Intrinsic energy of Lemaître-Tolman-Bondi models and cosmological implications

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Recently, some Lemaître-Tolman-Bondi metrics have been considered as models alternative to the dark energy within the Friedmann-Lemaître-Robertson-Walker universes. The vanishing of the intrinsic energy of these metrics is examined since such a vanishing, in the present case and in general, could be interpreted as a necessary condition to consider the possibility of the quantum creation of a metric. More specifically, this vanishing is examined in the particular case where the Lemaître-Tolman-Bondi metrics behave asymptotically like a Friedmann-Lemaître-Robertson-Walker universe. Finally, we deal with a particular model ruled out after being confronted with cosmic observations. In a minimal agreement with this negative result, leaving aside an unstable case, the value of the intrinsic energy of this particular model does not vanish and becomes in fact minus infinite.

I. INTRODUCTION

Out of the particular case where $A' = 0$ (see next section), the Lemaître-Tolman-Bondi (LTB) metric family is the most general family of spherically symmetric metrics in General Relativity, corresponding to a pressureless matter source [1,2].

Some of these metrics have been used, in a cosmological context, to describe large inhomogenous structures and the anisotropies they produce on the cosmic background radiation temperature [3–5] and, more recently, as models alternative to the dark energy Friedmann-Lemaître-Robertson-Walker (FLRW) universes [6–15] (see also the reviews [16,17] on the subject and beyond). The models have been confronted with cosmological observations, and the final conclusion of one of these papers [14] is that the confrontation has become sufficiently constraining to "rule out the whole class of adiabatic LTB models," by testing a particular LTB model hereafter called the constrained García-Bellido-Haugbølle (CGBH) model.

On the other hand, from the beginning of the 1970s, people have speculated on the possibility that the Universe could have arisen from a vacuum quantum fluctuation [18,19], an idea further developed by Vilenkin [20]. If this had been the case, we could expect that the energy of our Universe, P^0, and also the corresponding linear 3-momentum, P^i, and angular 4-momentum, $J^{\alpha\beta}$, would vanish ($i, j, ... = 1, 2, 3$ and $\alpha, \beta, ... = 0, 1, 2, 3$). But it is well known that the energy and momenta of a space-time in General Relativity are dramatically dependent on the coordinates used. So, which coordinates must be used in order to calculate the specific two 4-momenta, P^α, $J^{\alpha\beta}$, that

have to vanish in the case of a creatable universe, that is, in the case of a universe arising from a quantum fluctuation? Our answer in [21] is that these specific coordinates have to be *intrinsic* ones, defined as follows:

(a) First, they are Gauss coordinates in some region of the considered space-time, covering the three-space boundary. That is, in this region, the metric components $g_{0\alpha}$ take the values $g_{00} = -1$, $g_{0i} = 0$. As is well known, these are coordinates related to free-falling synchronized observers (Gauss coordinates have been also selected in [22] as the ones leading to a sound energy).

(b) Second, the corresponding linear 3-momentum, P^i, and angular 3-momentum, J^{ij}, vanish, the last one irrespective of the momentum origin.

(c) Third, let it be the spacelike three-surface $t = t_0 \equiv$ const., with t the time coordinate. We denote this three-surface by Σ_3. Then, asymptotically, the three-space metric, g_{ij}, approaches fast enough a manifestly conformally flat metric ($g_{ij} = G^2\delta_{ij}$) when we approach the boundary, Σ_2, of Σ_3. Of course, if the space-time is asymptotically Minkowskian, the conformal factor is the unity.

Notice that in Gauss coordinates the time coordinate, t, is the proper time of those falling observers. Furthermore, it is a *universal* time. This means that distant readings of this time, that are equal, correspond to events which are physically simultaneous (see [23], epigraph 84). This could be important in order to define a consistent energy, P^0, since, as is well known and we are going to see, its expression [see (1)] is a three-volume integral taken in a given time instant t. That is, the elementary partial contributions to this integral are taken for the same t value, and it would be a good thing that this common time instant labeled physically simultaneous events. Therefore, it would

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be good to deal with a proper and universal time, i.e., with Gauss coordinates, at least asymptotically, when calculating what we call here the intrinsic P^0 energy value: by definition, the one calculated in intrinsic coordinates. As far as the energy, P^0, is concerned, the term "intrinsic" refers to the fact that this energy is, first, calculated for coordinates whose corresponding linear and angular 3-momenta, P^i and J^{ij}, vanish and, second, since these coordinates are associated with free-falling observers (local inertial ones) that, in the present case of LTB metrics, are further comoving with the source pressureless matter. Thus, these observers add nothing extra to the considered space-time (compare this situation with, for example, the static observers in a static metric that would have to be prevented of free falling by some virtual nongravitational action). Therefore, we choose just these coordinates to define the specific P^0 energy and momenta, the intrinsic 4-momenta, which must vanish if a given metric has to be quantically creatable: that is, in our case, if some LTB metric had to be quantically creatable.

These intrinsic coordinates can be proved to exist always [21] for each constant value $t = t_0$ and, as mentioned above, the corresponding linear and angular 4-momenta P^α and $J^{\alpha\beta}$ will be called intrinsic momenta. In all, the above creatable universes would be space-times whose intrinsic linear and angular 4-momenta, the last one irrespective of the momentum origin, vanish.

We will make explicit the existence of intrinsic coordinates in the particular case of the LTB and FLRW models considered in the present paper. As we will see next, the closed and flat FLRW universes have vanishing intrinsic momenta, while the value of the intrinsic energy of the open nonflat FLRW universe is $-\infty$ [24]. Thus, in our parlance, the two first ones could be creatable universes, but the last one could not be. Furthermore, neither of the perturbed flat FLRW universes in the frame of standard inflation might be creatable, while the perturbed closed one could be creatable [25]. Notice that discarding, as we actually do, the physically perturbed flat FLRW model, because of its noncreatable character (and so discarding the unperturbed flat model itself) does not contradict the present standard cosmological model where inflation leads to a FLRW universe with a, perhaps, tremendously small curvature, but not necessarily a strictly vanishing one.

Let us point out that the starting point for our definition of the intrinsic momenta of a space-time is the Weinberg energy-momentum complex [26]. There are many complexes present in the literature on the subject, but in [27] we have explained why we have selected, among all of them, the one from Weinberg, from which some specific expressions for the energy, P^0, the linear 3-momentum, P^i, and angular 4-momentum, $J^{\alpha\beta}$, can be deduced. These expressions involve three-space volume integrals that in the particular case of P^0 read

$$P^0 = \frac{1}{16\pi} \int \partial_i(\partial_j g_{ij} - \partial_i g) d^3 x, \quad (1)$$

where the gravitational constant has been taken equal to 1, $g \equiv \delta_{ij} g_{ij}$ and the summation on repeated indices is performed with the Kronecker δ. This defined P^0 coincides (after writing it as a two-surface integral on the above Σ_2 boundary) with the well-known Arnowitt-Deser-Misner energy [28].

Assuming that an existing universe has to be creatable is an attempt at saying something about our Universe all along its existence from just after the big bang. It can also supply us with criteria for, from the beginning, tentatively discarding, as good candidates for good cosmological models, all metrics in General Relativity having any non-vanishing intrinsic moment component. This would be the case in the present paper with the observationally discarded CGBH model of the Universe. This discarding could have been suggested from the very beginning because of the physically noncreatable character of this model. In the present paper, we will characterize the LTB metrics that could be, in principle, creatable ones, by characterizing the ones whose two intrinsic 4-momenta vanish.

II. THE LTB METRICS AND THE GENERAL EXPRESSION FOR THEIR ENERGY

The LTB metric class [1,2] can be written as [10]

$$ds^2 = -dt^2 + \frac{A'(t,r)^2}{1-k(r)} dr^2 + A^2 d\sigma^2 \equiv -dt^2 + dl^2, \quad (2)$$

with $d\sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$ and where A and k are functions of the corresponding arguments satisfying the Einstein field equations, with $A' \equiv \partial_r A$ and $k < 1$. This family of metrics describes the spherical solutions of the Einstein field equations with a pressureless matter source [1,2]. We can see that the coordinates used are adapted to this spherical symmetry. Furthermore, they are Gauss coordinates; that is, for the metric components $g_{0\alpha}$ we have $g_{00} = -1$, $g_{0i} = 0$, such that the observers associated to these coordinates are synchronized free radially falling observers comoving with the pressureless matter source.

We will assume that A' exists and is different from zero everywhere, except perhaps for $r = 0$. Note that from (2) we recover the FLRW metrics by putting $A = a(t)r$, $k = \kappa r^2$, with a the corresponding expansion factor and κ the curvature index, $\kappa = 1, 0, -1$.

Because of the manifest spherical symmetry of the metric (2), its linear 3-momentum, P^i, and its angular 3-momentum, J^{ij}, relative to the center $r = 0$, vanish (see in detail the case of J^{ij} at the end of Sec. V). Thus, the coordinates used in (2) will be intrinsic coordinates, as defined above, provided that the r coordinate be such that dl^2 becomes in the boundary of the corresponding three-space a manifest conformally flat metric. Then, we are left

with the question of whether the energy, P^0, calculated in such coordinates, that is, the intrinsic energy, vanishes or not in order to conclude if a particular LTB metric could be creatable or not.

According to some general expressions given in [29], the expression for the P^0 energy of the metric (2) becomes

$$P^0 = \frac{1}{2} \lim_{r \to \infty} \left[\frac{(A - rA')^2}{r} + \frac{krA'^2}{1-k} \right], \qquad (3)$$

where we have put both the gravitational constant and the speed of light equal to 1.

To obtain this expression one must transform the three-space integral giving P^0 in (1) into a two-surface integral on the boundary of this three-space by applying Gauss theorem. To apply this theorem we need that the metric be regular enough: the three-space derivatives of the three-space metric must be continuous.[1] Thus, we will assume not only that, except for $r = 0$, A'' in (2) exists everywhere and is different from zero, but further that A'' is continuous, and that there is no intrinsic singularity at $r = 0$.

As remarked above, since we want this energy P^0 to be an intrinsic one, we must use in (3) an r coordinate such that dl^2 in (2) can be asymptotically conformally flat in a manifest way. This is what we are going to discuss in the next section.

III. LTB METRICS IN ASYMPTOTIC CONFORMALLY FLAT COORDINATES

Let us make a transformation of the radial coordinate r in (2), going to a new radial coordinate ρ, trough a time independent function, that is $r \to \rho = \rho(r)$. We will chose this function such that the three-space metric, dl^2, becomes in the new radial coordinate asymptotically conformally flat in a manifest way for any constant time t_0. That is,

$$dl^2(t = t_0, \rho \to \rho_b) \simeq G^2(t_0, \rho_b, \rho)\delta_{ij}d\rho^i d\rho^j$$
$$= G^2(d\rho^2 + \rho^2 d\sigma^2), \qquad (4)$$

where ρ^i is such that $\delta_{ij}\rho^i\rho^j = \rho^2$ and where $\rho = \rho_b \equiv \rho_b(t_0)$ is the equation of the radial boundary, Σ_2, of the space-time given by the metric ds^2 in (2) at t_0. We know such a radial coordinate to exist since in [21,24] it has been proved on general grounds for every such constant time

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$t = t_0$. This existence is obvious in the particular case of the LTB metrics which behave asymptotically like FLRW universes (see next Sec. IV).

Then, by simply comparing (4) with (2), we obtain

$$\frac{A'(t_0, \rho \to \rho_b)^2}{1 - k(\rho \to \rho_b)} \simeq G^2(t_0, \rho \to \rho_b), \qquad (5)$$

$$\rho_b^2 G^2(t_0, \rho \to \rho_b) \simeq A^2(t_0, \rho \to \rho_b). \qquad (6)$$

We will write these two equations more compactly by putting

$$\frac{A'^2}{1-k} \simeq G^2, \qquad A^2 \simeq \rho^2 G^2. \qquad (7)$$

These asymptotic equations give trivially

$$k \simeq 1 - \rho^2 \frac{A'^2}{A^2}, \qquad (8)$$

which, since k only depends on ρ, means that asymptotically the function $\rho^2 A'^2/A^2$ does not depend on t_0.

Then, let us calculate the intrinsic energy P^0 of any LTB metric that is P^0 calculated in coordinates such that we have (4). Since we have assumed at the beginning of Sec. II that our LTB metrics are regular enough, we can, using Gauss theorem, write (1) as the two-surface integral on the corresponding boundary,

$$P^0 = \frac{1}{16\pi} \lim_{\rho \to \rho_b} \int (\partial_j g_{ij} - \partial_i g) n^i \rho^2 \sin\theta d\theta d\phi, \qquad (9)$$

with $n^i \equiv \rho^i/\rho$.

Since now we have asymptotically $dl^2 \simeq G^2(t_0, \rho)\delta_{ij}dx^i dx^j$, P^0 becomes

$$P^0 = -\frac{1}{8\pi} \lim_{\rho \to \rho_b} \int \partial_i G^2 n^i \rho^2 \sin\theta d\theta d\varphi$$
$$= -\frac{1}{2} \lim_{\rho \to \rho_b} (\rho^2 \partial_\rho G^2), \qquad (10)$$

or what is equivalent, according to the second equation of (7),

$$P^0 = \frac{1}{2} \lim_{\rho \to \rho_b} (2\rho^{-3}A^2 - \partial_\rho A^2). \qquad (11)$$

Equalizing to zero, this P^0 expression allows us to fully characterize the family of LTB metrics whose two 4-momenta vanish for any constant time t_0. For instance, if $\rho_b = \infty$, then $G^2 \sim \frac{1}{\rho^p}$ with $p > 1$ gives $P^0 = 0$. This situation describes all the corresponding creatable LTB universes when $\rho_b = \infty$, although there is still another particular case in which $P^0 = 0$: i. e., the one in which G does not depend on ρ.

IV. THE LTB METRICS BEHAVING ASYMPTOTICALLY AS FLRW UNIVERSES

Imagine that function A factorizes as we approach the boundary, Σ_2, of the three-space. That is, we have near Σ_2,

$$A(t,\rho) \simeq a(t)f(\rho). \tag{12}$$

Having in mind this factorization, the Einstein field equations (see for example [10]),

$$\dot{A}^2 + 2A\ddot{A} + k = 0, \tag{13}$$

$$2\frac{\ddot{A}}{A} + \frac{\ddot{A}'}{A'} = -4\pi\rho_M, \tag{14}$$

become asymptotically

$$\dot{a}^2 + 2a\ddot{a} = \rho^2 \frac{f'^2}{f^4} - \frac{1}{f^2} \tag{15}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi\rho_M(t), \tag{16}$$

respectively, with $\rho_M(t) = \rho_M(t, \rho \to \rho_b)$, and $\rho_M(t,\rho)$ the matter density.

From (15) we obtain

$$\dot{a}^2 + 2a\ddot{a} = C, \qquad \rho^2 \frac{f'^2}{f^4} - \frac{1}{f^2} = C, \tag{17}$$

with C an arbitrary constant.

Equation (16) and the first equation of (17) are equivalent to the two dynamical cosmic equations for the expansion factor, $a(t)$, of a FLRW with $-C$ curvature, i.e., equivalent to (16) jointly with

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho_M}{3} + \frac{C}{a^2}. \tag{18}$$

As far as the second equation of (17) is concerned, its general solution is

$$f = \frac{2B\rho}{1 - B^2C\rho^2}, \tag{19}$$

with B an arbitrary constant. This gives for G the expression

$$G^2(t,\rho) = a^2(t) \frac{4B^2}{(1 - B^2C\rho^2)^2}. \tag{20}$$

Then, let us change to the new radial coordinate $\tilde{\rho} = -\frac{2}{BC\rho}$. We obtain for the asymptotic value of this ds^2 metric

$$ds^2 = -dt^2 + a^2(t) \frac{\delta_{ij}d\tilde{\rho}^i d\tilde{\rho}^j}{(1 - C\tilde{\rho}^2/4)^2}. \tag{21}$$

That is, all LTB metrics satisfying (12) reduce asymptotically to a FLRW universe: close, open or flat according to whether it is $C < 0$, $C > 0$, or $C = 0$, respectively. Actually, from (21), we obtain the corresponding standard form

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 + Cr^2} + r^2d\sigma^2\right), \tag{22}$$

by making the coordinate transformation

$$r = \frac{\tilde{\rho}}{1 - C\tilde{\rho}^2/4}. \tag{23}$$

According to previously obtained results (see [24] for example), the intrinsic energy, P^0, of a closed or flat FLRW universe vanishes, while it gets a $-\infty$ value for the open nonflat one. Then, it becomes obvious that $P^0 = 0$ for any LTB metric behaving asymptotically like a closed FLRW universe, and $P^0 = -\infty$ when it behaves asymptotically like an open nonflat one. But what about the case where the LTB metric approaches asymptotically a flat FLRW universe? We deal with this question in the next section.

V. THE PARTICULAR CASE OF THE LTB METRICS ASYMPTOTICALLY BEHAVING LIKE A FLAT FLRW UNIVERSE

In Refs. [13,14], the corresponding authors explore the possibility that we live close to the center of a large void, i.e., close to the center of a suitable LTB model, as an alternative to the prevailing interpretation of the Universe acceleration in terms of a ΛCDM model with a dominant dark energy component. They confront this possibility with a series of cosmological observations through two versions, the flat and the open ones, of the CGBH model cited above, the first (second) version becoming asymptotically a flat (an open nonflat) FLRW universe without cosmological constant. The CGBH model, in its two versions, is ruled out as a result of the confrontation.

Because of the asymptotic behaviour of the open nonflat version of this CGBH model, this intrinsic energy P^0 has to be $-\infty$. This result is in minimal accordance with the reported ruling out of this version by observational reasons (remember that, in our frame, universes with at least one of the components of their intrinsic 4-momenta different from zero would not be a good candidate to be quantically creatable). Of course, getting $P^0 = -\infty$ from this asymptotic behavior assumes that the metric of this open nonflat version is everywhere regular enough in order to apply the Gauss theorem. The same question will be raised below, in the present section, for the flat version. We will postpone to

the end of this section the proof that, in both cases, the theorem becomes applicable.

But, what about the asymptotic flat version of the CGBH model? In the present section, we will see that in this case the intrinsic energy P^0 vanishes or not depending on how fast the flat version of the CGBH model approaches asymptotically a flat FLRW universe.

Then, let us compare the LTB metric (2) with its flat FLRW universe limit for $r \to \infty$. Having in mind the expression $ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\sigma^2)$ for this flat FLRW metric, we easily obtain

$$A(t, r \to \infty) \simeq a(t)r, \qquad \lim_{r \to \infty} k(r) = 0. \qquad (24)$$

Notice that because of this asymptotic character, the coordinates in (2) are asymptotic conformally flat coordinates for dl^2. Consequently, they are intrinsic coordinates (they fulfill the three conditions $(a) - (c)$ defining the notion of intrinsic coordinates in the Introduction) and the corresponding P^0 energy we are going to calculate will be the intrinsic energy.

Because of (24), the partial contribution to P^0, in (3), coming from the term containing the k function is

$$\frac{1}{2} \lim_{r \to \infty} \frac{rkA'^2}{1-k} = \frac{a^2}{2} \lim_{r \to \infty} \frac{rk}{1-k} = \frac{a^2}{2} \lim_{r \to \infty} (rk), \qquad (25)$$

whose actual value depends on how fast k vanishes when $r \to \infty$. Then, in accordance with [13], let us define the function $\Omega_M(r)$, that generalizes the matter cosmic parameter, Ω, of the FLRW cosmology, by writing $k(r)$ like

$$k \equiv \dot{A}_0^2(\Omega_M - 1), \qquad (26)$$

with $\dot{A}_0 \equiv \partial_t A(t = t_0, r)$ and t_0 the cosmic present time.

The value of this function for the asymptotic flat version of the CGBH model is [13]

$$\Omega_M = \Omega_{out} + (\Omega_{in} - \Omega_{out}) \frac{1 - \tanh[(r - r_0)/2\Delta r]}{1 + \tanh(r_0/2\Delta r)}, \qquad (27)$$

with $\Omega_{out} = 1$ ($\Omega_{out} < 1$, for the open nonflat case), where Ω_{in}, r_0, and Δr are parameters to be fitted by cosmological observations. In particular, r_0 characterizes the void size, near whose center we are assumed to be placed, and Δr the transition to uniformity.

From (27), we easily obtain

$$\Omega_M(r \gg \Delta r) - 1 \simeq \lambda e^{-r/\Delta r}, \qquad (28)$$

with

$$\lambda \equiv \frac{2e^{r_0/\Delta r}}{1 + \tanh(r_0/2\Delta r)}(\Omega_{in} - 1). \qquad (29)$$

Then, from (26) and the first equation of (24) we have

$$k(r \gg \Delta r) \simeq \lambda \dot{a}_0^2 r^2 e^{-r/\Delta r}. \qquad (30)$$

According to (25), this asymptotic behavior of k means that the contribution to P^0 in (3) from the term involving k vanishes. Thus, in order to calculate P^0 for the asymptotic flat version of the CGBH model, we are left with the remaining contribution from the term dealing with the function A. To calculate this contribution we have to make explicit how fast A approaches $a(t)r$ [see the first equation of (24)] when $r \to \infty$. The easiest way to make this is to use one of the Einstein field equations (13) and (14) for the LTB metrics (2).

More precisely, we will consider Eq. (13) for $r \to \infty$, where, as we have just seen, k behaves like $k \sim r^2 e^{-r/\Delta r}$, jointly with the parametric A value for the asymptotically flat CGBH model [13,14], i.e.,

$$A(t, r) = \frac{\Omega_M}{2(1 - \Omega_M)}(\cosh \eta - 1)r, \qquad (31)$$

$$H_0(r)t = \frac{\Omega_M}{2(1 - \Omega_M)^{3/2}}(\sinh \eta - \eta), \qquad (32)$$

with η a positive real parameter, $\eta \in (0, +\infty)$, and

$$H_0(r) = \frac{3H_0}{2(1 - \Omega_M)}\left[1 - \frac{\Omega_M}{\sqrt{1 - \Omega_M}}\sinh^{-1}\sqrt{\frac{1 - \Omega_M}{\Omega_M}}\right], \qquad (33)$$

with the inserted factor $3/2$ allowing us to obtain the Hubble constant H_0, for $H_0(r \to \infty)$. Finally, Ω_M is given by (27) with $\Omega_{out} = 1$.

Then, in accordance with (24), let us write the asymptotic form of A as

$$A(t, r \to \infty) = a(t)r[1 + \epsilon(t, r)], \qquad (34)$$

where ϵ is a function of t and r such that $\lim_{r \to \infty} \frac{\epsilon}{r^p} = 0$ for any t value and for any $p > 1$. This asymptotic expression for A would guarantee the vanishing of the corresponding contribution to the intrinsic value of P^0. In the Appendix we show that A has actually such an asymptotic behavior since we obtain

$$\epsilon(t, r \to \infty) \sim e^{-r/\Delta r}. \qquad (35)$$

Then, the resulting asymptotic form for A when $r \to \infty$ leads to

$$\lim_{r \to \infty} \frac{(A - rA')^2}{r} = 0, \qquad (36)$$

and so to the final vanishing of the intrinsic energy of the asymptotically flat CGBH model.

It seems, then, that the asymptotically flat CGBH cosmological model could be quantically creatable. However, we are going to see in Sec. VI that this vanishing of P^0 is a dramatically nonstable result. We will account for this limitation by saying that the model is physically noncreatable.

But, according to the general considerations on the applicability of the Gauss theorem, made at the end of Sec. II (including footnote 1), we will show before going into Sec. VI that the metric of both versions of the CGBH model is regular enough to allow for the application of this theorem: we need to apply the theorem to go from (1) to (9) in order to calculate P^0. In our case, a sufficient condition for the theorem is that the first r derivatives of the three-space metric components of the LTB metric (2) be continuous functions everywhere. In the particular case of the CGBH model, this means that the second r derivatives of $A(t, r)$ given by (31), (32), (33), and (27), is continuous, and the same for the first r derivative of $k(r)$ given [13] by $k(r) = r^2 H_0^2(r)[\Omega_M(r) - 1]$.

But notice that to begin with, the function $\Omega_M(r)$ is infinitely derivable since it is essentially $\tanh r$. On the other hand, the first r derivative of the function $\sinh^{-1} \sqrt{(1 - \Omega_M)\Omega_M^{-1}}$ in (33), when calculated, gives the value $-(1/2)\Omega_M'\Omega_M^{-1}(1 - \Omega_M)^{-1/2}$. Then, it is straightforward to see that the function $H_0(r)$ given by (33) has a continuous first r derivative everywhere, out perhaps of $\Omega_M = 0$ and $\Omega_M = 1$. But, by definition $\Omega_M \neq 0$ (notice [14] that $\Omega_M \equiv \bar{\rho}(r)/\rho_c$, where $\rho_c \equiv 3H_0(r)/8\pi$ is the critical density and $\bar{\rho}(r) = (1/V) \int_0^r 4\pi r'^2 \rho(r', t_0)dr'$, with $V = 4\pi r^3/3$). Further, there is no physical singularity for $\Omega_M \to 1$, since in this case we simply recover asymptotically the corresponding flat FLRW limit. Then, in $H_0(r)$ given by (33), we can leave out the particular Ω_M values $\Omega_M = 0$, $\Omega_M = 1$, so that $H'(r)$ is continuous everywhere. In all, $k'(r)$ is a continuous function as we wanted to prove.

Now, let us prove the continuity of $A''(t, r)$. In order to do this, let us write Eq. (32) as

$$\sinh \eta - \eta = 2tH_0(r)\Omega_M^{-1}(1 - \Omega_M)^{3/2} \equiv t\chi(r), \quad (37)$$

where $\chi(r) \equiv 2H_0(r)\Omega_M^{-1}(1 - \Omega_M)^{3/2}$.

But the function $\sinh \eta - \eta$ of η is infinitely r derivable, monotonously increasing from 0, for $\eta \to 0$, to $+\infty$, for $\eta \to +\infty$. Then, the corresponding inverse function of r, $\eta = f(t\chi)$, is unique and infinitely derivable.

Thus, let us calculate $A'(t, r)$ from (31),

$$A'(t, r) = \frac{1}{2}\left\{\left[\frac{r\Omega_M}{1 - \Omega_M}\right]'(\cosh f - 1) + \frac{r\Omega_M f'}{1 - \Omega_M}\sinh f\right\}, \quad (38)$$

where $\cosh f$ and $\sinh f$ are infinitely derivable functions of r and the same for $f' \equiv \chi' \partial_\chi f$ according to the above reasoning. Then, by a mere inspection of (38) we can see that (out of $\Omega_M = 0$ and $\Omega_M = 1$, as it is due) A'' exists and is continuous.

Therefore, the sufficient regularity conditions to apply Gauss theorem to (1) are fulfilled.

Before finishing the present section, let us see in detail how the explicit spherical symmetry of a four-metric in Gauss coordinates leads to the vanishing of the corresponding J^{ij}, as announced at the beginning of Sec. II. Notice that, because of this symmetry, when using rectilinear coordinates at the space infinity $r \to \infty$, the three-space metric, g_{ij}, has the asymptotic form

$$g_{ij} = \alpha(t, r)\delta_{ij} + \beta(t, r)n_i n_j, \qquad n_i = \frac{x_i}{r} \qquad r = \sqrt{x_i x_i}, \quad (39)$$

where $\alpha(t, r)$ and $\beta(t, r)$ are two functions of t and r.

On the other hand, and similar to expression (9), starting from the Weinberg complex [26], we obtain for J^{ij} referred to the angular momentum origin $r = 0$, the following general expression as a two-surface integral on the boundary $r \to \infty$ of the three-space,

$$J^{ij} = \frac{1}{16\pi}\lim_{r\to\infty}\left[r^3 \int (n_j \dot{g}_{ki} - n_i \dot{g}_{kj})n_k \sin\theta d\theta d\phi\right]. \quad (40)$$

Of course, we need to apply the Gauss theorem to the corresponding three-volume integral in order to obtain (40). In our case, having just proved above that $k(r)$ is regular enough, this requires as a sufficient condition that $\dot{A}'$ be a continuous function of r. But from (38) we obtain

$$\dot{A}'(t, r) = \frac{r\Omega_M \dot{f}\sinh f}{2(1 - \Omega_M)} = \frac{r\Omega_M \chi \sinh f}{2(1 - \Omega_M)}\frac{df}{d(t\chi)}, \quad (41)$$

which, for reasons similar to the ones explained above to conclude the everywhere continuity of A'', is an everywhere continuous function of r. Therefore, we can write (40) for J^{ij} referred to the angular 3-momentum origin $r = 0$. Finally, we can substitute (39) in (40) and obtain $J^{ij} = 0$ since it is obvious that now the integrand vanishes identically.

But, what if we shift the angular 3-momentum origin from the above value $r = 0$ to $r = \sqrt{a_i a_i} \equiv a$, where a_i are the components of a constant three-vector? We will have in an evident notation

$$\bar{J}^{ij} = \frac{1}{16\pi}\lim_{r\to\infty}\left[r^2 \int (\bar{x}_j \dot{g}_{ki} - \bar{x}_i \dot{g}_{kj})n_k \sin\theta d\theta d\phi\right] \quad (42)$$

where $\bar{x}_j \equiv x_j + a_j$.

A sufficient condition to have $\bar{J}^{ij} = 0$ irrespective of the chosen angular 3-momentum origin, that is, irrespective of the constant value a_i, is to have:

$$I_i \equiv \int \dot{g}_{ki} n_k \sin\theta d\theta d\phi = 0. \qquad (43)$$

But, according to (39), this integral actually becomes

$$I_i = (\dot{\alpha} + \dot{\beta}) \int n_i \sin\theta d\theta d\phi = 0, \qquad (44)$$

since identically $\int n_i \sin\theta d\theta d\phi = 0$.

Thus, for any manifestly spherically symmetric four-metric in Gauss coordinates, J^{ij} vanish identically irrespective of the chosen angular 3-momentum origin.

For completeness we can report that J^{0i} vanish too, irrespective of the chosen origin of angular 4-momenta, simply because in the present case $P^0 = 0$ and $P^i = 0$.

This double vanishing happens too for the CGBH model in intrinsic coordinates, since in such a particular case, aside from being $P^0 = 0$ and $P^i = 0$, Eq. (39) reduces to $g_{ij} = \alpha(t, r)\delta_{ij}$, as it must according to the intrinsic coordinate definition (see point c of this definition in the Introduction).

The reader can see that asymptotically $g_{ij} = \alpha(t, r)\delta_{ij}$ for the open nonflat version of the CGBH model by consulting (21) for $C > 0$. For the flat version, this asymptotic behavior, $g_{ij} = \alpha\delta_{ij}$, becomes obvious by having in mind (30), (34), and (35).

VI. THE INSTABILITY OF THE ENERGY VANISHING FOR A LTB METRIC BEHAVING ASYMPTOTICALLY AS A FLAT FLRW UNIVERSE: DISCUSSION AND CONCLUSION

Let us come back to the open nonflat FLRW metric in the form (21) with C normalized to the corresponding value $C = 1$. The coordinates used are then intrinsic ones and, according to (10), the intrinsic value of P^0 becomes,

$$P^0 = -\frac{a^2}{2}\lim_{\rho\to 2}\left[\rho^2 \frac{d}{d\rho}(1 - \rho^2/4)^{-2}\right] = -\infty, \qquad (45)$$

since in this case it is $\rho_b = 2$.

Therefore, P^0, whose value for the asymptotically flat CGBH model (Sec. V) was zero, jumps to a minus infinite value, $P^0 = -\infty$, when an elementary shift of the C constant from its original value is performed, or what is the same, when a shift from $\Omega_{\text{out}} = 1$ (see (27)) to the new value $\Omega_{\text{out}} < 1$ is performed as close to 1 as we want.

Thus, in the frame of a hypothetical quantum creation of a universe (this universe being constrained to have vanishing energy in accordance with, for example, authors like Tryon [19] or Vilenkin [20]), the resulting vanishing of P^0 for this asymptotically flat CGBH model could be considered as an unstable result, and so the quantum creation of this asymptotically flat model could perhaps have a vanishing probability. All this is, of course, compatible with some inflationary process leading to a universe extremely (but not exactly) flat, as is the case in the present cosmological standard model. Then, though in our frame this flat model could be strictly speaking a creatable one, we can consider and denote it, jointly with the asymptotically nonflat open CGBH model, as physically noncreatable universes. All this could be seen in accordance with the fact that this model is ruled out by its confrontation with cosmological observations [13,14]. In conclusion: had this confrontation not taken place, in view of this physical noncreatable character, we could have predicted a subsequent negative result, not as a true prediction but as some plausible suggestion.

Furthermore, assuming a negative value for the constant C in (21) and normalizing to $C = -1$ we obtain the closed FLRW metric in three-space conformal flat coordinates:

$$ds^2 = -dt^2 + a^2(t)\frac{\delta_{ij}d\rho^i d\rho^j}{(1 + \rho^2/4)^2}. \qquad (46)$$

The boundary limit ρ_b is now $\rho_b = \infty$. Then, using the expression (10), we straightforwardly obtain $P^0 = 0$; that is, the LTB metric approaching asymptotically the closed FLRW model could be creatable. Thus, we could ask if this "closed" LTB model would not be better entitled than the CGBH one to be observationally tested as a model of the void universes considered in the present paper. Actually, aside with other models, the "closed" LTB model has been tested in [12], without fully conclusive results. But, obviously, the possible creatable character of this model, by itself, could not avoid that finally the model was ruled out by observations. In fact, as stated in [12], "in practice we could always approximate the correct answer by setting Ω_k to a small nonzero value."

Nevertheless, according to [30] (cf. [31] and references therein), it seems that the kinematic Sunyaev-Zel'dovich effect puts further severe limitations on the viability of simple LTB void models, irrespective of their asymptotic (closed, open, or flat) character and the constraining requirement of a homogeneous universe age.

APPENDIX A: CALCULATING THE $\epsilon(t,r)$ FUNCTION

We start from (13), that is, $\dot{A}^2 + 2A\ddot{A} + k = 0$, and from (30), that is, $k(r \gg \Delta r) \simeq \lambda \dot{a}_0^2 r^2 e^{-r/\Delta r}$. Then we write A as $A = a(t)r[1 + \epsilon(t, r)]$ with $a(t) = (t/t_0)^{2/3}$. After an

elementary calculation, neglecting quadratic terms in ϵ, we find

$$\frac{1}{r^2}(\dot{A}^2 + 2A\ddot{A}) \simeq (\dot{a}^2 + 2a\ddot{a})(1 + 2\varepsilon) + 2a(a\ddot{\epsilon} + 3\dot{a}\,\dot{\epsilon}). \tag{A1}$$

But we have $\dot{a}^2 + 2a\ddot{a} = 0$. Then, Eq. (13) becomes for large values of r

$$2a(a\ddot{\epsilon} + 3\dot{a}\,\dot{\epsilon}) \simeq -\dot{a}_0^2 \lambda e^{-r/\Delta r}, \tag{A2}$$

which in accordance with $a(t) = (t/t_0)^{2/3}$ can be written as

$$(a^3\dot{\epsilon})^{\cdot} \simeq -\frac{2}{9t_0^2}\lambda e^{-r/\Delta r}, \tag{A3}$$

whose general solution for large values of r ($r \gg \Delta r$) is

$$\epsilon(t, r) = g(r)\alpha(t) + h(r), \tag{A4}$$

with

$$g(r) = -\frac{2}{9t_0^2}\lambda e^{-r/\Delta r}, \tag{A5}$$

where $h(r)$ is an arbitrary function, and $\alpha(t)$ is the general solution of

$$(a^3\dot{a})^{\cdot} = a; \tag{A6}$$

that is to say,

$$\alpha = \frac{9}{10}t_0^{4/3}t^{2/3} + \mu t_0^2 t^{-1} + \nu, \tag{A7}$$

with μ and ν two arbitrary constants.

Substituting this expression of α in (A4), we obtain, for large values of r but for any time,

$$\epsilon(t, r) = -\left[\frac{1}{5}a + \frac{2}{9}\left(\frac{\mu}{t} + \frac{\nu}{t_0^2}\right)\right]\lambda e^{-r/\Delta r} + h(r). \tag{A8}$$

Then, in order to fix both arbitrary constants, μ and ν, and the arbitrary function $h(r)$, let us come back to the CGBH model, more precisely, to Eqs. (31) and (32). For small η values, that is, for small t values, we obtain

$$A(t, r) \simeq \left(\frac{3}{2}\right)^{2/3}\Omega_M^{1/3}(r)H_0^{2/3}(r)rt^{2/3}, \tag{A9}$$

where we can substitute Ω_M by its asymptotic expression from (28).

Furthermore, having in mind (27) with $\Omega_{\text{out}} = 1$ and (33), after an elementary calculation, we obtain the following asymptotic value:

$$H(r \gg \Delta r) \simeq H_0\left(1 - \frac{1}{5}\lambda e^{-r/\Delta r}\right). \tag{A10}$$

Carrying this expression to (A9) we obtain for large values of r and small values of t,

$$A(t, r) \simeq ar\left[1 + \frac{1}{5}(1 - a)\lambda e^{-r/\Delta r}\right]. \tag{A11}$$

Finally, through the relation $A = ar(1 + \epsilon)$, let us compare this approximated expression of $A(t, r)$ with $\epsilon(t, r)$, given by (A8). We obtain for large values of r, but for any time,

$$\epsilon \simeq \frac{1}{5}\lambda(1 - a)e^{-r/\Delta r}, \tag{A12}$$

which is in accordance with Eq. (35).

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