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INT-SOFT IDEALS OF PSEUDO MV-ALGEBRAS

Abstract

The notion of (implicative) int-soft ideal in a pseudo MV-algebra is introduced, and related properties are investigated. Conditions for a soft set to be an int-soft ideal are provided. Characterizations of (implicative) int-soft ideal are considered. The extension property for implicative int-soft ideal is established.

Keywords: int-soft ideal, implicative int-soft ideal.

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1. Introduction

MV-algebras have been introduced by Chang to prove the completeness theorem for the infinite-valued propositional calculus developed by Łukasiewicz. As a non-commutative generalization of MV-algebras, the pseudo MV-algebra has been introduced by Georgescu et al. [13] and Rachunek [19], respectively. Walendziak [20] studied (implicative) ideals in pseudo MV-algebras. A soft set theory is introduced by Molodtsov [18], and Çağman et al. [9] provided new definitions and various results on soft set theory. Jun et al. [14], [2], [3] have discussed soft set theory in residuated lattices. Jun and Park [17], Bordbar [1], [4], [5], [6], [7] and [8] studied applications of soft sets in ideal theory of BCK/BCI-algebras. Jun et al. [15, 16] introduced the notion of intersectional soft sets, and considered its applications to BCK/BCI-algebras.

In this paper, we introduce the notion of (implicative) int-soft ideal in a pseudo MV-algebra, and investigate the related properties. We provide

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conditions for a soft set to be an int-soft ideal. We consider characterizations of (implicative) int-soft ideal and establish the extension property for implicative int-soft ideal.

2. Preliminaries

Let $\mathcal{M} := (M, \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. We set a new binary operation $\odot$ on M via $x \odot y = (y^- \oplus x^-)^\sim$ for all $x, y \in M$. We will write $x \oplus y \odot z$ instead of $x \oplus (y \odot z)$, that is, the operation "$\odot$" is prior to the operation "$\oplus$".

A *pseudo MV-algebra* is an algebra $\mathcal{M} := (M, \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \tag{2.1}$$
$$x \oplus 0 = 0 \oplus x = x, \tag{2.2}$$
$$x \oplus 1 = 1 \oplus x = x, \tag{2.3}$$
$$1^\sim = 0,\ 1^- = 0, \tag{2.4}$$
$$(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-, \tag{2.5}$$
$$x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x, \tag{2.6}$$
$$x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y, \tag{2.7}$$
$$(x^-)^\sim = x \tag{2.8}$$

for all $x, y, z \in M$. If we define

$$(\forall x, y \in M) \left(x \leq y \ \Leftrightarrow \ x^- \oplus y = 1\right), \tag{2.9}$$

then $\leq$ is a partial order such that M is a bounded distributive lattice with the join $x \vee y$ and the meet $x \wedge y$ given by

$$x \vee y = x \oplus x^\sim \odot y = x \odot y^- \oplus y, \tag{2.10}$$
$$x \wedge y = x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y, \tag{2.11}$$

respectively.

For any pseudo MV-algebra $\mathcal{M}$, the following properties are valid (see [13]).

$$x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y, \tag{2.12}$$
$$(x \vee y)^- = x^- \wedge y^-, \tag{2.13}$$
$$x \leq y \;\Rightarrow\; z \odot x \leq z \odot y,\; x \odot z \leq y \odot z, \tag{2.14}$$
$$z \oplus (x \wedge y) = (z \oplus x) \wedge (z \oplus y), \tag{2.15}$$
$$z \odot (x \oplus y) \leq z \odot x \oplus y, \tag{2.16}$$
$$(x^{\sim})^- = x, \tag{2.17}$$
$$x \odot 1 = x = 1 \odot x, \tag{2.18}$$
$$x \oplus x^{\sim} = 1 = x^- \oplus x, \tag{2.19}$$
$$x \odot x^- = 0 = x^{\sim} \odot x, \tag{2.20}$$

for all $x, y, z \in M$.

A subset I of a pseudo MV-algebra $\mathcal{M}$ is called an *ideal* of $\mathcal{M}$ (see [20]) if it satisfies:

$$0 \in I, \tag{2.21}$$
$$(\forall x, y \in M)\,(x, y \in I \;\Rightarrow\; x \oplus y \in I)\,, \tag{2.22}$$
$$(\forall x, y \in M)\,(x \in I,\; y \leq x \;\Rightarrow\; y \in I)\,. \tag{2.23}$$

An ideal I of a pseudo MV-algebra $\mathcal{M}$ is said to be *implicative* (see [20]) if it satisfies:

$$(\forall x, y, z \in M)\,(x \odot y \odot z \in I,\; z^{\sim} \odot y \in I \;\Rightarrow\; x \odot y \in I)\,. \tag{2.24}$$

A soft set theory is introduced by Molodtsov [18]. Çağman et al. [9] provided new definitions and various results on soft set theory.

Let $\mathcal{P}(U)$ denote the power set of an initial universe set U and $A \subseteq E$ where E is a set of parameters.

A *soft set* $(\tilde{f}, A)$ over U in E (see [9, 18]) is defined to be a set of ordered pairs

$$(\tilde{f}, A) := \left\{ \left(x, \tilde{f}(x)\right) : x \in E,\; \tilde{f}(x) \in \mathcal{P}(U) \right\},$$

where $\tilde{f} : E \to \mathcal{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$.

The function $\tilde{f}$ is called an approximate function of the soft set $(\tilde{f}, A)$.

For a soft set $(\tilde{f}, A)$ over U in E, the set $(\tilde{f}, A)_\gamma = \left\{ x \in A \mid \gamma \subseteq \tilde{f}(x) \right\}$ is called the *γ-inclusive set* of $(\tilde{f}, A)$.

Assume that E has a binary operation $\hookrightarrow$. For any non-empty subset A of E, a soft set $(\tilde{f}, A)$ over U in E is said to be *intersectional* over U (see [15, 16]) if its approximate function $\tilde{f}$ satisfies:

$$(\forall x, y \in A)\left(x \hookrightarrow y \in A \Rightarrow \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x \hookrightarrow y)\right). \tag{2.25}$$

3. Int-soft ideals

In what follows, we take a pseudo MV-algebra $\mathcal{M}$ as a set of parameters.

DEFINITION 3.1. A soft set $(\tilde{f}, M)$ over U in a pseudo MV-algebra $\mathcal{M}$ is called an *int-soft ideal* of $\mathcal{M}$ if the following conditions hold

$$(\forall x, y \in M)\left(\tilde{f}(x \oplus y) \supseteq \tilde{f}(x) \cap \tilde{f}(y)\right), \tag{3.1}$$

$$(\forall x, y \in M)\left(y \leq x \Rightarrow \tilde{f}(y) \supseteq \tilde{f}(x)\right). \tag{3.2}$$

It is easily seen that (3.2) implies

$$(\forall x \in M)\left(\tilde{f}(0) \supseteq \tilde{f}(x)\right). \tag{3.3}$$

EXAMPLE 3.2. Let $M = \{(1, y) \in \mathbb{R}^2 \mid y \geq 0\} \cup \{(2, y) \in \mathbb{R}^2 \mid y \leq 0\}$. For any $(a, b), (c, d) \in M$, we define operations $\oplus$, $^-$ and $^\sim$ as follows:

$$(a, b) \oplus (c, d) = \begin{cases} (1, b + d) & \text{if } a = c = 1, \\ (2, ad + b) & \text{if } ac = 2 \text{ and } ad + b \leq 0, \\ (2, 0) & \text{otherwise,} \end{cases}$$

$$(a, b)^- = \left(\tfrac{2}{a}, -\tfrac{2b}{a}\right) \text{ and } (a, b)^\sim = \left(\tfrac{2}{a}, -\tfrac{b}{a}\right).$$

Then $\mathcal{M} := (M, \oplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ is a pseudo MV-algebra where $\mathbf{0} = (1, 0)$ and $\mathbf{1} = (2, 0)$ (see [11]). Let $A = \{(1, y) \in \mathbb{R}^2 \mid y > 0\}$ and $B = \{(2, y) \in \mathbb{R}^2 \mid y < 0\}$. Define a soft set $(\tilde{f}, M)$ over $U = \mathbb{R}$ in $\mathcal{M}$ by

$$\tilde{f} : M \to \mathcal{P}(U),\ x \mapsto \begin{cases} 3\mathbb{R} & \text{if } x = \mathbf{0}, \\ 3\mathbb{Z} & \text{if } x \in A, \\ 3\mathbb{N} & \text{if } x \in B \cup \{\mathbf{1}\}. \end{cases}$$

It is easily checked that $(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$.

EXAMPLE 3.3. For an ideal A of a pseudo MV-algebra $\mathcal{M}$, let $\left(\tilde{f}_A, M\right)$ be a soft set over $U = \mathbb{Z}$ in $\mathcal{M}$ given as follows:

$$\tilde{f}_A : M \to \mathcal{P}(U),\ x \mapsto \begin{cases} 2\mathbb{Z} & \text{if } x \in A, \\ 4\mathbb{N} & \text{otherwise.} \end{cases}$$

Then $\left(\tilde{f}_A, M\right)$ is an int-soft ideal of $\mathcal{M}$.

PROPOSITION 3.4. *For any int-soft ideal $(\tilde{f}, M)$ of a pseudo MV-algebra $\mathcal{M}$, we have the following properties.*

(1) $\tilde{f}(x \odot y) \supseteq \tilde{f}(x) \cap \tilde{f}(y)$,
(2) $\tilde{f}(x \wedge y) \supseteq \tilde{f}(x) \cap \tilde{f}(y)$,
(3) $\tilde{f}(x \oplus y) = \tilde{f}(x) \cap \tilde{f}(y)$

for all $x, y \in M$.

PROOF: Note that $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$ for all $x, y \in M$. Using (3.1) and (3.2), we have

$$\tilde{f}(x \odot y) \supseteq \tilde{f}(x \wedge y) \supseteq \tilde{f}(x \vee y) \supseteq \tilde{f}(x \oplus y) \supseteq \tilde{f}(x) \cap \tilde{f}(y).$$

Since $x \leq x \vee y \leq x \oplus y$ and $y \leq x \vee y \leq x \oplus y$ for all $x, y \in M$, it follows from (3.2) that $\tilde{f}(x \oplus y) \subseteq \tilde{f}(x)$ and $\tilde{f}(x \oplus y) \subseteq \tilde{f}(y)$. Hence $\tilde{f}(x \oplus y) \subseteq \tilde{f}(x) \cap \tilde{f}(y)$. This completes the proof. □

THEOREM 3.5. *Let $(\tilde{f}, M)$ be a soft set over U in a pseudo MV-algebra $\mathcal{M}$. Then $(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$ if and only if it satisfies* (3.1) *and*

$$(\forall x, y \in M) \left(\tilde{f}(x \wedge y) \supseteq \tilde{f}(x)\right). \tag{3.4}$$

PROOF: Let $(\tilde{f}, M)$ be an int-soft ideal of $\mathcal{M}$ and let $x, y \in M$. Since $x \wedge y \leq x$, it follows from (3.2) that $\tilde{f}(x \wedge y) \supseteq \tilde{f}(x)$. Suppose that $(\tilde{f}, M)$ satisfies (3.1) and (3.4). Let $x, y \in M$ be such that $y \leq x$. Then $x \wedge y = y$, and so $\tilde{f}(y) = \tilde{f}(x \wedge y) \supseteq \tilde{f}(x)$ by (3.4). Therefore $(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$. □

PROPOSITION 3.6. *Every int-soft ideal $(\tilde{f}, M)$ of a pseudo MV-algebra $\mathcal{M}$ satisfies the following inclusion.*

$$(\forall x, y \in M) \left(\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x^{\sim} \odot y)\right). \tag{3.5}$$

PROOF: Note that $y \leq x \vee y = x \oplus x^{\sim} \odot y$ for all $x, y \in M$. Using (3.1) and (3.2) imply that $\tilde{f}(y) \supseteq \tilde{f}(x \oplus x^{\sim} \odot y) \supseteq \tilde{f}(x) \cap \tilde{f}(x^{\sim} \odot y)$ for all $x, y \in M$. □

PROPOSITION 3.7. *Every int-soft ideal $(\tilde{f}, M)$ of a pseudo MV-algebra $\mathcal{M}$ satisfies the following inclusion.*

$$(\forall x, y \in M)\left(\tilde{f}(x \odot y) \supseteq \tilde{f}(x \odot y \odot y) \cap \tilde{f}(y \wedge y^{\sim})\right). \tag{3.6}$$

PROOF: Using (2.18), (2.19) and (2.16), we have $x \odot y = (x \odot y) \odot 1 = (x \odot y) \odot (y \oplus y^{\sim}) \leq (x \odot y) \odot y \oplus y^{\sim}$ for all $x, y \in M$. It follows from (2.15) that

$$\begin{aligned} x \odot y &\leq y \wedge (x \odot y \odot y \oplus y^{\sim}) \\ &\leq (x \odot y \odot y \oplus y) \wedge (x \odot y \odot y \oplus y^{\sim}) \\ &= x \odot y \odot y \oplus (y \wedge y^{\sim}). \end{aligned}$$

Using (3.2) and (3.1), we conclude that $\tilde{f}(x \odot y) \supseteq \tilde{f}(x \odot y \odot y \oplus (y \wedge y^{\sim})) \supseteq \tilde{f}(x \odot y \odot y) \cap \tilde{f}(y \wedge y^{\sim})$ for all $x, y \in M$. □

PROPOSITION 3.8. *Let $(\tilde{f}, M)$ be a soft set over U in a pseudo MV-algebra $\mathcal{M}$ satisfying two conditions* (3.3) *and* (3.5). *Then $(\tilde{f}, M)$ satisfies* (3.2) *and*

$$(\forall x, y \in M)\left(\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(y \odot x^{-})\right). \tag{3.7}$$

PROOF: Let $x, y \in M$ be such that $y \leq x$. Using (2.14) and (2.20), we get $x^{\sim} \odot y \leq x^{\sim} \odot x = 0$ and thus $x^{\sim} \odot y = 0$. It follows from (3.3) and (3.5) that

$$\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x^{\sim} \odot y) = \tilde{f}(x) \cap \tilde{f}(0) = \tilde{f}(x). \tag{3.8}$$

Hence (3.2) is valid. Since

$$(y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x) \leq (y \odot x^{-})^{\sim} \odot (y \odot x^{-}) \oplus x = 0 \oplus x = x \tag{3.9}$$

for all $x, y \in M$, we have $\tilde{f}(x) \subseteq \tilde{f}((y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x))$ by (3.2). Now since

$$x^{\sim} \odot y \leq x \oplus x^{\sim} \odot y = y \odot x^{-} \oplus x \tag{3.10}$$

for all $x, y \in M$, we get $\tilde{f}(x^{\sim} \odot y) \supseteq \tilde{f}(y \odot x^{-} \oplus x)$ by (3.2), and so

$$\begin{aligned} \tilde{f}(y) &\supseteq \tilde{f}(x) \cap \tilde{f}(x^{\sim} \odot y) \supseteq \tilde{f}(x) \cap \tilde{f}(y \odot x^{-} \oplus x) \\ &\supseteq \tilde{f}(x) \cap \left(\tilde{f}(y \odot x^{-}) \cap \tilde{f}((y \odot x^{-})^{\sim} \odot (y \odot x^{-} \oplus x))\right) \\ &\supseteq \tilde{f}(x) \cap \left(\tilde{f}(y \odot x^{-}) \cap \tilde{f}(x)\right) = \tilde{f}(x) \cap \tilde{f}(y \odot x^{-}) \end{aligned} \tag{3.11}$$

for all $x, y \in M$. □

We provide conditions for a soft set to be an int-soft ideal.

Proposition 3.9. *If a soft set $(\tilde{f}, M)$ over U in a pseudo MV-algebra $\mathcal{M}$ satisfies two conditions* (3.3) *and* (3.7), *then it is an int-soft ideal of $\mathcal{M}$.*

Proof: Let $x, y \in M$ be such that $y \leq x$. Then $y \odot x^- \leq x \odot x^- = 0$ by (2.14) and (2.20), and so $y \odot x^- = 0$. It follows from (3.3) and (3.7) that

$$\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(y \odot x^-) = \tilde{f}(x) \cap \tilde{f}(0) = \tilde{f}(x). \tag{3.12}$$

Note that $(x \oplus y) \odot y^- = (x \oplus (y^-)^\sim) \odot y^- = x \wedge y^- \leq x$ for all $x, y \in M$. Hence

$$\tilde{f}(x \oplus y) \supseteq \tilde{f}(y) \cap \tilde{f}((x \oplus y) \odot y^-) \supseteq \tilde{f}(y) \cap \tilde{f}(x). \tag{3.13}$$

Therefore $(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$. □

Combining Propositions 3.6, 3.8 and 3.9, we have the following characterization of an int-soft ideal of a pseudo MV-algebra.

Theorem 3.10. *For a soft set $(\tilde{f}, M)$ over U in a pseudo MV-algebra $\mathcal{M}$, the following are equivalent.*

(1) *$(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$.*
(2) *$(\tilde{f}, M)$ satisfies the conditions* (3.3) *and* (3.5).
(3) *$(\tilde{f}, M)$ satisfies the conditions* (3.3) *and* (3.7).

Theorem 3.11. *Let $(\tilde{f}, M)$ be a soft set over U in a pseudo MV-algebra $\mathcal{M}$ that satisfies* (3.3) *and*

$$(\forall x, y, z \in M)\left(\tilde{f}(x \odot y) \supseteq \tilde{f}(x \odot y \odot z) \cap \tilde{f}(z^\sim \odot y)\right). \tag{3.14}$$

Then $(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$, and satisfies the following conditions:

$$(\forall x, y \in M)\left(\tilde{f}(x \odot y) = \tilde{f}(x \odot y \odot y)\right), \tag{3.15}$$

$$(\forall x \in M)(\forall n \in \mathbb{N})(\tilde{f}(x) = \tilde{f}(x^n)) \tag{3.16}$$

where $x^n = x^{n-1} \odot x = x \odot x^{n-1}$ and $x^0 = 1$.

Proof: Taking $x = y$, $y = 1$ and $z = x^-$ in (3.14) and using (2.8) and (2.18), we have

$$\tilde{f}(y) = \tilde{f}(y \odot 1) \supseteq \tilde{f}(y \odot 1 \odot x^-) \cap \tilde{f}((x^-)^\sim \odot 1) = \tilde{f}(y \odot x^-) \cap \tilde{f}(x). \tag{3.17}$$

It follows from Theorem 3.10 that $(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$. If we put $z = y$ in (3.14) and use (2.20) and (3.3), then

$$\tilde{f}(x \odot y) \supseteq \tilde{f}(x \odot y \odot y) \cap \tilde{f}(y^\sim \odot y) = \tilde{f}(x \odot y \odot y) \cap \tilde{f}(0) = \tilde{f}(x \odot y \odot y). \tag{3.18}$$

Since $x \odot y \odot y \leq x \odot y$ for all $x, y \in M$, we get $\tilde{f}(x \odot y \odot y) \supseteq \tilde{f}(x \odot y)$ by (3.2). Therefore (3.15) is valid. If $n = 1$, then (3.16) is clearly true. If we take $x = 1$ and $y = x$ in (3.15), then

$$\tilde{f}(x) = \tilde{f}(1 \odot x) = \tilde{f}(1 \odot x \odot x) = \tilde{f}(x^2).$$

Now assume that (3.16) is valid for every positive integer $k > 2$. Then

$$\tilde{f}(x^{k+1}) = \tilde{f}(x^{k-1} \odot x \odot x) = \tilde{f}(x^{k-1} \odot x) = \tilde{f}(x^k) = \tilde{f}(x).$$

The mathematical induction shows that (3.16) is valid for every positive integer n. □

Lemma 3.12. *For any soft set $(\tilde{f}, M)$ over U in a pseudo MV-algebra $\mathcal{M}$, the condition* (3.14) *is equivalent to the following condition.*

$$(\forall x, y, z \in M)\left(\tilde{f}(x \odot y) \supseteq \tilde{f}(x \odot y \odot z^-) \cap \tilde{f}(z \odot y)\right). \tag{3.19}$$

Proof: Taking z^- instead of z in (3.14) induces (3.19). If we take $z^\sim$ instead of z in (3.19) and use (2.17), then we have the condition (3.14). □

For any soft set $(\tilde{f}, M)$ over U in a pseudo MV-algebra $\mathcal{M}$, consider the set

$$M_{\tilde{f}} := \{x \in M \mid \tilde{f}(x) = \tilde{f}(0)\}.$$

Theorem 3.13. *If $(\tilde{f}, M)$ is an int-soft ideal of a pseudo MV-algebra $\mathcal{M}$, then the set $M_{\tilde{f}}$ is an ideal of $\mathcal{M}$.*

Proof: Obviously, $0 \in M_{\tilde{f}}$. Let $x, y \in M_{\tilde{f}}$. Then $\tilde{f}(x) = \tilde{f}(0) = \tilde{f}(y)$, and so

$$\tilde{f}(x \oplus y) \supseteq \tilde{f}(x) \cap \tilde{f}(y) = \tilde{f}(0)$$

by (3.1). Combining this with (3.3) induces $\tilde{f}(x \oplus y) = \tilde{f}(0)$, that is, $x \oplus y \in M_{\tilde{f}}$. Let $x, y \in M$ be such that $x \in M_{\tilde{f}}$ and $y \leq x$. Then $\tilde{f}(y) \supseteq \tilde{f}(x) = \tilde{f}(0)$ by (3.2), and thus $\tilde{f}(y) = \tilde{f}(0)$ by (3.3). Hence $y \in M_{\tilde{f}}$. Therefore $M_{\tilde{f}}$ is an ideal of $\mathcal{M}$. □

The converse of Theorem 3.13 is not true in general as seen in the following example:

Example 3.14. Let $\mathcal{M} := (M, \oplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ be a pseudo MV-algebra in Example 3.2. Define a soft set $(\tilde{f}, M)$ over $U = \mathbb{N}$ in $\mathcal{M}$ by

$$\tilde{f} : M \to \mathcal{P}(U),\ x \mapsto \begin{cases} 4\mathbb{N} & \text{if } x = \mathbf{0}, \\ 2\mathbb{N} & \text{if } x \neq \mathbf{0}. \end{cases}$$

Then $M_{\tilde{f}} = \{\mathbf{0}\}$ is an ideal of $\mathcal{M}$ but $(\tilde{f}, M)$ is not an int-soft ideal of $\mathcal{M}$.

Proposition 3.15. *Let $(\tilde{f}, M)$ and $(\tilde{g}, M)$ be soft sets over U in a pseudo MV-algebra $\mathcal{M}$ such that $(\tilde{f}, M)\tilde{\subseteq}(\tilde{g}, M)$, that is, $\tilde{f}(x) \subseteq \tilde{g}(x)$ for all $x \in M$, and $\tilde{f}(0) = \tilde{g}(0)$. If $(\tilde{g}, M)$ satisfies the condition (3.3), then $M_{\tilde{f}} \subseteq M_{\tilde{g}}$.*

Proof: Let $x \in M_{\tilde{f}}$. Then $\tilde{g}(0) = \tilde{f}(0) = \tilde{f}(x) \subseteq \tilde{g}(x)$, which implies from (3.3) that $\tilde{g}(x) = \tilde{g}(0)$. Hence $x \in M_{\tilde{g}}$ and $M_{\tilde{f}} \subseteq M_{\tilde{g}}$. □

Corollary 3.16. *Let $(\tilde{f}, M)$ and $(\tilde{g}, M)$ be soft sets over U in a pseudo MV-algebra $\mathcal{M}$ such that $(\tilde{f}, M)\tilde{\subseteq}(\tilde{g}, M)$, that is, $\tilde{f}(x) \subseteq \tilde{g}(x)$ for all $x \in M$, and $\tilde{f}(0) = \tilde{g}(0)$. If $(\tilde{g}, M)$ is an int-soft ideal of $\mathcal{M}$, then $M_{\tilde{f}} \subseteq M_{\tilde{g}}$.*

Proposition 3.17. *If $(\tilde{f}, M)$ is an int-soft ideal of a pseudo MV-algebra $\mathcal{M}$, then the set*

$$P\left(M_{\tilde{f}}\right) := \{x \in M \mid \tilde{f}(x) \neq \emptyset\}$$

is an ideal of $\mathcal{M}$ when it is non-empty.

Proof: Assume that $P\left(M_{\tilde{f}}\right) \neq \emptyset$. Obviously, $0 \in P\left(M_{\tilde{f}}\right)$. Let $x, y \in P\left(M_{\tilde{f}}\right)$. Then $\tilde{f}(x) \neq \emptyset \neq \tilde{f}(y)$, and so $\tilde{f}(x \oplus y) \supseteq \tilde{f}(x) \cap \tilde{f}(y) \neq \emptyset$ by (3.1), that is, $x \oplus y \in P\left(M_{\tilde{f}}\right)$. Let $x, y \in M$ be such that $x \in P\left(M_{\tilde{f}}\right)$ and $y \leq x$. Then $\tilde{f}(y) \supseteq \tilde{f}(x) \neq \emptyset$ by (3.2), and thus $y \in P\left(M_{\tilde{f}}\right)$. Therefore, $P\left(M_{\tilde{f}}\right)$ is an ideal of $\mathcal{M}$. □

Definition 3.18. An int-soft ideal $(\tilde{f}, M)$ of a pseudo MV-algebra $\mathcal{M}$ is said to be *implicative* if it satisfies the condition (3.14).

Example 3.19. For an implicative ideal A of a pseudo MV-algebra $\mathcal{M}$, let $\left(\tilde{f}_A, M\right)$ be a soft set over $U = \mathbb{R}$ in $\mathcal{M}$ given as follows:

$$\tilde{f}_A : M \to \mathcal{P}(U),\ x \mapsto \begin{cases} 3\mathbb{R} & \text{if } x \in A, \\ 6\mathbb{Z} & \text{otherwise.} \end{cases}$$

Then $\left(\tilde{f}_A, M\right)$ is an implicative int-soft ideal of $\mathcal{M}$.

We consider characterizations of implicative int-soft ideals.

Theorem 3.20. *For an int-soft ideal $(\tilde{f}, M)$ of a pseudo MV-algebra $\mathcal{M}$, the following are equivalent*

(1) $(\tilde{f}, M)$ *is implicative.*

(2) $(\forall x, y \in M)\left(\tilde{f}(x \odot y) = \tilde{f}(x \odot y \odot y)\right)$.

(3) $(\forall x \in M)\left(x^2 = 0 \;\Rightarrow\; \tilde{f}(x) = \tilde{f}(0)\right)$.

(4) $(\forall x \in M)\left(\tilde{f}(x \wedge x^-) = \tilde{f}(0)\right)$.

(5) $(\forall x \in M)\left(\tilde{f}(x \wedge x^\sim) = \tilde{f}(0)\right)$.

Proof: (1) $\Rightarrow$ (2) follows from Theorem 3.11. Assume that $x^2 = 0$ for all $x \in M$. Taking $x = 1$ and $y = x$ in (2) and using (2.18) induces

$$\tilde{f}(x) = \tilde{f}(1 \odot x) = \tilde{f}(1 \odot x \odot x) = \tilde{f}(x^2) = \tilde{f}(0).$$

Suppose that the condition (3) is valid. Since

$$(x \wedge x^-)^2 = (x \wedge x^-) \odot (x \wedge x^-) \leq x \odot x^- = 0$$

by (2.14) and (2.20), we have $(x \wedge x^-)^2 = 0$, and so $\tilde{f}(x \wedge x^-) = \tilde{f}(0)$ by (3). Since $x \wedge x^\sim = x^\sim \wedge x = x^\sim \wedge (x^\sim)^-$ for all $x \in M$, it follows from (4) that $\tilde{f}(x \wedge x^\sim) = \tilde{f}(0)$ for all $x \in M$. Finally, assume that the condition (5) holds. By Proposition 3.7, (5) and (3.3), we have

$$\begin{aligned} \tilde{f}(x \odot y) &\supseteq \tilde{f}(x \odot y \odot y) \cap \tilde{f}(y \wedge y^\sim) \\ &= \tilde{f}(x \odot y \odot y) \cap \tilde{f}(0) = \tilde{f}(x \odot y \odot y) \end{aligned} \tag{3.20}$$

for all $x, y \in M$. Note that

$$x \odot y \odot y \leq x \odot y \odot (z \vee y) = x \odot y \odot (z \oplus z^\sim \odot y) \leq x \odot y \odot z \oplus z^\sim \odot y$$

for all $x, y, z \in M$ by (2.14) and (2.16). It follows from (3.20), (3.2) and (3.1) that

$$\tilde{f}(x \odot y) \supseteq \tilde{f}(x \odot y \odot y) \supseteq \tilde{f}(x \odot y \odot z \oplus z^\sim \odot y) \supseteq \tilde{f}(x \odot y \odot z) \cap \tilde{f}(z^\sim \odot y)$$

for all $x, y, z \in M$. Therefore, $(\tilde{f}, M)$ is an implicative int-soft ideal of $\mathcal{M}$. □

Theorem 3.20 is used in providing an example of implicative int-soft ideal.

Example 3.21. Let $\mathcal{M} := (M, \oplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ be a pseudo MV-algebra in Example 3.2. Define a soft set $(\tilde{f}, M)$ over $U = \mathbb{R}$ in $\mathcal{M}$ by

$$\tilde{f} : M \to \mathcal{P}(U),\ x \mapsto \begin{cases} 3\mathbb{R} & \text{if } x \in A \cup \{\mathbf{0}\}, \\ 3\mathbb{N} & \text{if } x \in B \cup \{\mathbf{1}\} \end{cases}$$

where $A = \{(1, y) \in \mathbb{R}^2 \mid y > 0\}$ and $B = \{(2, y) \in \mathbb{R}^2 \mid y < 0\}$. It is easy to verify that $(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$. Note that $x \wedge x^- \in A \cup \{\mathbf{0}\}$ for all $x \in M$. Hence $\tilde{f}(x \wedge x^-) = 3\mathbb{R} = \tilde{f}(\mathbf{0})$, and so $(\tilde{f}, M)$ is an implicative int-soft ideal of $\mathcal{M}$ by Theorem 3.20.

THEOREM 3.22. *For a soft set $(\tilde{f}, M)$ over U in a pseudo MV-algebra $\mathcal{M}$, the following are equivalent.*

(1) *$(\tilde{f}, M)$ is an implicative int-soft ideal of $\mathcal{M}$.*
(2) *The non-empty γ-inclusive set $(\tilde{f}, M)_\gamma$ is an implicative ideal of $\mathcal{M}$ for all $\gamma \in \mathcal{P}(U)$.*

PROOF: Suppose that $(\tilde{f}, M)$ is an implicative int-soft ideal of $\mathcal{M}$. Let $\gamma \in \mathcal{P}(U)$ be such that $(\tilde{f}, M)_\gamma \neq \emptyset$. Then there exists $x \in (\tilde{f}, M)_\gamma$, and so $\tilde{f}(x) \supseteq \gamma$. It follows from (3.3) that $\tilde{f}(0) \supseteq \tilde{f}(x) \supseteq \gamma$. Hence $0 \in (\tilde{f}, M)_\gamma$. Let $x, y \in (\tilde{f}, M)_\gamma$ for $x, y \in M$. Then $\tilde{f}(x) \supseteq \gamma$ and $\tilde{f}(y) \supseteq \gamma$, which implies from (3.1) that $\tilde{f}(x \oplus y) \supseteq \tilde{f}(x) \cap \tilde{f}(y) \supseteq \gamma$. Thus $x \oplus y \in (\tilde{f}, M)_\gamma$. Let $x, y \in M$ be such that $x \in (\tilde{f}, M)_\gamma$ and $y \leq x$. Then $\tilde{f}(y) \supseteq \tilde{f}(x) \supseteq \gamma$ by (3.2), and so $y \in (\tilde{f}, M)_\gamma$. Hence $(\tilde{f}, M)_\gamma$ is an ideal of $\mathcal{M}$. Let $x, y, z \in M$ be such that $x \odot y \odot z \in (\tilde{f}, M)_\gamma$ and $z^\sim \odot y \in (\tilde{f}, M)_\gamma$. Then $\tilde{f}(x \odot y \odot z) \supseteq \gamma$ and $\tilde{f}(z^\sim \odot y) \supseteq \gamma$. It follows from (3.14) that

$$\tilde{f}(x \odot y) \supseteq \tilde{f}(x \odot y \odot z) \cap \tilde{f}(z^\sim \odot y) \supseteq \gamma$$

and so that $x \odot y \in (\tilde{f}, M)_\gamma$. Therefore, $(\tilde{f}, M)_\gamma$ is an implicative ideal of $\mathcal{M}$.

Conversely, assume that the non-empty γ-inclusive set $(\tilde{f}, M)_\gamma$ is an implicative ideal of $\mathcal{M}$ for all $\gamma \in \mathcal{P}(U)$. For any $x \in M$, let $\tilde{f}(x) = \gamma$. Then $x \in (\tilde{f}, M)_\gamma$. Since $(\tilde{f}, M)_\gamma$ is an ideal of $\mathcal{M}$, we have $0 \in (\tilde{f}, M)_\gamma$ and so $\tilde{f}(0) \supseteq \gamma = \tilde{f}(x)$. For any $x, y \in M$, let $\tilde{f}(x) \cap \tilde{f}(y) = \gamma$. Then $x, y \in (\tilde{f}, M)_\gamma$, and so $x \oplus y \in (\tilde{f}, M)_\gamma$ by (2.22). Hence $\tilde{f}(x \oplus y) \supseteq \gamma = \tilde{f}(x) \cap \tilde{f}(y)$. Let $x, y \in M$ be such that $y \leq x$ and $\tilde{f}(x) = \gamma$. Then $x \in (\tilde{f}, M)_\gamma$, and so $y \in (\tilde{f}, M)_\gamma$ by (2.23). Thus $\tilde{f}(y) \supseteq \gamma = \tilde{f}(x)$. Hence $(\tilde{f}, M)$ is an int-soft ideal of $\mathcal{M}$. For any $x, y, z \in M$, let $\tilde{f}(x \odot y \odot z) \cap \tilde{f}(z^\sim \odot y) = \gamma$. Then $x \odot y \odot z \in (\tilde{f}, M)_\gamma$ and $z^\sim \odot y \in (\tilde{f}, M)_\gamma$. It follows from (2.24) that $x \odot y \in (\tilde{f}, M)_\gamma$ and so that $\tilde{f}(x \odot y) \supseteq \gamma$. Therefore, $(\tilde{f}, M)$ is an implicative int-soft ideal of $\mathcal{M}$. □

Lemma 3.23 ([20]). *An ideal I of a pseudo MV-algebra $\mathcal{M}$ is implicative if and only if the following assertion is valid.*

$$(\forall x \in M)\,(x \wedge x^{\sim} \in I)\,.$$

Theorem 3.24. *If $(\tilde{f}, M)$ is an implicative int-soft ideal of a pseudo MV-algebra $\mathcal{M}$, then the set*

$$M_a := \{x \in M \mid \tilde{f}(x) \supseteq \tilde{f}(a)\}$$

is an implicative ideal of $\mathcal{M}$ for all $a \in M$.

Proof: Since $\tilde{f}(0) \supseteq \tilde{f}(x)$ for all $x \in M$, we have $0 \in M_a$. Let $x, y \in M$ be such that $x \in M_a$ and $y \in M_a$. Then $\tilde{f}(x) \supseteq \tilde{f}(a)$ and $\tilde{f}(y) \supseteq \tilde{f}(a)$. It follows from (3.1) that $\tilde{f}(x \oplus y) \supseteq \tilde{f}(x) \cap \tilde{f}(y) \supseteq \tilde{f}(a)$ and so that $x \oplus y \in M_a$. Let $x, y \in M$ be such that $y \leq x$ and $x \in M_a$. Then $\tilde{f}(y) \supseteq \tilde{f}(x) \supseteq \tilde{f}(a)$ by (3.2), and so $y \in M_a$. Thus M_a is an ideal of $\mathcal{M}$. Note from Theorem 3.20 and (3.3) that $\tilde{f}(x \wedge x^{\sim}) = \tilde{f}(0) \supseteq \tilde{f}(x)$ for all $x \in M$. Hence $x \wedge x^{\sim} \in M_a$. Therefore, M_a is an implicative ideal of $\mathcal{M}$ by Lemma 3.23. □

Corollary 3.25. *If $(\tilde{f}, M)$ is an implicative int-soft ideal of a pseudo MV-algebra $\mathcal{M}$, then the set $M_{\tilde{f}}$ is an implicative ideal of $\mathcal{M}$.*

Proof: Since $\tilde{f}(0) \supseteq \tilde{f}(x)$ for all $x \in \mathcal{M}$, we have $M_{\tilde{f}} = M_0$ which is an implicative ideal of $\mathcal{M}$. □

Theorem 3.26. *If $(\tilde{f}, M)$ is an implicative int-soft ideal of a pseudo MV-algebra $\mathcal{M}$, then the set*

$$P\left(M_{\tilde{f}}\right) := \{x \in M \mid \tilde{f}(x) \neq \emptyset\}$$

is an implicative ideal of $\mathcal{M}$ when it is non-empty.

Proof: Suppose that $(\tilde{f}, M)$ is an implicative int-soft ideal of a pseudo MV-algebra $\mathcal{M}$. If $P\left(M_{\tilde{f}}\right)$ is non-empty, then it is an ideal of $\mathcal{M}$ by Proposition 3.17. Let $x, y, z \in M$ be such that $x \odot y \odot z \in P\left(M_{\tilde{f}}\right)$ and $z^{\sim} \odot y \in P\left(M_{\tilde{f}}\right)$. Then $\tilde{f}(x \odot y \odot z) \neq \emptyset$ and $\tilde{f}(z^{\sim} \odot y) \neq \emptyset$. It follows from (3.14) that

$$\tilde{f}(x \odot y) \supseteq \tilde{f}(x \odot y \odot z) \cap \tilde{f}(z^{\sim} \odot y) \neq \emptyset$$

and so that $\tilde{f}(x \odot y) \neq \emptyset$, that is, $x \odot y \in P\left(M_{\tilde{f}}\right)$. Therefore, $P\left(M_{\tilde{f}}\right)$ is an implicative ideal of $\mathcal{M}$. □

THEOREM 3.27. (Extension property for implicative int-soft ideal) *Let $(\tilde{f}, M)$ and $(\tilde{g}, M)$ be int-soft ideals of a pseudo MV-algebra $\mathcal{M}$ such that $(\tilde{f}, M)\tilde{\subseteq}(\tilde{g}, M)$, that is, $\tilde{f}(x) \subseteq \tilde{g}(x)$ for all $x \in M$, and $\tilde{f}(0) = \tilde{g}(0)$. If $(\tilde{f}, M)$ is an implicative int-soft ideal of $\mathcal{M}$, then so is $(\tilde{g}, M)$.*

PROOF: Assume that $x^2 = 0$ for any $x \in M$. Then

$$\tilde{g}(x) \supseteq \tilde{f}(x) = \tilde{f}(0) = \tilde{g}(0)$$

by the assumption and Theorem 3.20. Since $\tilde{g}(0) \supseteq \tilde{g}(x)$ for all $x \in M$, it follows that $\tilde{g}(x) = \tilde{g}(0)$ for all $x \in M$ with $x^2 = 0$. By Theorem 3.20, we conclude that $(\tilde{g}, M)$ is an implicative int-soft ideal of $\mathcal{M}$. □

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