GENERALIZATION ACROSS DOMAINS: THE RELATING-FORMING-EXTENDING GENERALIZATION FRAMEWORK

Amy Ellis
University of Georgia
amyellis@uga.edu

Erik Tillema
Indiana University Purdue University Indianapolis
etillima@iupui.edu

Elise Lockwood
Oregon State University
elise.lockwood@oregonstate.edu

Kevin Moore
University of Georgia
kvcmoore@uga.edu

Generalization is a critical aspect of doing mathematics, with policy makers recommending that it be a central component of mathematics instruction at all levels. This recommendation poses serious challenges, however, given researchers consistently identifying students' difficulties in creating and expressing normative mathematical generalizations. We address these challenges by introducing a comprehensive framework characterizing students' generalizing, the Relating-Forming-Extending framework. Based on individual interviews with 90 students, we identify three major forms of generalizing and address relationships between forms of abstraction and forms of generalization. This paper presents the generalization framework and discusses the ways in which different forms of generalizing can play out in activity.

Keywords: Cognition, Learning Theory, Reasoning and Proof

Introduction: The Importance of Mathematical Generalization

The act of generalizing is at the core of mathematical activity, serving as the means of constructing new knowledge. Researchers have argued that mathematical thought cannot occur in the absence of generalization (Sriraman, 2003; Vygotsky, 1986). As a result, "developing children's generalizations is regarded as one of the principal purposes of school instruction" (Davydov, 1972/1990, p. 10). Researchers have studied the importance of generalization for promoting algebraic reasoning (Cooper & Warren, 2008), mathematical modeling (Becker & Rivera, 2006), functional thinking (Ellis, 2011; Rivera & Becker, 2007), and probability (Sriraman, 2003), among other areas. Despite the importance of generalization to success in mathematical reasoning, research on students' abilities to generalize has identified pervasive student difficulties. For instance, Rivera (2008) reported results of 5 years of performance assessments on generalization given to more than 60,000 middle and early high school students; these findings revealed a stable ceiling value of only a 20% success rate in the construction of a general formula. Other researchers have similarly documented students' difficulties in creating correct general statements, shifting from pattern recognition to pattern generalization, and using generalized language (e.g., English & Warren, 1995; Mason, 1996).

Although student difficulties are well documented, the instructional conditions necessary for fostering more productive student generalizing are not well understood. Complicating the matter, the bulk of research on generalization has occurred with algebraic patterning tasks, situating generalization as a type of, and route to, algebraic reasoning (Becker & Rivera, 2006; Cooper & Warren, 2008). There remains a need to understand how students construct generality in more varied and more advanced mathematical domains. The goals of this study are to investigate students' mathematical generalizing from middle school through the undergraduate level in the topics of algebra, advanced algebra, and combinatorics. In particular, our aim is to elaborate the nature of students' generalizing, contributing to the field's knowledge base by extending the investigation of generalization up the grade bands. Based on clinical interviews with 90 students from 6th grade through the undergraduate level, we introduce a framework characterizing three major forms of generalizing activity: *relating*, *forming*, and *extending*. We also introduce and discuss relationships between forms of generalization and forms of abstraction.

Theoretical Framework

Forms of Generalization

Definitions of generalization vary, with the most prominent situating generalization as an individual, cognitive construct (e.g., Kaput, 1999). More recent sociocultural definitions position generalization within activity and context, as a collective act distributed across multiple agents (Tuomi-Gröhn & Engeström, 2003). These perspectives attend to how social interaction, tools, and history shapes people's generalizing, recognizing generalization as a social practice that is rooted in activity and discourse (Jurow, 2004). We borrow from both the cognitive and the sociocultural traditions to define generalizing as an activity in which learners in specific sociocultural and instructional contexts engage in at least one of the following three actions: (a) identifying commonality across cases (Dreyfus, 1991), (b) extending one's reasoning beyond the range in which it originated (Radford, 2006), and/or (c) deriving broader results from particular cases (Kaput, 1999). We use the term *generalizing* to refer to any of these processes, whereas *generalization* refers to the outcome(s) of these processes.

Borrowing from Lobato's (2003) transfer framework, we take an actor-oriented approach to studying students' processes of generalizing. This approach represents a shift from the observer's

(usually the researcher's) stance to the actor's (the student's) stance. In particular, it compels us to abandon normative notions of what should count as a generalization, instead seeking to understand the processes by which students construct relations of similarity that they experience as meaningful. Our framework also builds on Ellis' (2007) taxonomy of generalizations, which distinguishes between students' activity as they generalize, called *generalizing actions*, and students' final statements of generalization, called *reflection generalizations*.

Forms of Abstraction

The second line of research we rely on examines the role of abstraction in developing generalizations (e.g., Dorfler, 1991). Abstraction has been characterized in multiple ways, but we focus particularly on reflective abstraction and the interrelationships among the actions and operations that constitute students' construction of mental objects. In particular, we distinguish three types of reflective abstraction salient in informing students' generalizing activity: *pseudo-empirical abstraction*, *reflecting abstraction*, and *reflected abstraction* (Montangero & Maurice-Naville, 1997; Piaget, 2001). Pseudo-empirical abstraction is based on the observation of perceptible results, in which new knowledge is drawn not just from the properties of objects, but from how the student has organized the activities she has exerted on those objects. We further distinguish pseudo-empirical abstraction from other forms by noting that pseudo-empirical abstraction includes reflection on the *outcome* of one's activity. The focus is on the products of a learner's actions, rather than the coordination and transformation of actions themselves.
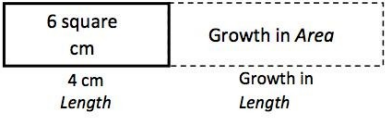
In contrast, reflecting abstraction includes reflection on one's actions, not merely on the outcomes of those actions. One can transfer to a higher plane what he or she has gleaned from lower levels of activity, leading to differentiations that imply new, generalizing compositions at that higher level. In reflected abstraction, one becomes conscious of his or her actions, bringing awareness of qualitative differences between his or her actions. Through reflected abstraction, one is able to formulate, formalize, and subsequently operate on his or her thought.

Methods

We conducted a series of individual semi-structured interviews with middle school (ages 12-14), high school (ages 14-17), and undergraduate students in the domains of algebra, advanced algebra, discrete mathematics, and combinatorics. The algebra and advanced-algebra topics included linear, quadratic, higher-order polynomial, and trigonometric functions, and the discrete mathematics and combinatorics topics included counting problems, combination and permutation problems, and the binomial theorem. We conducted 10 middle -school, 11 high-school, and 10 undergraduate algebra or advanced algebra interviews, and 19 middle-school, 13 high-school, and 27 undergraduate discrete mathematics (combinatorics) interviews.

During the interviews we presented the participants with domain-specific tasks to elicit both near and far generalizations, and we asked the participants to identify patterns and themes, discuss any elements of similarity they noticed, and, where reasonable, explain and discuss the generalizations they formed. All interviews were videotaped and we used gender -preserving pseudonyms for all participants. Table 1 presents a sample of the interview tasks across the mathematical domains.

Table 1: Sample Interview Tasks

Interview Task	Domain and grade level
The rectangle below grows along the dotted path as shown: 6 square cm Growth in *Area* 4 cm *Length* Growth in *Length* Complete the following statement: When the length of the rectangle grows by _____, the area grows by _____.	Algebra, middle school
You have a 1 cm by 1 cm by 1 cm cube, and all sides grow at the same rate. How much additional volume does the cube gain when the sides each increase by 1 cm?	Adv. algebra, high school
You have a deck of number cards numbered 1-6. You create a two-card hand by drawing a card from the deck, putting it back, and drawing a second card. Determine how many possible two-card hands you could get. How many times the number of two-card hands would you have if you had twice the number of cards?	Discrete math, middle school
Suppose passwords consist of (uppercase) As, Bs, and/or the number 1. How many such passwords are there that are *n* characters long?	Combinatorics, undergraduate

Analysis

We relied on the constant comparative method (Strauss & Corbin, 1990) to analyze the interview data in order to identify forms of generalization and abstraction. For the first round of analysis we drew on Ellis' (2007) analytic framework for categorizing students' generalizing actions and reflection generalizations, using open coding to infer categories of generalizing activity based on students' talk, gestures, and task responses. This first round led to an initial set of codes, which then guided subsequent rounds of analysis in which the project team met weekly to refine and adjust the codes in relation to one another. This iterative process continued until no new codes emerged. A final round of analysis was descriptive and supported the development of an emergent set of relationships between forms of abstraction and forms of generalizing, characterizing the evolving nature of students' mental activity as they generalized.

Results: The Relating-Forming-Extending Framework

Based on data analysis from the 90 interviews we developed an empirically-grounded framework capturing the broad range of generalizing activity across a variety of grade bands and domains. We present the results in two major sections. First we introduce the framework itself, which provides definitions, descriptions, and examples of each form of generalization demonstrated by the study participants (Tables 2-4). Due to space constraints, we do not elaborate on every form of generalizing, but we instead present a data episode identifying the interrelationships between the forms of abstraction and forms of generalizing. This episode is meant to be representative of the explanatory power of the framework, which we limit to one student due to space considerations. The Relating-Forming-Extending framework distinguishes between *inter-contextual* forms of generalizing, in which students established relations of similarity across problems or contexts, and *intra-contextual* forms of generalizing, in which

students formed and extended similarities and regularities within one task. Following the actor-oriented perspective, we made the inter/intra distinction based on evidence of whether the student perceived the establishment of similarity or regularity he or she formed to occur across different contexts or situations, or to occur within the same context.

Table 2: Inter-Contextual Forms of Generalizing (Relating)

Form of Generalizing		Example
Relating Situations: Forming a relation of similarity across contexts, problems, or situations	*Connecting Back*: Formation of a connection between a current and previous problem or situation.	*HS Adv. Algebra Student:* All the sides are the same length. The formula is generally the same [as the prior problem], you're just adding one more side for the 4-dimensional one.
	Analogy Invention: Creating a new situation or problem to be similar to the current one.	*MS Algebra Student*: The more seconds he has, he'll slow down. And the less seconds he has, he'll speed up faster. *Int*: Okay, and how come? *Student*: You know how, if you had less time to go into the grocery store to get the foods on the grocery list, you would go faster if you had like 1 second to do it in? You would, like, be in and out real quick. Same thing here.
Relating Ideas or Strategies (Transfer): Influence of a prior context or task is evident in student's current operating.		*HS Adv. Algebra Student*: So in this case it'd be P plus, let's do V for valence because that's one word I know for outer ring. *Int*: Cool, is that from chemistry? *Student*: Yep. Like the valence electrons…how much that equals plus the previous one, would equal your new answer.

The inter-contextual forms of generalizing all involved a type of *relating* activity. The intra-contextual generalizing, however, occurred in two major categories: (a) *forming* a similarity or regularity, in which students searched for and identified similar elements, patterns, and relationships (Table 3); and (b) *extending* or applying a similarity or regularity (Table 4). In the latter case, students extended established patterns or relationships to new cases.

Table 3: Intra-Contextual Forms of Generalizing (Forming)

Form of Generalizing		Example
Relating Objects:	*Operative*: Associating objects by isolating a	*Und. Adv. Algebra Student*: [Comparing $x = \sin(y)$ with $y = \sin(x)$ graphs] They're both representing the

Forming a relation of similarity between two or more present mathematical objects	similar property, function, or structure.	same thing…with equal changes of angle measures my vertical distance is increasing at a decreasing rate [tracing graph]…here it's doing the exact same thing.
	Figurative: Associating objects by isolating similarity in form.	*HS Adv. Algebra Student*: How does the volume equation relate to this cube? Well the three numbers are getting one bigger and the three sides got one bigger.
	Activity: Relating objects or ideas based on identifying one's activity as similar.	*MS Algebra Student*: I think it would be 2 more than the 6. *Int*: Two more than the 6? Okay, how come? *Student*: Because, like, same as this one [points to the prior problem] you're just adding it.
Search for similarity or regularity: Searching to find a stable pattern, regularity, or element of similarity across cases, numbers, or figures.		*HS Adv. Algebra Student*: I don't think it goes up by the same amount each time. Does it? That goes up by 3, and that goes up by 5, and that goes up by 7. Three, 5, 7. Yeah, it goes up by…okay.
Identify a regularity: Identification of a regularity or pattern across cases, numbers, or figures.	*Extracted*: Extracting regularity across multiple cases.	*MS Combinatorics Student*: For every addition problem that we do, like 6 plus 6 equals 12, it is always one more added to that every time.
	Projected: Describing a predicted or known stable feature.	*MS Algebra Student:* You could do, you could do 1.5 times growth and that would get you, times the growth in the length and then that would give you the growth in area.
Isolate constancy: Focusing on and isolating regularity – a stable feature – across varying features.		*HS Adv. Algebra Student*: This is like the one thing that you started off with [circles the original rectangle]. It's like the only constant really. And so each time it changes a little bit so it's really one of these is being added each time and so that's not really taking it into account, the 15 that was already there.

Table 4: Intra-Contextual Forms of Generalizing (Extending)

Form of Generalizing		Example
Continuing: Continuing an existing pattern or regularity to a new case.		*MS Combinatorics Student*: [Moves from a 7-card case to an 8-card case]: It is like the last time. You don't count (8, 8) twice.
Operating: Operating on an identified pattern, regularity, or relationship in order to extend it to a new case.	*Near*: Making a minor change to a regularity in order to extend it to a new case.	*HS Adv. Algebra Student*: [After having established a pattern of adding 8 square units for every additional rectangle]: And then plus 8, or I could just do plus, um, 8 times 5, right? And so that would be 40.
	Projection: Making a major change to a regularity in order to project it to a far case.	*Und. Combinatorics Student*: [After solving cases with 3 and 4 combinations]: So now I believe if you gave me something where if there was 20 combinations I could solve how many combinations there are without having to write them all out: 2^{20} and whatever that equals.
Transforming: Extending a generalization and, in doing so, changing	*Constructing a Quantity*: Constructing a new quantity or a relationship between quantities in order to extend a regularity to a new case.	*HS Adv. Algebra Student*: [Exploring the three sides of a rectangular prism, the interviewer asks the student to express one side in terms of the other.] So it's x plus 1, right?

the generalization that is being extended.	*Recursive Embedding:* Embedding a previous situation into a new one as a key component of the new task.	*Und. Combinatorics Student*: Okay, we're definitely using 1, so we're limiting ourselves to only 2 possible states for the entire password, A and B, which means it's basically no different than what we did in one of the earlier examples. So that I'll probably just figure, okay, 2 to the 3rd equals 8.
Removing particulars: Extending a specific relationship, pattern, or regularity by removing particular details to express the relationship more generally.		*MS Algebra Student*: I was thinking, like, trying to put it in an equation I guess, so it kind of makes sense…Well it could be, area equals 2 plus total length [writes A = 2 + T].

We illustrate several intra-contextual generalizations and their relationships to forms of abstraction by presenting the work of Willow, a middle-school algebra student, who worked on the growing rectangle task (Table 1). Willow initially established a numerical relationship between the length of 4 cm and the area of 6 cm^2:

> Well, the area is 2 more than the length so I would think if, however, if they grew like the same amounts of, if this (points to the area) grew by 2 in the area, so it would be 8 and this (points to the length) grew by 2 and it would be 6, then it would always be 2 more if they grew in the same, like, the same amount.

Willow *identified a regularity* by stating "It (the areas) would always be 2 more (than the length)". Although Willow's generalization is incorrect, it represented a pattern that she saw as valid. We also suspect Willow's generalization relied on a pseudo-empirical abstraction, not because her generalization was incorrect, but because she appeared to generalize based on the outcome of her activity. Specifically, Willow's operation was to take the difference of the numbers 4 and 6, and she generalized the difference remaining constant. She made an additive comparison between numerical values that did not appear to be based in quantitative operations relating length to area. When asked what would happen if the rectangle grew by another 4 cm, Willow responded, "So it grew by 4…would the area have grown by 4 too? It could be, like, 10." Here Willow *extended by continuing* the "area = length + 2" relationship she had established to a new case. She then further generalized by stating, "If the length grew by *x*, then the area would be 2 more than the total length," which she expressed as "A = 2 + T". Here Willow *extended by removing particulars* in order to algebraically express the relationship she had established. We maintain that this string of generalizations remained grounded in Willow's activity of pseudo-empirical abstraction. Her focus remained on the result of her operation, the difference of 2, and at no time did Willow coordinate the growth of the rectangle simultaneously with varying measures of length and area.

Right when the interviewer began to remove the task in order to transition to a new problem, Willow suddenly evidenced a shift in her thinking, saying, "Unless it will start at 0?"

> Because if you start it at 0…to find out the actual growth, then, say this is like the first they grew and this, kind of, so this grew by 4 first (gestures along the length) and then this grew by 6 (gestures to the whole figure, along the area). So this (the length) could grow by 4 again, and this (the area) could grow by 6 again.

Willow appeared to construct a dynamic image of the rectangle growing "from 0". She further explained, "Because it would always be plus 4 and plus 6, so if you said when the length grows by 8, the area grows by 12." Willow imagined the rectangle growing in chunks, iterating twice. Willow therefore *identified a regularity* that if the rectangle started growing from 0, then for every 4-cm increase in length, the rectangle would increase in area by 6 cm^2. This regularity, unlike the first one Willow identified, was based on an image of growth in which Willow was able to coordinate an increase in length with a corresponding increase in area. This image was informed

by the operations of forming a ratio and iterating it. It was also a product of reflecting abstraction in that Willow reflected on her activity in order to coordinate iterating her formed ratio with the number of times it was iterated. Therefore, she could then state that the length would increase by 4 again, resulting in another increase of 6 for the area. Willow *extended by continuing* the relationship, and she did so by relying on her ability to coordinate growth in one quantity with growth in the other.

We take further evidence that Willow engaged in reflecting abstraction by what occurred next. Namely, she was able to *extend by operating* on the relationship she had formed, multiplying each term in the 4:6 ratio by 4, then by 10, ½, ¼, ¾, and 5/4 in order to generate new length:area pairs. This extension was significant because it included the use of both whole number and fraction values. It also suggests that Willow had reflected on her operation of forming a ratio in order to develop a flexible, generalizable relationship with which she could meaningfully operate. Willow ultimately developed a unit ratio, explaining, "Each time the growth in length goes up by 1, the growth in area, I think the growth in area equals [writes A = 1.5 × L]." Thus Willow *identified a new regularity* and then *removed particulars* for this regularity. When she removed particulars, she reflectively abstracted a ratio from the phenomenological bounds in which it was created, and Willow's subsequent flexible use of this ratio with messy numbers is evidence that she could imagine it holding for any arbitrary value.

Discussion

The Relating-Forming-Extending framework identifies forms of generalizing based on data from multiple grade bands and mathematical domains, addressing the need to understand how students construct generality in more varied and advanced mathematical contexts. Willow's work provides evidence that students can and do generalize their reasoning on a variety of problems beyond typical patterning tasks. In particular, in contrast to much of the literature identifying how students inductively generalize patterns, Willow abductively (Peirce, 1931-1958; Radford, 2006) developed a generalization from just one case. Willow's reflective activity enabled her to develop, solidify and apply generalizations in two ways. Firstly, she generalized an additive comparison based on the numerical relationship she established between 4 and 6 (a pseudo-empirical abstraction). Secondly, she generalized by forming and operating on a ratio between quantities that was rooted in her image of the rectangle's length growing in tandem with its area (a reflecting abstraction).

The Relating-Forming-Extending framework extends prior work by distinguishing and characterizing three forms of generalizing activity and by coordinating these forms of generalizing with forms of abstracting. The case of Willow shows that students can engage in many forms of generalizing, such as *identifying regularities*, *extending by continuing*, and *removing particulars*, based on either pseudo-empirical or reflecting abstraction. Other forms of generalizing, such as *extending by operating* or *transforming*, appear to be more typically grounded in reflecting abstraction, as they often entail differentiations based on activity in order to support new compositions. By attending to both abstraction and generalization in students' sense-making, we can begin to characterize how students can leverage initial abstractions into first-pass generalizations that they can then reflect on and transform in further activity. Further analysis of these relationships between abstraction and generalization will inform a better understanding of the conceptual mechanisms driving generalizing activity in a variety of mathematical contexts.

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