ON AN OPERATIONAL MATRIX METHOD BASED ON GENERALIZED BERNOULLI POLYNOMIALS OF LEVEL m

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Abstract. An operational matrix method based on generalized Bernoulli polynomials of level m is introduced and analyzed in order to obtain numerical solutions of initial value problems. The most innovative component of our method comes, essentially, from the introduction of the generalized Bernoulli polynomials of level m, which generalize the classical Bernoulli polynomials. Computational results demonstrate that such operational matrix method can lead to very ill-conditioned matrix equations.

1. Introduction

It is well known that differential equations and their solutions play a prominent role in fields related with science and engineering; for example, when some boundary conditions are imposed, differential equations can be used to model many natural phenomena. Unfortunately, it might be difficult (or even impossible) to find the analytical solution to some initial value problems and, for this reason, some approximation method is often needed. The so-called operational matrix method is one of the numerical methods available to solve a wide class of differential equations. This technique turns the differential equation problem into a system of algebraic equations by means of a finite set of orthogonal basis functions, which simplifies the problem (see [1, 3–5, 7–9]). Furthermore, the requirement

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of orthogonality for the basis can be skipped when formulating those algebraic equations and, therefore, the problem is significantly simplified (cf. [10, 16, 19]).

This paper uses the generalized Bernoulli polynomials of level $m \in \mathbb{N}$ as basis functions in order to construct a numerical technique based on the operational matrix method mentioned above. This class of polynomials was firstly introduced by Natalini and Bernardini in [14], as a generalization of the classical Bernoulli polynomials and they represent a particular case of the so-called extensions of generalized Apostol-type polynomials [11]. The interested reader can find recent literature which contains a large number of new and interesting properties involving these polynomials (see for instance, [11] and the references therein).

The outline of the paper is as follows. In Section 2 some relevant properties of the generalized Bernoulli polynomials of level m are given. Section 3 contains the basic ideas to describe for a fixed $m \in \mathbb{N}$, the best projection of elements in $L^2[0,1]$ onto the subspace $\mathfrak{B}_N^{[m-1]}$, generated by the generalized Bernoulli polynomials of level m, $\left\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \ldots, B_N^{[m-1]}(x)\right\}$. In Section 4 the operational matrices of differentiation, integration and product are defined, and explicit representations of the entries of these matrices (see Theorems 4.1 and 4.2) are established. Finally, Section 5 provides examples to demonstrate the use of the proposed method and how it leads to very ill-conditioned matrix equations.

2. Some properties of the generalized Bernoulli polynomials of level m

For a fixed $m \in \mathbb{N}$, the generalized Bernoulli polynomials of level m are defined by means of the following generating function [17]

$$\frac{z^m e^{xz}}{e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{z^n}{n!}, \quad |z| < 2\pi \tag{2.1}$$

and, the generalized Bernoulli numbers of level m are defined by $B_n^{[m-1]} := B_n^{[m-1]}(0)$, for all $n \geq 0$. It is clear that if $m = 1$ in (2.1), then we obtain the definition of the classical Bernoulli polynomials $B_n(x)$, and classical Bernoulli numbers, respectively, i.e., $B_n(x) = B_n^{[0]}(x)$, and $B_n = B_n^{[0]}$, respectively, for all $n \geq 0$.

The generalized Bernoulli polynomials of level m and the generalized Bernoulli numbers of level m were introduced by Natalini and Bernardini in [14] as a generalization of the classical Bernoulli polynomials, and classical Bernoulli numbers, respectively. In that paper, some algebraic and differential properties satisfied by those polynomials are studied as well.

For example, the first four generalized Bernoulli polynomials of level m are:

$$\begin{aligned}
B_0^{[m-1]}(x) &= m!, \\
B_1^{[m-1]}(x) &= m!\left(x - \frac{1}{m+1}\right), \\
B_2^{[m-1]}(x) &= m!\left(x^2 - \frac{2}{m+1}x + \frac{2}{(m+1)^2(m+2)}\right), \\
B_3^{[m-1]}(x) &= m!\left(x^3 - \frac{3}{m+1}x^2 + \frac{6}{(m+1)^2(m+2)}x + \frac{6(m-1)}{(m+1)^3(m+2)(m+3)}\right).
\end{aligned}$$

The following proposition summarizes some properties of the generalized Bernoulli polynomials of level m (cf. [11, 14]).

PROPOSITION 2.1. *For a fixed $m \in \mathbb{N}$, let $\left\{B_n^{[m-1]}(x)\right\}_{n\geq 0}$ be the sequence of generalized Bernoulli polynomials of level m. Then the following statements hold:*

a) *Summation formula. For every $n \geq 0$,*

$$B_n^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]} x^{n-k}. \tag{2.2}$$

b) *Differential relations (Appell polynomial sequences). For $n, j \geq 0$ with $0 \leq j \leq n$, we have*

$$[B_n^{[m-1]}(x)]^{(j)} = \frac{n!}{(n-j)!} B_{n-j}^{[m-1]}(x). \tag{2.3}$$

c) *Inversion formula. [14, Equation (2.6)] For every $n \geq 0$,*

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(m+k)!} B_{n-k}^{[m-1]}(x). \tag{2.4}$$

d) *Recurrence relation. [14, Lemma 3.2] For every $n \geq 1$,*

$$B_n^{[m-1]}(x) = \left(x - \frac{1}{m+1}\right) B_{n-1}^{[m-1]}(x) - \frac{1}{n(m-1)!} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x).$$

e) *Integral formulas.*

$$\begin{aligned}
\int_{x_0}^{x_1} B_n^{[m-1]}(x)dx &= \frac{1}{n+1}\left[B_{n+1}^{[m-1]}(x_1) - B_{n+1}^{[m-1]}(x_0)\right] \\
&= \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k} B_k^{[m-1]}((x_1)^{n-k+1} - (x_0)^{n-k+1}).
\end{aligned} \tag{2.5}$$

$$B_n^{[m-1]}(x) = n\int_0^x B_{n-1}^{[m-1]}(t)dt + B_n^{[m-1]}. \tag{2.6}$$

f) *[14, Theorem 3.1] Differential equation. For every $n \geq 1$, the polynomial $B_n^{[m-1]}(x)$ satisfies the following differential equation*

$$\frac{B_n^{[m-1]}}{n!}y^{(n)} + \frac{B_{n-1}^{[m-1]}}{(n-1)!}y^{(n-1)} + \cdots + \frac{B_2^{[m-1]}}{2!}y'' + (m-1)!\left(\frac{1}{m+1} - x\right)y' + n(m-1)!y = 0.$$

In what follows, we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and degree less than or equal to n. Notice that the inversion formula (2.4) immediately implies that

Proposition 2.2. *For a fixed $m \in \mathbb{N}$ and each $n \geq 0$, the set $\left\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \ldots, B_n^{[m-1]}(x)\right\}$ is a basis for $\mathbb{P}_n$, i.e.,*

$$\mathbb{P}_n = \mathfrak{B}_n^{[m-1]} = \operatorname{span}\left\{B_0^{[m-1]}(x), B_1^{[m-1]}(x), \ldots, B_n^{[m-1]}(x)\right\}.$$

From the summation formula (2.2) we can obtain the following matrix form of $B_r^{[m-1]}(x)$, $r = 0, 1, \ldots, n$.

$$\begin{aligned}
B_r^{[m-1]}(x) &= \sum_{k=0}^{r}\binom{r}{k}B_k^{[m-1]}\,x^{r-k} \\
&= \binom{r}{r}B_r^{[m-1]} + \binom{r}{r-1}B_{r-1}^{[m-1]}\,x + \cdots + \binom{r}{0}B_0^{[m-1]}\,x^r \\
&= \begin{pmatrix}\binom{r}{r}B_r^{[m-1]} & \binom{r}{r-1}B_{r-1}^{[m-1]} & \cdots & \binom{r}{0}B_0^{[m-1]} & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}1\\ x\\ \vdots\\ x^r\\ \vdots\\ x^n\end{pmatrix} \\
&= \mathbf{M}_r^{[m-1]}\mathbf{T}(x),
\end{aligned}$$

where

$$\mathbf{M}_r^{[m-1]} = \begin{pmatrix}\binom{r}{r}B_r^{[m-1]} & \binom{r}{r-1}B_{r-1}^{[m-1]} & \cdots & \binom{r}{0}B_0^{[m-1]} & 0 & \cdots & 0\end{pmatrix}, \tag{2.7}$$

the null entries of the matrix $\mathbf{M}_r^{[m-1]}$ appear $(n-r)$-times and $\mathbf{T}(x) = \begin{pmatrix}1 & x & \cdots & x^r & \cdots & x^n\end{pmatrix}^T$.

Analogously, by (2.7) the matrix $\mathbf{B}^{[m-1]}(x) = \begin{pmatrix} B_0^{[m-1]}(x) & B_1^{[m-1]}(x) & \cdots & B_n^{[m-1]}(x) \end{pmatrix}^T$, can be expressed as follows:

$$\begin{aligned}
\mathbf{B}^{[m-1]}(x) &= \mathbf{M}^{[m-1]}\mathbf{T}(x) \\
&= \begin{pmatrix}
B_0^{[m-1]} & 0 & 0 & 0 & \cdots & 0 \\
\binom{1}{1}B_1^{[m-1]} & \binom{1}{0}B_0^{[m-1]} & 0 & 0 & \cdots & 0 \\
\binom{2}{2}B_2^{[m-1]} & \binom{2}{1}B_1^{[m-1]} & \binom{2}{0}B_0^{[m-1]} & 0 & \cdots & 0 \\
\binom{3}{3}B_3^{[m-1]} & \binom{3}{2}B_2^{[m-1]} & \binom{3}{1}B_1^{[m-1]} & \binom{3}{0}B_0^{[m-1]} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{n}{n}B_n^{[m-1]} & \binom{n}{n-1}B_{n-1}^{[m-1]} & \binom{n}{n-2}B_{n-2}^{[m-1]} & \binom{n}{n-3}B_{n-3}^{[m-1]} & \cdots & \binom{n}{0}B_0^{[m-1]}
\end{pmatrix}\mathbf{T}(x).
\end{aligned} \tag{2.8}$$

Notice that according to (2.7) the rows of the matrix $\mathbf{M}^{[m-1]}$ are precisely the matrices $\mathbf{M}_r^{[m-1]}$ for $r = 0, \ldots, n$. Furthermore, the matrix $\mathbf{M}^{[m-1]}$ is a lower triangular matrix, so that

$$\det\left(\mathbf{M}^{[m-1]}\right) = \left(B_0^{[m-1]}\right)^{n+1} \prod_{k=1}^{n} \binom{k}{0} = \left(B_0^{[m-1]}\right)^{n+1} = (m!)^{n+1}.$$

Therefore, $\mathbf{M}^{[m-1]}$ is an invertible matrix.

Let $\mathbf{W}^{[m-1]}$ be the inverse matrix of $\mathbf{M}^{[m-1]}$. For any $P(x) \in \mathbb{P}_n$ there exist $0 \leq s \leq n$ such that $\deg(P(x)) = s$, and a unique $(C_0, \ldots, C_s) \in \mathbb{R}^{s+1} \setminus \{0\}$ such that

$$P(x) = \sum_{k=0}^{s} C_k x^k = C\,\mathbf{T}(x), \tag{2.9}$$

where C is the matrix given by

$$C = \begin{pmatrix} C_0 & C_1 & \cdots & C_s & 0 & \cdots & 0 \end{pmatrix},$$

and the null entries of the matrix C appear $(n-s)$-times. According to (2.8) the expression (2.9) becomes

$$P(x) = C\,\mathbf{W}^{[m-1]}\mathbf{B}^{[m-1]}(x). \tag{2.10}$$

Using the inversion formula (2.4) it is possible to find an explicit expression for the matrix $\mathbf{W}^{[m-1]}$. In order to do this, we rewrite (2.4) in matrix form as follows:

$$\begin{aligned}\mathbf{T}(x) &= \mathbf{Q}^{[m-1]}\mathbf{B}^{[m-1]}(x)\\ &= \begin{pmatrix} \frac{1}{m!} & 0 & 0 & 0 & \cdots & 0\\ \frac{1}{(m+1)!} & \frac{1}{m!} & 0 & 0 & \cdots & 0\\ \frac{2!}{(m+2)!} & \frac{2!}{(m+1)!} & \frac{1}{m!} & 0 & \cdots & 0\\ \frac{3!}{(m+3)!} & \frac{3!}{(m+2)!} & \frac{3}{(m+1)!} & \frac{1}{m!} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{n!}{(m+n)!} & \frac{n!}{(m+n-1)!} & \frac{n!}{2!(m+n-2)!} & \frac{n!}{3!(m+n-3)!} & \cdots & \frac{1}{m!} \end{pmatrix}\mathbf{B}^{[m-1]}(x). \end{aligned} \tag{2.11}$$

Since the matrix $\mathbf{Q}^{[m-1]}$ is a lower triangular matrix, we have

$$\det\left(\mathbf{Q}^{[m-1]}\right) = \left(\frac{1}{m!}\right)^{n+1}.$$

Therefore, $\mathbf{Q}^{[m-1]}$ is an invertible matrix. Finally,

$$\mathbf{W}^{[m-1]}\mathbf{B}^{[m-1]}(x) = \mathbf{T}(x) = \mathbf{Q}^{[m-1]}\mathbf{B}^{[m-1]}(x),$$

and consequently,

$$\left(\mathbf{M}^{[m-1]}\right)^{-1} = \mathbf{W}^{[m-1]} = \mathbf{Q}^{[m-1]}.$$

It is well known that using the Euler-Maclaurin summation formula (cf. [2, 13, 18], and [15, Chap. 2, Sec. 3, p. 30]) it is possible to deduce the following formula for the integral of the product of two classical Bernoulli polynomials

$$\int_0^1 B_s(t)B_r(t)dt = (-1)^{s+1}\frac{s!r!}{(s+r)!}B_{s+r}, \quad r,s \geq 1. \tag{2.12}$$

We are interested in a similar formula for the integral of the product of two generalized Bernoulli polynomials of level m. In order to establish a formula like (2.12), we will deduce a summation formula of Euler-Maclaurin type based on generalized Bernoulli polynomials of level m.

For an integer $s \geq 0$ and a closed interval $[a,b]$, let $C^s[a,b]$ denote the set of all s-times continuously differentiable functions defined on $[a,b]$. The integration by parts formula asserts that the following result holds.

Lemma 2.1. *Let $s \geq 1$ and $f \in C^s[0,1]$. For a fixed $m \in \mathbb{N}$, we have*

$$\int_0^1 f(t)dt = \frac{1}{m!}\left[\sum_{k=1}^{s} A_k^{[m-1]}(f) + \frac{(-1)^s}{s!}\int_0^1 f^{(s)}(t)B_s^{[m-1]}(t)dt\right], \tag{2.13}$$

where

$$A_k^{[m-1]}(f) = \frac{(-1)^k}{k!}\left(f^{(k-1)}(0)B_k^{[m-1]} - f^{(k-1)}(1)B_k^{[m-1]}(1)\right), \quad k = 1, \dots, s.$$

Proof. Since the integral on the left-hand side of (2.13) can be expressed as

$$\int_0^1 f(t)dt = \frac{1}{m!}\int_0^1 f(t)B_0^{[m-1]}(t)dt,$$

it suffices to apply repeated integration by parts on the right-hand side of the equation above, using a suitable form of (2.3) in each step.

Applying the substitution $f(t) = B_{s+r}^{[m-1]}(t)$ into (2.13) and taking into account (2.3), (2.6) and some straightforward calculations, we can show that

(2.14)
$$\int_0^1 B_s^{[m-1]}(t)B_r^{[m-1]}(t)dt = \frac{(-1)^{s+1}s!r!m!}{(s+r)!}\left[\frac{B_{s+r+1}^{[m-1]} - B_{s+r+1}^{[m-1]}(1)}{s+r+1} + \frac{1}{m!}\sum_{k=1}^{s} A_k^{[m-1]}\right], \quad r, s \geq 1,$$

where

$$A_k^{[m-1]} = \frac{(-1)^k}{k}\binom{r+s}{k-1}\left(B_{s+r-k+1}^{[m-1]}B_k^{[m-1]} - B_{s+r-k+1}^{[m-1]}(1)B_k^{[m-1]}(1)\right), \quad k = 1, \dots, s.$$

The expression (2.14) is our desired analogue of (2.12) in the setting of the generalized Bernoulli polynomials of level m.

3. The best projection of elements in $L^2[0,1]$ onto the subspace $\mathfrak{B}_N^{[m-1]}$

Let $L^2[0,1]$ be the space of the square-integrable functions on $[0,1]$, endowed with the norm

$$\|f\|_2 := \left(\int_0^1 |f(t)|^2 dt\right)^{1/2} = \langle f, f\rangle^{1/2},$$

where

$$\langle f, g\rangle := \int_0^1 f(t)g(t)dt, \text{ for every } f, g \in L^2[0,1].$$

Let us consider $N \geq 0$. Since $L^2[0,1]$ is a uniformly convex Banach space and $\mathfrak{B}_N^{[m-1]}$ is a closed convex set in $L^2[0,1]$, for $f \in L^2[0,1]$ there exists a unique $\xi_N(f) \in \mathfrak{B}_N^{[m-1]}$ such that (see e.g., [6, Chap. 1, Sec. 6, p. 22]).

$$\inf\left\{\|f-p\|_2 : p \in \mathfrak{B}_N^{[m-1]}\right\} = \|f - \xi_N(f)\|_2. \tag{3.15}$$

The element $\xi_N(f) \in \mathfrak{B}_N^{[m-1]}$ is called the best projection of f onto $\mathfrak{B}_N^{[m-1]}$. Furthermore, if we define $g := f - \xi_N(f)$, then for every $p(x) \in \mathfrak{B}_N^{[m-1]} \setminus \{0\}$ we have

$$\left\| g - \frac{\langle g, p\rangle}{\|p\|_2^2} p \right\|^2 = \|g\|_2^2 - \frac{\langle g, p\rangle^2}{\|p\|_2^2} \leq \|g\|_2^2.$$

Thus, taking $q = \xi_N(f) + \frac{\langle g,p\rangle}{\|p\|_2^2} p \in \mathfrak{B}_N^{[m-1]}$ we see that

$$\|f - q\|_2 \leq \|f - \xi_N(f)\|_2 .$$

But, as a consequence of (3.15) we obtain that necessarily $\langle g, p\rangle = \langle f - \xi_N(f), p\rangle = 0$. More precisely,

$$\langle f - \xi_N(f), p\rangle = 0, \quad \text{for every } p \in \mathfrak{B}_N^{[m-1]},$$

or equivalently,

$$\left\langle f - \xi_N(f), B_r^{[m-1]}(x)\right\rangle = 0, \quad \text{for every } r = 0, \ldots, N. \tag{3.16}$$

Since $\xi_N(f) \in \mathfrak{B}_N^{[m-1]}$, there exists $(a_0, a_1, \ldots, a_N) \in \mathbb{R}^{N+1}$ such that

$$\xi_N(f)(x) = \sum_{k=0}^{N} a_k B_k^{[m-1]}(x) = A\mathbf{B}^{[m-1]}(x), \tag{3.17}$$

where A is the matrix given by

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_N \end{pmatrix}.$$

Thus,

$$\xi_N(f)(x) \left(\mathbf{B}^{[m-1]}(x)\right)^T = A \left[\mathbf{B}^{[m-1]}(x) \left(\mathbf{B}^{[m-1]}(x)\right)^T\right], \tag{3.18}$$

then

$$\int_0^1 \xi_N(f)(x) \left(\mathbf{B}^{[m-1]}(x)\right)^T dx = A \int_0^1 \left[\mathbf{B}^{[m-1]}(x) \left(\mathbf{B}^{[m-1]}(x)\right)^T\right] dx, \tag{3.19}$$

where the matrix $\int_0^1 \xi_N(f)(x) \left(\mathbf{B}^{[m-1]}(x)\right)^T dx$ is an $1 \times (N+1)$ matrix with entries given by

$$\int_0^1 \xi_N(f)(x) \left(\mathbf{B}^{[m-1]}(x)\right)^T dx = \begin{pmatrix} \int_0^1 \xi_N(f)(x) B_0^{[m-1]}(x) dx & \cdots & \int_0^1 \xi_N(f)(x) B_N^{[m-1]}(x) dx \end{pmatrix},$$

and the matrix $\int_0^1 \left[\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T\right]dx$ is an $(N+1)\times(N+1)$ matrix with entries given by

$$d_{s,r}^{[m-1]} = \int_0^1 B_s^{[m-1]}(x)B_r^{[m-1]}(x)dx, \quad 0 \leq s, r \leq N.$$

Therefore, the entries of the matrix A on the right-hand side of (3.19) can be obtained by means of the following expression

$$A = \left[\int_0^1 \xi_N(f)(x)\mathbf{B}^T(x)dx\right]\left[\int_0^1 \left[\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T\right]dx\right]^{-1}, \tag{3.20}$$

provided that the matrix $\int_0^1 \left[\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T\right]dx$ is an invertible matrix.

Remark 3.1 Notice that

(a) The matrix $\int_0^1 \left[\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T\right]dx$ is symmetric and its entries can be obtained by (2.14).

(b) If $m = 1$, by (2.12) and the properties of the classical Bernoulli numbers it is possible to check that the matrix $\int_0^1 \left[\mathbf{B}^{[0]}(x)\left(\mathbf{B}^{[0]}(x)\right)^T\right]dx$ is invertible (cf. [16]).

Now, we denote by

$$\mathbf{R}^{[m-1]} = \int_0^1 \left[\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T\right]dx. \tag{3.21}$$

Since $\mathbf{R}^{[m-1]}$ is a real symmetric matrix, the Principal Axis Theorem guarantees that $\mathbf{R}^{[m-1]}$ can be written as

$$\mathbf{R}^{[m-1]} = \mathbf{O}^{[m-1]}\Lambda^{[m-1]}\left(\mathbf{O}^{[m-1]}\right)^T,$$

where the matrix $\mathbf{O}^{[m-1]}$ is orthogonal and the matrix $\Lambda^{[m-1]} = \mathrm{diag}(\lambda_0^{[m-1]}, \lambda_1^{[m-1]}, \ldots, \lambda_N^{[m-1]})$, with $\lambda_r^{[m-1]} \in \mathbb{R}$ an eigenvalue of $\mathbf{R}^{[m-1]}$, for $r = 0, \ldots, N$.

Thus,

$$\det\left(\mathbf{R}^{[m-1]}\right) = \prod_{r=0}^{N} \lambda_r^{[m-1]}.$$

Hence, for determining the nonsingularity of the matrix $\mathbf{R}^{[m-1]}$ it suffices to see that $\lambda_r^{[m-1]} \neq 0$ for all $r = 0, \ldots, N$. To the best of our knowledge, this property is not available in the literature, not even for the case $m = 1$ (cf. [16]), and to find eigenvalues of the matrix $\mathbf{R}^{[m-1]}$ for a fixed $m \in \mathbb{N}$, and show that these eigenvalues are different from zero can be a tedious and cumbersome task, except possibly for $N = 1$ or $N = 2$.

A similar situation occurs when criteria involving diagonal dominance are imposed (cf., e.g., [12]). However, one simple way to explore the non-singularity of the matrix $\mathbf{R}^{[m-1]}$ is to carry out numerical simulations (with the help of MAPLE, for instance).

Example 3.1. *For $m=2$ and $N=9$, we have that*

$$\mathbf{R}^{[1]} \simeq \begin{pmatrix} 4 & 0.666 & 0.222 & 0.044 & -0.014 & -0.017 & -0.0007 & 0.011 & 0.007 & -0.009 \\ 0.666 & 0.044 & 0.148 & 0.029 & -0.009 & -0.011 & -0.0004 & 0.007 & 0.004 & -0.006 \\ 0.222 & 0.148 & 0.148 & 0.020 & -0.002 & -0.006 & -0.001 & 0.003 & 0.003 & -0.001 \\ 0.044 & 0.029 & 0.020 & 0.008 & 0.0008 & -0.002 & -0.001 & 0.0007 & 0.001 & 0.0001 \\ -0.014 & -0.009 & -0.002 & 0.0008 & 0.0009 & 0.0002 & -0.0003 & -0.0003 & 0.0001 & 0.0005 \\ -0.017 & -0.011 & -0.006 & -0.002 & 0.0002 & 0.0007 & 0.0002 & -0.0003 & -0.0004 & 0.0001 \\ -0.0007 & -0.0004 & -0.001 & -0.001 & -0.0003 & 0.0002 & 0.0003 & -0.159\times10^{-5} & -0.0002 & -0.0001 \\ 0.011 & 0.007 & 0.003 & 0.0007 & -0.0003 & -0.0003 & -0.159\times10^{-5} & 0.0002 & 0.0001 & -0.0002 \\ 0.007 & 0.004 & 0.003 & 0.001 & 0.0001 & -0.0004 & -0.0002 & 0.0001 & 0.0003 & 0.00004 \\ -0.009 & -0.006 & -0.001 & 0.0001 & 0.0005 & 0.0001 & -0.0001 & -0.0002 & 0.00004 & 0.0002 \end{pmatrix},$$

$\det\left(\mathbf{R}^{[1]}\right) \simeq 0.136427477242195\times 10^{-42}$, *and the eigenvalues of the matrix $\mathbf{R}^{[1]}$ are the following*

$$\textit{Eigenvalues of } \mathbf{R}^{[1]} \simeq \begin{pmatrix} 4.136818498 \\ 0.3733056185 \\ 0.08678164218 \\ 0.006714935295 \\ 0.0008408259139 \\ 0.00003826222613 \\ 0.00000246716679 \\ 0.000000050804729 \\ 0.000000002135169 \\ 0.000000000017607 \end{pmatrix}.$$

Example 3.2. *For $m=5$ and $N=8$, we have*

$$\mathbf{R}^{[4]} \simeq \begin{pmatrix} 14400 & 4800 & 2514.285 & 1400 & 771.156 & 394.104 & 167.438 & 39.256 & -21.283 \\ 4800 & 2800 & 1638.095 & 975.238 & 563.718 & 300.302 & 133.817 & 35.360 & -13.727 \\ 2514.285 & 1638.095 & 1052.335 & 663.492 & 400.043 & 221.117 & 102.780 & 29.905 & -8.209 \\ 1400 & 975.238 & 663.492 & 436.791 & 272.751 & 155.716 & 75.165 & 23.712 & -4.382 \\ 771.156 & 563.718 & 400.043 & 272.751 & 175.577 & 103.219 & 51.586 & 17.467 & -1.920 \\ 394.104 & 300.302 & 221.117 & 155.716 & 103.219 & 62.468 & 32.323 & 11.707 & -0.486 \\ 167.438 & 133.817 & 102.780 & 75.165 & 51.586 & 32.323 & 17.428 & 6.805 & 0.215 \\ 39.256 & 35.360 & 29.905 & 23.712 & 17.467 & 11.707 & 6.805 & 3.003 & 0.427 \\ -21.283 & -13.727 & -8.209 & -4.382 & -1.920 & -0.486 & 0.215 & 0.427 & 0.349 \end{pmatrix},$$

$\det\left(\mathbf{R}^{[4]}\right) \simeq 0.267549566246871\times 10^{-4}$, *and the eigenvalues of the matrix $\mathbf{R}^{[4]}$ are the following.*

$$\textit{Eigenvalues of } \mathbf{R}^{[4]} \simeq \begin{pmatrix} 16940.98768 \\ 1844.198384 \\ 150.3436662 \\ 11.51349907 \\ 0.8573769742 \\ 0.05105520057 \\ 0.001725633103 \\ 0.00002963569967 \\ 0.0000002209988277 \end{pmatrix}.$$

Notice that in the two previous examples $\mathbf{R}^{[m-1]}$ is close to a singular matrix and therefore it could be ill conditioned. This fact has a negative impact on the numerical simulations that involve that matrix.

4. Operational matrices of differentiation, integration and product

In this section, we are going to provide operational matrices of differentiation, integration and product in the setting of the generalized Bernoulli polynomials of level m. The outlines of the method are the same as that in [16]. The technical details, however, are much more demanding than in the Bernoulli case.

Definition 4.1. *For a fixed $m \in \mathbb{N}$ and $N \geq 0$, let us consider the matrix*

$$\mathbf{B}^{[m-1]}(x) = \begin{pmatrix} B_0^{[m-1]}(x) & B_1^{[m-1]}(x) & \cdots & B_N^{[m-1]}(x) \end{pmatrix}^T.$$

We say that the matrices $\mathcal{D}^{[m-1]}$ and $\mathcal{I}^{[m-1]}$ are the operational matrices of differentiation and integration, respectively, if and only if

$$\frac{d}{dx}\mathbf{B}^{[m-1]}(x) = \mathcal{D}^{[m-1]}\,\mathbf{B}^{[m-1]}(x), \tag{4.22}$$

$$\int_0^x \mathbf{B}^{[m-1]}(t)\,dt \simeq \mathcal{I}^{[m-1]}\,\mathbf{B}^{[m-1]}(x), \tag{4.23}$$

where

$$\frac{d}{dx}\mathbf{B}^{[m-1]}(x) := \begin{pmatrix} \frac{d}{dx}B_0^{[m-1]}(x) & \frac{d}{dx}B_1^{[m-1]}(x) & \cdots & \frac{d}{dx}B_N^{[m-1]}(x) \end{pmatrix}^T,$$

and

$$\int_0^x \mathbf{B}^{[m-1]}(t)\,dt := \begin{pmatrix} \int_0^x B_0^{[m-1]}(t)dt & \int_0^x B_1^{[m-1]}(t)dt & \cdots & \int_0^x B_N^{[m-1]}(t)dt \end{pmatrix}^T.$$

Definition 4.2. *For a fixed $m \in \mathbb{N}$ and $N \geq 0$, let us consider the matrix*

$$\mathbf{B}^{[m-1]}(x) = \begin{pmatrix} B_0^{[m-1]}(x) & B_1^{[m-1]}(x) & \cdots & B_N^{[m-1]}(x) \end{pmatrix}^T.$$

Suppose that $C \in \mathbb{R}^{(N+1)\times 1}$ is an arbitrary vector. We say that the matrix $\mathcal{C}^{[m-1]}$ is the operational matrix of the product for the vector C, if and only if

$$\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T C \simeq \mathcal{C}^{[m-1]}\,\mathbf{B}^{[m-1]}(x). \tag{4.24}$$

THEOREM 4.1. *For any $m \in \mathbb{N}$ and $N \geq 0$, let $\mathcal{D}^{[m-1]}$ and $\mathcal{I}^{[m-1]}$ be the operational matrices of differentiation and integration, respectively. Then, the entries of $\mathcal{D}^{[m-1]}$ are given by*

$$\mathcal{D}^{[m-1]}_{i,j} = \begin{cases} i, & i = j+1, \\ 0, & \textit{otherwise}, \end{cases} \tag{4.25}$$

with $0 \leq i, j \leq N$.

While the operational matrix $\mathcal{I}^{[m-1]}$ is given by

$$\mathcal{I}^{[m-1]} = \mathbf{U}^{[m-1]}\mathbf{Q}^{[m-1]}, \tag{4.26}$$

where $\mathbf{Q}^{[m-1]}$ is the lower triangular matrix (2.11) and $\mathbf{U}^{[m-1]}$ is the matrix whose rows are given by

$$\begin{aligned} U^{[m-1]}_s &= \begin{pmatrix} 0 & \frac{1}{s}\binom{s}{s-1}B^{[m-1]}_{s-1} & \frac{1}{s}\binom{s}{s-2}B^{[m-1]}_{s-2} & \cdots & \frac{1}{s}B^{[m-1]}_0 & 0 & \cdots & 0 \end{pmatrix}, \quad s = 1, \ldots, N, \\ U^{[m-1]}_{N+1} &= \begin{pmatrix} \beta^{[m-1]}_0 & \beta^{[m-1]}_1 & \cdots & \beta^{[m-1]}_N \end{pmatrix} \mathbf{M}^{[m-1]}, \end{aligned}$$

being $\beta^{[m-1]}_j$ the coefficients of the best projection of $\frac{B^{[m-1]}_{N+1}(x) - B^{[m-1]}_{N+1}}{N+1}$ onto $\mathfrak{B}^{[m-1]}_N$, $j = 0, \ldots, N$, and $\mathbf{M}^{[m-1]}$ the lower triangular matrix (2.7).

Proof. For the sake of clarity and readability, we have decided to include the details of the proof of this theorem. However, one can check that it suffices to follow the reasoning in [16, Subsections 4.1 and 4.2], making the appropriate modifications.

By (2.3) with $j = 1$, we have

$$\begin{aligned} \frac{d}{dx}\mathbf{B}^{[m-1]}(x) &= \begin{pmatrix} \frac{d}{dx}B^{[m-1]}_0(x) & \frac{d}{dx}B^{[m-1]}_1(x) & \cdots & \frac{d}{dx}B^{[m-1]}_N(x) \end{pmatrix}^T \\ &= \begin{pmatrix} 0 & B^{[m-1]}_0(x) & 2B^{[m-1]}_1(x) & \cdots & NB^{[m-1]}_{N-1}(x) \end{pmatrix}^T \\ &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2 & 0 & \cdots & N & 0 \end{pmatrix} \mathbf{B}^{[m-1]}(x). \end{aligned} \tag{4.27}$$

It is clear that the matrix on the right-hand side of (4.27) satisfies (4.22) and consequently, it also satisfies (4.25).

From (2.6) we can deduce that

$$\begin{aligned}\int_0^x \mathbf{B}^{[m-1]}(t)\,dt &= \begin{pmatrix}\int_0^x B_0^{[m-1]}(t)dt & \int_0^x B_1^{[m-1]}(t)dt & \cdots & \int_0^x B_N^{[m-1]}(t)dt\end{pmatrix}^T \\ &= \begin{pmatrix}B_1^{[m-1]}(x) - B_1^{[m-1]} & \frac{B_2^{[m-1]}(x)-B_2^{[m-1]}}{2} & \cdots & \frac{B_{N+1}^{[m-1]}(x)-B_{N+1}^{[m-1]}}{N+1}\end{pmatrix}^T.\end{aligned}$$

On the one hand, using the summation formula (2.2) for each $s = 1, \dots, N$, we get

$$\begin{aligned}\frac{B_s^{[m-1]}(x) - B_s^{[m-1]}}{s} &= \frac{1}{s}\sum_{k=0}^{s}\binom{s}{k}B_k^{[m-1]}\,x^{s-k} - \frac{1}{s}B_s^{[m-1]} \\ &= \frac{1}{s}\binom{s}{s-1}B_{s-1}^{[m-1]}\,x + \frac{1}{s}\binom{s}{s-2}B_{s-2}^{[m-1]}\,x^2 + \cdots + \frac{1}{s}B_0^{[m-1]}\,x^s \\ &= \begin{pmatrix}0 & \frac{1}{s}\binom{s}{s-1}B_{s-1}^{[m-1]} & \frac{1}{s}\binom{s}{s-2}B_{s-2}^{[m-1]} & \cdots & \frac{1}{s}B_0^{[m-1]} & 0 & \cdots & 0\end{pmatrix}\mathbf{T}(x) \\ &= U_s^{[m-1]}\mathbf{T}(x), \qquad (4.28)\end{aligned}$$

where the last null entries of the matrix $U_s^{[m-1]}$ appear $(N-s)$-times, and

$$\mathbf{T}(x) = \begin{pmatrix}1 & x & x^2 & \cdots & x^N\end{pmatrix}^T.$$

On the other hand, by taking the best projection of $\frac{B_{N+1}^{[m-1]}(x)-B_{N+1}^{[m-1]}}{N+1}$ onto $\mathfrak{B}_N^{[m-1]}$, we obtain that there exists a unique $\left(\beta_0^{[m-1]}, \dots, \beta_N^{[m-1]}\right) \in \mathbb{R}^{N+1}$ such that

$$\begin{aligned}\frac{B_{N+1}^{[m-1]}(x) - B_{N+1}^{[m-1]}}{N+1} &\simeq \sum_{k=0}^{N}\beta_k^{[m-1]}B_k^{[m-1]}(x) \\ &= \begin{pmatrix}\beta_0^{[m-1]} & \beta_1^{[m-1]} & \cdots & \beta_N^{[m-1]}\end{pmatrix}\mathbf{B}^{[m-1]}(x) \\ &= H^{[m-1]}\mathbf{B}^{[m-1]}(x). \qquad (4.29)\end{aligned}$$

Then in view of (2.11), (4.28) and (4.29), we see that

$$\begin{aligned}\int_0^x \mathbf{B}^{[m-1]}(t)\,dt &\simeq \begin{pmatrix} U_1^{[m-1]}\mathbf{Q}^{[m-1]}\mathbf{B}^{[m-1]}(x) \\ U_2^{[m-1]}\mathbf{Q}^{[m-1]}\mathbf{B}^{[m-1]}(x) \\ \vdots \\ U_N^{[m-1]}\mathbf{Q}^{[m-1]}\mathbf{B}^{[m-1]}(x) \\ H^{[m-1]}\mathbf{M}^{[m-1]}\mathbf{Q}^{[m-1]}\mathbf{B}^{[m-1]}(x) \end{pmatrix} = \begin{pmatrix} U_1^{[m-1]} \\ U_2^{[m-1]} \\ \vdots \\ U_N^{[m-1]} \\ H^{[m-1]}\mathbf{M}^{[m-1]} \end{pmatrix} \mathbf{Q}^{[m-1]}\mathbf{B}^{[m-1]}(x) \\ &= \mathbf{U}^{[m-1]}\mathbf{Q}^{[m-1]}\mathbf{B}^{[m-1]}(x), \qquad (4.30)\end{aligned}$$

where $\mathbf{U}^{[m-1]}$ is the matrix given by

$$\mathbf{U}^{[m-1]} = \begin{pmatrix} U_1^{[m-1]} & U_2^{[m-1]} & \cdots & U_N^{[m-1]} & H^{[m-1]}\mathbf{M}^{[m-1]} \end{pmatrix}^T .$$

Finally, from (4.30) we obtain (4.27).

Our next goal is to obtain an explicit formula for the operational matrix of the product (4.24).

THEOREM 4.2. *For a fixed $m \in \mathbb{N}$, $N \geq 0$ and $C \in \mathbb{R}^{(N+1)\times 1}$, let $\mathcal{C}^{[m-1]}$ be the operational matrix of the product for the vector C. Then $\mathcal{C}^{[m-1]}$ is given by*

$$\mathcal{C}^{[m-1]} = \mathbf{M}^{[m-1]}\mathbf{N}^{[m-1]}, \qquad (4.31)$$

where $\mathbf{N}^{[m-1]}$ is the $(N+1)\times(N+1)$ matrix whose rows are given by

$$N_s^{[m-1]} = \begin{pmatrix} \eta_{s,0}^{[m-1]} & \eta_{s,1}^{[m-1]} & \eta_{s,2}^{[m-1]} & \cdots & \eta_{s,N}^{[m-1]} \end{pmatrix}, \quad s = 0, \dots, N,$$

being $\eta_{s,r}^{[m-1]}$ the coefficients of the best projection of the s-th row of $\mathbf{T}(x)\mathbf{T}^T(x)\left(\mathbf{M}^{[m-1]}\right)^T C$ onto $\mathfrak{B}_N^{[m-1]}$, $0 \leq r \leq N$, and $\mathbf{M}^{[m-1]}$ the lower triangular matrix (2.7).

Remark 4.1 This theorem is virtually a reflex of [10, Subsection 2.3]. Although our approach is slightly different in this paper, the argument given here is based on the one appearing in [10, Subsection 2.3] and it uses again the best projection of elements in $L^2[0,1]$ onto the subspace $\mathfrak{B}_N^{[m-1]}$. Next, the details of the proof are provided for the readers convenience. However, we would like to emphasize that the formula for the operational matrix of the product obtained now is completely different from the results given in [16, Subsection 4.3].

Proof. Let $C = \begin{pmatrix} C_0 & C_1 & \cdots & C_N \end{pmatrix}^T \in \mathbb{R}^{(N+1)\times 1}$; from (2.7) we see that

$$\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T C = \mathbf{M}^{[m-1]}\mathbf{T}(x)\left(\mathbf{T}(x)\right)^T\left(\mathbf{M}^{[m-1]}\right)^T C. \tag{4.32}$$

Then, the factor $\mathbf{T}(x)\left(\mathbf{T}(x)\right)^T\left(\mathbf{M}^{[m-1]}\right)^T C$ on the right-hand side of (4.32) has the following form

$$\mathbf{T}(x)\left(\mathbf{T}(x)\right)^T\left(\mathbf{M}^{[m-1]}\right)^T C = \begin{pmatrix} \sum_{j=0}^{N}\alpha_j^{[m-1]}(C)x^j \\ \sum_{j=1}^{N+1}\alpha_{j-1}^{[m-1]}(C)x^j \\ \vdots \\ \sum_{j=N}^{2N}\alpha_{j-N}^{[m-1]}(C)x^j \end{pmatrix}, \tag{4.33}$$

where $\alpha_j^{[m-1]}(C)$ are determined by

$$\alpha_j^{[m-1]}(C) = \sum_{k=j}^{N}\binom{k}{k-j}C_k B_{k-j}^{[m-1]}, \quad j = 0, \ldots, N.$$

The rows of the matrix on the right-hand side of (4.33) are polynomials with degrees varying from N to $2N$. Therefore, for each row we can consider its corresponding best projection onto $\mathfrak{B}_N^{[m-1]}$, i.e.,

$$\sum_{j=s}^{2N}\alpha_{j-s}^{[m-1]}(C)x^j \simeq \sum_{r=0}^{N}\eta_{s,r}^{[m-1]}B_r^{[m-1]}(x) = \begin{pmatrix} \eta_{s,0}^{[m-1]} & \eta_{s,1}^{[m-1]} & \cdots & \eta_{s,N}^{[m-1]} \end{pmatrix}\mathbf{B}^{[m-1]}(x), \tag{4.34}$$

where $s = 0, \ldots, N$.

Substituting (4.34) into (4.33) yields

$$\mathbf{T}(x)\left(\mathbf{T}(x)\right)^T\left(\mathbf{M}^{[m-1]}\right)^T C \simeq \begin{pmatrix} \eta_{0,0}^{[m-1]} & \eta_{0,1}^{[m-1]} & \cdots & \eta_{0,N}^{[m-1]} \\ \eta_{1,0}^{[m-1]} & \eta_{1,1}^{[m-1]} & \cdots & \eta_{1,N}^{[m-1]} \\ \vdots & \vdots & \cdots & \vdots \\ \eta_{N,0}^{[m-1]} & \eta_{N,1}^{[m-1]} & \cdots & \eta_{N,N}^{[m-1]} \end{pmatrix}\mathbf{B}^{[m-1]}(x),$$

or equivalently,

$$\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T C \simeq \mathbf{M}^{[m-1]}\begin{pmatrix} \eta_{0,0}^{[m-1]} & \eta_{0,1}^{[m-1]} & \cdots & \eta_{0,N}^{[m-1]} \\ \eta_{1,0}^{[m-1]} & \eta_{1,1}^{[m-1]} & \cdots & \eta_{1,N}^{[m-1]} \\ \vdots & \vdots & \cdots & \vdots \\ \eta_{N,0}^{[m-1]} & \eta_{N,1}^{[m-1]} & \cdots & \eta_{N,N}^{[m-1]} \end{pmatrix}\mathbf{B}^{[m-1]}(x)$$

Summarizing, we have shown that

$$\mathcal{C}^{[m-1]} = \mathbf{M}^{[m-1]}\mathbf{N}^{[m-1]},$$

with

$$\mathbf{N}^{[m-1]} = \begin{pmatrix} \eta_{0,0}^{[m-1]} & \eta_{0,1}^{[m-1]} & \cdots & \eta_{0,N}^{[m-1]} \\ \eta_{1,0}^{[m-1]} & \eta_{1,1}^{[m-1]} & \cdots & \eta_{1,N}^{[m-1]} \\ \vdots & \vdots & \cdots & \vdots \\ \eta_{N,0}^{[m-1]} & \eta_{N,1}^{[m-1]} & \cdots & \eta_{N,N}^{[m-1]} \end{pmatrix},$$

then (4.31) follows.

5. Some examples

In this section, four numerical examples are examined to illustrate that the proposed method can lead to very ill-conditioned matrix equations. All the numerical experiments are performed using MAPLE 15. Also, we would like to point out that some of these examples have previously been used in [16, Section 5].

Example 5.1. *Consider the following Bessel differential equation of order 2.*

$$x\frac{d^2u(x)}{dx^2} + \frac{du(x)}{dx} + xu(x) = 0, \tag{5.35}$$

with initial conditions

$$u(0) = 1, \quad \left.\frac{du(x)}{dx}\right|_{x=0} = 0. \tag{5.36}$$

The exact solution of this problem is

$$u_{\text{es}}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n(n!)^2}x^{2n}.$$

To solve this problem we propose the following steps:

Step 1. Approximate $\frac{d^2u(x)}{dx^2}$ *by the generalized Bernoulli polynomials of level* m*. This step requires to choose specific values for* m *and* N

$$\frac{d^2u(x)}{dx^2} \simeq \sum_{k=0}^{N} A_k B_k^{[m-1]}(x) = A^T \mathbf{B}^{[m-1]}(x), \tag{5.37}$$

where $A = \begin{pmatrix} A_0 & A_1 & \cdots & A_N \end{pmatrix}^T$ *is a vector with unknown components that will need to be determined.*

Step 2. Use the initial conditions (5.36) *and the operational matrix of integration* (4.23) *in order to obtain approximate expressions for* $\frac{du(x)}{dx}$ *and* $u(x)$*.*

From (5.36) *we have*

$$\int_0^x \frac{d^2u(t)}{dt^2}\, dt = \frac{du(x)}{dx}.$$

Hence, from (5.37) *and* (4.23) *we have*

$$\frac{du(x)}{dx} \simeq \int_0^x A^T \mathbf{B}^{[m-1]}(t)\, dt = A^T \int_0^x \mathbf{B}^{[m-1]}(t)\, dt \simeq A^T \mathcal{I}^{[m-1]}\, \mathbf{B}^{[m-1]}(x). \tag{5.38}$$

Again, from (5.36) *we have*

$$\int_0^x \frac{du(t)}{dt}\, dt = u(x) - 1.$$

Hence, from (5.38) *and* (4.23) *we get*

$$\begin{aligned} u(x) = \int_0^x \frac{du(t)}{dt}\, dt + 1 \quad &\simeq \quad A^T \mathcal{I}^{[m-1]} \int_0^x \mathbf{B}^{[m-1]}(t)\, dt + V^T \mathbf{B}^{[m-1]}(x) \\ &\simeq \quad A^T \left(\mathcal{I}^{[m-1]}\right)^2 \mathbf{B}^{[m-1]}(x) + V^T \mathbf{B}^{[m-1]}(x), \end{aligned} \tag{5.39}$$

where $V = \begin{pmatrix} \frac{1}{m!} & 0 & \cdots & 0 \end{pmatrix}^T$.

Step 3. Transform the differential equation into a suitable matrix equation.

The substitution of (5.38) *and* (5.39) *into* (5.35) *yields*

$$xA^T \mathbf{B}^{[m-1]}(x) + A^T \mathcal{I}^{[m-1]}\, \mathbf{B}^{[m-1]}(x) + x \left(A^T \left(\mathcal{I}^{[m-1]}\right)^2 \mathbf{B}^{[m-1]}(x) + V^T \mathbf{B}^{[m-1]}(x) \right) = 0,$$

and this last equation can be rewritten as

$$\begin{aligned} 0 \quad = \quad & E^T \mathbf{B}^{[m-1]}(x) A^T \mathbf{B}^{[m-1]}(x) + A^T \mathcal{I}^{[m-1]}\, \mathbf{B}^{[m-1]}(x) \\ & + E^T \mathbf{B}^{[m-1]}(x) A^T \left(\mathcal{I}^{[m-1]}\right)^2 \mathbf{B}^{[m-1]}(x) + E^T \mathbf{B}^{[m-1]}(x) V^T \mathbf{B}^{[m-1]}(x), \end{aligned}$$

where $E = \begin{pmatrix} \frac{1}{(m+1)!} & \frac{1}{m!} & 0 & \cdots & 0 \end{pmatrix}^T$*, or equivalently,*

$$0 = E^T\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T A + A^T\mathcal{I}^{[m-1]}\,\mathbf{B}^{[m-1]}(x)$$
$$+E^T\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T\left[\left(\mathcal{I}^{[m-1]}\right)^2\right]^T A + E^T\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T V. \tag{5.40}$$

Step 4. Use the operational matrix of the product.

From (4.24) *the operational matrices of the product for the vectors* A, $G = \left[\left(\mathcal{I}^{[m-1]}\right)^2\right]^T A$, *and* V *are given by*

$$\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T A \simeq \mathcal{A}^{[m-1]}\mathbf{B}^{[m-1]}(x), \tag{5.41}$$
$$\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T G \simeq \mathcal{G}^{[m-1]}\mathbf{B}^{[m-1]}(x), \tag{5.42}$$
$$\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T V \simeq \mathcal{V}^{[m-1]}\mathbf{B}^{[m-1]}(x). \tag{5.43}$$

Hence, the substitution of (5.41)-(5.43) *into* (5.40) *yields*

$$\left(E^T\mathcal{A}^{[m-1]} + A^T\mathcal{I}^{[m-1]} + E^T\mathcal{G}^{[m-1]} + E^T\mathcal{V}^{[m-1]}\right)\mathbf{B}^{[m-1]}(x) = 0 \tag{5.44}$$

Step 5. Go back to the Galerkin method.

Now, we proceed as in the Galerkin method [8]: multiplying (5.44) *by* $\left(\mathbf{B}^{[m-1]}(x)\right)^T$ *and integrating on the interval* $[0,1]$, *we obtain*

$$\begin{aligned} 0 &= \int_0^1 \left(E^T\mathcal{A}^{[m-1]} + A^T\mathcal{I}^{[m-1]} + E^T\mathcal{G}^{[m-1]} + E^T\mathcal{V}^{[m-1]}\right)\mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T dx \\ &= \left(E^T\mathcal{A}^{[m-1]} + A^T\mathcal{I}^{[m-1]} + E^T\mathcal{G}^{[m-1]} + E^T\mathcal{V}^{[m-1]}\right)\int_0^1 \mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T dx \\ &= \left(E^T\mathcal{A}^{[m-1]} + A^T\mathcal{I}^{[m-1]} + E^T\mathcal{G}^{[m-1]} + E^T\mathcal{V}^{[m-1]}\right)\mathbf{R}^{[m-1]}, \end{aligned} \tag{5.45}$$

where $\mathbf{R}^{[m-1]}$ *is the matrix given in* (3.21).

Step 6. Determine the unknown components of the vector A *and calculate the approximate solution* $u_{\text{as}}(x)$.

Since $\mathbf{R}^{[m-1]}$ *is a nonsingular matrix (at least the numerical experimentation verifies this fact), we deduce from* (5.45) *that*

$$E^T\mathcal{A}^{[m-1]} + A^T\mathcal{I}^{[m-1]} + E^T\mathcal{G}^{[m-1]} + E^T\mathcal{V}^{[m-1]} = 0, \tag{5.46}$$

and this last matrix equation generates a linear system of $N+1$ *equations where, of course, the* $N+1$ *components of vector* A *are the unknowns. Consequently, solving the system generated by* (5.46) *and using* (5.39) *the approximate solution* $u_{\mathrm{as}}(x)$ *can be calculated.*

Following the previous six steps with $m = 1$ *and* $N = 10$*, we get*

$$A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \end{pmatrix} \simeq \begin{pmatrix} -0.4398 \\ 0.1771 \\ 0.1786 \\ 9.7191 \times 10^{-3} \\ 5.5216 \times 10^{-2} \\ 6.892 \times 10^{-2} \\ 5.5349 \times 10^{-2} \\ 4.7603 \times 10^{-2} \\ 3.2582 \times 10^{-2} \\ 1.3713 \times 10^{-2} \\ 8.8693 \times 10^{-3} \end{pmatrix}.$$

Finally, using (5.39) *we obtain the approximate solution of* (5.35)*:*

$$u_{\mathrm{as}}(x) = A^T \left(\mathcal{I}^{[0]}\right)^2 \mathbf{B}^{[0]}(x) + V^T \mathbf{B}^{[0]}(x) = B^T \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{10} \end{pmatrix}, \tag{5.47}$$

where

$$B = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \\ B_{10} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ 1.0387 \times 10^{-4} \\ -0.25 \\ -4.4402 \times 10^{-5} \\ 1.778 \times 10^{-2} \\ 3.6898 \times 10^{-3} \\ -2.1682 \times 10^{-3} \\ 1.3656 \times 10^{-3} \\ 4.9502 \times 10^{-4} \\ -1.6387 \times 10^{-3} \\ 4.5268 \times 10^{-5} \end{pmatrix}.$$

While for $m = 2$ *and* $N = 10$*, we have*

$$\tilde{A} = \begin{pmatrix} \tilde{A}_0 \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \\ \tilde{A}_4 \\ \tilde{A}_5 \\ \tilde{A}_6 \\ \tilde{A}_7 \\ \tilde{A}_8 \\ \tilde{A}_9 \\ \tilde{A}_{10} \end{pmatrix} \simeq \begin{pmatrix} -1.1535 \\ -4.9657 \\ -10.5765 \\ -12.5315 \\ -17.1534 \\ 16.8326 \\ -65.1876 \\ 128.704 \\ -141.573 \\ 81.7372 \\ -44.8865 \end{pmatrix},$$

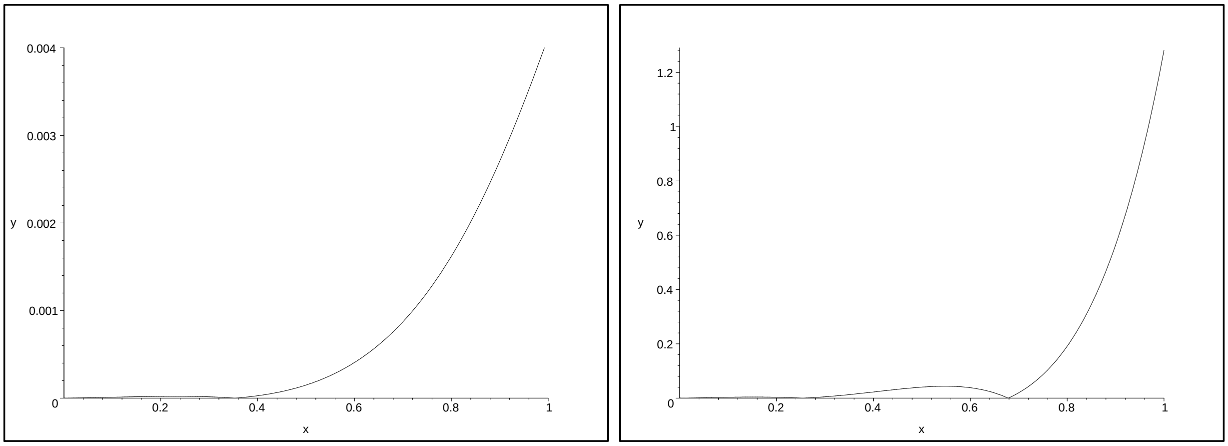
and the approximate solution of (5.35) *is given by*

$$u_{\mathrm{as}}(x) = \tilde{A}^T \left(\mathcal{I}^{[1]}\right)^2 \mathbf{B}^{[1]}(x) + V^T \mathbf{B}^{[1]}(x) = \tilde{B}^T \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{10} \end{pmatrix}, \tag{5.48}$$

where

$$\tilde{B} = \begin{pmatrix} \tilde{B}_0 \\ \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \\ \tilde{B}_4 \\ \tilde{B}_5 \\ \tilde{B}_6 \\ \tilde{B}_7 \\ \tilde{B}_8 \\ \tilde{B}_9 \\ \tilde{B}_{10} \end{pmatrix} \simeq \begin{pmatrix} 0.9996 \\ 4.1595 \times 10^{-2} \\ -0.1919 \\ -0.7265 \\ -0.9144 \\ 1.1818 \\ 1.3515 \\ 7.6945 \\ -17.2239 \\ 13.1319 \\ -3.2978 \end{pmatrix}.$$

Figure 1 shows the plots for the absolute error between the exact solution of the problem (5.35) *and the approximate solutions* (5.47) *and* (5.48), *respectively.*



(A) $m = 1$, $N = 10$ (B) $m = 2$, $N = 10$

Figure 1. Graphs of the absolute error between the exact solution of the problem (5.35) and the approximate solutions (5.47) (left) and (5.48) (right), respectively.

Example 5.2. *Consider the following Lane-Emden type equation*

$$x\frac{d^2u(x)}{dx^2} + 8\frac{du(x)}{dx} + x^2u(x) = x^6 - x^5 + 44x^3 - 30x^2, \tag{5.49}$$

with initial conditions

$$u(0) = 0, \qquad \left.\frac{du(x)}{dx}\right|_{x=0} = 0. \tag{5.50}$$

The exact solution of this problem is

$$u_{\text{es}}(x) = x^4 - x^3.$$

In order to solve this example, we take $m = 5$, $N = 6$ and follow the six steps as in Example 5.1. Making the suitable modifications, from the steps $1 - 3$ we obtain that (5.49) can be transformed into the following equation

$$xA^T\mathbf{B}^{[4]}(x) + 8A^T\mathcal{I}^{[4]}\,\mathbf{B}^{[4]}(x) + x^2A^T\left(\mathcal{I}^{[4]}\right)^2\mathbf{B}^{[4]}(x) = x^6 - x^5 + 44x^3 - 30x^2, \tag{5.51}$$

where $A = \begin{pmatrix} A_0 & A_1 & \cdots & A_6\end{pmatrix}^T$ is a vector with unknown components.

Also, the polynomials x, x^2, and $x^6 - x^5 + 44x^3 - 30x^2$ can be expressed in terms of $\mathbf{B}^{[4]}(x)$ as follows:

$$\begin{aligned} x &= E_1^T\,\mathbf{B}^{[4]}(x), \\ x^2 &= E_2^T\,\mathbf{B}^{[4]}(x), \\ x^6 - x^5 + 44x^3 - 30x^2 &= E_3^T\,\mathbf{B}^{[4]}(x), \end{aligned}$$

where

$$\begin{aligned} E_1 &= \left(\frac{1}{720}, \frac{1}{120}, 0, 0, 0, 0, 0\right)^T, \\ E_2 &= \left(\frac{1}{2520}, \frac{1}{360}, \frac{1}{120}, 0, 0, 0, 0\right)^T, \\ E_3 &= \left(-\frac{1787}{332640}, -\frac{47}{1512}, -\frac{677}{10080}, \frac{1843}{5040}, -\frac{1}{1008}, 0, \frac{1}{120}\right)^T. \end{aligned}$$

The substitution of $E_j^T\,\mathbf{B}^{[4]}(x)$ $(j = 1, 2, 3)$ into (5.51) yields

$$E_1^T\,\mathbf{B}^{[4]}(x)\left(\mathbf{B}^{[4]}(x)\right)^T A + 8A^T\mathcal{I}^{[4]}\,\mathbf{B}^{[4]}(x) + E_2^T\,\mathbf{B}^{[4]}(x)\left(\mathbf{B}^{[4]}(x)\right)^T\left[\left(\mathcal{I}^{[4]}\right)^2\right]^T A - E_3^T\,\mathbf{B}^{[4]}(x) = 0.$$

Next, the steps $4-6$ allow us to determine the unknown components of the vector A and calculate the approximate solution $u_{\text{as}}(x)$. Summarizing, it suffices to solve the linear system associated to the following matrix equation

$$E_1^T\mathcal{A}^{[4]} + 8A^T\mathcal{I}^{[4]} + E_2^T\mathcal{G}^{[4]} - E_3^T = 0, \tag{5.52}$$

where $\mathcal{A}^{[4]}$ and $\mathcal{G}^{[4]}$ are the operational matrices of the product for the vectors A and $G = \left[\left(\mathcal{I}^{[4]}\right)^2\right]^T A$, respectively.

So, computing the solution of the linear system associated to (5.52) with the aid of MAPLE we get

$$A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{pmatrix} \simeq \begin{pmatrix} -3.6439 \times 10^{-3} \\ -1.3536 \times 10^{-2} \\ 0.1134 \\ -6.4221 \times 10^{-2} \\ 0.1438 \\ -0.1705 \\ 0.1569 \end{pmatrix}.$$

And consequently,

$$u_{\text{as}}(x) = A^T\left(\mathcal{I}^{[4]}\right)^2\mathbf{B}^{[4]}(x) = B_0 + B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6,$$

where

$$B = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{pmatrix} \simeq \begin{pmatrix} 3.4049 \times 10^{-4} \\ 5.1303 \times 10^{-5} \\ 3.8604 \times 10^{-2} \\ -1.0199 \\ 1.4472 \\ -0.8341 \\ 0.4925 \end{pmatrix}.$$

If we take $m = 5$ and $N = 4$, then

$$u_{\text{as}}(x) = \tilde{A}^T \left(\mathcal{I}^{[4]}\right)^2 \mathbf{B}^{[4]}(x) = \tilde{B}_0 + \tilde{B}_1 x + \tilde{B}_2 x^2 + \tilde{B}_3 x^3 + \tilde{B}_4 x^4,$$

with

$$\tilde{A} = \begin{pmatrix} \tilde{A}_0 \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \\ \tilde{A}_4 \end{pmatrix} \simeq \begin{pmatrix} -3.9737 \times 10^{-3} \\ -1.0028 \times 10^{-2} \\ 8.2594 \times 10^{-2} \\ 3.9931 \times 10^{-2} \\ -7.8268 \times 10^{-2} \end{pmatrix}, \text{ and } \tilde{B} = \begin{pmatrix} \tilde{B}_0 \\ \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \\ \tilde{B}_4 \end{pmatrix} \simeq \begin{pmatrix} 6.1511 \times 10^{-8} \\ -4.8212 \times 10^{-4} \\ -9.8039 \times 10^{-2} \\ -0.7541 \\ 0.8288 \end{pmatrix}.$$

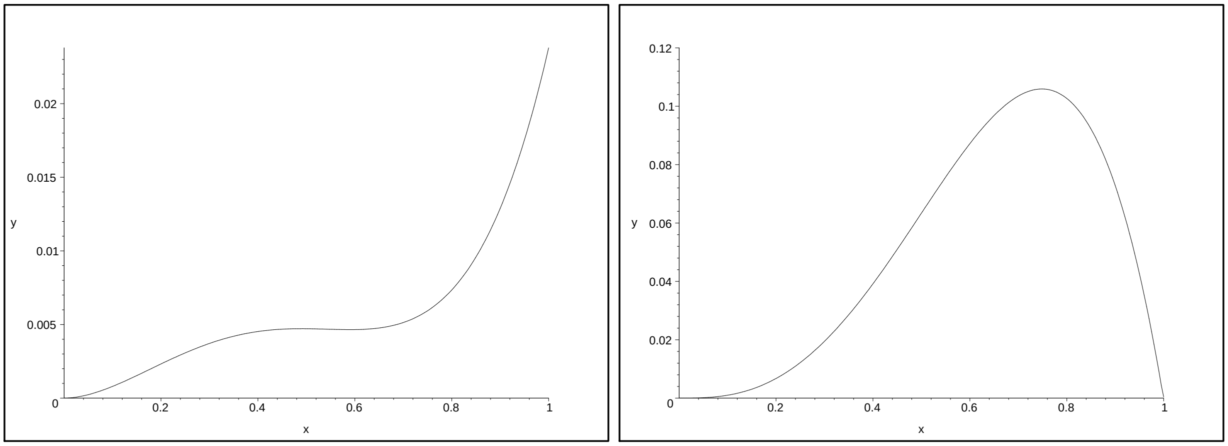
Finally, if we take $m = 1$ and $N = 4$, then

$$u_{\text{as}}(x) = \hat{A}^T \left(\mathcal{I}^{[0]}\right)^2 \mathbf{B}^{[0]}(x) = \hat{B}_0 + \hat{B}_1 x + \hat{B}_2 x^2 + \hat{B}_3 x^3 + \hat{B}_4 x^4,$$

with

$$\hat{A} = \begin{pmatrix} \hat{A}_0 \\ \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \\ \hat{A}_4 \end{pmatrix} \simeq \begin{pmatrix} -0.8557 \\ -2.5344 \\ -2.3844 \\ 4.7599 \times 10^{-3} \\ -7.5712 \end{pmatrix}, \text{ and } \hat{B} = \begin{pmatrix} \hat{B}_0 \\ \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ \hat{B}_4 \end{pmatrix} \simeq \begin{pmatrix} 8.3483 \times 10^{-18} \\ 3.7777 \times 10^{-5} \\ 1.3048 \times 10^{-2} \\ -2.5133 \times 10^{-2} \\ 1.161 \times 10^{-2} \end{pmatrix}.$$

Figure 2 shows the plots for the absolute error between the exact solution of the problem (5.49) and two approximate solutions for different levels.



(A) $m = 5$, $N = 4$ (B) $m = 1$, $N = 4$

FIGURE 2. Graphs of the absolute error between the exact solution of the problem (5.49) and the approximate solutions for different levels with $N = 4$ (left: $m = 5$ and right: $m = 1$, respectively).

EXAMPLE 5.3. *Consider the following nonlinear Riccati equation*

$$\frac{du(x)}{dx} = 2u(x) - u^2(x) + 1, \tag{5.53}$$

with initial condition

$$u(0) = 0. \tag{5.54}$$

The exact solution of this problem is

$$u_{\text{es}}(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right).$$

In order to solve this example, we take $m = 1$ $N = 7$ *and follow the six steps as in Example 5.1 with* $\frac{du(x)}{dx} \simeq A^T\mathbf{B}^{[0]}(x)$, *where* $A = \begin{pmatrix} A_0 & A_1 & \cdots & A_9 \end{pmatrix}^T$ *is a vector with unknown components. So,*

$$u(x) \simeq A^T\mathcal{I}[0]\mathbf{B}^{[0]}(x), \tag{5.55}$$

and we can deduce the following matrix equation:

$$-A^T + 2A^T\mathcal{I}^{[0]} - A^T\mathcal{I}[0]\mathcal{G}^{[0]} + e_1^T = 0, \tag{5.56}$$

where $\mathcal{G}^{[0]}$ *is the operational matrix of the product for the vector* $G = \left(\mathcal{I}^{[0]}\right)^T A$ *and* $e_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$.

The matrix equation (5.56) *generates a nonlinear system of* 8 *equations. With the aid of MAPLE, the solution of the nonlinear system associated to* (5.56) *is shown to be*

$$A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{pmatrix} \simeq \begin{pmatrix} 1.6609 \\ 0.26018 \\ -3.1014 \\ -2.7939 \\ -1.8719 \\ -1.1758 \\ -3.4717 \\ -0.9226 \end{pmatrix}.$$

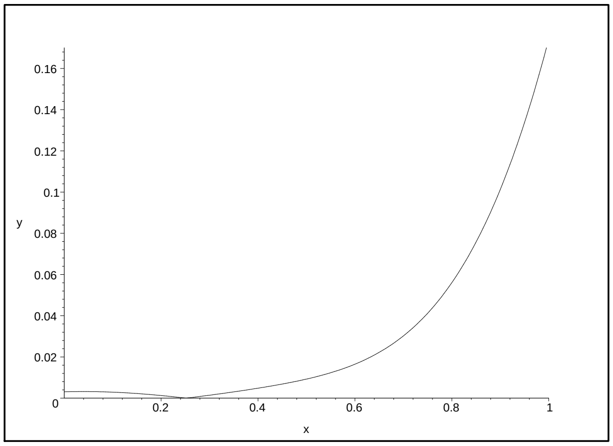
Finally, by (5.55) *the approximate solution* $u_{\text{as}}(x)$ *is given by*

$$u_{\text{as}}(x) = B^T \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \end{pmatrix}, \tag{5.57}$$

where

$$B = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \end{pmatrix} \simeq \begin{pmatrix} -3.0753 \times 10^{-3} \\ 0.9936 \\ 1.0808 \\ 0.3178 \\ -0.2193 \\ -1.5223 \\ 1.7121 \\ -0.4959 \end{pmatrix}.$$

Figure 3 shows the plot for the absolute error between the exact solution of the problem (5.53) *and the approximate solution* (5.57).



(A) $m = 1$, $N = 7$

FIGURE 3. Graph of the absolute error between the exact solution of the problem (5.53) and the approximate solution (5.57).

EXAMPLE 5.4. *Consider the following initial value problem*

$$\begin{aligned} \frac{du(x)}{dx} &= \begin{cases} x(1+2\ln|x|), & \text{if } x \neq 0, x \in [0,1], \\ 0, & \text{if } x = 0, \end{cases} \\ u(0) &= 0. \end{aligned} \tag{5.58}$$

This example obviously fulfills the assumptions of the existence and uniqueness theorem, so there is exactly one solution. It can be easily checked that the exact solution is given by

$$u_{\text{es}}(x) = \begin{cases} x^2 \ln|x|, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{5.59}$$

This function is non analytic in any neighborhood of $x = 0$. In order to solve this example, we approximate $\frac{du(x)}{dx}$ by the generalized Bernoulli polynomials of level m.

$$\frac{du(x)}{dx} \simeq A^T \mathbf{B}^{[m-1]}(x), \tag{5.60}$$

where $A = \begin{pmatrix} A_0 & A_1 & \cdots & A_N \end{pmatrix}^T$ is a vector with unknown components. So,

$$u(x) = \int_0^x \frac{du(t)}{dt} dt \simeq A^T \int_0^x \mathbf{B}^{[m-1]}(t) dt \simeq A^T \mathcal{I}^{[m-1]} \mathbf{B}^{[m-1]}(x). \tag{5.61}$$

Hence, the differential equation (5.60) *can be transformed into the following matrix equation*

$$A^T \mathbf{B}^{[m-1]}(x) = \left(E^T - 2\ln|x|E^T\right)\mathbf{B}^{[m-1]}(x),$$

where $E = \begin{pmatrix} \frac{1}{(m+1)!} & \frac{1}{m!} & 0 & \cdots & 0 \end{pmatrix}^T$. *The previous equation can be rewritten as*

$$\left(A^T - E^T\right)\mathbf{B}^{[m-1]}(x) - 2E^T\left(\mathbf{C}^{[m-1]}(x)\right) = 0, \tag{5.62}$$

where $\mathbf{C}^{[m-1]}(x) = \ln|x|\mathbf{B}^{[m-1]}(x)$. *Now, we proceed as in the Galerkin method [8]: multiplying* (5.62) *by* $\left(\mathbf{B}^{[m-1]}(x)\right)^T$ *and integrating on* $[0,1]$, *we obtain*

$$\begin{aligned} 0 &= \left(A^T - E^T\right)\int_0^1 \mathbf{B}^{[m-1]}(x)\left(\mathbf{B}^{[m-1]}(x)\right)^T dx - 2E^T\int_0^1 \left(\mathbf{C}^{[m-1]}(x)\right)\left(\mathbf{B}^{[m-1]}(x)\right)^T dx \\ &= \left(A^T - E^T\right)\mathbf{R}^{[m-1]} - 2E^T\tilde{\mathbf{R}}^{[m-1]}, \end{aligned} \tag{5.63}$$

where $\mathbf{R}^{[m-1]}$ *is the matrix given in* (3.21) *and* $\tilde{\mathbf{R}}^{[m-1]}$ *is the matrix given by*

$$\tilde{\mathbf{R}}^{[m-1]} = \int_0^1 \left(\mathbf{C}^{[m-1]}(x)\right)\left(\mathbf{B}^{[m-1]}(x)\right)^T dx.$$

Since $\mathbf{R}^{[m-1]}$ *is a nonsingular matrix (at least the numerical simulation satisfies this property), we deduce from* (5.63) *that*

$$A^T = 2E^T\tilde{\mathbf{R}}^{[m-1]}\left(\mathbf{R}^{[m-1]}\right)^{-1} + E^T.$$

For $m = 1$ *and* $N = 10$, *we get the condition number of the matrix* $2\tilde{\mathbf{R}}^{[m-1]}\left(\mathbf{R}^{[m-1]}\right)^{-1} + I$ *is*

$$\kappa_\infty\left(2\tilde{\mathbf{R}}^{[m-1]}\left(\mathbf{R}^{[m-1]}\right)^{-1} + I\right) = 1.254012426 \times 10^{11},$$

where I *is the identity matrix, and the approximate solution of* (5.58) *is given by*

$$u_{\mathrm{as}}(x) = B^T\mathbf{B}^{[m-1]}(x),$$

with

$$B = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \\ B_{10} \end{pmatrix} \simeq \begin{pmatrix} -20.7979163822070 \\ -124.884720872938 \\ -312.475202424591 \\ -414.941535634544 \\ -322.968758389248 \\ -100.298624469552 \\ -130.391215933618 \\ 75.1317393558092 \\ -286.444942862447 \\ 29.7443777765990 \\ -89.4231482966264 \end{pmatrix}.$$

6. Conclusion

A new operational matrix method based on generalized Bernoulli polynomials of level $m \in \mathbb{N}$ has been presented in order to obtain numerical solutions of initial value problems. We show that equations systems associated to the new scheme are very similar to those obtained from standard operational matrix methods. Unfortunately, the numerical evidence suggests that this operational matrix method can lead to very ill-conditioned matrix equations, even when $m = 1$, i.e., when the associated matrices are best handled from a computational point of view.

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