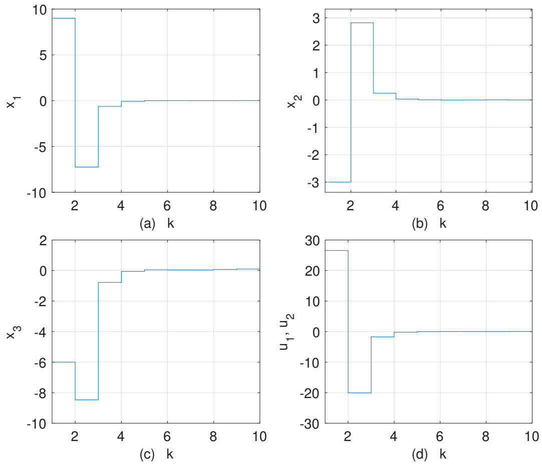

Fig. 3. Example 2: State and control trajectories.

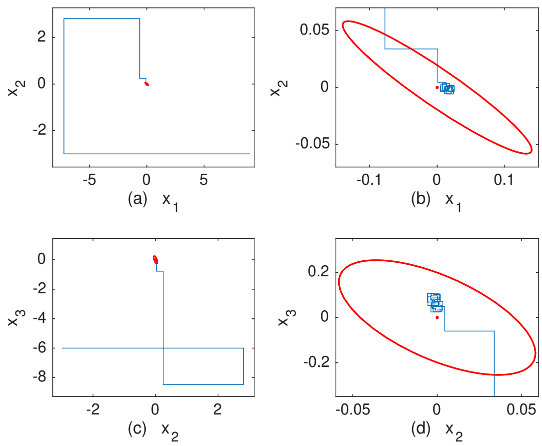

Fig. 4. Example 2: Projection onto the subspaces (x_1, x_2) and (x_2, x_3). The minimal size attractive ellipsoid is indicated with red color.

$\theta_1 = \frac{\theta_{\max} - \theta}{\theta_{\max} - \theta_{\min}}$, $\theta_2 = \frac{\theta - \theta_{\min}}{\theta_{\max} - \theta_{\min}}$, with $\theta_{\min} = -1$ and $\theta_{\max} = 1$. The convex coordinates θ_i, $i = 1, 2$, satisfy $0 \leq \theta_i \leq 1$, and $\theta_1 + \theta_2 = 1$. The feedback gains K_1, K_2, and the ellipsoidal matrices P_1, P_2 resulting from the solution of the SDP problem (24) are as follows,

$$K_1 = \begin{bmatrix} 2.8917 & -0.1381 & -0.1120 \\ -1.5954 & -0.4125 & -0.6762 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.5318 & 1.1772 & 0.4340 \\ -0.2418 & -0.7690 & -0.4158 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} .0198 & -.0077 & .0235 \\ -.0077 & .0034 & -.0092 \\ .0235 & -.0092 & .0643 \end{bmatrix}, P_2 = \begin{bmatrix} .0120 & -.0044 & .0211 \\ -.0044 & .0020 & -.0078 \\ .0211 & -.0078 & .0735 \end{bmatrix},$$

with $\alpha = 0.55$ and $\eta = 0.0875$ To illustrate the approach, consider an unstable system with $\theta_1 = 0.98$, $\theta_2 = 0.02$ and initial state $x_0 = [9\ -3\ -6]^\mathsf{T}$. Figs 3–4 show the obtained results. As can be seen in Fig. 3, the proposed method provides fast and smooth convergence to the origin. Figs 4(a), 4(c) indicate that the system states are driven inside the minimum size ellipsoid. Finally, Figs 4(b), 4(d) provide a zoom of the attractive ellipsoid to demonstrate that the state trajectories remain inside the attractive ellipsoid and converge to a small neighborhood of the origin despite the presence of the non-vanishing disturbances. It should be noted that in this example for any initial state inside $\mathcal{E}(P(\theta))$, the control constraint (9) are satisfied.

5. CONCLUSIONS

The attractive ellipsoid method has extended to cope with the problem of robust stabilization of discrete-time polytopic systems subject to control constraints and unknown but bounded non-vanishing perturbations. New LMI conditions were obtained with the important property of decoupling state-space matrices and Lyapunov matrix. These conditions allowed us to extend the proposed approach to systems with polytopic uncertainty and scheduling parameters. The obtained state-feedback robust control laws provide a minimal size ellipsoid for the closed-loop system. Two examples demonstrated the ability of the proposed AEM to keep the system states inside the attractive ellipsoid and to converge to a small neighborhood of the origin despite the presence of non-vanishing disturbances.

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