Robust Disturbance Rejection by the Attractive Ellipsoid Method – Part I: Continuous-time Systems ⋆

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Abstract: This paper develops sufficient conditions for the constrained robust stabilization of continuous-time polytopic linear systems with unknown but bounded perturbations. The attractive ellipsoid method (AEM) is employed to determine a *robustly controllable invariant set*, known as *attractive ellipsoid*, such that the state trajectories of the system asymptotically converge to a small neighborhood of the origin despite the presence of non-vanishing perturbations. To solve the stabilization problem, we employ the Finsler's lemma and derive new linear matrix inequality (LMI) conditions for robust state-feedback control design, ensuring convergence of state trajectories of the system to a minimal size ellipsoidal set. We also consider the state and control constrained problem and derive extended LMI conditions. Under certain conditions, the obtained LMIs guarantee that the attractive ellipsoid is nested inside the bigger ellipsoids imposed by the control and state constraints. Finally, we extend our AEM approach to the gain-scheduled state-feedback control problem, where the scheduling parameters governing the time-variant system are unknown in advance but can be measured in real-time. Two examples demonstrate the feasibility of the proposed AEM and its improvements over previous works.

Keywords: Robust control; Disturbance rejection; Polytopic uncertainty; Gain-scheduled control; Linear parameter-varying systems; Lyapunov methods.

1. INTRODUCTION

The influence of uncertainty on the performance of dynamical systems has attracted significant attention over the past several decades. Robust control is an efficient and systematic approach for control systems design under structured or unstructured uncertainties. The main objectives of robust control under uncertainty are stability, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance, multi-objective control, asymptotic disturbance rejection, passivity and plant model uncertainty (see e.g., Vidyasagar (1986), Scherer et al. (1997), Petersen et al. (2000), Han (2009)).

The attractive ellipsoid method (AEM), which initially developed for nonlinear systems (Gonzalez-Garcia et al., 2009; Poznyak et al., 2014) and later for linear systems (Nazin et al., 2007; Polyak et al., 2008), use the concept of the *asymptotically attractive* (invariant) ellipsoid. The main objective of the AEM is to determine a robust controller and a corresponding asymptotically attractive ellipsoid such that state trajectories of the system converge to a small (in a given sense) neighborhood of the origin. The attractive ellipsoid set has the appealing feature of being robustly controlled invariant, and, thus the state trajectories remain inside this minimal size set despite the presence of non-vanishing and unmatched perturbations.

Ellipsoids are probably the most used tools for control synthesis, possessing some "nice" properties that permits the use of LMI. In general, restricting the uncertainty to lie within ellipsoidal sets leads to a good trade-off between generality of the uncertainty and computational tractability. The complexity of an ellipsoidal representation is quadratic in the dimension of the state space, and linear in the number of time steps (needed to calculate). On the other hand, the complexity of methods associated with polytopes (e.g., vertex control (Gutman and Cwikel, 1986) or multi-parametric optimization) that involve the convex hull computation, increases exponentially with the number of vertices (Kurzhanskiy and Varaiya, 2006). The main drawback of ellipsoids is that having a fixed and symmetrical structure imposes conservativeness in the control design. Reducing conservatism in robust control problems has been an extensive research topic. Mainly devoted in the development of new LMI conditions for stability and $\mathcal{H}_2$, $\mathcal{H}_\infty$ performance of linear systems, and robust analysis and synthesis of linear parameter varying (LPV) systems (Gahinet and Apkarian, 1994; Peaucelle et al., 2000; de Oliveira and Skelton, 2001; Daafouz and Bernussou, 2001; Pipeleers, 2009).

The present paper, develops new state-feedback LMI conditions for the continuous-time version of the AEM. Conditions for the robust stabilization of discrete-time polytopic systems by the AEM are presented in García and Ampountolas (2018). These LMI conditions feature decoupled

⋆ A companion paper (García and Ampountolas, 2018) deals with the robust stabilization of discrete-time systems by the AEM.

state-space matrices and Lyapunov matrices. This property allows for the introduction of parameter-dependent Lyapunov functions, and thus for the application of the AEM to gain-scheduled control. The obtained conditions provide the minimum size of the corresponding attractive ellipsoid, solve the stabilization problem, and ensure convergence of system state trajectories to a minimal ellipsoidal set. To the best of our knowledge, this is the first work that extends the AEM to systems with polytopic uncertainty and scheduling parameters with decoupled matrices. The AEM literature is dominated by complex BMI or LMI conditions where matrices are not decoupled and the associated control gain matrices depend on the Lyapunov matrix, see e.g., Perez et al. (2015); Mera et al. (2016). The proposed methodology employs the celebrated Finsler's lemma (Finsler, 1937) and develops new state-feedback LMI conditions. Synthesis of the robust control law is reduced to a semidefinite optimization problem that can be readily solved using efficient interior-point methods (Nesterov and Nemirovskii, 1994; Boyd et al., 1994). Numerical examples are given to illustrate the feasibility of the proposed approach and its improvements over previous works.

2. PRELIMINARIES

Notation. For matrices and vectors $(\cdot)^\mathsf{T}$ indicates transpose and $A^\dagger := A + A^\mathsf{T}$ denotes the Hermitian operator on A. For matrix elements $\star$ denotes the transposed symmetric element. For symmetric matrices, $X \prec 0$ indicates that X is negative definite and $X \preceq 0$ indicates that X is negative semi-definite. $\mathbb{S}^n$ denotes the space of square and symmetric real matrices of dimension n. For square matrices trace$(\cdot)$ denotes the trace of $(\cdot)$.

Problem formulation. Consider a continuous-time linear time-invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t) + \omega(t), \qquad x_0 \text{ given}, \tag{1}$$

where the pair (A, B) is controllable; $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the state and control matrices, respectively; $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input. The disturbance vector $\omega \in \mathbb{R}^n$ is unknown but bounded (at each time instant) satisfying the following property,

$$\omega(t)^\mathsf{T} W \omega(t) \le 1, \qquad \forall\, t \ge 0, \tag{2}$$

where the matrix $W \in \mathbb{S}^n \succ 0$ is given. No other constraints are imposed on the perturbation ω, however it is not considered to be random.

The objective of this paper is to design a robust state-feedback controller of the form $u = Kx$, where $K \in \mathbb{R}^{m\times n}$ is a gain matrix, for the system (1) to compensate the influence of external perturbations (2) on the system state such that the closed-loop system trajectories

$$\dot{x}(t) = (A + BK)\, x(t) + \omega(t), \tag{3}$$

converge asymptotically to a minimal size ellipsoid, which includes the origin. This minimum size ellipsoid guarantees that the state trajectories of the system will remain within a neighborhood of the origin despite the presence of non-vanishing perturbations (2).

The following definition characterizes this minimal region.

Definition 1. (Ellipsoidal set). An ellipsoid $\mathcal{E}(P, \bar{x}) \subset \mathbb{R}^n$ with center $\bar{x}$ and shape matrix P is a set of the form,

$$\mathcal{E}(P, \bar{x}) := \left\{x \in \mathbb{R}^n : (x - \bar{x})^\mathsf{T} P^{-1}(x - \bar{x}) \le 1\right\}, \tag{4}$$

where $P \in \mathbb{S}^n$ is a positive definite matrix.

Definition 2. (Robustly controlled invariant set). The set $\Omega \subseteq \mathcal{X}$, where $\mathcal{X}$ is the set of admissible states, is *robustly controlled invariant* for the system (1) if for all $x(t) \in \Omega$, there exists a control value $u(t)$ such that, for all $\omega(t)$ in (2), with $W \in \mathbb{S}^n \succ 0$,

$$\dot{x}(t) = Ax(t) + Bu(t) + \omega(t) \in \Omega, \quad \forall\, t \ge 0$$

If the control value is constrained as $u(t) \in \mathcal{U}$, where $\mathcal{U}$ is the set of admissible controls, such a control action is called *admissible.* If $\bar{x} = 0$ then the ellipsoid can be written as $\mathcal{E}(P) := \left\{x \in \mathbb{R}^n : x^\mathsf{T} P^{-1} x \le 1\right\}$, $P \in \mathbb{S}^n \succ 0$, and we assume it is a *robustly controlled invariant set* of (1).

Also, consider the following distance metric from a point x to a set $\mathcal{E}$, $\|x\|_\mathcal{E} := \inf_{y\in\mathcal{E}} \|x - y\|, \forall\, x \in \mathbb{R}^n$.

Definition 3. (Asymptotically Attractive Ellipsoid). The set $\mathcal{E}(P)$ is an asymptotically attractive ellipsoid for the system (1) if $\|x(t, x_0)\|_{\mathcal{E}(P)} \to 0$, as $t \to \infty$, for any $x_0 \in \mathbb{R}^n$, where $x(t, x_0)$ is a trajectory of the system for a given admissible control.

For any initial condition x_0, convergence of state trajectories in (1) to a minimal size ellipsoid is guaranteed by the asymptotic attractivity of the set $\mathcal{E}(P)$.

Definition 4. (Support function). The support function of a compact set $\mathcal{S} \subseteq \mathbb{R}^n$, evaluated at $y \in \mathbb{R}^n$ is defined as,

$$f(\mathcal{S} \mid y) = \sup_{x\in\mathcal{S}} \{y^\mathsf{T} x\}. \tag{5}$$

In particular, the support function of the ellipsoid (4) is,

$$f(\mathcal{E}(P, \bar{x}) \mid y) = y^\mathsf{T}\bar{x} + \left(y^\mathsf{T} P\, y\right)^{1/2}.$$

Finsler's lemma (Finsler, 1937; de Oliveira and Skelton, 2001), will be used to derive the LMI characterization.

Lemma 5. (Finsler). Let $x \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$ and $\mathcal{B} \in \mathbb{R}^{m\times n}$ such that rank$(\mathcal{B}) < n$. The following statements are equivalent:

(1) If $x \neq 0$ lie in the null-space of $\mathcal{B}$ then: $x^\mathsf{T}\mathcal{Q}x \prec 0$.
(2) If $\mathcal{B}^\perp$ is a basis for the null-space of $\mathcal{B}$ then: ${\mathcal{B}^\perp}^\mathsf{T}\mathcal{Q}\mathcal{B}^\perp \prec 0$.
(3) There exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that: $\mathcal{Q} - \mu\mathcal{B}^\mathsf{T}\mathcal{B} \prec 0$.
(4) There exists a Lagrange multiplier $\mathcal{Y} \in \mathbb{R}^{n\times m}$ such that: $\mathcal{Q} + \mathcal{Y}\mathcal{B} + \mathcal{B}^\mathsf{T}\mathcal{Y}^\mathsf{T} \prec 0$.

3. MAIN RESULTS

3.1 The attractive ellipsoid method

Consider the Lyapunov function $V(x) = x^\mathsf{T} P^{-1} x, P \succ 0$. Its total derivative along the trajectories of system (3) is,

$$\dot{V}(x) = x^\mathsf{T}\left[Q^\mathsf{T} P^{-1} + P^{-1} Q\right] x + \omega^\mathsf{T} P^{-1} x + x^\mathsf{T} P^{-1}\omega < 0, \tag{6}$$

for all $(x, \omega) \neq 0$, where $Q := A + BK$ and $\omega(t)^\mathsf{T} W\omega(t) \le 1$ for all $t \ge 0$. Define the augmended vector $z^\mathsf{T} = \left[x^\mathsf{T}\ \omega^\mathsf{T}\right]$, then inequality (6) is equivalent to $\dot{V}(x) = z^\mathsf{T}\Omega_1 z < 0$, which holds for all $z \neq 0$ with,

$$\Omega_1 := \begin{bmatrix} Q^\mathsf{T} P^{-1} + P^{-1} Q & P^{-1} \\ P^{-1} & 0 \end{bmatrix}.$$

Adding and subtracting $\alpha x^{\mathsf{T}} P^{-1} x$ and $\alpha \omega^{\mathsf{T}} W \omega$ in (6), where $\alpha > 0$, we obtain,

$$\begin{aligned} \dot{V}(x) &= z^{\mathsf{T}} \Omega_2 z - \alpha x^{\mathsf{T}} P^{-1} x + \alpha \omega^{\mathsf{T}} W \omega \\ &\leq z^{\mathsf{T}} \Omega_2 z + \alpha \left(1 - V(x)\right), \end{aligned}$$

with $\Omega_2 := \begin{bmatrix} Q^{\mathsf{T}} P^{-1} + P^{-1} Q + \alpha P^{-1} & P^{-1} \\ \star & -\alpha W \end{bmatrix}$.

If $\Omega_2 \prec 0$, this implies that $z^{\mathsf{T}} \Omega_2 z < 0$ for all $z \neq 0$ and the corresponding Lyapunov function $V(x)$ satisfies the inequality $z^{\mathsf{T}} \Omega_2 z + \alpha(1 - V(x)) < \alpha(1 - V(x))$. Then $\dot{V}(x)$ is upper bounded as, $\dot{V}(x) < \alpha(1 - V(x)) \leq 0$, and $V(x) > 1$ guarantees that $\mathcal{E}(P)$ is an attractive ellipsoid of the closed-loop system (3).

The following LMI condition guarantees that the ellipsoid $\mathcal{E}(P)$ is an attractive ellipsoid of the closed-loop system (3) with gain matrix $K = LG^{-1}$, where L and G are design matrices of appropriate dimension.

Theorem 6. If there exists a positive definite matrix $P \in \mathbb{S}^n$, matrices $L \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times n}$ (non-singular), and a constant scalar $\alpha > 0$, such that the following condition is satisfied,

$$\begin{bmatrix} \alpha P + S^{\dagger} & P + S - G^{\mathsf{T}} & I + S \\ \star & -G - G^{\mathsf{T}} & -G \\ \star & \star & -\alpha W \end{bmatrix} \prec 0 \tag{7}$$

where $S := AG + BL$, then the ellipsoid $\mathcal{E}(P)$ is an attractive ellipsoid of the closed-loop system (3) with feedback gain matrix $K = LG^{-1}$.

Proof. Congruence transformation to Ω_2 with diag(P, I),

$$\Omega_3 = \begin{bmatrix} PQ^{\mathsf{T}} + QP + \alpha P & I \\ \star & -\alpha W \end{bmatrix} \prec 0.$$

Decomposing matrix Ω_3 as $\mathcal{B}^{\perp\mathsf{T}} \mathcal{Q} \mathcal{B}^{\perp} \prec 0$ with,

$$\mathcal{Q} = \begin{bmatrix} \alpha P & P & I \\ P & 0 & 0 \\ I & 0 & -\alpha W \end{bmatrix}, \qquad \mathcal{B}^{\perp} = \begin{bmatrix} I & 0 \\ Q^{\mathsf{T}} & 0 \\ 0 & I \end{bmatrix}$$

allows the application of Lemma 5 (Condition 2) with $\mathcal{Q} \in \mathbb{S}^{3n}$, where $\mathcal{B}^{\perp} \in \mathbb{R}^{3n \times 2n}$ is a basis for the null-space of $\mathcal{B} := \begin{bmatrix} Q^{\mathsf{T}} & -I & 0 \end{bmatrix} \in \mathbb{R}^{n \times 3n}$. By condition 4 of Lemma 5, there exists a multiplier $\mathcal{Y}^{\mathsf{T}} := [F_1 \ F_2 \ F_3] \in \mathbb{R}^{n \times 3n}$, such that $\mathcal{Q} + \mathcal{Y}\mathcal{B} + \mathcal{B}^{\mathsf{T}} \mathcal{Y}^{\mathsf{T}} \prec 0$, or equivalently,

$$\begin{bmatrix} \alpha P + QF_1 + F_1^{\mathsf{T}} Q^{\mathsf{T}} & P + QF_2 - F_1^{\mathsf{T}} & I + QF_3 \\ \star & -F_2 - F_2^{\mathsf{T}} & -F_3 \\ \star & \star & -\alpha W \end{bmatrix}. \tag{8}$$

Since matrix F_2 is nonsingular, we restrict matrices F_1 and F_3 to be equal to F_2, i.e., $F_1 = F_3 = F_2$ and set $G = F_2$. With these matrices, the resulted LMI is the LMI (7) with the change of variable, $L := KG$. Condition (7) guarantees that $\mathcal{E}(P)$ is the attractive ellipsoid of the closed-loop system (3). The control is given by

$$u(t) = Kx(t) = LG^{-1} x(t), \quad t \geq 0$$

for G non-singular. ■

Two important features of condition (7) are: (1) state-space matrices and Lyapunov matrix are separated, and (2) the feedback gain K does not depend on the Lyapunov matrix P. The slack variable matrix G can be seen as an additional degree of freedom.

Remark 7. Due to the presence of the decision variable $\alpha > 0$, condition (7) is not an LMI. However, for fixed α, this condition actually becomes an LMI. To find a suitable $\alpha > 0$, a line-search algorithm can be performed such that it keeps increasing the value of α until the problem becomes feasible, or stops when α reaches a certain threshold value. The Armijo rule is essentially a successive reduction rule, suitable for this line-search subproblem after some modification. The idea here is to find the maximum α that minimizes the trace(P) subject to feasibility of (7) and $\Omega_2 \succ 0$.

An optimal attractive ellipsoid will be found using the trace criterion due to linearity of the trace function. The following semi-definite programming (SDP) (with fixed α) problem provides LMI-based conditions for optimal robust stabilization,

$$\begin{aligned} \min_{P,L,G} \quad & \operatorname{trace}(P) \\ \text{subject to:} \quad & (7). \end{aligned} \tag{9}$$

The gain matrix $K = LG^{-1}$ minimizes the size of the attractive ellipsoid of the closed-loop system (3).

3.2 Handling state and control constraints

Control constraints. Suppose that inside an ellipsoid $\mathcal{E}(\Omega_u) \subseteq \mathcal{U}$, the magnitude of the control signal $u(t) = Kx(t)$ is constrained as,

$$\|u\|_R^2 := x^{\mathsf{T}} K^{\mathsf{T}} R^{-1} K x < \mu^2 \tag{10}$$

for all x satisfying $x^{\mathsf{T}} \Omega_u^{-1} x \leq 1$, where $\|\cdot\|_R^2$ is the weighted matrix 2-norm and μ is an appropriate positive constant scalar that restricts the inputs for given $R \in \mathbb{S}^m \succ 0$. Equivalently, we have, $\left(1/\mu^2\right) x^{\mathsf{T}} K^{\mathsf{T}} R^{-1} K x < x^{\mathsf{T}} \Omega_u^{-1} x$, which implies $\left(1/\mu^2\right) K^{\mathsf{T}} R^{-1} K \prec \Omega_u^{-1}$, for all $x \neq 0$, multiplying both sides by Ω_u, yields $\left(1/\mu^2\right) \Omega_u K^{\mathsf{T}} R^{-1} K \Omega_u \prec \Omega_u$. Applying Schur's lemma, this condition can be rewritten as,

$$\begin{bmatrix} -\Omega_u & \Omega_u K^{\mathsf{T}} \\ K\Omega_u & -\mu^2 R \end{bmatrix} \prec 0.$$

Decomposing this last inequality as,

$$\begin{bmatrix} -\Omega_u & 0 \\ 0 & -\mu^2 R \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} \begin{bmatrix} \Omega_u & 0 \end{bmatrix} + \begin{bmatrix} \Omega_u \\ 0 \end{bmatrix} \begin{bmatrix} 0 & K^{\mathsf{T}} \end{bmatrix} \prec 0.$$

will allow us to derive the main result of this section with the help of the following elimination lemma.

Lemma 8. Let us define a symmetric matrix Φ and U, M matrices with appropriate dimensions. Then the following conditions are equivalent:

(1) $\Phi \prec 0$ and $\Phi + UM^{\mathsf{T}} + MU^{\mathsf{T}} \prec 0$.

(2) The LMI problem $\begin{bmatrix} \Phi & M + UF \\ M^{\mathsf{T}} + F^{\mathsf{T}} U^{\mathsf{T}} & -F - F^{\mathsf{T}} \end{bmatrix} \prec 0$, is feasible with respect to F.

Proof. Proof is omitted; see Song and Yang (2011). ■

Using Lemma 8 with matrix assignments,

$$\Phi \leftarrow \begin{bmatrix} -\Omega_u & 0 \\ 0 & -\mu^2 R \end{bmatrix}, \quad U \leftarrow \begin{bmatrix} 0 \\ K \end{bmatrix}, \quad M \leftarrow \begin{bmatrix} \Omega_u \\ 0 \end{bmatrix}, \quad F \in \mathbb{S}^n,$$

we have to satisfy, $\begin{bmatrix} -\Omega_u & 0 & \Omega_u \\ \star & -\mu^2 R & KF \\ \star & \star & -F - F^{\mathsf{T}} \end{bmatrix} \prec 0, \Omega_u \succ 0.$

In particular, we can impose that $\Omega_u \succeq P$. In this case, the attractive ellipsoid corresponding to P is *nested* inside the bigger ellipsoid Ω_u when both are represented at the same level set $\mathcal{E}(\cdot)$ of the corresponding Lyapunov functions. If $\Omega_u = P$ the magnitude of the control signal inside the attractive ellipsoid is admissible. Also, F can be selected as $F := G$ and $L := KG$, so

$$\begin{bmatrix} -\Omega_u & 0 & \Omega_u \\ \star & -\mu^2 R & L \\ \star & \star & -G - G^\mathsf{T} \end{bmatrix} \prec 0. \tag{11}$$

State constraints. We assume symmetric state constraints

$$\mathcal{X} = \{x \in \mathbb{R}^n : |y_i^\mathsf{T} x| \le 1\}, \quad i = 1, \ldots, n_x, \tag{12}$$

where $y_i \in \mathbb{R}^n$ are given vectors and n_x is the number of state constraints. The support function of the ellipsoid $\mathcal{E}(\Omega_x)$, evaluated at $y_i \in \mathbb{R}^n$ is (see Definition 4),

$$f(\mathcal{E}(\Omega_x) \mid y_i) = \left(y_i^\mathsf{T} \Omega_x \, y_i\right)^{1/2}.$$

Then $\mathcal{E}(\Omega_x)$ is a subset of the polyhedral set of (12) if and only if $y_i^\mathsf{T} \Omega_x \, y_i \le 1$ for all $i = 1, \ldots, n_x$. Using now Schur's lemma this condition can be rewritten as,

$$\begin{bmatrix} 1 & y_i^T \Omega_x \\ \star & \Omega_x \end{bmatrix} \succeq 0, \quad i = 1, \ldots, n_x. \tag{13}$$

Concluding, the state constraints are satisfied if $\mathcal{E}(\Omega_x) \subseteq \mathcal{X}$, or if the LMI conditions (13) are satisfied.

It is generally desirable to have the largest ellipsoid among the ones satisfying conditions (11) and (13), while the attractive ellipsoid satisfying the condition (7) should be mimimized. We can again impose that $\Omega_x = \Omega_u \succeq P$. In this case, the attractive ellipsoid corresponding to P is *nested* inside the bigger ellipsoids Ω_u, Ω_x. Our ultimate goal is to reject the non-vanishing disturbances or keep them inside a minimal attractive ellipsoid in a small neighborhood of the origin. Therefore, we formulate the following SDP problem (for given α),

$$\begin{array}{cc} \min\limits_{P,L,G,\Omega_u,\Omega_x} & \operatorname{trace}(P) \\ \text{subject to:} & (7), (11), (13). \end{array} \tag{14}$$

3.3 Polytopic systems

The aim of this section is to derive a finite-dimensional set of LMI conditions for the design of static feedback controllers of linear parameter varying polytopic systems. Consider the class of continuous-time linear parameter varying (LPV) systems of the form

$$\dot{x}(t) = A(\theta)\, x(t) + B(\theta)\, u(t) + \omega(t), \quad t \ge 0 \tag{15}$$

where matrices $A(\theta)$ and $B(\theta)$ depend affinely on the unknown but *measurable* time-invariant vector of parameters θ. The vector θ takes values in the unit simplex Θ_N, $\theta \in \Theta_N \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, $N \ge 2$, where N is the number of vertices and Θ_N may be expressed as,

$$\Theta_N = \left\{\theta \in \mathbb{R}^N : \sum_{i=1}^N \theta_i = 1, \theta_i \ge 0, \forall\, i = 1, \ldots, N\right\}. \tag{16}$$

The affine assumption implies that matrices $A(\theta)$ and $B(\theta)$ are matrix polytopes and can be expressed as,

$$[A(\theta), B(\theta)] = \sum_{i=1}^N \theta_i (A_i, B_i), \tag{17}$$

where $\theta \in \Theta_N$ as above. Using a state feedback control $u(t) = Kx(t)$, the closed loop system is

$$\dot{x}(t) = (A(\theta) + B(\theta) K)\, x(t) + \omega(t). \tag{18}$$

Assume that $\mathcal{E}(P(\theta))$ given by (4) with $\bar{x} = 0$ and,

$$P(\theta)^{-1} := \left(\sum_{i=1}^N \theta_i P_i\right)^{-1}, \tag{19}$$

is a robustly controlled invariant set of (18).

Theorem 9. The ellipsoid $\mathcal{E}(P)$ is the attractive ellipsoid of the closed-loop system (18) with feedback gain matrix $K = LG^{-1}$ if and only if there exist positive definite matrices $P_i \in \mathbb{S}^n$, $i = 1, \ldots, N$, matrices $L \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times n}$, and $\alpha > 0$, such that the following conditions are satisfied,

$$\begin{bmatrix} \alpha P_i + S_i^\dagger & P_i + S_i - G^\mathsf{T} & I + S_i \\ \star & -G - G^\mathsf{T} & -G \\ \star & \star & -\alpha W \end{bmatrix} \prec 0 \tag{20}$$

for all vertices, $i = 1, \ldots, N$, where $S_i := A_i G + B_i L$.

Proof. Necessity proof is directly obtained from Theorem 6 and (7). Indeed, if we satisfy (7) for all $\theta \in \Theta_N$, i.e., for all $P(\theta)$, $A(\theta)$ and $B(\theta)$, then this implies that the LMI must hold on the vertices. Now suppose that (20) holds for all A_i, B_i and P_i, $i = 1, \ldots, N$. Multiplying each inequality matrix by θ_i and summing for all $i = 1, \ldots, N$, by convexity of the sets, we conclude that (20) must hold for all $\theta \in \Theta_N$. ∎

Minimization of trace $P(\theta)$ can be equivalently expressed as the minimization of a parameter $\eta > 0$, subject to,

$$\operatorname{trace}(P_i) \le \eta, \qquad \forall\, i = 1, \ldots, N \tag{21}$$

The following SDP problem (for given $\alpha > 0$) summarises,

$$\begin{array}{cc} \min\limits_{P_i,L,G} & \eta \\ \text{subject to:} & (20), (21). \end{array} \tag{22}$$

3.4 Gain-scheduled dynamic feedback control

This section studies the problem of designing gain-scheduled dynamic feedback controllers in the special case where $B_i = B$, for all $i = 1, 2, \ldots, N$. Assume gain-scheduled state-feedback control of the form,

$$u(x(t)) = K(\theta)\, x(t), \qquad K(\theta) = \sum_{i=1}^N \theta_i K_i, \tag{23}$$

where K_i, $i = 1, \ldots, N$, are gain matrices, such that the system $\dot{x}(t) = [A(\theta) + BK(\theta)]\, x(t) + \omega(t)$ converges asymptotically to a minimal size ellipsoid, which includes the origin for all $\theta \in \theta_N \subseteq \mathbb{R}^N$ in (16).

Theorem 10. The ellipsoid $\mathcal{E}(P)$ given by (4) with center $\bar{x} = 0$ is the attractive ellipsoid of the closed-loop system (18) with gain-scheduled control (23) and feedback gains $K_i = L_i G^{-1}$, $i = 1, \ldots, N$, or

$$u(t) = \left(\sum_{i=1}^N \theta_i L_i\right) G^{-1} x(t), \qquad t \ge 0, \tag{24}$$

if and only if there exist positive definite $P_i \in \mathbb{S}^n$, matrices $L_i \in \mathbb{R}^{m \times n}$, $i = 1, \ldots, N$, $G \in \mathbb{R}^{n \times n}$, and constant scalar $\alpha > 0$, such that the following LMI conditions are satisfied,

$$\begin{bmatrix} \alpha P_i + S_i^\dagger & P_i + S_i - G^\mathsf{T} & I + S_i \\ \star & -G - G^\mathsf{T} & -G \\ \star & \star & -\alpha W \end{bmatrix} \prec 0 \tag{25}$$

for all vertices, $i = 1, \ldots, N$, where $S_i := A_i G + BL_i$.

Proof. Direct application of Theorems 6 and 9. ∎

Finally, we obtain the following SDP problem (given α),

$$\begin{array}{cc} \min\limits_{P_i, L_i, G} & \eta \\ \text{subject to:} & (21)\,, (25)\,. \end{array} \tag{26}$$

The feedback gain-scheduled gains $L_i := K_i F$ minimize the size of the attractive ellipsoid of the closed-loop system $\dot{x}(t) = [A(\theta) + BK(\theta)]\, x(t) + \omega(t)$.

Remark 11. The LMI conditions in Theorems 9 and 10 can be readily extended by LMIs (11) and (13) to cope with constrained states and inputs as in subsection 3.2.

4. NUMERICAL EXAMPLES

4.1 Example 1

Consider the continuous-time system (1) with the following random matrices,

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1.2 & 1.5 & 1 \\ 0.8 & 1 & -0.3 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, W = \begin{bmatrix} 3.3 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.0004 \end{bmatrix} \times 10^5,$$

Let the exogenous disturbance input $\omega(t)$ be

$$\omega(t) = 0.2 + 0.4\sin(100t) + 0.4\sin(50t)\,(0.001, 0.01, 0.1)^{\mathsf{T}}.$$

The system with $u(t) = 0$ is unstable. The control signal (10) is bounded inside the ellipsoid $\mathcal{E}(P)$ by $\mu^2 = 1$ with $R = 1$. The linear feedback gain K and the ellipsoid matrix P resulting from the solution of the SDP problem (14) are,

$$K = [4.0615\ 6.8485\ 3.2546]\,, P = \begin{bmatrix} .0475 & -.0105 & -.009 \\ -.0105 & .0067 & -.0093 \\ -.009 & -.0093 & .0596 \end{bmatrix},$$

with $\alpha = 0.65$, $\operatorname{trace}(P) = 0.1138$ and $\Omega = P$. To assess the performance of the proposed AEM, we compared LMI condition (7) with the AEM as in Poznyak et al. (2014). For the AEM in Poznyak et al. (2014), the resulted feedback gain K, ellipsoidal matrix P and its size are,

$$K = [5.0334\ 13.0809\ 5.1816]\,, P = \begin{bmatrix} .1033 & -.0024 & -.0915 \\ -.0024 & .0055 & -.0080 \\ -.0915 & -.008 & .1346 \end{bmatrix},$$

with $\alpha = 0.05$ and $\operatorname{trace}(P) = 0.2433$. To illustrate and compare the two approaches, consider the initial state $x_0 = [-12\ 3\ 8]^{\mathsf{T}}$. Figs 1(a)–1(h) compare the proposed AEM approach in Section 3 with the AEM approach in Poznyak et al. (2014), denoted as AEM*. The first subfigure illustrates the results obtained by Theorem 6 and the solution of SDP problem (14), while second subfigure illustrates the results obtained from AEM* in Poznyak et al. (2014). As can be seen, the proposed approach indicates smooth convergence of the system states to the origin (cf. Fig. 1(a) with Fig. 1(b)) compared to AEM*. Remarkably, the obtained minimal size ellipsoid with Theorem 6 and SDP (14) is substantially smaller than the one in Poznyak et al. (2014) (cf. Fig. 1(c) with Fig. 1(d); and, Fig. 1(e) with Fig. 1(f)). This demonstrates that the proposed AEM approach is less conservative (compared to previous works) given that the feedback parameters of control design should be such that making the attractive ellipsoid of a smaller size. Also it needs less control effort to stabilise the system, where for the proposed approach $\|u\| = 9.12$, while for Poznyak et al. (2014) $\|u\| = 20.3$, cf. Fig. 1(g) and Fig. 1(h)).

Note that every state in state space can be taken inside P (red ellipsoid). In this example, Ω is equal to P but can be taken as $\Omega \succ P$ and any state inside Ω will satisfy the control constraint (10), see Section 3.2. The example in next subsection demonstrates this concept for clarity.

4.2 Example 2

Consider a continuous-time LPV system with the following affine parameter-dependent matrices,

$$A(\theta) = \begin{bmatrix} -1.6 + 0.4\theta & 2 & \theta \\ -2 + \theta & -2\theta & \theta \\ -\theta & 1 & -2\theta \end{bmatrix}, \quad B(\theta) = \begin{bmatrix} -1 & -0.5 \\ 0 & -1 \\ 1 & -2 \end{bmatrix},$$

and $W = \operatorname{diag}(3.333, 0.04, 0.0005) \times 10^5$, where θ is a time-invariant parameter with $|\theta| \le 1$. Matrix $A(\theta)$ is unstable for all $\theta \le 0$. The control input (10) is bounded by $\mu = 150$ with $R = I$ and $\Omega \succ P$. Let the disturbance input be,

$$\omega(t) = 0.5 + 0.2\sin(100t) + 0.2\sin(100t)\,(0.001, 0.01, 0.1)^{\mathsf{T}}.$$

To solve the stabilization problem, we consider a gain-scheduled feedback controller of the form (23) and express the LPV system into the polytopic form (15). The state matrix $A(\theta)$ can be expressed as in (17) with $N = 2$,

$$A_1 = \begin{bmatrix} -2 & 2 & -1 \\ -3 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1.2 & 2 & 1 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix},$$

where, $\theta_1 = \frac{\theta_{\max} - \theta}{\theta_{\max} - \theta_{\min}}, \theta_2 = \frac{\theta - \theta_{\min}}{\theta_{\max} - \theta_{\min}}$, with $\theta_{\min} = -1$ and $\theta_{\max} = 1$. The convex coordinates θ_i, $i = 1, 2$, satisfy $0 \le \theta_i \le 1$, and $\theta_1 + \theta_2 = 1$. The linear feedback gains K_1, K_2, the ellipsoidal matrices P_1, P_2, Ω_1 and Ω_2 resulting from the solution of the SDP problem (26) are as follows,

$$K_1 = \begin{bmatrix} -1.006 & 1.418 & -1.416 \\ 2.317 & -0.235 & 2.229 \end{bmatrix}, K_2 = \begin{bmatrix} 0.324 & 2.548 & 0.482 \\ -0.006 & 1.387 & 0.049 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} .0095 & .0443 & .0317 \\ .0443 & .2057 & .1471 \\ .0317 & .1471 & .1247 \end{bmatrix}, P_2 = \begin{bmatrix} .0678 & .1302 & -.0137 \\ .1302 & .2500 & -.0262 \\ -.0137 & -.0262 & .0221 \end{bmatrix},$$

$$\Omega_1 = \begin{bmatrix} 10.25 & -17.87 & -34.74 \\ -17.87 & 32.73 & 62.22 \\ -34.74 & 62.22 & 119.53 \end{bmatrix}, \Omega_2 = \begin{bmatrix} 10.28 & -17.93 & -34.84 \\ -17.93 & 32.82 & 62.4 \\ -34.84 & 62.4 & 119.87 \end{bmatrix},$$

with $\alpha = 0.92$ and $\eta = 0.34$. To illustrate the approach, consider the unstable system with $\theta_1 = 0.98$, $\theta_2 = 0.02$ and initial condition $x_0 = [-10\ -12\ 9]^{\mathsf{T}}$. Figs 1(i) and 1(j) show the obtained results. As can be seen, the proposed method provides fast convergence to the origin with maximum control effort $\|u\| = 19.9055$. In this example, $\Omega \succ P$ and any state inside Ω (blue ellipsoid) satisfies the control constraints. Thus, any initial state inside $\mathcal{E}(\Omega(\theta))$ can be taken to the origin for all θ while satisfying the control constraints (10).

5. CONCLUSIONS

The paper presented LMI conditions for the state-feedback constrained stabilization of continuous-time polytopic linear systems with unknown but bounded perturbations by the AEM. These LMI characterizations present the important property of decoupling state-space matrices with Lyapunov matrix. It allowed us to extend our approach to systems with structured uncertainty and scheduling parameters. For the constrained control case, the obtained LMIs guarantee that the attractive ellipsoid is nested inside the bigger ellipsoids imposed by the control and state constraints. Two numerical examples demonstrated the efficiency and advantages (in terms of convergence, control effort, and minimal size ellipsoidal sets) of the proposed approach compared to previous AEM-based works.

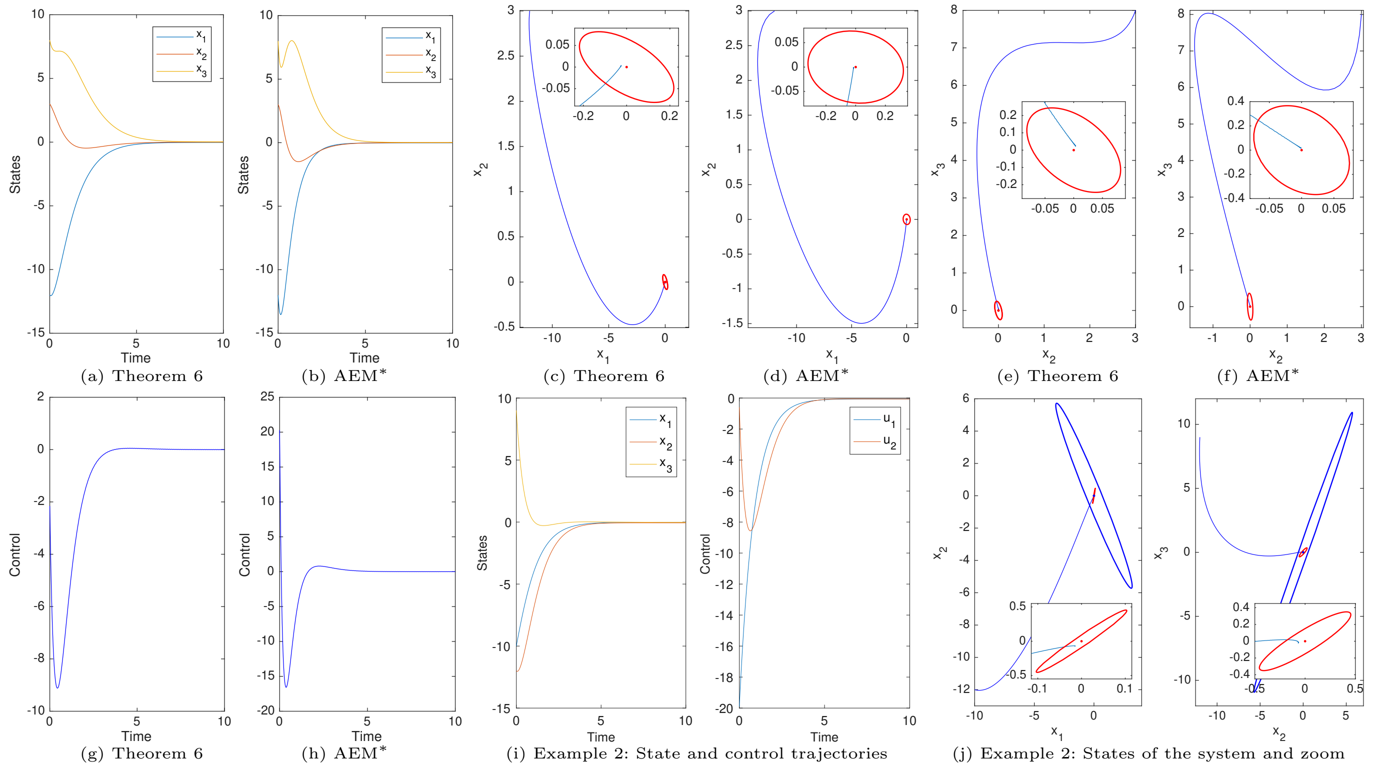

Fig. 1. Example 1: Subfigures (a)–(h). Example 2: Subfigures (i)–(j). The attractive ellipsoid P (indicated with red color) is nested inside the bigger ellipsoid Ω (indicated with blue color) imposed by the control constraints.

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