A NOTE ON THE STRONG LAW OF LARGE NUMBERS

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1. Introduction. Let $\{X_k\}$ denote a sequence of independent, identically distributed (i.i.d.) random variables. Let

$$S_n = \sum_{k=1}^{n} X_k \qquad (n = 1, 2, \cdots).$$

A long standing problem in probability theory has been to find necessary and sufficient conditions on the distribution function of X_k in order that $n^{-1}S_n$ converge almost surely to plus infinity. The purpose of this paper is to exhibit such conditions.

2. THEOREM. *Let $\{X_k\}$ denote a sequence of i.i.d. random variables with common characteristic function ϕ. Then $n^{-1}S_n \to +\infty$ a.s. if and only if*

$$(1) \qquad \lim_{b\to\infty} \int_{-1}^{1} \frac{e^{iub}-1}{iu} \log \left\{1 - \frac{e^{-iua}\phi(u)}{1+u^2}\right\}^{-1} du$$

is finite for each $a>0$.

The proof of the theorem is based on the following lemma.

LEMMA. *Let $\{X_k\}$ denote a sequence of i.i.d. random variables. Then*

(a) $n^{-1}S_n \to +\infty$ a.s.

if and only if, for each $a>0$,

(b) $\sum_{n=1}^{\infty} n^{-1}P(S_n < an) < \infty.$

PROOF OF THE LEMMA. We first show that (a) implies (b). Suppose there exists an $a>0$ such that $\sum n^{-1}P(S_n<an)=\infty$. Let

$$T_n = \sum_{k=1}^{n} (a - X_k).$$

Then $\sum n^{-1}P(T_n>0)=\infty$, and, by a theorem of Spitzer [2] it follows that lim sup $T_n=\infty$ a.s. However, lim $n^{-1}S_n=\infty$ a.s. certainly implies that lim sup $T_n=-\infty$ a.s. Thus (b) holds.

Conversely, suppose that (b) holds. Then, for each $a>0$,

$$\sum n^{-1}P(S_n - na < 0) < \infty$$

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and consequently, from the same work of Spitzer, we have that

$$\max_{k\geq 1} \quad (ka - S_k)^+ < \infty \text{ a.s.}$$

for all $a>0$. This clearly implies that $\lim n^{-1}S_n = \infty$ a.s., and therefore (a) holds.

Before proceeding to the proof of the theorem, we note that previous work (e.g. Derman and Robbins [1]) giving sufficient conditions that $\lim n^{-1}S_n = \infty$ a.s. follows quickly from the above lemma.

Proof of the Theorem. Let $\{Y_k\}$ denote a sequence of i.i.d. random variables, each with characteristic function $(1+u^2)^{-1}$, and, further, let $\{Y_k\}$ be independent of $\{X_k\}$. Write $Z_k = X_k + Y_k$ and

$$W_n = \sum_{k=1}^{n} Z_k \qquad (n = 1, 2, \cdots).$$

Since Y_k has expectation zero, $n^{-1}S_n \to \infty$ a.s. if and only if $n^{-1}W_n \to \infty$ a.s.

By means of a well-known inversion formula, we have that

$$P(an - b \leqq W_n < an) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuan}\left(\frac{e^{iub} - 1}{iu}\right)\left(\frac{\phi(u)}{1 + u^2}\right)^n du.$$

(Note that $\{\phi(u)/(1+u^2)\}^n$ is integrable and that W_n has an absolutely continuous distribution function.)

Therefore,

$$\sum_{n=1}^{\infty} n^{-1}P(an - b \leqq W_n < an)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iub} - 1}{iu} \log\left\{1 - \frac{e^{-iua}\phi(u)}{1 + u^2}\right\}^{-1} du,$$

the interchange of sum and integral being justified, since

$$\begin{aligned}
&\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\left|\frac{1}{n}e^{-iuan}\left(\frac{e^{iub} - 1}{iu}\right)\left(\frac{\phi(u)}{1 + u^2}\right)^n\right| du \\
&\leqq b\int_{-\infty}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left|\frac{\phi(u)}{1 + u^2}\right|^n\right\} du \\
&\leqq b\int_{-\infty}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{1 + u^2}\right)^n\right\} du \\
&= b\int_{-\infty}^{\infty}\log\left(1 + \frac{1}{u^2}\right) du \\
&< \infty.
\end{aligned}$$

From the Monotone Convergence Theorem, it follows that

$$\sum_{n=1}^{\infty} n^{-1}P(W_n < an) = \lim_{b\to\infty} \sum_{n=1}^{\infty} n^{-1}P(an - b \leqq W_n < an)$$
$$= \lim_{b\to\infty} \int_{-\infty}^{\infty} \frac{e^{iub} - 1}{iu} \log\left(1 - \frac{e^{-iua}\phi(u)}{1 + u^2}\right)^{-1} du.$$

By the Riemann-Lebesgue lemma,

$$\lim_{b\to\infty} \int_{|u|>1} \frac{e^{iub} - 1}{iu} \log\left(1 - \frac{e^{-iua}\phi(u)}{1 + u^2}\right)^{-1} du$$

exists and is finite. It follows that (1) is finite for each $a>0$ if and only if $\sum n^{-1}P(W_n<an)$ is finite for each $a>0$. The latter condition is equivalent to $n^{-1}W_n \to \infty$ a.s. which is, in turn, equivalent to $n^{-1}S_n \to \infty$ a.s. This completes the proof.

References

1. C. Derman and H. Robbins, *The strong law of large numbers when the first moment does not exist*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 586–587.

2. F. Spitzer, *A combinatorial lemma and its application to probability theory*, Trans. Amer. Math. Soc. **82** (1956), 323–339.

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