Helping Students Explore the Cartesian Coordinate System

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It is crucial to engage students in examining and discussing mathematical ideas while providing them with challenging tasks and a safe environment to present emergent thinking they can revise upon further exploration (Foote, Earnest, & Mukhopadhyay, 2014). One way to meet these goals is to provide instruction that integrates problem solving across many facets of the curriculum. Centering student voice is an important facet of creating more equitable classrooms (Gutiérrez, 2009), and implementing a problem solving approach within mathematics classrooms is one way to bring student voice to the forefront. Given these aims, this paper explores a problem-based approach to developing the Cartesian coordinate system as a set of whole number, integer number, or rational number ordered pairs.

Developing the Cartesian coordinate system as a set of whole number, integer number, and rational number ordered pairs is called for in the fifth- and sixth-grade Indiana Academic Standards (5.AT.6, 5.AT.7, 6.AF.7, 6.AF.8)—with whole number ordered pairs taught in fifth grade, and integer and rational number ordered pairs taught in sixth grade. Students are then expected to use the Cartesian coordinate system throughout middle and high school (e.g., 7.AF.6, 7.AF.8, 8.GM.6, 8.AF.8) and into advanced college mathematics courses. Given the centrality of this representation in mathematics, how we initially teach students to develop this representation is important. As former middle school teachers and current teacher educators, we have noticed—and research supports (Battista, 2007)—the fact that many middle school students have difficulty reading and plotting points in the Cartesian coordinate system. The most common mistake students make involves reversing the order of the points, reading the *y*-coordinate first and the *x*-coordinate second (Sarama, et. al., 2003). While we are aware of a range of "tricks" to help students remember that the *x*-coordinate comes before the *y*-coordinate, we decided to use a problem-based approach to develop and explore key concepts of the Cartesian coordinate system with middle grades students. We use this article to share our approach, discuss work collected from 14 sixth-grade students, and outline a sequence of problems and key conversations for classroom implementation.

THE APPROACH AND END GOAL

Since this approach departs from what might be considered a traditional approach to teaching the Cartesian coordinate system, we first outline the end goal in order to clarify the subsequent discussion—think "backwards planning" as teachers often do (i.e., know the end goal and then plan accordingly).

Our goal was to use problems like the *Digits Problem I* to develop whole number ordered pairs in the first quadrant of the Cartesian coordinate system, the *Digits Problem II* to develop integer ordered pairs in all four quadrants of the Cartesian coordinate system, and the *Digits Problem III* to develop rational ordered pairs of the Cartesian coordinate system.

> *Digits Problem I:* You have the number cards 1 through 7. You select one card, replace it, and draw a second card to create a coordinate point (e.g., (1, 2) is a coordinate point). Represent all possible coordinate points using an array (Figure 1a).

> *Digits Problem II:* You have the number cards -7 through 7. You select one card, replace it, and then draw a second card to create a coordinate point. Represent all possible coordinate points using an array (Figure 1b).

> *Digits Problem III:* You have the number cards 0, 1/7, 2/7, 3/7, … 6/7, 7/7. You select one card, replace it, and then draw a second card to create a coordinate point. Represent all possible coordinate points using an array.

The primary purpose of the solution and representation of the *Digits Problems* is to help students establish points in the Cartesian coordinate system as ordered pairs— a basic, but very important component of understanding the Cartesian coordinate system (Battista, 2007). Before giving students problems like the *Digits Problems*, we first worked with them on the solution and representation of problems that would prepare them for a discussion of the Cartesian coordinate system. The remainder of this paper unpacks this approach and highlights key conversations in the teaching and learning process, beginning with how students might first solve these problems and represent them using arrays before they are introduced to problems that explicitly involve making coordinate points.

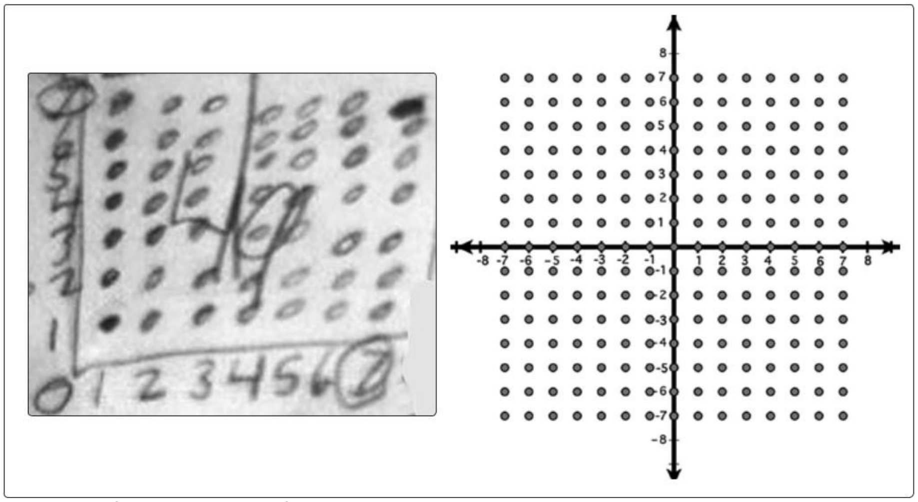

Figure 1a (left), 1b (right). Arrays for the Digit Problem I and II.

HOW WE STARTED

To start, we used problems like the *Outfits Problem*.

> *Outfits Problem:* You have 3 pairs of pants and 4 shirts. One outfit consists of one pair of pants and one shirt. How many possible outfits could you make?

Researchers have found that students as early as second and third grade can solve these problems when they are provided with manipulatives (Nunes & Bryant, 1996). For example, English (1991, 1993) asked students to dress a toy bear in as many different outfits as they could and found many second-grade students were successful at solving this problem. However, researchers have also found that when given only a written version of this type of problem, students in the elementary grades find it difficult to solve (Mulligan, & Mitchelmore, 1997; Outhred, 1996). Given these findings, we anticipated these problems might initially be difficult for students. Thus, we were not surprised when many of the sixth-grade students we worked with said the answer to the problem was three outfits (Tillema, under review). Sample reasoning for this solution would be to put a pair of pants with a different shirt, leaving one shirt remaining.

By prompting our sixth-grade students to imagine different colors of shirts and pants, and questioning them about whether they could wear the same colored shirt with different pants, all of them were able to adjust their thinking to arrive at the solution of twelve outfits. Furthermore, we asked students to create a representation of the problem on their own, and students most frequently used a list or tree diagram (Figure 2). In Figure 2, the student chose to label the shirts with the letters L, P, Z, and F, and label the pants with the letters C, D, and H. Once our students were familiar with lists and tree diagrams, we asked them to coordinate these with making an array. This coordination of representations—in addition to the initial listing of outcomes—was a crucial part of students organizing their solutions as well as being able to justify and explain their work. This connection across multiple mathematical representations supported students as they engaged in deep, meaningful mathematical conversations, which is a major advantage to this approach.

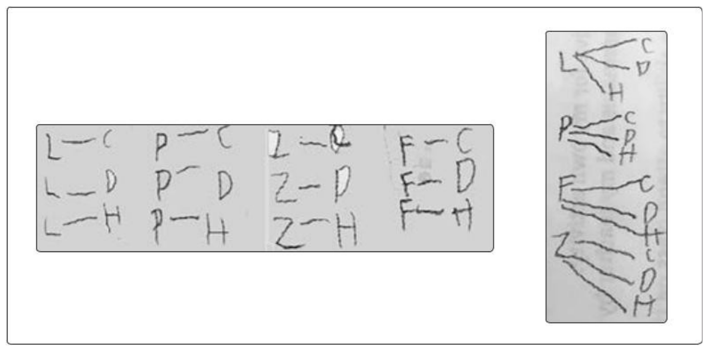

Figure 2. A student's list and tree diagram.

REPRESENTING THE OUTFITS PROBLEM USING AN ARRAY

These problems are well-suited for representation as arrays because the outfits students create are like coordinate points; pairing one shirt with one pair of pants parallels the process of creating a coordinate point by pairing one number with a second number (Vergnaud, 1983; Tillema, 2013; Tillema & Gatza, 2016). One major difference, however—and a key pedagogical reason for using this type of problem— is that students have to order the shirts and pants themselves in this context since no particular order is implied as in the case of problems involving digits. The open nature of this task, then, creates space for students to ask questions and make decisions as they structure how they will organize their creation of the set of outcomes.

The list in Figure 2 illustrates how one student made all of the outfits with shirt L first, all of the outfits with shirt P second, all of the outfits with shirt F third, and all of the outfits with shirt Z fourth. This implicitly established an order for the shirts: L, P, F, and Z. Similarly, the student always cycled through the pants in a consistent order: C first, D second, H third. We conjectured that students would keep this order consistent when they created arrays. While not all students were consistent on every array, the majority of them followed the same order in making this transition. Figure 3 shows what we mean by consistent: Along the vertical axis, shirt L is represented first at the top, shirt P is represented second below it, and so on; similarly, pants C is represented first on the horizontal axis furthest left, then pants D, and finally pants H.

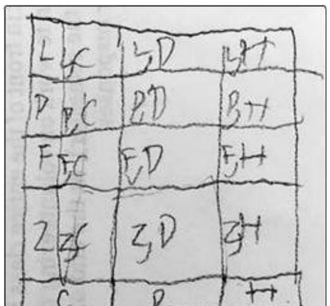

Figure 3. An array for the Outfits Problem.

To create an array, students had to make a decision about where to locate the first shirt (shirt L), which could be either on the top or bottom of the vertical axis, and where to locate the first pants (pants C), which could be either on the left or right of the horizontal axis. Depending on which choices they made, there were four possible arrays they could create while still keeping the order of the shirts and pants the same as in their lists. These four possible arrays are shown in Figure 4, and each has the first outfit located in a different place on the array.

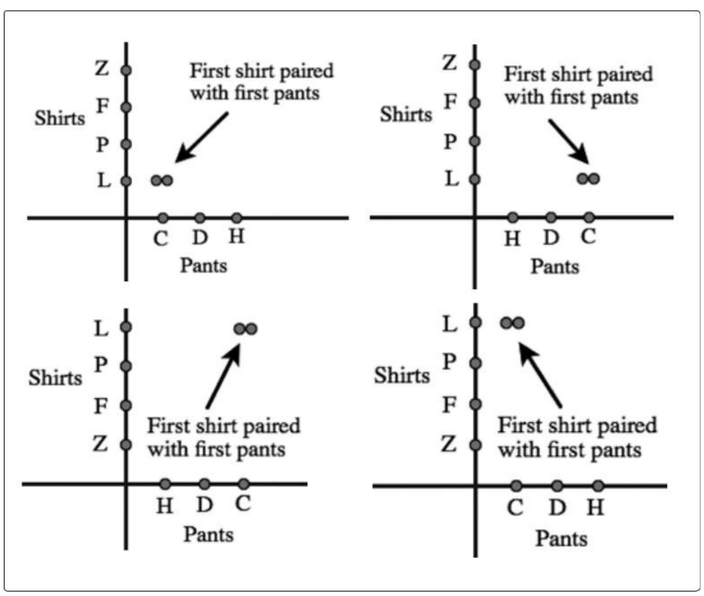

Figure 4. Four possible ways to organize arrays.

When we have used this approach in the classroom, we have had students share their work to highlight the different ways of organizing arrays. Specifically, we highlight the connection between the first outfit in their list or tree diagram and where it "ends up" in their array, depending on how they have chosen to organize the shirts and pants on the axes of the array. This explicit conversation helps them see the connection between where they placed the first shirt and first pants on the axes of the array and where the first outfit appears in the interior of their array. This conversation opens up questions for students such as: What is the connection between where the shirt and pants are represented on the axes and where the first outfit is represented in an array? Does it matter where the first outfit from a list is represented in an array? Why or why not?

ORDERING THE AXES: A MORE COMPLEX PROBLEM

Once students have represented several problems like the *Outfits Problem* with arrays, we have them work on problems like the *Password Problem*.

> *Password Problem:* You have a deck of cards with the letters A-N (14 total cards). You create a two-letter password by drawing a card, replacing it, and drawing a second card. How many different two-letter passwords are possible?

We introduce this problem to encourage students to think specifically about ordered outcomes, which parallels the idea of ordered coordinate points in the Cartesian coordinate system.

Again, many of the students we worked with found this type of problem initially challenging. One student, Charice, remarked, "I have a question. There is nothing to times it by…," meaning she did not think there was anything to multiply 14 by in the problem. This same issue arose while working with another student, Leonard, who remarked, "I know letters should go on one side (axis) of my array, but what about the other side (axis)?" Students were able to resolve these issues by creating lists for the solution of the problem. This helped them see they were pairing letters with letters; Charice noticed she could multiply 14 times 14, and Leonard realized he needed to place letters on each axis of the array. Here again, having students create lists proved to be a key pedagogical move in unpacking and understanding the task at hand. This was also helpful in clarifying the idea of *ordered* pairs as students could see the difference between, for example, the password "AB" and the password "BA" in their lists.

Once students were familiar with this type of problem, we asked them to represent the problem using an array (Figure 5). As part of this process, we asked them to "read" points in their array where the only difference between the points was the order of the letters in the password (e.g., "AB" and "BA"). Students usually read such points differently but were still unaware of how to actually show this difference on their array. Such a misunderstanding, in general, can lead to conversation about how to differentiate

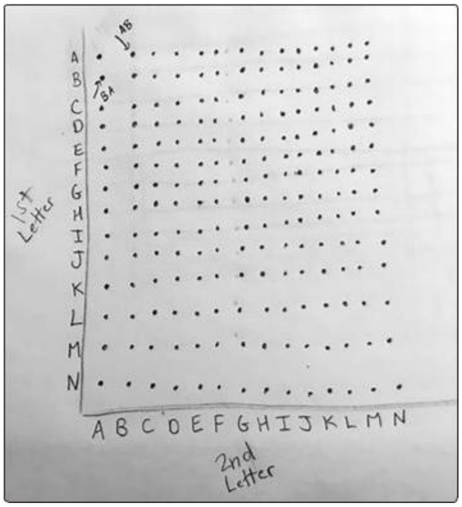

Figure 5. Charice's array for the Password Problem.

the axes based on whether the letter represents the first or second letter in the password (i.e., creating an order for the axes). Students could choose to do this by either having the horizontal axis represent the first letter or the vertical axis represent the first letter. This choice means that there are two possible ways to organize arrays for each of the four possible arrays shown in Figure 4. Figure 6, then, shows these eight possible ways to organize an array for problems like the *Password Problem*.

STUDENT GENERATED REPRESENTATIONS AND SOURCES FOR DISCUSSION

The numbers underneath each array in Figure 6 show the total number of students out of the14 sixth graders who used each way of organizing arrays. Interesting to note is that students were remarkably consistent in how they organized their arrays: eight of the 14 students used the same organization across all problems, and another five of the 14 used the same organization except on one or two problems. In addition to being consistent in how they organized their arrays, very few students, only three of the 14, used a way of organizing their arrays that was parallel to the Cartesian coordinate system (see the upper left array of Figure 6). These observations suggest that the students we worked with had a relatively stable way for organizing arrays across all problems that did not match the conventional way of organizing the Cartesian coordinate system.

There were also interesting consistencies amongst the students' decisions. First, the most common decision was selecting the y-axis as the first axis; ten of the 14 students made this decision. Deciding to have the vertical axis represent the first axis allowed these students to read points either left to right or right to left. Of the ten students who chose the vertical axis as the first axis, nine made organizations that enabled them to read the points from left to right (see the top and bottom right corners of Figure 6). We think a large number of students likely used this organization because of the standard English language convention that involves reading from left to right. However, to read points in the Cartesian coordinate system in the way it is conventionally

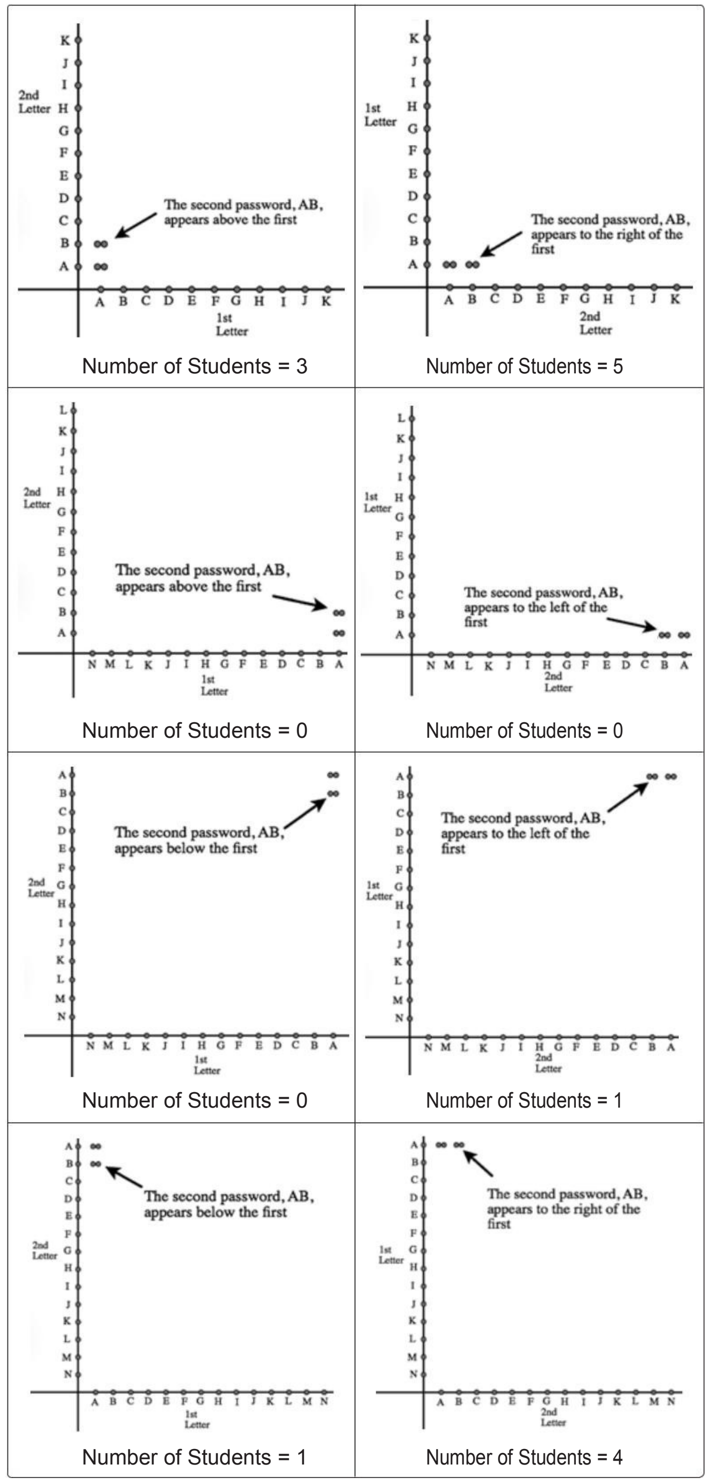

Figure 6: Eight possible ways to organize the 2-D arrays.

organized—and is consistent with how students typically list outcomes—means starting at the bottom left corner and moving upward: (1,1); (1,2); (1,3); (1,4) etc.

We used the different ways students organized their arrays to continue discussion about how different decisions regarding which axis is the first axis and which axis is the second axis leads to different organizations of the arrays. As part of these conversations, we specifically asked students questions like: Where is the first password from your list located in your array? Where are the second and third passwords from your list located in your array? In which direction are you reading your points (left-right)? These questions helped highlight the different structures of the arrays that students generated and allowed for discussion of how they were similar to or different from the structure of the Cartesian coordinate system.

WHAT NEXT?: MAKING A TRANSITION TO THE CARTESIAN COORDINATE SYSTEM

After working through problems like the *Outfits Problem* and *Password Problem*, students were prepared for problems like the *Digits Problem I, II*, and *III* previously discussed in this article. We have used problems like the *Digits Problem I*, *II*, and *III* in classrooms to facilitate discussion about the organizational conventions of the Cartesian coordinate system. Such discussion allows students to compare the way they organized ordered pairs in the *Outfits Problem* and *Password Problem* with the way ordered pairs are organized in the Cartesian coordinate system. This discussion supports students in understanding the difference between "correct" and "conventional" in mathematics, allowing them to see that there is a conventional way to organize arrays like the Cartesian coordinate system—but it is not the only way to organize arrays. These kinds of conversations make the organizational structure of the Cartesian coordinate system explicit because students have considered alternative possibilities for organizing arrays prior to working through problems that support them in developing the Cartesian coordinate system. These alternative possibilities are rarely considered in typical instructional approaches despite the fact that students routinely struggle to use correct conventions when naming points in the Cartesian coordinate system (Sarama, et. al., 2003). Although we, as teachers, find using these conventions to be "normal" or "natural," it is important we make this a point of conversation with students because they do not often consider these same features of mathematical representations to be "normal" or "natural" when they first develop them. The 14 sixth-grade students mentioned in this paper made comments indicating they had never thought about different ways to organize arrays. Also, they commented how conversations like those discussed in this article helped them think about how the Cartesian coordinate system was actually organized. Thus, a problem-based approach to developing the Cartesian coordinate system provides a rich context for *productive mathematical discourse* (Chapin, O'Connor, & Anderson, 2013) to unfold and also affords a thorough investigation of this fundamental mathematical representation.

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