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ON THE COMPUTATION OF SETS OF POINTS WITH LOW LEBESGUE CONSTANT ON THE UNIT DISK

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Abstract. In this paper of numerical nature, we test the Lebesgue constant of several pointsets on the disk Ω and propose new ones that enjoy low Lebesgue constant. Furthermore we extend some results in [15], analysing the case of Bos arrays whose radii are nonnegative Gegenbauer-Gauss-Lobatto nodes with exponent α, noticing that the optimal α still allow to achieve pointsets on Ω with low Lebesgue constant Λ_n for degrees $n \leq 30$. Next we introduce an algorithm that through optimization determines pointsets with the best known Lebesgue constants for $n \leq 25$. Finally, we determine theoretically a pointset with the best Lebesgue constant for the case $n = 1$.

Key words. Interpolation, Lebesgue constant, unit disk.

1. Introduction. In this paper we are interested in determining *good* sets of points for the interpolation of functions in the unit disk that is,

$$B_2 := \{(x, y) \,|\, x^2 + y^2 \leq 1\}.$$

where *good* means *with low Lebesgue constant*.

To explain this concept, let Ω be a compact domain of $\mathbb{R}^m$ and $\mathbb{P}_d$ be the space of algebraic polynomials of two variables on Ω whose total degree is equal to d. In general the dimension N of the vector space $\mathbb{P}_d$ is equal to or less than $\binom{d+2}{2}$. The equality $N = \binom{d+m}{m} = \mathcal{O}(d^m)$ holds if Ω is polynomial determining (i.e. a polynomial vanishing on Ω vanishes everywhere in $\mathbb{R}^m$), as in the case when Ω has nonempty interior.

Suppose that $X = \{\xi_k\}_{k=1,\dots,N} \subset \Omega$ is a *unisolvent pointset* at degree d for Ω, i.e. there exists a unique interpolant of f in X, and let ℓ_i be the i-th Lagrange polynomial w.r.t. the set X, that is

- $\ell_i \in \mathbb{P}_d$,
- $\ell_i(\xi_j) = \delta_{i,j}$

where $\delta_{i,j}$ is the Kronecker symbol. The Lebesgue function λ_X evaluated in ξ is defined as

$$\lambda_X(\xi) := \sum_{i=1}^{N} |\ell_i(\xi)|,$$

and the Lebesgue constant is the maximum of the Lebesgue function on the domain Ω,

$$\Lambda_X := \max_{\xi \in \Omega}(\lambda_X(\xi)).$$

Let

$$\|f\|_\infty = \max_{\xi \in \Omega} |f(\xi)|, \;\; f \in C(\Omega, \infty).$$

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One of the most important features of Λ_X is that if $p \in \mathbb{P}_d$ interpolates $f \in C(\Omega) \in X$, and $p^* \in \mathbb{P}_d$ is the best approximant of f relatively to the uniform norm, then

$$\|f - p\|_\infty \leq (1 + \Lambda_X)\|f - p^*\|_\infty$$

implying that if Λ_X is small then the interpolation error $\|f - p\|_\infty$ is comparable to that given by the best approximant, so explaining the meaning of *good* at the beginning of this section.

The interest on the subject is of theoretical and practical nature. Depending on the domain Ω, there is a vast literature in which the growth of Λ_X was numerically estimated for some pointsets X and possibly their location provided. Recently, such a problem has been tackled numerically by several authors (see, e.g. [11], [18], [27]), essentially resting on some optimization algorithms. In [11], it has been explored how to achieve pointsets with low Lebesgue constant, using Matlab built-in optimization routines *fmincon* and *fminsearch*, with a kind of multilevel approach, determining good pointsets on the square, simplex and disk for $d \leq 20$. Next in [23], in a similar fashion, the authors computed Lebesgue-like points on the triangle, with sixfold symmetries and Gauss-Legendre-Lobatto distribution on the sides, up to interpolation degree 18. These sets are available at the homepage given in reference [24]. In [18], a similar strategy has been suggested, by means of the Matlab built-in routine *fminimax*, improving some results in [11] for $d \leq 10$, also performing experiments on the cube and on the 3D ball. These pointsets are available at [19]. In [27] a very different approach has been used via two *greedy algorithm*s, the first one for determining a good initial pointset, the second one for refining the results via a leaving-out one point technique. The results were promising, also in non standard domains Ω and for large degrees d. Unfortunately, only the plot of the Lebesgue constant is available for $d \leq 30$, showing that the computed pointsets enjoy very low Lebesgue constants (often better than those in [11] and comparable to those in [18] where only low degrees were considered).

In the case Ω corresponds to the unit disk, there is a rather limited literature, though some sets have already been proposed and theoretical insights about unisolvency and the properties of the asymptotically optimal pointsets are available.

One of the aims of this paper is to determine pointsets on the unit disk $B_2 \subset \mathbb{R}^2$ with Lebesgue constants lower than those previously available and also to make the "good" poinsets X^* available to users. Futhermore we discuss the actual values of Λ_X for several well-known pointsets of the literature, pointing out some less known results from the literature.

The paper is organized as follows. In section 1, we explain how to efficiently compute the Lebesgue constant. In section 2, we discuss some unisolvent pointsets on the unit disk that are closely related to Bos arrays, introducing some new ones. In section 3, we evaluate the Lebesgue constants on some sets that have a different structure, showing that there exist at least a set with at most quadratic growth of the Lebesgue constant Λ_X w.r.t. the degree d. In section 4, we explain the methodology we used to obtain numerically sets with the best known Λ_X and compare them with the best available in literature. Finally, in the appendix we provide a new result about the pointsets with the minimum Lebesgue constant on the unit disk when $d = 1$ and a theoretical upper bound for $d = 2$ and 3.

2. The Lebesgue function and its computation. In view of the definition of the Lebesgue constant, depending on the compact domain $\Omega \subset \mathbb{R}^d$, it is important to define a suitable basis $\{\phi_k\}$ so that the evaluation of Λ_X is efficient.

In [27] the authors proposed the following approach. Let us assume that we have a compact set $\Omega \subset \mathbb{R}^m$ (e.g. the unit disk), where the dimension of the space of algebraic polynomials of total degree d is equal to N. As discussed in the introduction, $N \leq \binom{d+m}{m} = \mathcal{O}(d^m)$, with the equality holding if Ω is polynomial determining.

Define a discrete inner product defined by distinct points $\xi_j = (x_j, y_j) \in \Omega$, $j = 1, \ldots, N$ and positive weights w_j summing up to 1 (here we will use equal weights $w_i = 1/N$),

$$\langle f, g \rangle = \sum_{k=1}^{N} w_k f(\xi_k) g(\xi_k). \tag{2.1}$$

Associated with the discrete inner product defined by the points ξ_j and the weights, we have a basis of orthonormal polynomials φ_j, $j = 1, \ldots, N$ such that

$$\langle \varphi_i, \varphi_j \rangle = \sum_{k=1}^{N} w_k \varphi_i(\xi_k) \varphi_j(\xi_k) = \delta_{i,j}.$$

This can be written in matrix form as $\Phi^T W \Phi = I$ where

$$\Phi = \begin{pmatrix} \varphi_1(\xi_1) & \cdots & \varphi_N(\xi_1) \\ \vdots & & \vdots \\ \varphi_1(\xi_N) & \cdots & \varphi_N(\xi_N) \end{pmatrix},$$

is the transpose of a (generalized) Vandermonde matrix and W is the diagonal matrix of the weights. We will assume that $\varphi_1 \equiv 1$ (notice that $\varphi_1 = 1$ is an orthogonal polynomial w.r.t. the discrete scalar product since $\sum_i w_i = 1$).

Recalling that the N bivariate Lagrange polynomials ℓ_j of total degree d related to these points are such that the polynomial ℓ_j is equal to 1 at point ξ_j and zero at the other points, we can write ℓ_j w.r.t. the basis of the orthonormal polynomials,

$$\ell_j(\xi) = \sum_{k=1}^{N} \alpha_{k,j} \varphi_k(\xi), \qquad j = 1, \ldots, N.$$

The coefficients $\alpha_{k,j}$ can be found by writing the values of ℓ_j at the points ξ_i,

$$\Phi \begin{pmatrix} \alpha_{1,j} \\ \vdots \\ \alpha_{N,j} \end{pmatrix} = e_j,$$

where e_j is the j-th column of the identity matrix. If we group the coefficients $\alpha_{k,j}$ for all the Lagrange polynomials in a matrix Θ whose entries of column j are $\alpha_{k,j}$, $k = 1, \ldots, N$ we have the matrix equation

$$\Phi \Theta = I.$$

Multiplying this relation by $\Phi^T W$, using $\Phi^T W \Phi = I$, we obtain the solution $\Theta = \Phi^T W$. It yields

$$\begin{pmatrix} \alpha_{1,j} \\ \vdots \\ \alpha_{N,j} \end{pmatrix} = w_j \begin{pmatrix} \varphi_1(\xi_j) \\ \vdots \\ \varphi_N(\xi_j) \end{pmatrix}.$$

Therefore

$$\ell_j(\xi) = w_j \sum_{k=1}^{N} \varphi_k(\xi_j)\varphi_k(\xi). \tag{2.2}$$

Thus the Lebesgue function, relatively to the pointset $X = \{\xi_j\}_{j=1}^N$, evaluated in ξ, is

$$\lambda_X(\xi) = \sum_{j=1}^{N} |\ell_j(\xi)| = \sum_{j=1}^{N} w_j \left| \sum_{k=1}^{N} \varphi_k(\xi_j)\varphi_k(\xi) \right|. \tag{2.3}$$

The usual approach for computing an approximation of the Lebesgue constant is to replace the maximum of the Lebesgue function over $\xi \in \Omega$ by the maximum over a set of "well chosen" points η_j, $j = 1, \dots, K$ such that

$$\Lambda_\Omega \approx \max_{\{\eta_j\}} \lambda_X(\eta_j).$$

This is clearly a lower bound for the Lebesgue constant and once again we have to use enough carefully chosen test points to obtain a reliable approximation of the true Lebesgue constant. To this purpose, the use of weakly admissible meshes (shortened with the acronym *WAM* [13]), that are pointsets satisfying certain polynomial inequalities, had been advocated in several works, e.g. [7], [11]. In the case of the unit disk, a WAM of degree d consists of the points whose polar coordinates are (r_j, θ_k) where

$$r_j = \cos(j\pi/d), j = 0, \dots, d, \qquad \theta_k = k\pi/(d+1), k = 0, \dots, d.$$

It is easy to check that the cardinality of the WAM is d^2+d+1 if d is even, d^2+2d+1 if d is odd (see [10] for its use in least-squares approximation).

It is straightforward to notice that the lower bound of the Lebesgue constant can only be close to the actual value if the test points are cleverly chosen.

Observe that denoting

$$\ell(\xi)^T = \begin{pmatrix} \ell_1(\xi) & \cdots & \ell_N(\xi) \end{pmatrix}, \quad \varphi(\xi)^T = \begin{pmatrix} \varphi_1(\xi) & \cdots & \varphi_N(\xi) \end{pmatrix},$$

and using $\Phi\Theta = I$ we have

$$\ell(\xi)^T = \varphi(\xi)^T\Theta = \varphi(\xi)^T\Phi^{-1} \tag{2.4}$$

Let $\eta_1, \dots, \eta_K$ be the test points in which we want to compute the Lebesgue function and put

$$\mathcal{L}_\eta = \begin{pmatrix} \ell_1(\eta_1) & \cdots & \ell_N(\eta_1) \\ \vdots & & \vdots \\ \ell_1(\eta_K) & \cdots & \ell_N(\eta_K) \end{pmatrix}, \qquad V_\eta = \begin{pmatrix} \varphi_1(\eta_1) & \cdots & \varphi_N(\eta_1) \\ \vdots & & \vdots \\ \varphi_1(\eta_K) & \cdots & \varphi_N(\eta_K) \end{pmatrix}.$$

From (2.4), stacking the row vectors in matrices, we have $\mathcal{L}_\eta = V_\eta\Theta = V_\eta\Phi^{-1}$.

When the polynomial basis is orthonormal w.r.t. the discrete scalar product (2.1), from $\Phi^T W \Phi = I$, we get $\Phi^{-1} = \Phi^T W$, that yields

$$\mathcal{L}_\eta = V_\eta \Phi^T W. \tag{2.5}$$

and the approximation of the Lebesgue constant using the set of points $\{\eta_j\}_{j=1,\dots,K}$ is the 1-norm of the matrix $\mathcal{L}_\eta^T = W\Phi V_\eta^T$. We point out that if the basis is not orthonormal w.r.t. the discrete inner product (2.1), as when we use any system of orthonormal polynomials, e.g. the Koornwinder polynomials on the unit disk (see, for instance, [17]), we can similarly compute the Lebesgue constant by evaluating the 1-norm of the matrix ${\mathcal{L}_\eta}^T = \Phi^{-T} V_\eta^T$.

We point out that one of the key points for such evaluation of the Lebesgue constant is the determination of the orthonormal basis. A basic approach resorts on linear algebra but in the case of algebraic polynomials of total degree d, w.r.t. a discrete scalar product, one can use as alternative the Huhtanen-Larsen algorithm; see [20].

Now, we illustrate the difficulty of computing a reliable approximation of the Lebesgue constant Λ_X in Table 2.1. In our test, we use the approach previously described via the Huhtanen-Larsen algorithm, where $X = \{\xi_j\}_{j=1,\dots,153}$ is a prescribed unisolvent set for polynomial interpolation on the disk at total degree 16, and the test points are WAMs with an increasing degree (hence with an increasing cardinality). We see that to obtain a good result we need at least a WAM of degree 500 with 250501 test points. A WAM of degree 100 gives a rough approximation of the Lebesgue constant with a relative error 0.003 which seems small, but we have only one correct digit. This can be explained by the fact that the Lebesgue function is highly oscillating for large degrees.

It is numerically evident that the higher is the degree of the WAM, the closer is the numerical value of the computed Lebesgue constant to the actual one, as well as the more time consuming is the evaluation. In view of this, if not reported otherwise, we will use WAMs of degree 100 in the optimization algorithms and of degree 500 to test the final pointset obtained by the optimization process.

TABLE 2.1
Lebesgue constants for different WAMs

order	nb. of points	Leb. const.
50	2551	9.873
100	10101	9.884
200	40201	9.896
300	90301	9.903
400	160401	9.913
500	250501	9.916
600	360601	9.916
700	490701	9.915
800	640801	9.916

We finally point out that pointsets that are *close* in ∞-norm have *comparable* Lebesgue function. To be more precise on this point, we observe that on the unit disk Ω, since the domain is central-symmetric, the Markov inequality states that

$$|\nabla p(x)| \le d^2 \|p\|_\Omega, \forall p \in \mathbb{P}_d.$$

From [21, Cor. 1], valid in more general domains, we get that if for some $\alpha \in (0,1)$ we have $\|X - \tilde{X}\|_\infty \le \frac{\alpha}{\Lambda_X d^2}$ then also $\tilde{X}$ is unisolvent at degree d and $\Lambda_{\tilde{X}} \le \frac{1}{1-\alpha}\Lambda_X$. Since with the same argument, we can state that $\Lambda_X \le \frac{1}{1-\alpha}\Lambda_{\tilde{X}}$, we conclude that

$$(1-\alpha)\Lambda_X \le \Lambda_{\tilde{X}} \le \frac{1}{1-\alpha}\Lambda_X.$$

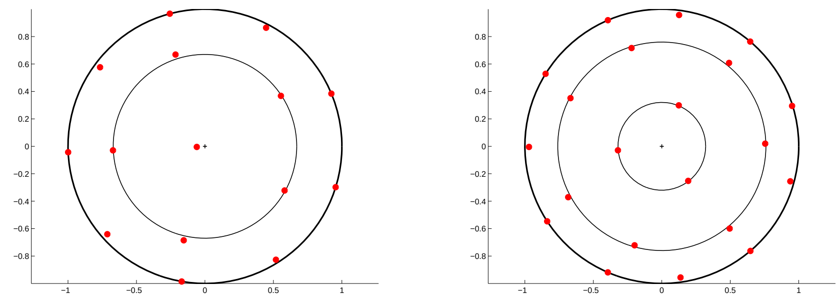

FIG. 3.1. *Almost optimal configurations at degree $d = 4$ (left) and $d = 5$ (right).*

3. Pointsets with special distributions. In the first studies on this topic, some authors proposed pointsets $X \subset \mathbb{R}^2$ of the unit disk Ω with low Lebesgue constant and enjoying a certain symmetry. This is suggested by the structure of the domain as well as proved in the seminal work [5] that Fekete points, whose Lebesgue constant grows at most as the dimension of $\mathbb{P}_d$, for low degrees d have such distribution.

Furthermore, as we will see later in Table 6.1, for low degrees d, every pointset proposed by different authors has approximatively the same Lebesgue constant and actually the same structure. Most of them have points distributed on $\lfloor d/2 \rfloor + 1$ concentric circles $S(r_l)$, centered in the origin $(0, 0)$ and having radius r_l, and are equispaced in the angular variable. For example, in Figure 3.1 we show 2 pointsets on the disk, obtained by numerical optimization, that enjoy low Lebesgue constant respectively for $d = 4$ and $d = 5$, In both cases the points are distributed very close to 3 concentric circles centered in the origin, with points almost equispaced in the angular variable. In the case $d = 4$, one of the points is close to the origin that can be interpreted as a circle with null radius.

Thus it was natural to introduce sets $\mathcal{B}_d$, sometimes called *Bos arrays*, that

- for even d, the points belong to $d/2+1$ circles, i.e. they are in $\cup_{l=1}^{d/2+1} S(r_l)$; in particular $(0, 0) \in \mathcal{B}_d$, and the i-th circle has $4(i-1)+1$ points equi-spaced on the angles;
- for odd d, the points belong to $(d + 1)/2$ circles, and the i-th circle has $4(i-1)+3$ points equi-spaced on the angles;

One observes that on the outer circle of a Bos array there are $2d + 1$ points, and that each circle has 4 points less than the closest outer one.

Y. Xu and collaborators, devoted some studies on the unisolvency of these and more general sets (e.g. see [4, Thm. 3.4] and the references therein). Using the notation that $S(\mathbf{a}, r)$ is the circle centered in $\mathbf{a}$ and radius r, and that

$$\Theta_{\alpha,m} = \{\theta_j^\alpha : \theta_j^\alpha = (2j + \alpha)\pi/(2m + 1), j = 0, 1, \dots, 2m\} \tag{3.1}$$

they proved the following result.

THEOREM 3.1. *Let d, σ, $\lambda_1, \dots, \lambda_\sigma$ be positive integers, such that $d > \lambda_i$. Let $\mathbf{a}_k = (a_{1,k}, a_{2,k})$ and let $r_{l,k}$ be distinct nonnegative real numbers, $1 \le l \le \lambda_k$, $1 \le k \le \sigma$. Define*

$$n_1 = d - \lambda_1 + 1 \text{ and } n_k = d - 2\lambda_1 - \dots - 2\lambda_{k-1} - \lambda_k + 1,\ k \ge 2.$$

Let $(x_{j,l,k}, y_{j,l,k})$, $0 \le j \le 2n_k$, *denote the equidistant points in the circle* $S(\mathbf{a}_k, r_{l,k})$ *i.e.*

$$(x_{j,l,k}, y_{j,l,k}) = (a_{1,k} + r_{l,k}\cos(\theta_j), a_{2,k} + r_{l,k}\sin(\theta_j)), \;\; \theta_j \in \Theta_{\alpha_k, n_k}, \tag{3.2}$$

and assume that all points are distinct. Then for any given data $\{f_{j,l,k}\}$, *there is a unique polynomial* $P \in \mathbb{P}_d$ *that satisfies*

$$P(x_{j,l,k}, y_{j,l,k}) = f_{j,l,k}, \;\; 0 \le j \le 2n_k, \quad 1 \le l \le \lambda_k, \quad 1 \le k \le \sigma.$$

The Bos arrays $\mathcal{B}_d$ satisfy the assumptions of this theorem by setting

- $\sigma = \lfloor d/2 \rfloor + 1$,
- $\mathbf{a}_k = (0,0)$, $k = 1, \ldots, \sigma$,
- $\lambda_k = 1$, $k = 1, \ldots, \sigma$,

and consequently they are an unisolvent set for polynomial interpolation on the unit disk at degree d. There is numerical evidence that the parameters α_k in (3.2), defined in (3.1), do not vary significantly the Lebesgue constant.

As first attempt of a Bos array, we generate a sort of uniform distribution. The points are obtained in polar coordinates and distributed on concentric circles with centers at the origin. Let d be the total degree of the bivariate polynomials and $\ell = \lfloor d/2 \rfloor + 1$. The radii are $r_i = 1 - (2(i-1)/d)$, $i = 1 \ldots, \ell$. For a given radius we have $p = 2d - 4i + 5$ points on that circle whose angles are $\theta_j = (2(j-1)+1)\pi/p$, $j = 1, \ldots p$. One can immediatly check that this set is a Bos array, since on the outer circle of a Bos array there are $2d+1$ points equispaced on the polar angular variable, and that each circle has 4 points less than the closest outer one.

The corresponding Lebesgue constants are reported in Table 3.1. We see that, asymptotically, they grow almost exponentially, approximately by a factor 2 when the degree is increased by 1. However, as it will be clear later in our numerical tests, the Lebesgue constant is almost optimal for $d \le 2$.

TABLE 3.1
Lebesgue constants Λ_d *for uniform Bos arrays, consisting of* N *points, at degree* d

d	1	2	3	4	5	6	7	8	9	10	15	20
N	1	6	10	15	21	28	36	45	55	66	136	231
Λ_d	1.67	1.99	2.81	3.96	5.64	7.85	11.65	17.33	28.78	46.25	83.38	178.49

In order to get better Lebesgue constants from Bos arrays, in [15, p. 303] it has been explored numerically what are the best choices of the relevant radial and angular parameters. As for the radii $r_1, \ldots, r_\sigma$, the authors used the nonnegative values of a *Gauss-Chebyshev-Lobatto* rule of degree $d-1$ on $[-1,1]$. We point out that Gauss-Chebyshev-like distributions were also considered in the theoretical studies of [2] (see also descriptions of these topics in [3]), to analyse the possible asymptotic optimality of these Bos Arrays, that is $\Lambda_{X_d}^{1/d} \to 1$ as $d \to +\infty$ (sometimes cited as *subexponential growth of* Λ_X).

In this paper we test the case of sets $\mathcal{B}_d$ based on circles $S((0,0), r_j)$ whose radii are the nonnegative values of a *Gauss-Gegenbauer-Lobatto* rule of degree $d-1$. We computed numerically the Lebesgue constants of several instances of these sets, say $\mathcal{B}_d(\alpha, \alpha)$, where $\alpha > -1$ is the Jacobi exponent. We report the values of the Lebesgue

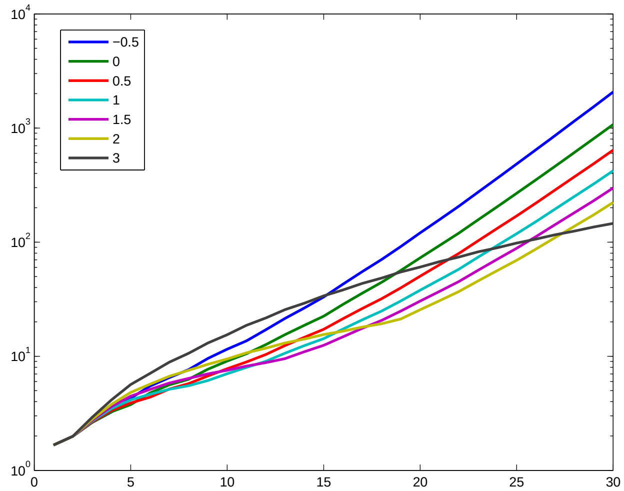

Fig. 3.2. *Growth of the Lebesgue constants of $\mathcal{B}_d(\alpha,\alpha)$ for $\alpha = -0.5, 0, 0.5, 1, 1.5, 2, 3$ and $d = 1, \ldots, 30$.*

constants $\Lambda_{\mathcal{B}_d(\alpha,\alpha)}$ for $\alpha = -0.5, 0, 0.5, 1, 1.5, 2, 3,\ d = 5, 10, \ldots, 30$ in Table 3.2 and their growth in Figure 3.2.

Next, in Table 3.3, we have computed using Matlab built-in routine `fminbnd` the almost optimal α^*, i.e. the value of α^* that minimizes the Lebesgue function on a WAM of degree 500. The numerical results show that the value of α for which $\min_{\alpha>-1} \Lambda_{\mathcal{B}_d(\alpha,\alpha)}$ is attained, is an increasingly monotone function of d.

Table 3.2
Lebesgue constants $\Lambda_{\mathcal{B}_n(\alpha,\alpha)}$ for $\alpha = -0.5, 0, 0.5, 1, 1.5, 2, 3,\ d = 5, 10, \ldots, 30$

d	-0.5	0	0.5	1.0	1.5	2.0	3.0
5	4.28	**3.76**	3.89	4.15	4.47	4.82	5.64
10	11.5	9.07	7.78	**7.02**	7.52	9.46	15.4
15	33.0	22.5	17.2	14.3	**12.5**	15.5	33.8
20	121	72.9	50.3	37.9	30.4	**25.5**	60.5
25	483	268	170	118	88.3	**69.4**	98.1
30	2070	1070	6410	4220	2990	2230	**146**

One immediatly notices that each column in Table 3.2 seems to have rapidly very high Lebesgue constants. This issue can also be seen in Figure 3.2.

By a result of Bos and collaborators [3], for a fixed value of $\alpha > -1$, none of these distributions is optimal.

If $d = 2s$ is even, one chooses $s+1$ radii $r_1 < \ldots < r_s = 1$ and $4(j-1)+1$ equally spaced points on the circle of radius r_j. One can prove that the Vandermonde determinant depends only on the radii $r_1 < \ldots < r_s = 1$ and if the asymptotic distribution of the radii on $[0,1]$ is given by a function $G : [0,1] \to [0,1]$, i.e., if $G((j-1)/(s+1)) = r_j^2$ then the growth of the Lebesgue constant $\Lambda_d := \Lambda_{X_d}$ is such that $\lim_{d\to} \Lambda_d^{1/d} \neq 1$, and thus not with subexponential growth. This happens also for $G(x) = (1-\cos(\pi x))/2$, where the radii distribute asymptotically like the Chebyshev distribution in $[0,1]$.

Table 3.3
Lebesgue constants $\Lambda_{\mathcal{B}_n(\alpha,\alpha)}$ for almost optimal α^*, for $d = 1, \ldots, 30$

d	Λ_d	α^*	d	Λ_d	α^*	d	Λ_d	α^*
1	1.667	10.00	11	7.567	1.338	21	28.77	2.193
2	1.989	10.00	12	8.326	1.402	22	33.53	2.277
3	2.624	−0.2522	13	9.526	1.508	23	40.02	2.368
4	3.234	−0.3126	14	10.72	1.602	24	47.36	2.458
5	3.727	0.04774	15	12.03	1.664	25	55.99	2.534
6	4.328	0.5968	16	13.82	1.773	26	67.16	2.628
7	4.967	0.8599	17	15.93	1.856	27	80.80	2.715
8	5.492	0.9852	18	18.14	1.933	28	96.68	2.801
9	6.061	1.082	19	21.09	2.022	29	116.3	2.882
10	6.778	1.238	20	24.65	2.119	30	141.1	2.975

In spite of that, from Table 3.3, the case of optimal α, up to degree 30, shows that the Lebesgue constants are still not too large.

Other pointsets in the family of Bos arrays $\mathcal{B}_d$ were obtained by D. Ramos-Lòpez and collaborators in [22] for which we will use the acronym *OCS* (optimal concentric sampling) and by Carnicer and Godés in [14], later shortened with "C-G".

We start describing the *OCS* pointset. Let d be the degree of the set, $\ell = \lfloor d/2 \rfloor + 1$. Let

$$\xi_j = \cos\left(\frac{(2j-1)\pi}{2(d+1)}\right),\ j = 1, \ldots, \ell.$$

Taking as radii $r_j = 1.1565\,\xi_j - 0.76535\,\xi_j^2 + 0.60517\,\xi_j^3$, the *OCS* set consists of the points $\{P_{j,i}\}_{j=1,\ldots,\ell,i=1,\ldots,2d+5-4j}$ whose polar coordinates are

$$P_{j,i} = \left(r_j, 2\pi\frac{i-1}{2d+5-4j}\right),\ i = 1, \ldots, 2d+5-4j,\ j = 1, \ldots, \ell.$$

Thus $\{P_{j,i}\}$ are distributed in $\ell = \lfloor d/2 \rfloor + 1$ circles and in the j-th one there are $n_j = 2d + 5 - 4j$ points equispaced w.r.t. the angle.

It is easy to notice that the *OCS* set is a Bos array since

- on the outer circle there are $2d+1$ points while on each other circle there are 4 points less than the closest outer one;
- the points lying on a certain circle are equispaced w.r.t. the angles;
- for even d, $\xi_l = 0$ and hence $r_l = 0$, implying that the origin is a point of the set.

It is interesting to point out that differently from the majority of Bos arrays in the literature, the circle of radius one does not contain any point.

The *OCS* set was constructed to optimize the condition number (in the 2 norm) of Vandermonde matrices in the Zernike basis (see also [28]), but as we shall see, its Lebesgue constant grows too rapidly and seems far from being optimal. This was observed also in the paper [22], where it is shown that in view of the results in [3], we have $\lim_{d\to} \Lambda_d^{1/d} \neq 1$, meaning a growth of Λ_d more than polynomial. Since the Fekete points of degree d have Lebesgue constants of polynomial growth in d, it turns out that the *OCS* set is not asymptotically optimal.

Better results are obtained with the *C-G* points proposed in [14] by Carnicer and Godés. Here, again, the points are located on concentric circles but the radii are

$$r_j = 1 - \left(\frac{2\,(j-1)}{d}\right)^{e_d},\ j = 1, \ldots, \ell,$$

where e_d is an ad-hoc exponent depending on the degree d obtained experimentally. The angles are

$$\theta_{j,i} = \frac{(2i-1)\pi}{2d+5-4j},\ i = 1, \dots, 2d+5-4j,\ j = 1, \dots, \ell.$$

Taking into account that for even d we have $r_\ell = 0$, for the same reasons used in *OCS* it is easy to prove that they form a Bos array. We observe that differently from *OCS*, $r_1 = 1$, and thus the unit circle contains $2d+1$ points.

The exponents e_d proposed in [14] are in $[1.3975, 1.67132]$, a generic value being 1.46 (though for large degrees this is not optimal). They will be used in the column labelled "C-G" of Table 3.5. The next column "C-G$_{\text{opt}}$" uses exponents obtained by minimizing the Lebesgue constant using `fminimax` and `fminbnd` as a function of the exponent. They are only slightly different from the *C-G* exponents and their Lebesgue constants are close.

In Table 3.4, we report the Lebesgue constants of the *C-G* sets for degrees $d = 1, \dots, 20, 23, 26, \dots, 47, 50$ and the best exponent γ_d^*. As theoretically suggested by the theorems about the asymptotical optimality of Bos arrays, though for low degrees their values are still acceptable, they seem far from being optimal when d is increased, even for γ_d^* as optimal exponent.

In Table 3.5 we compare for $d \leq 20$, the Lebesgue constants of the *OCS* and *C-G* sets. In particular the second column consists of their cardinality N, by "C-G" we refer to the sets proposed in [14], by "*C-G$_{opt}$*" those computed by us after some optimisation. For the last column, without using a certain exponent γ, we chose the radii as minimization variables, still having the distribution of the angles as in *C-G*. The Lebesgue constants in the last column, with optimal radii, are a little better, showing that the analytic formula in [14] is only nearly optimal when $d \leq 20$. As in the experiments with the other sets, we have evaluated the relevant Lebesgue functions on a WAM on the disk of degree 500 taking the maximum absolute values to determine an approximation of the Lebesgue function. We expect that at least two decimal digits are correct.

TABLE 3.4
Lebesgue constants for Carnicer-Godés pointsets with almost optimal exponent γ_d^*, for $d = 1, \dots, 20, 23, 26, \dots, 47, 50$.

d	Λ_d	γ_d^*	d	Λ_d	γ_d^*	d	Λ_d	γ_d^*
1	1.667	2.000	11	7.406	1.418	23	31.86	1.475
2	1.989	2.000	12	8.163	1.425	26	50.20	1.481
3	2.624	1.577	13	8.991	1.436	29	81.54	1.487
4	3.234	1.671	14	10.05	1.440	32	134.7	1.492
5	3.721	1.559	15	11.22	1.446	35	227.5	1.495
6	4.324	1.431	16	12.49	1.451	38	392.6	1.498
7	4.983	1.399	17	14.21	1.455	41	679.1	1.501
8	5.555	1.398	18	16.00	1.459	44	1173	1.504
9	6.182	1.407	19	18.15	1.464	47	2043	1.506
10	6.749	1.437	20	20.82	1.466	50	3661	1.507

4. Other sets. In the recent literature one may find some other sets suitable for interpolation on the unit disk.

In [15], A, Cuyt and collaborators, starting from pointset of the square, i.e. *Padua points*, tried to find good interpolation sets on the disk.

Table 3.5
Lebesgue constants for OCS and some C-G like sets.

d	N	OCS	C-G	C-$G_{\rm opt}$	opt radii
1	3	2.388	1.667	1.667	1.667
2	6	3.307	1.989	1.989	1.989
3	10	4.167	2.624	2.624	2.624
4	15	4.902	3.234	3.234	3.234
5	21	5.682	3.721	3.721	3.724
6	28	6.422	4.324	4.324	4.273
7	36	7.451	4.983	4.983	4.780
8	45	8.384	5.555	5.555	5.324
9	55	9.630	6.182	6.182	5.896
10	66	10.901	6.750	6.749	6.523
11	78	12.567	7.406	7.406	7.203
12	91	14.337	8.163	8.163	7.916
13	105	16.590	8.993	8.991	8.756
14	120	18.905	10.057	10.049	9.790
15	136	22.085	11.228	11.219	10.712
16	153	25.784	12.492	12.494	11.801
17	171	30.237	14.257	14.214	13.075
18	190	35.284	16.000	15.999	14.596
19	210	41.842	18.192	18.154	16.462
20	231	49.705	20.927	20.818	19.030

The *Padua points* (see, e.g. [6], [9], [12]), at degree d, are described as the union of the sets S_k for $k = 1, \ldots, d$ where

$$\begin{aligned} S_d &= \{(-1)^{l+1}\,(\cos(l\pi/d), 1)\,,\ l = 0, \ldots, d\} \\ &\cup \{(-1)^{l+1}\,(1, \cos(l\pi/(d+1)))\,,\ l = 1, \ldots, d\} \end{aligned} \tag{4.1}$$

and for $i = 1, \ldots, d-1$, setting $m = \lceil (i/2) \rceil$, if i is odd then,

$$S_{d-i} = \{(-1)^{l+m+1}\,(\cos(l\pi/d), \cos(m\pi/(n+1)))\,,\ l = m, \ldots, d-m\}$$

while for i even

$$S_{d-i} = \{(-1)^{l+m+1}\,(\cos(m\pi/d), \cos(l\pi/(d+1)))\,,\ l = m+1, \ldots, d-m\}.$$

Padua points $\cup_{i=1}^{d} S_i$ are, at degree d, unisolvent on the unit square $[-1,1]^2$ and it is shown that their Lebesgue constant increases like *log square* of the degree. In particular, it is easy to observe that the points of S_d lie on the boundary of $[-1,1]^2$ while $\cup_{i=1}^{d-1} S_i \subset (-1,1)^2$.

In the first set proposed in [15], the authors observe that if $P = (x, y) \in S_{d-i}$ then

$$\|P\|_\infty = \max(|x|, |y|) = \begin{cases} \cos(\lceil i/2 \rceil \pi/(d+1)), & \text{if } i \text{ is odd} \\ \cos(\lceil i/2 \rceil \pi/d), & \text{if } i \text{ is even.} \end{cases}$$

where as usual, $\lceil \cdot \rceil$ is the rounding of a number to the closer and larger integer. Furthermore, the set S_{d-k} contains k points for $k > 1$. In view of these observations they define by similitude a set on the unit disk determined by $2d+1$ equispaced points on the boundary and, from the boundary to the center of the disk, such that the union of the points on each pair of concentric circles are also distributed equidistantly as if the points were lying on only one circle. As an alternative, they also propose a second set based on taking the Padua points and mapping them to the unit disk via

$$t(x, y) = \frac{\|(x, y)\|_\infty}{\|(x, y)\|_2} (x, y) .$$

We observe that in [15] there are not too many informations about the Lebesgue constant of these sets with the exception of $d = 6$, and here we intend to fill this gap. In Table 4.1 we report the Lebesgue constants for the first and second set, for degrees $d = 1, 2, \ldots, 20$, respectively denoted as Λ_d^I, Λ_d^{II}.

Table 4.1

The Lebesgue constants of the first and second sets introduced in [15], based on Padua points, for $d = 1, \ldots, 20, 23, 26, \ldots, 47, 50$.

d	Λ_d^{II}	Λ_d^{II}	d	Λ_d^I	Λ_d^{II}	d	Λ_d^I	Λ_d^{II}
1	1.667	1.945	11	21.59	134.1	23	$5.002\ 10^2$	$2.380\ 10^5$
2	2.330	2.836	12	26.99	262.4	26	$1.190\ 10^3$	$1.765\ 10^6$
3	3.190	3.781	13	34.18	417.9	29	$2.866\ 10^3$	$1.335\ 10^7$
4	5.068	6.739	14	44.38	794.1	32	$6.972\ 10^3$	$1.086\ 10^8$
5	6.041	8.076	15	56.74	1371	35	$1.711\ 10^4$	$8.317\ 10^8$
6	7.796	12.56	16	73.03	2761	38	$4.221\ 10^4$	$7.226\ 10^9$
7	9.415	19.38	17	95.48	4870	41	$1.045\ 10^5$	$6.093\ 10^{10}$
8	11.30	31.11	18	125.0	9912	44	$2.604\ 10^5$	$5.256\ 10^{11}$
9	13.40	45.54	19	163.1	$1.729\ 10^4$	47	$6.498\ 10^5$	$4.569\ 10^{12}$
10	17.17	81.57	20	215.2	$3.289\ 10^4$	50	$1.628\ 10^6$	$4.169\ 10^{13}$

A quick comparison with the Lebesgue constants of the best Bos arrays shows that those in Table 4.1 are far from being optimal, even asymptotically much larger than *OCS* and *C-G*, and that the second set based on Padua points is worst than the first one.

In this work we tried two other sets that, at first glance, were looking promising but ended up to be quite disappointing. Their Lebesgue constants are given in Table 4.2.

In the first one, we followed the intuition coming from biology [1], where in some natural phenomena, points are distributed on spiral curves. Consequently, we considered $N = (d+1)(d+2)/2$ points $\{P_i\}_{i=1,\ldots,N}$, defined in polar coordinates by the radii $r_i = \sqrt{i/N}$, $i = 1, \ldots, N$ and the angles $\theta_i = \theta i$, $i = 1, \ldots, N$ with $\theta = (3 - \sqrt{5})\pi$ (i.e., in cartesian coordinates, $P_i = (x_i, y_i)$ where $x_i = r_i \cos(\theta_i), y_i = r_i \sin(\theta_i)$), [1].

In radial basis function (RBF) algorithms, in view of some trade-off principles, one often works with scattered data that possess good mesh norm and separation distance [16, p.135]. Having this in mind, we minimized a function $f = \sum_{i,j} p_{i,j}^6$ which is obtained by computing all the inverses $p_{i,j}$ of the points pairwise distances, so to homogenize the minimum distances. For instance, for the degree 10, the smallest minimum distance is 0.2212 and the largest minimum distance is 0.2564. The growth is not monotonic (this is due to the optimization algorithm) but we have very large Lebesgue constants, even though at the beginning they are smaller than with the "spiral" distribution.

After all this negative results one may think it is difficult to have pointsets X_d, unisolvent at degree d on the unit disk with low Lebesgue constant, meaning that Λ_{X_d} has a subexponential growth in d but, as we anticipated before, this is actually false.

To this purpose define an arbitrary basis $\{\phi_k\}_{k=1,\ldots,N}$ of $\mathbb{P}_d$ on the disk, and an

Table 4.2
Lebesgue constants for the spiral and minimum distance pointsets

d	N	spiral	distance
1	3	2.582	1.667
2	6	5.907	1.989
3	10	14.314	2.679
4	15	27.671	4.191
5	21	68.619	5.762
6	28	112.995	32.479
7	36	185.746	54.186
8	45	387.846	707.253
9	55	744.444	214.578
10	66	1460.313	361.177
11	78	2339.123	1680.033
12	91	4560.877	10088.754
13	105	9538.562	4847.249
14	120	17056.332	6924.656
15	136	36718.235	14603.544
16	153	72807.988	70758.250
17	171	114832.513	47751.926
18	190	213995.918	43603.608
19	210	315773.261	1508380.518
20	231	564236.490	260899.073

unisolvent set $\{\xi_1, \ldots, \xi_N\} \subset \Omega$ for polynomial interpolation at degree d. Define as

$$\mathrm{vdm}(\xi_1, \ldots, \xi_N, \phi_1, \ldots, \phi_N)$$

the determinant of the Vandermonde matrix relative to the chosen pointset and polynomial basis. Let

$$\ell_j(x) = \frac{\mathrm{vdm}(\xi_1, \ldots, \xi_{j-1}, x, \xi_{j+1}, \ldots, \xi_N, \phi_1, \ldots, \phi_N)}{\mathrm{vdm}(\xi_1, \ldots, \xi_{j-1}, \xi_j, \xi_{j+1}, \ldots, \xi_N, \phi_1, \ldots, \phi_N)}$$

define the cardinal basis. Next find the set $\Xi^* = \{\xi_1^*, \ldots, \xi_N^*\} \subset \Omega$ that between all the possible $\{\xi_1, \ldots, \xi_N\} \subset \Omega$ maximizes the denominator

$$\mathrm{vdm}(\xi_1, \ldots, \xi_{j-1}, \xi_j, \xi_{j+1}, \ldots, \xi_N, \phi_1, \ldots, \phi_N).$$

It is clear that this set exists and that $\|\ell_j\|_\infty \leq 1$ for all $j = 1, \ldots, N$. As a consequence the Lebesgue constant is smaller than the dimension N of $\mathbb{P}_d$ since

$$\Lambda_{\Xi^*} = \max_{x \in \Omega}(\sum_{i=1}^{N} |\ell_j(x)|) \leq N.$$

The points maximizing the denominator are known as Fekete points and in general these points are difficult to compute and known only in few cases. As shown in [7] and [8], a set named *Approximate Fekete Points* (there shortened as AFP), shares some of their asymptotical properties, i.e. they are asymptotically equidistributed with respect to the pluripotential-theoretic equilibrium measure of the compact set, with the advantage of being available from a WAM of degree d by using linear algebra algorithms.

Another remarkable class of pointsets are the so called *continuous Leja points*, that we briefly introduce. Suppose that $\Omega \subset \mathbb{C}^k$ be a continuous or discrete compact set and that we have an ordered basis $\Phi = \{\phi_1, \ldots, \phi_N\}$ of $\mathbb{P}_d(\Omega)$. The Leja points of degree d, w.r.t. Φ are defined as follows:

1. ξ_1 is the point in which the maximum of ϕ_1 in Ω is achieved;
2. once $\xi_1, \ldots, \xi_j$ are available, ξ_{j+1} is the point that maximizes

$$\mathrm{vdm}(\xi_1, \ldots, \xi_j, \xi_{j+1}, \phi_1, \ldots, \phi_{j+1}).$$

One interesting property of the continuous Leja points for multivariate interpolation is that they are nested, in the sense that the points used for interpolation at degree $k_1 \leq d$ are still used for each degree k_2 such that $k_1 \leq k_2 \leq d$. For some theoretical results in the complex plane see [26].

In [7, p.1992], the authors introduced by a greedy algorithm that determines via LU decomposition with partial pivoting a set named *Discrete Leja Points* (there shortened as DLP) that shares again the same asymptotical properties of the AFP, being again asymptotically equidistributed with respect to the pluripotential-theoretic equilibrium measure of the compact set. One of their peculiarities, similar to the continuous Leja sets, is that if the basis is chosen so that the first $N_j = dim(\mathbb{P}^k)$ basis elements span $\mathbb{P}^k$, then the first N_j Leja points are a candidate set of interpolation points for polynomials of degree j.

In spite of this, at the moment it is not known theoretically if the growth of the Lebesgue constant of AFP or DLP is asymptotically optimal.

Table 4.3
Lebesgue constants of AFP and DLP.

d	N	AFP	DLP
1	3	1.667	3.000
2	6	1.989	5.000
3	10	2.630	7.000
4	15	3.190	9.000
5	21	3.852	11.389
6	28	4.385	13.000
7	36	5.583	15.000
8	45	6.447	20.520
9	55	7.021	26.061
10	66	7.845	24.372
11	78	8.919	56.768
12	91	14.850	44.854
13	105	17.022	90.575
14	120	18.515	92.292
15	136	21.441	61.321
16	153	28.910	59.927
17	171	30.381	99.063
18	190	32.705	94.945
19	210	41.256	153.845
20	231	32.129	196.884

We observe in Table 4.3 that for the AFP the growth of the Lebesgue constant is smaller than for the DLP. In particular, there are some anomalies at degree $d = 19$ and $d = 20$, while for the DLP it is clear that the growth is not monotonic. This can be explained by the fact that already the true Fekete and Leja points are not defined to minimize the Lebesgue constant, but to maximize some Vandermonde determinants.

5. Our methodology. In this section we show how we computed unisolvent pointsets

$$X_d^* = \{(x_k^*, y_k^*)\}, k = 1, \ldots, (d+1)(d+2)/2$$

at degree d on the unit disk B_2, so that their Lebesgue constants $\Lambda_d^* = \Lambda_{X_d^*}$ are as small as possible.

First, we determine a set of Approximate Fekete Points or Discrete Leja Points of degree d on the disk.

Once such an initial set is at hand, we apply several optimization routines, i.e. sequentially *fminimax*, *fmincon*, *fminsearch*, *greedy algorithm* (a variant of the second algorithm in [27]). Each one of them, at each level L, if X is the current set, has to evaluate several times a target function, that depending on the method is the Lebesgue function (as for *fminimax*, *greedy algorithm*) or the Lebesgue constant (as for *fmincon*, *fminsearch*). In our case, at level L we evaluate the Lebesgue function λ_X on a WAM, say Y_δ, of degree δ, where $\delta = 30 \cdot \lceil d/5 \rceil L$ and if required by the method, compute its maximum.

Observe that the larger the level L, the larger is the cardinality M_L of the mesh Y_δ, and consequently we can expect that the more accurate is the approximation of the Lebesgue constant Λ_X, the more time consuming the evaluation of the target function. Relatively to Matlab built-in optimizers, the performance depends on some specific parameters. We used in *fminimax*, *fminsearch*, *fmincon*, for

- $L < 4$: `TolX`$=10^{-6}$, `TolFun`$=10^{-6}$,
- $L \geq 4$: `TolX`$=10^{-14}$, `TolFun`$=10^{-14}$.

where TolFun is the function tolerance and TolX is the step tolerance of the algorithm (for details see Matlab documentation on *Tolerances and Stopping Criteria*).

Furthermore, we also assigned independently of the level L, `DiffMaxChange`$=10^{-3}$, `MaxFunEvals`$= 25000$, `MaxIter`$=25000$. As for the routine *fmincon*, we specified *active-set* as the algorithm variable. About the *greedy-algorithm*, we started from the second *greedy-algorithm* introduced in [27], that is defined in its simpler version as follows. Suppose that $X = \{x_k\}_{k=1,\ldots,N} \subset \Omega$ is the initial point set and that $Y = \{y_k\} \subset \Omega$ is a fine discretization of the domain Ω. Let $X^{(0)} = X$. For $k = 1, \ldots, N$, first consider $\tilde{X}_k^{(0)} = X_{k-1}^{(0)} \backslash \{x_k\}$, and then determine $y_k \in Y$ such that

$$\Lambda_{X_k^{(0)}}(y_k) = \max_{y \in Y} \lambda_{X_k^{(0)}}(y).$$

Finally take $X_k^{(0)} = \tilde{X}_k^{(0)} \cup \{y_k\}$. In case that the final mesh $X_N^{(0)}$ is (sufficiently) better than the initial one $X^{(0)}$, one keeps iterating the procedure.

In our experiments we consider a variant, named in what follows *greedy algorithm*, that accepts the new point set $X_k^{(0)}$ if the approximation of the Lebesgue constant, i.e. the maximum of the Lebesgue function on a WAM of degree 750, is smaller than the best one previously computed, i.e. $\min_{j=0,\ldots,k-1} \Lambda_{X_j^{(0)}}$, otherwise it discards it by setting $X_k^{(0)} = X_{k-1}^{(0)}$.

Observe, that any method must determine at any level, the abscissae and the ordinates of each element of the point set X with low Lebesgue constant, and consequently the optimization problems involve in the case of the unit disk $2N = (d+1)(d+2)$ variables, showing how difficult its numerical solution may become for $d > 20$.

We finally stress that the function that maps X into Λ_X is not differentiable, and thus we cannot benefit of optimization algorithms typical of regular maps.

In Table 5.1 we report our numerical experiments for $d = 15$. First we run *fminimax* for levels $L = 2, 3, \ldots, 9$ and then sequentially *fmincon*, *fminsearch*, *greedy algorithm*.

The Lebesgue constant of each pointset proposed by any method and at any level, is estimated using a very fine mesh, i.e. a WAM of degree 750 (whose cardinality is of

≈ 563000 points). In this case, it is clear that after the application of several levels of *fminimax*, we achieve a pointset with Lebesgue constant approximately 8.08960. The further application of *fmincon*, *fminsearch*, *greedy algorithm* does not change much its value (we reach a final value of 8.0860).

In the tables we use the acronym *fmini.*, *fminc.*, *fmins.*, *greedy* respectively for the *fminimax*, *fmincon*, *fminsearch*, *greedy algorithm* optimizers.

TABLE 5.1

Degree $d = 15$. Initial DLP set has Lebesgue constant $\Lambda_{15}^{(0)} \approx 87.4475$, while the final set has Lebesgue constant $\Lambda_{15}^{(1)} \approx 8.08601$ (after the optimization process). The application of `fminimax` *and fminsearch for further levels L provided a point set with an almost optimal Lebesgue constant $\Lambda_{15}^{*} \approx 8.00$.*

L	*fmini.*	*fminc.*	*fmins.*	*greedy*
2	11.4582	8.6909	8.2740	8.0896
3	9.2039	8.2226	8.2368	8.0896
4	8.5221	8.2581	8.2368	8.0881
5	8.3096	8.2309	8.2099	8.0860
6	8.2723	8.1544	8.1967	8.0860
7	8.1603	8.1350	8.1801	8.0860
8	8.1302	8.1553	8.1267	8.0860
9	8.0896	8.1020	8.1142	8.0860

Though there is no actual mathematical reason but just numerical evidence, this suggested to us that somehow it is not much valuable to apply sequentially all the methods but just *fminimax*. We experimented that further improvements can be obtained with higher values of L, though the computations for large L may be much more time consuming. We point out that such values of the Lebesgue contants are only an approximation of the real ones, and very likely only three decimal digits are exact and consequently the value obtained by the refinement via the greedy algorithm may not even represent a better set.

Consequently, one can use exclusively *fminimax* (for $L = 2, \ldots$) followed by *fminsearch* or *greedy algorithm* hoping to find some minor refinements.

6. Comparisons with previous results. Some pointsets with low-Lebesgue constant were also computed in other works and we intend to compare our results with those. In what follows, unless otherwise stated, the final Lebesgue constant is computed on a WAM of order 500 that is, 250501 test points.

In Table 6.1 we have reported the values of the Lebesgue constants of some sets numerically determined. In particular,

- the third column stores the results in Gunzburger and Teckentrup [18], that uses an approach very similar to ours, with a different evaluation of the Lebesgue constant. The pointsets were available at [19] and tested. Some points were outside the domain by a very little quantity. In these instances, we modified the radii so that the new points were in the unit disk. We report here their Lebesgue constant on a WAM of degree 750. The results are practically equal to those presented in [18]. We used an asterisk to point out Lebesgue constants different from the original paper [18];
- the fourth column gives the results in Van Barel, Humet and Sorber [27]. The pointsets are not available but a plot of their Lebesgue constants is given in

Table 6.1

Table of Lebesgue constants of several sets, for degrees $d \leq 25$. In the last column we report the pointsets with the best Lebesgue constant computed in this work

d	N	[18]	[11]	[27]	**MS**
1	3	1.67		1.67	1.67
2	6	1.99		1.99	1.99
3	10	2.47	2.50	2.47	2.47
4	15	2.95	3.00	2.97	2.96
5	21	3.39	3.80	3.50	3.39
6	28	3.85	4.00	4.30	3.85
7	36	4.34*	4.40	5.08	4.32
8	45	4.85*	5.50	5.43	4.76
9	55	5.21*	5.60	6.73	5.21
10	66	5.70*	6.50	7.62	5.67
11	78	–	6.70	8.48	6.10
12	91	–	7.70	9.65	6.57
13	105	–	8.00	11.98	6.95
14	120	–	8.20	12.39	7.52
15	136	–	9.40	14.13	8.00
16	153	–	10.00	17.44	8.35
17	171	–	10.20	–	8.85
18	190	–	10.80	–	9.20
19	210	–	11.80	–	9.70
20	231	–	12.00	–	10.17
21	253	–	13.00	–	10.99
22	276	–	13.20	–	11.50
23	300	–	14.30	–	11.89
24	325	–	14.50	–	12.28
25	351	–	14.80	–	12.96

[27]. We report here a rough approximation of their values by extrapolating the results from the plot. Their control mesh is different from the one given in this paper, but due to the size of the pointsets there used, the results should be comparable to the other columns in this paper;

- the fifth column describes the Lebesgue constants of the pointsets in Briani, Sommariva, Vianello [11]. As reported in [27], the results were far from the optimal Lebesgue constant for degrees $d > 9$;
- the last column in which appears the acronym *MS* considers the results obtained by the procedure described in the previous section. As initial set we used the Discrete Leja Points at degree d or Approximate Fekete Points, reaching level $L = 20$. We stress for all $d \leq 25$, the Lebesgue constant so obtained is equal or less than the best previously known.

All the given pointsets are available at the address given in reference [24].

It is interesting to study, at least numerically, the growth of the sets *MS*. First, observe that the interpolation operator $\mathcal{I}_d$ on an unisolvent pointset X at degree d, is a linear projection operator on $\mathbb{P}_d$, i.e. it is surjective and $\mathcal{I}_d \circ \mathcal{I}_d = \mathcal{I}_d$.

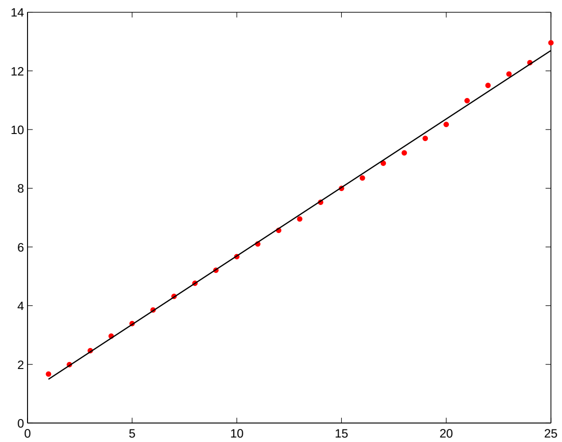

FIG. 6.1. *In red, the plot of the Lebesgue constants of the pointsets computed in this paper for $d = 1, 2, \ldots, 25$. In black, the straight line $y = 1.0244 + 0.4668d$.*

By a result due to Sündermann [25, p. 116], for every projection $L : C(\Omega, \infty) \to \mathbb{P}_d$ on the unit disk Ω, there exist two constants $c_1, c_2 > 0$ such that

$$c_1 d^{1/2} \leq \|L\|_\infty \leq c_2 d^{1/2}. \tag{6.1}$$

In view of (6.1), with $\|\mathcal{I}_d\|_\infty = \Lambda_X$, we get that the growth of the Lebesgue constant for any set is in the best case $\Lambda_X \leq c_1 d^{1/2}$. A plot of the values of Λ_{X_d}, where X_d are the unisolvent sets *MS* at degree d, shows that for $d \leq 25$ the Lebesgue constant lies close to the straight line $y = 1.0244 + 0.4668d$, thus we conjecture that the best Lebesgue constant is actually asymptotically proportional to d and hence not the optimal projection.

7. Conclusion. In this paper we tested the Lebesgue constant of several pointsets on the disk Ω. We started from some type of Bos-arrays, as those presented in [14], [15], [22] remembering that by a theorem in [3] they cannot be optimal. We extended the results in [15], analysing the case of Bos arrays whose radii are nonnegative Gegenbauer-Gauss-Lobatto nodes with exponent α. We noticed that the optimal α, in spite of the results in [3] still allow to achieve pointsets on Ω with low Lebesgue constant Λ_n for degrees $n \leq 30$.

Next we proposed an algorithm that through optimization allowed us to achieve the best Lebesgue constants now available (see [24] for the pointsets described as Matlab files).

In the Appendix we proved that a certain set is optimal for degree $d = 1$, as well as some considerations of the case $d = 2$, $d = 3$.

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Appendix: Cases of degrees one, two and three. When the total degree of the bivariate polynomials is $d = 1$ the best Lebesgue constant can be computed exactly. It is equal to $5/3 = 1.6666666666\ldots$ Let us see how we can prove that result. The Lagrange polynomials of degree 1 define planes in the three-dimensional space. It is possible to prove that the three control points must be as far as possible to each other, even though the proof is too lengthy to be given here. Therefore they are located on the unit circle and are vertices of an equilateral triangle. Any rotation of the three points yield the same Lebesgue constant. Let us choose the three points as

$$P_1 = (x_1, y_1) = (0, 1),\ P_2 = (x_2, y_2) = (\sqrt{3}/2, -1/2),\ P_3 = (x_3, y_3) = (-\sqrt{3}/2, -1/2).$$

The Lagrange polynomials ℓ_i, $i = 1, 2, 3$ can be written as $\ell_i(x, y) = \alpha_i x + \beta_i y + \delta_i$. The coefficients are obtained by writing that the value of ℓ_i is equal to 1 at point P_i and 0 at the other two points. For ℓ_1 they are given by the solution of the linear system

$$\begin{pmatrix} 0 & 1 & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The solution is the first column of the inverse of the matrix. The coefficients related to the other points are given by the other two columns of the inverse which is

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Hence, the three Lagrange polynomials are

$$\begin{aligned} \ell_1(x, y) &= \frac{2}{3}y + \frac{1}{3}, \\ \ell_2(x, y) &= \frac{1}{\sqrt{3}}x - \frac{1}{3}y + \frac{1}{3}, \\ \ell_3(x, y) &= -\frac{1}{\sqrt{3}}x - \frac{1}{3}y + \frac{1}{3}. \end{aligned}$$

The three edges of the equilateral triangle partition the unit disk in four regions that we denote by O, I, II, III. The region O is the triangle, I is the region opposed to P_1, II opposed to P_2 and III opposed to P_3.

The Lagrange polynomial ℓ_1 is positive in O, II, III and negative in I. The Lagrange polynomial ℓ_2 is positive in O, I, III and negative in II. The Lagrange polynomial ℓ_3 is positive in O, I, II and negative in III. Hence, we have the expressions of the Lebesgue function λ_X in each region,

$$\begin{aligned} O : \lambda_X &= \ell_1 + \ell_2 + \ell_3 = 1, \\ I : \lambda_X &= -\ell_1 + \ell_2 + \ell_3 = -\frac{4}{3}y + \frac{1}{3}, \\ II : \lambda_X &= \ell_1 - \ell_2 + \ell_3 = -\frac{2}{\sqrt{3}}x + \frac{2}{3}y + \frac{1}{3}, \\ III : \lambda_X &= \ell_1 + \ell_2 - \ell_3 = \frac{2}{\sqrt{3}}x + \frac{2}{3}y + \frac{1}{3}. \end{aligned}$$

The Lebesgue function is constant in the equilateral triangle O and linear in the other regions being zero on the edges of the triangle. It is increasing towards the unit circle. Hence, the maxima occurs on the boundary of the unit disk.

Let us first consider the intersection of the plane in region I ($z = -4/3y+1/3$) with the cylinder defined by the unit circle ($x^2+y^2 = 1$). The intersection is parameterized by $(x(\theta), y(\theta), z(\theta))$ with

$$x(\theta) = \cos(\theta), \quad y(\theta) = \sin(\theta), \quad z(\theta) = -\frac{4}{3}\sin(\theta) + \frac{1}{3}, \quad \theta \in [\frac{7\pi}{6}, \frac{11\pi}{6}].$$

The maximum occurs for $\cos(\theta) = 0$ that is, $\theta = \frac{3\pi}{2}$. Hence, $x = 0$ and $y = -1$. It gives the value of $z = 5/3$ which is the Lebesgue constant.

In region II we have

$$z(\theta) = -\frac{2}{\sqrt{3}}\cos(\theta) + \frac{2}{3}\sin(\theta) + \frac{1}{3}.$$

Computing the derivative we obtain that the maxima is given for $\tan(\theta) = \sqrt{3}$ that is, $\theta = 5\pi/6$. It yields

$$x = -\frac{\sqrt{3}}{2}, \quad y = \frac{1}{2}, \quad z = \frac{5}{3}.$$

By symmetry in region III we obtain

$$x = \frac{\sqrt{3}}{2}, \quad y = \frac{1}{2}, \quad z = \frac{5}{3}.$$

Therefore, the Lebesgue constant for $d = 1$ is $5/3$.

Things are a little more complicated for $d = 2$ but still manageable. Here we have $N = 6$ points. The best configuration is to put five equally spaced points on the boundary and one point inside the unit disk. The Lagrange polynomial corresponding to the interior point is positive everywhere. The best position of this point is at the origin because then, the maximum of the polynomial is equal to one.

Let us put the first point P_1 at $(0, 1)$. The other points $P_2, \dots, P_5$ are numbered clockwise and separated by an angle of $2\pi/5$. In polar coordinates their angles are

$$\frac{\pi}{10}, \quad \frac{17\pi}{10}, \quad \frac{13\pi}{10}, \quad \frac{9\pi}{10}.$$

The cartesian coordinates of the points are

	1	2	3	4	5	6
x	0	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$-x_3$	$-x_2$	0
y	1	$\frac{\sqrt{5}-1}{4}$	$-\frac{1+\sqrt{5}}{4}$	y_3	y_2	0

The Lagrange polynomials can be written as

$$\ell_i(x, y) = \alpha_i x^2 + \beta_i xy + \gamma_i y^2 + \delta_i x + \omega_i y + \nu_i.$$

The coefficients are given by the entries in the columns of the inverse of the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_3^2 & -x_3y_3 & y_3^2 & -x_3 & y_3 & 1 \\ x_2^2 & -x_2y_2 & y_2^2 & -x_2 & y_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It turns out that the inverse has the following structure

$$M^{-1} = \begin{pmatrix} -\frac{1}{5} & \alpha_2 & \alpha_3 & \alpha_3 & \alpha_2 & -1 \\ 0 & \beta_2 & \beta_3 & -\beta_3 & -\beta_2 & 0 \\ \frac{3}{5} & \gamma_2 & \gamma_3 & \gamma_3 & \gamma_2 & -1 \\ 0 & \delta_2 & \delta_3 & -\delta_3 & -\delta_2 & 0 \\ \frac{2}{5} & -\gamma_2 & -\gamma_3 & -\gamma_3 & -\gamma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It can be shown that

$$\alpha_2 = \frac{3+\sqrt{5}}{10}, \quad \alpha_3 = \frac{3-\sqrt{5}}{10},$$

$$\gamma_2 = \frac{1-\sqrt{5}}{10}, \quad \gamma_3 = \frac{1+\sqrt{5}}{10}.$$

We do not give the values of the coefficients β_i and δ_i because we will see that we do not need them.

We have already said that the Lagrange polynomial ℓ_6 is always positive. The other polynomials change sign inside the unit disk. For each point on the boundary the corresponding polynomial partitions the unit disk in three regions bounded by implicit second order polynomials in (x, y). As an example, consider the point P_1 in Figure 8.1. The Lagrange polynomial ℓ_1 is positive in regions I and III and negative in region II. The curve separating the regions is defined implicitly by $x^2+3y^2+2y = 0$.

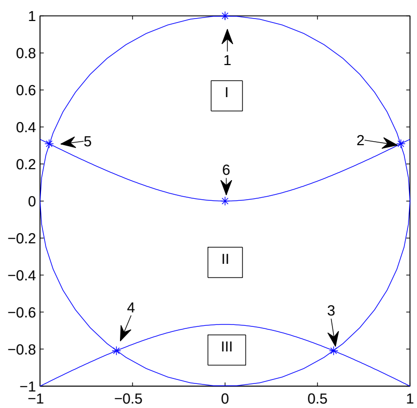

FIG. 8.1. *Regions for the point* P_1

By symmetry the maxima of the Lebesgue function occur on the boundary at points which are at mid distances of the points P_i, $i = 1, \ldots, 5$. In particular, there is a maximum at $(0, -1)$. By looking at the signs of the Lagrange polynomials we have that

$$\lambda_x(0,-1) = \ell_1(0,-1) - \ell_2(0,-1) + \ell_3(0,-1) + \ell_4(0,-1) - \ell_5(0,-1) + \ell_6(0,-1).$$

But

$$\ell_i(0,-1) = \gamma_i - \omega_i + \nu_i, \; i = 1, \ldots, 6.$$

From what we have seen above the Lebesgue constant is

$$\lambda_x(0,-1) = \frac{1}{5} - 4\gamma_2 + 4\gamma_3 = \frac{1}{5} + \frac{4}{\sqrt{5}} = 1.988854381999832.$$

The last value is the rounded value given by Matlab. Notice that, when rounded, this is the value found by the optimization algorithms.

It is much more difficult to find an optimal distribution of the $N = 10$ points for $d = 3$. Let us order the monomials of the bivariate polynomials as

$$x^3,\ x^2y,\ xy^2,\ y^3,\ x^2,\ xy,\ y^2,\ x,\ y,\ 1.$$

To be able to compute an interpolation polynomial the set of control points has to be unisolvent. It means that the generalized Vandermonde matrix whose rows are

$$V_{i,:} = \begin{pmatrix} x_i^3 & x_i^2 y_i & x_i y_i^2 & y_i^3 & x_i^2 & x_i y_i & y_i^2 & x_i & y_i & 1 \end{pmatrix}, \quad i = 1, \ldots, N,$$

where $\xi_i = (x_i, y_i)$, $i = 1, \ldots, N$ are the control points, must be nonsingular.

If we put the 10 points on the boundary of the unit disk we have $y_i^2 = 1 - x_i^2, \forall i$. Hence one column of V is a linear combination of two other columns and the matrix V is singular. This happens also if we put 9 points on the boundary and one point at the origin.

From our numerical experiments it seems that a good distribution is to have 7 points on the boundary and 3 points inside the unit disk. Let us put the 7 evenly distributed points on the boundary at angles

$$\frac{\pi}{2},\ \frac{3\pi}{14},\ \frac{27\pi}{14},\ \frac{23\pi}{14},\ \frac{19\pi}{14},\ \frac{15\pi}{14},\ \frac{11\pi}{14}.$$

The 3 points inside are located on a circle centered at the origin with a radius of 0.5 with angles

$$\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}.$$

Rotating these three inner points does not change too much the Lebesgue constant. Using the same techniques as above one can show that, with this distribution of points, a maximum of the Lebesgue function occurs exactly at the point $(0, -1/\sqrt{2})$. Its value is $2.62435957\ldots$ This gives an upper bound of the Lebesgue constant for $d = 3$. From our numerical computations (which give a lower bound) it yields that the exact Lebesgue constant is in [2.47 2.625) very probably close to 2.47.

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