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SPECTRAL STABILITY FOR A CLASS OF FOURTH ORDER STEKLOV PROBLEMS UNDER DOMAIN PERTURBATIONS

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ABSTRACT. We study the spectral stability of two fourth order Steklov problems upon domain perturbation. One of the two problems is the classical DBS - Dirichlet Biharmonic Steklov - problem, the other one is a variant. Under a comparatively weak condition on the convergence of the domains, we prove the stability of the resolvent operators for both problems, which implies the stability of eigenvalues and eigenfunctions. The stability estimates for the eigenfunctions are expressed in terms of the strong H^2-norms. The analysis is carried out without assuming that the domains are star-shaped. Our condition turns out to be sharp at least for the variant of the DBS problem. In the case of the DBS problem, we prove stability of a suitable Dirichlet-to-Neumann type map under very weak conditions on the convergence of the domains and we formulate an open problem. As bypass product of our analysis, we provide some stability and instability results for Navier and Navier-type boundary value problems for the biharmonic operator.

Keywords: Biharmonic operators, Steklov boundary conditions, spectral stability, domain perturbation, Dirichlet-to-Neumann maps.

2000 Mathematics Subject Classification: 35J40, 35B20, 35P15

1. INTRODUCTION

In this paper we consider a class of *Steklov problems* and related spectral properties. By Steklov problems we mean a class of elliptic problems with parameters in the boundary conditions. This denomination is motivated by [35] where the classical Steklov problem

$$
(1) \qquad \begin{cases} \Delta u = 0, & \text{in } \Omega\,, \\ u_\nu = \lambda u, & \text{on } \partial\Omega\,, \end{cases}
$$

was introduced. We refer to the recent paper [25] for a survey on this subject. Here we are interested in the study of a class of elliptic equations with the biharmonic operator subject to Steklov type boundary conditions. Such kind of conditions for the biharmonic operator can be already found in the following works [22, 27, 28, 29, 32]. In those papers the attention is mainly focused on the first eigenvalue of the fourth order Steklov problem and its dependence on the domain. To be more clear, let us introduce the classical DBS - Dirichlet Biharmonic Steklov - problem. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain (i.e., connected open set) with sufficiently regular boundary. Consider the fourth order problem

$$
(2) \qquad \begin{cases} \Delta^2 u = 0, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \\ \Delta u - d u_\nu = 0, & \text{on } \partial\Omega\,, \end{cases}
$$

where u_ν denotes the normal derivative on the boundary and d is a real parameter. By a solution of (2) we mean a function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$
(3) \qquad \int_\Omega \Delta u \Delta v\, dx = d \int_{\partial\Omega} u_\nu v_\nu\, dS\,, \qquad \text{for any } v \in H^2(\Omega) \cap H_0^1(\Omega)\,.
$$

Clearly (2) is a linear homogeneous problem and hence it always admits the trivial solution. Therefore, it appears reasonable to interpret (2) as an eigenvalue problem with respect to the parameter d: we say that d is an eigenvalue for (2) if (2) admits a nontrivial solution in the sense given in (3). Such a nontrivial solution

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is called eigenfunction for d. In the sequel, we will refer to (2) as the *Steklov problem.* It is well-known that the eigenvalues of (2) are strictly positive and that the first eigenvalue d_1 admits the following variational characterization

$$(4) \qquad d_1 = \inf_{v \in (H^2(\Omega) \cap H^1_0(\Omega)) \setminus H^2_0(\Omega)} \frac{\int_\Omega |\Delta v|^2 \, dx}{\int_{\partial\Omega} v_\nu^2 \, dS}.$$

In [29, 32] the authors study the isoperimetric properties of d_1 and in [22] it is proved that d_1 is the sharp constant for L^2 a priori estimates for solutions of the (second order) Laplace equation under nonhomogeneous Dirichlet boundary conditions, see also [27, 28].

From a physical point of view, it is known that in the two dimensional case $N = 2$, Steklov boundary conditions for the biharmonic operator have a natural interpretation in modeling the vibrations of a hinged plate. For more details see for example [14, 15, 21, 23, 24, 30].

In the last years, the Steklov boundary conditions for the biharmonic operator were studied in [8, 24] from the point of view of the positivity preserving property for nonhomogeneous biharmonic equations. Other recent results were focused on isoperimetric properties of d_1 and on related shape optimization problems, see [2, 13, 14, 21]. More precisely from these papers, among many other results, we know that, among all convex domains of fixed volume, there exists an optimal domain which minimizes d_1 and that this domain is not the ball: for example in dimension $N = 2$ the square is better than the disk. It is still an open question, even in dimension $N = 2$, which is the optimal shape among convex domains of fixed volume. We also quote the recent paper [7] where the eigenvalues of (2) are used to provide the singular value decomposition for the Poisson kernel of the Laplace operator on bounded domains.

For its physical interpretation and for its relevance in the study of positivity preserving properties, in our paper we also consider the following Steklov type problem

$$(5) \qquad \begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u - K(x)u_\nu - \delta u_\nu = 0, & \text{on } \partial\Omega, \end{cases}$$

where K denotes the mean curvature of the boundary, i.e. the sum of the principal curvatures and δ is a real parameter. As one can see for example from [24], when $N = 2$, the curvature of the boundary naturally comes out when we consider the model of a hinged plate. Problem (5) admits the following variational formulation: we say that $u \in H^2(\Omega) \cap H^1_0(\Omega)$ is a weak solution of (5) if

$$(6) \qquad \int_\Omega D^2 u : D^2 v \, dx = \delta \int_{\partial\Omega} u_\nu v_\nu \, dS \qquad \text{for any } v \in H^2(\Omega) \cap H^1_0(\Omega)$$

where $D^2 u$, $D^2 v$ denote the Hessian matrices of u, v respectively and $D^2 u : D^2 v := \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$. In the sequel we also write $|D^2 u|^2 := \sum_{i,j=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2$. For more details on the correlation between (5) and (6) see the proof of [19, Proposition 5] and combine it with the identity $\Delta u = u_{\nu\nu} + K(x)u_\nu$ on $\partial\Omega$ (see [34, (4.68), p. 62]) valid for any smooth enough function u vanishing on $\partial\Omega$. Here $u_{\nu\nu}$ denotes the second order normal derivative, i.e. $u_{\nu\nu} = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \nu_i \nu_j$ with $\nu = (\nu_1, \ldots, \nu_N)$.

Similarly to problem (2), also problem (5) may be interpreted as an eigenvalue problem with respect to the parameter δ: we say that δ is an eigenvalue of (5) if (5) admits a nontrivial solution in the sense given in (6). In the sequel, we refer to (5) as the *modified Steklov problem.*

It is known that the set of the eigenvalues of (2) is a countable set of isolated positive numbers which may be ordered in an increasing sequence diverging to $+\infty$. We choose the usual notation of repeating each eigenvalue as many times as its multiplicity:

$$(7) \qquad 0 < d_1 < d_2 \leq \cdots \leq d_n \leq \ldots$$

As one can see from (7) we wrote $d_1 < d_2$ in order to emphasize the fact that the first eigenvalue d_1 is simple. We quote the papers [13, 21] for more details on these and further properties of the eigenvalues of (2).

In a completely similar way, it can be proved that also the set of the eigenvalues of (5) is a countable set of isolated numbers which may be ordered in an increasing sequence diverging to $+\infty$ and moreover the least eigenvalue is strictly positive. For the eigenvalues of (5) we use a notation analogous to (7):

$$(8) \qquad\qquad 0 < \delta_1 \le \delta_2 \le \cdots \le \delta_n \le \ldots$$

Here we are interested in the study of the stability of the eigenvalues and eigenfunctions of (2) and (5) with respect to domain perturbation. We note that the stability of the eigenvalues of the classical second order problem (1) has been discussed in [10] as well as in [37] which is actually concerned with the stability of the so-called Dirichlet-to-Neumann maps.

More precisely if $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is a family of domains which converges to a domain Ω in a suitable sense, denoting by $\{d_n^\varepsilon\}_{n=1}^\infty$, $\{d_n\}_{n=1}^\infty$ the eigenvalues of the Steklov problem in Ω_ε and Ω respectively, and by $\{\delta_n^\varepsilon\}_{n=1}^\infty$ $\{\delta_n\}_{n=1}^\infty$ the eigenvalues of the modified Steklov problem in Ω_ε and Ω respectively, we say that we have stability for the eigenvalues of each of the two problems, if for any $n \ge 1$, $d_n^\varepsilon \to d_n$ and $\delta_n^\varepsilon \to \delta_n$ as $\varepsilon \to 0$. We are also interested in what we will call spectral convergence of $S_{\Delta,\varepsilon}$ to S_Δ and of $S_{D^2,\varepsilon}$ to S_{D^2}, where $S_{\Delta,\varepsilon}$, S_Δ denote the linear operators associated to (2) respectively in Ω_ε and Ω, and $S_{D^2,\varepsilon}$, S_{D^2} denote the linear operators associated to (5) respectively in Ω_ε and Ω. See Section 3.1 for the precise definitions of these operators.

Basically, these operators are the resolvent operators associated with problems (2), (5) and their spectral convergence is obtained by proving their compact convergence which is a well-known notion of convergence implying the convergence of eigenvalues and eigenfunctions. Unfortunately, since the underlying function spaces depend on ε, we cannot directly use the standard notion of compact convergence, but we need to use suitable 'connecting systems' which allow to pass from the varying Hilbert spaces defined on Ω_ε to the limiting fixed Hilbert space defined on Ω. Namely, we use a number of notions and results which go back to the works of F. Stummel [36] and G. Vainniko [38] and which have been further implemented in [3, 17]. See also the recent paper [9]. In particular we use the notion of E-compact convergence. These notions are discussed in detail in Section 2.1.

In order to prove the E-compact convergence of the operators under consideration, we consider domains Ω_ε, Ω belonging to uniform classes of domains with $C^{1,1}$ boundaries - see Definition 2.6 - and we require that the boundaries of Ω_ε converge to the boundary of Ω in the sense of (21). Condition (21) has been introduced in [4] and turns out to be a sharp condition in the analysis of domain perturbation problems for fourth order operators. Note that if the boundaries of Ω_ε have uniformly bounded $C^{1,1}$-norms and converge to the boundary of Ω in C^1 then condition (21) is satisfied. However, it is important to note that condition (21) allows to deal also with boundaries whose $C^{1,1}$-norms are not uniformly bounded with respect to ε.

Condition (21) allows to construct in Section 3.2 a linear continuous operator $E_\varepsilon : H^2(\Omega) \cap H_0^1(\Omega) \to H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ and to define the notion of E-convergence for a sequence $u_\varepsilon \in H^2(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ to a function $u \in H^2(\Omega) \cap H_0^1(\Omega)$, that is

$$\|u_\varepsilon - E_\varepsilon u\|_{H^2(\Omega_\varepsilon)} \to 0, \ \text{ as } \varepsilon \to 0.$$

The operators E_ε are constructed by pasting together suitable pull-back operators defined by means of appropriate local diffeomorphisms. Importantly, the operators E_ε enjoy the following property: for any $\varepsilon > 0$ there exists an open set $K_\varepsilon \subset \Omega \cap \Omega_\varepsilon$ such that $(E_\varepsilon u)(x) = u(x)$ for all $x \in K_\varepsilon$ and such that $|(\Omega_\varepsilon \cup \Omega) \setminus K_\varepsilon| \to 0$ as $\varepsilon \to 0$. Here and in the sequel $|A|$ denotes the Lebesgue measure of any measurable set $A \subset \mathbb{R}^N$. This implies the more familiar strong convergence

$$(9) \qquad\qquad \|u_\varepsilon - u\|_{H^2(\Omega_\varepsilon \cap \Omega)} \to 0, \ \text{ as } \varepsilon \to 0.$$

Moreover, it also implies

$$(10) \qquad\qquad \|\nabla u_\varepsilon - E_\varepsilon \nabla u\|_{L^2(\partial\Omega_\varepsilon)} \to 0, \ \text{ as } \varepsilon \to 0,$$

and we note that $E_\varepsilon \nabla u_{|\partial\Omega_\varepsilon}$ is nothing but a natural transplantation of u from $\partial\Omega$ to $\partial\Omega_\varepsilon$ (which differs from an exact pull-back by a minor deformation at the intersections of different local charts describing the boundaries, a deformation which vanishes as $\varepsilon \to 0$). For more details on (9)-(10) see Proposition 3.11.

One of the main results of the paper is Theorem 3.2 where it is proved that condition (21) implies the E-compact convergence of $S_{\Delta,\varepsilon}$ to S_Δ and of $S_{D^2,\varepsilon}$ to S_{D^2} as $\varepsilon \to 0$. On the base of the general Theorem 2.5, this implies the spectral convergence of $S_{\Delta,\varepsilon}$ to S_Δ and of $S_{D^2,\varepsilon}$ to S_{D^2}, hence the convergence of the eigenvalues and the E-convergence of the eigenfunctions in the sense of Theorem 2.5. In particular the eigenfunctions converge in the sense of (9) and (10).

We note that our domains are not required to be star-shaped as in [37] (which is concerned with second order operators) and that removing such an assumption is not straightforward. In fact, here we only assume that the domains under consideration belong to a uniform class of domains defined by means of the same atlas and this leads to a number of technical difficulties that are overcome in Section 3. We believe that Section 3 gives a partial answer (in the case of fourth order operators) to the question raised in the introduction of [37] where the authors suggest the problem of studying the convergence of the resolvent operators of second order Steklov problems on arbitrary Lipschitz domains.

In Section 4 we analyze the optimality of condition (21). In the spirit of [4], we consider the case of a domain of the form $\Omega = W \times (-1, 0)$ where W is a cuboid or a bounded domain in $\mathbb{R}^{N-1}$ of class $C^{1,1}$ and we assume that the perturbed domain Ω_ε is given by

$$\Omega_\varepsilon = \{(x', x_N) : x' \in W, -1 < x_N < g_\varepsilon(x')\}$$

where $g_\varepsilon(x') = \varepsilon^\alpha b(x'/\varepsilon)$ for any $x' \in W$ and $b : \mathbb{R}^{N-1} \to [0, +\infty)$ is a nonconstant Y-periodic function where $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. We denote by Γ_ε and Γ the sets

$$\Gamma_\varepsilon := \{(x', g_\varepsilon(x')) : x' \in W\} \quad \text{and} \quad \Gamma := W \times \{0\}.$$

It appears that for $\alpha > 3/2$ condition (21) is satisfied hence we have spectral convergence of $S_{\Delta,\varepsilon}$ to S_Δ and of $S_{D^2,\varepsilon}$ to S_{D^2}, see Corollary 4.1. For $\alpha < 3/2$ condition (21) is not satisfied and we have a degeneration phenomenon. Namely, for $1 \le \alpha < 3/2$ we have spectral convergence of the operator $S_{D^2,\varepsilon}$ to $S_{D^2,\Gamma}$ where $S_{D^2,\Gamma}$ is an operator encoding Dirichlet boundary conditions on Γ, that is $u = u_\nu = 0$ on $\partial\Gamma$, see Theorem 4.1. For $\alpha = 3/2$, we have spectral convergence to another operator which encodes the appearance of a strange term in the boundary conditions, see Theorem 4.5. We note that in order to simplify our analysis in the case $\alpha = 3/2$, we impose Dirichlet boundary conditions on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$ and $\partial\Omega \setminus \Gamma$.

We also note that imposing Dirichlet boundary conditions on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$ and $\partial\Omega \setminus \Gamma$ allows to view the trichotomy discussed above in a rather transparent way. Namely, under Dirichlet boundary conditions on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$ and $\partial\Omega \setminus \Gamma$ we have that:

  (i) If $\alpha > 3/2$ then $\delta_n(\varepsilon) \to \delta_n(0)$, as $\varepsilon \to 0$.
  (ii) If $\alpha = 3/2$ then $\delta_n(\varepsilon) \to \delta_n(0) + \gamma$ as $\varepsilon \to 0$, where $\gamma > 0$ is the strange term defined by (134).
  (iii) If $1 \le \alpha < 3/2$ then $\delta_n(\varepsilon) \to \infty$, as $\varepsilon \to 0$.

The strange term γ is the same term found in [4] in the analysis of the Babuska Paradox and is referred to as 'the strange curvature' in [4].

The analysis of the cases $\alpha \le 3/2$ is carried out in Section 4 only for problem (5) since problem (2) presents severe difficulties in these cases. In order to overcome these difficulties, in Section 5 we change our point of view and we recast the eigenvalue problem (2) as an eigenvalue problem for a suitable Dirichlet-to-Neumann-type map. Considering our boundary conditions, we find it natural to call such a map Neumann-to-Navier map, and actually we find it simpler to consider its inverse which we call Navier-to-Neumann map and we denote by $\mathcal{N}_\Omega$. Since we aim at relaxing the boundary regularity assumptions, we are forced to work with the space $H(\Delta, \Omega) \cap H_0^1(\Omega)$ and we find it convenient to consider the normal derivatives of the functions in such a space as elements of $H^{-1/2}(\partial\Omega)$ although they belong to $L^2(\partial\Omega)$ see [7, Thm. 3.2] (recall that the space $H(\Delta, \Omega)$ is the space of functions in $L^2(\Omega)$ with distributional Laplacian in $L^2(\Omega)$). In this way, our Navier-to-Neumann map $\mathcal{N}_\Omega$ can be considered as a map from $H^1(\Omega)/H_0^1(\Omega)$ to its dual and problem (2) can be recast in the form $\mathcal{N}_\Omega f = d^{-1} J_0 f$ where J_0 is defined by $\langle J_0 f, v \rangle := \int_{\partial\Omega} f v \, dS$ for all $f, v \in H^1(\Omega)/H_0^1(\Omega)$. It is interesting to note that this weak formulation of problem (2) makes sense under the sole assumption that Ω is a bounded domain of class $C^{0,1}$. We note that, in order to avoid some technicalities which would not add much to the results of Section 5, we shall actually avoid using the quotient

spaces and we shall consider the Navier-to-Neumann $\mathcal{N}_\Omega$ directly as map from $H^1(\Omega)$ to its dual, which has also its own interest, see Remark 6.

Thus we study the behaviour of the maps $\mathcal{N}_{\Omega_\varepsilon}$ as $\varepsilon \to 0$ and in Theorem 5.6 we prove their EE^*-convergence to $\mathcal{N}_\Omega$ under weak conditions on the convergence of domains. Here the EE^*-convergence is a natural type of convergence for operators acting from a family of spaces to the corresponding duals and is based on two connecting systems E_ε and E_ε^* (E_ε is now defined by means of a standard extension operator for the space $H^1(\Omega)$), see Definition 5.4. In Theorem 5.7 we also prove a compactness result for the convergence of $\mathcal{N}_{\Omega_\varepsilon}$ under an additional assumption on the convergence of Ω_ε.

Unfortunately, the compact convergence result obtained in Theorem 5.7 is not sufficient to prove a stability result for the spectrum of the Steklov problem (2) under such very weak assumptions on the convergence of Ω_ε. Indeed, in order to apply [38, Theorem 6.1], the regular convergence condition 1) in [38, Section 6] has to be satisfied by the family of operators $\{\mathcal{N}_{\Omega_\varepsilon} - \lambda(J_0)_\varepsilon\}_\varepsilon$ for any $\lambda \in \Lambda$, where Λ is a suitable bounded set in the complex plane, and this does not seem the case, not even if we consider these operators on the quotient spaces and adjust the corresponding connecting systems. Taking into account this issue we suggest the Open Problem 5.1.

As a bypass product of the analysis carried out in Section 5, we prove stability and instability results for Navier and Navier-type problems, see Theorems 5.3 and 5.8.

The paper is organized as follows. In Section 2 we discuss some preliminary results, in particular we discuss the notion of E-convergence and our functional setting. In Section 3 we prove the main stability result of the paper under the assumption that condition (21) holds. In Section 4 we discuss the optimality of condition (21). In Section 5 we relax the assumptions on the convergence of the domains and we prove a stability result for the above mentioned Navier-to-Neumann map; moreover, we discuss the stability and instability of Navier and Navier-type problems.

2. PRELIMINARIES AND NOTATION

2.1. **A general approach to spectral stabilty.** The study of all spectral problems considered in this paper will be reduced to the study of convenient non-negative compact self-adjoint operators in Hilbert spaces.

A natural way to do so is to consider appropriate modified versions of the classical Dirichlet-to-Neumann map densely defined in $L^2(\partial\Omega)$ and to consider their resolvents which are defined on the whole of $L^2(\partial\Omega)$. For our fourth order Steklov problems, it is natural to talk about Neumann-to-Navier map and its inverse will be called here Navier-to-Neuman map. Then one note that the eigenvalues of those Navier-to-Neumann maps are in fact the reciprocals of the eigenvalues of our differential problems. One drawback of using Navier-to-Neumann maps consists in the fact that eventually they provide information concerning the convergence of the (normal derivatives of the) eigenfunctions considered as elements of $L^2(\partial\Omega)$. In order to provide global information about the behaviour of the eigenfunctions in Ω, in the regular cases we shall actually consider appropriate realisations of the resolvents in $H^2(\Omega)$ which will allow to get stronger stability results involving the norm of $H^2(\Omega)$.

Thus, our spectral problems will be considered from two different points of view depending on the regularity of the domain perturbations. In any case, both methods lead to the study of suitable families $\{B_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ of non-negative compact self-adjoint operators defined in Hilbert spaces $\mathcal{H}_\varepsilon$ associated with the domains Ω_ε. It is then important to study the convergence of those operators B_ε as $\varepsilon \to 0$. To do so, we shall use the notion of E-convergence which we now recall. We follow the approach of [38], further developed in [3], [17].

In the spirit of [3] we denote by $\mathcal{H}_\varepsilon$ a family of Hilbert spaces for $\varepsilon \in [0,\varepsilon_0]$ and we assume that there exists a family of linear operators $E_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$ such that

$$\|E_\varepsilon u\|_{\mathcal{H}_\varepsilon} \xrightarrow{\varepsilon\to 0} \|u\|_{\mathcal{H}_0}, \qquad \text{for all } u \in \mathcal{H}_0. \tag{11}$$

Definition 2.1. *We say that a family $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$, with $u_\varepsilon \in \mathcal{H}_\varepsilon$, E-converges to $u \in \mathcal{H}_0$ if $\|u_\varepsilon - E_\varepsilon u\|_{\mathcal{H}_\varepsilon} \to 0$ as $\varepsilon \to 0$. We write this as $u_\varepsilon \xrightarrow{E} u$.*

Definition 2.2. *Let $\{B_\varepsilon \in \mathcal{L}(\mathcal{H}_\varepsilon) : \varepsilon \in (0, \varepsilon_0]\}$ be a family of linear and continuous operators. We say that $\{B_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ converges to $B_0 \in \mathcal{L}(\mathcal{H}_0)$ as $\varepsilon \to 0$ if $B_\varepsilon u_\varepsilon \xrightarrow{E} B_0 u$ whenever $u_\varepsilon \xrightarrow{E} u$. We write this as $B_\varepsilon \xrightarrow{EE} B_0$.*

Definition 2.3. *Let $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family such that $u_\varepsilon \in \mathcal{H}_\varepsilon$. We say that $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ is precompact if for any sequence $\varepsilon_n \to 0$ there exist a subsequence $\{\varepsilon_{n_k}\}$ and $u \in \mathcal{H}_0$ such that $u_{\varepsilon_{n_k}} \xrightarrow{E} u$.*

Definition 2.4. *We say that $\{B_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ with $B_\varepsilon \in \mathcal{L}(\mathcal{H}_\varepsilon)$ and B_ε compact, converges compactly to a compact operator $B_0 \in \mathcal{L}(\mathcal{H}_0)$ if $B_\varepsilon \xrightarrow{EE} B_0$ and for any family $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ such that $u_\varepsilon \in \mathcal{H}_\varepsilon$, $\|u_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$, we have that $\{B_\varepsilon u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ is precompact in the sense of Definition 2.3. We write this as $B_\varepsilon \xrightarrow{C} B_0$.*

Recall that given a non-negative compact self-adjoint operator B in a infinite dimensional Hilbert space $\mathcal{H}$, the spectrum is a finite or countably infinite set and all non-zero elements of the spectrum are positive eigenvalues of finite multiplicity. In case the spectrum is a countably infinite set, the eigenvalues can be represented by a non-increasing sequence μ_n, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \mu_n = 0$. As customary, we agree to repeat each eigenvalue μ_n in the sequence as many times as its multiplicity. Given a finite set of m eigenvalues $\mu_n, \ldots, \mu_{n+m-1}$ with $\mu_n \neq \mu_{n-1}$ and $\mu_{n+m-1} \neq \mu_{n+m}$, we call generalized eigenfunction (associated with $\mu_n, \ldots, \mu_{n+m-1}$) any linear combination of eigenfunctions associated with the eigenvalues $\mu_n, \ldots, \mu_{n+m-1}$. Then we have the following theorem which is a simplified rephrased version of [38, Theorem 6.3], see also [3, Theorem 4.10], [6, Theorem 5.1], [17, Theorem 3.3].

Theorem 2.5. *Let $\{B_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ and B_0 be non-negative compact self-adjoint operators in the Hilbert spaces $\mathcal{H}_\varepsilon$, $\mathcal{H}_0$ respectively. Assume that their eigenvalues are given by the sequences $\mu_n(\varepsilon)$ and $\mu_n(0)$, $n \in \mathbb{N}$, respectively. If $B_\varepsilon \xrightarrow{C} B_0$ then we have spectral convergence of B_ε to B_0 as $\varepsilon \to 0$ in the sense that the following statements hold:*

  (i) *For every $n \in \mathbb{N}$ we have $\mu_n(\varepsilon) \to \mu_n(0)$ as $\varepsilon \to 0$.*
  (ii) *If $u_n(\varepsilon)$, $n \in \mathbb{N}$, is an orthonormal sequence of eigenfunctions associated with the eigenvalues $\mu_n(\varepsilon)$ then there exists a sequence ε_k, $k \in \mathbb{N}$, converging to zero and orthonormal sequence of eigenfunctions $u_n(0)$, $n \in \mathbb{N}$ associated with $\mu_n(0)$, $n \in \mathbb{N}$ such that $u_n(\varepsilon_k) \xrightarrow{E} u_n(0)$.*
  (iii) *Given m eigenvalues $\mu_n(0), \ldots, \mu_{n+m-1}(0)$ with $\mu_n(0) \neq \mu_{n-1}(0)$ and $\mu_{n+m-1}(0) \neq \mu_{n+m}(0)$ and corresponding orthonormal eigenfunctions $u_n(0), \ldots, u_{n+m-1}(0)$, there exist m orthonormal generalized eigenfunctions $v_n(\varepsilon), \ldots, v_{n+m-1}(\varepsilon)$ associated with $\mu_n(\varepsilon), \ldots, \mu_{n+m-1}(\varepsilon)$ such that $v_{n+i}(\varepsilon) \xrightarrow{E} u_{n+i}(0)$ for all $i = 0, 1, \ldots, m-1$.*

Remark 1. *We mention that an alternative approach to the spectral analysis of operators defined on variable Hilbert spaces has been recently proposed in the book by O. Post [33]. In particular, that method allows to obtain stability estimates which can be expressed in terms of suitable estimates for the variation of the resolvent operators. It would be interesting to implement that approach in our setting in particular in the cases where we do not have degeneration phenomena but this would require heavier computations going beyond the scopes of the present paper.*

2.2. **Classes of domains.** In order study the spectral convergence for Steklov eigenvalue problems on varying domains, we shall consider uniform families of domains for which a number of parameters are prescribed. This is done by using the notion of atlas from [16], see also [4, Section 5]. According to [4, 16], given a set $V \subset \mathbb{R}^N$ and a number $\delta > 0$ we write

$$(12) \qquad\qquad V_\delta := \{x \in V : d(x, \partial V) > \delta\}.$$

Definition 2.6. [16, Definition 2.4] *Let $\rho > 0$, $s, s' \in \mathbb{N}$ with $s' < s$. Let $\{V_j\}_{j=1}^s$ be a family of open cuboids (i.e. rotations of rectangle parallelepipeds in $\mathbb{R}^N$) and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$. We say that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$ briefly an atlas in $\mathbb{R}^N$. Moreover, we say that a bounded domain Ω in $\mathbb{R}^N$ belongs to the class $C^{k,\gamma}(\mathcal{A})$ with $k \in \mathbb{N}$ and $\gamma \in [0,1]$ if the following conditions are satisfied:*

(i) $\Omega \subset \cup_{j=1}^{s}(V_j)_\rho$ and $(V_j)_\rho \cap \Omega \neq \emptyset$ where $(V_j)_\rho$ is meant in the sense given in (12) ;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ for $j = 1, \ldots, s'$ and $V_j \cap \partial\Omega = \emptyset$ for $s' + 1 \leq j \leq s$;

(iii) for $j = 1, \ldots, s$ we have

$$r_j(V_j) = \{x \in \mathbb{R}^N : a_{ij} < x_i < b_{ij}, i = 1, \ldots, N\}, \quad j = 1, \ldots, s;$$

$$r_j(V_j \cap \Omega) = \{x = (x', x_N) \in \mathbb{R}^N : x' \in W_j, a_{Nj} < x_N < g_j(x')\}, \quad j = 1, \ldots, s'$$

where $x' = (x_1, \ldots, x_{N-1})$, $W_j = \{x' \in \mathbb{R}^{N-1} : a_{ij} < x_i < b_{ij}, i = 1, \ldots, N-1\}$ and the functions $g_j \in C^{k,\gamma}(\overline{W_j})$ for any $j \in 1, \ldots, s'$ with $k \in \mathbb{N} \cup \{0\}$ and $0 \leq \gamma \leq 1$. Moreover, for $j = 1, \ldots, s'$ we assume that $a_{Nj} + \rho \leq g_j(x') \leq b_{Nj} - \rho$, for all $x' \in \overline{W_j}$.

Finally we say that Ω if of class $C^{k,\gamma}$ if it is of class $C^{k,\gamma}(\mathcal{A})$ for some atlas $\mathcal{A}$. In the sequel $C^{k,0}$ will be simply denoted by C^k.

2.3. **Function spaces.** Let Ω be a domain in $\mathbb{R}^N$. In this paper, we shall often use the standard Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ for $k = 1, 2$. Recall that $H^k(\Omega)$ is the space of those functions in $L^2(\Omega)$ with weak derivatives up to order k in $L^2(\Omega)$, and that $H_0^k(\Omega)$ is the closure in $H^k(\Omega)$ of $C_c^\infty(\Omega)$. Note that in this paper we assume that $L^2(\Omega)$ is made of real-valued functions.

Besides $H^2(\Omega)$ we shall also consider the larger space $H(\Delta, \Omega)$ defined by

$$(13) \qquad H(\Delta, \Omega) = \{u \in L^2(\Omega) : \Delta u \in L^2(\Omega)\},$$

where Δu denotes the distributional Laplacian of u. Thus, by definition $\int_\Omega u \Delta \varphi dx = \int_\Omega \Delta u \varphi dx$ for all $\varphi \in C_c^\infty(\Omega)$. The space $H(\Delta, \Omega)$ is a natural space associated with the Dirichlet Laplacian which we temporarily denote here by Δ_D. Recall that the operator Δ_D is the non-negative self-adjoint operator in $L^2(\Omega)$ canonically associated with the quadratic form $\int_\Omega \nabla u \nabla \varphi dx$ defined for all $u, \varphi \in H_0^1(\Omega)$. This means that a function $u \in L^2(\Omega)$ belongs to the domain $\mathrm{Dom}(\Delta_D)$ of Δ_D if and only if $u \in H_0^1(\Omega)$ and there exists $f \in L^2(\Omega)$ such that $\int_\Omega \nabla u \nabla \varphi dx = \int_\Omega f \varphi dx$ for all $\varphi \in H_0^1(\Omega)$, in which case $-\Delta_D u = f$. It follows by these definitions that $\mathrm{Dom}(\Delta_D) = H(\Delta, \Omega) \cap H_0^1(\Omega)$.

We have the following lemma where by Poincaré inequality we mean the classical inequality $\|u\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega)}$ for all $u \in H_0^1(\Omega)$ and by Poincaré constant we mean the best constant c in that inequality. Recall that the Poincaré inequality holds under general assumptions on Ω, for example if Ω has finite measure it holds with $c = c_N |\Omega|^{\frac{1}{N}}$, or if Ω is bounded in one direction it holds with $c = c_N d$ where d is the diameter in that direction (here the symbol c_N is used to denote constants depending only on N). Moreover, the definition of uniform exterior ball condition is standard, see e.g., [1].

Lemma 2.7. *Let Ω be a domain in $\mathbb{R}^N$ such that the Poincaré inequality holds. Then the following statements hold:*

(i) *For all $u \in H(\Delta, \Omega) \cap H_0^1(\Omega)$ we have*

$$(14) \qquad \|u\|_{L^2(\Omega)} \leq c_\mathcal{P}^2 \|\Delta u\|_{L^2(\Omega)}, \text{ and } \|\nabla u\|_{L^2(\Omega)} \leq c_\mathcal{P} \|\Delta u\|_{L^2(\Omega)},$$

where $c_\mathcal{P}$ denotes the Poincaré constant.

(ii) *The space $H(\Delta, \Omega) \cap H_0^1(\Omega)$ endowed with the scalar product defined by $\int_\Omega \Delta u \Delta v dx$ for all $u, v \in H(\Delta, \Omega) \cap H_0^1(\Omega)$ is a Hilbert space.*

(iii) *If Ω is a bounded open set of class $C^{0,1}$ which satisfies the uniform exterior ball condition then $H(\Delta, \Omega) \cap H_0^1(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof. The second inequality in (i) easily follows by the Poincaré inequality because if $u \in H(\Delta, \Omega) \cap H_0^1(\Omega)$ then

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega u \Delta u dx \leq \|u\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)} \leq c_\mathcal{P} \|\nabla u\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)}.$$

The first inequality in (i) follows by the second inequality combined again with the Poincaré inequality. We now prove statement (ii). It clearly suffices to prove the completeness of the space. Let u_n, $n \in \mathbb{N}$ be a sequence in $H(\Delta, \Omega) \cap H_0^1(\Omega)$ such that $f_n := \Delta u_n$ is a Cauchy sequence in $L^2(\Omega)$. By (i), also $u_n, n \in \mathbb{N}$

is a Cauchy sequence in $H_0^1(\Omega)$. Thus, there exists $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ such that $\Delta u_n \to f$ in $L^2(\Omega)$ and $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$. Passing to the limit in the equality $\int_\Omega \nabla u_n \nabla \varphi \, dx = -\int_\Omega f_n \varphi \, dx$ as $n \to \infty$, we get $\int_\Omega \nabla u \nabla \varphi \, dx = -\int_\Omega f \varphi \, dx$ for all $\varphi \in H_0^1(\Omega)$ which means that $-\Delta u = f$ hence $u \in H(\Delta, \Omega)$. This completes the proof of (ii). Statement (iii) is a consequence of regularity results for the solutions of the Dirichlet problem, see [1]. $\qquad\square$

Remark 2. *It follows by Lemma 2.7 that if Ω is a bounded domain of class $C^{1,1}$ then $H(\Delta, \Omega) \cap H_0^1(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$ and the two norms are equivalent. Indeed, it is well known that $C^{1,1}$ regularity implies the validity of the uniform outer ball condition.*

3. STRONG SPECTRAL CONVERGENCE FOR STEKLOV AND STEKLOV-TYPE PROBLEMS

3.1. **The functional setting.** Assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain of class $C^{0,1}$ satisfying the uniform exterior ball condition. To shorten our notation we set

$$V(\Omega) := H^2(\Omega) \cap H_0^1(\Omega).$$

Proceeding as in [13], we relate the spectrum of the Steklov problem to the spectrum of a suitable self-adjoint compact operator acting on the space $V(\Omega)$. The space $V(\Omega)$ inherits the scalar product of $H^2(\Omega)$ and being closed in $H^2(\Omega)$ it becomes a Hilbert space. The space $V(\Omega)$ may be endowed with the two alternative scalar products

$$(15) \qquad (u,v) \mapsto \int_\Omega \Delta u \Delta v \, dx \qquad \text{for any } u, v \in V(\Omega),$$

$$(16) \qquad (u,v) \mapsto \int_\Omega D^2 u : D^2 v \, dx \qquad \text{for any } u, v \in V(\Omega).$$

Recall that by Lemma 2.7 (iii) the two scalar products defined in (15) and (16) are equivalent to the one induced by the $H^2(\Omega)$-norm. In the sequel we denote by $V'(\Omega)$ the dual space of $V(\Omega)$.

Consider now the eigenvalue problem (2) and let us introduce a suitable characterization of its spectrum. We introduce the operator $T_\Delta : V(\Omega) \mapsto V'(\Omega)$ defined by

$$_{V'(\Omega)}\langle T_\Delta u, v \rangle_{V(\Omega)} := \int_\Omega \Delta u \Delta v \, dx \qquad \text{for any } u, v \in V(\Omega).$$

The operator T_Δ is clearly well-defined and continuous. Moreover by Riesz Theorem we also deduce that T_Δ is invertible and T_Δ^{-1} is continuous. Then we introduce the operator $J : V(\Omega) \mapsto V'(\Omega)$ defined by

$$_{V'(\Omega)}\langle J u, v \rangle_{V(\Omega)} := \int_{\partial\Omega} u_\nu v_\nu \, dS \qquad \text{for any } u, v \in V(\Omega).$$

Since the linear map

$$(17) \qquad\qquad V(\Omega) \mapsto (L^2(\partial\Omega))^N$$
$$u \mapsto \nabla u_{|\partial\Omega}$$

is well-defined and compact being $\partial\Omega$ Lipschitzian (see [31, Theorem 6.2, Chap. 2] for more details), then the operator J is also well-defined and compact.

Next we introduce the operator $S_\Delta : V(\Omega) \mapsto V(\Omega)$ defined by $S_\Delta := T_\Delta^{-1} \circ J$. Clearly S_Δ is a linear compact operator and moreover it is easy to see that it is also self-adjoint. Moreover one can show that $\mu \neq 0$ is an eigenvalue of S_Δ if and only if $d := \mu^{-1}$ is an eigenvalue of (2). See the proof of Lemma 2 in [13] for more details.

With a completely similar procedure we can treat problem (5) by introducing the operator $T_{D^2} : V(\Omega) \to V'(\Omega)$ defined as

$$_{V'(\Omega)}\langle T_{D^2} u, v \rangle_{V(\Omega)} := \int_\Omega D^2 u : D^2 v \, dx \qquad \text{for any } u, v \in V(\Omega).$$

Applying Riesz Theorem, we see that also the operator T_{D^2} is invertible and its inverse function is continuous. Then one can define $S_{D^2} : V(\Omega) \to V(\Omega)$ as $S_{D^2} := T_{D^2}^{-1} \circ J$ and prove that S_{D^2} is a compact

self-adjoint operator. Moreover any $\mu \neq 0$ is an eigenvalue of S_{D^2} if and only if $\delta = \mu^{-1}$ is an eigenvalue of (5). This also proves the validity of (8) and show that the first eigenvalue δ_1 admits the following characterization

$$(18) \qquad \delta_1 = \inf_{v \in V(\Omega) \setminus H_0^2(\Omega)} \frac{\int_\Omega |D^2 v|^2 \, dx}{\int_{\partial\Omega} v_\nu^2 \, dS} \, .$$

3.2. **Domain perturbations and definition of the operators E_ε.** Let $\mathcal{A}$ be an atlas. We shall consider a family $\{\Omega_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ of domains of class $C^{1,1}(\mathcal{A})$ converging in a suitable sense to a fixed domain Ω of class $C^{1,1}(\mathcal{A})$. For any $0 < \varepsilon \leq \varepsilon_0$ we introduce the Steklov operators

$$S_{\Delta,\varepsilon} : V(\Omega_\varepsilon) \to V(\Omega_\varepsilon) \quad \text{and} \quad S_{D^2,\varepsilon} : V(\Omega_\varepsilon) \to V(\Omega_\varepsilon)$$

according to the definition given in Section 3.1, and we study the convergence of $S_{\Delta,\varepsilon}$ to S_Δ and of $S_{D^2,\varepsilon}$ to S_{D^2}. To do so, we shall apply the general method described in Subsection 2.1 where the Hilbert spaces $\mathcal{H}_\varepsilon$ and $\mathcal{H}$ under consideration here are exactly $V(\Omega_\varepsilon)$ and $V(\Omega)$ endowed either with the scalar product (15) or with the scalar product (16).

In particular, we need to define a family of operators E_ε as required in Subsection 2.1, which satisfy condition (11). In the specific case under consideration, this means that we have to introduce linear operators $E_\varepsilon : V(\Omega) \to V(\Omega_\varepsilon)$ such that

$$(19) \qquad \int_{\Omega_\varepsilon} |\Delta(E_\varepsilon u)|^2 dx \to \int_\Omega |\Delta u|^2 dx \qquad \text{as } \varepsilon \to 0 \, , \quad \text{for any } u \in V(\Omega)$$

and

$$(20) \qquad \int_{\Omega_\varepsilon} |D^2(E_\varepsilon u)|^2 dx \to \int_\Omega |D^2 u|^2 dx \qquad \text{as } \varepsilon \to 0 \, , \quad \text{for any } u \in V(\Omega) \, .$$

To do so, we proceed as in [4]. Suppose that the following assumptions hold true: for any $\varepsilon \in (0, \varepsilon_0]$ there exists $\kappa_\varepsilon > 0$ such that for any $j = 1, \ldots, s'$

$$(21)$$

$$\|g_{\varepsilon,j} - g_j\|_{L^\infty(W_j)} < \kappa_\varepsilon \, , \quad \lim_{\varepsilon \to 0} \kappa_\varepsilon = 0 \, , \qquad \lim_{\varepsilon \to 0} \frac{\|D^\beta(g_{\varepsilon,j} - g_j)\|_{L^\infty(W_j)}}{\kappa_\varepsilon^{3/2 - |\beta|}} = 0 \quad \forall \beta \in \mathbb{N}^N \text{ with } |\beta| \leq 2 \, .$$

Here we denoted by $g_j, g_{\varepsilon,j} \in C^{1,1}(\overline{W_j})$ the functions corresponding respectively to Ω and Ω_ε according to Definition 2.6.

Remark 3. *One can prove that if*

$$(22) \qquad D^\alpha g_{\varepsilon,j}(x') \to D^\alpha g_j(x'), \quad \text{uniformly on } W_j$$

for all $|\alpha| \leq 1$ and $j = 1, \ldots, s'$ and if there exists $M > 0$ such that

$$(23) \qquad |D^\alpha g_{\varepsilon,j}(x')| \leq M, \quad \text{a.e. on } W_j$$

for all $|\alpha| = 2$ and $j = 1, \ldots, s'$, then condition (21) is satisfied, see [4, Prop. 6.17]. We note that condition (21) is much less restrictive than conditions (22), (23), since it does not require the L^∞-norms of the second order weak derivatives of functions $g_{\varepsilon,j}$ to be uniformly bounded. For example, in Corollary 4.1 if $3/2 < \alpha < 2$ then condition (21) is satisfied but the L^∞-norms of the second order weak derivatives of functions g_ε are not uniformly bounded.

We are going to define the operator E_ε by using a partition of unity and pasting together suitable pull-back operators associated with local diffeomorphisms defined on each cuboid of the atlas $\mathcal{A}$. Note that in the simplified setting of one single cuboid, partition of unity would not be required and the operator E_ε would be simply defined as in Remark 4 below.

We denote by $\hat{k}$ a fixed constant satisfying $\hat{k} > 6$ whose meaning will be clear below. We also put $k_\varepsilon := \hat{k}\kappa_\varepsilon$, $\tilde{g}_{\varepsilon,j} := g_{\varepsilon,j} - k_\varepsilon$ and $K_{\varepsilon,j} := \{(x', x_N) \in W_j \times (a_{Nj}, b_{Nj}) : a_{Nj} < x_N < \tilde{g}_{\varepsilon,j}(x')\}$ for any $j = 1, \ldots, s'$.

For any $1 \leq j \leq s'$ we define the map $h_{\varepsilon,j} : r_j(\overline{\Omega_\varepsilon \cap V_j}) \to \mathbb{R}$

$$
(24) \qquad h_{\varepsilon,j}(x', x_N) := \begin{cases} 0, & \text{if } a_{jN} \leq x_N \leq \tilde{g}_{\varepsilon,j}(x'), \\ (g_{\varepsilon,j}(x') - g_j(x')) \left(\frac{x_N - \tilde{g}_{\varepsilon,j}(x')}{g_{\varepsilon,j}(x') - \tilde{g}_{\varepsilon,j}(x')} \right)^3, & \text{if } \tilde{g}_{\varepsilon,j}(x') < x_N \leq g_{\varepsilon,j}(x') . \end{cases}
$$

We observe that $h_{\varepsilon,j} \in C^{1,1}(r_j(\overline{\Omega_\varepsilon \cap V_j}))$ and that the map $\Phi_{\varepsilon,j} : r_j(\overline{\Omega_\varepsilon \cap V_j}) \to r_j(\overline{\Omega \cap V_j})$ defined by $\Phi_{\varepsilon,j}(x', x_N) := (x', x_N - h_{\varepsilon,j}(x', x_N))$ is a diffeomorphism of class $C^{1,1}$. When $s' + 1 \leq j \leq s$ we define $\Phi_{\varepsilon,j} : r_j(\overline{V_j}) \to r_j(\overline{V_j})$ as the identity map.

Exploiting the definition of $\Phi_{\varepsilon,j}$ we may define the above mentioned linear map E_ε. Consider a partition of unity $\{\psi_j\}_{1 \leq j \leq s}$ subordinate to the open cover $\{V_j\}_{1 \leq j \leq s}$ of the compact set $\overline{\Omega \cup \bigcup_{\varepsilon \in (0,\varepsilon_0]} \Omega_\varepsilon}$, i.e., a set of functions $\{\psi_j\}_{1 \leq j \leq s}$ such that $\text{supp}(\psi_j) \subset V_j$ for any $j \in \{1, \ldots, s\}$ and $\sum_{j=1}^s \psi_j \equiv 1$ in $\overline{\Omega \cup \bigcup_{\varepsilon \in (0,\varepsilon_0]} \Omega_\varepsilon}$. Indeed one can apply for example [11, Lemma IX.3] with $\Gamma = \overline{\Omega \cup \bigcup_{\varepsilon \in (0,\varepsilon_0]} \Omega_\varepsilon}$. Note that this partition of unity is independent of ε. We also define the deformation $\Psi_{\varepsilon,j} : \overline{\Omega_\varepsilon \cap V_j} \to \overline{\Omega \cap V_j}$ by $\Psi_{\varepsilon,j} := r_j^{-1} \circ \Phi_{\varepsilon,j} \circ r_j$. In this way $\Psi_{\varepsilon,j}$ becomes a $C^{1,1}$ diffeomorphism from $\overline{\Omega_\varepsilon \cap V_j}$ onto $\overline{\Omega \cap V_j}$ for any $j \in \{1, \ldots, s\}$.

From the definition of $h_{\varepsilon,j}$ and the restriction $\hat{k} > 6$, we deduce that

$$
(25) \qquad \frac{1}{2} \leq \det(D\Psi_{\varepsilon,j}(x)) \leq \frac{3}{2} \qquad \text{for any } x \in \Omega_\varepsilon \cap V_j .
$$

In order to show this, we observe that $\det(D\Psi_{\varepsilon,j}) = \det(D\Phi_{\varepsilon,j}) = 1 - \frac{\partial h_{\varepsilon,j}}{\partial x_N}(x', x_N)$.

Given $u \in V(\Omega)$ we put $u_j = \psi_j u$ for any $j \in \{1, \ldots, s\}$ in such a way that $u = \sum_{j=1}^s u_j$. Then we define

$$
(26) \qquad E_\varepsilon u := \sum_{j=1}^{s'} \tilde{u}_{\varepsilon,j} + \sum_{j=s'+1}^s u_j \in V(\Omega_\varepsilon)
$$

where

$$
(27) \qquad \tilde{u}_{\varepsilon,j}(x) = \begin{cases} u_j(\Psi_{\varepsilon,j}(x)) & \text{if } x \in \Omega_\varepsilon \cap V_j , \\ 0 & \text{if } x \in \Omega_\varepsilon \setminus V_j . \end{cases}
$$

for any $j \in \{1, \ldots, s'\}$.

Remark 4. *If Ω and Ω_ε are in the form*

$$
\Omega = \{(x', x_N) \in \mathbb{R}^N : x' \in W \text{ and } a_N < x_N < g(x')\} ,
$$
$$
\Omega_\varepsilon = \{(x', x_N) \in \mathbb{R}^N : x' \in W \text{ and } a_N < x_N < g_\varepsilon(x')\} ,
$$

where W is a cuboid or a bounded domain in $\mathbb{R}^{N-1}$ of class $C^{1,1}$ then the operator E_ε can be defined in the following simple way

$$
E_\varepsilon u(x) = u(\Phi_\varepsilon(x)) \qquad \text{for any } x \in \Omega_\varepsilon
$$

where $\Phi_\varepsilon(x', x_N) = (x', x_N - h_\varepsilon(x', x_N))$ and h_ε is defined by (24) with g_ε and g in place of $g_{\varepsilon,j}$ and g_j respectively.

Lemma 3.1. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$, Ω a domain of class $C^{1,1}(\mathcal{A})$ and for any $\varepsilon \in (0, \varepsilon_0]$ let E_ε be the map defined in (26). Assume the validity of condition (21).*

Then the following assertions hold true:

(i) *the map $E_\varepsilon : V(\Omega) \to V(\Omega_\varepsilon)$ is continuous for any $\varepsilon \in (0, \varepsilon_0]$;*
(ii) *the family of operators $\{E_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ satisfies (19) and (20).*

Proof. Since the proof of (i) easily follows from the definition of E_ε, it is left to the reader.

We now prove (19). Let $u \in V(\Omega)$ and according to (27) introduce the functions $\tilde{u}_{\varepsilon,j}$. Throughout the proof of this lemma we denote by C a positive constant independent of ε which may vary from line to line. We observe that in order to prove (19), it is sufficient to show that

$$(28) \qquad \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \, dx = \int_\Omega \frac{\partial^2 u_i}{\partial x_k^2} \frac{\partial^2 u_j}{\partial x_l^2} \, dx$$

for any $i, j \in \{1, \ldots, s'\}$ and $k, l \in \{1, \ldots, N\}$. We first prove that (28) holds when $i = j$ and $k = l$. One can proceed similarly to the proof of [4, Lemma 6.2] and hence we only give here a sketch of it. By (21) and (6.7) in [4] we have that

$$(29) \qquad \|D^\alpha h_{\varepsilon,j}\|_{L^\infty(W_j)} \le C \sum_{0 \le \gamma \le \alpha} \frac{\|D^\gamma(g_{\varepsilon,j} - g_j)\|_{L^\infty(W_j)}}{\kappa_\varepsilon^{|\alpha| - |\gamma|}}$$

for any $\alpha \in \mathbb{N}^N$ with $|\alpha| \le 2$. By $\gamma \le \alpha$ we mean that the inequality holds component by component. For any $x \in \Omega_\varepsilon \cap V_i$ we have

$$(30)$$
$$\frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2}(x) = \sum_{j=1}^N \left\{ \left[\sum_{l=1}^N \frac{\partial^2 u_i}{\partial x_l \partial x_j}(\Psi_{\varepsilon,i}(x)) \frac{\partial[(\Psi_{\varepsilon,i}(x))_l]}{\partial x_k} \right] \frac{\partial[(\Psi_{\varepsilon,i}(x))_j]}{\partial x_k} + \frac{\partial u_i}{\partial x_j}(\Psi_{\varepsilon,i}(x)) \frac{\partial^2[(\Psi_{\varepsilon,i}(x))_j]}{\partial x_k^2} \right\}.$$

By (21), (29) and the definitions of $\Phi_{\varepsilon,i}, \Psi_{\varepsilon,i}$, we deduce that there exists $C > 0$ independent of ε such that

$$(31)$$
$$\left\| \frac{\partial[(\Psi_{\varepsilon,i}(x))_l]}{\partial x_k} \frac{\partial[(\Psi_{\varepsilon,i}(x))_j]}{\partial x_k} \right\|_{L^\infty(V_i \cap \Omega_\varepsilon)} \le C \quad \text{and} \quad \left\| \frac{\partial^2[(\Psi_{\varepsilon,i}(x))_j]}{\partial x_k^2} \right\|_{L^\infty(V_i \cap \Omega_\varepsilon)} = o(\kappa_\varepsilon^{-1/2}) \quad \text{as } \varepsilon \to 0.$$

Exploiting (30)-(31) and proceeding as in the proof of [4, Lemma 6.2] we deduce that

$$(32) \qquad \lim_{\varepsilon \to 0} \int_{(\Omega_\varepsilon \cap V_i) \setminus r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \right)^2 dx = 0.$$

We briefly give a proof of (32). First of all by (25) we have

$$(33) \qquad \int_{(\Omega_\varepsilon \cap V_i) \setminus r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial^2 u_i}{\partial x_l \partial x_j}(\Psi_{\varepsilon,i}(x)) \right)^2 dx \le C \int_{(\Omega \cap V_i) \setminus r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial^2 u_i}{\partial x_l \partial x_j} \right)^2 dx \to 0$$

as $\varepsilon \to 0$ since $|(\Omega \cap V_i) \setminus r_i^{-1}(K_{\varepsilon,i})| \to 0$ as $\varepsilon \to 0$.

On the other hand, putting $v_i(z) = u_i(r_i^{-1}(z))$ for any $z \in r_i(\Omega \cap V_i)$, by (21), (31) and changes of variables, we have

$$(34) \qquad \int_{(\Omega_\varepsilon \cap V_i) \setminus r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial u_i}{\partial x_j}(\Psi_{\varepsilon,i}(x)) \frac{\partial^2[(\Psi_{\varepsilon,i}(x))_j]}{\partial x_k^2} \right)^2 dx \le o(\kappa_\varepsilon^{-1}) \sum_{l=1}^N \int_{r_i(\Omega \cap V_i) \setminus K_{\varepsilon,i}} \left(\frac{\partial v_i}{\partial z_l} \right)^2 dz$$

$$= o(\kappa_\varepsilon^{-1}) \sum_{l=1}^N \int_{W_i} \left(\int_{\tilde{g}_{\varepsilon,i}(z')}^{g_i(z')} \left(\frac{\partial v_i}{\partial z_l}(z', z_N) \right)^2 dz_N \right) dz'$$

$$\le o(\kappa_\varepsilon^{-1}) \sum_{l=1}^N \int_{W_i} |g_i(z') - \tilde{g}_{\varepsilon,i}(z')| \left\| \frac{\partial v_i}{\partial z_l}(z', \cdot) \right\|_{L^\infty(a_{Ni}, g_i(z'))}^2 dz'$$

$$\le o(1) \sum_{l=1}^N \int_{W_i} \left\| \frac{\partial v_i}{\partial z_l}(z', \cdot) \right\|_{H^1(a_{Ni}, g_i(z'))}^2 dz'$$

$$\le o(1) \|v_i\|_{H^2(r_i(\Omega \cap V_i))} = o(1).$$

Note that here we have used the classical one dimensional estimate

$$(35) \qquad \|f\|_{L^\infty(a,b)} \le C \|f\|_{H^1(a,b)}$$

for all $f \in H^1(a,b)$ with a constant $C = C(d)$ uniformly bounded for $b - a > d$. Combining (33), (34) with (30) and (31), (32) follows.

On the other hand

$$
(36) \qquad \lim_{\varepsilon \to 0} \int_{r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \right)^2 dx = \lim_{\varepsilon \to 0} \int_{r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial^2 u_i}{\partial x_k^2} \right)^2 dx = \int_{\Omega \cap V_i} \left(\frac{\partial^2 u_i}{\partial x_k^2} \right)^2 dx = \int_{\Omega} \left(\frac{\partial^2 u_i}{\partial x_k^2} \right)^2 dx.
$$

Combining (32)-(36) we conclude the proof of (28) in the case $i = j$ and $k = l$.

It remains to prove (28) in the general case.

We observe that

$$
(37) \qquad \int_{\Omega_\varepsilon} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \, dx = \int_{\Omega_\varepsilon \cap V_i \cap V_j} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \, dx
$$

$$
= \int_{r_i^{-1}(K_{\varepsilon,i}) \cap r_j^{-1}(K_{\varepsilon,j})} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \, dx + \int_{(\Omega_\varepsilon \cap V_i \cap V_j) \setminus (r_i^{-1}(K_{\varepsilon,i}) \cap r_j^{-1}(K_{\varepsilon,j}))} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \, dx \, .
$$

We have that $\Psi_{\varepsilon,i}$ and $\Psi_{\varepsilon,j}$ are the identity maps if restricted to $r_i^{-1}(K_{\varepsilon,i}) \cap r_j^{-1}(K_{\varepsilon,j})$ so that by (21) we obtain

$$
(38) \qquad \lim_{\varepsilon \to 0} \int_{r_i^{-1}(K_{\varepsilon,i}) \cap r_j^{-1}(K_{\varepsilon,j})} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \, dx = \lim_{\varepsilon \to 0} \int_{r_i^{-1}(K_{\varepsilon,i}) \cap r_j^{-1}(K_{\varepsilon,j})} \frac{\partial^2 u_i}{\partial x_k^2} \frac{\partial^2 u_j}{\partial x_l^2} \, dx = \int_{\Omega} \frac{\partial^2 u_i}{\partial x_k^2} \frac{\partial^2 u_j}{\partial x_l^2} \, dx \, .
$$

In order to estimate the other term in the right hand side of (37), we split the domain of integration in the following way

$$
(39) \quad (\Omega_\varepsilon \cap V_i \cap V_j) \setminus (r_i^{-1}(K_{\varepsilon,i}) \cap r_j^{-1}(K_{\varepsilon,j})) = [(\Omega_\varepsilon \cap V_i \cap V_j) \setminus r_i^{-1}(K_{\varepsilon,i})] \cup [(r_i^{-1}(K_{\varepsilon,i}) \setminus r_j^{-1}(K_{\varepsilon,j})) \cap V_j] \, .
$$

Then

$$
(40) \qquad \left| \int_{(\Omega_\varepsilon \cap V_i \cap V_j) \setminus r_i^{-1}(K_{\varepsilon,i})} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \, dx \right|
$$

$$
\leq \left(\int_{(\Omega_\varepsilon \cap V_i) \setminus r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \right)^2 dx \right)^{1/2} \left(\int_{\Omega_\varepsilon \cap V_j} \left(\frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \right)^2 dx \right)^{1/2} \to 0 \qquad \text{as } \varepsilon \to 0
$$

in view of (32) and (28) in the case $i = j$ and $k = l$. On the other hand

$$
(41) \qquad \left| \int_{(r_i^{-1}(K_{\varepsilon,i}) \setminus r_j^{-1}(K_{\varepsilon,j})) \cap V_j} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \, dx \right|
$$

$$
\leq \left(\int_{r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k^2} \right)^2 dx \right)^{1/2} \left(\int_{(\Omega_\varepsilon \cap V_j) \setminus r_j^{-1}(K_{\varepsilon,j})} \left(\frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_l^2} \right)^2 dx \right)^{1/2} \to 0 \qquad \text{as } \varepsilon \to 0
$$

in view of (32) and (36).

Combining (38)-(41) with (37), the proof of (28) follows also in the general case.

We now prove the validity of (20). We observe that it is sufficient to show that

$$
(42) \qquad \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{\partial^2 \tilde{u}_{\varepsilon,i}}{\partial x_k \partial x_l} \frac{\partial^2 \tilde{u}_{\varepsilon,j}}{\partial x_k \partial x_l} \, dx = \int_{\Omega} \frac{\partial^2 u_i}{\partial x_k \partial x_l} \frac{\partial^2 u_j}{\partial x_k \partial x_l} \, dx \, .
$$

The procedure that one has to follow to prove (42) is essentially the same adopted in Step 2: first one has to prove (42) in the case $i = j$ with $k, l \in \{1, \dots, N\}$ not necessarily equal and then one has to pass to the general case by arguing as above. $\qquad \square$

3.3. **The stability result.** In this subsection we prove the following result on spectral convergence for problems (2) and (5). One of the main assumptions is condition (21) which, as we can see from Lemma 3.1, guarantees the validity of the properties (19) and (20).

Theorem 3.2. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and let Ω be a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition (21). Then $S_{\Delta,\varepsilon} \xrightarrow{C} S_\Delta$ and $S_{D^2,\varepsilon} \xrightarrow{C} S_{D^2}$ with respect to the operators E_ε defined in (26). In particular, the spectra of (2) and (5) behave continuously at $\varepsilon = 0$ in the sense of Theorem 2.5.*

Proof. The proof of Theorem 3.2 follows directly by Theorem 2.5 and Lemmas 3.9-3.10 below.   □

Remark 5. *In the case of a single cuboid we can state a simplified version of Theorem 3.2 the proof of which would follow the same lines of the proof of Theorem 3.2 with obvious modifications. Namely, if Ω, Ω_ε are as in Remark 4 with g_ε and g satisfying (21) with g_ε and g in place of $g_{\varepsilon,j}$ and g_j respectively, then $S_{\Delta,\varepsilon} \xrightarrow{C} S_\Delta$ and $S_{D^2,\varepsilon} \xrightarrow{C} S_{D^2}$ with respect to the operators E_ε defined in Remark 4. In particular, the spectra of (2) and (5) behave continuously at $\varepsilon = 0$ in the sense of Theorem 2.5.*

Note that although in this specific case Ω_ε and Ω are not of class $C^{1,1}$ but only piecewise $C^{1,1}$, each of them is of class $C^{0,1}$ and satisfies the uniform outer ball condition.

In order to prove Lemma 3.9 and Lemma 3.10, we need a number of preliminary technical results.

To begin with, we give the definition of a map which acts between the spaces $V(\Omega)$, $V(\Omega_\varepsilon)$ in a reversed way with respect to E_ε. For any $w \in V(\Omega_\varepsilon)$ put

$$(43) \qquad \widehat{w}_{\varepsilon,j}(x) = \begin{cases} w_j(\Psi_{\varepsilon,j}^{-1}(x)), & \text{if } x \in \Omega \cap V_j \\ 0, & \text{if } x \in \Omega \setminus V_j. \end{cases}$$

for any $j \in \{1,\dots,s'\}$ and $w_j := \psi_j w$ for any $j \in \{1,\dots,s\}$. We define

$$(44) \qquad B_\varepsilon w := \sum_{j=1}^{s'} \widehat{w}_{\varepsilon,j} + \sum_{j=s'+1}^{s} w_j.$$

In this way we have constructed a map $B_\varepsilon : V(\Omega_\varepsilon) \to V(\Omega)$.

Lemma 3.3. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and Ω a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition (21). Then, up to shrink ε_0 if necessary,*

$$\|B_\varepsilon w\|_{H^2(\Omega)} \leq C\|\Delta w\|_{L^2(\Omega_\varepsilon)} \qquad \text{for any } w \in V(\Omega_\varepsilon) \text{ and } \varepsilon \in (0,\varepsilon_0]$$

where $C > 0$ is a constant independent of ε.

Proof. Throughout the proof we denote by C a general constant independent of ε which may vary from line to line. Let $w \in V(\Omega_\varepsilon)$. Since by (21) $|\Omega_\varepsilon| \to |\Omega|$ as $\varepsilon \to 0$ and the Poincaré constant depends only on the dimension of the space and the volume of the domain (see the comment before Lemma 2.7), we deduce by (14) that possibly shrinking ε_0

$$(45) \qquad \int_{\Omega_\varepsilon} |\nabla w|^2 dx \leq C\|\Delta w\|_{L^2(\Omega_\varepsilon)}^2 \text{ and } \int_{\Omega_\varepsilon} w^2 \, dx \leq C\|\Delta w\|_{L^2(\Omega_\varepsilon)}^2$$

for all $\varepsilon \in (0,\varepsilon_0]$. Define $w_j := \psi_j w$. Then by (45) and the fact that $\Delta w_j = w\Delta\psi_j + 2\nabla\psi_j \cdot \nabla w + \psi_j \Delta w$, it follows

$$(46) \qquad \|\Delta w_j\|_{L^2(\Omega_\varepsilon)} \leq C\|\Delta w\|_{L^2(\Omega_\varepsilon)}.$$

In the rest of the proof we fix $j \in \{1,\dots,s'\}$. By direct computation one can show that the function $\widehat{w}_{\varepsilon,j}$ defined in (43) satisfies

$$\text{div}\,(A_\varepsilon(y)\nabla\widehat{w}_{\varepsilon,j}(y)) = [\det(D\Psi_{\varepsilon,j}(\Psi_{\varepsilon,j}^{-1}(y)))]^{-1}\,\Delta w_j(\Psi_{\varepsilon,j}^{-1}(y)) \qquad \text{for any } y \in \Omega \cap V_j$$

where $A_\varepsilon(y) := [\det(D\Psi_{\varepsilon,j}(\Psi_{\varepsilon,j}^{-1}(y)))]^{-1}[D\Psi_{\varepsilon,j}(\Psi_{\varepsilon,j}^{-1}(y))][D\Psi_{\varepsilon,j}(\Psi_{\varepsilon,j}^{-1}(y))]^T$. We may think to the function $\Phi_{\varepsilon,j}$ as defined over all the set $\{(x', x_N) \in \mathbb{R}^N : x' \in W_j, x_N \le g_{\varepsilon,j}(x')\}$ since we may extend trivially the function $h_{\varepsilon,j}$ over the same set. Let us introduce the operator $L_\varepsilon v := \operatorname{div}(A_\varepsilon(y)\nabla v)$ for any function $v \in V(\Omega)$. Then we have

$$(47) \qquad \int_{\Omega \cap V_j} |L_\varepsilon \widehat{w}_{\varepsilon,j}|^2 dy = \int_{\Omega_\varepsilon \cap V_j} |\Delta w_j|^2 [\det(D\Psi_{\varepsilon,j})]^{-1} dx$$

Combining (46), (47) and (25), we deduce that

$$(48) \qquad \int_{\Omega \cap V_j} |L_\varepsilon \widehat{w}_{\varepsilon,j}|^2 dy \le C\|\Delta w_j\|^2_{L^2(\Omega_\varepsilon)} \le C\|\Delta w\|^2_{L^2(\Omega_\varepsilon)} \qquad \text{for any } \varepsilon \in (0, \varepsilon_0].$$

In order to complete the proof of the lemma we prove the following claim: there exists a constant C independent of ε such that

$$(49) \qquad \|u\|_{H^2(\Omega)} \le C\|L_\varepsilon u\|_{L^2(\Omega \cap V_j)} \qquad \text{for any } u \in V(\Omega), \operatorname{supp}(u) \subset V_j, \text{ and } \varepsilon \in (0, \varepsilon_0].$$

In order to do that, we proceed by using a standard method from the theory of regularity.

For any $u \in V(\Omega)$ with $\operatorname{supp}(u) \subset V_j$, we define the function $f_\varepsilon \in L^2(r_j^{-1}(\Sigma_j))$ by setting $f_\varepsilon(y) = -L_\varepsilon u(y)$ if $y \in \Omega \cap V_j$ and $f_\varepsilon(y) = 0$ if $y \in r_j^{-1}(\Sigma_j) \setminus (\Omega \cap V_j)$ where

$$\Sigma_j := \{(x', x_N) \in \mathbb{R}^N : x' \in W_j, x_N \le g_j(x')\}.$$

We define the deformation

$$\widetilde{\Phi}_j : \Sigma_j \to W_j \times (-\infty, 0]$$
$$\widetilde{\Phi}_j(x', x_N) := (x', x_N - g_j(x'))$$

and the function $\tilde{u} : W_j \times (-\infty, 0] \to \mathbb{R}$ which is defined by $\tilde{u}(y) = u(r_j^{-1}(\widetilde{\Phi}_j^{-1}(y)))$ if $y \in \widetilde{\Phi}_j(r_j(\Omega \cap V_j))$ and $\tilde{u}(y) = 0$ if $y \in (W_j \times (-\infty, 0)) \setminus \widetilde{\Phi}_j(r_j(\Omega \cap V_j))$.

We observe that $\tilde{u} \in H^2(W_j \times (-\infty, 0)) \cap H_0^1(W_j \times (-\infty, 0))$ and its support is a compact set contained in $W_j \times (-\infty, 0]$ thus allowing to extend it trivially over all $\mathbb{R}^{N-1} \times (-\infty, 0)$; we still denote by $\tilde{u}$ this trivial extension. Moreover, we set $\widetilde{\Psi}_j(z) := \widetilde{\Phi}_j(r_j(z))$ for all $z \in r_j^{-1}(\Sigma_j)$, and we note that $\tilde{u}$ solves the equation

$$(50) \qquad -\operatorname{div}\left(\tilde{A}_\varepsilon(y)\nabla\tilde{u}\right) = \tilde{f}_\varepsilon \qquad \text{in } W_j \times (-\infty, 0)$$

where

$$\tilde{A}_\varepsilon(y) := [\det(D\widetilde{\Psi}_j(\widetilde{\Psi}_j^{-1}(y)))]^{-1}[D\widetilde{\Psi}_j(\widetilde{\Psi}_j^{-1}(y))]A_\varepsilon(\widetilde{\Psi}_j^{-1}(y))[D\widetilde{\Psi}_j(\widetilde{\Psi}_j^{-1}(y))]^T$$

and $\tilde{f}_\varepsilon(y) := [\det(D\widetilde{\Psi}_j(\widetilde{\Psi}_j^{-1}(y)))]^{-1} f_\varepsilon(\widetilde{\Psi}_j^{-1}(y))$ for any $y \in W_j \times (-\infty, 0)$.

We point out that, in the definition of $\tilde{A}_\varepsilon$, the function A_ε has been extended by zero over the set $r_j^{-1}(\Sigma_j)$.

Then, we perform the Nirenberg translation method. In order to estimate second derivatives of $\tilde{u}$, we fix arbitrarily $i \in \{1, \ldots, N-1\}$ and we define $\tilde{u}_h(x) := \tilde{u}(x + he_i)$ for any $h \in \mathbb{R}$ where we have omitted for simplicity the index i. We use the same notation for any function $v \in H_0^1(\mathbb{R}^{N-1} \times (-\infty, 0))$. Similarly, we denote by $(\tilde{A}_\varepsilon)_h$ the translation of $\tilde{A}_\varepsilon$ where $\tilde{A}_\varepsilon$ is extended by zero to the whole of $\mathbb{R}^{N-1} \times (-\infty, 0)$ (the fact that this extension may not be Lipschitz continuous in $\mathbb{R}^{N-1} \times (-\infty, 0)$ is not relevant in what follows).

Fix $v \in H_0^1(W_j \times (-\infty, 0))$ such that $\operatorname{supp}(v)$ is a compact set contained in $W_j \times (-\infty, 0]$. Then we have for $|h|$ small enough $v_{-h} \in H_0^1(W_j \times (-\infty, 0))$ and

$$(51) \qquad \int_{W_j \times (-\infty,0)} \left(\tilde{A}_\varepsilon \nabla(\tilde{u}_h - \tilde{u})\right) \cdot \nabla v\, dy$$
$$= \int_{W_j \times (-\infty,0)} \tilde{f}_\varepsilon(v_{-h} - v)\, dy - \int_{W_j \times (-\infty,0)} \left[\left((\tilde{A}_\varepsilon)_h - \tilde{A}_\varepsilon\right)\nabla\tilde{u}_h\right] \cdot \nabla v\, dy$$

Now we choose $v = \tilde{u}_h - \tilde{u}$ in (51) and we proceed by estimating the term on the right hand side of it.

In the rest of proof we denote by C positive constants independent of h and ε which may vary from line to line.

Let $\omega \subset \mathbb{R}^{N-1}$ and $\delta > 0$ be such that ω is open,

$$\text{supp}(\tilde{u}) + he_i \subset \omega \times (-\infty, 0] \quad \text{and} \quad (\overline{\omega} \times (-\infty, 0]) + he_i \subset W_j \times (-\infty, 0]$$

for any $h \in (-\delta, \delta)$. In this way we can write

$$(52) \quad |(\tilde{A}_\varepsilon)_h(y) - \tilde{A}_\varepsilon(y)| \leq C|h| \max_{n,m,l \in \{1,\dots,N\}} \left\| \frac{\partial(\tilde{A}_\varepsilon)_{nm}}{\partial y_l} \right\|_{L^\infty(W_j \times (-\infty,0))} \quad \forall y \in \omega \times (-\infty, 0), \forall |h| < \delta$$

where $|A| := \sup_{x \in \mathbb{R}^N \setminus \{0\}} |Ax|/|x|$ for any $N \times N$ matrix. With this choice of ω and δ, exploiting (52), (21), the definitions of $\tilde{A}_\varepsilon$, $\Psi_{\varepsilon,j}$, $\Phi_{\varepsilon,j}$, the fact that $\tilde{A}_\varepsilon(y', y_N) = I_N$ if $y_N < -(\hat{k}+1)\kappa_\varepsilon < \tilde{g}_{\varepsilon,j}(y') - g_j(y')$ (here I_N denotes the $N \times N$ identity matrix), and (35) we have

(53)

$$\left| \int_{W_j \times (-\infty,0)} \left[\left((\tilde{A}_\varepsilon)_h - \tilde{A}_\varepsilon \right) \nabla \tilde{u}_h \right] \cdot \nabla(\tilde{u}_h - \tilde{u}) \, dy \right| = \left| \int_{\omega \times (-\infty,0)} \left[\left((\tilde{A}_\varepsilon)_h - \tilde{A}_\varepsilon \right) \nabla \tilde{u}_h \right] \cdot \nabla(\tilde{u}_h - \tilde{u}) \, dy \right|$$

$$= \left| \int_\omega \left(\int_{-(\hat{k}+1)\kappa_\varepsilon}^0 \left[\left((\tilde{A}_\varepsilon)_h - \tilde{A}_\varepsilon \right) \nabla \tilde{u}_h \right] \cdot \nabla(\tilde{u}_h - \tilde{u}) \, dy_N \right) dy' \right|$$

$$\leq C|h| \, o(\kappa_\varepsilon^{-1/2}) \left[\int_\omega \left(\int_{-(\hat{k}+1)\kappa_\varepsilon}^0 \left| \nabla \left(\tilde{u}_h - \tilde{u} \right) \right|^2 dy_N \right) dy' \right]^{1/2} \times$$

$$\times \left[\int_\omega (\hat{k}+1)\kappa_\varepsilon \, \| |\nabla \tilde{u}_h(y', \cdot)| \|_{L^\infty(-\infty,0)}^2 \, dy' \right]^{1/2}$$

$$\leq C|h| \, o(1) \, \|\tilde{u}_h - \tilde{u}\|_{H^1(W_j \times (-\infty,0))} \left[\int_\omega \| |\nabla \tilde{u}_h(y', \cdot)| \|_{H^1(-\infty,0)}^2 \, dy' \right]^{1/2}$$

$$\leq C|h| \, o(1) \cdot \|\tilde{u}_h - \tilde{u}\|_{H^1(W_j \times (-\infty,0))} \|\tilde{u}\|_{H^2(W_j \times (-\infty,0))}$$

where here and in the rest of the proof, $o(\kappa_\varepsilon^{-1/2})$ and $o(1)$ denote functions depending only on ε but independent of h which have the prescribed asymptotic behavior as $\varepsilon \to 0$. We point out that the estimate $\| |\nabla \tilde{u}_h(y', \cdot)| \|_{L^\infty(-\infty,0)}^2 \leq C \| |\nabla \tilde{u}_h(y', \cdot)| \|_{H^1(-\infty,0)}^2$ has been obtained by exploiting the fact that there exists $M > 0$ such that $\text{supp}(\tilde{u}_h) \subset W_j \times (-M, 0]$ for any $h \in (-\delta, \delta)$ and applying (35) on the interval $(-M, 0)$. On the other hand we also have that

$$(54) \quad \left| \int_{W_j \times (-\infty,0)} \tilde{f}_\varepsilon[(\tilde{u}_h - \tilde{u})_{-h} - (\tilde{u}_h - \tilde{u})] \, dy \right| \leq C|h| \, \|\tilde{f}_\varepsilon\|_{L^2(W_j \times (-\infty,0))} \|\tilde{u}_h - \tilde{u}\|_{H^1(W_j \times (-\infty,0))}$$

Let us consider the left hand side of (51). Since by (21), $\tilde{A}_\varepsilon \to I_N$ in $L^\infty(W_j \times (-\infty, 0))$, then up to shrink ε_0 if necessary we have

$$(55) \quad \sum_{n,m=1}^N (\tilde{A}_\varepsilon(y))_{nm} \, \xi_n \xi_m \geq \frac{1}{2}|\xi|^2 \qquad \text{for any } y \in W_j \times (-\infty, 0), \xi \in \mathbb{R}^N \text{ and } \varepsilon \in (0, \varepsilon_0].$$

Therefore we have that

$$(56) \quad \int_{W_j \times (-\infty,0)} \left(\tilde{A}_\varepsilon(y) \nabla(\tilde{u}_h - \tilde{u}) \right) \cdot \nabla(\tilde{u}_h - \tilde{u}) \, dy$$

$$\geq \frac{1}{2} \int_{W_j \times (-\infty,0)} |\nabla(\tilde{u}_h - \tilde{u})|^2 dy \geq C \|\tilde{u}_h - \tilde{u}\|_{H^1(W_j \times (-\infty,0))}^2 \, .$$

Combining (53)-(56) with (51) we obtain

$$\left\| \frac{\tilde{u}_h - \tilde{u}}{h} \right\|_{H^1(W_j \times (-\infty, 0))} \leq o(1) \cdot \|\tilde{u}\|_{H^2(W_j \times (-\infty, 0))} + C\|\tilde{f}_\varepsilon\|_{L^2(W_j \times (-\infty, 0))} \quad \forall 0 < |h| < \delta\,, \forall \varepsilon \in (0, \varepsilon_0]\,.$$

Passing to the limit as $h \to 0$ we obtain

$$(57) \qquad \left\| \frac{\partial \tilde{u}}{\partial x_i} \right\|_{H^1(W_j \times (-\infty, 0))} \leq o(1) \cdot \|\tilde{u}\|_{H^2(W_j \times (-\infty, 0))} + C\|\tilde{f}_\varepsilon\|_{L^2(W_j \times (-\infty, 0))} \quad \forall \varepsilon \in (0, \varepsilon_0]$$

for any $i \in \{1, \ldots, N-1\}$.

Since $\tilde{u}$ solves equation (50), using a standard argument (see e.g. page 319 in [20]) one can show that the validity of (57) for any $i \in \{1, \ldots, N-1\}$ implies the validity of (57) also for $i = N$. Therefore we may conclude that up to shrink ε_0 if necessary, for any $\varepsilon \in (0, \varepsilon_0]$ we have

$$\|D^\alpha \tilde{u}\|_{L^2(W_j \times (-\infty, 0))} \leq o(1) \cdot \|\tilde{u}\|_{H^2(W_j \times (-\infty, 0))} + C\|\tilde{f}_\varepsilon\|_{L^2(W_j \times (-\infty, 0))} \, \forall \alpha \in \mathbb{N}^N, |\alpha| = 2,\ \forall \varepsilon \in (0, \varepsilon_0]$$

and, in turn,

$$(58) \qquad \|\tilde{u}\|_{H^2(W_j \times (-\infty, 0))} \leq C \left(\|\tilde{u}\|_{H^1(W_j \times (-\infty, 0))} + \|\tilde{f}_\varepsilon\|_{L^2(W_j \times (-\infty, 0))} \right).$$

Testing (50) with $\tilde{u}$, using (55) and the Poincaré inequality in $W_j \times (-\infty, 0)$, we deduce that

$$\|\tilde{u}\|_{H^1(W_j \times (-\infty, 0))} \leq C\|\tilde{f}_\varepsilon\|_{L^2(W_j \times (-\infty, 0))} \quad \forall \varepsilon \in (0, \varepsilon_0]$$

which combined with (58) gives

$$\|\tilde{u}\|_{H^2(W_j \times (-\infty, 0))} \leq C\|\tilde{f}_\varepsilon\|_{L^2(W_j \times (-\infty, 0))} \quad \forall \varepsilon \in (0, \varepsilon_0]\,.$$

Coming back to V_j the last estimate proves (49).

Applying (49) to $\widehat{w}_{\varepsilon,j}$ and using (48) it follows that

$$(59) \qquad \|\widehat{w}_{\varepsilon,j}\|_{H^2(\Omega)} \leq C\|L_\varepsilon \widehat{w}_{\varepsilon,j}\|_{L^2(\Omega \cap V_j)} \leq C\|\Delta w\|_{L^2(\Omega_\varepsilon)}$$

for any $\varepsilon \in (0, \varepsilon_0]$.

The proof of the lemma then follows from (43). □

As a consequence of Lemma 3.3 we prove the following

Lemma 3.4. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and Ω a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition (21). Then, up to shrink ε_0 if necessary,*

$$\|w\|_{H^2(\Omega_\varepsilon)} \leq C\|\Delta w\|_{L^2(\Omega_\varepsilon)} \qquad \text{for any } w \in V(\Omega_\varepsilon) \text{ and } \varepsilon \in (0, \varepsilon_0]$$

where $C > 0$ is a constant independent of w and ε.

Proof. Let $w \in V(\Omega_\varepsilon)$. In this proof we use the same notations introduced in the proof of Lemma 3.3. By C we denote positive constants independent of w and ε which may vary from line to line.

For any $j \in \{1, \ldots, s'\}$ we have

$$(60) \qquad \frac{\partial^2 w_j}{\partial x_n \partial x_m}(x) = \sum_{k=1}^N \left\{ \left[\sum_{l=1}^N \frac{\partial^2 \widehat{w}_{\varepsilon,j}}{\partial x_l \partial x_k}(\Psi_{\varepsilon,j}(x)) \frac{\partial[(\Psi_{\varepsilon,j}(x))_l]}{\partial x_m} \right] \frac{\partial[(\Psi_{\varepsilon,j}(x))_k]}{\partial x_n} \right.$$

$$\left. + \frac{\partial \widehat{w}_{\varepsilon,j}}{\partial x_k}(\Psi_{\varepsilon,j}(x)) \frac{\partial^2[(\Psi_{\varepsilon,j}(x))_k]}{\partial x_n \partial x_m} \right\}.$$

By (21), (60), with an argument similar to the one exploited in the proof of Lemma 3.1, we infer

$$\|w_j\|_{H^2(\Omega_\varepsilon)} \leq C\|\widehat{w}_{\varepsilon,j}\|_{H^2(\Omega)}\,.$$

The proof of the lemma now follows combining this estimate with (59). □

Thanks to Lemma 3.4 we deduce that, under the assumption (21), the two norms

$$u \mapsto \left(\int_{\Omega_\varepsilon} |\Delta u|^2 dx \right)^{1/2}, \qquad u \mapsto \left(\int_{\Omega_\varepsilon} |D^2 u|^2 dx \right)^{1/2}$$

defined on the space $V(\Omega_\varepsilon)$, are *uniformly equivalent* in the sense that there exists a constant $C > 0$ independent of u and ε such that

$$\left(\int_{\Omega_\varepsilon} |\Delta u|^2 dx \right)^{1/2} \leq C \left(\int_{\Omega_\varepsilon} |D^2 u|^2 dx \right)^{1/2} \qquad \text{and} \qquad \left(\int_{\Omega_\varepsilon} |D^2 u|^2 dx \right)^{1/2} \leq C \left(\int_{\Omega_\varepsilon} |\Delta u|^2 dx \right)^{1/2}$$

for any $u \in V(\Omega_\varepsilon)$.

For this reason, for any $u \in V(\Omega_\varepsilon)$, we put

$$(61) \qquad \|u\|_{V(\Omega_\varepsilon)} := \begin{cases} \left(\int_{\Omega_\varepsilon} |\Delta u|^2 dx \right)^{1/2}, & \text{for problem (2)}, \\ \left(\int_{\Omega_\varepsilon} |D^2 u|^2 dx \right)^{1/2}, & \text{for problem (5)}. \end{cases}$$

For the same reason, given a set A, we denote by $Q_A(\cdot, \cdot)$ and $Q_A(\cdot)$ the bilinear form and the quadratic form respectively defined by

$$(62) \qquad Q_A(u, v) := \int_A \Delta u \Delta v \, dx \qquad \text{and} \qquad Q_A(u) = Q_A(u, u)$$

or by

$$(63) \qquad Q_A(u, v) := \int_A D^2 u : D^2 v \, dx \qquad \text{and} \qquad Q_A(u) = Q_A(u, u)$$

depending again on the problem we are treating among (2) or (5).

The next lemma deals with a uniform estimate with respect to domain perturbation of the $H^{1/2}$-norm on the boundary in terms of the H^1-norm of the domain. For simplicity, here and in the sequel we denote the trace of a function and the function itself with the same symbol.

Lemma 3.5. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and Ω a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition (21). Then, up to shrink ε_0 if necessary,*

$$\|w\|_{H^{1/2}(\partial\Omega_\varepsilon)} \leq C \|w\|_{H^1(\Omega_\varepsilon)} \qquad \text{for any } w \in H^1(\Omega_\varepsilon) \text{ and } \varepsilon \in (0, \varepsilon_0]$$

where $C > 0$ is a constant independent of ε.

Proof. In this proof C will denote a positive constant independent of w and ε which may vary from line to line. The norm in $H^{1/2}(\partial\Omega_\varepsilon)$ can be defined locally using the partition of unity $\{\psi_j\}_{1 \leq j \leq s}$ introduced in Subsection 3.2. In other words if $v \in H^{1/2}(\partial\Omega_\varepsilon)$, for any $j = 1, \dots, s'$, one can define $v_j = \psi_j v$ on $\partial\Omega_\varepsilon \cap V_j$ and the function

$$(64) \qquad \bar{v}_{\varepsilon,j}(y') := v_j(r_j^{-1}(y', g_{\varepsilon,j}(y'))) \qquad \text{for any } y' \in W_j \subset \mathbb{R}^{N-1}.$$

Let now $w \in H^1(\Omega_\varepsilon)$ and still denote by w its trace on $\partial\Omega_\varepsilon$ so that w can be also seen as an element of $H^{1/2}(\partial\Omega_\varepsilon)$. Let $\bar{w}_{\varepsilon,j}$ be the corresponding function defined on W_j according to (64).

Exploiting in an appropriate way the maps $\Phi_{\varepsilon,j}, \Psi_{\varepsilon,j}$ defined in Subsection 3.2 and the maps $\widetilde{\Phi}_{\varepsilon,j}, \widetilde{\Psi}_{\varepsilon,j}$ defined in the proof of Lemma 3.3, since $w \in H^1(\Omega_\varepsilon)$, the function $\bar{w}_{\varepsilon,j}$ can be also interpreted as a function defined on the whole $W_j \times (-\infty, 0)$ which belongs to $H^1(W_j \times (-\infty, 0))$ and its support is a compact set contained in $W_j \times (-\infty, 0]$.

Now it is clear that the extension-by-zero of $\bar{w}_{\varepsilon,j}$ to the whole $\mathbb{R}^{N-1} \times (-\infty, 0)$, that we still denote by $\bar{w}_{\varepsilon,j}$, belongs to $H^1(\mathbb{R}^{N-1} \times (-\infty, 0))$ and its trace belongs to $H^{1/2}(\mathbb{R}^{N-1})$. Since $\bar{w}_{\varepsilon,j}$ is a compactly supported $H^{1/2}(\mathbb{R}^{N-1})$-function, its norm may be defined in the usual way by mean of the Fourier transform.

Hence, as possible choice (see [31, Page 83]) for $H^{1/2}(\partial\Omega_\varepsilon)$-norm of w, we put

$$(65) \qquad \|w\|_{H^{1/2}(\partial\Omega_\varepsilon)} = \left(\sum_{j=1}^{s'} \|\overline{w}_{\varepsilon,j}\|^2_{H^{1/2}(\mathbb{R}^{N-1})} \right)^{1/2}.$$

Applying a well-known trace theorem for functions in $H^1(\mathbb{R}^{N-1} \times (-\infty, 0))$ (see Theorem 5.1, Chap. 2 in [31]) we obtain

$$\|\overline{w}_{\varepsilon,j}\|_{H^{1/2}(\mathbb{R}^{N-1})} \le C\|\overline{w}_{\varepsilon,j}\|_{H^1(\mathbb{R}^{N-1}\times(-\infty,0))}.$$

By (21) and direct computation, one can verify that

$$\|\overline{w}_{\varepsilon,j}\|_{H^1(\mathbb{R}^{N-1}\times(-\infty,0))} \le C\|w_j\|_{H^1(\Omega_\varepsilon)}$$

The proof of the lemma now follows from (65) and the definition of w_j. $\qquad\square$

Lemma 3.6. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and Ω a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition (21). Then*

$$\liminf_{\varepsilon\to 0} d_1^\varepsilon > 0$$

where d_1^ε is given by (4) with Ω_ε in place of Ω.

Proof. In this proof we denote by C a general constant independent of ε which may vary from estimate to estimate. For any $0 < \varepsilon \le \varepsilon_0$, let w_ε an eigenfunction of (2) corresponding to d_1^ε satisfying $\int_{\partial\Omega_\varepsilon}(w_\varepsilon)^2_\nu dS = 1$ and let $u_\varepsilon := B_\varepsilon w_\varepsilon$ with B_ε as in Subsection 3.3. Suppose by contradiction that $\liminf_{\varepsilon\to 0} d_1^\varepsilon = 0$ so that along a sequence of values of ε converging to zero we may assume that $d_1^\varepsilon \to 0$. For simplicity we write this as $d_1^\varepsilon \to 0$ as $\varepsilon \to 0$. Then we have that along the same sequence $\|w_\varepsilon\|_{V(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$ since $\int_{\Omega_\varepsilon}|\Delta w_\varepsilon|^2 dx = d_1^\varepsilon \int_{\partial\Omega_\varepsilon}(w_\varepsilon)^2_\nu dS$. Applying Lemma 3.3 we deduce that $u_\varepsilon \to 0$ in $V(\Omega)$.

According to (43), for any $1 \le j \le s'$, we can define $\widehat{w}_{\varepsilon,j}$. Moreover for any $1 \le j \le s$ we put $w_{\varepsilon,j} := \psi_j w_\varepsilon$. In this way we have that $u_\varepsilon = \sum_{j=1}^{s'} \widehat{w}_{\varepsilon,j} + \sum_{j=s'+1}^s w_{\varepsilon,j}$. By direct computation one sees that

$$(66) \qquad \frac{\partial\widehat{w}_{\varepsilon,j}}{\partial x_k}(x) = \sum_{n=1}^N \frac{\partial w_{\varepsilon,j}}{\partial x_n}(\Psi_{\varepsilon,j}^{-1}(x))\frac{\partial[(\Psi_{\varepsilon,j}^{-1}(x))_n]}{\partial x_k} \qquad \text{for any } x \in \Omega \cap V_j.$$

We observe that from the definition of $\Psi_{\varepsilon,j}$ and (21) we have

$$(67) \qquad \frac{\partial[(\Psi_{\varepsilon,j}^{-1}(x))_n]}{\partial x_k} \to \delta_{nk} \qquad \text{uniformly in } V_j \cap \Omega \text{ as } \varepsilon \to 0$$

for any $n, k \in \{1, \dots, N\}$.

On the other hand, since $\int_{\partial\Omega_\varepsilon}|\nabla w_\varepsilon|^2\, dS = \int_{\partial\Omega_\varepsilon}(w_\varepsilon)^2_\nu\, dS = 1$ being $w_\varepsilon \equiv 0$ on $\partial\Omega_\varepsilon$, for any $j \in \{1, \dots, s'\}$ and $k \in \{1, \dots, N\}$, $\int_{\partial\Omega_\varepsilon}\left(\frac{\partial w_{\varepsilon,j}}{\partial x_k}\right)^2 dS$ remains bounded as $\varepsilon \to 0$. Moreover, using the local parametrizations of $\partial\Omega_\varepsilon$ and $\partial\Omega$ combined with (21), we also have

$$(68) \qquad \int_{\partial\Omega\cap V_j}\left(\frac{\partial w_{\varepsilon,j}}{\partial x_k}(\Psi_{\varepsilon,j}^{-1}(x))\right)^2 dS = \int_{\partial\Omega_\varepsilon\cap V_j}\left(\frac{\partial w_{\varepsilon,j}}{\partial x_k}\right)^2 dS + o(1) = O(1) \qquad \text{as } \varepsilon \to 0.$$

Combining (68) with (67), for any $j \in \{1, \dots, s'\}$ and $n, k \in \{1, \dots, N\}$, we obtain

$$(69) \qquad \int_{\partial\Omega\cap V_j}\left(\frac{\partial w_{\varepsilon,j}}{\partial x_n}(\Psi_{\varepsilon,j}^{-1}(x))\right)^2\left(\frac{\partial[(\Psi_{\varepsilon,j}^{-1}(x))_n]}{\partial x_k}\right)^2 dS \to 0 \quad \text{as } \varepsilon \to 0$$

if $n \ne k$ and

$$(70)$$

$$\int_{\partial\Omega\cap V_j}\left(\frac{\partial w_{\varepsilon,j}}{\partial x_k}(\Psi_{\varepsilon,j}^{-1}(x))\right)^2\left(\frac{\partial[(\Psi_{\varepsilon,i}^{-1}(x))_k]}{\partial x_k}\right)^2 dS = \int_{\partial\Omega\cap V_j}\left(\frac{\partial w_{\varepsilon,j}}{\partial x_k}(\Psi_{\varepsilon,j}^{-1}(x))\right)^2 dS + o(1) \quad \text{as } \varepsilon \to 0.$$

From the proof of Lemma 3.3 one can deduce that not only $u_\varepsilon \to 0$ in $V(\Omega)$ but also $\widehat{w}_{\varepsilon,j} \to 0$ in $V(\Omega)$, for any $j \in \{1, \dots, s'\}$. Hence from the continuity of the trace map (17) we also have that $\frac{\partial \widehat{w}_{\varepsilon,j}}{\partial x_k} \to 0$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$. From this and (66), (69), it follows that $\frac{\partial w_{\varepsilon,j}}{\partial x_k}(\Psi_{\varepsilon,j}^{-1}(x)) \frac{\partial[(\Psi_{\varepsilon,j}^{-1}(x))_k]}{\partial x_k} \to 0$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$. This, together with (68) and (70) yields

$$\int_{\partial\Omega_\varepsilon} \left(\frac{\partial w_{\varepsilon,j}}{\partial x_k}\right)^2 dS \to 0 \qquad \text{as } \varepsilon \to 0 \quad \text{for any } j \in \{1, \dots, s'\} \text{ and } k \in \{1, \dots, N\}$$

and, in turn, $\int_{\partial\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dS = o(1)$ as $\varepsilon \to 0$, a contradiction. $\qquad\square$

Now, we introduce some notations which will be fundamental in the proofs of the next lemmas. For any $y \in \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)$ we put $\Theta_{\varepsilon,i,j}(y) := \Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y))$ in order to define

$$(71) \qquad\qquad \Theta_{\varepsilon,i,j} : \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j) \to \Theta_{\varepsilon,i,j}(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j))$$

as a diffeomorphism between two open subsets of the manifold $\partial\Omega$.

For any $k \in \{1, \dots, s'\}$ we define the local charts $\Gamma_k : \partial\Omega \cap V_k \to W_k \subset \mathbb{R}^{N-1}$ where $\Gamma_k(y) := P(r_k(y))$ and $P : \mathbb{R}^N \to \mathbb{R}^{N-1}$ is the projection $(x', x_N) \mapsto x'$. We observe that $\Gamma_k^{-1} : W_k \to \partial\Omega \cap V_k$ satisfies $\Gamma_k^{-1}(z') = r_k^{-1}(z', g_k(z'))$ for any $z' \in W_k$. Next we introduce the map

$$(72) \qquad\qquad \Upsilon_{\varepsilon,i,j} : \Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)) \to \Upsilon_{\varepsilon,i,j}(\Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)))$$

where $\Upsilon_{\varepsilon,i,j}(z') := \Gamma_i(\Theta_{\varepsilon,i,j}(\Gamma_j^{-1}(z')))$ for any $z' \in \Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j))$.

Lemma 3.7. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and Ω a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition* (21).

(i) *Let $i, j \in \{1, \dots, s'\}$. Let $\{f_\varepsilon\}_{0<\varepsilon\le\varepsilon_0} \subset L^2(\mathbb{R}^{N-1})$ be such that $\mathrm{supp}(f_\varepsilon) \subset \Gamma_i(\partial\Omega \cap V_i)$ for any $\varepsilon \in (0, \varepsilon_0]$ and let $f \in L^2(\mathbb{R}^{N-1})$ be such that $f_\varepsilon \to f$ in $L^2(\mathbb{R}^{N-1})$ as $\varepsilon \to 0$. Let $\Upsilon_{\varepsilon,i,j}$ be as in* (72). *Define*

$$\tilde{f}_\varepsilon(z') := \begin{cases} f_\varepsilon(\Upsilon_{\varepsilon,i,j}(z')) & \text{if } z' \in \Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)), \\ 0 & \text{if } z' \in \mathbb{R}^{N-1} \setminus \Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)), \end{cases}$$

for any $\varepsilon \in (0, \varepsilon_0]$ and

$$\tilde{f}(z') := \begin{cases} f(\Gamma_i(\Gamma_j^{-1}(z'))) & \text{if } z' \in \Gamma_j(\partial\Omega \cap V_i \cap V_j), \\ 0 & \text{if } z' \in \mathbb{R}^{N-1} \setminus \Gamma_j(\partial\Omega \cap V_i \cap V_j). \end{cases}$$

Then $\tilde{f}_\varepsilon \to \tilde{f}$ in $L^2(\mathbb{R}^{N-1})$ as $\varepsilon \to 0$.

(ii) *Let $\{\omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0} \subset L^2(\partial\Omega)$ be such that $\mathrm{supp}(\omega_\varepsilon) \subset \partial\Omega \cap V_i$ for any $\varepsilon \in (0, \varepsilon_0]$, for some $i \in \{1, \dots, s'\}$. Suppose that there exists $\omega \in L^2(\partial\Omega)$ such that $\omega_\varepsilon \to \omega$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$. For $j \in \{1, \dots, s'\}$ let $\Theta_{\varepsilon,i,j}$ be as in* (71). *For any $\varepsilon \in (0, \varepsilon_0]$ define the function*

$$\tilde{\omega}_\varepsilon(y) := \begin{cases} \omega_\varepsilon(\Theta_{\varepsilon,i,j}(y)) & \text{if } y \in \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j), \\ 0 & \text{if } y \in \partial\Omega \setminus \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j). \end{cases}$$

Then $\tilde{\omega}_\varepsilon \to \omega \chi_{\partial\Omega \cap V_i \cap V_j}$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$.

Proof. We divide the proof of the lemma into three steps. Throughout this proof, C denotes a constant independent of ε which may vary from line to line.

Step 1. In this step we prove (i) when $f_\varepsilon \equiv f$ for any $\varepsilon \in (0, \varepsilon_0]$.

For any $\sigma > 0$ let $\varphi_\sigma \in C^0(\mathbb{R}^{N-1})$ be such that $\|f - \varphi_\sigma\|_{L^2(\mathbb{R}^{N-1})} < \sigma$ and $\mathrm{supp}(\varphi_\sigma) \subset \Gamma_i(\partial\Omega \cap V_i)$.

Similarly to the statement of the lemma we define

$$\tilde{\varphi}_{\sigma,\varepsilon}(z') := \begin{cases} \varphi_\sigma(\Upsilon_{\varepsilon,i,j}(z')) & \text{if } z' \in \Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)), \\ 0 & \text{if } z' \in \mathbb{R}^{N-1} \setminus \Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)). \end{cases}$$

and

$$\tilde{\varphi}_\sigma(z') := \begin{cases} \varphi_\sigma(\Gamma_i(\Gamma_j^{-1}(z'))) & \text{if } z' \in \Gamma_j(\partial\Omega \cap V_i \cap V_j)\,, \\ 0 & \text{if } z' \in \mathbb{R}^{N-1} \setminus \Gamma_j(\partial\Omega \cap V_i \cap V_j)\,. \end{cases}$$

By (21) we deduce that $\tilde{\varphi}_{\varepsilon,\sigma} \to \tilde{\varphi}_\sigma$ almost everywhere in $\mathbb{R}^{N-1}$ and that up to shrink ε_0 if necessary, there exists a compact set in $\mathbb{R}^{N-1}$ which contains the support of $\tilde{\varphi}_{\sigma,\varepsilon}$ for any $\varepsilon \in (0, \varepsilon_0]$. Therefore by the Dominated Convergence Theorem we have that

$$(73) \qquad\qquad \tilde{\varphi}_{\sigma,\varepsilon} \to \tilde{\varphi}_\sigma \qquad \text{in } L^2(\mathbb{R}^{N-1}) \text{ as } \varepsilon \to 0\,.$$

On the other hand with a change of variable, by (21) we obtain

$$(74) \qquad \|\tilde{f}_\varepsilon - \tilde{\varphi}_{\sigma,\varepsilon}\|_{L^2(\mathbb{R}^{N-1})}^2 = \int_{\Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j))} |f(\Upsilon_{\varepsilon,i,j}(z')) - \varphi_\sigma(\Upsilon_{\varepsilon,i,j}(z'))|^2 dz'$$
$$\le C \int_{\Gamma_i(\partial\Omega \cap V_i)} |f(x') - \varphi_\sigma(x')|^2 dx' \le C\sigma^2$$

and

$$(75) \qquad \|\tilde{f} - \tilde{\varphi}_\sigma\|_{L^2(\mathbb{R}^{N-1})}^2 = \int_{\Gamma_j(\partial\Omega \cap V_i \cap V_j)} |f(\Gamma_i(\Gamma_j^{-1}(z'))) - \varphi_\sigma(\Gamma_i(\Gamma_j^{-1}(z')))|^2 dz'$$
$$\le C \int_{\Gamma_i(\partial\Omega \cap V_i)} |f(x') - \varphi_\sigma(x')|^2 dx' \le C\sigma^2\,.$$

Combining (74), (75) we have

$$\|\tilde{f}_\varepsilon - \tilde{f}\|_{L^2(\mathbb{R}^{N-1})}$$
$$\le \|\tilde{f}_\varepsilon - \tilde{\varphi}_{\sigma,\varepsilon}\|_{L^2(\mathbb{R}^{N-1})} + \|\tilde{\varphi}_{\sigma,\varepsilon} - \tilde{\varphi}_\sigma\|_{L^2(\mathbb{R}^{N-1})} + \|\tilde{\varphi}_\sigma - \tilde{f}\|_{L^2(\mathbb{R}^{N-1})} \le 2\sqrt{C}\sigma + \|\tilde{\varphi}_{\sigma,\varepsilon} - \tilde{\varphi}_\sigma\|_{L^2(\mathbb{R}^{N-1})}$$

and the proof of Step 1 follows by (73) and the arbitrariness of σ.

Step 2. In this step we prove (i) in the general case. To this purpose we define

$$\tilde{g}_\varepsilon(z') := \begin{cases} f(\Upsilon_{\varepsilon,i,j}(z')) & \text{if } z' \in \Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j))\,, \\ 0 & \text{if } z' \in \mathbb{R}^{N-1} \setminus \Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)) \end{cases}$$

for any $\varepsilon \in (0, \varepsilon_0]$. By Step 1 we have

$$(76) \qquad\qquad \tilde{g}_\varepsilon \to \tilde{f} \qquad \text{in } L^2(\mathbb{R}^{N-1}) \text{ as } \varepsilon \to 0\,.$$

On the other hand by (21),

$$(77) \qquad \|\tilde{f}_\varepsilon - \tilde{g}_\varepsilon\|_{L^2(\mathbb{R}^{N-1})}^2 = \int_{\Gamma_j(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j))} |f_\varepsilon(\Upsilon_{\varepsilon,i,j}(z')) - f(\Upsilon_{\varepsilon,i,j}(z'))|^2 dz'$$
$$\le C \int_{\Gamma_i(\partial\Omega \cap V_i)} |f_\varepsilon(x') - f(x')|^2 dx' = C\|f_\varepsilon - f\|_{L^2(\mathbb{R}^{N-1})}\,.$$

Combining (76) and (77) we obtain

$$\|\tilde{f}_\varepsilon - \tilde{f}\|_{L^2(\mathbb{R}^{N-1})} \le \|\tilde{f}_\varepsilon - \tilde{g}_\varepsilon\|_{L^2(\mathbb{R}^{N-1})} + \|\tilde{g}_\varepsilon - \tilde{f}\|_{L^2(\mathbb{R}^{N-1})}$$

the proof of Step 2 follows.

Step 3. In this step we prove (ii). We set

$$f_\varepsilon(z') = \begin{cases} \omega_\varepsilon(\Gamma_i^{-1}(z')) & \text{if } z' \in \Gamma_i(\partial\Omega \cap V_i)\,, \\ 0 & \text{if } z' \in \mathbb{R}^{N-1} \setminus \Gamma_i(\partial\Omega \cap V_i)\,, \end{cases}$$

$$f(z') = \begin{cases} \omega(\Gamma_i^{-1}(z')) & \text{if } z' \in \Gamma_i(\partial\Omega \cap V_i)\,, \\ 0 & \text{if } z' \in \mathbb{R}^{N-1} \setminus \Gamma_i(\partial\Omega \cap V_i)\,, \end{cases}$$

and

$$
\phi_\varepsilon(y) = \begin{cases} f_\varepsilon(\Gamma_i(\Theta_{\varepsilon,i,j}(y))) & \text{if } y \in \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)\,, \\[2mm] 0 & \text{if } y \in \partial\Omega \setminus \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j), \end{cases}
$$

$$
\phi(y) = \begin{cases} f(\Gamma_i(y)) & \text{if } y \in \partial\Omega \cap V_i \cap V_j\,, \\[2mm] 0 & \text{if } y \in \partial\Omega \setminus (V_i \cap V_j)\,. \end{cases}
$$

We observe that, using the notation of statement (i), $\tilde{f}_\varepsilon(z') = \phi_\varepsilon(\Gamma_j^{-1}(z'))$ and $\tilde{f}(z') = \phi(\Gamma_j^{-1}(z'))$ for any $z' \in \Gamma_j(\partial\Omega \cap V_j)$. Moreover,

$$
\int_{\partial\Omega} |\phi_\varepsilon - \phi|^2 dS = \int_{\partial\Omega \cap V_j} |\phi_\varepsilon - \phi|^2 dS = \int_{\Gamma_j(\partial\Omega \cap V_j)} |\phi_\varepsilon(\Gamma_j^{-1}(z')) - \phi(\Gamma_j^{-1}(z'))|^2 \sqrt{1 + |\nabla_{z'} g_j(z')|^2}\, dz'
$$

$$
\leq C \int_{\mathbb{R}^{N-1}} |\tilde{f}_\varepsilon(z') - \tilde{f}(z')|^2\, dz' = C \|\tilde{f}_\varepsilon - \tilde{f}\|_{L^2(\mathbb{R}^{N-1})}\,.
$$

Then by (i) we have that $\phi_\varepsilon \to \phi$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$. In order to conclude it suffices to observe that $\tilde{\omega}_\varepsilon$ coincides with ϕ_ε and $\omega \chi_{\partial\Omega \cap V_i \cap v_j}$ coincides with ϕ. $\qquad\square$

Next we prove

Lemma 3.8. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and Ω a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition (21). Let $w_\varepsilon \in V(\Omega_\varepsilon)$ with $0 < \varepsilon \leq \varepsilon_0$, and $w \in V(\Omega)$ be such that $w_\varepsilon \xrightarrow{E} w$. If we put $u_\varepsilon := S_{\Delta,\varepsilon} w_\varepsilon$ and $u := S_\Delta w$ or $u_\varepsilon := S_{D^2,\varepsilon} w_\varepsilon$ and $u := S_{D^2} w$, then $B_\varepsilon u_\varepsilon \rightharpoonup u$ in $V(\Omega)$ as $\varepsilon \to 0$.*

Proof. We prove the lemma by using an argument which works for both problems (2) and (5). In doing this, we use the notation introduced in (61), (62), (63). We divide the proof of the lemma in several steps.

Step 1. In this step we prove that the $V(\Omega_\varepsilon)$-norm of u_ε is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_0]$. First of all we observe that

$$
(78) \quad \|u_\varepsilon\|^2_{V(\Omega_\varepsilon)} = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu\, dS \leq \left(\int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu^2\, dS \right)^{1/2} \left(\int_{\partial\Omega_\varepsilon} (u_\varepsilon)_\nu^2\, dS \right)^{1/2}
$$

$$
\leq (d_1^\varepsilon)^{-1} \left(\int_{\Omega_\varepsilon} |\Delta w_\varepsilon|^2 dx \right)^{1/2} \left(\int_{\Omega_\varepsilon} |\Delta u_\varepsilon|^2 dx \right)^{1/2} \leq N (d_1^\varepsilon)^{-1} \|w_\varepsilon\|_{V(\Omega_\varepsilon)} \|u_\varepsilon\|_{V(\Omega_\varepsilon)}\,.
$$

from which it follows that $\|u_\varepsilon\|_{V(\Omega_\varepsilon)} \leq N (d_1^\varepsilon)^{-1} \|w_\varepsilon\|_{V(\Omega_\varepsilon)}$.

Therefore, by Lemma 3.6 and the fact that $\varepsilon \mapsto \|w_\varepsilon\|_{V(\Omega_\varepsilon)}$ is bounded in view of Lemma 3.1 (ii), it follows that there exists a constant $C > 0$ independent of ε such that

$$
(79) \qquad\qquad \|u_\varepsilon\|_{V(\Omega_\varepsilon)} \leq C \qquad \text{for any } \varepsilon \in (0, \varepsilon_0]\,.
$$

Step 2. We note that $\{B_\varepsilon u_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ is bounded in $V(\Omega)$ and, in particular, that it is weakly convergent in $V(\Omega)$ along a sequence $\varepsilon_n \downarrow 0$ as $n \to +\infty$. Indeed, applying Lemma 3.3 to $\{u_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$, it follows that $\{B_\varepsilon u_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ is bounded in $V(\Omega)$ up to shrink ε_0 if necessary. Thus, for a sequence $\varepsilon_n \downarrow 0$ we have that there exists $\tilde{u} \in V(\Omega)$ such that $B_{\varepsilon_n} u_{\varepsilon_n} \rightharpoonup \tilde{u}$ in $V(\Omega)$. For simplicity in the sequel we only write $B_\varepsilon u_\varepsilon \rightharpoonup \tilde{u}$ as $\varepsilon \to 0$ for denoting this convergence along the sequence $\{\varepsilon_n\}$. The purpose of the next two steps is to prove that $\tilde{u} = u$. This will be done passing to the limit in the following identity

$$
(80) \qquad\qquad Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (E_\varepsilon \varphi)_\nu\, dS \qquad \text{for any } \varphi \in V(\Omega)\,.
$$

Step 3. In this step we pass to the limit in the left hand side of (80).

We define $K_\varepsilon := \left(\bigcup_{j=1}^{s'} r_j^{-1}(K_{\varepsilon,j}) \right) \cup \left(\bigcup_{j=s'+1}^{s} V_j \right)$ and we split the left hand side of (80) into two parts

$$
(81) \qquad\qquad Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) = Q_{K_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) + Q_{\Omega_\varepsilon \setminus K_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi)\,.
$$

Here $K_{\varepsilon,j}$ denotes the set defined in Subsection 3.2.

We recall that $\varphi_j = \psi_j \varphi$ and that $E_\varepsilon \varphi = \sum_{j=1}^{s'} \tilde{\varphi}_{\varepsilon,j} + \sum_{j=s'+1}^{s} \varphi_j$. Thus

$$Q_{K_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) = \sum_{j=1}^{s'} Q_{K_\varepsilon}(u_\varepsilon, \tilde{\varphi}_{\varepsilon,j}) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_\varepsilon, \varphi_j) = \sum_{j=1}^{s'} Q_{K_\varepsilon \cap V_j}(u_\varepsilon, \tilde{\varphi}_{\varepsilon,j}) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_\varepsilon, \varphi_j)$$

$$= \sum_{j=1}^{s'} Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(u_\varepsilon, \tilde{\varphi}_{\varepsilon,j}) + \sum_{j=1}^{s'} Q_{r_j^{-1}(K_{\varepsilon,j})}(u_\varepsilon, \tilde{\varphi}_{\varepsilon,j}) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_\varepsilon, \varphi_j) \,.$$

But

$$|Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(u_\varepsilon, \tilde{\varphi}_{\varepsilon,j})| \leq \left(Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(u_\varepsilon) \right)^{1/2} \left(Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(\tilde{\varphi}_{\varepsilon,j}) \right)^{1/2} = o(1)$$

since $Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(u_\varepsilon) = O(1)$ thanks to (79) and $Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(\tilde{\varphi}_{\varepsilon,j}) = o(1)$ as one can see by proceeding as for (32) . Similarly we have $Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(u_\varepsilon, \varphi_j) = o(1)$. Hence

$$(82) \qquad Q_{K_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) = \sum_{j=1}^{s'} Q_{r_j^{-1}(K_{\varepsilon,j})}(u_\varepsilon, \tilde{\varphi}_{\varepsilon,j}) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_\varepsilon, \varphi_j) + o(1)$$

$$= \sum_{j=1}^{s'} Q_{r_j^{-1}(K_{\varepsilon,j})}(u_\varepsilon, \varphi_j) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_\varepsilon, \varphi_j) + o(1)$$

$$= \sum_{j=1}^{s'} Q_{K_\varepsilon}(u_\varepsilon, \varphi_j) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_\varepsilon, \varphi_j) + o(1) = Q_{K_\varepsilon}(u_\varepsilon, \varphi) + o(1) \,.$$

In a similar way one can treat $Q_{K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi)$:

(83)

$$Q_{K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi) = \sum_{j=1}^{s'} Q_{K_\varepsilon}(\hat{u}_{\varepsilon,j}, \varphi) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_{\varepsilon,j}, \varphi) = \sum_{j=1}^{s'} Q_{K_\varepsilon \cap V_j}(\hat{u}_{\varepsilon,j}, \varphi) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_{\varepsilon,j}, \varphi)$$

$$= \sum_{j=1}^{s'} Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(\hat{u}_{\varepsilon,j}, \varphi) + \sum_{j=1}^{s'} Q_{r_j^{-1}(K_{\varepsilon,j})}(\hat{u}_{\varepsilon,j}, \varphi) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_{\varepsilon,j}, \varphi)$$

$$= \sum_{j=1}^{s'} Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(\hat{u}_{\varepsilon,j}, \varphi) + \sum_{j=1}^{s'} Q_{r_j^{-1}(K_{\varepsilon,j})}(u_{\varepsilon,j}, \varphi) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_{\varepsilon,j}, \varphi) \,.$$

From the end of the proof of Lemma 3.3 we infer that $\{\hat{u}_{\varepsilon,j}\}_{0 < \varepsilon \leq \varepsilon_0}$ is bounded in $H^2(\Omega)$.

Moreover $Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(\varphi) = o(1)$. Therefore, as $\varepsilon \to 0$, we have

$$|Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(\hat{u}_{\varepsilon,j}, \varphi)| \leq (Q_\Omega(\hat{u}_{\varepsilon,j}))^{1/2} \left(Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(\varphi) \right)^{1/2} = o(1) \,,$$

$$|Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(u_{\varepsilon,j}, \varphi)| \leq (Q_{\Omega_\varepsilon}(u_\varepsilon))^{1/2} \left(Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(\varphi) \right)^{1/2} = o(1) \,.$$

Combining this with (83), we obtain

(84)

$$Q_{K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi) = \sum_{j=1}^{s'} Q_{(K_\varepsilon \cap V_j) \backslash r_j^{-1}(K_{\varepsilon,j})}(u_{\varepsilon,j}, \varphi) + \sum_{j=1}^{s'} Q_{r_j^{-1}(K_{\varepsilon,j})}(u_{\varepsilon,j}, \varphi) + \sum_{j=s'+1}^{s} Q_{K_\varepsilon}(u_{\varepsilon,j}, \varphi) + o(1)$$

$$= Q_{K_\varepsilon}(u_\varepsilon, \varphi) + o(1) \qquad \text{as } \varepsilon \to 0 \,.$$

By (82) and (84) it follows

$$(85) \qquad Q_{K_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) = Q_{K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi) + o(1) \qquad \text{as } \varepsilon \to 0.$$

Now, let us consider the second term on the right hand side of (81). We have

$$(86) \qquad |Q_{\Omega_\varepsilon \setminus K_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi)| \le (Q_{\Omega_\varepsilon}(u_\varepsilon))^{1/2} (Q_{\Omega_\varepsilon \setminus K_\varepsilon}(E_\varepsilon \varphi))^{1/2}.$$

But, as $\varepsilon \to 0$, we have

$$(87) \qquad Q_{\Omega_\varepsilon \setminus K_\varepsilon}(E_\varepsilon \varphi) \le s' \sum_{j=1}^{s'} Q_{\Omega_\varepsilon \setminus K_\varepsilon}(\tilde{\varphi}_{\varepsilon,j}) \le s' \sum_{j=1}^{s'} Q_{(\Omega_\varepsilon \cap V_j) \setminus r_j^{-1}(K_{\varepsilon,j})}(\tilde{\varphi}_{\varepsilon,j}) = o(1).$$

By (86) and (87) we arrive to $Q_{\Omega_\varepsilon \setminus K_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) = o(1)$ as $\varepsilon \to 0$.

Similarly we also have $Q_{\Omega \setminus K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi) = o(1)$ as $\varepsilon \to 0$. Inserting these two last asymptotic estimates and (85) into (81) we infer

$$(88) \quad Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) = Q_{K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi) + Q_{\Omega \setminus K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi) + o(1) = Q_\Omega(B_\varepsilon u_\varepsilon, \varphi) + o(1) \qquad \text{as } \varepsilon \to 0.$$

Since $B_\varepsilon u_\varepsilon \rightharpoonup \tilde{u}$ in $V(\Omega)$ then

$$(89) \qquad Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon \varphi) \to Q_\Omega(\tilde{u}, \varphi) \qquad \text{as } \varepsilon \to 0.$$

Step 4. The next purpose is to pass to the limit in the right hand side of (80).

First of all we observe that thanks to Lemma 3.1

$$(90) \qquad \left| \int_{\partial \Omega_\varepsilon} (w_\varepsilon)_\nu (E_\varepsilon \varphi)_\nu \, dS - \int_{\partial \Omega_\varepsilon} (E_\varepsilon w)_\nu (E_\varepsilon \varphi)_\nu \, dS \right|$$
$$\le 2(d_1^\varepsilon)^{-1} \|w_\varepsilon - E_\varepsilon w\|_{V(\Omega_\varepsilon)} \cdot (\|\varphi\|_{V(\Omega)} + o(1)) = o(1),$$

as $\varepsilon \to 0$. We claim that

$$(91) \qquad \int_{\partial \Omega_\varepsilon} (E_\varepsilon w)_\nu (E_\varepsilon \varphi)_\nu \, dS \to \int_{\partial \Omega} w_\nu \varphi_\nu \, dS \qquad \text{as } \varepsilon \to 0.$$

In order to estimate $\int_{\partial \Omega_\varepsilon} (E_\varepsilon w)_\nu (E_\varepsilon \varphi)_\nu \, dS$ we proceed as follows:

$$(92) \qquad \int_{\partial \Omega_\varepsilon} (E_\varepsilon w)_\nu (E_\varepsilon \varphi)_\nu \, dS = \sum_{n,m=1}^{N} \int_{\partial \Omega_\varepsilon} \frac{\partial (E_\varepsilon w)}{\partial x_n} \nu_{\varepsilon,n} \frac{\partial (E_\varepsilon \varphi)}{\partial x_m} \nu_{\varepsilon,m} \, dS$$

$$= \sum_{n,m=1}^{N} \sum_{i,j=1}^{s'} \int_{\partial \Omega_\varepsilon \cap V_i \cap V_j} \frac{\partial (w_i(\Psi_{\varepsilon,i}(x)))}{\partial x_n} \frac{\partial (\varphi_j(\Psi_{\varepsilon,j}(x)))}{\partial x_m} \nu_{\varepsilon,n}(x) \nu_{\varepsilon,m}(x) \, dS$$

$$= \sum_{n,m,k,l=1}^{N} \sum_{i,j=1}^{s'} \int_{\Psi_{\varepsilon,j}(\partial \Omega_\varepsilon \cap V_i \cap V_j)} \frac{\partial w_i}{\partial y_k} (\Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y))) \frac{\partial \varphi_j}{\partial y_l}(y) W_{\varepsilon,n,m,k,l,i,j}(y) \, dS.$$

where we denoted by $\nu_\varepsilon = (\nu_{\varepsilon,1}, \ldots, \nu_{\varepsilon,N})$ the unit normal to $\partial \Omega_\varepsilon$ and we put

$$(93)$$
$$W_{\varepsilon,n,m,k,l,i,j}(y) := \nu_{\varepsilon,n}(\Psi_{\varepsilon,j}^{-1}(y)) \nu_{\varepsilon,m}(\Psi_{\varepsilon,j}^{-1}(y)) \frac{\partial (\Psi_{\varepsilon,i})_k}{\partial x_n}(\Psi_{\varepsilon,j}^{-1}(y)) \frac{\partial (\Psi_{\varepsilon,j})_l}{\partial x_m}(\Psi_{\varepsilon,j}^{-1}(y)) \frac{\sqrt{1 + |\nabla_{x'} g_{\varepsilon,j}(\Gamma_j(y))|^2}}{\sqrt{1 + |\nabla_{x'} g_j(\Gamma_j(y))|^2}}.$$

By (21) we deduce that the trivial extension of $W_{\varepsilon,n,m,k,l,i,j}$ to the whole $\partial \Omega$ converges almost everywhere to the function $\nu_n \nu_m \delta_{kn} \delta_{lm} \chi_{\partial \Omega \cap V_i \cap V_j}$ and it remains uniformly bounded as $\varepsilon \to 0$. Then, applying Lemma 3.7 (ii) to w, by (92) we obtain as $\varepsilon \to 0$

$$\int_{\partial \Omega_\varepsilon} (E_\varepsilon w)_\nu (E_\varepsilon \varphi)_\nu \, dS \to \sum_{n,m=1}^{N} \sum_{i,j=1}^{s'} \int_{\partial \Omega \cap V_i \cap V_j} \frac{\partial w_i}{\partial y_n} \frac{\partial \varphi_j}{\partial y_m} \nu_n \nu_m \, dS = \int_{\partial \Omega} w_\nu \varphi_\nu , \, dS.$$

This completes the proof of (91). In turn, by (90)-(91), we obtain

$$(94) \qquad \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (E_\varepsilon\varphi)_\nu \, dS \to \int_{\partial\Omega} w_\nu \varphi_\nu \, dS \qquad \text{as } \varepsilon \to 0 \, .$$

Step 5. In this last step we complete the proof of the lemma.

Inserting (89) and (94) into (80) we deduce that

$$Q_\Omega(\tilde{u}, \varphi) = \int_{\partial\Omega} w_\nu \varphi_\nu \, dS \qquad \text{for any } \varphi \in V(\Omega) \, .$$

On the other hand, also the function u is a solution of the same variational problem and hence, by uniqueness of the solution of such a problem, we conclude that $\tilde{u} = u$. In particular this means that the weak limit $\tilde{u}$ does not depend on the choice of the sequence $\varepsilon \downarrow 0$, thus proving that the convergence $B_\varepsilon u_\varepsilon \rightharpoonup u$ does not occur only along a special sequence but as $\varepsilon \to 0$ in the usual sense. This completes the proof of the lemma. $\square$

Lemma 3.9. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0 < \varepsilon \le \varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and Ω a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition (21). Let $w_\varepsilon \in V(\Omega_\varepsilon)$ with $0 < \varepsilon \le \varepsilon_0$, and $w \in V(\Omega)$ be such that $w_\varepsilon \xrightarrow{E} w$. If we put $u_\varepsilon := S_{\Delta,\varepsilon} w_\varepsilon$ and $u := S_\Delta w$ or $u_\varepsilon := S_{D^2,\varepsilon} w_\varepsilon$ and $u := S_{D^2} w$ then $u_\varepsilon \xrightarrow{E} u$. In particular this implies that $S_{\Delta,\varepsilon} \xrightarrow{EE} S_\Delta$ and $S_{D^2,\varepsilon} \xrightarrow{EE} S_{D^2}$ as $\varepsilon \to 0$ in the sense of Definition 2.2.*

Proof. We use the notation of (61), (62), (63) in order to treat the problems (2) and (5) simultaneously.

Let us consider

$$(95) \qquad \|u_\varepsilon - E_\varepsilon u\|_{V(\Omega_\varepsilon)}^2 = \|u_\varepsilon\|_{V(\Omega_\varepsilon)}^2 - 2Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon u) + \|E_\varepsilon u\|_{V(\Omega_\varepsilon)}^2 \, .$$

Proceeding as in (88) and (89) we obtain

$$(96) \qquad Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon u) = Q_\Omega(B_\varepsilon u_\varepsilon, u) + o(1) \to Q_\Omega(u, u) \, .$$

The fact that $Q_\Omega(B_\varepsilon u_\varepsilon, u) \to Q_\Omega(u, u)$ is a consequence of the weak convergence $B_\varepsilon u_\varepsilon \rightharpoonup u$ in $V(\Omega)$ proved in Lemma 3.8. On the other hand by Lemma 3.1 we have

$$(97) \qquad \|E_\varepsilon u\|_{V(\Omega_\varepsilon)}^2 \to \|u\|_{V(\Omega)}^2 = Q_\Omega(u, u) \, .$$

It remains to prove that $\|u_\varepsilon\|_{V(\Omega_\varepsilon)}^2 \to \|u\|_{V(\Omega)}^2$. We proceed as follows

$$(98) \qquad Q_{\Omega_\varepsilon}(u_\varepsilon, u_\varepsilon) = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS = \int_{\partial\Omega_\varepsilon} (E_\varepsilon w)_\nu (u_\varepsilon)_\nu \, dS + o(1)$$

where the second identity can be obtained by proceeding exactly as in (90) and exploiting the fact that $\|u_\varepsilon\|_{V(\Omega_\varepsilon)} = O(1)$ as $\varepsilon \to 0$ as we have shown in (79).

We claim that

$$(99) \qquad \int_{\partial\Omega_\varepsilon} (E_\varepsilon w)_\nu (u_\varepsilon)_\nu \, dS \to \int_{\partial\Omega} w_\nu u_\nu \, dS = Q_\Omega(u, u) \, .$$

Inserting (96)-(99) into (95) we obtain the proof of the lemma. It remains to prove (99).

In order to estimate $\int_{\partial\Omega_\varepsilon} (E_\varepsilon w)_\nu (u_\varepsilon)_\nu \, dS$ we proceed as follows:

$$(100) \qquad \int_{\partial\Omega_\varepsilon} (E_\varepsilon w)_\nu (u_\varepsilon)_\nu \, dS = \sum_{n,m=1}^{N} \int_{\partial\Omega_\varepsilon} \frac{\partial (E_\varepsilon w)}{\partial x_n} \nu_{\varepsilon,n} \frac{\partial u_\varepsilon}{\partial x_m} \nu_{\varepsilon,m} \, dS$$

$$= \sum_{n,m=1}^{N} \sum_{i,j=1}^{s'} \int_{\partial\Omega_\varepsilon \cap V_i \cap V_j} \frac{\partial (w_i(\Psi_{\varepsilon,i}(x)))}{\partial x_n} \frac{\partial u_{\varepsilon,j}}{\partial x_m}(x) \, \nu_{\varepsilon,n}(x) \nu_{\varepsilon,m}(x) \, dS$$

$$= \sum_{n,m,k,l=1}^{N} \sum_{i,j=1}^{s'} \int_{\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)} \frac{\partial w_i}{\partial y_k}(\Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y))) \frac{\partial \widehat{u}_{\varepsilon,j}}{\partial y_l}(y) W_{\varepsilon,n,m,k,l,i,j}(y) \, dS \, .$$

with $\nu_{\varepsilon,n}$, $\nu_{\varepsilon,m}$ and $W_{\varepsilon,n,m,k,l,i,j}$ as in (93).

Applying Lemma 3.7 (ii) to w, applying (59) to u_ε, exploiting the fact that $B_\varepsilon u_\varepsilon \rightharpoonup u$ in $V(\Omega)$, and proceeding as at the end of the proof of Lemma 3.8, as $\varepsilon \to 0$, we have

$$\int_{\partial\Omega_\varepsilon} (E_\varepsilon w)_\nu (u_\varepsilon)_\nu \, dS = \sum_{n,m=1}^N \sum_{i,j=1}^{s'} \int_{\partial\Omega \cap V_i \cap V_j} \frac{\partial w_i}{\partial y_n} \frac{\partial \widehat{u}_{\varepsilon,j}}{\partial y_m} \nu_n \nu_m \, dS + o(1)$$

$$= \sum_{n,m=1}^N \sum_{i=1}^{s'} \int_{\partial\Omega \cap V_i} \frac{\partial w_i}{\partial y_n} \frac{\partial (B_\varepsilon u_\varepsilon)}{\partial y_m} \nu_n \nu_m \, dS + o(1)$$

$$= \sum_{n,m=1}^N \sum_{i=1}^{s'} \int_{\partial\Omega \cap V_i} \frac{\partial w_i}{\partial y_n} \frac{\partial u}{\partial y_m} \nu_n \nu_m \, dS + o(1) = \int_{\partial\Omega} w_\nu u_\nu \, dS + o(1) \, .$$

This completes the proof of (99) and of the lemma. $\qquad\square$

Lemma 3.10. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$ and Ω a domain of class $C^{1,1}(\mathcal{A})$. Assume the validity of condition (21). Let $w_\varepsilon \in V(\Omega_\varepsilon)$ with $0 < \varepsilon \leq \varepsilon_0$ be such that $\|w_\varepsilon\|_{V(\Omega_\varepsilon)} = 1$. Then $\{S_{\Delta,\varepsilon} w_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ and $\{S_{D^2,\varepsilon} w_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ are precompact in the sense of Definition 2.3. In particular, by Definition 2.4 and Lemma 3.9 we have that $S_{\Delta,\varepsilon} \xrightarrow{C} S_\Delta$ and $S_{D^2,\varepsilon} \xrightarrow{C} S_{D^2}$ as $\varepsilon \to 0$.*

Proof. According to the notations of Lemma 3.8 we put $u_\varepsilon := S_{\Delta,\varepsilon} w_\varepsilon$ for problem (2) and $u_\varepsilon := S_{D^2,\varepsilon} w_\varepsilon$ for problem (5). Exploiting the fact that $\|w_\varepsilon\|_{V(\Omega_\varepsilon)}$ is bounded we may proceed as in the proof of Lemma 3.8 and prove that $B_\varepsilon u_\varepsilon \rightharpoonup \widetilde{u}$ along a sequence, for some $\widetilde{u} \in V(\Omega)$. We divide the remaining part of the proof into four steps. We recall that as in the proof of Lemma 3.8, for any $\varepsilon \in (0, \varepsilon_0]$ we have that u_ε satisfies (80).

Step 1. In this step we pass to the limit in the right hand side of (80).

We put $h_{\varepsilon,n,i} := \psi_i \frac{\partial w_\varepsilon}{\partial x_n}$ for $n = 1, \ldots, N$ and $i = 1, \ldots, s'$, and we proceed as follows:

$$(101) \qquad \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (E_\varepsilon \varphi)_\nu \, dS = \sum_{n,m=1}^N \int_{\partial\Omega_\varepsilon} \frac{\partial w_\varepsilon}{\partial x_n} \nu_{\varepsilon,n} \frac{\partial (E_\varepsilon \varphi)}{\partial x_m} \nu_{\varepsilon,m} \, dS$$

$$= \sum_{n,m=1}^N \sum_{i,j=1}^{s'} \int_{\partial\Omega_\varepsilon \cap V_i \cap V_j} h_{\varepsilon,n,i}(x) \frac{\partial (\varphi_j(\Psi_{\varepsilon,j}(x)))}{\partial x_m} \nu_{\varepsilon,n}(x) \nu_{\varepsilon,m}(x) \, dS$$

$$= \sum_{n,m,l=1}^N \sum_{i,j=1}^{s'} \int_{\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)} h_{\varepsilon,n,i}(\Psi_{\varepsilon,j}^{-1}(y)) \frac{\partial \varphi_j}{\partial y_l}(y) W_{\varepsilon,n,m,l,j}(y) \, dS \, .$$

where we denoted by $\nu_\varepsilon = (\nu_{\varepsilon,1}, \ldots, \nu_{\varepsilon,N})$ the unit normal to $\partial\Omega_\varepsilon$ and we put

$$W_{\varepsilon,n,m,l,j}(y) := \nu_{\varepsilon,n}(\Psi_{\varepsilon,j}^{-1}(y)) \nu_{\varepsilon,m}(\Psi_{\varepsilon,j}^{-1}(y)) \frac{\partial (\Psi_{\varepsilon,j})_l}{\partial x_m}(\Psi_{\varepsilon,j}^{-1}(y)) \frac{\sqrt{1+|\nabla_{x'} g_{\varepsilon,j}(\Gamma_j(y))|^2}}{\sqrt{1+|\nabla_{x'} g_j(\Gamma_j(y))|^2}} \, .$$

By (21) we deduce that the trivial extension of $W_{\varepsilon,n,m,l,j}$ to the whole $\partial\Omega$ converges almost everywhere to the function $\nu_n \nu_m \delta_{lm} \chi_{\partial\Omega \cap V_i \cap V_j}$ and it remains uniformly bounded as $\varepsilon \to 0$.

Now we observe that by Lemma 3.4 and Lemma 3.5, there exists a positive constant C such that

$$\|h_{\varepsilon,n,i}\|_{H^{1/2}(\partial\Omega_\varepsilon)} < C \qquad \text{for any } \varepsilon \in (0, \varepsilon_0] \, .$$

Hence, if we put $\omega_{\varepsilon,n,i} = h_{\varepsilon,n,i} \circ \Psi_{\varepsilon,i}^{-1}$, by (65), we also deduce that $\{\omega_{\varepsilon,n,i}\}_{0<\varepsilon\leq\varepsilon_0}$ is bounded in $H^{1/2}(\partial\Omega)$ and by the compact embedding $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ we also have that $\{\omega_{\varepsilon,n,i}\}_{0<\varepsilon\leq\varepsilon_0}$ is precompact in $L^2(\partial\Omega)$. Then, along a sequence $\varepsilon_k \downarrow 0$ we may assume that $\omega_{\varepsilon_k,n,i} \to F_{n,i}$ in $L^2(\partial\Omega)$ as $k \to +\infty$. For simplicity, in the rest of the proof of the lemma we will omit the subindex k and we simply write $\omega_{\varepsilon,n,i} \to F_{n,i}$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$.

But $\omega_{\varepsilon,n,i}(\Theta_{\varepsilon,i,j}(y)) = h_{\varepsilon,n,i}(\Psi_{\varepsilon,j}^{-1}(y))$ for any $y \in \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)$.

Then, applying Lemma 3.7 (ii) to $\omega_{\varepsilon,n,i}$, by (101) we obtain as $\varepsilon \to 0$

$$(102) \qquad \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (E_\varepsilon \varphi)_\nu \, dS \to \sum_{n,m=1}^{N} \sum_{i,j=1}^{s'} \int_{\partial\Omega \cap V_i \cap V_j} F_{n,i} \frac{\partial \varphi_j}{\partial y_m} \nu_n \nu_m \, dS = \int_{\partial\Omega} (F \cdot \nu) \, \varphi_\nu \, dS$$

where we put $F = \left(\sum_{i=1}^{s'} F_{1,i}, \ldots, \sum_{i=1}^{s'} F_{N,i} \right)$.

Step 2. In this step we pass to the limit in the left hand side of (80).

We can proceed exactly as in the proof of Step 3 in Lemma 3.8 since the only fact exploited there, is that the $V(\Omega_\varepsilon)$-norm of u_ε can be estimated uniformly with respect to ε. In this way, one proves that (89) holds true.

Inserting (89) and (102) into (80) we obtain

$$(103) \qquad Q_\Omega(\widetilde{u}, \varphi) = \int_{\partial\Omega} (F \cdot \nu) \, \varphi_\nu \, dS \qquad \text{for any } \varphi \in V(\Omega).$$

Step 3. The purpose of this step is to pass to the limit in the right hand side of the following identity

$$(104) \qquad Q_{\Omega_\varepsilon}(u_\varepsilon, u_\varepsilon) = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS.$$

Similarly to (101) we can write

$$(105) \qquad \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS = \sum_{n,m,l=1}^{N} \sum_{i,j=1}^{s'} \int_{\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)} h_{\varepsilon,n,i}(\Psi_{\varepsilon,j}^{-1}(y)) \frac{\partial \widehat{u}_{\varepsilon,j}}{\partial y_l}(y) W_{\varepsilon,n,m,l,j}(y) \, dS$$

with $h_{\varepsilon,n,i}$ and $W_{\varepsilon,n,m,l,j}$ as in Step 1, and $\widehat{u}_{\varepsilon,j}$ as in (43). By (59) we deduce that, passing to the limit along a subsequence of the sequence $\{\varepsilon_k\}$ introduced in Step 1, for any $j \in \{1, \ldots, s'\}$ there exists a function $U_j \in V(\Omega)$ such that $\widehat{u}_{\varepsilon,j} \rightharpoonup U_j$ in $V(\Omega)$. Using the same notations of Step 1, we simply write $\varepsilon \to 0$ to denote the convergence along a sequence. By (105), we then have as $\varepsilon \to 0$

$$(106) \qquad \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to \sum_{n,m,l=1}^{N} \sum_{i,j=1}^{s'} \int_{\partial\Omega \cap V_i \cap V_j} F_{n,i} \frac{\partial U_j}{\partial y_l} \nu_n \nu_m \, dS$$

Since $B_\varepsilon u_\varepsilon = \sum_{j=1}^{s'} \widehat{u}_{\varepsilon,j}$ on $\partial\Omega$, from the uniqueness of the weak limit in $V(\Omega)$, one immediately obtains $\widetilde{u} = \sum_{j=1}^{s'} U_j$ which inserted into (106) gives

$$(107) \qquad \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to \int_{\partial\Omega} (F \cdot \nu) \, \widetilde{u}_\nu \, dS.$$

Step 4. In this step we conclude the proof of the lemma.

Choosing $\varphi = \widetilde{u}$ in (103) and combining this with (104), (107), we obtain as $\varepsilon \to 0$ along a proper sequence $\varepsilon_k \downarrow 0$,

$$(108) \quad \|u_\varepsilon\|_{V(\Omega_\varepsilon)}^2 = Q_{\Omega_\varepsilon}(u_\varepsilon, u_\varepsilon) = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to \int_{\partial\Omega} (F \cdot \nu) \, \widetilde{u}_\nu \, dS = Q_\Omega(\widetilde{u}, \widetilde{u}) = \|\widetilde{u}\|_{V(\Omega)}^2.$$

On the other hand, by (80), (102), (103) with $\varphi = \widetilde{u}$, we obtain as $\varepsilon \to 0$ along the same sequence converging to zero,

$$(109) \qquad Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon \widetilde{u}) \to Q_\Omega(\widetilde{u}, \widetilde{u}) = \|\widetilde{u}\|_{V(\Omega)}^2.$$

Combining (108) and (109) with Lemma 3.1 (ii), we obtain

$$\|u_\varepsilon - E_\varepsilon \widetilde{u}\|_{V(\Omega_\varepsilon)}^2 = \|u_\varepsilon\|_{V(\Omega_\varepsilon)}^2 - 2Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon \widetilde{u}) + \|E_\varepsilon \widetilde{u}\|_{V(\Omega_\varepsilon)}^2 \to 0.$$

This proves that along a sequence we have $u_\varepsilon \xrightarrow{E} \widetilde{u}$ as $\varepsilon \to 0$ or equivalently that $\{S_{\Delta,\varepsilon} w_\varepsilon\}_{0 < \varepsilon \le \varepsilon_0}$ and $\{S_{D^2,\varepsilon} w_\varepsilon\}_{0 < \varepsilon \le \varepsilon_0}$ are precompact in the sense of Definition 2.3. The proof of the lemma now follows from Lemma 3.9 and Definition 2.4. $\qquad\square$

As a bypass product of a number of results proved in this section, we have the following proposition which we believe has its own interest since it clarifies even more the meaning of E-convergence with respect to the operators E_ε used above.

Proposition 3.11. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{1,1}(\mathcal{A})$, Ω a domain of class $C^{1,1}(\mathcal{A})$ and for any $\varepsilon \in (0,\varepsilon_0]$ let E_ε be the map defined in (26). Assume the validity of condition (21). If $u_\varepsilon \in V(\Omega_\varepsilon)$, $u \in V(\Omega)$ is such that $u_\varepsilon \xrightarrow{E} u$ as $\varepsilon \to 0$ then*

$$(110) \qquad \|u_\varepsilon - u\|_{H^2(\Omega_\varepsilon\cap\Omega)} \to 0, \quad \text{and} \quad \|\nabla u_\varepsilon - E_\varepsilon\nabla u\|_{L^2(\partial\Omega_\varepsilon)} \to 0,$$

as $\varepsilon \to 0$.

Proof. Let $K_\varepsilon \subset \Omega_\varepsilon \cap \Omega$ be as in the proof of Lemma 3.8. Recall that $E_\varepsilon u = u$ on K_ε. In order to prove the first limit in (3.11) it suffices to note that by Lemma 3.4 and the proof of Lemma 3.1 we have

$$
\begin{aligned}
\|u_\varepsilon - u\|_{H^2(\Omega_\varepsilon\cap\Omega)} &\le\ \|u_\varepsilon - E_\varepsilon u\|_{H^2(\Omega_\varepsilon\cap\Omega)} + \|E_\varepsilon u - u\|_{H^2(\Omega_\varepsilon\cap\Omega)} \\
&\le\ \|u_\varepsilon - E_\varepsilon u\|_{H^2(\Omega_\varepsilon)} + \|E_\varepsilon u - u\|_{H^2((\Omega_\varepsilon\cap\Omega)\setminus K_\varepsilon)} \\
&\le\ C\|u_\varepsilon - E_\varepsilon u\|_{V(\Omega_\varepsilon)} + \|E_\varepsilon u - u\|_{H^2((\Omega_\varepsilon\cap\Omega)\setminus K_\varepsilon)} \\
&=\ C\|u_\varepsilon - E_\varepsilon u\|_{V(\Omega_\varepsilon)} + o(1).
\end{aligned}
$$

In order to prove the second limit in (3.11) we note that by Lemma 3.5 the trace constant can be taken uniform with the respect to ε hence $\|\nabla u_\varepsilon - \nabla(E_\varepsilon u)\|_{L^2(\partial\Omega_\varepsilon)} \le C\|u_\varepsilon - E_\varepsilon u\|_{H^2(\Omega_\varepsilon)} = o(1)$. Finally, by standard computations one can see that $\|\nabla u_\varepsilon - E_\varepsilon\nabla u\|_{L^2(\partial\Omega_\varepsilon)} = \|\nabla u_\varepsilon - \nabla(E_\varepsilon u)\|_{L^2(\partial\Omega_\varepsilon)} + o(1)$. $\qquad\square$

4. OPTIMALITY OF CONDITION (21): DEGENERATIONS AND STRANGE TERMS

We plan to discuss the optimality of condition (21). We ask whether spectral stability is still verified if we remove condition (21) from the statement of Theorem 3.2. In order to give an answer to this question we assume particular conditions on the domain Ω and on its perturbations Ω_ε. We follow closely the approach introduced in [4]. We assume that Ω is in the form $\Omega = W \times (-1,0)$ where W is a cuboid or a bounded domain in $\mathbb{R}^{N-1}$ of class $C^{1,1}$. We assume that the perturbed domain Ω_ε is given by

$$\Omega_\varepsilon = \{(x', x_N) : x' \in W, -1 < x_N < g_\varepsilon(x')\}$$

where $g_\varepsilon(x') = \varepsilon^\alpha b(x'/\varepsilon)$ for any $x' \in W$ and $b : \mathbb{R}^{N-1} \to [0,+\infty)$ is a Y-periodic function where $Y = \left(-\frac{1}{2},\frac{1}{2}\right)^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$.

We denote by Γ_ε and Γ the sets

$$(111) \qquad \Gamma_\varepsilon := \{(x', g_\varepsilon(x')) : x' \in W\} \quad \text{and} \quad \Gamma := W \times \{0\}.$$

In this case, condition (21) can be used provided $\alpha > 3/2$. In fact, we have the following straightforward application of Theorem 3.2 in the simplified form discussed in Remark 5.

Corollary 4.1. *Suppose that the function b defined above belongs to $C^{1,1}(\mathbb{R}^{N-1})$. Let Ω and Ω_ε be as above. Let $S_{\Delta,\varepsilon}$, S_Δ be the Steklov operators for (2) in Ω_ε and Ω respectively, and let $S_{D^2,\varepsilon}$, S_{D^2} be the Steklov operators for (5) in Ω_ε and Ω respectively, see Subsection 3.1. If $\alpha > 3/2$ then $S_{\Delta,\varepsilon} \xrightarrow{C} S_\Delta$ and $S_{D^2,\varepsilon} \xrightarrow{C} S_{D^2}$ as $\varepsilon \to 0$.*

Proof. For fixed $\tilde\alpha \in]3/2, \alpha[$, we set $\kappa_\varepsilon = \varepsilon^{2\tilde\alpha/3}$ and we note that condition (21) is satisfied with $g_{\varepsilon,j}$, g_j replaced by g_ε, $g \equiv 0$ respectively. The proof follows by Theorem 3.2 (actually, by the simplified version of it concerning the case of domains defined as subgraphs of functions defined in one single cuboid, see Remark 5). $\qquad\square$

We shall see that Corollary 4.1 no longer holds for problem (5) if $\alpha \le 3/2$. The cases $\alpha < 3/2$ and $\alpha = 3/2$ are considerably different. In the first case we have degeneration to Dirichlet boundary conditions, in the second case we have the appearance of a strange term in the limiting boundary conditions. We discuss the two cases in the following subsections.

4.1. **Case $\alpha < 3/2$: degeneration.** In this case it appears that the energy spaces $V(\Omega_\varepsilon) = H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ degenerate as $\varepsilon \to 0$ to a suitable energy space which encodes Dirichlet boundary conditions on Γ. Namely, let us define the space

(112) $$H_{0,\Gamma}^2(\Omega) := \{u \in H^2(\Omega) \cap H_0^1(\Omega) : u_\nu = 0 \text{ on } \Gamma\}\,.$$

According to Section 3.1, we define $T_{D^2,\Gamma} : H_{0,\Gamma}^2(\Omega) \to (H_{0,\Gamma}^2(\Omega))'$ by

$$_{(H_{0,\Gamma}^2(\Omega))'}\langle T_{D^2,\Gamma} u, v \rangle_{H_{0,\Gamma}^2(\Omega)} := \int_\Omega D^2 u : D^2 v \, dx \qquad \text{for any } u, v \in H_{0,\Gamma}^2(\Omega)\,.$$

We also define the operator $J_\Gamma : H_{0,\Gamma}^2(\Omega) \to (H_{0,\Gamma}^2(\Omega))'$ by

$$_{(H_{0,\Gamma}^2(\Omega))'}\langle J_\Gamma u, v \rangle_{H_{0,\Gamma}^2(\Omega)} := \int_{\partial\Omega\setminus\Gamma} u_\nu v_\nu \, dS \qquad \text{for any } u, v \in H_{0,\Gamma}^2(\Omega)\,.$$

Finally we define $S_{D^2,\Gamma} : H_{0,\Gamma}^2(\Omega) \to H_{0,\Gamma}^2(\Omega)$ as $S_{D^2,\Gamma} := T_{D^2,\Gamma}^{-1} \circ J_\Gamma$.

We shall prove that $S_{D^2,\varepsilon}$ converges compactly to $S_{D^2,\Gamma}$ as $\varepsilon \to 0$. To do so we need to clarify which family of operators $E_\varepsilon : H_{0,\Gamma}^2(\Omega) \to V(\Omega_\varepsilon)$ we consider. In the case $\alpha < 3/2$ we can no longer use the operators E_ε defined in Subsection 3.2. However, here it is possible to consider a much simpler family of operators given by the extension-by-zero operators.

Let us define the extension-by-zero operator $\mathcal{E}_0 : H_{0,\Gamma}^2(\Omega) \to V(\Omega_\varepsilon)$ which maps any function $u \in H_{0,\Gamma}^2(\Omega)$ to $\mathcal{E}_0 u$ where $\mathcal{E}_0 u : W \times (-1, +\infty) \to \mathbb{R}$ coincides with u over Ω and vanishes in $W \times [0, +\infty)$. We observe that $\mathcal{E}_0$ is well-defined as an operator from $H_{0,\Gamma}^2(\Omega)$ to $V(\Omega_\varepsilon)$ since for any $u \in H_{0,\Gamma}^2(\Omega)$ we have $\mathcal{E}_0 u \in H^2(W \times (-1, +\infty))$ being $u = u_\nu = 0$ on Γ and hence its restriction to Ω_ε belongs indeed to $V(\Omega_\varepsilon)$.

Since we are dealing with problem (5), the space $V(\Omega_\varepsilon)$ and the space $H_{0,\Gamma}^2(\Omega)$ will be endowed with the second scalar product in (15), i.e.

$$\|u\|_{V(\Omega_\varepsilon)} := \left(\int_{\Omega_\varepsilon} |D^2 u|^2 dx\right)^{1/2} \qquad \text{for any } u \in V(\Omega_\varepsilon)\,,$$

$$\|u\|_{H_{0,\Gamma}^2(\Omega)} := \left(\int_\Omega |D^2 u|^2 dx\right)^{1/2} \qquad \text{for any } u \in H_{0,\Gamma}^2(\Omega)\,.$$

According to Definitions 2.1-2.4, the following notions of convergence are well defined

$$u_\varepsilon \xrightarrow{E} u\,, \qquad B_\varepsilon \xrightarrow{EE} B_0\,, \qquad B_\varepsilon \xrightarrow{C} B_0$$

once we put $\mathcal{H}_\varepsilon = V(\Omega_\varepsilon)$, $\mathcal{H}_0 = H_{0,\Gamma}^2(\Omega)$ and $E_\varepsilon = \mathcal{E}_0$. We observe that property (11) is trivially satisfied.

Then the following result holds

Theorem 4.1. *Suppose that the function b defined above belongs to $C^{1,1}(\mathbb{R}^{N-1})$ and is not constant. Let Ω and Ω_ε be as above. Let $S_{D^2,\varepsilon}$, S_{D^2} be the Steklov operators for (5) in Ω_ε and Ω respectively, see Section 3.1. Finally, let $S_{D^2,\Gamma} : H_{0,\Gamma}^2(\Omega) \to H_{0,\Gamma}^2(\Omega)$ be as above. If $\alpha \in \left[1, \frac{3}{2}\right)$ then $S_{D^2,\varepsilon} \xrightarrow{C} S_{D^2,\Gamma}$ as $\varepsilon \to 0$ and in particular the eigenvalues of (5) in Ω_ε converge, as $\varepsilon \to 0$, to the eigenvalues of the following problem*

(113) $$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \\ u_\nu = 0, & \text{on } \Gamma\,, \\ \Delta u - K(x)u_\nu - \delta u_\nu = 0, & \text{on } \partial\Omega \setminus \Gamma\,. \end{cases}$$

The proof of Theorem 4.1 is based on a number of technical results. First of all, we prove that the $V(\Omega_\varepsilon)$-norm is uniformly equivalent to the $H^2(\Omega_\varepsilon)$-norm.

Lemma 4.2. *Suppose that all the assumptions of Theorem 4.1 are satisfied. Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

(i) *for any $u \in V(\Omega_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0]$ we have*

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq C\|u\|_{V(\Omega_\varepsilon)};$$

(ii) *for any $v \in H^1(\Omega_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0]$ we have*

$$\|v\|_{L^2(\partial\Omega_\varepsilon)} \leq C\|v\|_{H^1(\Omega_\varepsilon)}.$$

Proof. The proof of (i) is a consequence of Lemma 2.7, see also (45). The proof of (ii) follows from direct computation exploiting the fact that under the assumptions of Theorem 4.1, the first order derivatives of g_ε remain uniformly bounded as $\varepsilon \to 0$. $\qquad\square$

As an immediate consequence of Lemma 4.2 we obtain the following uniform estimate for the first eigenvalue of (5).

Lemma 4.3. *Suppose that all the assumptions of Theorem 4.1 are satisfied. Then*

$$\liminf_{\varepsilon \to 0} \delta_1^\varepsilon > 0$$

where δ_1^ε is defined by (18) with Ω_ε in place of Ω.

Proof. We denote by C a positive constant independent of ε and u which may vary from line to line. Let $u \in V(\Omega_\varepsilon)$ and for $i \in \{1, \ldots, N\}$ let us apply Lemma 4.2 to $\frac{\partial u}{\partial x_i}$:

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\partial\Omega_\varepsilon)} \leq C\left\|\frac{\partial u}{\partial x_i}\right\|_{H^1(\Omega_\varepsilon)} \leq C\|u\|_{H^2(\Omega_\varepsilon)} \leq C\|u\|_{V(\Omega_\varepsilon)} \qquad \text{for any } u \in V(\Omega_\varepsilon) \text{ and } \varepsilon \in (0, \varepsilon_0].$$

Thus

$$\int_{\partial\Omega_\varepsilon} u_\nu^2 \, dS \leq C \int_{\Omega_\varepsilon} |D^2 u|^2 dx \qquad \text{for any } u \in V(\Omega_\varepsilon) \text{ and } \varepsilon \in (0, \varepsilon_0].$$

The proof of the lemma then follows from (18). $\qquad\square$

Next we prove that the family of operators $\{S_{D^2,\varepsilon}\}$ E-converges to $S_{D^2,\Gamma}$ as $\varepsilon \to 0$.

Lemma 4.4. *Suppose that all the assumptions of Theorem 4.1 are satisfied. Then $S_{D^2,\varepsilon} \xrightarrow{EE} S_{D^2,\Gamma}$ as $\varepsilon \to 0$.*

Proof. Let $w_\varepsilon \in V(\Omega_\varepsilon)$ and $w \in H^2_{0,\Gamma}(\Omega)$ be such that $w_\varepsilon \xrightarrow{E} w$ as $\varepsilon \to 0$. Put $u_\varepsilon := S_{D^2,\varepsilon} w_\varepsilon$. We claim that $\|u_\varepsilon\|_{V(\Omega_\varepsilon)}$ remains bounded as $\varepsilon \to 0$. Using the same argument as in (78) we have $\|u_\varepsilon\|^2_{V(\Omega_\varepsilon)} \leq (\delta_1^\varepsilon)^{-1}\|w_\varepsilon\|_{V(\Omega_\varepsilon)}\|u_\varepsilon\|_{V(\Omega_\varepsilon)}$ and this together with Lemma 4.3 proves the claim being $\|w_\varepsilon\|_{V(\Omega_\varepsilon)}$ bounded as $\varepsilon \to 0$ as a consequence of the fact that $w_\varepsilon \xrightarrow{E} w$.

In particular by Lemma 4.2 (i) we also have that $\{(u_\varepsilon)_{|\Omega}\}_{0 < \varepsilon \leq \varepsilon_0}$ is bounded in $H^2(\Omega)$ as $\varepsilon \to 0$ and hence along a sequence we have that there exists $u \in H^2(\Omega)$, possibly depending on the choice of the sequence, such that $(u_\varepsilon)_{|\Omega} \rightharpoonup u$ in $H^2(\Omega)$ as $\varepsilon \to 0$.

By taking the trivial extension of u_ε to the whole $\mathbb{R}^N$, one sees that it is weakly convergent in $H^1(\mathbb{R}^N)$ and pointwise convergent almost everywhere (up to a subsequence) to a function belonging to $H^1(\mathbb{R}^N)$, whose restriction to Ω coincides with u. This shows that $u \in H^1_0(\Omega)$.

We claim that $u_\nu = \frac{\partial u}{\partial x_N} = 0$ on Γ thus showing that $u \in H^2_{0,\Gamma}(\Omega)$. In order to prove the claim we apply [18, Lemma 4.3] or more precisely its extension to the N-dimensional case. Indeed, for any $i \in \{1, \ldots, N-1\}$, proceeding as in proof of Theorem 7.4 in [4], we define the family of vector fields depending on the parameter ε

$$V_\varepsilon^{(i)} = \left(0, \ldots, 0, -\frac{\partial u_\varepsilon}{\partial x_N}, 0, \ldots, 0, \frac{\partial u_\varepsilon}{\partial x_i}\right)$$

where the only two nontrivial components are the i-th and N-th ones. Since $u_\varepsilon(x', g_\varepsilon(x')) = 0$ for any $x' \in W$ it follows that $V_\varepsilon^{(i)} \cdot \nu = 0$ on Γ_ε and hence, since $\alpha < \frac{3}{2}$, we can apply [18, Lemma 4.3 (i)] to this vector field and obtain

$$-\frac{\partial u}{\partial x_N}(x', 0)\frac{\partial b}{\partial y_i}(y') = 0 \qquad \text{for any } x' \in W \text{ and } y' \in \mathbb{R}^{N-1}.$$

Since b is not constant we deduce that $\frac{\partial u}{\partial x_N} \equiv 0$ in Γ thus proving the claim.

We now prove that u does not depend on the sequence converging to zero. Letting $\varphi \in H^2_{0,\Gamma}(\Omega)$ we obtain

$$(114) \qquad \int_\Omega D^2 u_\varepsilon : D^2 \varphi \, dx = \int_{\Omega_\varepsilon} D^2 u_\varepsilon : D^2(\mathcal{E}_0 \varphi) \, dx = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (\mathcal{E}_0\varphi)_\nu \, dS = \int_{\partial\Omega\setminus\Gamma} (w_\varepsilon)_\nu \varphi_\nu \, dS \,.$$

From the weak convergence $(u_\varepsilon)_{|\Omega} \rightharpoonup u$ in $H^2(\Omega)$, boundedness of $\{(w_\varepsilon)_{|\Omega}\}_{0<\varepsilon\leq\varepsilon_0}$ in $H^2(\Omega)$ as $\varepsilon \to 0$, and compactness of the trace map, we can pass to the limit in (114) as $\varepsilon \to 0$ along a sequence and obtain

$$(115) \qquad \int_\Omega D^2 u : D^2 \varphi \, dx = \int_{\partial\Omega\setminus\Gamma} w_\nu \varphi_\nu \, dS \qquad \text{for any } \varphi \in H^2_{0,\Gamma}(\Omega) \,.$$

We observe that the first function appearing in the integral at the right hand side of (115) is exactly w since, by Lemma 4.2 (i), $w_\varepsilon \xrightarrow{E} w$ implies $(w_\varepsilon)_{|\Omega} \to w$ in $H^2(\Omega)$.

The variational identity (115) proves that $u \in H^2_{0,\Gamma}(\Omega)$ is a weak solution of the problem

$$(116) \qquad \begin{cases} \Delta^2 u = 0, & \text{in } \Omega \,, \\ u = 0, & \text{on } \partial\Omega \,, \\ u_\nu = 0, & \text{on } \Gamma \,, \\ \Delta u - K(x) u_\nu = w_\nu, & \text{on } \partial\Omega \setminus \Gamma \,. \end{cases}$$

Since it is easy to check that this problem admits a unique solution, we deduce that u depends only on w and not on the sequence converging to zero. Moreover, we also have that $u = S_{D^2,\Gamma} w$.

Then we prove that $u_\varepsilon \xrightarrow{E} u$. Since $\mathcal{E}_0 u$ vanishes outside Ω and $(u_\varepsilon)_{|\Omega} \rightharpoonup u$ in $H^2(\Omega)$, we have

$$(117) \qquad \|u_\varepsilon - \mathcal{E}_0 u\|^2_{V(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} |D^2 u_\varepsilon|^2 dx - 2\int_\Omega D^2 u_\varepsilon : D^2 u \, dx + \int_\Omega |D^2 u|^2 dx$$

$$= \int_{\Omega_\varepsilon} |D^2 u_\varepsilon|^2 dx - \int_\Omega |D^2 u|^2 dx + o(1) \,.$$

Now we observe that

$$(118) \qquad \int_{\Omega_\varepsilon} |D^2 u_\varepsilon|^2 dx = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \,.$$

We prove that

$$(119) \qquad \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to \int_{\partial\Omega} w_\nu u_\nu \, dS = \int_{\partial\Omega\setminus\Gamma} w_\nu u_\nu \, dS \,.$$

Indeed we have

$$\left| \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS - \int_{\partial\Omega} w_\nu u_\nu \, dS \right|$$

$$\leq \left| \int_{\partial\Omega_\varepsilon} (w_\varepsilon - \mathcal{E}_0 w)_\nu (u_\varepsilon)_\nu \, dS \right| + \left| \int_{\partial\Omega_\varepsilon} (\mathcal{E}_0 w)_\nu (u_\varepsilon)_\nu \, dS - \int_{\partial\Omega} w_\nu u_\nu \, dS \right|$$

$$\leq (\delta_1^\varepsilon)^{-1} \|w_\varepsilon - \mathcal{E}_0 w\|_{V(\Omega_\varepsilon)} \|u_\varepsilon\|_{V(\Omega_\varepsilon)} + \left| \int_{\partial\Omega} w_\nu (u_\varepsilon)_\nu \, dS - \int_{\partial\Omega} w_\nu u_\nu \, dS \right| = o(1) \quad \text{as } \varepsilon \to 0$$

since $(\delta_1^\varepsilon)^{-1}$ is bounded as $\varepsilon \to 0$ in view of Lemma 4.3, $\|u_\varepsilon\|_{V(\Omega_\varepsilon)}$ is bounded as $\varepsilon \to 0$ as already observed, $\|w_\varepsilon - \mathcal{E}_0 u\|_{V(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$ since $w_\varepsilon \xrightarrow{E} w$ and $(u_\varepsilon)_\nu \to u_\nu$ in $L^2(\partial\Omega)$ since $(u_\varepsilon)_{|\Omega} \rightharpoonup u$ in $H^2(\Omega)$ as $\varepsilon \to 0$. This proves the validity of (119).

Finally we observe that, by (115), we also have

$$(120) \qquad \int_\Omega |D^2 u|^2 dx = \int_{\partial\Omega\setminus\Gamma} w_\nu u_\nu \, dS \,.$$

Combining (118), (119) and (120) we infer that $\int_{\Omega_\varepsilon} |D^2 u_\varepsilon|^2 dx \to \int_\Omega |D^2 u|^2 dx$ as $\varepsilon \to 0$ which combined with (117) proves that $u_\varepsilon \xrightarrow{E} u$.

We just proved that $S_{D^2,\varepsilon} w_\varepsilon \xrightarrow{E} S_{D^2,\Gamma} w$ as $\varepsilon \to 0$ and this, according to Definition 2.2, implies $S_{D^2,\varepsilon} \xrightarrow{EE} S_{D^2,\Gamma}$ as $\varepsilon \to 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

End of the proof of Theorem 4.1. Having already proved Lemma 4.4, it suffices to prove the validity of the compactness property stated in Definition 2.4. Let $w_\varepsilon \in V(\Omega_\varepsilon)$ and $\|w_\varepsilon\|_{V(\Omega_\varepsilon)} = 1$ for any $\varepsilon > 0$. As in Lemma 4.4 we put $u_\varepsilon = S_{D^2,\varepsilon} w_\varepsilon$. By Lemma 4.2 (i) we may assume that, along a sequence converging to zero, there exists $w \in H^2(\Omega)$ such that $(w_\varepsilon)_{|\Omega} \rightharpoonup w$ in $H^2(\Omega)$. Proceeding as in the proof of Lemma 4.4 we obtain

$$\|u_\varepsilon\|_{V(\Omega_\varepsilon)} \le (\delta_1^\varepsilon)^{-1} \|w_\varepsilon\|_{V(\Omega_\varepsilon)} \le C \qquad \text{for any } \varepsilon \in (0, \varepsilon_0]$$

for some constants ε_0 and C independent of ε. By Lemma 4.2 (i) we have that $\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \le C$ for any $\varepsilon \in (0, \varepsilon_0]$ and hence $\{(u_\varepsilon)_{|\Omega}\}_{0<\varepsilon\le\varepsilon_0}$ is bounded in $H^2(\Omega)$; in particular there exists $\widetilde{u} \in H^2(\Omega)$ such that $(u_\varepsilon)_{|\Omega} \rightharpoonup \widetilde{u}$ in $H^2(\Omega)$ as $\varepsilon \to 0$ along a sequence. Proceeding as in the proof of Lemma 4.4 one can show that $\widetilde{u} \in H^2_{0,\Gamma}(\Omega)$ and it solves (116). In particular, we have that $\widetilde{u} = u := S_{D^2,\Gamma} w$.

We claim that $u_\varepsilon \xrightarrow{E} u$ as $\varepsilon \to 0$ along a sequence. Proceeding exactly as in the proof of Lemma 4.4 one can verify that (117), (118) and (120) still hold true here. It remains to prove (119) but here we cannot exploit the E-convergence of w_ε as we did in the proof of Lemma 4.4. Since $\alpha \ge 1$ we have that $\partial\Omega_\varepsilon$ is Lipschitz uniformly with respect to ε converging to zero. If we look at the proof of [31, Theorem 5.5, Chapter 2] with $k = 1$ and $p = 2$, we deduce the following uniform estimate

$$(121) \qquad\qquad \|v\|_{H^{1/2}(\partial\Omega_\varepsilon)} \le C\|v\|_{H^1(\Omega_\varepsilon)} \qquad \text{for any } v \in H^1(\Omega_\varepsilon)$$

where C is a positive constant independent of v and ε small. Introducing a ε-dependent family of extension linear operators from $H^1(\Omega_\varepsilon)$ to $H^1(W \times (-1, +\infty))$ whose operatorial norms remain bounded as $\varepsilon \to 0$ (we recall that $\alpha \ge 1$ in the definition of g_ε), we deduce that if $\{v_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is a family of functions whose $H^1(\Omega_\varepsilon)$-norms remain bounded as $\varepsilon \to 0$ then

$$(122) \qquad\qquad \int_{\partial\Omega_\varepsilon \setminus (\Gamma_\varepsilon \cup \partial\Omega)} v_\varepsilon^2 \, dS \to 0 \qquad \text{as } \varepsilon \to 0$$

since the surface measure of $\partial\Omega_\varepsilon \setminus (\Gamma_\varepsilon \cup \partial\Omega)$ converges to zero as $\varepsilon \to 0$.

Applying (122) to $\frac{\partial w_\varepsilon}{\partial x_i}$ and $\frac{\partial u_\varepsilon}{\partial x_i}$ and recalling that $\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)}, \|w_\varepsilon\|_{H^2(\Omega_\varepsilon)}$ remain bounded as $\varepsilon \to 0$, we infer that

$$(123) \qquad\qquad \int_{\partial\Omega_\varepsilon \setminus (\Gamma_\varepsilon \cup \partial\Omega)} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to 0 \qquad \text{as } \varepsilon \to 0.$$

On the other hand, by (121) applied to $\frac{\partial w_\varepsilon}{\partial x_i}$ and $\frac{\partial u_\varepsilon}{\partial x_i}$ combined with the compact embedding $H^{1/2}(\partial\Omega \setminus \Gamma) \subset L^2(\partial\Omega \setminus \Gamma)$, we have that

$$(124) \qquad\qquad \int_{\partial\Omega \setminus \Gamma} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to \int_{\partial\Omega \setminus \Gamma} w_\nu u_\nu \, dS$$

along a sequence converging to zero.

It remains to consider $\int_{\Gamma_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS$. By Lemma 4.3, the fact that $\alpha \geq 1$ and $b \in C^{1,1}(\mathbb{R}^{N-1})$, we obtain for ε small enough

(125)

$$\left| \int_{\Gamma_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon) \, dS \right| \leq \left(\int_{\Gamma_\varepsilon} (w_\varepsilon)_\nu^2 \, dS \right)^{1/2} \left(\int_{\Gamma_\varepsilon} (u_\varepsilon)_\nu^2 \, dS \right)^{1/2}$$

$$\leq (\delta_1^\varepsilon)^{-1/2} \|w_\varepsilon\|_{V(\Omega_\varepsilon)} \left(\int_{\Gamma_\varepsilon} (u_\varepsilon)_\nu^2 \, dS \right)^{1/2} \leq C \left(\sum_{i=1}^N \int_{\Gamma_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^2 dS \right)^{1/2}$$

$$= C \left(\sum_{i=1}^N \int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(x', g_\varepsilon(x')) \right|^2 \sqrt{1 + |\nabla_{x'} g_\varepsilon(x')|^2} \, dx' \right)^{1/2} \leq C \left(\sum_{i=1}^N \int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(x', g_\varepsilon(x')) \right|^2 dx' \right)^{1/2}$$

where C denotes a constant independent of ε which may vary in every estimate.

We claim that $\int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(x', g_\varepsilon(x')) \right|^2 dx' \to 0$ as $\varepsilon \to 0$. Indeed, we have

(126) $\quad \int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(x', g_\varepsilon(x')) \right|^2 dx' \leq 2 \int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(x', g_\varepsilon(x')) - \frac{\partial u_\varepsilon}{\partial x_i}(x', 0) \right|^2 dx' + 2 \int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(x', 0) \right|^2 dx' .$

The second term at the right hand side of (126) converges to zero as one can verify combining the fact that $(u_\varepsilon)_{|\Omega} \rightharpoonup u$ in $H^2(\Omega)$ and $u \in H^2_{0,\Gamma}(\Omega)$, with compactness of the trace map. With the first term at the right hand side of (126) we proceed in this way

(127) $\quad \int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(x', g_\varepsilon(x')) - \frac{\partial u_\varepsilon}{\partial x_i}(x', 0) \right|^2 dx' = \int_W \left| \int_0^{g_\varepsilon(x')} \frac{\partial^2 u_\varepsilon}{\partial x_N \partial x_i}(x', x_N) \, dx_N \right|^2 dx'$

$$\leq \int_W |g_\varepsilon(x')| \left| \int_0^{g_\varepsilon(x')} \left| \frac{\partial^2 u_\varepsilon}{\partial x_N \partial x_i}(x', x_N) \right|^2 dx_N \right| dx' \leq \varepsilon^\alpha \|b\|_{L^\infty(\mathbb{R}^{N-1})} \int_{\Omega_\varepsilon \setminus \Omega} \left| \frac{\partial^2 u_\varepsilon}{\partial x_N \partial x_i} \right|^2 dx$$

$$\leq \varepsilon^\alpha \|b\|_{L^\infty(\mathbb{R}^{N-1})} \|u_\varepsilon\|_{H^2(\Omega_\varepsilon)}^2 \to 0 \qquad \text{as } \varepsilon \to 0 .$$

The proof of the claim follows by combining (126) and (127). In turn, the claim combined with (125) implies

(128) $$\int_{\Gamma_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to 0 \qquad \text{as } \varepsilon \to 0 .$$

Now, combining (123), (124) and (128), we obtain

(129) $$\int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to \int_{\partial\Omega \setminus \Gamma} w_\nu u_\nu \, dS$$

as $\varepsilon \to 0$ along a sequence. Since u_ε satisfies (118) and $u \in H^2_{0,\Gamma}(\Omega)$ is a weak solution of (116), from (129) we obtain

(130) $$\int_{\Omega_\varepsilon} |D^2 u_\varepsilon|^2 dx = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu (u_\varepsilon)_\nu \, dS \to \int_{\partial\Omega \setminus \Gamma} w_\nu u_\nu \, dS = \int_\Omega |D^2 u|^2 dx$$

as $\varepsilon \to 0$ along a sequence.

Since (117) still holds true for u_ε, by (130), it follows that $u_\varepsilon \xrightarrow{E} u$ as $\varepsilon \to 0$ along a sequence. We have proved that $\{S_{D^2,\varepsilon} w_\varepsilon\}$ is precompact in the sense of Definition 2.3. The proof of the theorem is now a consequence of Lemma 4.4 and Definition 2.4.

4.2. **Case $\alpha = 3/2$: strange term.** Assume that Ω, Ω_ε, Γ and Γ_ε are as above. We set $\Sigma_\varepsilon = \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$ and $\Sigma = \partial\Omega \setminus \Gamma$. In this subsection we discuss the following modified Steklov problem:

(131)
$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega_\varepsilon\,, \\ u = 0, & \text{on } \partial\Omega_\varepsilon\,, \\ u_\nu = 0, & \text{on } \Sigma_\varepsilon\,, \\ \Delta u - K(x)u_\nu = \delta u_\nu, & \text{on } \Gamma_\varepsilon\,. \end{cases}$$

The third condition $u_\nu = 0$ on Σ_ε is used here to simplify our arguments as well as to avoid a few annoying technicalities which seem not particularly interesting for the specific case under discussion.

The eigenvalues of problem (131) are the reciprocals of the eigenvalues of a Navier-to-Neumann map N_{Ω_ε} that we now define. Namely, given $f \in L^2(\Gamma_\varepsilon)$ consider the solution $u_f = u$ to the following modified Navier problem

(132)
$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega_\varepsilon\,, \\ u = 0, & \text{on } \partial\Omega_\varepsilon\,, \\ u_\nu = 0, & \text{on } \Sigma_\varepsilon\,, \\ \Delta u - K(x)u_\nu = f, & \text{on } \Gamma_\varepsilon\,, \end{cases}$$

which means that $u \in H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon)$ is the solution to the weak problem

(133)
$$\int_{\Omega_\varepsilon} D^2 u : D^2 \varphi\, dx = \int_{\Gamma_\varepsilon} f \frac{\partial\varphi}{\partial\nu}\, dS\,, \quad \forall\, \varphi \in H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon).$$

Here $H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$ is defined as in (112) with Ω replaced by Ω_ε and Γ replaced by Σ_ε. Then we consider the function N_{Ω_ε} from $L^2(\Gamma_\varepsilon)$ to itself defined by $N_{\Omega_\varepsilon} f = \frac{\partial u_f}{\partial\nu}$ for all $f \in L^2(\Gamma_\varepsilon)$ where u_f is the solution to (133). Here we analyse the behaviour of the map N_{Ω_ε} as $\varepsilon \to 0$ in the case $\alpha = 3/2$.

The limiting problem involves a strange factor γ, which is called 'strange curvature' in [4]. We recall that γ is defined by

(134)
$$\gamma = \int_{Y \times (-\infty, 0)} |D^2 V|^2 dy,$$

and the function V is Y-periodic in the variables y' and satisfies the following microscopic problem

(135)
$$\begin{cases} \Delta^2 V = 0, & \text{in } Y \times (-\infty, 0), \\ V(y', 0) = b(y'), & \text{on } Y, \\ \frac{\partial^2 V}{\partial y_N^2}(y', 0) = 0, & \text{on } Y. \end{cases}$$

It turns out that the limiting functional for the maps N_{Ω_ε} as $\varepsilon \to 0$ is the map S_Ω defined as follows. Given $f \in L^2(\Gamma)$, consider the solution $\tilde{u}_f = u$ to the problem

(136)
$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \\ u_\nu = 0, & \text{on } \Sigma\,, \\ \Delta u - K(x)u_\nu + \gamma u_\nu = f, & \text{on } \Gamma\,. \end{cases}$$

Then S_Ω is the map from $L^2(\Gamma)$ to itself defined by $S_\Omega f = \frac{\partial \tilde{u}_f}{\partial\nu}$ for all $f \in L^2(\Gamma)$. Since the trace operator is compact we have that both operators N_{Ω_ε} and S_Ω are compact.

We now consider a family of operators E_ε from $L^2(\Gamma)$ to $L^2(\Gamma_\varepsilon)$ defined by $E_\varepsilon(f)(x', g_\varepsilon(x')) = f(x', 0)$ for all $x' \in W$ and all $f \in L^2(\Gamma)$. It is obvious that if $\alpha = 3/2$ then g_ε converges to zero uniformly together with its first derivatives hence

(137)
$$\|E_\varepsilon f\|_{L^2(\Gamma_\varepsilon)} \to \|f\|_{L^2(\Gamma)}$$

as $\varepsilon \to 0$, which means that condition (11) is satisfied. With reference to these operators E_ε, the following theorem holds.

Theorem 4.5. *If $\alpha = 3/2$ then $N_{\Omega_\varepsilon} \xrightarrow{C} S_\Omega$ as $\varepsilon \to 0$, hence the spectrum of problem* (131) *converges to the spectrum of* (136) *in the sense of Theorem 2.5.*

Proof. Let $f_\varepsilon \in L^2(\Gamma_\varepsilon)$ be E-convergent to $f \in L^2(\Gamma)$ as $\varepsilon \to 0$ in the sense of Definition 2.1. Let u_ε be the solution to problem (133) with f replaced by f_ε. Using u_ε as a test function in (133), we get that $\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)}$ is uniformly bounded, hence there exists $u \in H^2_{0,\Sigma}(\Omega) \cap H^1_0(\Omega)$ such that possibly passing to a subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $H^2(\Omega)$ and strongly in $H^1(\Omega)$. We now prove that u satisfies problem (136).

To do so, following [4, § 8.1] we define a diffeomorphism Φ_ε from $\overline{\Omega}_\varepsilon$ onto $\overline{\Omega}$. Namely, $\Phi_\varepsilon(x', x_N) = (x', x_N - h_\varepsilon(x', x_N))$ for all $(x', x_N) \in \overline{\Omega}_\varepsilon$ where

$$
(138) \qquad h_\varepsilon(x', x_N) = \begin{cases} 0, & \text{if } -1 \le x_N \le -\varepsilon \\[2mm] g_\varepsilon(x') \left(\frac{x_N+\varepsilon}{g_\varepsilon(x')+\varepsilon} \right)^3, & \text{if } -\varepsilon < x_N \le g_\varepsilon(x'). \end{cases}
$$

It is convenient to denote by K_ε the set where Φ_ε coincides with the identity, that is $K_\varepsilon = \{(x', x_N) \in W \times \mathbb{R} : -1 < x_N \le -\varepsilon\}$.

Let $\varphi \in H^2_{0,\Sigma}(\Omega) \cap H^1_0(\Omega)$ be fixed. We set $\varphi_\varepsilon = \varphi \circ \Phi_\varepsilon$ and we note that $\varphi_\varepsilon \in H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon)$. Using φ_ε as a test function in (133) we get

$$
(139) \qquad \int_{K_\varepsilon} D^2 u_\varepsilon : D^2 \varphi \, dx + \int_{\Omega_\varepsilon \setminus K_\varepsilon} D^2 u_\varepsilon : D^2 \varphi_\varepsilon \, dx = \int_{\Omega_\varepsilon} D^2 u_\varepsilon : D^2 \varphi_\varepsilon \, dx = \int_{\Gamma_\varepsilon} f_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial \nu} \, dS \, .
$$

It is proved in [4, Theorem 8.72] that

$$
(140) \qquad \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon \setminus K_\varepsilon} D^2 u_\varepsilon : D^2 \varphi_\varepsilon \, dx = \gamma \int_W \frac{\partial u}{\partial x_N}(x', 0) \frac{\partial \varphi}{\partial x_N}(x', 0) \, dx' \, ,
$$

where γ is as in the statement. Note that a careful inspection of the arguments in [4, § 8] shows that the definition of γ is not affected by our Dirichlet boundary conditions on Σ_ε. Thus

$$
(141) \qquad \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} D^2 u_\varepsilon : D^2 \varphi_\varepsilon \, dx = \int_\Omega D^2 u : D^2 \varphi \, dx + \gamma \int_W \frac{\partial u}{\partial x_N}(x', 0) \frac{\partial \varphi}{\partial x_N}(x', 0) \, dx' \, .
$$

We now consider the integral in the right-hand side of (139). We have

(142)
$$
\left| \int_{\Gamma_\varepsilon} f_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial \nu} \, dS - \int_\Gamma f \frac{\partial \varphi}{\partial \nu} \, dS \right| \le \int_{\Gamma_\varepsilon} |f_\varepsilon - E_\varepsilon f| \left| \frac{\partial \varphi_\varepsilon}{\partial \nu} \right| dS + \left| \int_{\Gamma_\varepsilon} E_\varepsilon f \frac{\partial \varphi_\varepsilon}{\partial \nu} \, dS - \int_\Gamma f \frac{\partial \varphi}{\partial \nu} \, dS \right|
$$

$$
\le C \|f_\varepsilon - E_\varepsilon f\|_{L^2(\Gamma_\varepsilon)} + \left| \int_{\Gamma_\varepsilon} E_\varepsilon f \frac{\partial \varphi_\varepsilon}{\partial \nu} \, dS - \int_\Gamma f \frac{\partial \varphi}{\partial \nu} \, dS \right|
$$

$$
= o(1) + \left| \int_W f(x', 0) \frac{\partial \varphi_\varepsilon}{\partial \nu}(x', g_\varepsilon(x')) \sqrt{1 + |\nabla_{x'} g_\varepsilon(x')|^2} \, dx' - \int_W f(x', 0) \frac{\partial \varphi}{\partial \nu}(x', 0) \, dx' \right| = o(1),
$$

where $C > 0$ is independent of ε. It follows by (141)-(142) that function u satisfies the problem

$$
(143) \qquad \int_\Omega D^2 u : D^2 \varphi \, dx + \gamma \int_W \frac{\partial u}{\partial x_N}(x', 0) \frac{\partial \varphi}{\partial x_N}(x', 0) \, dx' = \int_\Gamma f \frac{\partial \varphi}{\partial \nu} \, dS \, .
$$

We have now to prove that $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}$ is E-convergent to $\frac{\partial u}{\partial \nu}$. We note that

$$\int_{\partial\Omega_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial \nu} - E_\varepsilon \frac{\partial u}{\partial \nu} \right|^2 dS = \int_W \left| \frac{\partial u_\varepsilon}{\partial \nu}(x', g_\varepsilon(x')) - \frac{\partial u}{\partial \nu}(x', 0) \right|^2 \sqrt{1 + |\nabla g_\varepsilon(x')|^2} dx'$$

$$(144) \hspace{3cm} = \int_W \left| \frac{\partial u_\varepsilon}{\partial \nu}(x', g_\varepsilon(x')) - \frac{\partial u}{\partial \nu}(x', 0) \right|^2 dx' + o(1) \, .$$

By the Fundamental Theorem of Calculus we have that

$$(145) \hspace{2cm} \| \nabla u_\varepsilon(x', g_\varepsilon(x')) - \nabla u_\varepsilon(x', 0) \|_{L^2(W)} \le c\varepsilon^{\alpha/2} \| u_\varepsilon \|_{H^2(\Omega_\varepsilon \setminus \Omega)} = O(\varepsilon^{\alpha/2})$$

as $\varepsilon \to 0$. On the other hand, by the compactness of the trace operator we have that possibly passing to a subsequence

$$(146) \hspace{2cm} \| \nabla u_\varepsilon(x', 0) - \nabla u(x', 0) \|_{L^2(W)} \to 0$$

$\varepsilon \to 0$. By combining (144)-(146) we conclude that $\int_{\partial\Omega_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial \nu} - E\frac{\partial u}{\partial \nu} \right|^2 dS \to 0$ as $\varepsilon \to 0$ hence $\frac{\partial u_\varepsilon}{\partial \nu}$ is E-convergent to $\frac{\partial u}{\partial \nu}$.

In order to conclude the proof, it suffices to observe that if f_ε is a sequence with $\|f_\varepsilon\|_{L^2(\Gamma_\varepsilon)} = 1$ for all $\varepsilon > 0$, then the corresponding solutions u_ε have uniformly bounded H^2 norms, hence exploiting the compactness of the trace operator as above one can easily prove that $\frac{\partial u_\varepsilon}{\partial \nu}$ has an E-convergent subsequence. $\square$

4.3. **Another way of viewing the trichotomy.**

In this subsection we consider the eigenvalue problem (131) and we re-interpret the results of this section adapting the analysis of Subsection 4.1 to this specific case. This allows to state a trichotomy result for the eigenvalues which we believe is quite transparent. Namely, we have the following theorem.

Theorem 4.6. *Let Ω_ε, $\varepsilon \ge 0$ be as above, with $\Omega_0 = \Omega$. Let $\lambda_n(\varepsilon)$, $n \in \mathbb{N}$, be the eigenvalues of problems (131) for $\varepsilon \ge 0$. Then the following statements hold:*

  (i) *If $\alpha > 3/2$ then $\lambda_n(\varepsilon) \to \lambda_n(0)$, as $\varepsilon \to 0$.*
  (ii) *If $\alpha = 3/2$ then $\lambda_n(\varepsilon) \to \lambda_n(0) + \gamma$ as $\varepsilon \to 0$, where γ is as in (134).*
  (iii) *If $1 \le \alpha < 3/2$ then $\lambda_n(\varepsilon) \to \infty$, as $\varepsilon \to 0$.*

Proof. The proof of (i) is exactly the same proof of Corollary 4.1 with obvious minor modifications. Statement (ii) is a straightforward application of Theorem 4.5. We now prove statement (iii). We proceed as in Subsection 4.1 replacing the energy spaces $H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ and $H_{0,\Gamma}^2(\Omega)$ by $H_{0,\Sigma_\varepsilon}^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ and $H_0^2(\Omega)$ respectively. Accordingly, the operators $S_{D^2,\varepsilon}$ and $S_{D^2,\Gamma}$ used in Subsection 4.1 have to be re-defined in an obvious way (formally, it is the same way) taking into account the new energy spaces. Let's call $\tilde{S}_{D^2,\varepsilon}$ and $\tilde{S}_{D^2,\Gamma}$ the new operators replacing $S_{D^2,\varepsilon}$ and $S_{D^2,\Gamma}$, respectively. Having a closer look at the definition of $\tilde{S}_{D^2,\Gamma}$, one realises that actually $\tilde{S}_{D^2,\Gamma}$ is the operator identically equal to zero: indeed, the map J_Γ used in the definition of $S_{D^2,\Gamma}$ vanishes on $H_0^2(\Omega)$. Thus, by the same argument used in the proof of Theorem 4.1 we have that $\tilde{S}_{D^2,\varepsilon} \xrightarrow{C} 0$ as $\varepsilon \to 0$. Assume now by contradiction that there exists a sequence $\varepsilon_k \to 0$ such that for some $n \in \mathbb{N}$ we have $\sup_{k \in \mathbb{N}} \lambda_n(\varepsilon_k) < \infty$. Using the the Poincaré inequality, one can see that $\inf_{k \in \mathbb{N}} \lambda_n(\varepsilon_k) > 0$, see also Lemma 4.3. Now, we set $\mu_n(\varepsilon_k) = 1/\lambda_n(\varepsilon_k)$ and we observe that the sequence $\mu_n(\varepsilon_k)$, $k \in \mathbb{N}$ is bounded away from zero and infinity. Thus, possibly passing to a subsequence, there exists $\mu > 0$ such that $\mu_n(\varepsilon_k) \to \mu$ as $k \to \infty$. Since $\mu_n(\varepsilon_k)$ is an eigenvalue of $\tilde{S}_{D^2,\varepsilon_k}$ and $\tilde{S}_{D^2,\varepsilon_k} \xrightarrow{C} 0$, by Vainikko [38, Theorem 6.1] it follows that μ is an eigenvalue of the zero operator, a contradiction. $\square$

5. NAVIER-TYPE PROBLEMS AND NAVIER-TO-NEUMANN MAPS

The main aim of this section is to study the stability of the inverse of a Dirichlet-to-Neumann-type map associated with problem (2) under weak assumptions on the convergence of the varying domains. As we

have mentioned in the introduction, this map is called here Navier-to-Neumann map. To do so, we begin with discussing the stability of the Navier problem under domain perturbations of general type.

5.1. **Navier problems.** Given a sufficiently smooth bounded domain Ω in $\mathbb{R}^N$, the classical Navier problem for the biharmonic operator reads

$$(147) \qquad \begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = f, & \text{on } \partial\Omega. \end{cases}$$

In a classical setting, it would be natural to require at least that Ω is of class $C^{0,1}$ and that $f \in H^{1/2}(\partial\Omega)$. Actually, since the trace operator is surjective, one could directly assume that $f \in H^1(\Omega)$.

As it is well-known, problem (147) could be recast in the form of a system by setting $v := \Delta u$ and solving

$$(148) \qquad \begin{cases} \Delta v = 0, & \text{in } \Omega \\ v = f, & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta u = v, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Accordingly, in order to prove that problem (147) is stable under domain perturbations, one may think of exploiting the properties of both Poisson and Dirichlet problems in (148) which are stable under sufficiently regular boundary perturbations. However, since we aim at considering a rather general class of domain perturbations as well as at comparing problem (147) with problem (184), we prefer to adopt another point of view which has its own interest, and which is based on the energy quadratic form naturally associated with (147). This point of view allows to relax the boundary regularity assumptions on Ω even more.

Namely, we interpret problem (147) in a weak sense by formulating it as follows

$$(149) \qquad \int_\Omega \Delta u \Delta \varphi \, dx = \int_\Omega (f \Delta \varphi + \nabla f \nabla \varphi) \, dx, \quad \text{for all } \varphi \in H(\Delta, \Omega) \cap H_0^1(\Omega),$$

where the unknown u has to be considered in $H(\Delta, \Omega) \cap H_0^1(\Omega)$. Note that our weak formulation is motivated by the fact that if Ω is regular enough, for example as in Lemma 2.7 (iii), then $H(\Delta, \Omega) \cap H_0^1(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$ hence equality (149) could be re-written as

$$(150) \qquad \int_\Omega \Delta u \Delta \varphi \, dx = \int_{\partial\Omega} f \varphi_\nu \, dS \,,$$

which is a more familiar way of writing the variational formulation for (147). In fact the right-hand side of (149) is the standard way of defining the normal derivative for a function in $H(\Delta, \Omega)$ as an element of $H^{-1/2}(\partial\Omega)$, see (173) below, see also e.g., [26].

The following lemma shows that problem (149) is well-posed even without any boundary regularity assumption on Ω.

Lemma 5.1. *Let Ω be a domain in $\mathbb{R}^N$ such that the Poincaré inequality holds and let $f \in H^1(\Omega)$. Then problem (149) is well-posed in the sense of Hadamard, that is there exists a unique solution u in $H(\Delta, \Omega) \cap H_0^1(\Omega)$ which depends continuously on f.*

Proof. We simply observe that by Lemma 2.7, the map $\varphi \mapsto \int_\Omega (f \Delta \varphi + \nabla f \nabla \varphi) \, dx$ is an element of the dual of $H(\Delta, \Omega) \cap H_0^1(\Omega)$ which depends continuously on $f \in H^1(\Omega)$. Then the proof easily follows by the Riesz Theorem applied to the Hilbert space $H(\Delta, \Omega) \cap H_0^1(\Omega)$. $\qquad\square$

We now consider a family of bounded domains Ω_ε, $\varepsilon > 0$ and the corresponding boundary value problems

$$(151) \qquad \begin{cases} \Delta^2 u_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega_\varepsilon, \\ \Delta u_\varepsilon = f_\varepsilon, & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

in the unknown $u_\varepsilon \in H(\Delta, \Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$, where $f_\varepsilon \in H^1(\Omega_\varepsilon)$, with the understanding that problem (151) has to be interpreted in the weak sense as above.

We shall assume that the domains Ω_ε compactly converge to Ω and that Ω is regular (stable) in the sense of the following definition which goes back to Keldysh (cf. [12, § 4.9]).

Definition 5.2. *We say that the domains Ω_ε, $\varepsilon > 0$, compactly converge to a domain Ω as $\varepsilon \to 0$ if for any compact set $K \subset \Omega \cup (\mathbb{R}^N \setminus \overline{\Omega})$ there exists $\varepsilon_K > 0$ such that $K \subset \Omega_\varepsilon \cup (\mathbb{R}^N \setminus \overline{\Omega}_\varepsilon)$ for all $\varepsilon \in]0, \varepsilon_K[$. Moreover, we say that Ω is stable if any function in $H^1(\mathbb{R}^N)$ vanishing almost everywhere on $\mathbb{R}^N \setminus \overline{\Omega}$ belongs to $H_0^1(\Omega)$.*

It is well-known that if Ω is of class $C^{0,1}$ then Ω is stable. We refer to [12] for more information. Then we have the following stability result. Here we denote by v_0 the extension-by-zero of a function v outside its natural domain of definition.

Theorem 5.3. *Let D be a fixed bounded set and let Ω_ε, $\varepsilon > 0$, and Ω be domains contained in D. Assume that the sets Ω_ε compactly converge to Ω as $\varepsilon \to 0$ and that Ω is stable. If the data f_ε converge to f in the sense that*

$$\text{(152)} \qquad \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)}, \;\; \|\nabla f_\varepsilon - (\nabla f)_0\|_{L^2(\Omega_\varepsilon)} \to 0, \;\; \text{as } \varepsilon \to 0$$

then the solutions u_ε of problems (151) converge to the solution u of problem (147) in the sense that

$$\text{(153)} \qquad \|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon)}, \;\; \|\nabla u_\varepsilon - (\nabla u)_0\|_{L^2(\Omega_\varepsilon)}, \;\; \|\Delta u_\varepsilon - (\Delta u)_0\|_{L^2(\Omega_\varepsilon)} \to 0, \;\; \text{as } \varepsilon \to 0.$$

Proof. By assumption, it follows that

$$\text{(154)} \qquad \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \;\; \|\nabla f_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C,$$

for all $\varepsilon > 0$ sufficiently small. Using u_ε as a test function in (149) we have

$$\text{(155)} \qquad \int_{\Omega_\varepsilon} |\Delta u_\varepsilon|^2 dx = \int_{\Omega_\varepsilon} (f_\varepsilon \Delta u_\varepsilon + \nabla f_\varepsilon \nabla u_\varepsilon) \, dx,$$

which, combined with (14) and (154), allows to conclude that

$$\text{(156)} \qquad \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \|\Delta u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C,$$

for all $\varepsilon > 0$ sufficiently small. Since D is bounded, by the Rellich-Kondrakov Theorem, $H_0^1(D)$ is compactly embedded in $L^2(D)$. This combined with the fact that $(u_\varepsilon)_0 \in H_0^1(D)$ allows to conclude that there exists $u \in H_0^1(D)$ such that possibly passing to a subsequence

$$\text{(157)} \qquad (u_\varepsilon)_0 \to u, \;\; \text{in } L^2(D), \;\; \text{and } (u_\varepsilon)_0 \rightharpoonup u, \;\; \text{in } H^1(D)$$

as $\varepsilon \to 0$. By the compact convergence of the domains Ω_ε to the set Ω it follows that $u = 0$ almost everywhere on $D \setminus \overline{\Omega}$. Since Ω is stable, this implies that $u \in H_0^1(\Omega)$ and that $u = 0$ almost everywhere on $D \setminus \Omega$, see [12, p. 117]. By (156) it follows that possibly passing to a subsequence there exists $v \in L^2(D)$ such that $(\Delta u_\varepsilon)_0 \rightharpoonup v$ in $L^2(D)$. Let $\psi \in C_c^\infty(\Omega)$ be fixed. By the compact convergence of the domains Ω_ε to the set Ω it follows that for all $\varepsilon > 0$ sufficiently small, $\psi \in C_c^\infty(\Omega_\varepsilon)$ hence $\int_{\Omega_\varepsilon} u_\varepsilon \Delta \psi \, dx = \int_{\Omega_\varepsilon} \Delta u_\varepsilon \psi \, dx$ and $\int_\Omega u_\varepsilon \Delta \psi \, dx = \int_\Omega \Delta u_\varepsilon \psi \, dx$. Passing to the limit in the last equality as $\varepsilon \to 0$, it follows that $\int_\Omega u \Delta \psi dx = \int_\Omega v \psi dx$ which, by the arbitrary choice of ψ, implies that $u \in H(\Delta, \Omega) \cap H_0^1(\Omega)$ and $\Delta u = v$. In particular

$$\text{(158)} \qquad ((\Delta u_\varepsilon)_0)_{|\Omega} \rightharpoonup \Delta u, \;\; \text{in } L^2(\Omega),$$

as $\varepsilon \to 0$.

Let $\varphi \in H(\Delta, \Omega) \cap H_0^1(\Omega)$ be fixed. For every $\varepsilon > 0$ we consider the function $\varphi_\varepsilon \in H(\Delta, \Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ uniquely defined as the solution to the problem

$$\text{(159)} \qquad \begin{cases} \Delta \varphi_\varepsilon = (\Delta \varphi)_0, & \text{in } \Omega_\varepsilon, \\ \varphi_\varepsilon = 0, & \text{on } \partial \Omega_\varepsilon. \end{cases}$$

This means that φ_ε satisfies the weak equation

$$(160) \qquad \int_{\Omega_\varepsilon} \nabla\varphi_\varepsilon \nabla\eta \, dx = \int_{\Omega_\varepsilon} (\Delta\varphi)_0 \eta \, dx, \quad \text{for all } \eta \in H_0^1(\Omega_\varepsilon).$$

We claim that

$$(161) \qquad ((\nabla\varphi_\varepsilon)_0)_{|\Omega} \rightharpoonup \nabla\varphi, \quad \text{in } L^2(\Omega),$$

as $\varepsilon \to 0$. Indeed, using φ_ε as a test function in (160) and the Poincaré inequality we obtain

$$(162) \qquad \|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \le \|\Delta\varphi\|_{L^2(\Omega)}\|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le c_N |\Omega_\varepsilon|^{\frac{1}{N}} \|\Delta\varphi\|_{L^2(\Omega)}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$

by which we deduce that $\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is uniformly bounded for $\varepsilon > 0$ sufficiently small. Thus, possibly passing to a subsequence, it turns out that $(\varphi_\varepsilon)_0$ is strongly convergent in $L^2(D)$ and weakly in $H^1(D)$ to a function ψ. As above, the compact convergence of Ω_ε implies that $\psi = 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$. Thus, since Ω is stable, $\psi \in H_0^1(\Omega)$. Fixing now $\eta \in C_c^\infty(\Omega)$ and observing that $\eta \in C_c^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$ sufficiently small, we can test η in (160) and pass to the limit to conclude that $\psi = \varphi$ in Ω. Thus (161) holds.

Before proceeding with using function φ_ε, we observe that by assumptions (152) we have that

$$(163) \qquad \int_{\Omega_\varepsilon \setminus \Omega} |f_\varepsilon|^2 dx, \ \int_{\Omega_\varepsilon \setminus \Omega} |\nabla f_\varepsilon|^2 dx \to 0$$

as $\varepsilon \to 0$, and by the compact convergence of domains we have

$$(164) \qquad |\Omega \setminus \Omega_\varepsilon| \to 0, \qquad \text{as } \varepsilon \to 0.$$

Testing φ_ε in the weak formulation of problem (151) we have

$$\int_\Omega (\Delta u_\varepsilon)_0 \Delta\varphi \, dx = \int_{\Omega \cap \Omega_\varepsilon} \Delta u_\varepsilon \Delta\varphi \, dx = \int_{\Omega \cap \Omega_\varepsilon} \Delta u_\varepsilon (\Delta\varphi)_0 \, dx$$

$$= \int_{\Omega_\varepsilon} \Delta u_\varepsilon \Delta\varphi_\varepsilon dx = \int_{\Omega_\varepsilon} (f_\varepsilon \Delta\varphi_\varepsilon + \nabla f_\varepsilon \nabla\varphi_\varepsilon) \, dx$$

$$= \int_{\Omega_\varepsilon} (f_\varepsilon (\Delta\varphi)_0 + \nabla f_\varepsilon \nabla\varphi_\varepsilon) \, dx = \int_{\Omega \cap \Omega_\varepsilon} (f_\varepsilon \Delta\varphi + \nabla f_\varepsilon \nabla\varphi_\varepsilon) \, dx$$

$$(165) \qquad + \int_{\Omega_\varepsilon \setminus \Omega} (\nabla f_\varepsilon \nabla\varphi_\varepsilon) \, dx.$$

Now we note that by the triangle inequality

$$\left| \int_{\Omega_\varepsilon \cap \Omega} \nabla f_\varepsilon \nabla\varphi_\varepsilon \, dx - \int_{\Omega_\varepsilon \cap \Omega} \nabla f \nabla\varphi \, dx \right|$$

$$(166) \qquad \le C\|\nabla f_\varepsilon - \nabla f\|_{L^2(\Omega_\varepsilon \cap \Omega)} + \left| \int_{\Omega_\varepsilon \cap \Omega} \nabla f (\nabla\varphi_\varepsilon - \nabla\varphi) \, dx \right|.$$

By combining (152), (161), (163) and (166), it follows that

$$(167) \qquad \int_{\Omega_\varepsilon \cap \Omega} \nabla f_\varepsilon \nabla\varphi_\varepsilon \, dx = \int_{\Omega_\varepsilon \cap \Omega} \nabla f \nabla\varphi \, dx + o(1), \quad \int_{\Omega \cap \Omega_\varepsilon} f_\varepsilon \Delta\varphi \, dx = \int_{\Omega \cap \Omega_\varepsilon} f \Delta\varphi \, dx + o(1),$$

and $\int_{\Omega_\varepsilon \setminus \Omega} \nabla f_\varepsilon \nabla\varphi_\varepsilon \, dx \to 0$, as $\varepsilon \to 0$. Thus, (165) can be re-written as follows

$$(168) \qquad \int_\Omega (\Delta u_\varepsilon)_0 \Delta\varphi dx = \int_{\Omega \cap \Omega_\varepsilon} (f\Delta\varphi + \nabla f \nabla\varphi) \, dx + o(1).$$

By passing to the limit in (168) and using (158), (164) we conclude that u is the solution to problem (147) as required in the statement.

Now, the first limit in (153) follows by the first limit in (157) and by observing that u vanishes almost everywhere outside Ω.

We now prove that the third limit in (153) holds. To do so, we consider the identity

$$
\begin{aligned}
\|\Delta u_\varepsilon - (\Delta u)_0\|^2_{L^2(\Omega_\varepsilon)} &= \|\Delta u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - 2(\Delta u_\varepsilon, \Delta u)_{L^2(\Omega_\varepsilon \cap \Omega)} + \|\Delta u\|^2_{L^2(\Omega_\varepsilon \cap \Omega)} \\
&= \|\Delta u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - 2(\Delta u_\varepsilon, \Delta u)_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} + o(1),
\end{aligned}
\tag{169}
$$

where the second equality in (169) is deduced by (164).

We note that arguing as above we can prove that

$$
\int_{\Omega_\varepsilon} (f_\varepsilon \Delta u_\varepsilon + \nabla f_\varepsilon \nabla u_\varepsilon)\, dx \to \int_\Omega (f \Delta u + \nabla f \nabla u)\, dx,
\tag{170}
$$

as $\varepsilon \to 0$. By passing to the limit in (155) as $\varepsilon \to 0$, and by (170), we deduce that

$$
\|\Delta u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \to \|\Delta u\|_{L^2(\Omega)},
\tag{171}
$$

as $\varepsilon \to 0$. Finally, using (171) and passing to the limit in (169) allows to conclude that the third limit in (153) holds.

It remains to prove that the second limit in (153) holds. To do so, it is convenient to set $F_\varepsilon := -\Delta u_\varepsilon$ and $F := -\Delta u$. Thus, u_ε and u are solutions to two Poisson problems for the Dirichlet Laplacian in Ω_ε and Ω with data F_ε and F respectively, and $\|F_\varepsilon - F_0\|_{L^2(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$. Proceeding as above, using the identity

$$
\|\nabla u_\varepsilon - (\nabla u)_0\|^2_{L^2(\Omega_\varepsilon)} = \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - 2(\nabla u_\varepsilon, \nabla u)_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + o(1)
\tag{172}
$$

and exploiting the weak formulations of the Poisson problems for the Dirichlet Laplacian and the weak convergence of the gradients of ∇u_ε, we conclude that the second limit in (153) holds. □

5.2. Stability of the Navier-to-Neumann map.
It is well-known that if Ω is a bounded domain of class $C^{0,1}$, given a function $u \in H(\Delta, \Omega) \cap H^1_0(\Omega)$, it is possible to define the normal derivative u_ν of u as an element of $H^{-1/2}(\partial\Omega)$ by setting

$$
{}_{H^{-1/2}(\partial\Omega)} \langle u_\nu, \varphi \rangle_{H^{1/2}(\partial\Omega)} = \int_\Omega (\Delta u \varphi + \nabla u \nabla \varphi)\, dx
\tag{173}
$$

for all $\varphi \in H^{1/2}(\partial\Omega)$, where it is meant that φ is extended to the whole of Ω as an element of $H^1(\Omega)$. Recall that the trace operator acts from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$ and note that definition (173) does not depend on the specific extension of φ since the above integral vanishes for functions φ belonging to $H^1_0(\Omega)$.

Since we aim at studying a domain perturbation problem, we find it convenient to consider u_ν as an element of the dual of $H^1(\Omega)$ defined by formula (173) for all $\varphi \in H^1(\Omega)$.

We call *Navier-to-Neumann map* the function $\mathcal{N}_\Omega$ from $H^1(\Omega)$ to $(H^1(\Omega))'$ defined by $\mathcal{N}_\Omega(f) = u_\nu$ where u is the solution to (147).

We note that if Ω is sufficiently smooth, say Ω is a domain of class $C^{1,1}$, d is an eigenvalue of (2) with eigenfunction $u \in V(\Omega)$ if and only

$$
\mathcal{N}_\Omega f = \mu J_0 f
\tag{174}
$$

where $\mu = \frac{1}{d}$ and f is any function in $H^1(\Omega)$ such that $f_{|\partial\Omega} = d u_\nu \in H^{1/2}(\partial\Omega)$ and $J_0 : H^1(\Omega) \to (H^1(\Omega))'$ is defined by

$$
{}_{(H^1(\Omega))'} \langle J_0 f, v \rangle_{H^1(\Omega)} := \int_{\partial\Omega} f v\, dS \qquad \text{for any } f, v \in H^1(\Omega).
$$

Remark 6. *If one wishes to consider equation (174) as an eigenvalue problem, it would be clearly appropriate to formulate it on $H^1(\Omega)/H^1_0(\Omega)$ by considering $\mathcal{N}_\Omega$ and J_0 as functions from $H^1(\Omega)/H^1_0(\Omega)$ to its dual. Indeed, any function $f \in H^1_0(\Omega)$ satisfies (174) for any $\mu \in \mathbb{R}$. However, as we have mentioned in the introduction, using the quotient space would not bypass the main obstructions to the proof of the spectral stability of the classical DBS problem under weak conditions on the convergence of the domains. For this reason, we prefer to work directly on the space $H^1(\Omega)$ to avoid further technical complications.*

In Theorems 5.6, 5.7 below we establish convergence results for the family of maps $\mathcal{N}_{\Omega_\varepsilon}$ associated with a family of domains Ω_ε compactly convergent to Ω. For the sake of simplicity, in order to avoid imposing extra assumptions on Ω_ε or further technicalities in the proofs, we assume that $\Omega \subseteq \Omega_\varepsilon \subset D$ for any $\varepsilon > 0$, where D is a fixed bounded domain.

Since $\mathcal{N}_{\Omega_\varepsilon}$ takes values in the dual of $H^1(\Omega_\varepsilon)$, in order to establish the vicinity of $\mathcal{N}_{\Omega_\varepsilon}$ to $\mathcal{N}_\Omega$ we introduce a notion of E-convergence for families of functionals on H^1.

Definition 5.4. *Let* $\Omega, \Omega_\varepsilon$, *with* $\varepsilon > 0$, *be bounded domains in* $\mathbb{R}^N$ *with* Ω *of class* $C^{0,1}$ *and* $\Omega \subseteq \Omega_\varepsilon$ *for all* $\varepsilon > 0$. *Let* $\mathcal{E} : H^1(\Omega) \to H^1(\mathbb{R}^N)$ *be a fixed linear continuous extension operator.*

   (i) *Let* $\{f_\varepsilon\}_{\varepsilon>0}$ *be a family of functions* $f_\varepsilon \in H^1(\Omega_\varepsilon)$. *We say that* $f_\varepsilon \xrightarrow{E} f$ *with* $f \in H^1(\Omega)$ *if* $\|f_\varepsilon - \mathcal{E}f\|_{H^1(\Omega_\varepsilon)} \to 0$ *as* $\varepsilon \to 0$.

   (ii) *Let* $\{\Lambda_\varepsilon\}_{\varepsilon>0}$ *be a family of functionals* $\Lambda_\varepsilon \in (H^1(\Omega_\varepsilon))'$. *We say that* $\Lambda_\varepsilon \xrightarrow{E^*} \Lambda$ *with* $\Lambda \in (H^1(\Omega))'$ *if*

(175) $$\|\Lambda_\varepsilon - E_\varepsilon^*\Lambda\|_{(H^1(\Omega))'} \to 0 \qquad \textit{as } \varepsilon \to 0$$

   where $E_\varepsilon^* : (H^1(\Omega))' \to (H^1(\Omega_\varepsilon))'$ *is the family of linear continuous operators defined by*

   $$_{(H^1(\Omega_\varepsilon))'}\langle E_\varepsilon^*\Lambda, v\rangle_{H^1(\Omega_\varepsilon)} := {}_{(H^1(\Omega))'}\langle \Lambda, v_{|\Omega}\rangle_{H^1(\Omega)} \qquad \textit{for any } v \in H^1(\Omega_\varepsilon)$$

   for any $\Lambda \in (H^1(\Omega))'$.

   (iii) *Let* $\{B_\varepsilon\}_{\varepsilon>0}$ *be a family of linear continuous operators such that* $B_\varepsilon : H^1(\Omega_\varepsilon) \to (H^1(\Omega_\varepsilon))'$. *We say that* $B_\varepsilon \xrightarrow{EE^*} B$ *as* $\varepsilon \to 0$, *with* $B \in \mathcal{L}(H^1(\Omega); (H^1(\Omega))')$, *if* $B_\varepsilon f_\varepsilon \xrightarrow{E^*} Bf$ *whenever* $f_\varepsilon \xrightarrow{E} f$.

Remark 7. *The family of operators* $\{E_\varepsilon^*\}_{\varepsilon>0}$ *represents a "connecting system" in the sense of* [38, Section 1]. *In order to see this, one can first show that* $\|E_\varepsilon^*\Lambda\|_{(H^1(\Omega_\varepsilon))'} \leq \|\Lambda\|_{(H^1(\Omega))'}$ *and then, by using the extension operator* $\mathcal{E}$ *defined above and the fact that* $|\Omega_\varepsilon \setminus \Omega| \to 0$ *as* $\varepsilon \to 0$, *one shows that for any* $\sigma > 0$ *there exists* $\bar{\varepsilon} > 0$ *such that for any* $\varepsilon \in (0, \bar{\varepsilon})$ *one has* $\|\Lambda\|_{(H^1(\Omega))'} \leq (1 + \sigma)\|E_\varepsilon^*\Lambda\|_{(H^1(\Omega_\varepsilon))'} + \sigma$; *the conclusion follows by letting* $\varepsilon \to 0$ *and by exploiting the arbitrariness of* $\sigma > 0$.

As a consequence of Theorem 5.3 we have

Lemma 5.5. *Let* Ω_ε, Ω *and* D *be as in Theorem 5.3. Assume that* Ω *is of class* $C^{0,1}$, $\Omega \subseteq \Omega_\varepsilon$ *for all* $\varepsilon > 0$ *and* Ω_ε *compactly converges to* Ω *as* $\varepsilon \to 0$. *Let* $f_\varepsilon \in H^1(\Omega_\varepsilon)$ *for any* $\varepsilon > 0$ *and* $f \in H^1(\Omega)$. *Let* u_ε, u *be the solutions of problems* (151), (147) *respectively. If* $f_\varepsilon \xrightarrow{E} f$ *as* $\varepsilon \to 0$ *as above then* (153) *holds.*

Proof. Under the assumptions of the lemma we have that $|\Omega_\varepsilon \setminus \Omega| = |\Omega_\varepsilon \setminus \overline{\Omega}| \to 0$ as $\varepsilon \to 0$ hence (152) holds true and Theorem 5.3 applies. $\qquad\square$

Thanks to Lemma 5.5 we can prove the following

Theorem 5.6. *Let* Ω_ε, Ω *and* D *be as in Theorem 5.3. Assume that* Ω *is of class* $C^{0,1}$, $\Omega \subseteq \Omega_\varepsilon$ *for all* $\varepsilon > 0$ *and* Ω_ε *compactly converges to* Ω *as* $\varepsilon \to 0$. *Let* $\mathcal{N}_{\Omega_\varepsilon}$ *and* $\mathcal{N}_\Omega$ *be the Navier-to-Neumann maps corresponding to* Ω_ε *and* Ω *respectively. Then* $\mathcal{N}_{\Omega_\varepsilon} \xrightarrow{EE^*} \mathcal{N}_\Omega$ *as* $\varepsilon \to 0$.

Proof. We have to prove that

(176) $$\|\mathcal{N}_{\Omega_\varepsilon}f_\varepsilon - E_\varepsilon^*\mathcal{N}_\Omega f\|_{(H^1(\Omega_\varepsilon))'} \to 0 \qquad \text{as } \varepsilon \to 0$$

whenever $f_\varepsilon \xrightarrow{E} f$. To do this, we define u_ε and u respectively as the solutions of (151) and (147) and we observe that for any $v \in H^1(\Omega_\varepsilon)$ we have

$$_{(H^1(\Omega_\varepsilon))'}\langle \mathcal{N}_{\Omega_\varepsilon}f_\varepsilon - E_\varepsilon^*\mathcal{N}_\Omega f, v\rangle_{H^1(\Omega_\varepsilon)} = {}_{(H^1(\Omega_\varepsilon))'}\langle (u_\varepsilon)_\nu, v\rangle_{H^1(\Omega_\varepsilon)} - {}_{(H^1(\Omega))'}\langle u_\nu, v_{|\Omega}\rangle_{H^1(\Omega)}$$

$$= \int_{\Omega_\varepsilon} (\Delta u_\varepsilon\, v + \nabla u_\varepsilon \nabla v)\, dx - \int_\Omega (\Delta u\, v + \nabla u \nabla v)\, dx$$

$$= \int_\Omega [(\Delta u_\varepsilon - \Delta u)v + (\nabla u_\varepsilon - \nabla u)\nabla v]\, dx + \int_{\Omega_\varepsilon \setminus \Omega} (\Delta u_\varepsilon\, v + \nabla u_\varepsilon \nabla v)\, dx\,.$$

This implies

(177)
$$\left| {}_{(H^1(\Omega_\varepsilon))'}\langle \mathcal{N}_{\Omega_\varepsilon} f_\varepsilon - E_\varepsilon^* \mathcal{N}_\Omega f, v \rangle_{H^1(\Omega_\varepsilon)} \right|$$
$$\leq \left[\|\Delta u_\varepsilon - \Delta u\|_{L^2(\Omega)} + \|\nabla u_\varepsilon - \nabla u\|_{L^2(\Omega)} + \|\Delta u_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega)} \right] \|v\|_{H^1(\Omega_\varepsilon)}.$$

The proof of (176) then follows from Lemma 5.5. This completes the proof of the lemma. □

Finally we prove the compact convergence $\mathcal{N}_{\Omega_\varepsilon} \xrightarrow{C^*} \mathcal{N}_\Omega$, i.e. $\mathcal{N}_{\Omega_\varepsilon} \xrightarrow{EE^*} \mathcal{N}_\Omega$ and $\{\mathcal{N}_{\Omega_\varepsilon} f_\varepsilon\}_{\varepsilon>0}$ is precompact in the sense of Definition 2.3 with $\mathcal{H}_\varepsilon = (H^1(\Omega_\varepsilon))'$ and $\mathcal{H}_0 = (H^1(\Omega))'$ whenever $\|f_\varepsilon\|_{H^1(\Omega_\varepsilon)} = 1$. To do so, we shall require that the following condition (A) is satisfied.

(A): For any family $\{f_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ of functions $f_\varepsilon \in H^1(\Omega_\varepsilon)$ such that $\|f_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ is uniformly bounded in ε, we have that $\|f_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega)} \to 0$ as $\varepsilon \to 0$.

This condition has been extensively used by J.M. Arrieta and his co-authors in the analysis of the spectral stability of second order elliptic operators subject to Neumann boundary conditions on dumbbell domains, see [3] for references . We note that in our case, this condition will be used also to control the behaviour of the data f_ε of the problems.

Remark 8. *If $\Omega_\varepsilon, \Omega$ are of class $C^0(\mathcal{A})$ for all ε, where $\mathcal{A}$ is a fixed atlas, then the compact convergence of Ω_ε to Ω is equivalent to the uniform convergence of the functions describing the boundaries of Ω_ε and that using the boundedness of functions in Sobolev spaces along lines as in Lemma 3.1 we have that the compact convergence implies the validity of condition (A).*

Theorem 5.7. *Let Ω_ε, Ω and D be as in Theorem 5.3. Assume that Ω is of class $C^{0,1}$, $\Omega \subseteq \Omega_\varepsilon$ for all $\varepsilon > 0$, Ω_ε compactly converges to Ω as $\varepsilon \to 0$, and condition (A) is satisfied. Let $\mathcal{N}_{\Omega_\varepsilon}$ and $\mathcal{N}_\Omega$ be the Navier-to-Neumann maps corresponding to Ω_ε and Ω respectively. Then $\mathcal{N}_{\Omega_\varepsilon} \xrightarrow{C^*} \mathcal{N}_\Omega$ as $\varepsilon \to 0$.*

Proof. Let $f_\varepsilon \in H^1(\Omega_\varepsilon)$ be such that $\|f_\varepsilon\|_{H^1(\Omega_\varepsilon)} = 1$ for all $\varepsilon > 0$. Let $u_\varepsilon \in H(\Delta, \Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ be the solutions to problem (151). Proceeding as in the proof of Theorem 5.3 one can show that possibly passing to a subsequence (156), (157), (158) hold true for some $u \in H^1(D)$ such that $u_{|\Omega} \in H(\Delta, \Omega) \cap H_0^1(\Omega)$, u vanishing outside Ω. Writing

(178)
$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \int_{\Omega_\varepsilon} -u_\varepsilon \Delta u_\varepsilon \, dx = \int_\Omega -u_\varepsilon \Delta u_\varepsilon \, dx + \int_{\Omega_\varepsilon \setminus \Omega} -u_\varepsilon \Delta u_\varepsilon \, dx,$$

and passing to the limit as $\varepsilon \to 0$, we have by (156), (157), (158) and condition (A)

(179)
$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \lim_{\varepsilon \to 0} \int_\Omega -u_\varepsilon \Delta u_\varepsilon \, dx = \int_\Omega -u \Delta u \, dx = \int_\Omega |\nabla u|^2 dx.$$

Moreover,

(180)
$$\int_\Omega |\nabla u|^2 dx \leq \liminf_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^2 dx \leq \limsup_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^2 dx \leq \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \int_\Omega |\nabla u|^2 dx$$

hence

(181)
$$\int_\Omega |\nabla u_\varepsilon|^2 dx \to \int_\Omega |\nabla u|^2 dx, \quad \int_{\Omega_\varepsilon \setminus \Omega} |\nabla u_\varepsilon|^2 dx \to 0, \quad (u_\varepsilon)_{|\Omega} \to u \quad \text{in } H^1(\Omega),$$

as $\varepsilon \to 0$.

By our assumptions, in particular by condition (A), we have that possibly passing to a subsequence there exists $f \in H^1(\Omega)$ such that

(182)
$$f_\varepsilon \to f \quad \text{in } L^2(\Omega), \qquad f_\varepsilon \rightharpoonup f \quad \text{in } H^1(\Omega), \qquad \|f_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega)} \to 0$$

as $\varepsilon \to 0$. We claim that function u is a solution to problem (147) with this datum f. To prove this, one can use the same argument as in the proof of Theorem 5.3. Here we cannot use the strong convergence of the

gradients of f_ε, not even the second limit in (163). However, by looking closely at the proof of Theorem 5.3, one can see that here it suffices to use the fact that the family of functions φ_ε used in that proof satisfies the conditions $\varphi_\varepsilon \to \varphi$ in $H^1(\Omega)$ and $\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon\setminus\Omega)} \to 0$ as $\varepsilon \to 0$, which can be proved by proceeding exactly in the same way as in (178)-(181).

By (156), (158), (181), (182), we conclude that

$$\int_{\Omega_\varepsilon} |\Delta u_\varepsilon|^2 dx = \int_\Omega (f_\varepsilon \Delta u_\varepsilon + \nabla f_\varepsilon \nabla u_\varepsilon)\, dx + \int_{\Omega_\varepsilon\setminus\Omega} (f_\varepsilon \Delta u_\varepsilon + \nabla f_\varepsilon \nabla u_\varepsilon)\, dx$$

$$= \int_\Omega (f\Delta u + \nabla f\nabla u)\, dx + o(1) = \int_\Omega |\Delta u|^2 dx + o(1).$$

In particular, proceeding as we have done above for ∇u_ε, we have that

(183) $$(\Delta u_\varepsilon)_{|\Omega} \to \Delta u \quad \text{in } L^2(\Omega), \qquad \int_{\Omega_\varepsilon\setminus\Omega} |\Delta u_\varepsilon|^2 dx \to 0$$

as $\varepsilon \to 0$. Inserting (181) and (183) into (177) we obtain $\mathcal{N}_{\Omega_\varepsilon} f_\varepsilon \xrightarrow{E^*} \mathcal{N}_\Omega f$ as $\varepsilon \to 0$ along a sequence. We have proved that $\{\mathcal{N}_{\Omega_\varepsilon} f_\varepsilon\}$ is precompact in the sense of Definition 2.3. The proof of the theorem now follows combining this with Theorem 5.6. $\square$

Unfortunately, as we have mentioned in the introduction the compact convergence result obtained in Theorem 5.7 is not sufficient to prove a stability result for the spectrum of the Steklov problem (2). Thus we suggest the following open problem

Open Problem 5.1. *Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ and Ω be such that condition (21) is not satisfied. For simplicity one may think to a family of domains as in Section 4 and choose $1 \leq \alpha \leq \frac{3}{2}$ is such a way that (21) is not satisfied. Study the stability of the spectrum of (2).*

We recall that, in a situation like the one proposed in Open Problem 5.1, we proved that the spectrum of the modified Steklov problem (5) does not behave continuously as $\varepsilon \to 0$. The question proposed in Open Problem 5.1 aims at clarifying what happens to the spectrum of (2) under the very same assumptions.

5.3. **Instability of a modified Navier problem.** Given a sufficiently smooth bounded domain Ω in $\mathbb{R}^N$ we consider the boundary value problem

(184) $$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u - K(x) u_\nu = f, & \text{on } \partial\Omega, \end{cases}$$

in the unknown u. The natural way of writing the weak formulation of (184) is

(185) $$\int_\Omega D^2 u : D^2\varphi\, dx = \int_{\partial\Omega} f\varphi_\nu\, dS, \ \text{ for all } \varphi \in H^2(\Omega) \cap H_0^1(\Omega),$$

where the unknown u has to be considered in $H^2(\Omega) \cap H_0^1(\Omega)$. This formulation allows to consider the datum f in $L^2(\partial\Omega)$. However, in order to emphasize the peculiar behaviour of this problem with respect to problem (149), we prefer to write (185) in a form similar to (149). Thus we assume that f is the trace of function $f \in H^1(\Omega)$ and rewrite (185) as

(186) $$\int_\Omega D^2 u : D^2\varphi\, dx = \int_\Omega (f\Delta\varphi + \nabla f\nabla\varphi)\, dx, \ \text{ for all } \varphi \in H^2(\Omega) \cap H_0^1(\Omega).$$

As in the previous section, we now consider a family of domains Ω_ε, $\varepsilon > 0$ and the corresponding boundary value problems

$$(187) \qquad \begin{cases} \Delta^2 u_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega_\varepsilon, \\ \Delta u_\varepsilon - K_\varepsilon(x)(u_\varepsilon)_\nu = f_\varepsilon, & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

in the unknown $u_\varepsilon \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega)$, where $f_\varepsilon \in H^1(\Omega_\varepsilon)$, with the understanding that problem (187) has to be interpreted as in (186).

We assume that Ω and Ω_ε are as in Section 4. As in [4], the limiting behaviour of problem (187) depends on the value of α. In particular, we have stability for $\alpha > 3/2$ and degeneration for $\alpha < 3/2$. For $\alpha = 3/2$, the limiting problem involves the strange factor γ defined by (134).

Remark 9. *Theorem 5.8 below and Theorem 5.3 show that problem* (147) *is much more stable than problem* (184) *under domain perturbations. Note that the main step in the proof of Theorem 5.3, is the definition of the test function φ_ε in* (159)*. Unless the convergence of the domains Ω_ε to Ω is stronger, this function cannot be used in the analysis of problem* (184)*, where more information on the Hessian of φ_ε would be required.*

Theorem 5.8. *Assume that Ω and Ω_ε, $\varepsilon > 0$, are as in Section 4, and that the datum f_ε converges to f in the sense of* (152)*. Let u_ε be the solution of problem* (187)*. Then the following statements hold:*

(i) *If $\alpha > 3/2$ then u_ε converges to the solution of problem* (184) *in the sense that for every multi-index $\beta \in \mathbb{N}_0^N$ with $0 \le |\beta| \le 2$ we have $\|D^\beta u_\varepsilon - (D^\beta u)_0\|_{L^2(\Omega_\varepsilon)} \to 0$, as $\varepsilon \to 0$.*

(ii) *If $\alpha = 3/2$ then u_ε converges to the solution u of the problem*

$$(188) \qquad \begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0 \text{ and } \Delta u - K(x)u_\nu = f, & \text{on } \partial\Omega \setminus \Gamma, \\ u = 0 \text{ and } \Delta u - K(x)u_\nu + \gamma u_\nu = f, & \text{on } \Gamma, \end{cases}$$

in the sense that for every multi-index $\beta \in \mathbb{N}_0^N$ with $0 \le |\beta| \le 1$ we have $\|D^\beta u_\varepsilon - (D^\beta u)_0\|_{L^2(\Omega_\varepsilon)} \to 0$, and $u_\varepsilon \rightharpoonup u$ weakly in $H^2(\Omega)$ as $\varepsilon \to 0$.

(iii) *If $0 < \alpha < 3/2$ then u_ε converges to the solution u of the problem*

$$(189) \qquad \begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0 \text{ and } \Delta u - K(x)u_\nu = f, & \text{on } \partial\Omega \setminus \Gamma, \\ u = 0 \text{ and } u_\nu = 0, & \text{on } \Gamma, \end{cases}$$

in the sense that for every multi-index $\beta \in \mathbb{N}_0^N$ with $0 \le |\beta| \le 2$ we have $\|D^\beta u_\varepsilon - (D^\beta u)_0\|_{L^2(\Omega_\varepsilon)} \to 0$, as $\varepsilon \to 0$.

Proof. In each of the three cases, the solution $u_\varepsilon \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ of problem (187) satisfies the weak equation

$$(190) \qquad \int_{\Omega_\varepsilon} D^2 u_\varepsilon : D^2\varphi \, dx = \int_{\Omega_\varepsilon} (f_\varepsilon \Delta\varphi + \nabla f_\varepsilon \nabla\varphi) \, dx,$$

for all $\varphi \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$. Thus, by using u_ε as test function in (190) and arguing as in the proof of Theorem 5.3, we get that $\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)}$ is uniformly bounded, hence there exists $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that possibly passing to a subsequence, $u_\varepsilon \rightharpoonup u$ in $H^2(\Omega)$ and $u_\varepsilon \to u$ in $H^1(\Omega)$. We have now to identify the limiting problem satisfied by u and this depends on the value of α.

We begin with statement (i). For every $\varepsilon > 0$ sufficiently small, say $\varepsilon \in]0, \varepsilon_0[$, we consider a diffeomorphism Φ_ε from $\overline{\Omega}_\varepsilon$ onto $\overline{\Omega}$ defined exactly as the diffeomorphism $\Phi_{\varepsilon,j}$ introduced in Subsection 3.2. To

do so, for fixed $\tilde{\alpha} \in]3/2, \alpha[$, we set $\kappa_\varepsilon = \varepsilon^{2\tilde{\alpha}/3}$ and we note that condition (21) is satisfied for ε_0 sufficiently small. Then, Φ_ε is defined by $\Phi_\varepsilon(x', x_N) = (x', x_N - h_\varepsilon(x', x_N))$ for all $(x', x_N) \in \overline{\Omega}_\varepsilon$ where h_ε is defined as $h_{\varepsilon,j}$ by formula (24) with $g_{\varepsilon,j}$ replaced by g_ε, g_j replaced by the zero function, and $a_{N,j}$ replace by -1. It is also convenient to recall that $\Phi_\varepsilon(x', x_N) = (x', x_N)$ for all $(x', x_N) \in K_\varepsilon$ where $K_\varepsilon = \{(x', x_N) \in W \times \mathbb{R} : -1 < x_N < \tilde{g}_\varepsilon(x', x_N)\}$ and $\tilde{g}_\varepsilon = -k_\varepsilon = -\hat{k}\kappa_\varepsilon$ with $\hat{k} > 6$ as in Subsection 3.2.

Let $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$. We set $\varphi_\varepsilon = \varphi \circ \Phi_\varepsilon$ and we note that $\varphi_\varepsilon \in H^2(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon)$, hence it can be used as test function in (190). Thus, since Φ_ε coincides with the identity on K_ε we have

$$\int_{K_\varepsilon} D^2 u_\varepsilon : D^2\varphi \, dx + \int_{\Omega_\varepsilon \setminus K_\varepsilon} D^2 u_\varepsilon : D^2\varphi_\varepsilon \, dx$$

$$= \int_{\Omega_\varepsilon} D^2 u_\varepsilon : D^2\varphi_\varepsilon \, dx = \int_{\Omega_\varepsilon} (f_\varepsilon \Delta\varphi_\varepsilon + \nabla f_\varepsilon \nabla\varphi_\varepsilon) \, dx$$

$$(191) \qquad = \int_{K_\varepsilon} (f_\varepsilon \Delta\varphi + \nabla f_\varepsilon \nabla\varphi) \, dx + \int_{\Omega_\varepsilon \setminus K_\varepsilon} (f_\varepsilon \Delta\varphi_\varepsilon + \nabla f_\varepsilon \nabla\varphi_\varepsilon) \, dx .$$

By Lemma 3.1 we have that (20) holds hence $\int_{\Omega_\varepsilon \setminus K_\varepsilon} |D^2\varphi_\varepsilon|^2 dx \to 0$ and $\int_{\Omega_\varepsilon \setminus K_\varepsilon} D^2 u_\varepsilon : D^2\varphi_\varepsilon dx \to 0$ as $\varepsilon \to 0$. By a similar argument, one can also prove that $\int_{\Omega_\varepsilon \setminus K_\varepsilon} (f_\varepsilon \Delta\varphi_\varepsilon + \nabla f_\varepsilon \nabla\varphi_\varepsilon) \, dx \to 0$ as $\varepsilon \to 0$ (in which case one should use a limiting relation for the norms of the gradients analogous to (20), see the proof of Theorem 5.7 for more details).

Thus, passing to the limit in (191) we obtain

$$(192) \qquad \int_\Omega D^2 u : D^2\varphi \, dx = \int_\Omega (f\Delta\varphi + \nabla f \nabla\varphi) \, dx .$$

Using the same argument as in the proof of Theorem 5.3 with identity (169) replaced by

$$(193) \qquad \|D^2 u_\varepsilon - (D^2 u)_0\|^2_{L^2(\Omega_\varepsilon)} = \|D^2 u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - 2\int_\Omega D^2 u_\varepsilon : D^2 u \, dx + \|D^2 u\|^2_{L^2(\Omega)} ,$$

we deduce that $\|D^2 u_\varepsilon - (D^2 u)_0\|_{L^2(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$. In order to conclude, it suffices to use the strong convergence in $H^1(\Omega)$ stated above and to note that for any multi-index $|\beta| \leq 1$ we have $\|D^\beta u_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega)} \to 0$ as $\varepsilon \to 0$ which follows from the fact that for $|\beta| \leq 1$ and for almost all $x' \in W$ we have that $D^\beta u_\varepsilon(x', x_N)$ is bounded in x_N, see (34) for more details.

We now prove statement (ii). In this case, we need a different diffeomorphism from $\overline{\Omega}_\varepsilon$ onto $\overline{\Omega}$, which for convenience we denote with the same symbol Φ_ε. Namely, $\Phi_\varepsilon(x', x_N) = (x', x_N - h_\varepsilon(x', x_N))$ for all $(x', x_N) \in \overline{\Omega}_\varepsilon$ where h_ε is defined as in (138).

It is convenient to denote again by K_ε the set where Φ_ε coincides with the identity, that is $K_\varepsilon = \{(x', x_N) \in W \times \mathbb{R} : -1 < x_N \leq -\varepsilon\}$. Let $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$. We set $\varphi_\varepsilon = \varphi \circ \Phi_\varepsilon$ and we note that $\varphi_\varepsilon \in H^2(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon)$, hence it can be used as test function in (190). Clearly, equality (191) holds. Concerning the second term in the right-hand side of (191), we note that

$$(194) \qquad \left| \int_{\Omega_\varepsilon \setminus K_\varepsilon} f_\varepsilon \Delta\varphi_\varepsilon \, dx \right| \leq \|f_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)} \|\Delta\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)} = o(1) \|\Delta\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)}$$

and

$$(195) \qquad \left| \int_{\Omega_\varepsilon \setminus K_\varepsilon} \nabla f_\varepsilon \nabla\varphi_\varepsilon \, dx \right| \leq \|\nabla f_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)} \|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)} = o(1) \|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)}$$

where we have used (163). On the other hand, one can see that

$$(196) \qquad \|D^2\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)} = O(1), \text{ and } \|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)} = o(1) .$$

Indeed, by the chain rule it follows that

$$(197) \qquad \frac{\partial^2 \varphi_\varepsilon}{\partial x_n \partial x_m}(x) = \sum_{k=1}^{N} \left[\sum_{l=1}^{N} \frac{\partial^2 \varphi}{\partial x_l \partial x_k}(\Phi_\varepsilon(x)) \frac{\partial[(\Phi_\varepsilon(x))_l]}{\partial x_m} \right] \frac{\partial[(\Phi_\varepsilon(x))_k]}{\partial x_n}$$

$$+ \frac{\partial \varphi}{\partial x_k}(\Phi_\varepsilon(x)) \frac{\partial^2[(\Phi_\varepsilon(x))_k]}{\partial x_n \partial x_m} \,.$$

Now the L^2-norm of the first term in the right-hand side of (197) is clearly $O(\varepsilon^{2\alpha-2})$ hence it is infinitesimal for $\alpha = 3/2$ as $\varepsilon \to 0$. As for the second term, we note that $\frac{\partial^2[(\Phi_\varepsilon(x))_k]}{\partial x_n \partial x_m} = O(\varepsilon^{\alpha-2})$ while $\|\frac{\partial \varphi_\varepsilon}{\partial x_k}(\Phi_\varepsilon(x))\|_{L^2(\Omega \setminus K_\varepsilon)} = O(\varepsilon^{1/2})$ which simply follows by the fact that for almost all $x' \in W$, $\frac{\partial \varphi_\varepsilon}{\partial x_k}(x', x_N)$ is bounded in the variable x_N. Thus, for $\alpha = 3/2$ we have that the L^2-norm of the second term in the right-hand side of (197) is $O(1)$ as $\varepsilon \to 0$, hence the first equality in (196) is proved. The second equality in (196) can be proved in the same way. The proof of statement (ii) follows by passing to the limit in (191) as $\varepsilon \to 0$ and using (140), (196).

We now prove statement (iii). It follows from the proof of [5, Theorem 7.4] that since $\alpha < 3/2$ the function u belongs to $H_{0,\Gamma}^2(\Omega) \cap H_0^1(\Omega)$ where $H_{0,\Gamma}^2(\Omega) = \{u \in H^2(\Omega) : \ u = u_\nu = 0 \text{ on } \Gamma\}$, see the proof of Lemma 4.4 for more details. Let $\varphi \in H_{0,\Gamma}^2(\Omega)$. Since $\varphi = \varphi_\nu = 0$ on Γ, the extension-by-zero φ_0 of φ belongs to $H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$, hence it can uses ad test function in (190) to obtain
(198)
$$\int_\Omega D^2 u_\varepsilon : D^2 \varphi \, dx = \int_{\Omega_\varepsilon} D^2 u_\varepsilon : D^2 \varphi_0 \, dx = \int_{\Omega_\varepsilon} (f_\varepsilon \Delta \varphi_0 + \nabla f_\varepsilon \nabla \varphi_0) \, dx = \int_\Omega (f_\varepsilon \Delta \varphi + \nabla f_\varepsilon \nabla \varphi) \, dx \,.$$

Passing to the limit in (198) as $\varepsilon \to 0$ we conclude that u is the solution to problem (189). The proof of the convergence of u_ε to u in the H^2 norms follows the same lines of the analogous proof in statement (i).

$\square$

REFERENCES

[1] V. Adolfsson, *L^2-integrability of second order derivatives for Poisson's equation in nonsmooth domains*, Math. Scand. **70**, 146-160, (1992).

[2] P. R. S. Antunes, F. Gazzola, *Convex shape optimization for the least biharmonic Steklov eigenvalue*, ESAIM COCV **19**, 385-403, (2013).

[3] J. M. Arrieta, A. N. Carvalho, G. Losada-Cruz, *Dynamics in dumbell domains I. Continuity of the set of equilibria*, J. Diff. Eq. **231**, 551-597, (2006).

[4] J. M. Arrieta, P. D. Lamberti, *Higher order elliptic operators on variable domains. Stability results and boundary oscillations for intermediate problems*, J. Differential Equations **263**, no. 7, 4222-4266, (2017).

[5] J. M. Arrieta, P. D. Lamberti, *Spectral stability results for higher order operators under perturbations of the domain*, C. R. Acad. Sci. Paris, ser. I **351**, 725-730, (2013).

[6] J.M. Arrieta, M. López-Fernández, E. Zuazua, *Enrique Approximating travelling waves by equilibria of non-local equations*, Asymptot. Anal. **78**, no. 3, 145-186, (2012).

[7] G. Auchmuty, *The S.V.D. of the Poisson Kernel*, J. Fourier Anal. Appl. **23**, no. 6, 1517-1536, (2017).

[8] E. Berchio, F. Gazzola, E. Mitidieri, *Positivity preserving property for a class of biharmonic elliptic problems*, J. Differ. Equa. **229**, 1-23, (2006).

[9] S. Bögli, *Convergence of sequences of linear operators and their spectra*, Integral Equations Operator Theory **88**, no. 4, 559-599, (2017).

[10] B. Bogosel, *The Steklov Spectrum on Moving Domains*, Appl. Math. Optim. **75**, 1-25, (2017).

[11] H. Brezis, *Analisi funzionale*, Liguori Editore, (1986)

[12] D. Bucur, G. Buttazzo, *Variational methods in shape optimization problems*, Progress in Nonlinear Differential Equations and their Applications **65**, Birkhäuser Boston, Boston, (2005).

[13] D. Bucur, A. Ferrero, F. Gazzola, *On the first eigenvalue of a fourth order Steklov problem*, Calc. Var. PDE **35**, 103-131, (2009).

[14] D. Bucur, F. Gazzola, *The first biharmonic Steklov eigenvalue: positivity preserving and shape optimization*, Milan J. Math. **79**, 247-258, (2011).

[15] D. Buoso, L. Provenzano, A few shape optimization results for a biharmonic Steklov problem. J. Differential Equations **259**, no. 5, 1778-1818, (2015).

[16] V. Burenkov, P. D. Lamberti, *Sharp spectral stability estimates via the Lebesgue measure of domains for higher order elliptic operators*, Rev. Mat. Complut. **25**, 435-457, (2012).

[17] A. Carvalho, S. Piskarev, A general approximation scheme for attractors of abstract parabolic problems. Numer. Funct. Anal. Optim. **27**, no. 7-8, 785-829, (2006).

[18] J. Casado-Díaz, M. Luna-Laynez, F. J. Suárez-Grau, *Asymptotic behavior of a viscous fluid with slip boundary conditions on a slightly rough wall*, Math Models Methods Appl. Sci. **20**, 121-156, (2010).

[19] L. M. Chasman, *An isoperimetric inequality for fundamental tones of free plates*, Comm. Math. Phys. **303**, 421-449, (2011).

[20] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, (1998).

[21] A. Ferrero, F. Gazzola, T. Weth, *On a fourth order Steklov eigenvalue problem*, Analysis **25**, 315-332, (2005).

[22] G. Fichera. Su un principio di dualit per talune formole di maggiorazione relative alle equazioni differenziali. Atti Accad. Naz. Lincei, **19**, 411-418, (1955).

[23] F. Gazzola, H.-Ch. Grunau, G. Sweers, *Polyharmonic boundary value problems*, Springer-Verlag, Berlin, (2010).

[24] F. Gazzola, G. Sweers, *On positivity for the biharmonic operator under Steklov boundary conditions*, Arch. Rat. Mech. Anal. **188**, 399-427, (2008).

[25] A. Girouard, I Polterovich, *Spectral geometry of the Steklov problem*, J. Spectr. Theory **7**, no. 2, 321-359, (2017).

[26] B. Helffer, *Spectral theory and its applications*, Cambridge Studies in Advanced Mathematics, 139. Cambridge University Press, Cambridge, (2013).

[27] J. R. Kuttler, *Remarks on a Stekloff eigenvalue problem*, SIAM J. Numer. Anal. **9**, 1-5, (1972).

[28] J. R. Kuttler, *Dirichlet eigenvalues*, SIAM J. Numer. Anal. **16**, 332-338, (1979).

[29] J. R. Kuttler, V. G. Sigillito, *Inequalities for membrane and Stekloff eigenvalues*, J. Math. Anal. Appl. **23**, 148-160, (1968).

[30] P.D. Lamberti, L., Provenzano, *Neumann to Steklov eigenvalues: asymptotic and monotonicity results*, Proc. Roy. Soc. Edinburgh Sect. A **147**, no. 2, 429-447, (2017).

[31] J. Necas, *Direct Methods in the Theory of Elliptic Equations*, Springer-Verlag, (2012).

[32] L. E. Payne, *Some isoperimetric inequalities for harmonic functions*, SIAM J. Math. Anal. **1**, 354-359, (1970).

[33] O. Post, *Spectral analysis on graph-like spaces*, LNM 2039, Springer, Heidelberg, (2012).

[34] R. Sperb, *Maximum principles and applications*, Academic Press, (1981).

[35] W. Stekloff, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. Ecol. Norm. Sup. **19**, 455-490, (1902).

[36] F. Stummel, *Perturbation of domains in elliptic boundary-value problems*, in: Lecture Notes Math., Vol. 503, Springer-Verlag, Berlin-Heidelberg-New York (1976), pp. 110-136.

[37] A. F. M. ter Elst, E.M. Ouhabaz, *Convergence of the Dirichlet-to-Neumann operator on varying domains*. Operator semigroups meet complex analysis, harmonic analysis and mathematical physics, 147-154, Oper. Theory Adv. Appl., 250, Birkhuser/Springer, Cham, 2015.

[38] G. M. Vainikko, *Regular convergence of operators and the approximate solution of equations*, Mathematical analysis **16** (Russian), 5-53, (1979).

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