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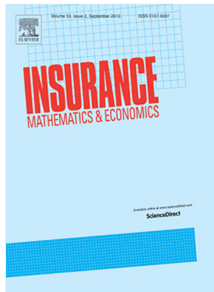

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BUDGET-CONSTRAINED OPTIMAL INSURANCE WITHOUT THE NONNEGATIVITY CONSTRAINT ON INDEMNITIES

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Abstract. In a problem of Pareto-efficient insurance contracting (bilateral risk sharing) with expected-utility preferences, Gollier [28] relaxes the nonnegativity constraint on indemnities and argues that the existence of a deductible is only due to the variability in the cost of insurance, not the nonnegativity constraint itself. In this paper, we find support for a similar statement in problems of budget-constrained optimal insurance (i.e., demand for insurance). Specifically, we consider a setting of ambiguity (unilateral and bilateral) and a setting of belief heterogeneity. We drop the nonnegativity constraint and assume no cost (or a fixed cost) to the insurer, and we derive closed-form solutions to the problems that we formulate. In particular, we show that optimal indemnities no longer include a deductible provision; and they can be negative for small values of the loss, or in case of no loss.

Key Words and Phrases: Optimal Insurance, Deductible Contract, Nonnegativity Constraint, Ambiguity, Knightian Uncertainty, Non-Additive Probability, Probability Distortion, Choquet Integral.

JEL Classification: C02, D86, G22.

2010 Mathematics Subject Classification: 91B30.

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1. Introduction

The literature on budget-constrained optimal insurance design follows Arrow's [4] classical formulation of the insurance demand problem. In the latter, an Expected-Utility (EU) maximizing decision maker (DM) is subject to an insurable random loss. He seeks an insurance indemnification against this loss so as to maximize his expected utility of terminal wealth, subject to the constraint that, in each state of the world, the indemnity is nonnegative and does not exceed the value of the loss, and subject to a budget constraint. The latter is typically formulated as a premium constraint: the price of insurance (measured through a premium principle) is at most equal to the DM's available insurance budget ($\Pi > 0$, fixed *ex ante*). Arrow [4] shows that, for the expected value premium principle, if the DM and the insurer share the same probabilistic beliefs about the realizations of the random loss, then an optimal indemnification schedule for a risk-averse EU-maximizing DM is a linear deductible schedule[1].

The subsequent actuarial literature on budget-constrained optimal insurance or reinsurance extended this classical model in several directions. For instance, Goovaerts, Van Heerwaarden, and Kaas [29], as well as Denuit and Vermandele [18], show that under the expected-value premium principle and a fixed budget constraint, a deductible contract is still optimal for a large class of stop-loss-order preserving preferences for the DM. Cai and Wei [10] extend the results of Denuit and Vermandele [18] to account for dependence between individual risks in an insurance portfolio. Young [53] provides an analytical characterization of the optimal indemnity, in a problem of maximizing expected utility of wealth with a fixed total insurance budget, but with a Wang premium principle (that is, a Choquet integral with respect to a concave distortion function - Definition 2.3). Gajek and Zagrodny [21, 22] and Kaluszka [33, 34, 35] study a problem of optimal reinsurance with a fixed total insurance budget, in which the premium principle is a mean-and-variance premium principle, and the optimization criterion relates to minimizing a convex risk measure of the insurer's retained risk. Kaluszka and Okolewski [36] extend Arrow's result to the case of a fixed total insurance budget and a maximal-possible-claims premium principle. Cheung et al. [12] extend the setting of Kaluszka and Okolewski [36] to the case of a fixed total insurance budget and when the DM's behaves according to Disappointment theories of choice, rather than expected-utility theory. Bernard and Tian [6] consider a setting similar to that of Kaluszka [33, 34, 35], with a fixed insurance budget, but assuming different optimization criteria related to trail risk measures. Bernard and Tian [7] extend Arrow's setting to account for an additional regulatory constraint related to insurer insolvency, while assuming a fixed insurance budget. Zhou et al. [54] and Cai et al. [9] consider Arrow's setting, with a fixed insurance budget, but impose in addition a fixed upper limit on the indemnity function. Tan et al. [48] examine the problem of determining the indemnity function that minimizes the conditional tail expectation (CTE) risk measure of the insurer's total risk, assuming a fixed total reinsurance budget and an expected-value premium principle. Sung et al. [47] extend Arrow's setting to the case of a fixed total insurance budget and an expected-value premium principle, but where the DM behaves according to Cumulative Prospect Theory [32, 49]. Bernard et al. [5], Xu et al. [51], and Ghossoub [24] extend Arrow's setting to the case of a fixed total insurance budget and an expected-value premium principle, but where the DM behaves according to Rank-Dependent Expected-Utility [42, 52]. Amarante et al. [3] and Amarante and Ghossoub [2] consider the case of a fixed total insurance budget and an EU-maximizing DM, but distortion premium principles and more general Choquet premium principles. Cui et al. [16], Zhuang et al. [55], Cheung et al. [11], Cheung and

[1] Note that Arrow's work and the subsequent literature on budget-constrained optimal (re)insurance focused on a budget constraint given by the expected value premium principle because, by the Law of Large Numbers, an insurer with EU preferences is essentially asymptotically risk neutral with vanishing risk premia. As shown in Knispel, Laeven, and Svindland [37], this broadly remains true for an insurer with ambiguity-averse preferences. I am grateful to the Associate Editor for pointing this out.

Lo [13], and Lo [38, 39] examine the problem of minimizing a distortion risk measure of the insurer's retained risk, with a fixed reinsurance budget, and under a distortion reinsurance premium principle.

Raviv [43] re-examines Arrow's [4] problem, but in a setting of bilateral risk sharing, rather than a setting of demand for insurance indemnification. He concludes that the presence of a deductible is due to both the nonnegativity constraint on the indemnification function and the variability in the cost of insurance. In an effort to test this statement, Gollier [28] relaxes the nonnegativity constraint and argues that the existence of a deductible is due to the variability in the cost of insurance, not the nonnegativity constraint. In this paper, we ask a similar question, but in a setting of budget-constrained optimal insurance design (that is, a problem of demand for insurance indemnification), rather than a setting of Pareto-optimal bilateral risk sharing. Specifically, we ask the following question: If we relax both the nonnegativity constraint on indemnities and the variability in the cost to the insurer (by assuming a fixed cost, or - without loss of generality - no cost), is it true that an optimal indemnity no longer includes a deductible provision? It turns out that the answer is positive, in several settings.

We first consider a setting where the DM is a Rank-Dependent Expected-Utility (RDEU)-maximizer (as in [42, 52]), who distorts the true probability distribution of the random loss, due to some ambiguity on his side, and where the premium principle is an expected-value premium principle. We relax the nonnegativity constraint and we assume that there are no costs associated with handling an insurance claim[2]. This allows us to test whether the existence of a deductible is due to the variable cost of insurance under ambiguity on the side of the DM. We give an analytical characterization of the optimal indemnity and find that if the DM distorts the true probabilities then the optimal indemnity for the DM does not include a deductible provision when there are no (or constant) insurance costs and no nonnegativity constraint. Moreover, the optimal indemnity can be negative for small values of the loss, or in case of no loss. This, as Gollier [28] notes, can be intuitively understood as the DM agreeing to pay an additional premium in case of no loss or small losses.

We then examine some special cases. In particular, we find that when the DM is ambiguity-averse, having a convex distortion function, the optimal indemnity is a linear function of the realizations of the random loss, and does not include a deductible provision. Moreover, the optimal indemnity can take negative values for small losses, but it is bounded below by a constant that depends on the DM's distortion function and on the difference between the premium and the expected loss under the insurer's belief. When this difference is zero, full insurance is optimal and hence the optimal indemnity is nonnegative. This result essentially implies that when the DM is risk-averse, full insurance is optimal in the absence of insurance cost, that is, that Mossin's Theorem [41] holds in our setting. Indeed, in RDEU, strong risk aversion (i.e., aversion to mean-preserving increases in risk) is jointly characterized by a concave utility function and a convex distortion function (e.g., [14]), whereas in EUT concavity of the utility function fully characterizes risk-aversion. In the case of an ambiguity seeking DM, with a concave distortion function, the optimal indemnity function is a nonlinear function of the realizations of the random loss, but does not include a deductible provision. Depending on the curvature of the distortion function at zero and one, the optimal indemnity function could be full insurance (and hence nonnegative), it could fully insure only small losses, or it could never fully insure losses of any value. In the case of ambiguity-neutrality, i.e., when the DM does not distort probabilities, the optimal indemnity is a linear function of the realizations of the random loss, taking negative values, but it does not include a deductible provision and it is bounded below by a constant that depends only on the difference between the premium and the expected loss under the insurer's belief. When this difference is zero, full insurance is optimal and hence the optimal indemnity is nonnegative.

[2] This assumption could be replaced with an assumption of constant cost of insurance, without changing any of this paper's results.

We subsequently extend the previous setup to a problem with a distortion premium principle, with a different distortion function than that of the DM. Under no additional assumptions on the probability distortion functions used, we give a closed-form characterization of the optimal retention in the absence of the nonnegativity constraint. We then examine several special cases, and in particular the case in which the two probability distortion functions coincide. In the latter case, we find that if this common distortion function is convex (indicating ambiguity aversion on both sides), the optimal retention is a constant function of the random loss that is not equal to the realized loss. Moreover, if the premium is equal to the distorted expected loss, then a zero retention (full insurance) is optimal. This, again, is intuitive in light of Mossin's Theorem, since strong risk aversion in RDEU is jointly characterized by a concave utility function and a convex distortion function. If the premium is less than the distorted expected loss, then the optimal retention can take values higher than the realized loss (optimal indemnity can take negative values). In the case of a common distortion function that is concave (indicating ambiguity seeking on both sides), the optimal retention function is a nonlinear function of the realizations of the random loss that we characterize in closed form.

Finally, we consider a setting with no ambiguity but belief heterogeneity, in which the DM and the insurer have non-ambiguous but diverging beliefs about the realizations of the insurable loss X, represented by two different probability measures on the underlying space. Under no additional assumptions (in particular, no monotonicity assumptions) about the likelihood ratio, we provide an analytical characterization of the optimal indemnity function, and we show how it does not include a deductible provision. As a special case, we examine the case of belief homogeneity and show that the optimal indemnity is a linear function of the realized loss, but does not include a deductible provision.

Outline. The rest of this paper is organized as follows. Section 2 provides the setup for the problems examined in this paper, as well as the necessary technical background. In Section 3, we examine the problem in the presence of ambiguity on the DM's side, first with an expected value premium principle, and second with a distortion premium principle. Section 4 studies the problem in case of no ambiguity on either side, but with differing beliefs about the realizations of the insurable loss. Finally, Section 5 concludes. Omitted proofs can be found in the Appendices.

2. Setup and Preliminaries

2.1. **Setup.** Let S be a nonempty collection of states of the world equipped with a σ-algebra $\mathcal{F}$ of events. The DM is facing a random loss represented by a random variable X on the measurable space $(S, \mathcal{F})$. Let $\Sigma = \sigma\{X\}$ be the sub-σ-algebra of $\mathcal{F}$ on S generated by X. We assume that the measurable space (S, Σ) is endowed with a probability measure P, such that the image measure of X under P is nonatomic[3] on the range of X with Borel σ-algebra, and such that X is essentially bounded.

Assumption 2.1. *We make the following assumptions on X:*

(1) $X \in L^\infty(S, \Sigma, P)$; *and*

(2) *X is a continuous random variable[4] for P. That is, the Borel probability measure $P \circ X^{-1}$ is nonatomic.*

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[3]A finite nonnegative measure η on a measurable space $(\Omega, \mathcal{A})$ is said to be *nonatomic* if for any $A \in \mathcal{A}$ with $\eta(A) > 0$, there is some $B \in \mathcal{A}$ such that $B \subsetneq A$ and $0 < \eta(B) < \eta(A)$.

[4]This assumption can be dropped, but one would have to use the Distributional Transform approach of Rüschendorf [44]. All the results of this paper would still hold, with adequate modifications.

Let $B(\Sigma)$ denote the vector space of all bounded, $\mathbb{R}$-valued, and Σ-measurable functions on (S, Σ), and $B^+(\Sigma)$ its positive cone. When endowed with the supnorm[5], $B(\Sigma)$ is a Banach space [19, IV.5.1]. By Doob's measurability theorem [1, Theorem 4.41], for any $Y \in B(\Sigma)$ there exists a bounded, Borel-measurable map $I : \mathbb{R} \to \mathbb{R}$ such that $Y = I \circ X$. Moreover, $Y \in B^+(\Sigma)$ if and only if the function I is nonnegative.

The DM has access to a competitive insurance market in which he can transfer the risk associated with the random loss X by purchasing insurance indemnification against X for a premium $\Pi > 0$ determined by the insurer, based on his beliefs about the realizations of X. An indemnity function is a random variable $Y = I(X)$ on (S, Σ), for some bounded, Borel-measurable map $I : X(S) \to \mathbb{R}$ that pays off the amount $I(X(s)) \in \mathbb{R}$ in state of world $s \in S$. By Doob's measurability theorem, we will hereafter identify the collection of possible indemnity functions with $B(\Sigma)$.

The DM has initial wealth $W_0 > \Pi$ and his total state-contingent wealth is the Σ-measurable, $\mathbb{R}$-valued and essentially bounded function on S defined by

$$W(s) := W_0 - \Pi - X(s) + Y(s), \quad \forall s \in S.$$

We assume that the DM expects the loss to be less than his initial wealth. This can be interpreted as stating that the DM is well-diversified so that the particular loss exposure X against which he is seeking an insurance coverage is expected to be sufficiently small compared to his total initial wealth.

Assumption 2.2. $\int X dP \leqslant W_0$.

Note that Assumption 2.2 is weaker than similar-type assumptions used in the literature, such as in [5, 51], for instance.

Additionally, as in Arrow's [4] framework, we suppose that the DM is risk averse, having a utility index u that satisfies the following.

Assumption 2.3. *The DM's utility function u is strictly increasing, strictly concave, continuously differentiable, and satisfies* $\lim\limits_{x \to +\infty} (u')^{-1}(x) < 0$.

Remark 2.4. Assumption 2.3 is weaker than the usual Inada-type assumptions, commonly used in the literature. Assuming that u is strictly concave and continuously differentiable implies that u' is both continuous and strictly decreasing. This then implies that $(u')^{-1}$ is continuous and strictly decreasing, by the Inverse Function Theorem. Moreover, the continuity of u implies that u is bounded on every closed and bounded subset of $\mathbb{R}$.

The DM's problem is that of finding an indemnity function that maximizes a functional of the form $V : B(\Sigma) \to \mathbb{R}$ that represents the DM's expected utility of terminal wealth, or a distorted expected utility (in the sense of CEU), etc., subject to a premium constraint and the constraint that the indemnity does not exceed the total loss.

2.2. Probability Distortions and the Choquet Integral.

Definition 2.5. A *probability distortion (or weighting) function* is a strictly increasing function $T : [0, 1] \to [0, 1]$ such that $T(0) = 0$ and $T(1) = 1$.

[5]Any $Y \in B(\Sigma)$ is bounded, and its supnorm is defined by $\|Y\|_{sup} := \sup\{|Y(s)| : s \in S\} < +\infty$.

Definition 2.6. Let (S, Σ, P) be a probability space and T a distorting function. Define the set function $\upsilon = T \circ P$ by $\upsilon(A) = T(P(A))$ for all $A \in \Sigma$. Then υ is called a *distorted probability measure.*

Proposition 2.7. *Let (S, Σ, P) be a probability space and $\upsilon = T \circ P$ a distorted probability measure on (S, Σ). Then:*

(1) *$\upsilon(\varnothing) = 0$ and $\upsilon(S) = 1$;*
(2) *υ is monotone: for any $A, B \in \Sigma$, $A \subseteq B \Rightarrow \upsilon(A) \leqslant \upsilon(B)$.*
(3) *υ is additive if and only if T is linear.*

Definition 2.8. Let (S, Σ, P) be a probability space and $\upsilon = T \circ P$ a distorted probability measure on (S, Σ). The *Choquet integral* with respect to υ of an integrable random variable Y is defined by

$$\int Y\, d\upsilon := \int_0^{+\infty} \upsilon(\{s \in S : Y(s) \geqslant t\})\ dt + \int_{-\infty}^{0} [\upsilon(\{s \in S : Y(s) \geqslant t\}) - 1]\ dt,$$

where the integrals are taken in the sense of Riemann.

When the function T is the identity function, so that $\upsilon = P$, the Choquet integral coincides with the usual Lebesgue integral. Some properties of the Choquet integral are listed below.

Proposition 2.9. *Let $\upsilon = T \circ P$ be a distorted probability measure on (S, Σ).*

(1) *If $A \in \Sigma$ then $\int \mathbf{1}_A\, d\upsilon = \upsilon(A)$.*
(2) *If $a \geqslant 0$, then $\int a\ Y\ d\upsilon = a\ \int Y\ \upsilon$; and,*
(3) *If $Y_1 \leqslant Y_2$, then $\int Y_1\ d\upsilon \leqslant \int Y_2\ d\upsilon$.*
(4) *If Y_1 and Y_2 are comonotonic*[6]*, then $\int (Y_1 + Y_2)\ d\upsilon = \int Y_1\ d\upsilon + \int Y_2\ d\upsilon$.*

In particular, if $Y \geqslant 0$ then $\int Y\ d\upsilon \geqslant 0$, and $\int (Y + c)\ d\upsilon = \int Y\ d\upsilon + c$ for all $c \in \mathbb{R}$. We refer to Denneberg [17] and Marinacci and Montrucchio [40] for proofs and additional results.

2.3. **Robust Representation of the Choquet Integral.** Let $ba(\Sigma)$ denote the linear space of all bounded finitely additive set functions on (S, Σ), endowed with the usual mixing operations. When endowed with the variation norm $\|.\|_v$, $ba(\Sigma)$ is a Banach space. By a classical result [19, IV.5.1], $(ba(\Sigma), \|.\|_v)$ is isometrically isomorphic to the norm-dual of the Banach space $(B(\Sigma), \|.\|_{sup})$ via the duality $<\phi, \lambda> = \int \phi\, d\lambda$, $\forall \lambda \in ba(\Sigma)$, $\forall \phi \in B(\Sigma)$. Consequently, we can endow $ba(\Sigma)$ with the weak* topology $\sigma(ba(\Sigma), B(\Sigma))$. If $ca(\Sigma)$ denotes the collection of all countably additive elements of $ba(\Sigma)$, then $ca(\Sigma)$ is a $\|.\|_v$-closed linear subspace of $ba(\Sigma)$. Hence, $ca(\Sigma)$ is $\|.\|_v$-complete, i.e. $(ca(\Sigma), \|.\|_v)$ is a Banach space. Henceforth, a collection of probability measures will be called weak*-compact if it is compact in the topology $\sigma(ba(\Sigma), B(\Sigma))$.

By a classical result of Huber and Strassen [31] and Schmeidler [45, 46], we have the following representations of the Choquet integral.

Proposition 2.10. *Let $\upsilon = T \circ P$ be a distorted probability measure on (S, Σ).*

[6]Two functions $Y_1, Y_2 \in B(\Sigma)$ are said to be comonotonic if $\left[Y_1(s) - Y_1(s')\right]\left[Y_2(s) - Y_2(s')\right] \geqslant 0$, for all $s, s' \in S$. For instance any $Y \in B(\Sigma)$ is comonotonic with any $c \in \mathbb{R}$. Moreover, if $Y_1, Y_2 \in B(\Sigma)$, and if Y_2 is of the form $Y_2 = I \circ Y_1$, for some Borel-measurable function I, then Y_2 is comonotonic with Y_1 if and only if the function I is nondecreasing.

(1) *If T is convex, then there exists a non-empty, convex, and weak*-compact collection $\Pi \subset ca(\Sigma)$ of probability measures, called the core of ν, such that for all $Y \in B(\Sigma)$,*

$$\int Y d\upsilon = \min_{\mu \in \Pi} \int Y d\mu.$$

(2) *If T is concave, then there exists a non-empty, convex, and weak*-compact collection $\mathcal{A} \subset ca(\Sigma)$ of probability measures, called the anti-core of ν, such that for all $Y \in B(\Sigma)$,*

$$\int Y d\upsilon = \max_{\mu \in \mathcal{A}} \int Y d\mu.$$

(3) *If T is linear, then T is the identity function and $\Pi = \mathcal{A} = \{P\}$. In this case, for all $Y \in B(\Sigma)$,*

$$\int Y d\upsilon = \int Y dP.$$

In Schmeidler's [46] CEU model, a DM's ambiguous beliefs are represented by a nonadditive set function υ on the state space. In the special case where $\upsilon = T \circ P$, for some probability weighting function T, ambiguity aversion (resp., ambiguity seeking) is equivalent to convexity (resp., concavity) of the distortion function T. Hence, in light of Proposition 2.10, ambiguity aversion (resp., ambiguity-seeking) implies a worst-case (resp., best-case) expectation with respect to a collection of (additive) priors. Ambiguity-neutrality is equivalent to linearity of the distortion function T and therefore yields the classical EU-representation of preferences.

3. Relaxing the Non-Negativity Constraint on Indemnities: Probability Weighting

In this section, we examine the problem of optimal insurance design when the DM has ambiguous beliefs represented by a distortion of the physical probability measure P, and in the absence of the nonnegativity constraint on indemnities. We first consider the case of a standard premium constraint of the form $\int Y dP \leqslant \Pi$, or equivalently, a retention constraint of the form $\int R dP \geqslant R_0$. We then consider the case of a more general retention constraint of the form $\int R dT_2 \circ P \geqslant \widetilde{R}_0$ for some distortion function T_2 that is not necessarily identical to that of the DM.

3.1. Ambiguity on the DM's Side.

We now consider the case in which the insurer experiences no ambiguity about the realizations of the insurable loss X, but the DM does and hence distorts the probability measure P. Specifically, the DM's problem, is the following.

Problem 3.1.

$$\sup_{Y \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - X + Y\Big)\ dT \circ P : Y \leqslant X, \int Y dP \leqslant \Pi \right\}.$$

Letting $R := X - Y$ be the retention random variable, the problem can now be restated as

Problem 3.2.

$$\sup_{R \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - R\Big)\ dT \circ P : R \geqslant 0, \int R dP \geqslant R_0 := \int X dP - \Pi \right\},$$

Clearly, R^* is optimal for Problem (3.2) if and only if $Y^* = X - R^*$ is optimal for Problem (3.1). Therefore, we focus on solving Problem (3.2). Now, the monotonicity of the Lebesgue integral implies that for each $R \geqslant 0$, $\int R dP \geqslant 0$. Consequently, if $R_0 < 0$, then the feasibility set of Problem (3.2) is empty. Thus, we will make the following assumption in order to rule out trivial situations.

Assumption 3.3. $R_0 \geqslant 0$.

Hence, Assumption 2.2 and Assumption 3.3 imply that $0 \leqslant R_0 \leqslant W_0 - \Pi$.

Recall[7] that for a continuous real-valued function f on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0,1]$, the convex envelope of f on the interval $[0,1]$ is an absolutely continuous real-valued function g such that:

(1) $g(0) = f(0)$ and $g(1) = f(1)$;

(2) g is convex on $[0,1]$;

(3) For all $x \in [0,1]$, $g(x) \leqslant f(x)$; and,

(4) g is affine on $\{x \in [0,1] : g(x) < f(x)\}$.

Moreover,

(5) If f is increasing, then so is g;

(6) If f is differentiable on $(0,1)$, then g is continuously differentiable on $(0,1)$.

The following result gives an analytical characterization of the optimal solution to Problem (3.1) under very mild assumptions on the DM's distortion function T.

Theorem 3.4. *The function* $Y^* := X - q^*\Big(T\big(1 - F_X(X)\big)\Big)$ *is optimal for Problem* (3.1) *and comonotonic with* X*, where:*

- *For all* $t \in [0,1]$, $q^*(t) = \max\Big[0, W_0 - \Pi - (u')^{-1}\Big(\lambda^* \delta'(t)\Big)\Big]$;
- δ *is the convex envelope of* $v = T^{-1}$ *on* $[0,1]$*; and,*
- λ^* *is chosen such that* $\int_0^1 q^*(t)\, \delta'(t)\, dt = R_0$.

Theorem 3.4 holds regardless of the ambiguity aversion or ambiguity seeking attitude of the DM. The following two results examine these special cases.

Corollary 3.5. *If the DM is either ambiguity neutral (T is the identity function) or ambiguity averse (T is convex), then an optimal solution for Problem* (3.1) *is given by* $Y^* = X - R_0$*, where* $R_0 = \int X dP - \Pi \in \mathbb{R}^+$*. Hence, in particular:*

- *Both* Y^* *and* $X - Y^*$ *are comonotonic with* X*;*
- *If the premium is equal to the expected loss, then full insurance is optimal;*
- *If the premium is less than the expected loss, then the optimal indemnity can take negative values, but it is bounded below by the constant* R_0.

[7] See, for instance, He et al. [30, Appendix B].

Note that Corollary 3.5 essentially implies that when the DM is risk-averse, full insurance is optimal in the absence of insurance cost, that is, that Mossin's Theorem [41] holds in our setting. Indeed, in RDEU, strong risk aversion (i.e., aversion to mean-preserving increases in risk) is jointly characterized by a concave utility function and a convex distortion function (e.g., [14]), whereas in EUT concavity of the utility function fully characterizes risk-aversion.

Corollary 3.6. *If the DM is ambiguity seeking (T is concave), then an optimal solution for Problem (3.1) that is comonotonic with X is given by*

$$Y^* = X - \max\left[0, W_0 - \Pi - \left(u'\right)^{-1}\left(\frac{\lambda^*}{T'\left(1 - F_X\left(X\right)\right)}\right)\right],$$

where λ^* is chosen such that

$$\int_0^1 \max\left[0, W_0 - \Pi - \left(u'\right)^{-1}\left(\frac{\lambda^*}{T'\left(T^{-1}\left(t\right)\right)}\right)\right]\left(\frac{1}{T'\left(T^{-1}\left(t\right)\right)}\right) dt = R_0.$$

Moreover,

(1) *If $\frac{\lambda^*}{u'(W_0 - \Pi)} > T'(0)$, then $\left\{s \in S : Y^*(s) = X(s)\right\} = \varnothing$, and so full insurance is never optimal.*

(2) *If $\frac{\lambda^*}{u'(W_0 - \Pi)} < T'(1)$, then $\left\{s \in S : Y^*(s) = X(s)\right\} = S$. In other words, full insurance is optimal.*

(3) *If $\frac{\lambda^*}{u'(W_0 - \Pi)} \in \left[T'(1), T'(0)\right]$, then $\left\{s \in S : Y^*(s) = X(s)\right\} = \left\{s \in S : X(s) \leqslant L\right\}$, where $L := F_X^{-1}\left(1 - \left(T'\right)^{-1}\left(\frac{\lambda^*}{u'(W_0 - \Pi)}\right)\right) \geqslant 0$. In other words, small losses are fully insured, and the optimal indemnity is nonnegative.*

Bernard et al. [5], Xu et al. [51], and Ghossoub [24] study the problem of optimal insurance design with a retention constraint and in the presence of the nonnegativity constraint on indemnities, and with ambiguity on the side of the DM, represented by a distortion T of the probability measure P. Bernard et al. [5] and Ghossoub [24] show that when the DM is ambiguity averse (T is convex), the optimal indemnity is a straight (linear) deductible contract. Moreover, when the DM is ambiguity-seeking (T is concave), the optimal indemnity is a variable deductible schedule, with a state-contingent deductible that depends on the state of the world only through the distortion function. Additionally, when the DM's distortion function is inverse S-shaped, Ghossoub [24] provides a closed-form characterization of the optimal indemnity and shows that it is a linear deductible schedule up to a cut-off loss severity, beyond which the optimal indemnity is a disappearing variable deductible schedule. Corollary 3.5 and Corollary 3.6 show that in the absence of variable insurance costs to the insurer, the straight deductible indemnity schedule in the case of ambiguity aversion and the variable deductible indemnity schedule in the case of ambiguity seeking are no longer optimal when we relax the nonnegativity constraint on indemnities.

An Illustration. We now consider a simple numerical example to illustrate the previous results. Suppose that the DM's distortion function T is given by an inverse S-shaped distortion function, such as the one used in Cumulative Prospect Theory [32, 49]. That is, for all $t \in [0, 1]$,

$$T(t) = \frac{t^\gamma}{\left(t^\gamma + (1-t)^\gamma\right)^{1/\gamma}}. \tag{3.1}$$

We take $\gamma = 0.5$, so that for all $t \in [0,1]$, $T(t) = \frac{\sqrt{t}}{\left(\sqrt{t}+\sqrt{(1-t)}\right)^2}$. Then T is strictly increasing on $[0,1]$. Moreover, one can easily verify that there is a point $t_0 \approx 0.3845$ such that T is strictly concave on $[0,t_0]$ and strictly convex on $[t_0,1]$. Therefore, T^{-1} is strictly increasing on $[0,1]$, strictly convex on $[0,t_0]$, and strictly concave on $[t_0,1]$. Let δ be the convex envelope of $v = T^{-1}$ on $[0,1]$. Then $v(0) = \delta(0) = 0$ and $v(1) = \delta(1) = 1$. Moreover, since δ is affine on the set $\{t \in [0,1] : \delta(t) < v(t)\}$, there exists some $z_0 \in (0,t_0)$ such that δ is given by

$$\delta(t) = \begin{cases} v(t) & \text{if } t \leq z_0; \\ v(z_0) + \left(\frac{v(z_0)-1}{z_0-1}\right)(t-z_0) & \text{if } t \geq z_0. \end{cases}$$

Note that since δ is continuously differentiable by continuity of v, we have $v'(z_0) = \frac{v(z_0)-1}{z_0-1}$. Numerical computation gives $z_0 \approx 0.17215$, $T(z_0) \approx 0.2364$, $v(z_0) = \delta(z_0) \approx 0.06654$, $\frac{v(z_0)-1}{z_0-1} \approx 1.12757$, $T(t_0) \approx 0.31429$, $v(t_0) \approx 0.58312$,and $\delta(t_0) \approx 0.30597$. Figure 1 plots the graph of the functions T, v, and δ.

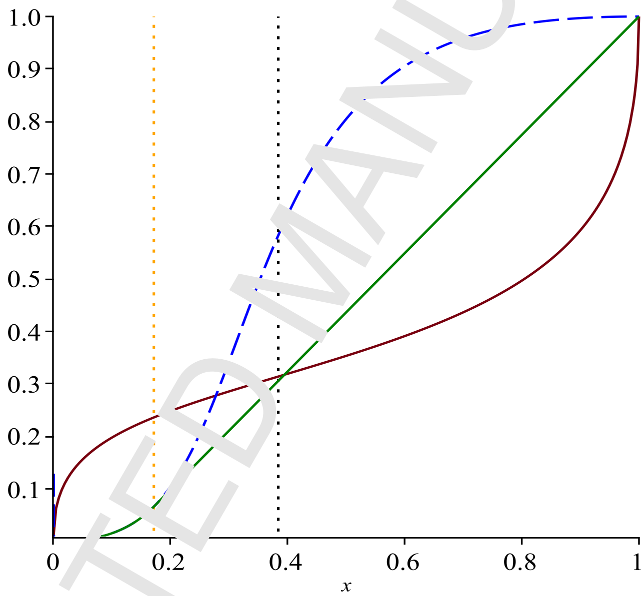

FIGURE 1. This graph plots the function T (solid red line), the function $v = T^{-1}$ (dashed blue line), and the convex envelope δ of v (solid dark green line). The dotted vertical black line is the graph of the function $f(t) := t_0$, and the dotted vertical orange line is the graph of the function $g(t) := z_0$.

We will assume that the loss random variable X follows a truncated exponential distribution on the interval $[0,M]$, with a probability density function f_X given by $f_X(x) = \frac{\eta e^{-\eta x}}{1-e^{-\eta M}}$, for $x \in [0,M]$, where η and M are constants. Then the expected value of X under P is given by

$$E[X] = \int X dP = \frac{1-(1+\eta M)e^{-\eta M}}{\eta\left(1-e^{-\eta M}\right)},$$

the cumulative distribution function of X is given by $F_X(x) = \frac{1-e^{-\eta x}}{1-e^{-\eta M}}$, for $x \in [0,M]$, and the quantile of X is given by

$$F_X^{-1}(t) = \frac{-1}{\eta} \ln\left(1 - t\left[1 - e^{-\eta M}\right]\right),$$

for $t \in [0,1]$. We take $W_0 = 50$, $\Pi = 5$, $M = 1{,}000$, and $\eta = 0.1$, so that $E[X] = 10$ and $R_0 = E[X] - \Pi = 5$. Hence, $0 \leqslant R_0 \leqslant W_0 - \Pi$ and so Assumption 2.2 and Assumption 3.3 hold. Now, assume that $u(x) = x^{\alpha}$, and take $\alpha = 0.5$. Then u satisfies the conditions of Assumption 2.3, and $(u')^{-1}(x) = \frac{1}{4x^2}$. Consequently, an optimal indemnity that is comonotonic with X is given by

$$Y^* = X - q^*\Big(T\big(1 - F_X(X)\big)\Big),$$

where the function q^* is given by

$$q^*(t) = \begin{cases} \max\left[0, W_0 - \Pi - \frac{1}{(2\lambda^*)^2 (v'(t))^2}\right] & \text{if } t \leqslant z_0; \\ \max\left[0, W_0 - \Pi - \frac{1}{(2\lambda^*)^2 \left(\frac{v(z_0)-1}{z_0-1}\right)^2}\right] & \text{if } t \geqslant z_0; \end{cases}$$

and λ^* is chosen such that $\int_0^1 q^*(t)\, v'(t)\, dt = R_0$. Figure 2 below illustrates the optimal indemnity and retention in this simple example. In this case, with an inverse-S-shaped distortion function for the DM, it turns out that the optimal indemnity function does not include a deductible provision, but mandates a negative reimbursement for small values of the loss. This can be intuitively understood as the DM agreeing to pay an additional premium in case of no loss or small losses. Moreover, indemnification is a linearly increasing function of the loss, and medium to high severity losses are fully insured.

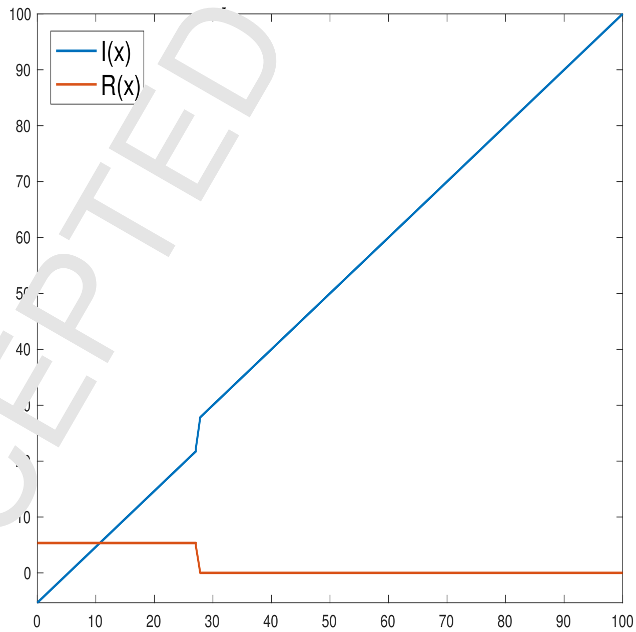

FIGURE 2. This graph plots the optimal indemnity function $I(X)$ (blue line) and the optimal retention function $R(X)$ (red line).

3.2. **Ambiguity on the DM's and the Insurer's side.** We now consider the case of a more general retention constraint of the form $\int R dT_2 \circ P \geqslant \widetilde{R}_0$ for some distortion function T_2 that is not necessarily identical to that of the DM. This reflects that fact that the insurer also experiences some ambiguity about the realizations of the insurable loss X, and such ambiguity is represented by a distortion of the baseline probability measure P.

Let $R := X - Y$ be the retention random variable, and consider the following problem.

Problem 3.7.
$$\sup_{R \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - R\Big)\, dT_1 \circ P : R \geqslant 0, \int R\, dT_2 \circ P \geqslant \widetilde{R}_0 := \int X dT_2 \circ P - \Pi \right\}.$$

Here, instead of Assumption 2.2 and Assumption 3.3, we use the following assumptions.

Assumption 3.8. $\int X dT_2 \circ P \leqslant W_0$.

Assumption 3.9. $\widetilde{R}_0 \geqslant 0$.

Hence, Assumption 3.8 and Assumption 3.9 imply that $0 \leqslant \widetilde{R}_0 \leqslant W_0 - \Pi$. By a proof similar to that of Theorem 3.4, we obtain the following result.

Theorem 3.10. *The function $R^* := q^*\Big(T_1\big(1 - F_X(X)\big)\Big)$ is optimal for Problem (3.7), where:*

- *For all $t \in [0,1]$, $q^*(t) = \max\left[0, W_0 - \Pi - (u')^{-1}\Big(\lambda^* \delta'(t)\Big)\right]$;*
- *δ is the convex envelope on $[0,1]$ of the function Ψ defined on $[0,1]$ by*
$$\Psi(t) := \int_0^t \left(\frac{T_2'\left(1 - T_1^{-1}(x)\right)}{T_1'\left(T_1^{-1}(x)\right)} \right) dx = 1 - T_2\left(1 - T_1^{-1}(t)\right);$$
- *λ^* is chosen such that $\int_0^1 q^*(t)\, \Psi'(t)\, dt = \widetilde{R}_0$.*

Theorem 3.10 gives a analytic characterization of the optimal solution to Problem (3.7) under very mild assumptions about the distortion functions T_1 and T_2. The following two corollaries examine some special cases of interest.

Corollary 3.11. *If the distortion functions T_1 and T_2 are such that, for all $t \in [0,1]$,*
$$\frac{T_2''(1-t)}{T_2'(1-t)} \geqslant -\frac{T_1''(t)}{T_1'(t)}, \tag{3.2}$$
then an optimal solution for Problem (3.7) is given by the constant function $R^* = \widetilde{R}_0$, where $\widetilde{R}_0 = \int X dT_2 \circ P - \Pi \in \mathbb{R}^+$. Hence, in particular:

- *Both R^* and $X - R^*$ are comonotonic with X;*
- *If the premium is equal to the distorted expected loss (under the insurer's distortion function), then a zero retention (full insurance) is optimal;*

- *If the premium is less than the distorted expected loss (under the insurer's distortion function), then the optimal retention can take values higher than the realized loss (i.e., the indemnity can be negative).*

One immediate case in which eq. (3.2) holds is when T_1 and T_2 are both linear or both convex.

Corollary 3.12. *If the distortion functions T_1 and T_2 are such that, for all $t \in [0,1]$,*

$$\frac{T_2''(1-t)}{T_2'(1-t)} \leqslant -\frac{T_1''(t)}{T_1'(t)}, \tag{3.3}$$

then an optimal solution for Problem (3.7) is given by the function

$$R^* := \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda^*\left(\frac{T_2'(F_X(X))}{T_1'(1-F_X(X))}\right)\right)\right],$$

where λ^* is chosen such that

$$\widetilde{R}_0 = \int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda^*\left(\frac{T_2'\left(1-T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}\right)\right)\right]\left(\frac{T_2'\left(1-T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}\right) dt.$$

One immediate case in which eq. (3.3) holds is when T_1 and T_2 are both linear or both concave.

Amarante and Ghossoub [2] study the problem of optimal insurance design with a retention constraint and in the presence of the nonnegativity constraint on indemnities, but with ambiguity only on the side of the insurer, represented by a distortion T_2 of the probability measure P. Ghossoub [24] extends the setting of Amarante and Ghossoub [2] to also account for ambiguity on the side of the DM, represented by a distortion T_1 of the probability measure P. He shows that the optimal indemnity is a variable deductible schedule, with a state-contingent deductible that depends on the sate of the world only through T_1 and T_2. The above results show that in the absence of ambiguity on the DM's side (T_1 is the identity function) and variable insurance costs to the insurer, the variable deductible indemnity schedule is no longer optimal when we relax the nonnegativity constraint on indemnities.

Theorem 3.10, Corollary 3.11, and Corollary 3.12 characterize of the optimal solution to Problem (3.7) when the DM and the insurer have different distortions of the baseline probability. As an immediate implication of Theorem 3.10, we obtain the following result, which characterizes the optimal solution in case the DM and insurer have the same distortion function $T := T_1 = T_2$.

Corollary 3.13. *If $T_1 = T_2 := T$, then the function $R^* := q^*\Big(T\Big(1-F_X(X)\Big)\Big)$ is optimal for Problem (3.7), where:*

- *For all $t \in [0,1]$, $q^*(t) := \max\Big[0, W_0 - \Pi - (u')^{-1}\Big(\lambda^*\delta'(t)\Big)\Big]$;*
- *δ is the convex envelope on $[0,1]$ of the function Ψ defined on $[0,1]$ by*
$$\Psi(t) := \int_0^t \left(\frac{T'\left(1-T^{-1}(x)\right)}{T'\left(T^{-1}(x)\right)}\right) dx = 1 - T\left(1-T^{-1}(t)\right);$$
- *λ^* is chosen such that $\int_0^1 q^*(t)\,\Psi'(t)\,dt = \widetilde{R}_0$, where $\widetilde{R}_0 = \int X dT_2 \circ P - \Pi \in \mathbb{R}^+$.*

Corollary 3.13 holds regardless of the concavity/convexity of T. The following two results examine these special cases.

Corollary 3.14. *If the distortion function $T := T_1 = T_2$ is either linear or convex, then an optimal solution for Problem* (3.7) *is given by the constant function $R^* = \widetilde{R}_0$. Hence, in particular:*

- *Both R^* and $X - R^*$ are comonotonic with X;*
- *If the premium is equal to the distorted expected loss, then a zero retention (full insurance) is optimal;*
- *If the premium is less than the distorted expected loss, then the optimal retention can take values higher than the realized loss (i.e., the indemnity can be negative).*

Note that Corollary 3.14 essentially implies that full insurance is optimal in the absence of insurance cost, when the DM is risk averse. This is intuitive in light of Mossin's Theorem [41], since strong risk aversion in RDEU is jointly characterized by a concave utility function and a convex distortion function [14].

Corollary 3.15. *If the distortion function $T := T_1 = T_2$ is concave, then an optimal solution for Problem* (3.7) *is given by the function*

$$R^* := \max\left[0, W_0 - \Pi - \left(u'\right)^{-1}\left(\lambda^*\left(\frac{T'\left(F_X\left(X\right)\right)}{T'\left(1 - F_X\left(X\right)\right)}\right)\right)\right],$$

where λ^* is chosen such that

$$\widetilde{R}_0 = \int_0^1 \max\left[0, W_0 - \Pi - \left(u'\right)^{-1}\left(\lambda^*\left(\frac{T'\left(1 - T_1^{-1}\left(t\right)\right)}{T'\left(T_1^{-1}\left(t\right)\right)}\right)\right)\right]\left(\frac{T'\left(1 - T^{-1}\left(t\right)\right)}{T'\left(T^{-1}\left(t\right)\right)}\right) dt.$$

An Illustration. We now consider a simple numerical example to illustrate the previous results. Suppose, as in the example of Section 3.1, that the DM's distortion function T_1 is given by an inverse S-shaped distortion function, such as the one used in Cumulative Prospect Theory [32, 49]. That is, for all $t \in [0, 1]$,

$$T_1\left(t\right) = \frac{t^{\gamma}}{\left(t^{\gamma} + (1 - t)^{\gamma}\right)^{1/\gamma}}, \tag{3.4}$$

withe $\gamma = 0.5$. Similarly, we assume that the insurer's distortion function T_2 is inverse S-shaped, with

$$T_2\left(t\right) = \frac{t^{\zeta}}{\left(t^{\zeta} + (1 - t)^{\zeta}\right)^{1/\zeta}}, \tag{3.5}$$

for all $t \in [0, 1]$, with $\zeta = 0.6$. Then one can easily verify that there is $t_0 \in [0, 1]$ such that the function Ψ defined on $[0, 1]$ by $\Psi\left(t\right) = 1 - T_2\left(1 - T_1^{-1}\left(t\right)\right)$, is convex on the interval $[0, t_0]$ and concave on the interval $[t_0, 1]$. Let δ be the convex envelope of Ψ on $[0, 1]$. Then $\Psi\left(0\right) = \delta\left(0\right) = 0$ and $\Psi\left(1\right) = \delta\left(1\right) = 1$. Moreover, since δ is affine on the set $\{t \in [0, 1] : \delta\left(t\right) < \Psi\left(t\right)\}$, there exists some $z_0 \in (0, t_0)$ such that δ is given by

$$\delta\left(t\right) = \begin{cases} \Psi\left(t\right) & \text{if } t \leqslant z_0; \\ \Psi\left(z_0\right) + \left(\frac{\Psi(z_0) - 1}{z_0 - 1}\right)\left(t - z_0\right) & \text{if } t \geqslant z_0. \end{cases}$$

Note that since δ is continuously differentiable by continuity of Ψ, we have $\Psi'\left(z_0\right) = \frac{\Psi(z_0) - 1}{z_0 - 1}$. Numerical computation gives $z_0 \approx 0.02414$. Figure 3 plots the graph of the functions T_1, T_2, ψ, and δ.

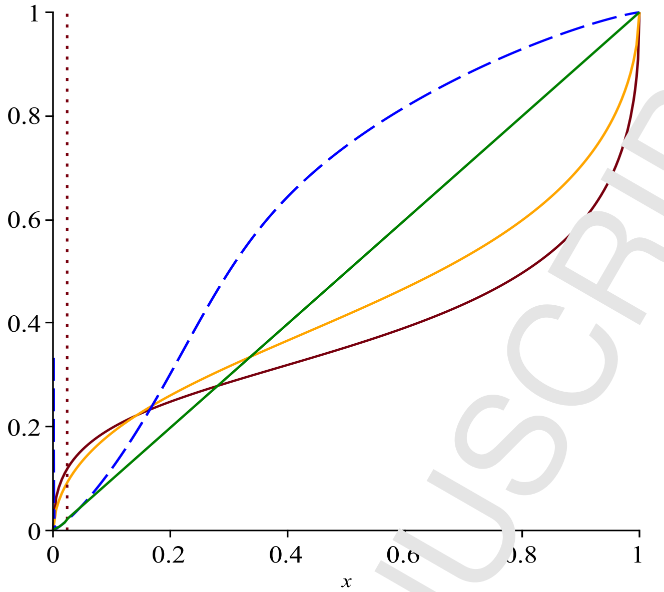

FIGURE 3. This graph plots the function T_1 (solid red line), the function T_2 (solid orange line), the function Ψ (dashed blue line), and the convex envelope δ of Ψ (solid dark green line). The dotted vertical burgundy line is the graph of the function $g(t) := z_0$.

Assuming, as in the example of Section 3.1, that $W_0 = 50$, $\Pi = 5$, and the loss random variable X follows a truncated exponential distribution on the interval $[0, 1000]$, with a probability density function f_X given by $f_X(x) = \frac{\eta e^{-\eta x}}{1-e^{-1,000\eta}}$, for $x \in [0, 1000]$, where $\eta = 0.1$, we have $E[X] = 10$ and the cumulative distribution function of X is given by $F_X(x) = \frac{1-e^{-\eta x}}{1-e^{-1,000\eta}}$, for $x \in [0, 1000]$. Therefore,

$$
\begin{aligned}
\int X dT_2 \circ P &= \int_0^{+\infty} T_2 \circ P(\{s \in S : X(s) \geqslant t\})\ dt = \int_0^{+\infty} T_2 \left[P(\{s \in S : X(s) > t\})\right]\ dt \\
&= \int_0^{+\infty} T_2\left[1 - F_X(t)\right]\ dt = \int_0^{+\infty} T_2\left(\frac{e^{-\eta t} - e^{-1,000\eta}}{1 - e^{-1,000\eta}}\right)\ dt \\
&= \int_0^{+\infty} \frac{\left(\frac{e^{-0.1t}-e^{-100}}{1-e^{-100}}\right)^{0.6}}{\left(\left(\frac{e^{-0.1t}-e^{-100}}{1-e^{-100}}\right)^{0.6} + \left(\frac{1-e^{-0.1t}}{1-e^{-100}}\right)^{0.6}\right)^{1/0.6}}\ dt \\
&\approx 12.047176.
\end{aligned}
$$

Thus, $\widetilde{R}_0 = \int X dT_2 \circ P - \Pi \approx 7.047176$, and so $0 \leqslant \widetilde{R}_0 \leqslant W_0 - \Pi$. Thus, Assumption 3.8 and Assumption 3.9 hold. Now, assume that $u(x) = x^{\alpha}$, and take $\alpha = 0.5$. Then u satisfies the conditions of Assumption 2.3, and $(u')^{-1}(x) = \frac{1}{4x^2}$. Consequently, an optimal retention is given by

$$
R^* = q^*\Big(T_1\big(1 - F_X(X)\big)\Big),
$$

where the function q^* is given by

$$q^*(t) = \begin{cases} \max\left[0, W_0 - \Pi - \frac{1}{(2\lambda^*)^2 (\Psi'(t))^2}\right] & \text{if } t \leqslant z_0; \\ \max\left[0, W_0 - \Pi - \frac{1}{(2\lambda^*)^2 \left(\frac{\Psi(z_0)-1}{z_0-1}\right)^2}\right] & \text{if } t > z_0; \end{cases}$$

and λ^* is chosen such that $\int_0^1 q^*(t)\,\Psi'(t)\,dt = \widetilde{R}_0$. Figure 4 below illustrates the optimal indemnity and retention in this simple example. In this case, with an inverse-S shaped distortion function for the DM and for the distortion premium principle, it turns out that the optimal indemnity function does not include a deductible provision, but mandates a negative reimbursement for small values of the loss. This can be intuitively understood as the DM agreeing to pay an additional premium in case of no loss or small losses. Moreover, indemnification is a linearly increasing function of the loss, and medium to high severity losses are fully insured.

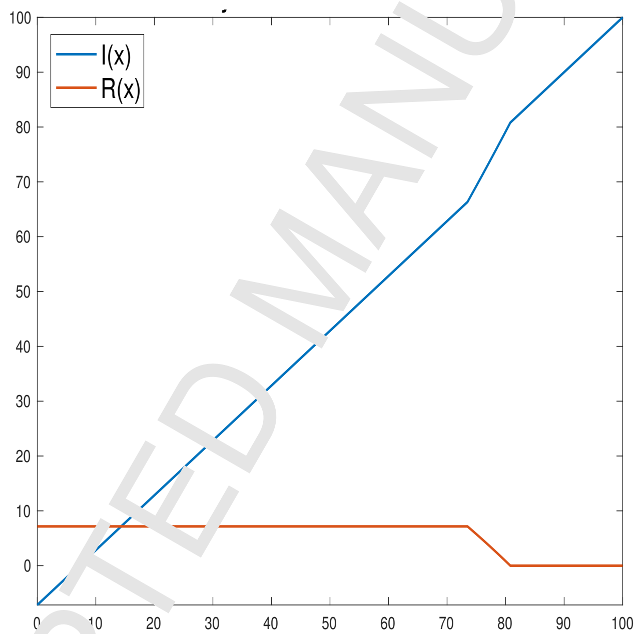

FIGURE 4. This graph plots the optimal indemnity function $I(X)$ (blue line) and the optimal retention function $R(X)$ (red line).

4. Relaxing the Non-Negativity Constraint on Indemnities: Belief Heterogeneity

In this section, we examine the problem of optimal insurance design in the absence of the nonnegativity constraint on indemnities, when the DM and the insurer experience no ambiguity about the realizations of the insurable loss X, but they disagree about the probability distribution of X. Specifically, we assume that the DM has a subjective probability measure P on the measurable space (S, Σ), whereas the insurer has a subjective probability measure Q on the same space. The DM's problem can then be formulated as follows.

Problem 4.1.

$$\sup_{Y \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - X + Y\Big)\, dP : Y \leqslant X, \int Y dQ \leqslant \Pi \right\}.$$

Letting $R := X - Y$ be the retention random variable, the problem can now be restated as

Problem 4.2.

$$\sup_{R \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - R\Big)\, dP : R \geqslant 0, \int R dQ \geqslant \overline{R}_0 := \int X dQ - \Pi \right\}.$$

Here, instead of Assumption 2.2 and Assumption 3.3, we use the following assumptions.

Assumption 4.3. $\int X dQ \leqslant W_0$.

Assumption 4.4. $\overline{R}_0 \geqslant 0$.

Note that Assumption 4.4 implies that the feasibility set of Problem (4.2) is non-empty.

Now suppose that the probability measure Q is absolutely continuous[8] with respect to P. Then, by the Radon-Nikodým Theorem [1, Theorem 13.20] there exists a P-a.s. unique, Σ-measurable, and P-integrable function $h : S \to [0, +\infty)$ such that $Q(C) = \int_C h\, dP$, for all $C \in \Sigma$. Moreover, since $h : S \to [0, +\infty)$ is Σ-measurable and P-integrable, there exists a Borel-measurable and $P \circ X^{-1}$-integrable map $\Gamma : X(S) \to [0, +\infty)$ such that $h = dQ/dP = \Gamma \circ X$. The function h can be interpreted as a likelihood ratio. We will assume that the Radon-Nikodým derivative h is continuous for P:

Assumption 4.5. *$Q \ll P$, with Radon-Nikodým derivative $h = {}^{dQ}/{}_{dP}$ such that $P \circ h^{-1}$ is nonatomic*[9].

Assumption 4.5 then implies that the random variable $\overline{U} = F_h(h)$ is uniformly distributed on $[0, 1]$, where F_h is the CDF of h under P. The Radon-Nikodým derivative h can be interpreted as a likelihood ratio. Note that in this section we do not make use of the assumption of nonatomicity of $P \circ X^{-1}$. Problem 4.2 can now be restated as follows.

Problem 4.6.

$$\sup_{R \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - R\Big)\, dP : R \geqslant 0, \int R h dP \geqslant \overline{R}_0 \right\}.$$

The following result gives an analytical characterization of the optimal solution to Problem (4.1) under no additional assumptions (in particular, no monotonicity assumptions) about the likelihood ratio h.

---

[8]Let μ_1 and μ_2 be two probability measures on a measurable space $(\Omega, \mathcal{G})$. The probability measure μ_2 is said to be absolutely continuous with respect to the probably measure μ_1 (denoted by $\mu_2 \ll \mu_1$) if for all $C \in \mathcal{G}$ with $\mu_1(C) = 0$, one has $\mu_2(C) = 0$. This does not rule out the existence of some $D \in \mathcal{G}$ such that $\mu_2(D) = 0$ but $\mu_1(D) > 0$.

[9]The assumption of nonatomicity of $P \circ h^{-1}$ can be dropped, but one would have to use the Distributional Transform approach of Rüschendorf [44]. All the results of this section would still hold, with adequate modifications. Moreover, the assumption of absolute continuity of Q with respect to P can be dropped, and one can use the technique developed in Ghossoub [23, 26] and Amarante and Ghossoub [2], based on a Lebesgue decomposition of P with respect to Q.

Theorem 4.7. *The function* $Y^* := X - f^*\Big(F_h(h)\Big)$ *is optimal for Problem* (4.1), *where:*

- *For all* $t \in [0,1]$, $f^*(t) = \max\Big[0, W_0 - \Pi - (u')^{-1}\Big(\lambda^* F_h^{-1}(t)\Big)\Big]$;
- λ^* *is chosen such that* $\int_0^1 f^*(t) F_h^{-1}(t)\, dt = \overline{R}_0$.

That is,

$$Y^* = \min\Big[X, X - \Big(W_0 - \Pi - (u')^{-1}(\lambda^* h)\Big)\Big]$$

As a special case, the following result characterizes the optimal solution assuming a monotone likelihood ratio.

Corollary 4.8. *Assuming a monotone likelihood ratio (MLR), i.e., that the function* Γ *in* $h = \Gamma \circ X$ *is nonincreasing, the optimal solution* $Y^* = X - f^*\Big(F_h(h)\Big)$ *given in Theorem 4.7 is comonotonic with* X.

Proof. The function f^* defined on $[0,1]$ by $f^*(t) = \max\Big[0, W_0 - \Pi - (u')^{-1}\left(\lambda^* F_h^{-1}(t)\right)\Big]$ is nondecreasing. If Γ is nonincreasing, then $-f^*(F_h(h))$ is nondecreasing in X. Hence, Y^* is comonotonic with X. $\square$

The problem of optimal insurance design with belief heterogeneity was studied by Ghossoub [23, 26, 27], Boonen [8], and Chi [15], in the presence of the nonnegativity constraint on indemnities. Among other results, Ghossoub [27] shows that when the likelihood ratio is monotone, the optimal indemnity is a variable deductible schedule, with a state-contingent deductible given by the random variable $d := W_0 - \Pi - (u')^{-1}(\lambda^* h)$, where h is the (monotone) likelihood ratio and λ^* is chosen so that the constraint binds. Under a condition of compatibility between the two beliefs, Ghossoub [26] fully characterizes the class of all optimal indemnity schedules that are nondecreasing in the loss, in terms of their distribution under the DM's probability measure, and he obtains Arrow's classical result as well as one of the results of Ghossoub [27] as corollaries. However, Ghossoub [26] does not provide a closed-form characterization of the optimal indemnity in the general case, which is done by Ghossoub [23]. The latter does not impose conditions on the type or level of disagreement about probabilities. He characterizes the optimal indemnity for any type or level of belief heterogeneity, and shows that it has a simple two-part structure: full insurance on an event to which the insurer assigns zero probability, and a variable deductible on the complement of this event. Chi [15] considers a similar setting to Ghossoub [26, 27] but imposes the *no sabotage* condition. That is, he restricts the set of admissible indemnities to those that are such that the indemnity and the retention function are both nondecreasing functions of the loss. Under an assumption of a Monotone Hazard Ratio (MHR), which is weaker than the MLR assumption, he shows optimality of a linear deductible schedule. Boonen [8] provides an implicit characterization of the optimal indemnity that relies on the hazard ratio, similarly to Chi [15]. Theorem 4.7 and Corollary 4.8 show that in the absence of variable insurance costs to the insurer, the optimal indemnity no longer contains a variable deductible provision when we relax the nonnegativity constraint on indemnities.

As a special case of Theorem 4.7, the following result characterizes the solution in the absence of belief heterogeneity.

Corollary 4.9. *In the absence of belief heterogeneity, i.e. when $P = Q$, an optimal solution to Problem (4.1) is given by*

$$Y^* = \min \Big[X, X - d \Big],$$

where $d = W_0 - \Pi - (u')^{-1}(\lambda^*)$ and λ^* is chosen such that the retention constraint binds.

Proof. If $P = Q$ then h is the constant function equal to 1 for each $s \in S$. The rest follows from Theorem 4.7. □

Corollary 4.9 shows that, in the absence of variable insurance costs to the insurer, a deductible indemnity schedule is not optimal when we relax the nonnegativity constraint on indemnities in Arrow's classical setting.

5. Conclusion

In this paper, we dropped the nonnegativity constraint on indemnities in several problems of budget-constrained optimal insurance (i.e., insurance demand). Specifically, we assumed no cost (or a fixed cost) to the insurer and considered three different settings: (i) a problem in which the DM experiences ambiguity about the realizations of the insurable loss, and distorts the underlying probability distribution while the insurer does not; (ii) a setting in which both the DM and the insurer experience ambiguity and distort the underlying probability measure using different distortion functions; and (iii) a setting in which the DM and the insurer experience no ambiguity but differ in their beliefs about the realizations of the insurable loss, and hence assign different probability distributions to that loss.

In all three settings, we derived closed-form analytical solutions to the problems that we formulated, and we showed that an optimal indemnity no longer includes a deductible provision. This is in line with the intuition behind Gollier's [28] finding in the case of belief homogeneity and no ambiguity, but in a setting of Pareto-efficient insurance contracting (bilateral risk sharing).

Future work on this topic will address the question of determining the optimal indemnity in the absence of the nonnegativity constraints on indemnities in each of the aforementioned three settings, but in a context of Pareto-optimal insurance design, in which the joint determination of the premium and the indemnity is required. Moreover, various cost structures for the insurer will be accounted for.

Appendix A. Proof of Theorem 3.4

Recall Problem (3.2):

$$\sup_{R \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - R\Big)\, dT \circ P : R \geqslant 0, \int R\, dP \geqslant R_0 := \int X\, dP - \Pi \right\}.$$

Clearly, R^* is optimal for Problem (3.2) if and only if $Y^* = X - R^*$ is optimal for Problem (3.1). Therefore, we focus on solving Problem (3.2).

Let $U := 1 - F_X(X)$ and $V := F_X(X)$. By assumption of nonatomicity of $P \circ X^{-1}$, U and V are uniformly distributed random variables on $(0,1)$ [20, Lemma A.25]. Now for all $R \in B(\Sigma)$, the fact that u is increasing and U is uniformly distributed implies that

$$\begin{aligned}
\int u(W_0 - \Pi - R)\, dT \circ P &= \int T'(1-U)\, F^{-1}_{u(W_0 - \Pi - R)}(U)\, dP = \int T'(1-U)\, u\left(F^{-1}_{W_0 - \Pi - R}(U)\right) dP \\
&= \int T'(1-U)\, u\left(W_0 - \Pi + F^{-1}_{-R}(U)\right) dP \\
&= \int T'(1-U)\, u\left(W_0 - \Pi - F^{-1}_{R}(1-U)\right) dP \\
&= \int T'(U)\, u\left(W_0 - \Pi - F^{-1}_{R}(U)\right) dP = \int_0^1 T'(t)\, u\left(W_0 - \Pi - F^{-1}_{R}(t)\right) dt.
\end{aligned}$$

Moreover,

$$\int R dP = \int F_R^{-1}(U)\, dP = \int_0^1 F_R^{-1}(t)\, dt,$$

and $R \geqslant 0$ whenever $F_R^{-1}(t) \geqslant 0$, for all $t \in (0,1)$.

Let $\mathcal{Q}$ denote the collection of all quantile functions, and let $\mathcal{Q}^*$ denote the collection of all quantile functions f that satisfy $f(t) \geqslant 0$, for all $t \in (0,1)$. Then

$$\mathcal{Q} = \Big\{ f : (0,1) \to \mathbb{R} \;\Big|\; f \text{ is nondecreasing and left-continuous} \Big\},$$

and

$$\mathcal{Q}^* = \Big\{ f \in \mathcal{Q} : f(t) \geqslant 0, \text{ for each } 0 < t < 1 \Big\}. \tag{A.1}$$

Consider the following problem:

Problem A.1.

$$\sup_{f \in \mathcal{Q}^*} \left\{ \int_0^1 u\Big(W_0 - \Pi - f(t)\Big) T'(t)\, dt : \int_0^1 f(t)\, dt \geqslant R_0 \right\}.$$

Lemma A.2. *If f^* is optimal for Problem (A.1), then both $R_1^* := f^*(1 - F_X(X))$ and $R_2^* := f^*(F_X(X))$ are optimal for Problem (3.2). Moreover, R_1^* is anti-comonotonic with X and R_2^* is comonotonic with X.*

Proof. Let f^* be optimal for Problem (A.1). Then, by definition of $\mathcal{Q}^*$, f^* is the quantile function of some $Z \in B(\Sigma)$ such that $Z \geqslant 0$. By assumption of nonatomicity of $P \circ X^{-1}$, $U = 1 - F_X(X)$ and $V = F_X(X)$ are uniformly distributed random variables on $(0,1)$ [20, Lemma A.25]. Therefore, $R_1^* = f^*(U) = F_Z^{-1}(U)$ is the nonincreasing equimeasurable rearrangement of Z with respect to X, and hence $R_1^* \geqslant 0$ and $F_{R_1^*} = F_Z$ (see Ghossoub [25] and references therein). Similarly, $R_2^* = f^*(V) = F_Z^{-1}(V)$ is the nondecreasing equimeasurable rearrangement of Z with respect to X, and hence $R_2^* \geqslant 0$ and $F_{R_2^*} = F_Z$. Moreover,

$$\int R_1^*\, dP = \int f^*(U)\, dP = \int f^*(V)\, dP = \int R_2^*\, dP = \int_0^1 f^*(t)\, dt \geqslant R_0,$$

where the last inequality follows from the feasibility of f^* for Problem (A.1). Hence, both R_1^* and R_2^* are feasible for Problem (3.2).

To show optimality of R_1^* and R_2^* for Problem (3.2), let R be any other feasible solution for Problem (3.2) and F_R^{-1} its quantile function. Then F_R^{-1} is feasible for Problem (A.1), and hence

$$\begin{aligned}
\int u\left(W_0 - \Pi - R\right) dT \circ P &= \int_0^1 T'(t)\, u\left(W_0 - \Pi - F_R^{-1}(t)\right) dt \\
&\leqslant \int_0^1 u\Big(W_0 - \Pi - f^*(t)\Big) T'(t)\, dt = \int_0^1 u\Big(W_0 - \Pi - F_Z^{-1}(t)\Big) T'(t)\, dt \\
&= \int_0^1 u\Big(W_0 - \Pi - F_{R_1^*}^{-1}(t)\Big) T'(t)\, dt = \int u\left(W_0 - \Pi - R_1^*\right) dT \circ P \\
&= \int_0^1 u\Big(W_0 - \Pi - F_{R_2^*}^{-1}(t)\Big) T'(t)\, dt = \int u\left(W_0 - \Pi - R_2^*\right) dT \circ P.
\end{aligned}$$

Therefore, R_1^* and R_2^* are optimal or Problem (3.2). $\square$

Now, letting $v(t) = T^{-1}(t)$ and using the change of variable $z = v^{-1}(t)$ gives

$$\begin{aligned}
\int_0^1 u\Big(W_0 - \Pi - f(t)\Big) T'(t)\, dt &= \int_0^1 u\Big(W_0 - \Pi - f(t)\Big) dT(t) = \int_0^1 u\Big(W_0 - \Pi - f(t)\Big) dv^{-1}(t) \\
&= \int_0^1 u\Big(W_0 - \Pi - f(v(z))\Big) dz = \int_0^1 u\Big(W_0 - \Pi - q(t)\Big) dt,
\end{aligned}$$

where $q(t) := f(v(t))$, for all $t \in (0,1)$. Moreover,

$$\int_0^1 f(t)\, dt = \int_0^1 f(v(z))\, dv(z) = \int_0^1 q(t)\, dv(t) = \int_0^1 q(t)\, v'(t)\, dt.$$

Consider the following problem:

Problem A.3.

$$\sup_{q \in \mathcal{Q}^*} \left\{ \int_0^1 u\Big(W_0 - \Pi - q(t)\Big) dt : \int_0^1 q(t)\, v'(t)\, dt \geqslant R_0 \right\}.$$

Lemma A.4. *If q^* is optimal for Problem (A.3), then $f^* := q^* \circ T$ is optimal for Problem (A.1). Moreover, $Y_1^* := X - f^*(1 - F_X(X))$ is optimal for Problem (3.1) and comonotonic with X, and $Y_2^* := X - f^*(F_X(X))$ is optimal for Problem (3.1) and anti-comonotonic with X.*

Proof. Suppose q^* is optimal for Problem (A.3), and let $f^* := q^* \circ T$. Then $q^*(t) = f^*(v(t))$, for all $t \in (0,1)$. Since q^* is feasible for Problem (A.3), we have that for all $t \in (0,1)$, $q^*(t) \geqslant 0$ and q^* is nondecreasing and left-continuous. Therefore, since T is increasing and continuous, it follows that f^* is nondecreasing and left-continuous. That is, $f^* \in \mathcal{Q}$. Moreover, for all $t \in (0,1)$, $f^*(t) = q^*(T(t)) \geqslant 0$. Hence, $f^* \in \mathcal{Q}^*$. Furthermore,

$$\int_0^1 f^*(t)\,dt = \int_0^1 f^*(v(z))\,dv(z) = \int_0^1 q^*(z)\,v'(z)\,dz \geqslant R_0,$$

where the last inequality follows from the feasibility of q^* for Problem (A.3). Therefore, f^* is feasible for Problem (A.1).

To show optimality of f^* for Problem (A.1), let f be any other feasible solution for Problem (A.1). Then:

$$\begin{aligned}\int_0^1 u\Big(W_0 - \Pi - f(t)\Big)T'(t)\,dt &= \int_0^1 u\Big(W_0 - \Pi - f(t)\Big)dT(t) = \int_0^1 u\Big(W_0 - \Pi - f(t)\Big)dv^{-1}(t)\\ &= \int_0^1 u\Big(W_0 - \Pi - f(v(z))\Big)dz = \int_0^1 u\Big(W_0 - \Pi - q(z)\Big)dz,\end{aligned}$$

where $q := f \circ v$. Therefore, to show optimality of f^* for Problem (A.1), it remains to show that q is feasible for Problem (A.3). Since f is feasible for Problem (A.1), it is nondecreasing, left-continuous, and satisfies, for all $t \in (0,1)$, $f(t) \geqslant 0$. Therefore, since v is increasing and continuous (by the inverse function theorem), q is nondecreasing, left-continuous, and satisfies, for all $t \in (0,1)$, $q(t) = f(v(t)) \geqslant 0$. Therefore, $q \in \mathcal{Q}^*$. Furthermore,

$$\int_0^1 q^*(t)\,v'(t)\,dt = \int_0^1 f^*(v(t))\,dv(t) = \int_0^1 f(z)\,dz \geqslant R_0,$$

where the last inequality follows from the feasibility of f for Problem (A.1). Thus, q is feasible for Problem (A.3), which concludes the proof that f^* is optimal for Problem (A.1).

We now show that $Y_1^* := X - f^*(1 - F_X(X))$ is optimal for Problem (3.1) and comonotonic with X. Since F_X is increasing, Y_1^* is clearly comonotonic with X. To show that Y_1^* is optimal for Problem (3.1), it suffices to show that $R_1^* := f^*(1 - F_X(X))$ is optimal for Problem (3.2). Since f^* is optimal for Problem (A.1), it is optimal for Problem (A.1) (by monotonicity of u). The rest follows from Lemma A.2.

We now show that $Y_2^* := X - f^*(F_X(X))$ is optimal for Problem (3.1) and anti-comonotonic with X. Since F_X is increasing, Y_2^* is clearly anti-comonotonic with X. To show that Y_2^* is optimal for Problem (3.1), it suffices to show that $R_2^* := f^*(F_X(X))$ is optimal for Problem (3.2). Since f^* is optimal for Problem (A.1), it is optimal for Problem (A.1) (by monotonicity of u). The rest follows from Lemma A.2. $\square$

In light of Lemma A.4, we turn our attention to solving Problem (A.3). In order to do that, we will use a similar methodology to the one used by Xu [50], but adapted to the present setting. Recall that for a continuous real-valued function f on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0,1]$, the convex envelope of f on the interval $[0,1]$ is an absolutely continuous real-valued function g such that:

(1) $g(0) = f(0)$ and $g(1) = f(1)$;

(2) g is convex on $[0,1]$;

(3) For all $x \in [0,1]$, $g(x) \leqslant f(x)$; and,
(4) g is affine on $\{x \in [0,1] : g(x) < f(x)\}$.

Moreover,

(5) If f is increasing, then so is g;
(6) If f is differentiable on $(0,1)$, then g is continuously differentiable on $(0,1)$.

Lemma A.5. *Let δ be the convex envelope of $v = T^{-1}$ on $[0,1]$. Then for any $q \in \mathcal{Q}^*$,*

$$\int_0^1 q(t)\, v'(t)\, dt \leqslant \int_0^1 q(t)\, \delta'(t)\, dt.$$

Proof. Let δ be the convex envelope of the function $v = T^{-1}$ on $[0,1]$. Since $\delta(t) \leqslant v(t)$, for all $t \in [0,1]$, and $v(1) = \delta(1)$, it follows from Fubini's Theorem that

$$0 \geqslant \int_0^1 \left[(v(1) - \delta(1)) - (v(y) - \delta(y))\right] dq(y) = \int_0^1 \int_y^1 \left[v'(x) - \delta'(x)\right] dx\, dq(y)$$
$$= \int_0^1 \int_0^x \left[v'(x) - \delta'(x)\right] dq(y)\ dx = \int_0^1 \left[\int_0^x dq(y)\right] \left[v'(x) - \delta'(x)\right] dx = \int_0^1 q(x) \left[v'(x) - \delta'(x)\right] dx.$$

□

Now consider the following problem:

Problem A.6.

$$\sup_{q \in \mathcal{Q}^*} \left\{ \int_0^1 u\Big(W_0 - \Pi - q(t)\Big) dt : \int_0^1 q(t)\, \delta'(t)\, dt \geqslant R_0 \right\}.$$

We first solve Problem (A.6) and then show that the solution is also optimal for Problem (A.3).

Lemma A.7. *If $q^* \in \mathcal{Q}^*$ satisfies:*

(1) $\int_0^1 \delta'(t)\, q^*(t)\, dt = R_0$ *and,*
(2) *There exists some $\lambda \geqslant 0$ such that for all $t \in (0,1)$,*

$$q^*(t) = \arg\max_{y \geqslant 0} \Big\{ u(W_0 - \Pi - y) + \lambda y \delta'(t) \Big\},$$

then q^* is optimal for Problem (A.6).

Proof. Let $q^* \in \mathcal{Q}^*$ be such that the two conditions above are satisfied. Then q^* is feasible for Problem (A.6). To show optimality, let $q \in \mathcal{Q}^*$ be any feasible solution for Problem (A.6). Then, by definition of q^*, it follows that for each t,

$$u\Big(W_0 - \Pi - q^*(t)\Big) - u\Big(W_0 - \Pi - q(t)\Big) \geqslant \lambda \left[\delta'(t)\, q(t) - \delta'(t)\, q^*(t)\right].$$

Hence,

$$\int_0^1 u\Big(W_0 - \Pi - q^*(t)\Big)dt - \int_0^1 u\Big(W_0 - \Pi - q(t)\Big)dt \geqslant \lambda\left[\int_0^1 \delta'(t)\, q(t)\, dt - \int_0^1 \delta'(t)\, q^*(t)\, dt\right]$$
$$= \lambda\left[\int_0^1 \delta'(t)\, q(t)\, dt - R_0\right] \geqslant 0.$$

Therefore, $\int_0^1 u(W_0 - \Pi - q^*(t))\, dt \geqslant \int_0^1 u(W_0 - \Pi - q(t))\, dt$. □

Lemma A.8. *For each $\lambda \geqslant 0$, define the function q^*_λ by*

$$q^*_\lambda(t) := \max\Big[0, W_0 - \Pi - (u')^{-1}\Big(\lambda\delta'(t)\Big)\Big]. \tag{A.2}$$

Then:

1. *For each $\lambda \geqslant 0$, $q^*_\lambda \in \mathcal{Q}^*$;*
2. *There exists $\lambda^* \geqslant 0$ such that $\int_0^1 \delta'(t)\, q^*_{\lambda^*}(t)\, dt = R_0$; and*
3. *For all $t \in (0,1)$, $q^*_\lambda(t) = \arg\max_{y \geqslant 0}\Big\{u(W_0 - \Pi - y) + \lambda y \delta'(t)\Big\}$.*

Proof. Follows from Remark 2.4, from the monotonicity and continuity properties of δ', from Assumption 2.2 and Assumption 3.3, and from the Intermediate Value Theorem. □

Therefore, lemmata A.5, A.7, and A.8 imply that for any $\lambda \geqslant 0$ and any $q \in \mathcal{Q}^*$,

$$\int_0^1 \Big[u\Big(W_0 - \Pi - q(t)\Big) + \lambda q(t)\, v'(t)\Big]\, dt = \int_0^1 u\Big(W_0 - \Pi - q(t)\Big)dt + \lambda \int_0^1 q(t)\, v'(t)\, dt$$
$$\leqslant \int_0^1 u\Big(W_0 - \Pi - q(t)\Big)dt + \lambda \int_0^1 q(t)\, \delta'(t)\, dt$$
$$= \int_0^1 \Big[u\Big(W_0 - \Pi - q(t)\Big) + \lambda q(t)\, \delta'(t)\Big]\, dt$$
$$\leqslant \int_0^1 \Big[u\Big(W_0 - \Pi - q^*_\lambda(t)\Big) + \lambda q^*_\lambda(t)\, \delta'(t)\Big]\, dt,$$

where q^*_λ is as in eq. (A.2). Now, for all $\lambda \geqslant 0$, since q^*_λ is monotone, it is differentiable a.e., and we have:

$$q^*_\lambda(t) = \begin{cases} 0 & \text{if} \quad W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)) \leqslant 0, \\ W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)) & \text{if} \quad 0 < W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)), \end{cases}$$

and

$$dq^*_\lambda(t) = \begin{cases} 0 & \text{if} \quad W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)) \leqslant 0, \\ -\lambda\Big((u')^{-1}\Big)'(\lambda\delta'(t))\, d\delta'(t) & \text{if} \quad 0 < W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)), \end{cases} \tag{A.3}$$

Now, define the subsets $\mathcal{A}$ and $\mathcal{B}$ of $[0,1]$ by

$$\mathcal{A} := \Big\{t \in [0,1] : \delta(t) = v(t)\Big\} \text{ and } \mathcal{B} := \Big\{t \in [0,1] : \delta(t) \neq v(t)\Big\} = \Big\{t \in [0,1] : \delta(t) < v(t)\Big\}.$$

Then for any $\lambda > 0$,

$$\int_0^1 [v(t) - \delta(t)]\, dq^*_\lambda(t) = \int_{\mathcal{A}} [v(t) - \delta(t)]\, dq^*_\lambda(t) + \int_{\mathcal{B}} [v(t) - \delta(t)]\, dq^*_\lambda(t) = \int_{\mathcal{B}} [v(t) - \delta(t)]\, dq^*_\lambda(t).$$

But, since δ is affine on $\mathcal{B}$, $d\delta' = 0$ on $\mathcal{B}$, and it follows from eq. (A.6) that $dq^*_\lambda(t) = 0$ on $\mathcal{B}$. Consequently,

$$\int_0^1 [v(t) - \delta(t)]\, dq^*_\lambda(t) = 0.$$

Therefore, applying Fubini's theorem as in the proof of Lemma A.5 gives

$$\begin{aligned} 0 &= \int_0^1 [v(t) - \delta(t)]\, dq^*_\lambda(t) = \int_0^1 [(v(1) - \delta(1)) - (v(y) - \delta(y))]\, dq^*_\lambda(y) = \int_0^1 \int_y^1 \left[v'(x) - \delta'(x)\right] dx\, dq^*_\lambda(y) \\ &= \int_0^1 \int_0^x \left[v'(x) - \delta'(x)\right] dq^*_\lambda(y)\ dx = \int_0^1 \left[\int_0^x dq^*_\lambda(y)\right] \left[v'(x) - \delta'(x)\right] dx = \int_0^1 q^*_\lambda(x) \left[v'(x) - \delta'(x)\right] dx. \end{aligned}$$

Consequently, $\int_0^1 q^*_\lambda(t)\, v'(t)\, dt = \int_0^1 q^*_\lambda(t)\, \delta'(t)\, dt$. Therefore, for all $\lambda \geqslant 0$ and all $q \in \mathcal{Q}^*$,

$$\begin{aligned} \int_0^1 \Big[u\Big(W_0 - \Pi - q(t)\Big) + \lambda q(t)\, v'(t)\Big] dt &\leqslant \int_0^1 \Big[u\Big(W_0 - \Pi - q^*_\lambda(t)\Big) + \lambda q^*_\lambda(t)\, \delta'(t)\Big] dt \\ &= \int_0^1 \Big[u\Big(W_0 - \Pi - q^*_\lambda(t)\Big) + \lambda q^*_\lambda(t)\, v'(t)\Big] dt. \end{aligned}$$

Hence, if λ^* is chosen such that $\int_0^1 q^*_{\lambda^*}(t)\, v'(t)\, dt = R_0$, then the optimal solution to Problem (A.3) is given by $q^*_{\lambda^*}$. Thus, By lemmata A.4, A.7, and A.8, the function $Y_1^* := X - q^*\Big(T\Big(1 - F_X(X)\Big)\Big)$ is optimal for Problem (3.1) and comonotonic with X, and the function $Y_2^* := X - q^*\Big(T\Big(F_X(X)\Big)\Big)$ is optimal for Problem (3.1) and anti-comonotonic with X, where:

- For all $t \in [0,1]$, $q^*(t) = \max\Big[0, W_0 - \Pi - (u')^{-1}\Big(\lambda^* \delta'(t)\Big)\Big]$;
- δ is the convex envelope of $v = T^{-1}$ on $[0,1]$; and,
- λ^* is chosen such that $\int_0^1 q^*(t)\, v'(t)\, dt = \int_0^1 q^*(t)\, \delta'(t)\, dt = R_0$.

This concludes the proof of Theorem 3.4. □

Appendix B. Proof of Corollary 3.5

If the DM is ambiguity neutral, that is, $T(t) = t$, for all $t \in [0,1]$, then $T^{-1}(t) = v(t) = \delta(t) = t$, for all $t \in [0,1]$, and so $\delta'(t) = v'(t) = 1$. If the DM is ambiguity averse, that is, T is convex (and strictly increasing) on $[0,1]$, then T^{-1} is concave and strictly increasing on $[0,1]$, and so δ is affine on $[0,1]$. Since $T(0) = 0$ and $T(1) = 1$, this implies that $\delta(t) = t$, for all $t \in [0,1]$. Consequently, $\delta'(t) = 1$ on $[0,1]$.

In both cases, Theorem 3.4 implies that the function $Y^* = X - \max\left[0, W_0 - \Pi - (u')^{-1}(\lambda^*)\right]$ is optimal for Problem (3.1) and comonotonic with X, where λ^* is chosen such that

$$\int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}(\lambda^*)\right] dt = R_0. \tag{B.1}$$

If $R_0 = 0$, that is $\Pi = \int X\,dP$, then eq. (B.1) implies that $\lambda^* \leqslant u'(W_0 - \Pi)$. For this choice of λ^*, the retention is zero, and so $Y^* = X$ (full insurance) is optimal. If $R_0 > 0$, then eq. (B.1) implies that $W_0 - \Pi - (u')^{-1}(\lambda^*) > 0$, and that $R_0 = W_0 - \Pi - (u')^{-1}(\lambda^*)$. Hence, $Y^* = X - R_0$. □

Appendix C. Proof of Corollary 3.6

Suppose that the DM is ambiguity seeking, that is, T is concave (and strictly increasing) on $[0,1]$. It then follows that T^{-1} is convex and strictly increasing on $[0,1]$, and so $\delta(t) = T^{-1}(t) = v(t)$, for all $t \in [0,1]$. Consequently, for all $t \in [0,1]$, $\delta'(t) = \frac{1}{T'(T^{-1}(t))}$. Therefore, Theorem 3.4 implies that the function $Y^* = X - \max\left[0, W_0 - \Pi - (u')^{-1}\left(\frac{\lambda^*}{T'(1-F_X(X))}\right)\right]$ is optimal for Problem (3.1) and comonotonic with X, where λ^* is chosen such that

$$\begin{aligned} R_0 &= \int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}\left(\frac{\lambda^*}{T'(T^{-1}(t))}\right)\right] v'(t)\,dt \\ &= \int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}\left(\frac{\lambda^*}{T'(T^{-1}(t))}\right)\right] \delta'(t)\,dt \\ &= \int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}\left(\frac{\lambda^*}{T'(T^{-1}(t))}\right)\right] \left(\frac{1}{T'(T^{-1}(t))}\right) dt \end{aligned}$$

Now, for any $s \in S$, $Y^*(s) = X(s)$ if and only if $\max\left[0, W_0 - \Pi - (u')^{-1}\left(\frac{\lambda^*}{T'(1-F_X(X(s)))}\right)\right] = 0$, that is, if and only if $W_0 - \Pi - (u')^{-1}\left(\frac{\lambda^*}{T'(1-F_X(X(s)))}\right) \leqslant 0$. Hence, by strict concavity of u and T, $Y^*(s) = X(s)$ if and only if

$$1 - (T')^{-1}\left(\frac{\lambda^*}{u'(W_0 - \Pi)}\right) \leqslant F_X(X(s)). \tag{C.1}$$

Therefore, since $F_X(X(s)) \in [0,1]$, for all $s \in S$, it follows that:

(1) If $(T')^{-1}\left(\frac{\lambda^*}{u'(W_0-\Pi)}\right) < 0$, i.e., if $\frac{\lambda^*}{u'(W_0-\Pi)} > T'(0)$, then $\left\{s \in S : Y^*(s) = X(s)\right\} = \varnothing$. In other words, the optimal indemnity is always less than full insurance.

(2) If $(T')^{-1}\left(\frac{\lambda^*}{u'(W_0-\Pi)}\right) > 1$, i.e., if $\frac{\lambda^*}{u'(W_0-\Pi)} < T'(1)$, then $F_X(X(s)) > 1 - (T')^{-1}\left(\frac{\lambda^*}{u'(W_0-\Pi)}\right)$, for all $s \in S$, and so $\left\{s \in S : Y^*(s) = X(s)\right\} = S$. In other words, full insurance is optimal.

(3) If $(T')^{-1}\left(\frac{\lambda^*}{u'(W_0-\Pi)}\right) \in [0,1]$, i.e., if $\frac{\lambda^*}{u'(W_0-\Pi)} \in [T'(1), T'(0)]$, then eq. (C.1) yields $X(s) \geqslant F_X^{-1}\left(1-(T')^{-1}\left(\frac{\lambda^*}{u'(W_0-\Pi)}\right)\right)$, and so $\Big\{s \in S : Y^*(s) = X(s)\Big\} = \Big\{s \in S : X(s) \leqslant L\Big\}$, where $L := F_X^{-1}\left(1-(T')^{-1}\left(\frac{\lambda^*}{u'(W_0-\Pi)}\right)\right) \geqslant 0$. In other words, small losses are fully insured.

This concludes the proof of Corollary 3.6. □

Appendix D. Proof of Theorem 3.10

Recall Problem (3.7):

$$\sup_{Y\in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - R\Big)\, dT_1\circ P : R \geqslant 0, \int R\, dT_2\circ P \geqslant \tilde{R}_0 \right\}.$$

Now, for all $R \in B(\Sigma)$, the fact that u is increasing and $U := F_X(X)$ is uniformly distributed implies that

$$\int u\left(W_0 - \Pi - R\right) dT_1\circ P = \int T_1'(U)\, u\left(W_0 - \Pi - F_R^{-1}(U)\right) dP = \int_0^1 T_1'(t)\, u\left(W_0 - \Pi - F_R^{-1}(t)\right) dt.$$

Moreover,

$$\int R dT_2\circ P = \int T_2'(1-U)\, F_R^{-1}(U)\, dP = \int_0^1 T_2'(1-t)\, F_R^{-1}(t)\, dt,$$

and $R \geqslant 0$ whenever $F_R^{-1}(t) \geqslant 0$, for all $t \in (0,1)$.

As before, let $\mathcal{Q}$ denote the collection of all quantile functions and let $\mathcal{Q}^*$ be as in eq. (A.1). That is, $\mathcal{Q}^*$ denotes the collection of all quantile functions f that satisfy $f(t) \geqslant 0$, for all $t \in (0,1)$. Consider the following problem:

Problem D.1.

$$\sup_{f\in\mathcal{Q}^*} \left\{ \int_0^1 u\Big(W_0 - \Pi - f(t)\Big) T_1'(t)\, dt : \int_0^1 T_2'(1-t)\, f(t)\, dt \geqslant \tilde{R}_0 \right\}.$$

By a proof similar to that of Lemma A.2, we obtain the following result.

Lemma D.2. *If f^* is optimal for Problem (D.1), then both $R_1^* := f^*\left(1-F_X(X)\right)$ and $R_2^* := f^*\left(F_X(X)\right)$ are optimal for Problem (3.7). Moreover, R_1^* is anti-comonotonic with X and R_2^* is comonotonic with X.*

Now, letting $v(t) = T_1^{-1}(t)$ and using the change of variable $z = v^{-1}(t)$ gives

$$\begin{aligned}\int_0^1 u\Big(W_0 - \Pi - f(t)\Big) T_1'(t)\, dt &= \int_0^1 u\Big(W_0 - \Pi - f(t)\Big) dT_1(t) = \int_0^1 u\Big(W_0 - \Pi - f(t)\Big) dv^{-1}(t)\\ &= \int_0^1 u\Big(W_0 - \Pi - f(v(z))\Big) dz = \int_0^1 u\Big(W_0 - \Pi - q(t)\Big) dt,\end{aligned}$$

where $q(t) := f(v(t))$, for all $t \in (0,1)$. Moreover,

$$\begin{aligned}\int_0^1 f(t)\, T_2'(1-t)\, dt &= \int_0^1 f(v(z))\, T_2'(1-v(z))\, dv(z) = \int_0^1 q(z)\, T_2'(1-v(z))\, v'(z)\, dz \\ &= \int_0^1 q(t)\, T_2'\left(1-T_1^{-1}(t)\right)\left(T_1^{-1}\right)'(t)\, dt = \int_0^1 q(t)\left[\frac{T_2'\left(1-T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}\right] dt \\ &= \int_0^1 q(t)\, \Psi'(t)\, dt,\end{aligned}$$

where the function Ψ is defined on $[0,1]$ by

$$\Psi(t) := \int_0^t \left[\frac{T_2'\left(1-T_1^{-1}(x)\right)}{T_1'\left(T_1^{-1}(x)\right)}\right] dx = 1 - T_2\left(1 - T_1^{-1}(t)\right). \tag{D.1}$$

Now, consider the following problem:

Problem D.3.

$$\sup_{q \in \mathcal{Q}^*} \left\{ \int_0^1 u\Big(W_0 - \Pi - q(t)\Big) dt : \int_0^1 q(t)\, \Psi'(t)\, dt \geqslant \widetilde{R}_0 \right\}.$$

By a proof similar to that of Lemma A.4, we obtain the following result.

Lemma D.4. *If q^* is optimal for Problem (D.3), then $f^* := q^* \circ T_1$ is optimal for Problem (D.1).*

In light of Lemma D.4, we turn our attention to solving Problem (D.3). By a proof similar to that of Lemma A.5, we obtain the following result.

Lemma D.5. *Let δ be the convex envelope of Ψ on $[0,1]$. Then for any $q \in \mathcal{Q}^*$,*

$$\int_0^1 q(t)\, \Psi'(t)\, dt \leqslant \int_0^1 q(t)\, \delta'(t)\, dt.$$

Now, consider the following problem.

Problem D.6.

$$\sup_{q \in \mathcal{Q}^*} \left\{ \int_0^1 u\Big(W_0 - \Pi - q(t)\Big) dt : \int_0^1 q(t)\, \delta'(t)\, dt \geqslant \widetilde{R}_0 \right\}.$$

We first solve Problem (D.6) and then show that the solution is also optimal for Problem (D.3). By a proof similar to that of Lemma A.7, we obtain the following result.

Lemma D.7. *If $q^* \in \mathcal{Q}^*$ satisfies:*

(1) *$\int_0^1 \delta'(t)\, q^*(t)\, dt = \widetilde{R}_0$; and,*

(2) *There exists some $\lambda \geqslant 0$ such that for all $t \in (0,1)$,*

$$q^*(t) = \arg\max_{y \geqslant 0} \left\{ u\left(W_0 - \Pi - y\right) + \lambda y \delta'(t) \right\},$$

then q^* is optimal for Problem (D.6).

By a proof similar to that of Lemma A.8, we obtain the following result.

Lemma D.8. *For each $\lambda \geqslant 0$, define the function q_λ^* by*

$$q_\lambda^*(t) := \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda\delta'(t)\right)\right]. \tag{D.2}$$

Then:

(1) *For each $\lambda \geqslant 0$, $q_\lambda^* \in \mathcal{Q}^*$;*

(2) *There exists $\lambda^* \geqslant 0$ such that $\int_0^1 \delta'(t)\, q_{\lambda^*}^*(t)\, dt = \widetilde{R}_0$; and*

(3) *For all $t \in (0,1)$, $q_{\lambda^*}^*(t) = \arg\max_{y \geqslant 0}\left\{u(W_0 - \Pi - y) + \lambda y \delta'(t)\right\}$.*

Therefore, lemmata D.5, D.7, and D.8 imply that for any $\lambda \geqslant 0$ and any $q \in \mathcal{Q}^*$,

$$\begin{aligned}
\int_0^1 \left[u\Big(W_0 - \Pi - q(t)\Big) + \lambda q(t)\Psi'(t)\right] dt &= \int_0^1 u\Big(W_0 - \Pi - q(t)\Big) dt + \lambda \int_0^1 q(t)\Psi'(t)\, dt \\
&\leqslant \int_0^1 u\Big(W_0 - \Pi - q(t)\Big) dt + \lambda \int_0^1 q(t)\delta'(t)\, dt \\
&= \int_0^1 \left[u\Big(W_0 - \Pi - q(t)\Big) + \lambda q(t)\delta'(t)\right] dt \\
&\leqslant \int_0^1 \left[u\Big(W_0 - \Pi - q_\lambda^*(t)\Big) + \lambda q_\lambda^*(t)\delta'(t)\right] dt,
\end{aligned}$$

where q_λ^* is as in eq. (D.2). Now, for all $\lambda \geqslant 0$, since q_λ^* is monotone, it is differentiable a.e., and we have:

$$q_\lambda^*(t) = \begin{cases} 0 & \text{if } \; W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)) \leqslant 0, \\ W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)) & \text{if } \; 0 < W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)), \end{cases}$$

and

$$dq_\lambda^*(t) = \begin{cases} 0 & \text{if } \; W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)) \leqslant 0, \\ -\lambda\left((u')^{-1}\right)'(\lambda\delta'(t))\, d\delta'(t) & \text{if } \; 0 < W_0 - \Pi - (u')^{-1}(\lambda\delta'(t)), \end{cases} \tag{D.3}$$

Now, define the subsets $\mathcal{A}$ and $\mathcal{B}$ of $[0,1]$ by:

$$\mathcal{A} := \Big\{t \in [0,1] : \delta(t) = \Psi(t)\Big\} \text{ and } \mathcal{B} := \Big\{t \in [0,1] : \delta(t) \neq \Psi(t)\Big\} = \Big\{t \in [0,1] : \delta(t) < \Psi(t)\Big\}.$$

Then for any $\lambda \geqslant 0$,

$$\int_0^1 [\Psi(t) - \delta(t)]\, dq_\lambda^*(t) = \int_{\mathcal{A}} [\Psi(t) - \delta(t)]\, dq_\lambda^*(t) + \int_{\mathcal{B}} [\Psi(t) - \delta(t)]\, dq_\lambda^*(t) = \int_{\mathcal{B}} [\Psi(t) - \delta(t)]\, dq_\lambda^*(t).$$

But, since δ is affine on $\mathcal{B}$, $d\delta' = 0$ on $\mathcal{B}$, and it follows from eq. (D.3) that $dq_\lambda^*(t) = 0$ on $\mathcal{B}$. Consequently,

$$\int_0^1 [\Psi(t) - \delta(t)]\, dq_\lambda^*(t) = 0.$$

Therefore, applying Fubini's theorem, as in the proof of Lemma A.5, gives

$$
\begin{aligned}
0 &= \int_0^1 \left[\Psi(t) - \delta(t)\right] dq^*_\lambda(t) = \int_0^1 \left[(\Psi(1) - \delta(1)) - (\Psi(y) - \delta(y))\right] dq^*_\lambda(y) = \int_0^1 \int_y^1 \left[\Psi'(x) - \delta'(x)\right] dx\; dq^*_\lambda(y) \\
&= \int_0^1 \int_0^x \left[\Psi'(x) - \delta'(x)\right] dq^*_\lambda(y)\; dx = \int_0^1 \left[\int_0^x dq^*_\lambda(y)\right] \left[\Psi'(x) - \delta'(x)\right] dx = \int_0^1 q^*_\lambda(x) \left[\Psi'(x) - \delta'(x)\right] dx.
\end{aligned}
$$

Consequently, $\int_0^1 q^*_\lambda(t)\, \Psi'(t)\, dt = \int_0^1 q^*_\lambda(t)\, \delta'(t)\, dt$. Therefore, for all $\lambda \geqslant 0$ and all $q \in \mathcal{Q}^*$,

$$
\begin{aligned}
\int_0^1 \left[u\Big(W_0 - \Pi - q(t)\Big) + \lambda q(t)\, \Psi'(t)\right] dt &\leqslant \int_0^1 \left[u\Big(W_0 - \Pi - q^*_\lambda(t)\Big) + \lambda q^*_\lambda(t)\, \delta'(t)\right] dt \\
&= \int_0^1 \left[u\Big(W_0 - \Pi - q^*_\lambda(t)\Big) + \lambda q^*_\lambda(t)\, \Psi'(t)\right] dt.
\end{aligned}
$$

Hence, if λ^* is chosen such that $\int_0^1 q^*_{\lambda^*}(t)\, \Psi'(t)\, dt = \widetilde{R}_0$, then the optimal solution to Problem (D.3) is given by $q^*_{\lambda^*}$. Thus, By lemmata D.2, D.4, D.7, and D.8, the function $R_1^* := f^*(1 - F_X(X)) = q^*(T_1(1 - F_X(X)))$ is optimal for Problem (3.7) and anti-comonotonic with X, and the function $R_2^* := f^*(F_X(X)) = q^*(T_1(F_X(X)))$ is optimal for Problem (3.7) and comonotonic with X, where:

- For all $t \in [0,1]$, $q^*(t) = \max\left[0, W_0 - \Pi - (u')^{-1}\Big(\lambda^* \delta'(t)\Big)\right]$;
- δ is the convex envelope of Ψ on $[0,1]$; and,
- λ^* is chosen such that $\int_0^1 q^*(t)\, \Psi'(t)\, dt = \widetilde{R}_0$.

This concludes the proof of Theorem 3.10. $\square$

Appendix E. Proof of Corollary 3.11

Suppose that for each $t \in [0,1]$, we have

$$
\frac{T_2''(1-t)}{T_2'(1-t)} \geqslant -\frac{T_1''(t)}{T_1'(t)}.
$$

Then, for each each $t \in [0,1]$, we have

$$
\frac{T_2''\left(1 - T_1^{-1}(t)\right)}{T_2'\left(1 - T_1^{-1}(t)\right)} \geqslant -\frac{T_1''\left(T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}.
$$

Consequently, for each each $t \in [0,1]$, we have

$$
T_1'\left(T_1^{-1}(t)\right) T_2''\left(1 - T_1^{-1}(t)\right) + T_2'\left(1 - T_1^{-1}(t)\right) T_1''\left(T_1^{-1}(t)\right) \geqslant 0.
$$

Therefore, for each each $t \in [0,1]$, we have

$$
\Psi''(t) = -\left(\frac{T_1'\left(T_1^{-1}(t)\right) T_2''\left(1 - T_1^{-1}(t)\right) + T_2'\left(1 - T_1^{-1}(t)\right) T_1''\left(T_1^{-1}(t)\right)}{\left[T_1'\left(T_1^{-1}(t)\right)\right]^3}\right) \leqslant 0,
$$

That is, Ψ is concave on $[0,1]$, and hence δ is affine on $[0,1]$. Since $\Psi(0) = 0$ and $\Psi(1) = 1$, this implies that $\delta(t) = t$, for all $t \in [0,1]$. Consequently, $\delta'(t) = 1$ on $[0,1]$.

Theorem 3.10 then implies that the function $R^* = \max\left[0, W_0 - \Pi - (u')^{-1}(\lambda^*)\right]$ is optimal for Problem (3.7) and comonotonic with X, where λ^* is chosen such that

$$\int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}(\lambda^*)\right] dt = \widetilde{R}_0. \tag{E.1}$$

If $\widetilde{R}_0 = 0$, that is $\Pi = \int X dP$, then eq. (E.1) implies that $\lambda^* \leqslant u'(W_0 - \Pi)$. For this choice of λ^*, the retention is zero: $R^* = 0$. If $\widetilde{R}_0 > 0$, then eq. (E.1) implies that $W_0 - \Pi - (u')^{-1}(\lambda^*) > 0$, and that $\widetilde{R}_0 = W_0 - \Pi - (u')^{-1}(\lambda^*)$. Hence, in this case, $R^* = \widetilde{R}_0 > 0$, a constant. □

Appendix F. Proof of Corollary 3.12

Suppose that for each $t \in [0,1]$, we have

$$\frac{T_2''(1-t)}{T_2'(1-t)} \leqslant -\frac{T_1''(t)}{T_1'(t)}.$$

Then, for each each $t \in [0,1]$, we have

$$\frac{T_2''\left(1 - T_1^{-1}(t)\right)}{T_2'\left(1 - T_1^{-1}(t)\right)} \leqslant -\frac{T_1''\left(T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}.$$

Consequently, for each each $t \in [0,1]$, we have

$$T_1'\left(T_1^{-1}(t)\right) T_2''\left(1 - T_1^{-1}(t)\right) + T_2'\left(1 - T_1^{-1}(t)\right) T_1''\left(T_1^{-1}(t)\right) \leqslant 0.$$

Therefore, for each each $t \in [0,1]$, we have

$$\Psi''(t) = -\left(\frac{T_1'\left(T_1^{-1}(t)\right) T_2''\left(1 - T_1^{-1}(t)\right) + T_2'\left(1 - T_1^{-1}(t)\right) T_1''\left(T_1^{-1}(t)\right)}{\left[T_1'\left(T_1^{-1}(t)\right)\right]^3}\right) \geqslant 0,$$

That is, Ψ is convex on $[0,1]$, and hence $\delta = \Psi$ on $[0,1]$. Consequently, for all $t \in [0,1]$,

$$\delta'(t) = \Psi'(t) = \frac{T_2'\left(1 - T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}.$$

Theorem 3.10 then implies that the function $R^* := \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda^*\left(\frac{T_2'(F_X(X))}{T_1'(1-F_X(X))}\right)\right)\right]$ is optimal for Problem (3.7), where λ^* is chosen such that

$$\begin{aligned}
\widetilde{R}_0 &= \int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda^*\left(\frac{T_2'\left(1 - T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}\right)\right)\right] \Psi'(t)\, dt \\
&= \int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda^*\left(\frac{T_2'\left(1 - T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}\right)\right)\right]\left(\frac{T_2'\left(1 - T_1^{-1}(t)\right)}{T_1'\left(T_1^{-1}(t)\right)}\right) dt.
\end{aligned}$$

□

Appendix G. Proof of Corollary 3.14

If T is linear, then $T(t)=t$ for all $t \in [0,1]$, and so $\Psi(t) = 1 - T\left(1-T^{-1}(t)\right) = t$, for all $t \in [0,1]$. Therefore, $\Psi = \delta$ and so $\delta'(t) = \Psi'(t) = 1$, for all $t \in [0,1]$. Similarly, if T is convex, then Ψ is concave on $[0,1]$, and hence δ is affine on $[0,1]$. Since $\Psi(0)=0$ and $\Psi(1)=1$, this implies that $\delta(t)=t$, for all $t \in [0,1]$. Consequently, $\delta'(t)=1$ on $[0,1]$.

Corollary 3.13 then implies that the function $R^* = \max\left[0, W_0 - \Pi - (u')^{-1}(\lambda^*)\right]$ is optimal for Problem (3.7) and comonotonic with X, where λ^* is chosen such that

$$\int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}(\lambda^*)\right] dt = \widetilde{R}_0. \tag{G.1}$$

If $\widetilde{R}_0 = 0$, that is $\Pi = \int X dT_2 \circ P$, then eq. (G.1) implies that $\lambda^* \leqslant u'(W_0 - \Pi)$. For this choice of λ^*, the retention is zero: $R^* = 0$. If $\widetilde{R}_0 > 0$, then eq. (G.1) implies that $W_0 - \Pi - (u')^{-1}(\lambda^*) > 0$, and that $\widetilde{R}_0 = W_0 - \Pi - (u')^{-1}(\lambda^*)$. Hence, in this case, $R^* = \widetilde{R}_0 > 0$, a constant. □

Appendix H. Proof of Corollary 3.15

If T is concave, then Ψ is convex on $[0,1]$, and hence $\delta = \Psi$ on $[0,1]$. Consequently, for all $t \in [0,1]$,

$$\delta'(t) = \Psi'(t) = \frac{T'\left(1-T^{-1}(t)\right)}{T'\left(T^{-1}(t)\right)}.$$

Corollary 3.13 then implies that the function $R^* := \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda^*\left(\frac{T'(F_X(X))}{T'(1-F_X(X))}\right)\right)\right]$ is optimal for Problem (3.7), where λ^* is chosen such that

$$\begin{aligned}
\widetilde{R}_0 &= \int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda^*\left(\frac{T'\left(1-T^{-1}(t)\right)}{T'\left(T^{-1}(t)\right)}\right)\right)\right]\Psi'(t)\, dt \\
&= \int_0^1 \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda^*\left(\frac{T'\left(1-T^{-1}(t)\right)}{T'\left(T^{-1}(t)\right)}\right)\right)\right]\left(\frac{T'\left(1-T^{-1}(t)\right)}{T'\left(T^{-1}(t)\right)}\right) dt.
\end{aligned}$$

□

Appendix I. Proof of Theorem 4.7

Recall Problem (4.2):

$$\sup_{R \in B(\Sigma)} \left\{ \int u\Big(W_0 - \Pi - R\Big)\, dP : R \geqslant 0, \int R dQ \geqslant \overline{R}_0 := \int X dQ - \Pi \right\}.$$

Clearly, R^* is optimal for Problem (4.2) if and only if $Y^* = X - R^*$ is optimal for Problem (4.1). Therefore, we focus on solving Problem (4.2).

Proposition I.1. *For any $R \in B(\Sigma)$ that is feasible for Problem (4.2), there exists $\widetilde{R} \in B(\Sigma)$, also feasible for Problem (4.2) such that:*

- *$\widetilde{R}$ is comonotonic with h;*

- $\int u\,(W_0 - \Pi - R)\,dP = \int u\left(W_0 - \Pi - \widetilde{R}\right)dP$; *and,*
- $\int \widetilde{R} dQ \geqslant \int R dQ$.

Proof. Let $\overline{U} := F_h(h)$, where $h = {}^{dQ}/_{dP}$. By Assumption 4.5, $\overline{U}$ is a uniformly distributed random variable on $(0,1)$ and $h = F_h^{-1}\left(\overline{U}\right)$, P-a.s. [20, Lemma A.25]. Now, for all $R \in B(\Sigma)$, the random variable $\widetilde{R} := F_R^{-1}\left(\overline{U}\right)$ is the nondecreasing rearrangement of R with respect to h (see Ghossoub [25] and references therein), and hence R and $\widetilde{R}$ are identically distributed under P. Therefore, for all $R \in B(\Sigma)$, we have

$$\begin{aligned}\int u\,(W_0 - \Pi - R)\,dP &= \int u\left(W_0 - \Pi - F_R^{-1}(U)\right)dP \\ &= \int_0^1 u\left(W_0 - \Pi - F_R^{-1}(t)\right)dt = \int u\left(W_0 - \Pi - \widetilde{R}\right)dP.\end{aligned}$$

Moreover, by the Hardy-Littlewood inequality (e.g., [20, Theorem A.28]), we have for all $R \in B(\Sigma)$ that are feasible for Problem (4.2),

$$R_0 \leqslant \int R dQ = \int R h dP \leqslant \int F_R^{-1}\left(\overline{U}\right) F_h^{-1}\left(\overline{U}\right) dP = \int F_R^{-1}\left(\overline{U}\right) h dP = \int \widetilde{R} dQ,$$

and $R \geqslant 0$ whenever $F_R^{-1}(t) \geqslant 0$, for all $t \in (0,1)$. □

Hence, in light of Proposition I.1, we can focus on obtaining solutions to Problem (4.2) that are of the form $f\left(\overline{U}\right)$, where f is a quantile function of some random variable $R \in B^+(\Sigma)$. We denote by $\mathcal{Q}^*$ the collection of all such quantile functions. That is,

$$\mathcal{Q}^* = \left\{f \in \mathcal{Q} : f(t) \geqslant 0, \text{ for each } 0 < t < 1\right\},$$

where $\mathcal{Q}$ denotes the collection of all quantile functions. That is,

$$\mathcal{Q} = \left\{f : (0,1) \to \mathbb{R} \;\middle|\; f \text{ is nondecreasing and left-continuous}\right\}.$$

Consider the following problem.

Problem I.2.

$$\sup_{f \in \mathcal{Q}^*} \left\{\int_0^1 u\left(W_0 - \Pi - f(t)\right)dt : \int_0^1 f(t)\,F_h^{-1}(t)\,dt \geqslant \overline{R}_0\right\}.$$

Lemma I.3. *If f^* is optimal for Problem (I.2), then $R^* := f^*\left(\overline{U}\right)$ is optimal for Problem (4.2) and comonotonic with h.*

Proof. Let f^* be optimal for Problem (I.2). Then, by definition of $\mathcal{Q}^*$, f^* is the quantile function of some $Z \in B(\Sigma)$ such that $Z \geqslant 0$. By assumption of nonatomicity of $P \circ h^{-1}$, $\overline{U} = F_h(h)$ is uniformly distributed random variable on $(0,1)$ and $h = F_h^{-1}\left(\overline{U}\right)$, P-a.s. [20, Lemma A.25]. Therefore,

$R^* = f^*\left(\overline{U}\right)$ is the nondecreasing equimeasurable rearrangement of Z with respect to h, and hence $R^* \geqslant 0$ and $F_{R^*} = F_Z$ (e.g., [25]). Moreover,

$$\int R^* dQ = \int R^* h dP = \int f^*\left(\overline{U}\right) F_h^{-1}\left(\overline{U}\right) dP = \int_0^1 f^*(t)\, F_h^{-1}(t)\, dt \geqslant \overline{R}_0,$$

where the last inequality follows from the feasibility of f^* for Problem (I.2). Hence, both R^* is feasible for Problem (4.2).

To show optimality of R^* for Problem (4.2), let R by any other feasible solution for Problem (4.2) and F_R^{-1} its quantile function. then $F_R^{-1} \in \mathcal{Q}^*$, and the Hardy-Littlewood inequality implies that

$$\begin{aligned}\int_0^1 F_R^{-1}(t)\, F_h^{-1}(t)\, dt &= \int F_R^{-1}\left(\overline{U}\right) F_h^{-1}\left(\overline{U}\right) dP \\ &\geqslant \int R F_h^{-1}\left(\overline{U}\right) dP = \int R h dP = \int R dQ \geqslant \overline{R}_0,\end{aligned}$$

where the last inequality follows form the feasibility of R for Problem (4.2). Thus, F_R^{-1} is feasible for Problem (I.2), and hence

$$\begin{aligned}\int u\left(W_0 - \Pi - R\right) dP &= \int_0^1 u\left(W_0 - \Pi - F_R^{-1}(t)\right) dt \\ &\leqslant \int_0^1 u\Big(W_0 - \Pi - f^*(t)\Big) dt = \int_0^1 u\Big(W_0 - \Pi - F_Z^{-1}(t)\Big) dt \\ &= \int_0^1 u\Big(W_0 - \Pi - F_{R^*}^{-1}(t)\Big) dt = \int u\left(W_0 - \Pi - R^*\right) dP.\end{aligned}$$

Therefore, R^* is optimal or Problem (4.2). □

Lemma I.4. *If $f^* \in \mathcal{Q}^*$ satisfies:*

(1) $\int_0^1 F_h^{-1}(t)\, f^*(t)\, dt = \overline{R}_0$*; and,*

(2) *There exists some $\lambda \geqslant 0$ such that for all $t \in (0,1)$,*

$$f^*(t) = \arg\max_{y \geqslant 0} \left\{ u\left(W_0 - \Pi - y\right) + \lambda y F_h^{-1}(t) \right\},$$

then f^* is optimal for Problem (I.2).

Proof. Let $f^* \in \mathcal{Q}^*$ be such that the two conditions above are satisfied. Then f^* is feasible for Problem (I.2). To show optimality, let $f \in \mathcal{Q}^*$ be any feasible solution for Problem (I.2). Then, by definition of f^*, it follows that for each t,

$$u\Big(W_0 - \Pi - f^*(t)\Big) - u\Big(W_0 - \Pi - f(t)\Big) \geqslant \lambda \left[F_h^{-1}(t)\, f(t) - F_h^{-1}(t)\, f^*(t)\right].$$

Hence,

$$\begin{aligned}\int_0^1 u\Big(W_0 - \Pi - f^*(t)\Big) dt - \int_0^1 u\Big(W_0 - \Pi - f(t)\Big) dt &\geqslant \lambda \left[\int_0^1 F_h^{-1}(t)\, f(t)\, dt - \int_0^1 F_h^{-1}(t)\, f^*(t)\, dt\right] \\ &= \lambda \left[\int_0^1 F_h^{-1}(t)\, f(t)\, dt - \overline{R}_0\right] \geqslant 0.\end{aligned}$$

Therefore, $\int_0^1 u\left(W_0 - \Pi - f^*(t)\right) dt \geqslant \int_0^1 u\left(W_0 - \Pi - f(t)\right) dt$. □

Lemma I.5. *For each $\lambda \geqslant 0$, define the function f^*_λ by*

$$f^*_\lambda(t) := \max\left[0, W_0 - \Pi - (u')^{-1}\left(\lambda F_h^{-1}(t)\right)\right]. \tag{I.1}$$

Then:

(1) *For each $\lambda \geqslant 0$, $f^*_\lambda \in \mathcal{Q}^*$;*

(2) *There exists $\lambda^* \geqslant 0$ such that $\int_0^1 F_h^{-1}(t)\, f^*_{\lambda^*}(t)\, dt = \overline{R}_0$; and*

(3) *For all $t \in (0,1)$, $f^*_{\lambda^*}(t) = \arg\max_{y \geqslant 0}\left\{u(W_0 - \Pi - y) + \lambda y F_h^{-1}(t)\right\}$.*

Proof. Follows from Remark 2.4, from the monotonicity and continuity properties of the quantile function F_h^{-1}, from Assumption 4.3 and Assumption 4.4, from the fact that $\int_0^1 F_h^{-1}(t)\, dt = \int h dP = 1$, and from the Intermediate Value Theorem. □

Hence, by lemmata I.4 and I.5, if λ^* is chosen such that $\int_0^1 f^*_{\lambda^*}(t)\, F_h^{-1}(t)\, dt = R_0$, then the optimal solution to Problem (I.2) is given by $f^*_{\lambda^*}$, defined as in eq. (I.1). Consequently, by Lemma I.3, the function $R^* := f^*_{\lambda^*}\left(\widetilde{U}\right) = f^*_{\lambda^*}(F_h(h))$ is optimal for Problem (4.2) and comonotonic with h. This concludes the proof of Theorem 4.7. □

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