Alvarez et al. *Advances in Difference Equations* (2018) 2018:311
https://doi.org/10.1186/s13662-018-1770-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Strongly damped wave equations with Stepanov type nonlinear forcing term

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Abstract

In this paper we investigate the existence and uniqueness of weighted pseudo almost automorphic mild solution for a class of strongly damped wave equations where the semilinear forcing term is a Stepanov weighted pseudo almost automorphic function.

MSC: 65R20; 35L72; 35A01; 35A02; 35L05

Keywords: Weighted Stepanov pseudo almost automorphic; Integral equations; Strongly damped wave; Abstract Cauchy problems

1 Introduction

Let X be a Banach reflexive space, let us assume that $A : D(A) \subset X \to X$ is a closed densely defined operator, $\eta > 0$, and $\theta \in [1/2, 1]$. Let us suppose that (η, θ, A) is an admissible triple in the sense of [4]. That is, there exist $\psi \in (0, \pi/2)$ and $M > 0$ such that

$$\|(\lambda - A)^{-1}\| \le \frac{M}{1 + |\lambda|}$$

for all $\lambda \in S_{0,\psi} := \sum_\psi \cup \{0\}$, where $\sum_\psi := \{\lambda \in \mathbb{C} : \psi \le |\arg\lambda| \le \pi\}$ and either

(i) $\theta \in (1/2, 1]$, or

(ii) $\theta = 1/2$ and $\pi/2 > \psi/2 + \arg(\eta + \sqrt{\eta^2 - 1})$.

If $\theta = 1/2$, then we denote $(\eta, 1/2, A)$ by (η, A).

For the admissible triple $(\eta, 1/2, A) = (\eta, A)$, we consider the power space X^α associated with A, following [14]. The main purpose of this paper is the study of existence and uniqueness of weighted pseudo almost automorphic mild solutions of the abstract Cauchy problem

$$\begin{cases} u_{tt} + 2\eta A^{1/2} u_t + Au = f(t, u, u_t), & t > 0, \\ u(0) = u_0 \in X^{1/2}, & u_t(0) = v_0 \in X, \end{cases} \tag{1.1}$$

where f is a Stepanov weighted pseudo almost automorphic function. Note that the wave equation with structural damping

$$u_{tt} + 2\eta(-\Delta)^{1/2} u_t + \beta u_t - \Delta u = f(u), \quad \eta, \beta > 0,$$

can be represented in the abstract form (1.1). In this case, one can verify that (η, A) is admissible.

The space of weighted pseudo almost automorphic functions was introduced by Blot et al. in [3]. They gave some basic properties of this space and proved the existence and uniqueness of weighted pseudo almost automorphic mild solutions for the problem $u'(t) = Au(t) + f(t, u(t))$, when f is a weighted pseudo almost automorphic function. After that, several papers about the existence of weighted pseudo almost automorphic mild solutions to semilinear evolution equations when the nonlinearity is of weighted pseudo almost automorphic type were written, see for example [11, 12] and the references therein.

On the other hand, the concept of Stepanov weighted pseudo almost automorphic functions, which is more general than the weighted pseudo almost automorphic one, was proposed by Xia and Fan in [19]. This time, the authors studied weighted pseudo almost automorphic mild solutions of partial neutral functional differential equations and integral equations with delay when the nonlinear term f is a Stepanov weighted pseudo almost automorphic function. Later, Zhang et al. proved composition theorems of Stepanov weighted pseudo almost automorphic functions and applied their results to the existence and uniqueness of weighted pseudo almost automorphic mild solutions to a class of nonautonomous evolution equations with S^p-weighted pseudo almost automorphic coefficients in [25]. In [16] the authors showed the existence and uniqueness of a weighted pseudo almost automorphic solution of an integro-differential equation with weighted S^p-pseudo almost automorphic forcing term in a Banach space X. Alvarez and Lizama in [1] proved the existence and uniqueness of weighted pseudo almost automorphic mild solutions for the two-term fractional order differential equation $D_t^\alpha u'(t) + \mu D_t^\beta u(t) = Au(t) + D^\alpha f(t, u(t))$, where $0 < \alpha \leq \beta < 1$, $\mu \geq 0$ and the nonlinear forcing term is S^p-weighted pseudo almost automorphic. Otherwise, when the weight is replaced by a suitable measure μ, there are works with similar results, see for example [6, 7]. Also, Xia et al. in [20] worked with two suitable measures and they investigated the existence and uniqueness of (μ, ν)-pseudo almost automorphic mild solutions to a semilinear fractional differential equation with Riemann–Liouville derivative in a Banach space, where the nonlinear perturbation is of (μ, ν)-pseudo almost automorphic type or Stepanov-like (μ, ν)-pseudo almost automorphic type. Yang and Zhu investigated the properties of the p-mean Stepanov-like doubly weighted pseudo almost automorphic processes and their application to Sobolev-type stochastic differential equations driven by G-Brownian motion, see [24]. Cuevas et al. in [2] studied the asymptotic periodicity, L^p-boundedness, classical (resp., strong) solutions, and the topological structure of solutions set of strongly damped semilinear wave equations. Recently, Xia et al. in [22] introduced and investigated the (μ, ν)-pseudo $\mathcal{S}$-asymptotically ω-periodic functions of class r and gave some applications. For related works, see [8, 13, 21] and the references therein.

Cuevas et al. in [9] studied asymptotic almost periodic mild solutions of problem (1.1). However, to the best of our knowledge, the fact that the existence and uniqueness of weighted pseudo almost automorphic mild solutions to (1.1) with the forcing term f belong to the space of Stepanov weighted pseudo almost automorphic functions is an untreated original problem, which constitutes one of the main motivations of this work.

In this article, we succeed in proving the existence of a unique weighted pseudo almost automorphic mild solution for (1.1) when the nonlinear term f is a Stepanov weighted pseudo almost automorphic function writing (1.1) as a first order abstract Cauchy problem. Then, we use the fact that the associated semigroup $(e^{-t\mathcal{A}_{1/2}})_{t\geq 0}$ is exponentially bounded in a suitable Banach space in order to prove that the mild solutions satisfy the required regularity.

We point out that the definition of weight carries implicitly a measure μ-absolutely continuous with respect to the Lebesgue measure and its Radom–Nikodym derivative is ρ since $d\mu(t) = \rho(t)\,dt$. Then, the results of this paper can be extended to more general and recent concepts of weighted pseudo almost automorphic functions in the context of measure theory [6, 7].

This paper is organized as follows. In Sect. 2 we recall some definitions, lemmas, and preliminary results. In Sect. 3 we recall some composition theorems. Furthermore, we give two convolution theorems. In Sect. 4 we give our main result and an application.

2 Preliminaries

In this section, we present some concepts and properties in order to develop the following sections. Let X be a reflexive complex Banach space. For an interval $I \subset \mathbb{R}$, $C_b(I,X)$ denotes the space formed by the bounded and continuous functions from I into X, endowed with the norm of uniform convergence. When $X = \mathbb{R}$, we denote $C_b(I)$ instead of $C_b(I,\mathbb{R})$. The notation $C_0(\mathbb{R}^+,X)$ stands for the subspace of $C_b(\mathbb{R}^+,X)$ consisting of functions that vanish at infinity. We denote by $\mathfrak{L}(X)$ the Banach algebra of bounded linear operators defined on X.

We recall some definitions about sectorial operators and fractional powers associated with this type of operators.

Definition 2.1 (Sectorial operator [14]) We call a linear operator A in Banach space X a sectorial operator if it is a closed densely defined operator such that, for some $\phi \in (0,\pi/2)$ and some $M \geq 1$ and real a, the sector $\sum_{a,\phi} := \{\lambda \in \mathbb{C} : \phi \leq |\arg(\lambda - a)| \leq \pi\} \subset \rho(A)$ and

$$\|(\lambda - A)^{-1}\| \leq M/|\lambda - a|$$

for $\lambda \in \sum_{a,\phi} \setminus \{a\}$.

Example 2.2 ([14]) If A is a bounded linear operator on a Banach space, then A is sectorial.

Definition 2.3 (Fractional powers of operators [14]) Suppose that A is a sectorial operator and $\operatorname{Re}\sigma(A) > 0$; then for any $\alpha > 0$

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At}\,dt.$$

Define A^α as the inverse of $A^{-\alpha}(\alpha > 0)$, $D(A^\alpha) = \mathcal{R}(A^{-\alpha})$.

Definition 2.4 (Fractional power space [14]) If A is a sectorial operator on a Banach space X, define for each $\alpha \geq 0$

$$X^\alpha = D(A_1^\alpha),$$

with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|, x \in X^\alpha$. Here $A_1 = A + aI$ and a is chosen such that $\operatorname{Re}\sigma(A_1) > 0$.

On the other hand, note that problem (1.1) can be written as

$$\begin{cases} \begin{bmatrix} u \\ v \end{bmatrix}_t + \mathcal{A}_{1/2}\begin{bmatrix} u \\ v \end{bmatrix} = F(t, \begin{bmatrix} u \\ v \end{bmatrix}), & t > 0, \\ \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \end{cases} \tag{2.1}$$

where

$$\mathcal{A}_{1/2} = \begin{bmatrix} 0 & -I \\ A & 2\eta A^{1/2} \end{bmatrix} : D(\mathcal{A}_{1/2}) \subseteq X \times X^{1/2} \to X \times X^{1/2} \tag{2.2}$$

is defined by

$$\begin{cases} \mathcal{A}_{1/2}\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ A\varphi + 2\eta A^{1/2}\psi \end{bmatrix}, & \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\mathcal{A}_{1/2}) \subset X \times X^{1/2}, \\ F(t, \begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 0 \\ f(t,u,v) \end{bmatrix}. \end{cases} \tag{2.3}$$

If (η, A) is admissible, then by [4, Proposition 2.1] the following properties hold:

(i) $\mathcal{A}_{1/2}$ is a closed operator;
(ii) $0 \in \rho(\mathcal{A}_{1/2})$;
(iii) If A has a compact resolvent, then $\mathcal{A}_{1/2}$ has a compact resolvent.

Moreover, by [4, Theorem 2.3] the operator $\mathcal{A}_{1/2}$ is sectorial in $X^{1/2} \times X$ and the semigroup $\{e^{-\mathcal{A}_{1/2}t} : t \geq 0\}$ is exponentially bounded, that is, there exist $K \geq 1$ and $C > 0$ such that

$$\left\| e^{-\mathcal{A}_{1/2}(t)} \right\|_{\mathfrak{L}(X^{1/2} \times X)} \leq Ke^{-Ct}, \quad t \geq 0. \tag{2.4}$$

Now, we present some definitions and main results of Stepanov weighted pseudo almost automorphic functions.

Definition 2.5 Let $f : \mathbb{R} \to X$ be a continuous function. We say that f is almost automorphic if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the convergence in the above definition is uniform on $\mathbb{R}$, we get almost periodicity in the sense of Bochner and von Neumann.

Let Y be a Banach space. We have the following concept of parameter-dependent almost automorphic function.

Definition 2.6 A continuous function $F : \mathbb{R} \times Y \to X$ is said to be almost automorphic if $F(t, u)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $u \in K$ in any bounded subset of Y.

Almost automorphic functions (denoted by $AA(\mathbb{R},X)$) (resp. $AA(\mathbb{R}\times Y,X)$) constitute a Banach space when it is endowed with the sup norm. They naturally generalize the concept of almost periodic functions. But the inverse is not true. A typical example of an almost automorphic function which is not almost periodic (see [18]) is given by

$$\psi(t)=\frac{2+e^{it}+e^{\sqrt{2}t}}{|2+e^{it}+e^{\sqrt{2}t}|}.$$

Lemma 2.7 ([17]) *If $f,f_1,f_2\in AA(\mathbb{R},X)$, then*

(i) $f_1+f_2\in AA(\mathbb{R},X)$;

(ii) $\lambda f\in AA(\mathbb{R},X)$;

(iii) $f_\alpha\in AA(\mathbb{R},X)$, *where $f_\alpha:\mathbb{R}\to X$ is defined by $f_\alpha(\cdot)=f(\cdot+\alpha),\alpha\in\mathbb{R}$*;

(iv) *The range $\mathcal{R}_f=\{f(t):t\in\mathbb{R}\}$ is relatively compact in X, thus f is bounded in norm*;

(v) *If $f_n\to f$ uniformly on $\mathbb{R}$, where each $f_n\in AA(\mathbb{R},X)$, then $f\in AA(\mathbb{R},X)$.*

Definition 2.8 ([23]) A function $f\in C(\mathbb{R},X)$ (resp. $C(\mathbb{R}\times Y,X)$) is called pseudo almost automorphic if it can be decomposed as $f=g+\varphi$, where $g\in AA(\mathbb{R},X)$ (resp. $AA(\mathbb{R}\times Y,X)$) and $\varphi\in BC(\mathbb{R},X)$ with

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|\varphi(s)\|\,ds=0$$

(resp. $\varphi\in BC(\mathbb{R}\times Y,X)$ with

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|\varphi(s,u)\|\,ds=0$$

uniformly for u in any bounded subset of Y).

Denote by $PAA(\mathbb{R},X)$ (resp. $PAA(\mathbb{R}\times Y,X)$) the collection of such functions, and $(PAA(\mathbb{R},X),\|\cdot\|_{\mathrm{PAA}})$ (resp. $(PAA(\mathbb{R}\times Y,X),\|\cdot\|_{\mathrm{PAA}})$) is a Banach space when endowed with the sup norm. The function

$$f_{\alpha,\beta,\gamma}(t)=\cos\left(\frac{1}{3-\sin t-2\sin\beta t}\right)+\frac{e^{-|t|^\gamma}}{(1+|t|^\gamma)^\alpha},\quad t\in\mathbb{R},$$

is a pseudo almost automorphic function, where $\alpha\in(1,\infty)$, $\beta\in\mathbb{R}/\mathbb{Q}$ and $\gamma\in[0,\infty)$ (see [19]).

Let U be the set of all functions $\rho:\mathbb{R}\to(0,\infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $T>0$ and each $\rho\in U$, set

$$\mu(T,\rho)=\int_{-T}^{T}\rho(t)\,dt.$$

Define

$$U_\infty=\Bigl\{\rho\in U:\lim_{T\to\infty}\mu(T,\rho)=\infty\Bigr\},$$

$$U_\beta=\Bigl\{\rho\in U_\infty:\rho\text{ is bounded and }\inf_{x\in\mathbb{R}}\rho(x)>0\Bigr\}.$$

It is clear that $U_\beta \subset U_\infty \subset U$.
For $\rho \in U_\infty$, define

$$PAA_0(\mathbb{R}, X, \rho) = \left\{ f \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(s) \| f(s) \| \, ds = 0 \right\},$$

$$PAA_0(\mathbb{R} \times Y, X, \rho) = \left\{ f \in BC(\mathbb{R} \times Y, X) : \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(s) \| f(s,u) \| \, ds = 0 \right. \\ \left. \text{uniformly in } u \in Y \right\}.$$

Definition 2.9 ([3]) Let $\rho \in U_\infty$. A function $f \in C(\mathbb{R}, X)$ (resp. $C(\mathbb{R} \times Y, X)$) is called weighted pseudo almost automorphic if it can be decomposed as $f = g + \varphi$, where $g \in AA(\mathbb{R}, X)$ (resp. $AA(\mathbb{R} \times Y, X)$) and $\varphi \in PAA_0(\mathbb{R}, X, \rho)$ (resp. $PAA_0\mathbb{R} \times Y, X, \rho)$. Denote by $WPAA(\mathbb{R} \times Y, X, \rho)$ (resp. $WPAA(\mathbb{R} \times Y, X, \rho)$) the set of such functions.

Definition 2.10 The Bochner transform $f^b(t,s), t \in \mathbb{R}, s \in [0,1]$, of a function $f : \mathbb{R} \to X$ is defined by $f^b(t,s) = f(t+s)$.

Definition 2.11 Let $p \in [1,\infty)$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^p(\mathbb{R}, L^p([0,1], X))$. It is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \right)^{1/p}.$$

Definition 2.12 ([5]) The space $S^pAA(\mathbb{R}, X)$ of Stepanov-like almost automorphic functions consists of all $f \in BS^p(\mathbb{R}, X)$ such that $f^b \in AA(\mathbb{R}, L^p([0,1], X))$.

Definition 2.13 ([19]) Let $\rho \in U_\infty$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be weighted Stepanov-like pseudo almost automorphic if it can be decomposed as $f = g + \varphi$, where $g^b \in AA(\mathbb{R}, L^p([0,1], X))$ and $\varphi^b \in PAA_0(\mathbb{R}, L^p([0,1], X), \rho)$. Denote by $S^pWPAA(\mathbb{R}, X, \rho)$ the collection of such functions.

Definition 2.14 ([19]) Let $\rho \in U_\infty$. A function $F : \mathbb{R} \times Y \to X, (t,u) \mapsto F(t,u)$ with $F(\cdot, u) \in L^p_{\text{loc}}(\mathbb{R}, X)$ for each $u \in Y$ is said to be weighted S^p-pseudo almost automorphic if it can be decomposed as $F = G + H$, where $G^b \in AA(\mathbb{R} \times Y, L^p([0,1], X))$ and $H^b \in PAA_0(\mathbb{R} \times Y, L^p([0,1], X), \rho)$. The collection of such functions will be denoted by $S^pWPAA(\mathbb{R} \times Y, X, \rho)$.

3 Convolution and composition results

In this section, we recall some well-known composition results in the context of almost automorphic and weighted pseudo almost automorphic functions. Also, we give two convolution results, which are part of the new contributions of this work.

Lemma 3.1 *If $h : \mathbb{R} \to X^{1/2} \times X$ is an almost automorphic function and*

$$u(t) = \int_{-\infty}^{t} e^{-\mathcal{A}_{1/2}(t-s)} h(s) \, ds, \quad t \in \mathbb{R}, \tag{3.1}$$

then u is almost automorphic.

Proof Let (s'_n) be a sequence in $\mathbb{R}$. From Definition 2.5 there exists a subsequence (s_n) such that $g(t) = \lim_{n\to\infty} h(t + s_n)$ is well defined. Now, by (3.1),

$$\|u(t+s_n)\| \le \int_0^\infty Ke^{-C\beta}\|h(t-\beta+s_n)\|\,d\beta.$$

It follows from Lemma 2.7 that $u(t + s_n)$ is well defined. On the other hand, we define

$$\widetilde{g}(t) = \int_0^\infty e^{-\mathcal{A}_{1/2}(\beta)}g(t-\beta)\,d\beta.$$

Since $g(t) = \lim_{n\to\infty} h(t + s_n)$, given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\|h(t+s_n)-g(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$, whenever $n \ge N_\varepsilon$. Then, for $n \ge N_\varepsilon$, we obtain

$$\begin{aligned}\|u(t+s_n)-\widetilde{g}(t)\| &\le \int_0^\infty \|e^{-\mathcal{A}_{1/2}(\beta)}[h(t-\beta+s_n)-g(t-\beta)]\|\,d\beta\\ &\le \int_0^\infty Ke^{-C\beta}\varepsilon\,d\beta = \frac{K}{C}\varepsilon.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} u(t+s_n) = \widetilde{g}(t).$$

Analogously, we can show that

$$\lim_{n\to\infty} \widetilde{g}(t-s_n) = \int_0^\infty e^{-\mathcal{A}_{1/2}(\beta)}h(t-\beta)\,d\beta = u(t).$$

It follows that u is almost automorphic. □

Next, we recall the following composition theorems (see [15, Lemma 2.2] and [25, Theorem 3.2], respectively).

Lemma 3.2 *If $f : \mathbb{R}\times X \to X$ is almost automorphic, and assume that $f(t,\cdot)$ is uniformly continuous on each subset $K \subset X$ uniformly for $t \in \mathbb{R}$, that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in K$ and $\|x - y\| < \delta$ imply that $\|f(t,x) - f(t,y)\| < \varepsilon$ for all $t \in \mathbb{R}$. Let $\phi : \mathbb{R} \to X$ be almost automorphic. Then the function $F : \mathbb{R} \to X$ defined by $F(t) = f(t,\phi(t))$ is almost automorphic.*

Lemma 3.3 *Let $\rho \in U_\infty$ and let $f = g + \varphi \in S^pWPAA(\mathbb{R}\times X, X, \rho)$ with $g^b \in AA(\mathbb{R}\times X, L^p([0,1], X))$, $\varphi^b \in PAA_0(\mathbb{R}\times X, L^p([0,1], X), \rho)$. Suppose that f and g satisfy the following conditions:*

(i) $f(t,x)$ is uniformly continuous in any bounded subset $K' \subset X$ uniformly for $t \in \mathbb{R}$;
(ii) $g(t,x)$ is uniformly continuous in any bounded subset $K' \subset X$ uniformly for $t \in \mathbb{R}$;
(iii) For every bounded subset $K' \subset X$, $\{f(\cdot,x) : x \in K'\}$ is bounded in $S^pWPAA(\mathbb{R}, X)$.

If $x = \alpha + \beta \in S^p WPAA(\mathbb{R}, X, \rho)$, with $\alpha \in S^p A(\mathbb{R}, X)$, $\beta^b \in PAA_0(\mathbb{R}, L^p([0,1], X), \rho)$ and $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, then the function $f(\cdot, x(\cdot))$ belongs to $S^p WPAA(\mathbb{R}, X)$.

The next lemma is one of our main results.

Lemma 3.4 *If $h : \mathbb{R}^+ \to X^{1/2} \times X$ is a S^p-weighted pseudo almost automorphic function and*

$$u(t) = \int_0^t e^{-\mathcal{A}_{1/2}(t-s)} h(s)\, ds, \quad t \geq 0,$$

then u is weighted pseudo almost automorphic.

Proof Let $\mathbf{X} := X^{1/2} \times X$ and $h(s) = \psi_1(s) + \psi_2(s)$, where $\psi_1^b \in AA(\mathbb{R}, L^p([0,1], \mathbf{X}))$ and $\psi_2^b \in PAA_0(\mathbb{R}, L^p([0,1], \mathbf{X}), \rho)$. Consider the integrals

$$u(t) = \int_0^t e^{-\mathcal{A}_{1/2}(t-s)} h(s)\, ds = \int_0^t e^{-\mathcal{A}_{1/2}(t-s)} \psi_1(s)\, ds + \int_0^t e^{-\mathcal{A}_{1/2}(t-s)} \psi_2(s)\, ds.$$

Thus

$$\begin{aligned} u(t) = {} & \left[\int_{-\infty}^t e^{-\mathcal{A}_{1/2}(t-s)} \psi_1(s)\, ds - \int_{-\infty}^0 e^{-\mathcal{A}_{1/2}(t-s)} \psi_1(s)\, ds \right] \\ & + \left[\int_{-\infty}^t e^{-\mathcal{A}_{1/2}(t-s)} \psi_2(s)\, ds - \int_{-\infty}^0 e^{-\mathcal{A}_{1/2}(t-s)} \psi_2(s)\, ds \right]. \end{aligned}$$

By [10, Lemma 11.2] and by [19, Lemma 4.1], we have that

$$\int_{-\infty}^t e^{-\mathcal{A}_{1/2}(t-s)} \psi_1(s)\, ds \in AA(\mathbb{R}, \mathbf{X})$$

and

$$\int_{-\infty}^t e^{-\mathcal{A}_{1/2}(t-s)} \psi_2(s)\, ds \in PAA_0(\mathbb{R}, \mathbf{X}, \rho),$$

respectively. Next, let us show that

$$G_1(t) = \int_{-\infty}^0 e^{-\mathcal{A}_{1/2}(t-s)} \psi_1(s)\, ds \in AA(\mathbb{R}, \mathbf{X})$$

and

$$G_2(t) = \int_{-\infty}^0 e^{-\mathcal{A}_{1/2}(t-s)} \psi_2(s)\, ds \in PAA_0(\mathbb{R}, \mathbf{X}, \rho).$$

Set

$$X_n(t) = \int_{-n}^{-n+1} e^{-\mathcal{A}_{1/2}(t-s)} \psi_1(s)\, ds, \qquad Y_n(t) = \int_{-n}^{-n+1} e^{-\mathcal{A}_{1/2}(t-s)} \psi_2(s)\, ds.$$

Next we show that $X_n \in AA(\mathbb{R}, \mathbf{X})$. By Holder's inequality, we obtain

$$\|X_n(t)\| \leq K \int_{-n}^{-n+1} e^{-C(t-s)} \|\psi_1(s)\|\, ds$$

$$\begin{aligned}
&\le K\left(\int_{-n}^{-n+1} e^{-qC(t-s)}\,ds\right)^{\frac{1}{q}}\left(\int_{-n}^{-n+1}\|\psi_1(s)\|^p\,ds\right)^{\frac{1}{p}} \\
&\le \frac{K}{\sqrt[q]{qC}}\left(e^{-qC(t+n-1)}-e^{-qC(t+n)}\right)^{\frac{1}{q}}\|\psi_1\|_{S^p} \\
&\le Ke^{-Ct}e^{-Cn}\sqrt[q]{\frac{e^{qC}-1}{qC}}\|\psi_1\|_{S^p},
\end{aligned} \tag{3.2}$$

where $q = p/(p-1)$. Since the series $K\sqrt[q]{\frac{e^{qC}-1}{qC}}\sum_{n=1}^{\infty} e^{-Cn}$ is convergent, by the Weierstrass test, $\sum_{n=1}^{\infty} X_n(t)$ is uniformly convergent on $\mathbb{R}^+$.

Let $X(t) = \sum_{n=1}^{\infty} X_n(t), t \in \mathbb{R}^+$, then

$$X(t) = \int_{-\infty}^{0} e^{-\mathcal{A}_{1/2}(t-s)}\psi_1(s), \quad t \in \mathbb{R}^+.$$

Fix $n \in \mathbb{N}$. For each $t \in \mathbb{R}^+$, we have

$$\begin{aligned}
\|X_n(t+h)-X_n(t)\| &\le \int_{-n}^{-n+1}\left\|e^{-\mathcal{A}_{1/2}(t-s)}\left[\psi_1(s+h)-\psi_1(s)\right]\right\|\,ds \\
&\le K\int_{-n}^{-n+1} e^{-C(t-s)}\|\psi_1(s+h)-\psi_1(s)\|\,ds \longrightarrow 0, \quad \text{as } h \to 0.
\end{aligned}$$

Thus, $X_n(t)$ is continuous for each $t \in \mathbb{R}^+$.

Next, we prove that $X_n \in AA(\mathbb{R}, \mathbf{X})$ for $n \in \mathbb{N}$. Let $(s'_m)_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $\psi_1^b \in AA(\mathbb{R}, L^p([0,1], \mathbf{X}), \rho)$, then there exist a subsequence $(s_{m_k})_{k\in\mathbb{N}}$ and a function $v \in L^p_{\mathrm{loc}}(\mathbb{R},)$ such that, for any $t \in \mathbb{R}^+$,

$$\left(\int_t^{t+1}\|\psi_1(s+s_{m_k})-v(s)\|^p\,ds\right)^{\frac{1}{p}} \longrightarrow 0 \quad \text{as } k \to \infty.$$

Note that

$$X_n(t) = \int_{-n}^{-n+1} e^{-\mathcal{A}_{1/2}(t-s)}\psi_1(s)\,ds = \int_{-n}^{-n+1} e^{-\mathcal{A}_{1/2}(\xi)}\psi_1(t-\xi)\,d\xi$$

and define $w_n(t) = \int_{-n}^{-n+1} e^{-\mathcal{A}_{1/2}(\xi)} w_1(t-\xi)\,d\xi$. Then, by Holder's inequality, we have

$$\begin{aligned}
&\|X_n(t+s_{m_k})-w_n(t)\| \\
&\quad= \left\|\int_{-n}^{-n+1} e^{-\mathcal{A}_{1/2}(\xi)}\left[\psi_1(t+s_{m_k}-\xi)-v(t-\xi)\right]\right\|\,d\xi \\
&\quad\le K\int_{-n}^{-n+1} e^{-C\xi}\|\psi_1(t+s_{m_k}-\xi)-v(t-\xi)\|\,d\xi \\
&\quad\le K\left(\int_{-n}^{-n+1} e^{-qC\xi}\,d\xi\right)^{\frac{1}{q}}\left(\int_{-n}^{-n+1}\|\psi_1(t+s_{m_k}-\xi)-v(t-\xi)\|^p\,d\xi\right)^{\frac{1}{p}} \\
&\quad\le C_q(K,C)\left(\int_{-n}^{-n+1}\|\psi_1(t+s_{m_k}-\xi)-v(t-\xi)\|^p\,d\xi\right)^{\frac{1}{p}} \to 0,
\end{aligned}$$

as $k \longrightarrow \infty$. Similarly, we obtain that $\|w_n(t-s_{m_k}) - X_n(t)\| \to 0$, when $k \to \infty$. Therefore, $X_n(t) \in AA(\mathbb{R}, \mathbf{X})$ for $n \in \mathbb{N}$. By Lemma 2.7, we have $X(t) = \sum_{n=1}^{\infty} X_n(t) \in AA(\mathbb{R}, \mathbf{X})$. To complete the proof, we shall prove $Y_n \in PAA_0(\mathbb{R}, \mathbf{X}, \rho)$. By carrying out similar arguments as above, we know that $Y_n \in BC(\mathbb{R}, \mathbf{X})$ and $\sum_{n=1}^{\infty} Y_n(t)$ is uniformly convergent on $\mathbb{R}$. Let $Y(t) = \sum_{n=1}^{\infty} Y_n(t)$, then

$$Y(t) = \int_{-\infty}^{0} e^{-\mathcal{A}_{1/2}(t-s)} \psi_2(s)\, ds, \quad t \in \mathbb{R}^+.$$

It is obvious that $Y(t) \in BC(\mathbb{R}, \mathbf{X})$. So, we only need to show that

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) \| Y(t) \| \, dt = 0.$$

In fact, first of all, we have the following estimate:

$$\begin{aligned}
\| Y_n(t) \| &\le K \int_{-n}^{-n+1} e^{-C(t-s)} \| \psi_2(s) \| \, ds \\
&\le K \left(\int_{-n}^{-n+1} e^{-qC(t-s)} \, ds \right)^{\frac{1}{q}} \left(\int_{-n}^{-n+1} \| \psi_2(s) \|^p \, ds \right)^{\frac{1}{p}} \\
&\le K e^{-Ct} e^{-Cn} \sqrt[q]{\frac{e^{qC} - 1}{qC}} \left(\int_{-n}^{-n+1} \| \psi_2(s) \|^p \, ds \right)^{\frac{1}{p}} \\
&\le C_q(K, C) \left(\int_{-n}^{-n+1} \| \psi_2(s) \|^p \, ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Then

$$\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) \| Y(t) \| \, dt \le \frac{C_q(K,C)}{\mu(T,\rho)} \int_{-T}^{T} \left(\int_{-n}^{-n+1} \| \psi_2(s) \|^p \, ds \right)^{\frac{1}{p}} dt.$$

Consequently, $Y_n \in PAA_0(\mathbb{R}, \mathbf{X}, \rho)$ since $\psi_2^b \in PAA_0(\mathbb{R}, L^p([0,1], \mathbf{X}), \rho)$. From $Y_n \in PAA_0(\mathbb{R}, \mathbf{X}, \rho)$ and

$$\begin{aligned}
\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) \| Y(t) \| \, dt &\le \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t) \left\| Y(t) - \sum_{n=1}^{N} Y_n \right\| dt \\
&\quad + \sum_{n=1}^{N} \frac{1}{\mu(T,\rho)} \int_{-T}^{t} \rho(t) \| Y_n(t) \| \, dt,
\end{aligned}$$

it follows that $Y(t) \in PAA_0(\mathbb{R}, \mathbf{X}, \rho)$. Therefore, $u \in WPAA(\mathbb{R}, \mathbf{X}, \rho)$. □

4 Weighted pseudo almost automorphic mild solutions

We introduce the definition of mild solution to problem (2.1) (or (1.1)).

Definition 4.1 Let $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ be in $X^{1/2} \times X$. We say that $\left[\begin{smallmatrix} u(\cdot) \\ v(\cdot) \end{smallmatrix}\right] : \mathbb{R}^+ \to X^{1/2} \times X$ is a mild solution to (2.1) if it satisfies

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = e^{-\mathcal{A}_{1/2}(t)} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t e^{-\mathcal{A}_{1/2}(t-s)} F\left(s, \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} \right) ds \quad (t \ge 0).$$

The next theorem is the main result of this paper.

Theorem 4.2 *Let $f : \mathbb{R}^+ \times X^{1/2} \times X \to X$ be a weighted Stepanov-like pseudo almost automorphic function as in Lemma 3.3, and assume that there exists a locally integrable function $L_f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\|_X \le L_f(t)\big[\|u_1 - u_2\|_{X^{1/2}} + \|v_1 - v_2\|_X\big] \tag{4.1}$$

for all $[\begin{smallmatrix} u_i \\ v_i \end{smallmatrix}] \in X^{1/2} \times X$, $i = 1,2$, and each $t \ge 0$. If

$$K \sup_{t\ge 0} \int_0^t e^{-C(t-s)} L_f(s)\, ds < 1, \tag{4.2}$$

where K and C are given in (2.4), then (2.1) (and hence (1.1)) has a unique weighted pseudo almost automorphic mild solution.

Proof We define the map $\mathfrak{F} : WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X) \to WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X)$ by the expression

$$\mathfrak{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)(t) = e^{-\mathcal{A}_{1/2}t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t e^{-\mathcal{A}_{1/2}(t-s)} f\left(s, \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}\right) ds, \quad t \in \mathbb{R}^+, \tag{4.3}$$

where $[\begin{smallmatrix} u(\cdot) \\ v(\cdot) \end{smallmatrix}]$ is a weighted pseudo almost automorphic function.

Since $f \in WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X) \subset S^p WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X)$, by Lemmas 3.3 and 3.4 we have that

$$\mathfrak{G}(t) = \int_0^t e^{-\mathcal{A}_{1/2}(t-s)} f\left(s, \begin{bmatrix} u_{aa}(s) \\ v_{aa}(s) \end{bmatrix}\right) ds \in WPAA\big(\mathbb{R}^+ \times \big(X^{1/2} \times X\big), X\big). \tag{4.4}$$

On the other hand, by (2.4) we have that

$$\mathfrak{B}(t) = T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in C_0\big(\mathbb{R}^+, X^{1/2} \times X\big). \tag{4.5}$$

Hence $\mathfrak{B} \in WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X)$. In consequence, we get that $\mathfrak{F}(WPAA(\mathbb{R}^+ \times (X^{1/2} \times X)) \subset WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X)$. It is enough to show that the operator $\mathfrak{F}$ has a unique fixed point in $WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X)$. For this, we consider that $[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}], [\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}] \in WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X)$. We can deduce that

$$\begin{aligned}
&\left\| \mathfrak{F}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \mathfrak{F}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_\infty \\
&\quad = \sup_{t>0} \left\| \mathfrak{F}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}(t) - \mathfrak{F}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}(t) \right\|_{X^{1/2}\times X} \\
&\quad \le K \sup_{t\ge 0} \int_0^t e^{-C(t-s)} \cdot \big\|f\big(s, u_1(s), v_1(s)\big) - f\big(s, u_2(s), v_2(s)\big)\big\|_X \, ds \\
&\quad \le K \sup_{t\ge 0} \int_0^t e^{-C(t-s)} L_f(s) \cdot \big[\big\|u_1(s) - u_2(s)\big\|_{X^{1/2}} + \big\|v_1(s) - v_2(s)\big\|_X\big]\, ds
\end{aligned}$$

$$\leq \left(K \sup_{t\geq 0} \int_0^t e^{-C(t-s)} L_f(s)\, ds\right) \left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_\infty .$$

By the contraction principle, $\mathfrak{F}$ has a unique fixed point in $WPAA(\mathbb{R}^+ \times (X^{1/2} \times X), X)$. This completes the proof. □

Next, we give an application to our main result.

Example 4.3 Suppose that $h, g \in C(\mathbb{R}, \mathbb{R})$, $a : \mathbb{R} \to \mathbb{R}$ $(b : \mathbb{R}^+ \to \mathbb{R})$ is a bounded continuous function, $\nu, \mu \in \mathbb{R}$, $\delta > 0$ and $\rho_1, \rho_2 \in (1, +\infty)$. In a bounded smooth domain $\Omega \subseteq \mathbb{R}^N$, we consider the following partial differential equation:

$$\begin{aligned} &u_{tt} + \nu b(t)u_t + \Delta^2 u - \delta u_t = \mu a(t)\bigl(h(u)\nabla \cdot u + g(u)\Delta u\bigr), \qquad x \in \Omega, \quad t \geq 0, \\ &u = \Delta u = 0, \qquad x \in \partial\Omega, \qquad t > 0, \\ &u(0,x) = u_0(x), \qquad u_t(0,x) = v_0(x), \quad x \in \Omega, \end{aligned} \tag{4.6}$$

where h and g satisfy the following growth conditions:

$$\begin{aligned} &\bigl|h(s_1) - h(s_2)\bigr| \leq C_h|s_1 - s_2|\bigl(1 + |s_1|^{\rho_1 - 1} + |s_2|^{\rho_1 - 1}\bigr), \quad s_1, s_2 \in \mathbb{R}, \\ &\bigl|g(s_1) - g(s_2)\bigr| \leq C_g|s_1 - s_2|\bigl(1 + |s_1|^{\rho_2 - 1} + |s_2|^{\rho_1 - 1}\bigr), \end{aligned} \tag{4.7}$$

where C_h and C_g are positive constants. To model this problem in the abstract form (1.1), we set that $\eta = \delta/2$, $p > N/2$, the operator A is defined in $L^p(\Omega)$ by $Au = \Delta_D^2$ (Δ_D is the Dirichlet Laplacian in Ω) on the domain

$$D\bigl(\Delta_D^2\bigr) = \bigl\{\phi \in H_p^4(\Omega) : \phi = \Delta\phi = 0 \text{ on } \partial\Omega\bigr\}, \tag{4.8}$$

where $H_p^4(\Omega) = W^{4,p}(\Omega)$ is the standard Sobolev space. With this specification, problem (4.6) will fall into the abstract formulation (1.1). Since $A^{1/2} = -\Delta_D$, we can choose the angle ψ for the sector

$$\Sigma_{\psi/2} = \left\{\lambda \in \mathbb{C} : \frac{\psi}{2} \leq |\arg\lambda| \leq \pi \text{ with } \psi \in (0, \pi/2)\right\} \tag{4.9}$$

as small as needed, and therefore (see [2], Example 4.3) (η, A) will be an admissible pair for any $\eta > 0$. From [21, Sect. 3], we get that

$$\bigl[L^p(\Omega)\bigr]^{1/2} = \bigl\{\phi \in H_p^4(\Omega) : \phi = \Delta\phi = 0 \text{ on } \partial\Omega\bigr\}. \tag{4.10}$$

We define $f : \mathbb{R}^+ \times [L^p(\Omega)]^{1/2} \times L^p(\Omega) \longrightarrow L^p(\Omega)$ by

$$\begin{aligned} &f(t, \varphi_1, \varphi_2) = \mu a(t)\bigl(h^e(\varphi_1)\nabla \cdot \varphi_1 + g^e(\varphi_1)\Delta\varphi_1\bigr) - \nu b(t)\varphi_2 \\ &\quad t \in \mathbb{R}^+, \varphi_1 \in \bigl[L^p(\Omega)\bigr]^{1/2}, \varphi \in L^p(\Omega), \end{aligned} \tag{4.11}$$

where θ^e is the Nemytskii operator associated with θ, and $\nabla\varphi$ represents the divergence of φ. For $\varphi_1 \in [L^p(\Omega)]^{1/2}$ and $\varphi_2 \in L^p(\Omega)$, using Minkowski's and Sobolev embedding, we have the estimate

$$\begin{aligned}\|f(t,\varphi_1,\varphi_2)\|_{L^p(\Omega)} &\le |\mu|\big|a(t)\big|\big(\|h^e(\varphi_1)\|_{L^\infty(\Omega)} + \|g^e(\varphi_1)\|_{L^\infty(\Omega)}\big)\\ &\quad\times \|\varphi_1\|_{H_p^2(\Omega)} + |\nu|\big|b(t)\big|\|\varphi_2\|_{L^p(\Omega)},\end{aligned}$$

whence f is well defined. Let K be a compact subset of $[L^p(\Omega)]^{1/2} \times L^p(\Omega)$. For $\varphi_1 \in [L^p(\Omega)]^{1/2}$ and $\varphi_2 \in L^p(\Omega)$, we set that

$$\begin{aligned}&f_{aa}(t,\varphi_1,\varphi_2) = \mu a(t)\big(h^e(\varphi_1)\nabla\cdot\varphi_1 + g^e(\varphi_1)\Delta\varphi_1\big),\\ &\Phi_f(t,\varphi_1,\varphi_2) = \nu b(t)\varphi_2,\\ &M_K := \sup\left\{\left\|\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}\right\|_{[L^p(\Omega)]^{1/2}\times L^p(\Omega)} : \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} \in K\right\}.\end{aligned}$$

We have the following estimates:

$$\|\Phi_f(t,\varphi_1,\varphi_2)\|_{L^p(\Omega)} \le |\nu|\big|b(t)\big|M_K, \tag{4.12}$$

$$\begin{aligned}&\|f_{aa}(t+\tau,\varphi_1,\varphi_2) - f_{aa}(t,\varphi_1,\varphi_2)\|_{L^p(\Omega)}\\ &\quad\le |\mu|\big|a(t+\tau) - a(t)\big|\\ &\qquad\times\big(\|h^e(\varphi_1)\|_{L^\infty(\Omega)} + \|g^e(\varphi_1)\|_{L^\infty(\Omega)}\big)\|\varphi_1\|_{H_p^2(\Omega)}\\ &\quad\le \widetilde{C}|\mu|\big|a(t+\tau) - a(t)\big|\\ &\qquad\times\big(1 + \|\varphi\|_{H_p^2(\Omega)} + \|\varphi_1\|^{\rho_1}_{H_p^2(\Omega)} + \|\varphi_1\|^{\rho_2}_{H_p^2(\Omega)}\big)\|\varphi_1\|_{H_p^2(\Omega)}\\ &\quad\le \widetilde{C}(K)\big|a(t+\tau) - a(t)\big|,\end{aligned} \tag{4.13}$$

where $\widetilde{C}(K)$ is a constant depending on K. Now, if $a \in AA(\mathbb{R})$, from (4.13) we get $f_{aa} \in AA(\mathbb{R} \times [L^p(\Omega)]^{1/2} \times L^p(\Omega), L^p(\Omega))$. On the other hand, if $b \in PAA_0(\mathbb{R}^+,\mathbb{R})$, from (4.12) we get

$$\Phi_f \in PAA_0\big(\mathbb{R}^+ \times \big[L^p(\Omega)\big]^{1/2} \times L^p(\Omega), L^p(\Omega), \rho\big). \tag{4.14}$$

Hence

$$f \in WPAA\big(\mathbb{R}^+ \times \big[L^p(\Omega)\big]^{1/2} \times L^p(\Omega), L^p(\Omega), \rho\big). \tag{4.15}$$

Applying Theorem 4.2, we have the following result.

Proposition 4.4 *Under the previous conditions, if $a \in AA(\mathbb{R})$, $b \in PAA_0(\mathbb{R}^+,\mathbb{R})$, and $|\mu| + |\nu|$ is small enough, then problem (4.6) has a weighted pseudo almost automorphic mild solution.*

Proof For the previous results, it suffices to show that f satisfies (4.1). In fact, for $\varphi_1, \varphi_2 \in [L^p(\Omega)]^{1/2}$ and $\widetilde{\varphi}_1, \widetilde{\varphi}_2 \in L^p(\Omega)$, we have

$$\|f(t,\varphi_1,\widetilde{\varphi}_1) - f(t,\varphi_2,\widetilde{\varphi}_2)\|_{L^p(\Omega)}$$

$$\begin{aligned}
&\leq \left|\mu a(t)\right| \left\| \left[\left(h^e(\varphi_1)\nabla \cdot \varphi_1 - h^e(\varphi_2)\nabla \cdot \varphi_2\right)\right]\right\|_{L^p(\Omega)} \\
&\quad + \left|\mu a(t)\right| \left\| \left[\left(g^e(\varphi_1)\Delta \cdot \varphi_1 - g^e(\varphi_2)\Delta \cdot \varphi_2\right)\right]\right\|_{L^p(\Omega)} \\
&\quad + \left|\nu b(t)\right| \left\| (\widetilde{\varphi}_1 - \widetilde{\varphi}_2)\right\|_{L^p(\Omega)}. \qquad (4.16)
\end{aligned}$$

However, by (4.7) we have

$$\begin{aligned}
&\left\| \left(h^e(\varphi_1) - h^e(\varphi_2)\right)\nabla \cdot \varphi_1 \right\|_{L^p(\Omega)} \\
&\quad \leq C_h \|\varphi_1 - \varphi_2\|_{H^2_p(\Omega)} \\
&\qquad \times \left(1 + \|\varphi_1\|^{\rho_1 - 1}_{H^2_p(\Omega)} + \|\varphi_2\|^{\rho_1 - 1}_{H^2_p(\Omega)}\right) \cdot \|\varphi_1\|_{H^2_p(\Omega)}, \qquad (4.17)\\
&\left\| h^e(\varphi_2)\nabla \cdot (\varphi_1 - \varphi_2)\right\|_{L^p(\Omega)} \leq C_h\left(1 + \|\varphi_2\|_{H^2_p(\Omega)} + \|\varphi_2\|^{\rho_1}_{H^2_p(\Omega)}\right) \cdot \|\varphi_1 - \varphi_2\|_{H^2_p(\Omega)}, \qquad (4.18)\\
&\left\| \left(g^e(\varphi_1) - g^e(\varphi_2)\right)\Delta \cdot \varphi_1 \right\|_{L^p(\Omega)} \\
&\quad \leq C_g \|\varphi_1 - \varphi_2\|_{H^2_p(\Omega)} \\
&\qquad \times \left(1 + \|\varphi_1\|^{\rho_2 - 1}_{H^2_p(\Omega)} + \|\varphi_2\|^{\rho_2 - 1}_{H^2_p(\Omega)}\right) \cdot \|\varphi_1\|_{H^2_p(\Omega)} \qquad (4.19)
\end{aligned}$$

and

$$\left\| g^e(\varphi_2)\Delta \cdot (\varphi_1 - \varphi_2)\right\|_{L^p(\Omega)} \leq C_g\left(1 + \|\varphi_2\|_{H^2_p(\Omega)} + \|\varphi_2\|^{\rho_2}_{H^2_p(\Omega)}\right) \cdot \|\varphi_1 - \varphi_2\|_{H^2_p(\Omega)}. \qquad (4.20)$$

Now, if

$$\|\varphi_i\|_{[L^p(\Omega)]^{1/2}} + \|\widetilde{\varphi}_i\|_{L^p(\Omega)} \leq r, \quad i = 1, 2,$$

and

$$L_f(r) = \widetilde{C}\left(|\mu| + |\nu|\right)\left(\|a\|_\infty + \|b\|_\infty\right) \times \left(1 + 2r + 3\left(r^{\rho_1} + r^{\rho_2}\right)\right),$$

where $\widetilde{C}$ is a positive constant independent of $\varphi_1, \varphi_2 \in [L^p(\Omega)]^{1/2}$.

By (4.16) to (4.20) we have that f satisfies ((4.1) with the L_f give above. Hence the conclusion of proposition follows. □

Acknowledgements
Thanks to Colciencias.

Funding
This work is partially supported by Colciencias, Grant No. 121556933876 with contract No. 0793-2013.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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Received: 25 May 2018 Accepted: 21 August 2018 Published online: 05 September 2018

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