Hindawi
Abstract and Applied Analysis
Volume 2018, Article ID 2937950, 13 pages
https://doi.org/10.1155/2018/2937950

Research Article

An Extended Generalized q-Extensions for the Apostol Type Polynomials

Letelier Castilla,[1] William Ramírez,[2] and Alejandro Urieles[3]

[1]*Universidad del Atlántico, Barranquilla, Colombia*
[2]*GICBAS, Universidad de la Costa, Barranquilla, Colombia*
[3]*Programa de Matemática, Universidad del Atlántico, Km 7, Vía Pto. Colombia, Barranquilla, Colombia*

Correspondence should be addressed to Alejandro Urieles; alejandrourieles@mail.uniatlantico.edu.co

Received 5 February 2018; Revised 3 May 2018; Accepted 22 May 2018; Published 2 July 2018

Academic Editor: Allan Peterson

Through a modification on the parameters associated with generating function of the q-extensions for the Apostol type polynomials of order α and level m, we obtain some new results related to a unified presentation of the q-analog of the generalized Apostol type polynomials of order α and level m. In addition, we introduce some algebraic and differential properties for the q-analog of the generalized Apostol type polynomials of order α and level m and the relation of these with the q-Stirling numbers of the second kind, the generalized q-Bernoulli polynomials of level m, the generalized q-Apostol type Bernoulli polynomials, the generalized q-Apostol type Euler polynomials, the generalized q-Apostol type Genocchi polynomials of order α and level m, and the q-Bernstein polynomials.

1. Introduction

With the development of q-calculus in the mid-19th century, many authors made generalizations to special functions and polynomial families based on the q-analogs (cf. [1–7]). During this process, properties and relations have been demonstrated and contributed to solving different kinds of problems in other subjects (see, [8–10]).

In 2003, Natalini P. and Bernardini A. [11] introduced a class the polynomials $B_n^{[m-1]}(x)$, considering a class of Appell polynomials defined by using a generating function linked to the Mittag-Leffler function (see, [12, p. 204, Eq. (2)])

$$E_{1,m+1}(z) := \frac{z^m}{e^z - \sum_{h=0}^{m-1}(z^h/h!)}, \quad m \in \mathbb{N}. \tag{1}$$

Kurt B. [13] did the generalization of the Bernoulli $B_n^{[m-1,\alpha]}(x)$, Euler $E_n^{[m-1,\alpha]}(x)$, and Genocchi $G_n^{[m-1,\alpha]}(x)$ polynomials of order α and level m. Tremblay R. et al. [14, 15] defined the generalized Apostol-Bernoulli polynomials $B_n^{[m-1,\alpha]}(x;\lambda)$ and their properties. In [12] which has studied the unification of Bernoulli, Euler, and Genocchi polynomials they considered the following Mittag-Leffler type function (see, [12, p. 209, Eq. (12)]):

$$E_{1,m+1}^{(c,a;\lambda;u;v)}(z) := \frac{(2^u z^v)^m}{\lambda c^z + \sum_{l=0}^{m-1}((z\ln a)^l/l!)}, \quad m \in \mathbb{N},\ a,c \in \mathbb{R}^+,\ \lambda,\mu,\nu \in \mathbb{C}, \tag{2}$$

to define a extension of the generalized Apostol type polynomials in x, parameters c, a, μ, ν, order α, and level m through the following generating function:

$$\left(E_{1,m+1}^{(c,a;\lambda;u;v)}(z)\right)^\alpha c^{xz} := \sum_{n=0}^{\infty} \mathcal{Q}_n^{[m-1,\alpha]}(x;c,a;\lambda,\mu;\nu)\frac{z^n}{n!}, \tag{3}$$

where $m \in \mathbb{N}$, $a, c \in \mathbb{R}^+$, and $\alpha, \lambda, \mu, \nu \in \mathbb{C}$. The numbers were given by

$$\mathcal{Q}_n^{[m-1,\alpha]}(c,a;\lambda,\mu;\nu) := \mathcal{Q}_n^{[m-1,\alpha]}(0;c,a;\lambda,\mu;\nu). \tag{4}$$

Recently a new class of polynomials has been introduced in [16] and it provides a unification of three families of polynomials through the following generating function (see, [16, p. 923, Eq. (3)]):

$$\left(\frac{2^{1-k}z^k}{\beta^b e_q^z - a^b}\right)^\alpha e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} \mathscr{P}_{n,\beta,q}^{(\alpha)}(x,y;k,a,b)\frac{z^n}{[n]_q!}, \quad (5)$$

where $k \in \mathbb{N}_0$, $a,b \in \mathbb{R} \setminus \{0\}$, and $\alpha,\beta \in \mathbb{C}$. The author named it the unified q-Apostol-Bernoulli, Euler, and Genocchi polynomials of order α and proved some properties for these unification.

On the other hand, in the definition of the q-Mittag-Leffler function when $\alpha = 1$, $\beta = m+1$, and $\gamma = 1$ lead us to (see, [17, p. 614, Eq (1.5)])

$$E_{1,m+1}(z;q) := \frac{z^m}{e_q^z - \sum_{h=0}^{m-1}\left(z^h/[h]_q!\right)}, \quad m \in \mathbb{N}, \quad (6)$$

which corresponds to the q-analog of the generating function defined in (1) and with this, new research emerged about other polynomial families based on the q-analogs.

In [18], the authors introduced the generalized q-Apostol-Bernoulli polynomials, the generalized q-Apostol Euler polynomials, and generalized q-Apostol Genocchi polynomials in variable x, y, order α, and level m through the following generating functions, defined in a suitable neighborhood of $z=0$ (see, [18, p. 2 Eq (8), (9), (10)])

$$\left(\frac{z^m}{\lambda e_q^z - T_{m-1,q}(z)}\right)^\alpha e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} B_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{z^n}{[n]_q!}, \quad (7)$$

$$\left(\frac{2^m}{\lambda e_q^z + T_{m-1,q}(z)}\right)^\alpha e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{z^n}{[n]_q!}, \quad (8)$$

$$\left(\frac{2^m z^m}{\lambda e_q^z + T_{m-1,q}(z)}\right)^\alpha e_q^{xz} E_q^{yz} = \sum_{n=0}^{\infty} G_{n,q}^{[m-1,\alpha]}(x,y;\lambda)\frac{z^n}{[n]_q!}, \quad (9)$$

where $q,\alpha \in \mathbb{C}$, $m \in \mathbb{N}$, $0 < |q| < 1$, and $T_{m-1,q}(z) = e_q^z - \sum_{h=0}^{m-1}(z^h/[h]_q!)$.

Based on the previous result, we focus our attention on a new unification of the q-analog of the generalized Apostol type polynomials of order α and level m, defined in [18], by doing some modifications to the generating function linked to the q-Mittag-Leffler function (6) through new parameters following the same scheme or procedure applied by [12].

The paper is organized as follows. Section 2 contains some notations, definitions, and properties of the q-analogs and some results about q-analogs of the Apostol type polynomials. In Section 3, we introduce the unification q-analog of the generalized Apostol type polynomials in x, y, parameters $\lambda,\mu,\nu \in \mathbb{C}$, order $\alpha \in \mathbb{C}$, and level $m \in \mathbb{N}$ and their algebraic and differential properties. Finally in the Section 4, we show relations between the q-analog of the generalized Apostol type polynomials and the q-Stirling numbers of the second kind, the generalized q-Bernoulli polynomials of level m, the generalized q-Apostol type Bernoulli polynomials, the generalized q-Apostol type Euler polynomials, the generalized q-Apostol type Genocchi polynomials of order α and level m, and the q-Bernstein polynomials.

2. Background and Previous Results

Throughout this paper, we use the following standard notions: $\mathbb{N} = \{1,2,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. The q-numbers and q-factorial numbers are defined, respectively, by

$$[z]_q = \frac{1-q^z}{1-q} = \frac{q^z-1}{q-1}, \quad z \in \mathbb{C},\ q \in \mathbb{C}\setminus\{1\},\ q^z \neq 1,$$
$$[n]_q! = \prod_{k=1}^{n}[k]_q = [1]_q[2]_q[3]_q\cdots[n]_q, \quad [0]_q! = 1,\ n \in \mathbb{N}. \quad (10)$$

El q-shifted factorial is defined as

$$(a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n \in \mathbb{N}. \quad (11)$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!}, \quad \left(n,k \in \mathbb{N}_0;\ 0 \le k \le n\right). \quad (12)$$

For more information about the q-standard definitions and properties see [8, 9, 19].

Furthermore, the q-binomial coefficient satisfies the following identity (see [10, p. 483, Eq. (41)]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q \quad 0 \le j \le k. \quad (13)$$

The q-analog of the function $(x+y)^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1/2)k(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0,$$
$$(1-a)_q^n = (a;q)_n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1/2)k(k-1)}(-1)^k a^k = \prod_{j=0}^{n-1}\left(1-q^j a\right). \quad (14)$$

The q-derivative of a function $f(z)$ is defined by

$$D_q f(z) = \frac{d_q f(z)}{d_q z} = \frac{f(qz) - f(z)}{(q-1)z}, \quad 0 < |q| < 1, \; 0 \neq z \in \mathbb{C}. \tag{15}$$

The q-analog of the exponential function is defined in two ways

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \; z \in \mathbb{C}, \; |z| < \frac{1}{|1-q|}, \tag{16}$$

$$E_q^z = \sum_{n=0}^{\infty} \frac{q^{(1/2)n(n-1)} z^n}{[n]_q!} = \prod_{n=0}^{\infty} \left(1 + (1-q) q^k z\right), \quad 0 < |q| < 1, \; z \in \mathbb{C};$$

we can see that

$$e_q^z \cdot E_q^{-z} = 1, \tag{17}$$

$$e_q^{z+w} = e_q^z \cdot E_q^w. \tag{18}$$

Therefore,

$$D_q e_q^z = e_q^z, \qquad D_q E_q^z = E_q^{qz}. \tag{19}$$

For any $t > 0$ (see, e.g., [19, p. 76, Eq (21.6)])

$$\Gamma_q(t) = \int_0^{\infty} x^{t-1} E_q^{-qx} d_q x \tag{20}$$

is called the q-gamma function.

The Jackson's q-gamma function is defined by (see [10, p. 490, Eq. (2)])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \quad 0 < |q| < 1. \tag{21}$$

In (21) we have

$$\Gamma_q(n+1) = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty} (1-q)^{-n} = (q;q)_n (1-q)^{-n} = [n]_q!, \quad n \in \mathbb{N}. \tag{22}$$

For $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $\mathrm{Re}(\gamma) > 0$ y $|q| < 1$ the function $E_{\alpha,\beta}^{\gamma}(z;q)$ is defined as (see, e.g., [17, p. 614, Eq. (1.5)])

$$E_{\alpha,\beta}^{\gamma}(z;q) = \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n}{(q;q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)}. \tag{23}$$

Notice that, when $\gamma = 1$ the previous equation is expressed as

$$E_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n + \beta)}. \tag{24}$$

Setting $\alpha = 1$ y $\beta = m + 1$, we can deduce

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+m+1)} = \sum_{n=0}^{\infty} \frac{z^n}{[n+m]_q!} = \frac{1}{z^m} \sum_{h=m}^{\infty} \frac{z^h}{[h]_q!} = \frac{\left(e_q^z - \sum_{h=0}^{m-1} \left(z^h / [h]_q!\right)\right)}{z^m}. \tag{25}$$

The q-Stirling numbers of the second kind $S(n,k)_q$ are defined through the following expansion (see, e.g., [20, p. 173, Equ (5.18)]):

$$x^n = \sum_{k=0}^{n} S(n,k)_q (x)_{k,q}, \tag{26}$$

where

$$(x)_{k,q} = \prod_{n=0}^{k-1} \left(x - [n]_q\right). \tag{27}$$

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For any $f \in C[0,1]$, the q- $\mathbb{B}_n(f;x)$ is called q-Bernstein operator of order n for f and is defined as (see, e.g., [21, p. 2, Eq (1.1)])

$$\mathbb{B}_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x) = \sum_{r=0}^{n} f_r B_r^n(x), \tag{28}$$

where $f_r = f([r]_q/[n]_q)$. For $n, r \in \mathbb{N}_0$, the q-Bernstein polynomials of degree n or q-Bernstein basis are defined by

$$B_r^n(x) = \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x). \tag{29}$$

We know that $\sum_{k=0}^{n-j} B_k^{n-j}(x) = 1$, then

$$x^j = \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix}_q x^{j+k} \prod_{t=0}^{n-j-k-1} \left(1 - q^t x\right), \tag{30}$$

and using the identity (13), we have (see [21, p. 6, Eq (2.3)])

$$x^j = \sum_{k=j}^{n} \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q} B_k^n(x). \tag{31}$$

Mahmudov N.I. [22] made a relation between the q-Bernstein basis with the q-Stirling numbers of the second kind and the q-Bernoulli polynomials of order α, $\alpha = k$ as follows:

$$B_k^n(x;q) = x^k \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q S(j,k;q) \mathfrak{B}_{n-j}^{(k)}(1,-x;q). \tag{32}$$

Proposition 1. *For a fixed $m \in \mathbb{N}$, $n, k \in \mathbb{N}_0$, and $0 < |q| < 1$, let $\{B_n^{[m-1,\alpha]}(x, y; q)\}_{n=0}^{\infty}$ be the sequence of generalized q-Bernoulli polynomials in x, y of level m. Then the following identities are satisfied:*

(1) [23, Lemma 10, Eq. (1)]

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-k}^{[m-1,1]}(x, 0; q). \tag{33}$$

(2) [23, Lemma 10, Eq. (2)]

$$y^n = \frac{1}{q^{(1/2)n(n-1)}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} B_{n-k}^{[m-1,1]}(0, y; q). \tag{34}$$

Now, setting $k = n - k$, we have

$$y^n = \frac{[n]_q!}{q^{(1/2)n(n-1)} [n+m]_q!} \cdot \sum_{k=0}^{n} \begin{bmatrix} n+m \\ k \end{bmatrix}_q B_k^{[m-1,1]}(0, y; q). \tag{35}$$

(3)

$$x^n = \sum_{k=0}^{n} \frac{[n]_q! B_k^{[m-1,1]}(x, 0; q)}{[k]_q! \Gamma_q(n-k+m+1)}. \tag{36}$$

Proof (see (36)). Setting $\alpha = \lambda = 1$, $y = 0$ in (7) and using (25), we have

$$\begin{aligned} \frac{e_q^{xz}}{\sum_{n=0}^{\infty} \left(z^n / \Gamma_q(n+m+1)\right)} &= \sum_{n=0}^{\infty} B_n^{[m-1,1]}(x, 0; q) \frac{z^n}{[n]_q!} \\ e_q^{xz} &= \sum_{n=0}^{\infty} B_n^{[m-1,1]}(x, 0; q) \frac{z^n}{[n]_q!} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+m+1)} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{[n]_q! B_k^{[m-1,1]}(x, 0; q)}{[k]_q! \Gamma_q(n-k+m+1)} \cdot \frac{z^n}{[n]_q!} \\ \sum_{n=0}^{\infty} \frac{x^n z^n}{[n]_q!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{[n]_q! B_k^{[m-1,1]}(x, 0; q)}{[k]_q! \Gamma_q(n-k+m+1)} \cdot \frac{z^n}{[n]_q!}. \end{aligned} \tag{37}$$

Comparing the coefficients of $z^n/[n]_q!$ we obtain

$$x^n = \sum_{k=0}^{n} \frac{[n]_q! B_k^{[m-1,1]}(x, 0; q)}{[k]_q! \Gamma_q(n-k+m+1)}. \tag{38}$$

□

Notice that in (36), as $\Gamma_q(n-k+m+1) = [n-k+m]_q!$ we can get (33).

Based on the results of (2), (6), and (7)–(9) and following the methodology given in [12], we consider the following q-Mittag-Leffler type function:

$$E_{1,m+1}^{(\lambda;\mu;\nu)}(z; q) := \frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{h=0}^{m-1} \left(z^h / [h]_q!\right)}, \tag{39}$$

where $m \in \mathbb{N}$, $\lambda, \mu, \nu \in \mathbb{C}$.

3. The Polynomials $\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)$ and Their Properties

Definition 2. Let $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\alpha, \lambda, \mu, \nu, q \in \mathbb{C}$, and $0 < |q| < 1$; the q-analog of the generalized Apostol type polynomials in x, y, with parameters λ, μ, ν, order α, and level m is defined by means of the following generating function, in a suitable neighborhood of $z = 0$,

$$\begin{aligned} &\left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)}\right)^\alpha e_q^{xz} E_q^{yz} \\ &\quad = \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}, \end{aligned} \tag{40}$$

where $|z| < |\log_q(-\lambda)|$ when $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ and $1^\alpha := 1$. The numbers are given by

$$\begin{aligned} &\left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)}\right)^\alpha \\ &\quad = \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]}(\lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}. \end{aligned} \tag{41}$$

Notice that

$$\mathcal{T}_n^{[m-1,\alpha]}(0, 0; \lambda; \mu; \nu; q) := \mathcal{T}_n^{[m-1,\alpha]}(\lambda; \mu; \nu; q). \tag{42}$$

Therefore,

$$\begin{aligned} \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) &:= \mathcal{T}_n^{[m-1,\alpha]}(x, \lambda; \mu; \nu; q), \\ \mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) &:= \mathcal{T}_n^{[m-1,\alpha]}(y, \lambda; \mu; \nu; q). \end{aligned} \tag{43}$$

We will use this notation $\mathcal{T}_n^{[m-1,\alpha]}(\lambda; \mu; \nu; q)$ instead of $\mathcal{T}_{n,q}^{[m-1,\alpha]}(\lambda; \mu; \nu)$ through the article. By comparing Definition 2 with (7)–(9), we have

$$\begin{aligned} (-1)^\alpha \mathcal{T}_n^{[m-1,\alpha]}(x, y; -\lambda; 0; 1; q) &= B_{n,q}^{[m-1,\alpha]}(x, y; \lambda), \\ \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; 1; y; q) &= E_{n,q}^{[m-1,\alpha]}(x, y; \lambda), \\ \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; 1; 1; q) &= G_{n,q}^{[m-1,\alpha]}(x, y; \lambda). \end{aligned} \tag{44}$$

Therefore,

$$\begin{aligned} &\lim_{q \to 1^-} \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) \\ &\quad = \mathcal{T}_n^{[m-1,\alpha]}(x+y; \lambda, \mu; \nu), \\ &\lim_{q \to 1^-} \mathcal{T}_n^{[m-1,\alpha]}(0, 0; \lambda; \mu; \nu; q) = \mathcal{T}_n^{[m-1,\alpha]}(\lambda, \mu; \nu). \end{aligned} \tag{45}$$

Clearly for $m = 1$, we have

$$\mathcal{T}_n^{[0,\alpha]}(x, y; \lambda; \mu; \nu; q) = \mathcal{T}_n^{(\alpha)}(x, y; \lambda; \mu; \nu; q). \tag{46}$$

For $\alpha = 1$, we have

$$\mathcal{T}_n^{[m-1,1]}(x, y; \lambda; \mu; \nu; q) = \mathcal{T}_n^{[m-1]}(x, y; \lambda; \mu; \nu; q). \tag{47}$$

For $m = \alpha = 1$, we have

$$\mathcal{T}_n^{[0,1]}(x, y; \lambda; \mu; \nu; q) = \mathcal{T}_n(x, y; \lambda; \mu; \nu; q). \tag{48}$$

Example 3. When $\alpha = 1, \mu = 0$, and $\nu = 1$ we can take $\lambda = -1$.

$$
\begin{aligned}
&- \mathcal{T}_0^{[m-1,1]}(x, 0; -1; 0; 1; q) = B_0^{[m-1]}(x; q) = [m]_q!,\\
&- \mathcal{T}_1^{[m-1,1]}(x, 0; -1; 0; 1; q) = B_1^{[m-1]}(x; q)\\
&\quad = [m]_q!\left(x - \frac{1}{[m+1]_q}\right),\\
&- \mathcal{T}_2^{[m-1,1]}(x, 0; -1; 0; 1; q) = B_2^{[m-1]}(x; q)\\
&\quad = [m]_q!\left(x^2 - \frac{[2]_q x}{[m+1]_q} + \frac{[2]_q q^{m+1}}{[m+2]_q [m+1]_q^2}\right).
\end{aligned} \tag{49}
$$

And the numbers are as follows:

$$
\begin{aligned}
&- \mathcal{T}_0^{[m-1,1]}(0, 0; -1; 0; 1; q) = B_0^{[m-1]}(q) = [m]_q!,\\
&- \mathcal{T}_1^{[m-1,1]}(0, 0; -1; 0; 1; q) = B_1^{[m-1]}(q) = -\frac{[m]_q!}{[m+1]_q},\\
&- \mathcal{T}_2^{[m-1,1]}(0, 0; -1; 0; 1; q) = B_2^{[m-1]}(q)\\
&\quad = \frac{[2]_q [m]_q! q^{m+1}}{[m+2]_q [m+1]_q^2}.
\end{aligned} \tag{50}
$$

Example 4. For any $\lambda \in \mathbb{C} \setminus \{-1, 1\}$, $m = 1$, $\alpha = 1$, $\mu = 1$, and $\nu = 2$

$$
\begin{aligned}
&\mathcal{T}_j^{[0,1]}(x, 0; \lambda; 1; 2; q) = 0, \quad j = 0, 1,\\
&\mathcal{T}_2^{[0,1]}(x, 0; \lambda; 1; 2; q) = \frac{2[2]_q}{\lambda + 1},\\
&\mathcal{T}_3^{[0,1]}(x, 0; \lambda; 1; 2; q) = \frac{2[3]_q! x}{\lambda + 1} - \frac{2[3]_q! \lambda}{\lambda^2 + 2\lambda + 1}.
\end{aligned} \tag{51}
$$

Example 5. For any $\lambda \in \mathbb{C} \setminus \{-1, 1\}$, $m = 2$, $\alpha = 1$, $\mu = 3$, and $\nu = 1$

$$
\begin{aligned}
&\mathcal{T}_j^{[1,1]}(x, 0; \lambda; 3; 1; q) = 0, \quad j = 0, 1,\\
&\mathcal{T}_2^{[1,1]}(x, 0; \lambda; 3; 1; q) = \frac{64[2]_q}{\lambda + 1},\\
&\mathcal{T}_3^{[1,1]}(x, 0; \lambda; 3; 1; q) = \frac{64[3]_q! x}{\lambda + 1} - \frac{64[3]_q!}{\lambda + 1}.
\end{aligned} \tag{52}
$$

The following proposition summarizes some properties of the polynomials $\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)$ which are a consequence of (40). Therefore, we will only show the details of its proofs (7) and (8).

Proposition 6. *For a fixed $m \in \mathbb{N}$, $0 < |q| < 1$, let $\{\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)\}_{n=0}^{\infty}$ be the sequence of the q-analog of the generalized Apostol type polynomials in x, y, parameters $\alpha, \lambda, \mu, \nu \in \mathbb{C}$, order α, and level m. Then the following statements hold:*

(1) Special values: for every $n \in \mathbb{N}_0$

$$\mathcal{T}_n^{[m-1,0]}(x, 0; \lambda; \mu; \nu; q) = x^n, \tag{53}$$

$$\mathcal{T}_n^{[m-1,0]}(0, y; \lambda; \mu; \nu; q) = q^{(1/2)n(n-1)} y^n. \tag{54}$$

(2) Summation formulas: for every $n \in \mathbb{N}_0$

$$
\begin{aligned}
\mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q\\
&\cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) x^k,
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
\mathcal{T}_n^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q\\
&\cdot q^{(1/2)k(k-1)} \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) y^k,
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
\mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q\\
&\cdot \mathcal{T}_{n-k}^{[m-1,\alpha-1]}(\lambda; \mu; \nu; q) \mathcal{T}_k^{[m-1,1]}(x, 0; \lambda; \mu; \nu; q),
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
\mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q\\
&\cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda; \mu; \nu; q) \mathcal{T}_k^{[m-1,0]}(x, 0; \lambda; \mu; \nu; q).
\end{aligned} \tag{58}
$$

(3) Differential relations: for $m \in \mathbb{N}$, fixed α, λ and $n, j \in \mathbb{N}_0$ with $0 \le j \le n$, we have

$$
\begin{aligned}
&D_q \mathcal{T}_{n+1}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q)\\
&\quad = [n+1]_q \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q),
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
&D_q^{(j)} \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q)\\
&\quad = \frac{[n]_q!}{[n-j]_q!} \mathcal{T}_{n-j}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q).
\end{aligned} \tag{60}
$$

(4) Integral formulas: for $m \in \mathbb{N}$, fixed α, λ, we have

$$
\begin{aligned}
&\int_{x_0}^{x_1} \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q)\, d_q x\\
&= \frac{\mathcal{T}_{n+1}^{[m-1,\alpha]}(x_1, 0; \lambda; \mu; \nu; q) - \mathcal{T}_{n+1}^{[m-1,\alpha]}(x_0, 0; \lambda; \mu; \nu; q)}{[n+1]_q},
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
&\int_{x_0}^{x_1} \mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q)\, d_q x\\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_k^{[m-1,\alpha]}(\lambda; \mu; \nu; q) \left(\frac{x_1^{n+1-k} - x_0^{n+1-k}}{[n+1-k]_q}\right),
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
&\mathcal{T}_n^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q)\\
&= [n]_q \int_0^x \mathcal{T}_{n-1}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q)\, d_q x\\
&\quad + \mathcal{T}_n^{[m-1,\alpha]}(\lambda; \mu; \nu; q).
\end{aligned} \tag{63}
$$

(5) Addition theorems:

$$\mathcal{T}_n^{[m-1,\alpha]}(x,y;\lambda;\mu;\nu;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \cdot q^{(1/2)k(k-1)}\mathcal{T}_{n-k}^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q)\,y^k, \tag{64}$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x,y;\lambda;\mu;\nu;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \cdot q^{(1/2)k(k-1)}\mathcal{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\,x^k, \tag{65}$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x,y;\lambda;\mu;\nu;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda;\mu;\nu;q)\,(x+y)_q^k, \tag{66}$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x,y;\lambda;\mu;\nu;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\,x^k, \tag{67}$$

$$\mathcal{T}_n^{[m-1,\alpha+\beta]}(x,y;\lambda;\mu;\nu;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \cdot \mathcal{T}_{n-k}^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q)\,\mathcal{T}_{k}^{[m-1,\beta]}(0,y;\lambda;\mu;\nu;q). \tag{68}$$

Clearly, setting $x=0$ in (64), we have

$$\mathcal{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(1/2)k(k-1)}\mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda;\mu;\nu;q)\,y^k. \tag{69}$$

Setting $y=0$ in (67), we obtain (55)

$$\mathcal{T}_n^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(\lambda;\mu;\nu;q)\,x^k. \tag{70}$$

Setting $y=1$ and $x=1$ in (64) and (67), respectively, we have

$$\begin{aligned}\mathcal{T}_n^{[m-1,\alpha]}(x,1;\lambda;\mu;\nu;q)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(1/2)k(k-1)}\mathcal{T}_{n-k}^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q),\\ \mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q).\end{aligned} \tag{71}$$

(6) If $a\in N$, we have

$$\mathcal{T}_n^{[m-1,\alpha]}(x,ax-x;\lambda;\mu;\nu;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \cdot q^{(1/2)k(k-1)}\mathcal{T}_{n-k}^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q)\,(a-1)^k x^k. \tag{72}$$

(7) The q-analog of the generalized Apostol type polynomials satisfies the following relations:

$$\lambda\mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)+\mathcal{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)=2\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathcal{T}_{k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\times\mathcal{T}_{n-k}^{[0,-1]}(\lambda;1;0;q), \tag{73}$$

$$\lambda\mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)=2^{\mu m}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \mathcal{T}_{k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\cdot\mathcal{T}_{n-k}^{[m-1,-1]}(\lambda;\mu;0;q), \tag{74}$$

$$\lambda\mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)=2^{\mu m}\cdot\frac{[n]_q!}{[n-\nu m]_q!}T_{n-\nu m}^{[m-1,\alpha-1]}(0,y;\lambda;\mu;\nu;q),\quad n\ge\nu m, \tag{75}$$

$$\lambda\mathcal{T}_n^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q)+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(x,-1;\lambda;\mu;\nu;q)=2^{\mu m}\cdot\frac{[n]_q!}{[n-\nu m]_q!}T_{n-\nu m}^{[m-1,\alpha-1]}(x,-1;\lambda;\mu;\nu;q),\quad n\ge\nu m. \tag{76}$$

Proof (see (73)). Considering the expression $\lambda\mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)+\mathcal{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)$ and using (40), we have

$$\begin{aligned}\mathbf{I}&:=\sum_{n=0}^{\infty}\left(\lambda\mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)+\mathcal{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\right)\frac{z^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\lambda\mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!}+\sum_{n=0}^{\infty}\mathcal{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!}\end{aligned}$$

$$= \lambda \left(\frac{(2^{\mu} z^{\nu})^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^{\alpha} e_q^z E_q^{yz} + \left(\frac{(2^{\mu} z^{\nu})^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^{\alpha} E_q^{yz}. \quad (77)$$

Now, factoring the previous expression, we get

$$\begin{aligned} \mathbf{I} &= 2 \left(\frac{(2^{\mu} z^{\nu})^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^{\alpha} E_q^{yz} \left(\frac{\lambda e_q^z + 1}{2} \right) \\ &= 2 \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ &\cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[0,-1]} (\lambda; 1; 0; q) \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} 2 \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \\ &\cdot \mathcal{T}_k^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \mathcal{T}_{n-k}^{[0,-1]} (\lambda; 1; 0; q) \frac{z^n}{[n]_q!}. \end{aligned} \quad (78)$$

Comparing the coefficients of $z^n/[n]_q!$ in both sides gives the result.

This completes the proof.

Proof (see (74)). By using the relation (see [23, p.5, Lemma 6])

$$\begin{aligned} &\sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ &= \sum_{n}^{\infty} \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \end{aligned} \quad (79)$$

and (40), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1,\alpha]} (1, y; \lambda; \mu; \nu; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \right) \frac{z^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1,\alpha]} (1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\ &\cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ &= \lambda \left(\frac{(2^{\mu} z^{\nu})^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^{\alpha} e_q^z E_q^{yz} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\ &\cdot \left(\frac{(2^{\mu} z^{\nu})^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^{\alpha} E_q^{yz} \end{aligned} \quad (80)$$

Now, factoring the above expression, we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1,\alpha]} (1, y; \lambda; \mu; \nu; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \right) \frac{z^n}{[n]_q!} \\ &= 2^{\mu m} \left(\frac{(2^{\mu} z^{\nu})^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)} \right)^{\alpha} \\ &\cdot E_q^{yz} \left(\frac{\lambda e_q^z + \sum_{l=0}^{m-1} (z^l / [l]_q!)}{2^{\mu m}} \right) \\ &= 2^{\mu m} \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ &\cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,-1]} (\lambda; \mu; 0; q) \frac{z^n}{[n]_q!} = 2^{\mu m} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \\ &\cdot \mathcal{T}_k^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \mathcal{T}_{n-k}^{[m-1,-1]} (\lambda; \mu; 0; q) \\ &\cdot \frac{z^n}{[n]_q!}. \end{aligned} \quad (81)$$

Comparing the coefficients of $z^n/[n]_q!$ in both sides gives the result.

This completes the proof.

Proof (see (75)). Let

$$\mathbf{I} := \sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1,\alpha]} (1, y; \lambda; \mu; \nu; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \right) \frac{z^n}{[n]_q!}. \quad (82)$$

Using (40) and (79), we have

$$\begin{aligned} \mathbf{I} &= \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1,\alpha]} (1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ &+ \sum_{n=0}^{\infty} \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1,\alpha]} (1, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \\ &\cdot \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]} (0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} \end{aligned}$$

$$= \lambda \left(\frac{\left(2^{\mu} z^{\nu}\right)^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)} \right)^{\alpha} e_q^z E_q^{yz} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \cdot \left(\frac{\left(2^{\mu} z^{\nu}\right)^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)} \right)^{\alpha} E_q^{yz} = \left(2^{\mu} z^{\nu}\right)^{m} \cdot \left(\frac{\left(2^{\mu} z^{\nu}\right)^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)} \right)^{\alpha} \cdot E_q^{yz} \left(\frac{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)}{\left(2^{\mu} z^{\nu}\right)^{m}} \right) = 2^{\mu m} z^{\nu m} \left(\frac{\left(2^{\mu} z^{\nu}\right)^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)} \right)^{\alpha-1} E_q^{yz}. \tag{83}$$

Therefore, we get

$$\mathbf{I} = 2^{\mu m} \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha-1]} (0, y; \lambda; \mu; \nu; q) \frac{z^{n+\nu m}}{[n]_q!} = 2^{\mu m} \sum_{n=0}^{\infty} \frac{[n+\nu m]_q!}{[n]_q!} \mathcal{T}_n^{[m-1,\alpha-1]} (0, y; \lambda; \mu; \nu; q) \cdot \frac{z^{n+\nu m}}{[n+\nu m]_q!} = 2^{\mu m} \sum_{n-\nu m=0}^{\infty} \frac{[n]_q!}{[n-\nu m]_q!} \cdot \mathcal{T}_{n-\nu m}^{[m-1,\alpha-1]} (0, y; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}. \tag{84}$$

Notice that

$$\mathcal{T}_0^{[m-1,\alpha-1]} (0, y; \lambda; \mu; \nu; q) = 0, \ldots, \mathcal{T}_{n-1-\nu m}^{[m-1,\alpha-1]} (0, y; \lambda; \mu; \nu; q) = 0. \tag{85}$$

Comparing the coefficients of $z^n/[n]_q!$ in both sides gives the result.

This completes the proof.

Proof (see (76)). Let

$$\mathbf{I} := \sum_{n=0}^{\infty} \left(\lambda \mathcal{T}_n^{[m-1,\alpha]} (x, 0; \lambda; \mu; \nu; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]} (x, -1; \lambda; \mu; \nu; q) \right) \cdot \frac{z^n}{[n]_q!}. \tag{86}$$

Using (40) and (79), we have

$$\mathbf{I} = \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1,\alpha]} (x, 0; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} + \sum_{n=0}^{\infty} \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]} (x, -1; \lambda; \mu; \nu; q) \cdot \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} \lambda \mathcal{T}_n^{[m-1,\alpha]} (x, 0; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha]} (x, -1; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!} = \lambda \left(\frac{\left(2^{\mu} z^{\nu}\right)^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)} \right)^{\alpha} e_q^{xz} + \sum_{l=0}^{m-1} \frac{z^l}{[l]_q!} \cdot \left(\frac{\left(2^{\mu} z^{\nu}\right)^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)} \right)^{\alpha} e_q^{xz} E_q^{-z}. \tag{87}$$

then, factoring the above equation and using (17), we have

$$\mathbf{I} = 2^{\mu m} z^{\nu m} \left(\frac{\left(2^{\mu} z^{\nu}\right)^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)} \right)^{\alpha} \cdot e_q^{xz} \left(\frac{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)}{\left(2^{\mu} z^{\nu}\right)^{m}} \right) \cdot \frac{1}{e_q^z} = 2^{\mu m} z^{\nu m} \left(\frac{\left(2^{\mu} z^{\nu}\right)^{m}}{\lambda e_q^z + \sum_{l=0}^{m-1} \left(z^l / [l]_q!\right)} \right)^{\alpha-1} e_q^{xz} E_q^{-z}. \tag{88}$$

Therefore, we get

$$\mathbf{I} = 2^{\mu m} \sum_{n=0}^{\infty} \mathcal{T}_n^{[m-1,\alpha-1]} (x, -1; \lambda; \mu; \nu; q) \frac{z^{n+\nu m}}{[n]_q!} = 2^{\mu m} \sum_{n=0}^{\infty} \frac{[n+\nu m]_q!}{[n]_q!} \mathcal{T}_n^{[m-1,\alpha-1]} (x, -1; \lambda; \mu; \nu; q) \cdot \frac{z^{n+\nu m}}{[n+\nu m]_q!} = 2^{\mu m} \sum_{n-\nu m=0}^{\infty} \frac{[n]_q!}{[n-\nu m]_q!} \cdot \mathcal{T}_{n-\nu m}^{[m-1,\alpha-1]} (x, -1; \lambda; \mu; \nu; q) \frac{z^n}{[n]_q!}. \tag{89}$$

Notice that

$$\mathcal{T}_0^{[m-1,\alpha-1]} (x, -1; \lambda; \mu; \nu; q) = 0, \ldots, \mathcal{T}_{n-1-\nu m}^{[m-1,\alpha-1]} (x, -1; \lambda; \mu; \nu; q) = 0. \tag{90}$$

Comparing the coefficients of $z^n/[n]_q!$ in both sides gives the desired result.

This completes the proof.

Remarks 7. Setting $\mu = 1$, $\nu = 0$, $\lambda = 1$ in (75), we obtain

$$\mathcal{T}_n^{[m-1,\alpha]}(1, y; 1; 1; 0; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; 1; 1; 0; q) = 2^m \mathcal{T}_n^{[m-1,\alpha-1]}(0, y; 1; 1; 0; q), \tag{91}$$

Note that (91) is equivalent to [23, Lemma 6, Eq.2].

Substituting $\nu = 0$ in (75), we obtain

$$\lambda \mathcal{T}_n^{[m-1,\alpha]}(1, y; \lambda; \mu; 0; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0, y; \lambda; \mu; 0; q) = 2^{\mu m} \mathcal{T}_n^{[m-1,\alpha-1]}(0, y; \lambda; \mu; 0; q). \tag{92}$$

(8) For $n \in \mathbb{N}_0$, $n \geq \nu m$, we have

$$y^{n-\nu m} = \frac{[n-\nu m]_q!}{2^{\mu m}[n]_q! q^{(1/2)(n-\nu m)(n-\nu m-1)}} \times \left(\lambda \mathcal{T}_n^{[m-1,1]}(1, y; \lambda; \mu; \nu; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,1]}(0, y; \lambda; \mu; \nu; q) \right). \tag{93}$$

Proof (see (93)). Putting $\alpha = 1$ in (75) and using (54), we obtain

$$\begin{aligned} &\lambda \mathcal{T}_n^{[m-1,1]}(1, y; \lambda; \mu; \nu; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,1]}(0, y; \lambda; \mu; \nu; q) \\ &= 2^{\mu m} \frac{[n]_q!}{[n-\nu m]_q!} T_{n-\nu m}^{[m-1,0]}(0, y; \lambda; \mu; \nu; q) \\ &= 2^{\mu m} \frac{[n]_q!}{[n-\nu m]_q!} q^{(1/2)(n-\nu m)(n-\nu m-1)} y^{n-\nu m}, \end{aligned} \tag{94}$$

then

$$y^{n-\nu m} = \frac{[n-\nu m]_q!}{2^{\mu m}[n]_q! q^{(1/2)(n-\nu m)(n-\nu m-1)}} \times \left(\lambda \mathcal{T}_n^{[m-1,1]}(1, y; \lambda; \mu; \nu; q) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,1]}(0, y; \lambda; \mu; \nu; q) \right). \tag{95}$$

This completes the proof.

4. Some Connection Formulas for the Polynomials $\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)$

In this section, we introduce some formulas of connection between the q-analog of the generalized Apostol type polynomials $\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q)$ and the generalized q-Bernoulli polynomials of level m, the q-Stirling numbers of the second kind, the generalized q-Apostol type Bernoulli, q-Apostol type Euler, q-Apostol type Genocchi polynomials of order α and level m, and the q-Bernstein polynomials.

Proposition 8. *For $n, j, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$, the q-analog of the generalized Apostol type polynomials of level m is related to the generalized q-Bernoulli polynomials of level m and the q-gamma function*

$$\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{j=0}^{n} \sum_{k=0}^{j} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} \cdot q^{(1/2)j(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \cdot B_{j-k}^{[m-1,1]}(x, 0; q), \tag{96}$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{j=0}^{n} \sum_{k=0}^{j} \begin{bmatrix} n \\ j \end{bmatrix}_q \cdot \frac{[j]_q! q^{(1/2)j(j-1)}}{[k]_q!} \cdot \frac{\mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) B_k^{[m-1,1]}(x, 0; q)}{\Gamma_q(j-k+m+1)}, \tag{97}$$

$$\mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) = \sum_{j=0}^{n} \sum_{k=0}^{j} \begin{bmatrix} n \\ j \end{bmatrix}_q \cdot \frac{[j]_q! \mathcal{T}_{n-j}^{[m-1,\alpha]}(x, 0; \lambda; \mu; \nu; q) B_k^{[m-1,1]}(0, y; q)}{[j+m-k]_q! [k]_q!}. \tag{98}$$

Proof. We only prove (97). Substituting (36) into the right-hand side of (65), we have

$$\begin{aligned} \mathcal{T}_n^{[m-1,\alpha]}(x, y; \lambda; \mu; \nu; q) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \cdot q^{(1/2)j(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \cdot \sum_{k=0}^{j} \frac{[j]_q! B_k^{[m-1,1]}(x, 0; q)}{[k]_q! \Gamma_q(j-k+m+1)} = \sum_{j=0}^{n} \sum_{k=0}^{j} \begin{bmatrix} n \\ j \end{bmatrix}_q \\ &\cdot q^{(1/2)j(j-1)} \mathcal{T}_{n-j}^{[m-1,\alpha]}(0, y; \lambda; \mu; \nu; q) \end{aligned}$$

$$\cdot \frac{[j]_q! B_k^{[m-1,1]}(x,0;q)}{[k]_q! \Gamma_q(j-k+m+1)} = \sum_{j=0}^{n}\sum_{k=0}^{j}\begin{bmatrix} n \\ j \end{bmatrix}_q \cdot \frac{[j]_q! q^{(1/2)j(j-1)}}{[k]_q!} \cdot \frac{\mathscr{T}_{n-j}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q) B_k^{[m-1,1]}(x,0;q)}{\Gamma_q(j-k+m+1)}. \tag{99}$$

□

Proposition 9. *For $n, j, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$, the q-analog of the generalized Apostol type polynomials is related to the q-Stirling numbers of the second kind $S(n,k;q)$ by means the following identities:*

$$\mathscr{T}_n^{[m-1,\alpha]}(x,y;\lambda;\mu;\nu;q) = \sum_{j=0}^{n}\sum_{k=0}^{j}\begin{bmatrix} n \\ j \end{bmatrix}_q \cdot q^{(1/2)j(j-1)}\mathscr{T}_{n-j}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q) S(j,k;q) \cdot (x)_{q;k}, \tag{100}$$

$$\mathscr{T}_n^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q) = \sum_{j=0}^{n}\sum_{k=0}^{j}\begin{bmatrix} n \\ j \end{bmatrix}_q \cdot \mathscr{T}_{n-j}^{[m-1,\alpha]}(\lambda;\mu;\nu;q) S(j,k;q)(x)_{q;k}. \tag{101}$$

Proof. Substituting (26) into the right-hand side of (55) and (65) gives the results. □

Proposition 10. *For $n, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$, the q-analog of the generalized Apostol type polynomials is related to the generalized q-Apostol-Bernoulli polynomials, the generalized q-Apostol-Euler polynomials, and the generalized q-Apostol-Genocchi polynomials by means the following identities:*

$$\lambda\mathscr{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q) - \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q) = \sum_{k=0}^{n-m}\begin{bmatrix} n-m \\ k \end{bmatrix}_q \frac{[n]_q!}{[n-m]_q!} \cdot \mathscr{T}_k^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\mathfrak{B}_{n-m-k}^{[m-1,-1]}(\lambda;q), \quad n \geq m, \tag{102}$$

$$\lambda\mathscr{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q) + \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q) = 2^m\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{T}_k^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^k}{[k]_q!} \cdot \mathfrak{E}_{n-k}^{[m-1,-1]}(\lambda;q), \tag{103}$$

$$\lambda\mathscr{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q) + \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot \mathscr{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q) = 2^m\sum_{k=0}^{n-m}\begin{bmatrix} n-m \\ k \end{bmatrix}_q \cdot \frac{[n]_q!}{[n-m]_q!}\mathscr{T}_k^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q) \cdot \mathfrak{G}_{n-m-k}^{[m-1,-1]}(\lambda;q), \quad n \geq m. \tag{104}$$

We will only show the details of the demonstrations of (102) and (103).

Proof (see (102)). Using (40) and (79), we have

$$\begin{aligned}\mathbf{I} &:= \sum_{n=0}^{\infty}\left(\lambda\mathscr{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q) - \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\right)\frac{z^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\lambda\mathscr{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!} - \sum_{l=0}^{m-1}\frac{z^l}{[l]_q!} \cdot \sum_{n=0}^{\infty}\mathscr{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!} \\ &= \lambda\left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1}(z^l/[l]_q!)}\right)^\alpha e_q^z E_q^{yz} - \sum_{l=0}^{m-1}\frac{z^l}{[l]_q!} \cdot \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1}(z^l/[l]_q!)}\right)^\alpha E_q^{yz}.\end{aligned} \tag{105}$$

Now, factoring the above equation and using (7), (40), we get

$$\mathbf{I} = z^m\left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1}(z^l/[l]_q!)}\right)^\alpha \cdot E_q^{yz}\left(\frac{\lambda e_q^z - \sum_{l=0}^{m-1}(z^l/[l]_q!)}{z^m}\right) = \sum_{n=0}^{\infty}\mathscr{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!}$$

$$\cdot \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,-1]}(\lambda;q) \frac{z^{n+m}}{[n]_q!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{T}_k^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^k}{[k]_q!}$$
$$\cdot \mathfrak{B}_{n-k}^{[m-1,-1]}(\lambda;q)\frac{z^{n-k+m}}{[n-k]_q!}. \tag{106}$$

Then, we get

$$\mathbf{I} = \sum_{n=m}^{\infty}\sum_{k=0}^{n-m} \mathcal{T}_k^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^k}{[k]_q!}$$
$$\cdot \mathfrak{B}_{n-m-k}^{[m-1,-1]}(\lambda;q)\frac{z^{n-k}}{[n-m-k]_q!}$$
$$= \sum_{n=m}^{\infty}\sum_{k=0}^{n-m} \begin{bmatrix} n-m \\ k \end{bmatrix}_q \frac{[n]_q!}{[n-m]_q!}$$
$$\cdot \mathcal{T}_k^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\,\mathfrak{B}_{n-m-k}^{[m-1,-1]}(\lambda;q)\frac{z^n}{[n]_q!}. \tag{107}$$

Comparing the coefficients of $z^n/[n]_q!$ in both sides gives the result. □

Proof (see (103)). Using (40) and (79), we have

$$\mathbf{I} := \sum_{n=0}^{\infty}\left(\lambda \mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)\right.$$
$$\left. + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\right)\frac{z^n}{[n]_q!}$$
$$= \sum_{n=0}^{\infty}\lambda \mathcal{T}_n^{[m-1,\alpha]}(1,y;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!} + \sum_{l=0}^{m-1}\frac{z^l}{[l]_q!}$$
$$\cdot \sum_{n=0}^{\infty}\mathcal{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!}$$
$$= \lambda\left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1}(z^l/[l]_q!)}\right)^\alpha e_q^z E_q^{yz} + \sum_{l=0}^{m-1}\frac{z^l}{[l]_q!}$$
$$\cdot \left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1}(z^l/[l]_q!)}\right)^\alpha E_q^{yz}. \tag{108}$$

Now, factoring the previous equation and using (8), (40), we get

$$\mathbf{I} = 2^m\left(\frac{(2^\mu z^\nu)^m}{\lambda e_q^z + \sum_{l=0}^{m-1}(z^l/[l]_q!)}\right)^\alpha$$
$$\cdot E_q^{yz}\left(\frac{\lambda e_q^z + \sum_{l=0}^{m-1}(z^l/[l]_q!)}{2^m}\right)$$
$$= 2^m \sum_{n=0}^{\infty}\mathcal{T}_n^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!}$$
$$\cdot \sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,-1]}(\lambda;q)\frac{z^n}{[n]_q!}$$
$$= 2^m \sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathcal{T}_k^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^k}{[k]_q!}$$
$$\cdot \mathfrak{E}_{n-k}^{[m-1,-1]}(\lambda;q)\frac{z^{n-k}}{[n-k]_q!}. \tag{109}$$

Therefore, we get

$$\mathbf{I} = \sum_{n=0}^{\infty}2^m\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_k^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\frac{z^k}{[k]_q!}$$
$$\cdot \mathfrak{E}_{n-k}^{[m-1,-1]}(\lambda;q)\frac{z^n}{[n]_q!}. \tag{110}$$

Comparing the coefficients of $z^n/[n]_q!$ in both sides gives the result. □

Proposition 11. *For $n, j, k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $0 < |q| < 1$, the q-analog of the generalized Apostol type polynomials is related to q-Bernstein basis $B_k^n(x;q)$ by means of the following identities:*

$$\mathcal{T}_n^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q) = \sum_{j=0}^{n}\sum_{k=0}^{n-j}\begin{bmatrix} k+j \\ j \end{bmatrix}_q$$
$$\cdot \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda;\mu;\nu;q)\,B_{k+j}^n(x;q), \tag{111}$$
$$\mathcal{T}_n^{[m-1,\alpha]}(x,y;\lambda;\mu;\nu;q) = \sum_{j=0}^{n}\sum_{k=0}^{n-j}\begin{bmatrix} k+j \\ j \end{bmatrix}_q$$
$$\cdot q^{(1/2)j(j-1)}\mathcal{T}_{n-j}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\,B_{k+j}^n(x;q). \tag{112}$$

Proof (see (111)). Substituting (31) into (55) we have

$$\mathcal{T}_n^{[m-1,\alpha]}(x,0;\lambda;\mu;\nu;q)$$
$$= \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda;\mu;\nu;q)\sum_{k=j}^{n}\frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n \\ j \end{bmatrix}_q}B_k^n(x;q)$$
$$= \sum_{j=0}^{n}\sum_{k=j}^{n}\begin{bmatrix} k \\ j \end{bmatrix}_q \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda;\mu;\nu;q)\,B_k^n(x;q)$$
$$= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\begin{bmatrix} k+j \\ j \end{bmatrix}_q \mathcal{T}_{n-j}^{[m-1,\alpha]}(\lambda;\mu;\nu;q)\,B_{k+j}^n(x;q). \tag{113}$$

□

Proof (see (112)). Substituting (31) into (65) we have

$$\mathcal{T}_{n}^{[m-1,\alpha]}(x,y;\lambda;\mu;\nu;q)=\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q \cdot q^{(1/2)(j)(j-1)}\mathcal{T}_{n-j}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)\sum_{k=j}^{n}\frac{\begin{bmatrix}k\\ j\end{bmatrix}_q}{\begin{bmatrix}n\\ j\end{bmatrix}_q} \cdot B_k^n(x;q)=\sum_{j=0}^{n}\sum_{k=j}^{n}\begin{bmatrix}k\\ j\end{bmatrix}_q \cdot q^{(1/2)(j)(j-1)}\mathcal{T}_{n-j}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)B_k^n(x;q)=\sum_{j=0}^{n}\sum_{k=0}^{n-j}\begin{bmatrix}k+j\\ j\end{bmatrix}_q \cdot q^{(1/2)(j)(j-1)}\mathcal{T}_{n-j}^{[m-1,\alpha]}(0,y;\lambda;\mu;\nu;q)B_{k+j}^n(x;q). \quad (114)$$

□

Proposition 12. *For* $n,k,j\in\mathbb{N}_0$, $m\in\mathbb{N}$, *and* $0<|q|<1$

$$B_k^n(x;q)=x^k\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\ j\end{bmatrix}_q\begin{bmatrix}n\\ k\end{bmatrix}_q\mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda;\mu;\nu;q) \cdot \mathcal{T}_{j}^{[m-1,1]}(1,-x;\lambda;\mu;\nu;q). \quad (115)$$

Proof (see (115)). The q-Bernstein basis is defined by means of following generating function:

$$\frac{x^kz^k}{[k]_q!}e_q^zE_q^{-xz}=\sum_{n=k}^{\infty}B_k^n(x;q)\frac{z^n}{[n]_q!}. \quad (116)$$

Using the left-hand side of the previous equation and (40), we have

$$\frac{x^kz^k}{[k]_q!}e_q^zE_q^{-xz}=\frac{x^kz^k}{[k]_q!}\left(\frac{e_q^z-\sum_{h=0}^{m-1}\left(z^h/[h]_q!\right)}{(2^\mu z^\nu)^m}\right) \cdot \left(\frac{(2^\mu z^\nu)^m}{e_q^z-\sum_{h=0}^{m-1}\left(z^h/[h]_q!\right)}\right)e_q^zE_q^{-xz}=\frac{x^kz^k}{[k]_q!} \cdot \sum_{n=0}^{\infty}\mathcal{T}_{n}^{[m-1,-1]}(\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!} \cdot \sum_{j=0}^{\infty}\mathcal{T}_{j}^{[m-1,1]}(1,-x;\lambda;\mu;\nu;q)\frac{z^j}{[j]_q!}=\frac{x^k}{[k]_q!} \cdot \sum_{n=0}^{\infty}\mathcal{T}_{n}^{[m-1,-1]}(\lambda;\mu;\nu;q)\frac{z^{n+k}}{[n]_q!} \cdot \sum_{j=0}^{\infty}\mathcal{T}_{j}^{[m-1,1]}(1,-x;\lambda;\mu;\nu;q)\frac{z^j}{[j]_q!}. \quad (117)$$

Therefore, we obtain

$$\frac{x^kz^k}{[k]_q!}e_q^zE_q^{-xz}=\frac{x^k}{[k]_q!}\sum_{n=k}^{\infty}\sum_{j=0}^{n-k}\mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda;\mu;\nu;q) \cdot \frac{z^{n-j}}{[n-k-j]_q!}\mathcal{T}_{j}^{[m-1,1]}(1,-x;\lambda;\mu;\nu;q)\frac{z^j}{[j]_q!}=\sum_{n=k}^{\infty}x^k\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\ j\end{bmatrix}_q\begin{bmatrix}n\\ k\end{bmatrix}_q\mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda;\mu;\nu;q) \cdot \mathcal{T}_{j}^{[m-1,1]}(1,-x;\lambda;\mu;\nu;q)\frac{z^n}{[n]_q!}. \quad (118)$$

Comparing coefficients the $z^n/[n]_q!$, we obtain

$$B_k^n(x;q)=x^k\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\ j\end{bmatrix}_q\begin{bmatrix}n\\ k\end{bmatrix}_q\mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda;\mu;\nu;q) \cdot \mathcal{T}_{j}^{[m-1,1]}(1,-x;\lambda;\mu;\nu;q). \quad (119)$$

□

Corollary 13. *For* $n,j,k\in\mathbb{N}_0$, $0\le k\le j\le n$, $m\in\mathbb{N}$, *and* $0<|q|<1$, *one has*

$$\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_q S(j,k;q)\mathfrak{B}_{n-j}^{(k)}(1,-x;q)=\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\ j\end{bmatrix}_q\begin{bmatrix}n\\ k\end{bmatrix}_q\mathcal{T}_{n-k-j}^{[m-1,-1]}(\lambda;\mu;\nu;q) \cdot \mathcal{T}_{j}^{[m-1,1]}(1,-x;\lambda;\mu;\nu;q). \quad (120)$$

Proof (see (120)). Substituting (32) into the left-hand side of (115), we obtain the result. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, pp. 987–1000, 1948.

[2] J. L. Cieslinski, "Improved q-exponential and q-trigonometric functions," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 2110–2114, 2011.

[3] T. Ernst, "q-Bernoulli and q-Euler polynomials, an umbral approach," *International Journal of Difference Equations*, vol. 1, no. 1, pp. 31–80, 2006.

[4] M. Garg and S. Alha, "Una nueva clase de polinomios q-Apostol-Bernoulli de orden a /A new class of q-Apostol-Bernoulli polynomials of order a," *Revista Tecnocientífica URU*, vol. 6, pp. 67–76, 2014.

[5] T. H. Koornwinder, "Special functions and q-commuting variables," https://arxiv.org/abs/q-alg/9608008, 1996.

[6] Q. M. Luo, "q-Extensions of Some Results Involving the Luo-Srivastava Generalizations," *Filomat*, vol. 28, no. 2, pp. 329–351, 2014.

[7] Q. M. Luo, "Some results for the q-Bernoulli andq-Euler polynomials," *Journal of Mathematical Analysis and Applications*, vol. 363, no. 1, pp. 7–18, 2010.

[8] T. Ernst, *The History of q-Calculus And A New Method*, Department of Mathematics, Uppsala University, Uppsala, Sweden, 2000.

[9] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, vol. 96, Cambridge university press, Cambridge, UK, 2004.

[10] H. M. Srivastava and Ch. Junesang, *Zeta And q-Zeta Functions And Associated Series And Integrals*, Editorial Elsevier, Boston, Massachusetts, 2012.

[11] P. Natalini and A. Bernardini, "A generalization of the Bernoulli polynomials," *Journal of Applied Mathematics*, no. 3, pp. 155–163, 2003.

[12] P. Hernández-Llanos, Y. Quintana, and A. Urieles, "About Extensions of Generalized Apostol-Type Polynomials," *Results in Mathematics*, vol. 68, no. 1-2, pp. 203–225, 2015.

[13] B. Kurt, "Some Relationships between the Generalized Apostol-Bernoulli and Apostol-Euler Polynomials," *Turkish Journal of Analysis and Number Theory*, vol. 1, no. 1, pp. 54–58, 2013.

[14] R. Tremblay, S. Gaboury, and J. Fugere, "A further generalization of Apostol-Bernoulli polynomials and related polynomials," *Honam Mathematical Journal*, vol. 34, no. 3, pp. 311–326, 2012.

[15] R. Tremblay, S. Gaboury, and B. J. Fugere, "A new class of generalized Apostol–Bernoulli polynomials and some analogues of the Srivastava–Pintér addition theorem," *Applied Mathematics Letters*, vol. 24, no. 11, pp. 1888–1893, 2011.

[16] B. Kurt, "Notes on Unified q-Apostol-Type Polynomials," *Filomat*, vol. 30, no. 4, pp. 921–927, 2016.

[17] S. K. Sharma and R. Jain, "On some properties of generalized q-Mittag leffler," *Mathematica Aeterna*, vol. 4, no. 6, pp. 613–619, 2014.

[18] N. I. Mahmudov and M. E. Keleshteri, "q-extensions for the Apostol type polynomials," *Journal of Applied Mathematics*, Art. ID 868167, 8 pages, 2014.

[19] V. Kac and P. Cheung, *Quantum Calculus*, Science and Business Media, Berlin, Germany, 2001.

[20] T. Ernst, *A Comprehensive Treatment of q-Calculus*, Springer Science and Business Media, Berlin, Germany, 2012.

[21] H. Oruç and G. M. Phillips, "Bernstein polynomials and Bézier curves," *Journal of Computational and Applied Mathematics*, vol. 151, no. 1, pp. 1–12, 2003.

[22] N. I. Mahmudov, "On a class of q-Bernoulli andq-Euler polynomials," *Advances in Difference Equations*, vol. 1, p. 108, 2013.

[23] N. I. Mahmudov and M. E. Keleshteri, "On a class of generalized q-Bernoulli and q-Euler polynomials," *Advances in Difference Equations*, vol. 1, 2013.