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Enumerative formulas of de Jonquières type on algebraic curves

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Abstract

This thesis is dedicated to the study of two enumerative geometry problems in the context of linear series on algebraic curves.

The first problem is that of settling the issue of the validity of the de Jonquières formulas. These formulas compute the number of divisors with prescribed multiplicity, or de Jonquières divisors, contained in a linear series on a smooth projective curve. To do so, we construct the space of de Jonquières divisors as a determinantal cycle of the symmetric product of the curve and prove that, for a general curve with a general linear series, it is of expected dimension. In doing so, we describe the degenerations of de Jonquières divisors to nodal curves using both limit linear series and compactified Picard schemes.

The second problem deals with cycles of subordinate or, more generally, secant divisors to a given linear series on a curve. We consider the intersection of two such cycles corresponding to secant divisors of two different linear series on the same curve and investigate the validity of the enumerative formulas counting the number of divisors in the intersection. We study some interesting cases, with unexpected transversality properties, and establish a general method to verify when this intersection is empty.

Zusammenfassung

Diese Arbeit widmet sich der Untersuchung von zwei Problemen der abzählenden Geometrie im Zusammenhang mit linearen Systemen auf algebraischen Kurven.

Das erste Problem besteht darin, die Frage der Gültigkeit der Jonquières-Formeln zu klären. Diese Formeln berechnen die Anzahl von Divisoren mit vorgeschriebener Multiplizität, genannt de Jonquières-Divisoren, die in einem linearen System auf einer glatten projektiven Kurve enthalten sind. Um dies zu tun, konstruieren wir den Raum der de Jonquières-Divisoren als einen Determinantenzyklus des symmetrischen Produkts der Kurve und beweisen, dass er für eine allgemeine Kurve die erwartete Dimension hat. Dabei beschreiben wir die Degenerationen der Jonquières-Divisoren zu den Knotenkurven sowohl mit linearen Systemen als auch mit kompaktifizierten Picard-Schemata.

Das zweite Problem behandelt Zyklen von Untergeordneten-, oder allgemeiner, Sekanten-Divisoren zu einem gegebenen linearen System auf einer Kurve. Wir betrachten den Durchschnitt zweier solcher Zyklen, die Sekanten-Divisoren von zwei verschiedenen linearen Sytemen auf der gleichen Kurve entsprechen, und untersuchen die Gültigkeit der enumerativen Formeln, die die Anzahl der Teiler im Durchschnitt zählen. Wir untersuchen einige interessante Fälle mit unerwarteten Transversalitätseigenschaften und etablieren eine allgemeine Methode, um zu überprüfen, wann dieser Durchschnitt leer ist.

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Conventions

Curve C	complete reduced algebraic curve over $\mathbb{C}$
C_d	d-th symmetric product of the curve C
$h^i(C, L)$	dimension of $H^i(C, L)$ for a line bundle L on C
$\mathrm{Pic}^d(C)$	Picard group of line bundles of degree d on C
$W^r_d(C)$	subvariety of $\mathrm{Pic}^d(C)$ of line bundles L with $h^0(C, L) \geqslant r + 1$
Linear series on C	pair (L, V) where V is a vector subspace of $H^0(C, L)$
g^r_d	linear series of degree d and dimension r on a curve
$\lvert L \rvert$	for a line bundle L, stands for the linear series $(L, H^0(C, L))$
$\lvert D \rvert$	for a divisor $D \in C_d$, stands for the linear series $\lvert \mathcal{O}_C(D) \rvert$
$G^r_d(C)$	variety of g^r_d-s on C
$\rho(g, r, d)$	Brill-Noether number
K_C	canonical line bundle on C; used also for canonical divisor and canonical linear series
$DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$	space of de Jonquières divisors of length N of the series l of type g^r_d determined by the partitions μ_1 and μ_2 of length k
$\Gamma_e(l)$	space of effective divisors of degree e subordinate to the linear series l
$V^{e-f}_e(l)$	space of effective divisors of degree e imposing at most $e - f$ independent conditions on l

Chapter 1

Introduction

The goal of algebraic geometry is the understanding of geometric manifestations of solutions to systems of polynomial equations, or in other words, the study of the properties of algebraic varieties. One of the oldest avenues of research in algebraic geometry is enumerative geometry, whose aim is to compute the number of objects satisfying certain geometric conditions. The subject saw significant development towards the end of the 19th century due to Hermann Schubert [Sch79], who introduced Schubert calculus for enumerative problems in projective geometry. This is a powerful tool, some of whose more computational aspects are still of interest today, and acted as a precursor to both characteristic classes and intersection theory. As his 15th problem, Hilbert asked at the beginning of the 20th century that a rigorous foundation for Schubert calculus, and more generally for enumerative geometry, be established. While Schubert calculus and its use of the "principle of conservation of number" has been put on firm ground via topology and intersection theory on Grassmannians by Kleiman and Laksov [KL72b], the broader question about enumerative geometry is still open. Perhaps the best way to understand where the issues come from is to describe the anatomy of an enumerative geometry problem. The first important observation is that in algebraic geometry the objects of study fall naturally into families that are themselves parametrised by other schemes, called *moduli* or *parameter spaces*. Thus, the objects of interest in the enumerative problem are parametrised by some moduli space and the conditions that they have to satisfy cut certain subschemes within it. Hence the enumerative question becomes an intersection theory problem on the moduli space and the sought-after number is given by the class of the intersection of the subschemes giving the conditions. However, in order for the resulting class computation to give a sensible answer, one needs to verify that the intersection is indeed non-empty, of expected dimension, and transverse, so that the counts occur without multiplicity.

In this thesis we study the issues of expected dimension and transversality for two beautiful enumerative geometry problems concerning algebraic curves

in the context of Brill-Noether theory. In the remainder of this chapter we first explain what we mean by "Brill-Noether theory" and how we understand its place in the wider framework of algebraic geometry, and more specifically, in the theory of algebraic curves. We then describe the problems of interest, namely that of de Jonquières divisors and that of intersections of secant varieties on an algebraic curve and state our main results. In Chapter 2 we give a summary of the techniques used in approaching our problems, i.e. degenerations to nodal curves using limit linear series and compactified Picard schemes. Finally, Chapter 3 and Chapter 4 are dedicated to the proofs of our results concerning de Jonquières divisors and intersections of secant varieties, respectively.

1.1 Brill-Noether theory: a brief summary

The developments in curve theory that led to the understanding of Brill-Noether theory that we have today mirror those that occurred in other fields of mathematics between the 19th and the 20th centuries. The significant paradigm shift that took place in this time was characterised by the change in the way mathematicians thought about their fundamental objects of study: from sub-objects of some ambient object to an abstract entity with extra structure. Indeed, this change can be recognized, for example, in the study of groups: while in the 19th century they were studied as subsets of GL_n, closed under matrix multiplication and inversion, in the 20th century the notion of abstract group was introduced. This prompted the separation of group theory into two complementary subjects: the study of the structure of abstract groups on the one hand, and representation theory, i.e. the study of the ways in which a given abstract group can be realised as a subgroup of GL_n on the other.

A similar development took place in algebraic geometry and in particular in the study of algebraic curves. From the classical, 19th century, point of view, a curve is a subset of projective space defined by polynomial equations, with three associated discrete invariants: its genus g, its degree d, and the dimension of the projective space r. The classification question is therefore equivalent to establishing which triples (d, g, r) may occur. In modern language, the question is reformulated by means of a moduli space, namely the *Hilbert scheme* $H^0_{d,g,r}$ parametrising all such smooth non-degenerate curves in $\mathbb{P}^r$ of degree d and genus g and the classification is reduced to describing all components of the Hilbert scheme.

In the 20th century, the modern notion of an abstract curve took hold, with only one discrete invariant, the genus g, and parametrised by the moduli space $\mathcal{M}_g$. The classification of all curves in this case becomes the problem of describing the geometry of $\mathcal{M}_g$.

Much like representation theory for groups, Brill-Noether theory relates these two points of view by attempting to describe all the ways in which an

abstract curve C may be mapped to the projective space $\mathbb{P}^r$. This entails the study of the set of all non-degenerate maps $C \to \mathbb{P}^r$ which has the structure of a scheme, denoted $G^r_d(C)$. More precisely, the questions seen as being in the purview of Brill-Noether theory are whether $G^r_d(C)$ is empty or not, what is its dimension, whether it is irreducible, or smooth and if not, what its singular locus is, and so on.

In what follows we make a summary of the main results of Brill-Noether theory, with an emphasis on those that are relevant to our enumerative problems.

Let C be a smooth curve of genus g. To avoid redundancies, we restrict our attention to *non-degenerate* maps, i.e. maps $f : C \to \mathbb{P}^r$ whose image does not lie in a hyperplane. The *degree* of the map f denotes the degree of the pullback divisor f^*H for any hyperplane $H \subset \mathbb{P}^r$. In Brill-Noether theory, this setup corresponds to the following data:

1. a line bundle L of degree d on C,
2. an $(r+1)$-dimensional vector space $V \subseteq H^0(C, L)$ of sections of L with no common zeroes.

We call the pair $l := (L, V)$ a *linear series of degree* d *and dimension* r *on* C and we denote it g^r_d for short. The condition that the sections in V have no common zeroes is expressed by saying that l is *base point free*. Unfortunately, it turns out that the set of base point free g^r_d-s is not a complete variety, so in what follows we consider the set $G^r_d(C)$ of *all* g^r_d-s, base point free or not. We also have the related spaces

$$W^r_d(C) = \{L \in \mathrm{Pic}^d(C) \mid h^0(C, L) \geqslant r+1\} \subseteq \mathrm{Pic}^d(C)$$

and

$$C^r_d = \{D \in C_d \mid h^0(C, \mathcal{O}_C(D)) \geqslant r+1\},$$

where $\mathrm{Pic}^d(C)$ denotes the Picard variety of degree d line bundles on C and C_d is the d-th symmetric product of the curve. For the construction of the above spaces we refer the reader to Chapter IV of [ACGH85]. We note here only that $W^r_d(C)$ is obtained as a degeneracy locus in $\mathrm{Pic}^d(C)$ and C^r_d is its preimage under the *Abel-Jacobi map*

$$\begin{aligned} u : C_d &\to \mathrm{Pic}^d(C) \\ D &\mapsto \mathcal{O}_C(D). \end{aligned}$$

The space $G^r_d(C)$ is constructed as the canonical blow-up of the determinantal variety $W^r_d(C)$.

Finally, we define the Brill-Noether number $\rho(g, r, d)$ by

$$\rho(g, r, d) = g - (r+1)(g - d + r).$$

Consider now the forgetful map

$$\phi : H^0_{d,g,r} \to \mathcal{M}_g.$$

The main results of Brill-Noether theory from the point of view of moduli of curves state that

1. if $\rho(g, r, d) < 0$, then the map ϕ is not dominant and
2. if $\rho(g, r, d) \geqslant 0$, then there exists a unique irreducible component $\mathcal{H}_{d,g,r}$ of the Hilbert scheme $H^0_{d,g,r}$ such that the restriction $\phi|_{\mathcal{H}_{d,g,r}}$ is dominant.

We call this unique component $\mathcal{H}_{d,g,r}$ the *Brill-Noether component* of the Hilbert scheme and its general point a *Brill-Noether general curve*.

In other words, we have the three main theorems concerning the non-emptiness, dimension and smoothness of the spaces of linear series:

Theorem 1.1 ([ACGH85, Proposition 1.1, Chapter V]). *For any smooth curve* C *of genus* g*, if* $\rho(g, r, d) \geqslant 0$*, then* $W^r_d(C) \neq \emptyset$ *and* $G^r_d(C) \neq \emptyset$*. Furthermore, every component of* $G^r_d(C)$ *has dimension at least equal to* $\rho(g, r, d)$*. The same is true for* $W^r_d(C)$ *provided* $r \geqslant d - g$*.*

Theorem 1.2 ([ACGH85, Theorem 1.5, Chapter V]). *If* C *is a general curve of genus* g *and* $\rho(g, r, d) < 0$*, then* $G^r_d(C)$ *is empty. If* $\rho(g, r, d) \geqslant 0$*, the* $G^r_d(C)$ *is reduced and of pure dimension* $\rho(g, r, d)$*.*

Theorem 1.3 ([ACGH85, Theorem 1.6, Chapter V]). *For a general curve* C *of genus* g*, the space* $G^r_d(C)$ *is smooth of dimension* $\rho(g, r, d)$*.*

Another important concept in the theory of linear series is that of ramification, as it simultaneously encapsulates various phenomena of classical algebraic geometry. For a curve C equipped with a linear series l of type g^r_d, we define the *vanishing sequence* at a point $p \in C$

$$a(l, p) = 0 \leqslant a_0(l, p) < \ldots < a_r(l, p) \leqslant d$$

to be the sequence of distinct orders of vanishing of sections in l at p. The *ramification sequence of type* (r, d):

$$\alpha(l, p) = 0 \leqslant \alpha_0(l, p) \leqslant \ldots \leqslant \alpha_r(l, p) \leqslant d - r$$

is given by $\alpha_i(l, p) = a_i(l, p) - i$.

When a linear series defines a morphism $f : C \to \mathbb{P}^r$, we may interpret the vanishing sequence geometrically in terms of the possible multiplicities at p of preimages under f of hyperplanes in $\mathbb{P}^r$. Furthermore, the ramification sequence gives information about the singularities of the image curve $f(C)$ at $f(p)$.

It is important to note that there are only finitely many ramification points of l and we have the *Plücker formula*

$$\sum_{p\in C}\left(\sum_{i=0}^{r}\alpha_i(l,p)\right) = (r+1)d + \binom{r+1}{2}(2g-2).$$

The refinement of Theorem 1.2 including ramification states that on a general pointed curve $(C, p_1, \ldots, p_n)$ of genus g the ramification $\alpha(l, p_j)$ of a linear series l of type g^r_d at the points p_j must satisfy:

$$\rho(g,r,d) \geqslant \sum_{j=1}^{n}\sum_{i=0}^{r}\alpha_i(l,p_j). \tag{1.1}$$

On the one hand, there are easy examples that show that this condition is not sufficient. On the other, degeneration techniques can be used to show that a slightly stronger version of the inequality (1.1) is both a a necessary and a sufficient condition for the existence of linear series with prescribed ramification:

Proposition 1.4 ([EH87, Proposition 1.2]). *A general pointed curve (C, p) of genus g possesses a linear series l of type g^r_d with ramification sequence at p given by $\alpha(l,p) = (\alpha_0, \ldots, \alpha_r)$ if and only if*

$$\sum_{i=0}^{r}(\alpha_i(l,p) + g - d + r)_+ \leqslant g, \tag{1.2}$$

where $(x)_+ = \max\{x, 0\}$ denotes the positive part of the integer x.

One possible direction of further study is to attempt to describe the geometry of a Brill-Noether general curve. Note that once the curve is embedded in projective space, we may start to ask questions about its extrinsic properties, such as for example the relations of points on the curve to each other, inflectionary points, secant planes, and so on. The enumerative problems we consider in this thesis are of this flavour and we describe them in the next section.

1.2 Enumerative questions for embedded curves

Although the statement and original formulation of the enumerative problems considered in this thesis are expressed in terms of extrinsic properties of curves, both problems and their degenerations to nodal curves are related and elucidate various aspects in the study of the moduli spaces of abstract curves $\mathcal{M}_g$, $\mathcal{M}_{g,n}$, and their compactified counterparts $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$.

1.2.1 De Jonquières divisors on algebraic curves

The first problem we deal with is that of *de Jonquières divisors* on algebraic curves. A toy example of de Jonquières' question is that of counting the number of flex points of a plane cubic. An easy computation of the intersection of the curve with its Hessian shows that there are 9 such inflectionary points. More generally, using the Plücker formula one can show that a plane curve of degree d has $3d(d-2)$ flex points.

The notion of de Jonquières divisors can be understood as a natural generalisation of the situation above. In his 1866 memoir [Jon66], de Jonquières set himself the "repellant but interesting task" [1] of computing the number of divisors with prescribed multiplicities that are contained in a fixed linear series on a given plane algebraic curve. In other words, one computes the number of lines in the projective plane intersecting a given curve with prescribed multiplicities at the points of intersection. Allen [All19] later generalised de Jonquières' formula to curves embedded in projective spaces of arbitrary dimension.

Almost a century after de Jonquières, using modern techniques of topology and intersection theory of cycles on the symmetric product on a curve, MacDonald [Mac62] and Mattuck [Mat65] recovered the original formula in characteristic zero and arbitrary characteristic, respectively. However, their work does not address the vagueness of the classical statements, assuming either that the linear series in question is sufficiently generic, or that the multiplicities are counted correctly. To address this issue, Vainsencher [Vai81] described the locus of divisors with prescribed multiplicities as the vanishing locus of a section of a bundle of the appropriate rank. Using a natural filtration of this bundle, he computed its Chern classes without making use of the Grothendieck-Riemann-Roch theorem, and established the enumerative validity of the de Jonquières formula for plane curves and for some higher dimensional cases.

Our aim in this thesis is twofold. On the one hand, we settle the issue of the validity of the de Jonquières formula for linear series of arbitrary degree and dimension on a general curve by studying the geometry of the respective moduli space. On the other hand, we develop a theory of degenerations for de Jonquières divisors to nodal curves, which plays a central role as the main tool in the study of the aforementioned moduli space.

Before stating our results, we must first make precise the notion of a de Jonquières divisor: for a fixed smooth curve C of genus g with a fixed linear series $l = (L, V) \in G^r_d(C)$, a *de Jonquières divisor of length* N is a divisor

$$a_1 D_1 + \ldots + a_k D_k \in C_d$$

such that

$$a_1 D_1 + \ldots + a_k D_k \in \mathbb{P}V,$$

[1] cf. J.L.Coolidge's *Treatise on algebraic plane curves* [Coo31, Chapter 3, §3]

where $k \leqslant d$ is a positive integer and the D_i are effective divisors of degree d_i for $i = 1, \ldots, k$ such that $N = \sum_{i=1}^{k} d_i$. If l is complete (so if $g - d + r \geqslant 0$), the definition of a de Jonquières divisor is equivalent to

$$L \simeq \mathcal{O}_C(a_1 D_1 + \ldots + a_k D_k).$$

Furthermore, if we let $\mu_1 = (a_1, \ldots, a_k)$ and $\mu_2 = (d_1, \ldots, d_k)$ be two positive partitions such that $\sum_{i=1}^{k} a_i d_i = d$, then we denote the set of de Jonquières divisors of length N determined by μ_1 and μ_2 by $DJ_{k,N}^{r,d}(\mu_1, \mu_2, C, l)$.

In the particular case when $d_i = 1$ for all $i = 1, \ldots, k$, let $n := N = k$ and it follows that the de Jonquières divisor is of the form

$$a_1 p_1 + \ldots + a_n p_n,$$

for some distinct points $p_1, \ldots, p_n \in C$. Here we simplify the notation to

$$DJ_{k,N}^{r,d}(\mu_1, \mu_2, C, l) = DJ_n^{r,d}(\mu_1, C, l).$$

Aside from being interesting objects in their own right, de Jonquières divisors and their degenerations are natural generalisations of the concept of strata of abelian differentials which were first introduced in the context of Teichmüller dynamics and flat surfaces - see the works of Masur [Mas82] and Veech [Vee82], and more recently, of Bainbridge, Chen, Gendron, Grushevsky, and Möller [BCGG+16]. These strata are, however, interesting objects also from the point of view of algebraic geometry, as can be seen in the work of Farkas and Pandharipande [FP18], Chen and Tarasca [CT16], or Mullane [Mul16]. In fact, the result of Polishchuk [Pol06] concerning the dimension of the strata in $\mathcal{M}_{g,n}$ provides an important clue towards the validity of the de Jonquières formulas.

It turns out that the space $DJ_{k,N}^{r,d}(\mu_1, \mu_2, C, l)$ has the structure of a determinantal variety and its expected dimension (or, equivalently, lower bound for its dimension) is

$$\exp\dim DJ_{k,N}^{r,d}(\mu_1, \mu_2, C, l) = N - d + r.$$

The de Jonquières formula (cf. [Mat65] §5) states that, if we expect there to be a finite number of de Jonquières divisors of length N (so if $N - d + r = 0$), then this virtual number is given by the coefficient of the monomial $t_1^{d_1} \cdot \ldots \cdot t_k^{d_k}$ in

$$(1 + a_1^2 t_1 + \ldots + a_k^2 t_k)^g (1 + a_1 t_1 + \ldots + a_k t_k)^{d-r-g}. \tag{1.3}$$

Substituting $r = 1$ and $d_1 = \ldots = d_k = 1$ in formula (1.3) recovers the number of ramification points of a Hurwitz cover of C obtained from the Plücker formula. In addition, if C is the plane cubic, then $g = 1$, $d - r - g = 1$ and we recover its 9 flex points. Lastly, taking the linear series to be the canonical one, we recover the number of odd theta characteristics on a general curve. Hence

we expect these counts to be true. To settle the issue, in Chapter 3 we study the space $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$, establish whether it is empty when the expected dimension is negative, and when non-empty whether it is smooth, reduced, and of expected dimension.

Luckily, we are able to settle these questions in the affirmative. In fact, the non-emptiness of the space of de Jonquières when the expected dimension is non-negative follows from an easy class computation, which we explain in Section 3.3. The questions regarding the dimension of the space $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ and whether it is empty when the expected dimension is negative are less straightforward and require the degeneration techniques described in Chapter 2. Using limit linear series on nodal curves of compact type, we prove

Theorem 1.5 (Dimension theorem)**.** *Fix a general curve* C *of genus* g *equipped with a general complete linear series* $l \in G^r_d(C)$*. If* $N - d + r \geqslant 0$*, the space* $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ *is of expected dimension i.e.*

$$\dim DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l) = N - d + r.$$

A direct consequence of the dimension theorem is the non-existence statement for complete linear series:

Corollary 1.6. *Let* C *be a general curve equipped with a general complete linear series* $l \in G^r_d(C)$*. If* $N - d + r < 0$*, the variety* $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ *is empty.*

The validity of de Jonquières' counts is a direct consequence of Theorem 1.5, and of the determinantal variety structure of the space of de Jonquières divisors. The latter implies that $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ is in fact a Cohen-Macaulay variety (Proposition 4.1, Chapter II, [ACGH85]). As such, if it is zero-dimensional, it consists of a finite number of discrete closed points. This yields

Corollary 1.7. *Let* C *be a general curve equipped with a general complete linear series* $l \in G^r_d(C)$*. If* $N - d + r = 0$*, the variety* $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ *is a finite collection of reduced points.*

We address the issue of the smoothness of $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$, by expressing the space as an intersection of subvarieties inside the symmetric product C_d and obtaining a transversality condition from the study of the relevant tangent spaces.

Theorem 1.8 (Smoothness result)**.** *Let* C *be a general curve of genus* g*. For any complete linear series* $l \in G^r_d(C)$*, the space* $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ *is smooth whenever* $N - d + r > 0$*.*

The proof is also by degeneration to nodal curves and limit linear series, however this time using a strategy developed in [Far08] .

Finally, we prove the non-existence result for non-complete linear series using a different degeneration technique, namely compactified Picard schemes for moduli of stable pointed curves. We obtain:

Theorem 1.9 (Non-existence statement)**.** *Let C be a general curve equipped with a general linear series $l \in G^r_d(C)$ satisfying $g - d + r < 0$ and let μ be a positive partition of d length n. If $n - d + r < 0$, the variety $DJ^{r,d}_n(\mu, C, l)$ is empty.*

These degenerations are only suitable for treating the case of de Jonquières divisors with $d_1 = \ldots = d_k = 1$, as they rely on the fact that the points in the support of the divisor are distinct.

1.2.2 Secant varieties to linear series on algebraic curves

The study of de Jonquières divisors may give a lot of information on the singularities of an embedded curve. For example, a consequence of the non-existence result (Corollary 1.6) is that a general space curve of degree 8 (i.e. embedded by a general g^3_8) cannot have two flex points, as that would entail the existence of a de Jonquières divisor $4p_1 + 4p_2$ of length two, and $2 - 8 + 3 < 0$.

In what follows we concentrate on an enumerative problem stemming from an alternative perspective on the topic of singularities of a curve embedded in projective space. An elementary example thereof is the calculation of the number of double points of a curve C contained in the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Assuming the curve C has arithmetic genus g and bidegree (d_1, d_2), the adjunction formula tells us that there are exactly

$$\nu = (d_1 - 1)(d_2 - 1) - g$$

ordinary double points.

We can reformulate this problem from the point of view of *intersections of incidence varieties* as follows: the embedding

$$C \to \mathbb{P}^1 \times \mathbb{P}^1$$

is given by a pair of pencils $l_1 = g^1_{d_1}$ and $l_2 = g^1_{d_2}$ on C and the double points correspond to pairs of points (p_1, p_2) common to both linear series, i.e. a divisor $D = p_1 + p_2 \in C_2$ such that

$$\dim(l_1 - D) \geqslant 0,$$
$$\dim(l_2 - D) \geqslant 0.$$

The enumerative problem becomes that of counting the number of divisors of degree two that are common to the two pencils, or more precisely the number of divisors in the intersection of the two incidence varieties corresponding to the series l_1 and l_2.

In this thesis we study the geometry of such intersections of incidence (or more generally secant) varieties to algebraic curves, with a focus on issues of transversality of intersection.

Before stating the precise results, we introduce some terminology. Let C be a general curve of genus g equipped with a linear series $l = (L, V) = g^r_d$

such that the Brill-Noether number $\rho(g, r, d)$ is non-negative and let $e \leqslant d$ be a positive integer. We set

$$\Gamma_e(l) := \{D \in C_e \mid D' - D \geqslant 0 \text{ for some } D' \in l\} \subset C_e$$

to be the *incidence variety* of all effective divisors of degree e that are subordinate to the linear series l.

As a subspace of C_e, the space $\Gamma_e(l)$ has the structure of a degeneracy locus so it is indeed a variety and it is easy to see that it has expected dimension r. We explain this in more detail in Chapter 4.

Consider the following setup: equip the smooth general curve C of genus g with two complete linear series $l_1 = g^{r_1}_{d_1}$ and $l_2 = g^{r_2}_{d_2}$ with positive Brill-Noether numbers $\rho(g, r_1, d_1)$ and $\rho(g, r_2, d_2)$.
Let

$$\Gamma_e(l_1) = \{D \in C_e \mid l_1 - D \geqslant 0\},$$
$$\Gamma_e(l_2) = \{D \in C_e \mid l_2 - D \geqslant 0\},$$

be the respective incidence varieties. We therefore expect to have finitely many divisors D in the intersection $\Gamma_e(l_1) \cap \Gamma_e(l_2)$ if

$$\dim \Gamma_e(l_1) + \dim \Gamma_e(l_2) = r_1 + r_2 = e.$$

In fact, in Chapter VIII, §3 of [ACGH85], a class computation shows that in this case, the number is expected to be the coefficient of the monomial $t_1^{e-r_1} t_2^{e-r_2}$ in

$$(1 + t_1)^{d_1 - g - r_1}(1 + t_2)^{d_2 - g - r_2}(1 + t_1 + t_2)^g. \tag{1.4}$$

Using this formula we immediately recover the number of double points of a curve C of genus g and bidegree (d_1, d_2) contained in the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, in this case $r_1 = r_2 = 1$ and $e = 2$. Thus, according to formula (1.4), the number we are after is the coefficient of $t_1 t_2$ in

$$(1 + t_1)^{d_1 - g - 1}(1 + t_2)^{d_2 - g - 1}(1 + t_1 + t_2)^g.$$

But this is exactly $(d_1 - 1)(d_2 - 1) - g$, i.e. the same count we obtained by geometric methods.

Unlike in the case of de Jonquières divisors, formula (1.4) yields unexpected zero counts that correspond to the case when the intersection

$$\Gamma_e(l_1) \cap \Gamma_e(l_2)$$

is not transverse.

We study this behaviour in Section 4.2 and, using the dimension theorem for de Jonquières divisors (Theorem 1.5), we obtain in Section 4.2.1 some examples where this intersection is actually empty. Using a tangent space computation, we prove in Section 4.2.2 our main non-transversality result:

Theorem 1.10. *Consider a general curve* C *of genus* g *equipped with arbitrary linear series* $l_1 = g^{r_1}_{d_1}$ *and* $l_2 = g^{r_2}_{d_2} = K_C - l_1$ *such that* $\rho(g, r_1, d_1)$ *is non-negative. If non-empty, the intersection* $\Gamma_e(l_1) \cap \Gamma_e(l_2)$ *is not transverse for* $e = r_1 + r_2$.

Another related direction of study is to consider a generalisation of the notion of incidence varieties, namely that of *secant varieties*: if C is a smooth general curve of genus g endowed with a linear series l of type g^r_d and if e and f are positive integers such that $0 \leqslant f < e \leqslant d$, then let

$$V_e^{e-f}(l) = \{D \in C_e \mid \dim(l - D) \geqslant r - e + f\}$$

be the *secant variety* of effective divisors of degree e which impose at most $e - f$ independent conditions on l. Equivalently, this space parametrises the e-secant $(e - f - 1)$-planes to the curve C embedded in $\mathbb{P}^r$ via l.

The cycle $V_e^{e-f}(l)$ of C_e is also endowed with a degeneracy locus structure (so it is an actual variety) and it was proven by Farkas [Far08] that, if non-empty, it does indeed have expected dimension

$$\dim V_e^{e-f}(l) = e - f(r + 1 - e + f),$$

for a general curve C with a general series l of type g^r_d. We remark here that incidence varieties are special cases of secant varieties, namely $\Gamma_e(l) = V_e^r(l)$ and $f = e - r$.

Furthermore, secant varieties are interesting objects not just from the point of view of classical algebraic geometry, but also from a modern perspective. For example, one may generalise the notion of secant varieties to nonsingular projective surfaces S with a line bundle L. If $|L|$ is a linear system of dimension $3m-2$ inducing a map $S \to \mathbb{P}^{3m-2}$, then the number of m-chords of dimension $m - 2$ to the image of S (so the cardinality of the secant variety $V_m^{m-1}(|L|)$ is given by the integral of the top Segre class

$$\int_{S^{[m]}} s_{2m}(H^{[m]}),$$

where $S^{[m]}$ is the Hilbert scheme of points of S carrying a tautological rank-m bundle $H^{[m]}$. Such Segre classes play a basic role in the Donaldson-Thomas counting of sheaves and appeared first in the algebraic study of Donaldson invariants via the moduli space of rank-2 bundles on S [Tyu93]. The exact result of the integral is the subject of Lehn's conjecture [Leh99] that states that it can be expressed as a polynomial of degree m in the four variables

$$H^2,\ H \cdot K_S,\ K_S^2,\ c_2(S).$$

For a proof of this conjecture, see [Tik94] and for a generalisation to K3 surfaces see [MOP17].

We shall therefore consider the more general case of the intersection of an incidence variety and a secant variety on a smooth general curve C, namely

$$\Gamma_e(l_1) \cap V_e^{e-f}(l_2),$$

where l_1 and l_2 are linear series on C and e and f are integers such that $0 \leqslant f < e \leqslant \min(d_1, d_2)$. Here we investigate the complementary problem to that studied in Theorem 1.10, i.e. the expected emptiness of the intersection when the sum of the dimensions of the two varieties $\Gamma_e(l_1)$ and $V_e^{e-f}(l_2)$ inside C_e is less than e. As in Theorem 1.10, we again focus on the case $l_2 = K_C - l_1$. To get the correct dimensional estimate when we allow for the series l_1 to vary in moduli (so now we do not take just the general series of type $g^{r_1}_{d_1}$), consider the correspondence

$$\Lambda = \{(D, l_1) \in C_e \times G^{r_1}_{d_1}(C) \mid D \in \Gamma_e(l_1) \cap V_e^{e-f}(K_C - l_1)\} \subset C_e \times G^{r_1}_{d_1}.$$

By construction, Λ has expected dimension

$$\exp\dim \Lambda = \rho(g, r_1, d_1) + \dim \Gamma_e(l_1) + \dim V_e^{e-f}(K_C - l_1) - e,$$

so if this number is negative, we expect Λ to be empty. Our main result in this context is:

Theorem 1.11. *Let C be a smooth general curve of genus g equipped with a complete linear series $l_1 = g^{r_1}_{d_1}$ such that $\rho(g, r_1, d_1) \geqslant 0$. If $f = 1$ and*

$$\dim \Gamma_e(l_1) + \dim V_e^{e-f}(K_C - l_1) \leqslant e - \rho(g, r_1, d_1) - 1,$$

then the intersection $\Gamma_e(l_1) \cap V_e^{e-f}(K_C - l_1)$ is empty for an arbitrary linear series $l_1 \in G^{r_1}_{d_1}(C)$.

Note that if $f = r_1 + 1 + \rho(g, r_1, d_1)$, then $V_e^{e-f}(K_C - l_1) = \Gamma_e(K_C - l_1)$ and we are back to the degenerate case of Theorem 1.10.

We prove this in Section 4.4 by degeneration to a nodal curve using limit linear series and by exploiting an ingenious construction of [Far08]. Furthermore, we in fact provide a method to check the emptiness of such intersections for any $f \neq r_1 + 1 + \rho(g, r_1, d_1)$, but the case $f = 1$ seems to be the one with the most tractable computations.

In the course of the proof of Theorem 1.11 we also find an interesting example that contradicts the expectation of non-emptiness of secant varieties as stated in Theorem 0.5 of [Far08]. We explain this in Remark 4.4.

Chapter 2

Degeneration techniques

Although the results of Brill-Noether theory concern a single curve, what distinguishes them from more elementary theorems such as Riemann-Roch is that most of them are not true for every curve C of genus g. Instead, they apply to an open dense subset of $\mathcal{M}_g$, and moreover they concern conditions that are open on proper, smooth, families of curves. Hence to prove such a theorem, it would be enough to exhibit a single smooth curve satisfying it. Unfortunately, up until recently (see [ABFS16]), it was not known how to write down a smooth curve of large genus satisfying any of the Brill-Noether theorems. One resolution of this problem is to work on families of curves, instead of just fixed curves, so as to be able to use variational tools in the proofs. In this framework, one needs to find degenerations to curves that are sufficiently special so that the required analysis can be carried out explicitly, but that are at the same time general from the point of view of the Brill-Noether theorems. It turns out that most known examples of such curves are singular and highly reducible. Indeed, the proofs of the fact that $G^r_d(C)$ is empty if $\rho(g, r, d)$ is negative and that $G^r_d(C)$ is smooth and of dimension $\rho(g, r, d)$ (Theorems 1.2 and 1.3) both involve taking the limit of a family of g^r_d-s on a degenerating family of smooth curves with nodal central fibres.

Since the enumerative problems considered in this thesis are of a similar nature, we shall also use degenerations arguments to approach them. These techniques allow us to understand what happens to the limit of a line bundle (or linear series) on a family of smooth curves as it degenerates to a singular special fibre. In the remainder of this chapter we make a summary of the two methods employed: *limit linear series* in Section 2.1 and *compactified Picard schemes* in Section 2.2.

Before we start, we fix some terminology. The *dual graph* of a curve consists of one vertex for each irreducible component and one edge for each node. A curve is *of compact type* if its dual graph is a tree and is *tree-like* if, after deleting self-edges, the dual graph becomes a tree. We call a curve X *stable* if each smooth rational component Y meets $\overline{X \setminus Y}$ in at least three points. It

is *semi-stable* if any smooth rational component $Y \subset X$ intersects $\overline{X \setminus Y}$ in at least two points. A smooth rational component $Y \subset X$ is *destabilising* if $\#\left(Y \cap \overline{X \setminus Y}\right) \leqslant 2$. Furthermore, assume X is an n-pointed curve of genus g and that $2g - 2 + n > 0$. Then X is *stable* (or *semi-stable*) if on each smooth rational component the number of nodes and marked points is at least three (or two). A smooth rational component is *destabilising* if the number of nodes and marked points is at most two.

2.1 Limit linear series

Consider a smooth 1-parameter family $\pi : \mathscr{X} \to B$ of curves of genus g over a smooth curve B such that the fibres over $B^* = B \setminus 0$ are smooth curves, while the special fibre is given by a nodal curve of compact type $\mathscr{X}_0$. Let $\mathscr{X}^*$ be the restriction of the family to B^*. Suppose that $\mathscr{L}^*$ is a line bundle on $\mathscr{X}^*$ such that the restriction $\mathscr{L}_t$ to each fibre $\mathscr{X}_t$ is of degree d for all $t \in B^*$. Then we can extend $\mathscr{L}^*$ to a limit line bundle $\mathscr{L}$ over the whole family $\mathscr{X}$. If Y is any irreducible component of $\mathscr{X}_0$, then

$$\mathscr{L} \otimes \mathcal{O}_{\mathscr{X}}(Y) \tag{2.1}$$

is also an extension of $\mathscr{L}^*$.

Given a line bundle $\mathscr{L}^*$ on $\mathscr{X}^*$, for each $t \in B^*$ fix a non-zero subvector space $V_t \subseteq H^0(\mathscr{X}_t, \mathscr{L}_t)$ of dimension $r+1$. If $\mathscr{L}$ is an extension of $\mathscr{L}^*$ to the whole $\mathscr{X}$, let $\mathscr{V}$ be a free module of rank $r+1$ over B with

$$\mathscr{V}_t := V_t \cap H^0(\mathscr{X}, \mathscr{L}) \text{ with } t \in B,$$

where the intersection is taken in $H^0(\mathscr{X}_t, \mathscr{L}_t)$. We denote by $\mathscr{V}^*$ the corresponding module over B^*. Note that the induced homomorphism

$$\mathscr{V}_0 \to (\pi_* \mathscr{L})_0 \to H^0(\mathscr{X}_0, \mathscr{L}_0)$$

is injective. To summarise, we call $\mathcal{L}^* = (\mathscr{L}^*, \mathscr{V}^*)$ a *linear series on* $\mathscr{X}^*$. Given a limit line bundle $\mathscr{L}$ on $\mathscr{X}$, $\mathcal{L}^*$ extends to a linear series $\mathcal{L} := (\mathscr{L}, \mathscr{V})$ on $\mathscr{X}$. Its restriction $\mathcal{L}_0 = (\mathscr{L}_0, \mathscr{V}_0)$ is a linear series of degree d and dimension r on $\mathscr{X}_0$. Unfortunately, twisting the bundle $\mathscr{L}$ by different components of $\mathscr{X}_0$ as in (2.1) yields infinitely many possible extensions of the linear series $\mathcal{L}^*$. Hence any geometric information that one may extract from the degeneration is lost.

In [EH86], Eisenbud and Harris explain that in order to get the most information about the limiting behaviour of the linear series on the central fibre we should only focus on some particular extensions of the line bundle in question. Thus, for each component Y of $\mathscr{X}_0$, we denote by $\mathscr{L}_Y$ the unique extension of the line bundle $\mathscr{L}^*$ that has degree d on Y and degree 0 on all other components of $\mathscr{X}_0$ and by $\mathscr{V}_Y$ the corresponding free module of rank $r+1$ over B defined as above. The advantage of this is that the sections belonging to

$(\mathscr{V}_Y)_0$ vanish on all components of $\mathscr{X}_0$ except for Y. Hence each $\mathcal{L}_Y = (\mathscr{L}_Y, \mathscr{V}_Y)$ induces on the component Y a linear series l_Y of type g^r_d, which is called an *aspect* of $\mathcal{L}^*$.

The relationship between the various aspects of $\mathcal{L}^*$ is best described in terms of the vanishing sequence at the point $p \in Y$

$$0 \leqslant a_0(l_Y, p) < a_1(l_Y, p) < \cdots < a_r(l_Y, p) \leqslant d.$$

If Z is another component of $\mathscr{X}_0$ such that $Y \cap Z = p$, then for all $i = 0, \ldots, r$,

$$a_i(l_Y, p) + a_{r-i}(l_Z, p) \geqslant d - r. \tag{2.2}$$

To sum up, a collection l of aspects of $\mathcal{L}^*$ satisfying (2.2) is called a *crude limit linear series* and it was proved in [EH86] that it indeed arises as a limit of ordinary linear series on smooth curves. If all inequalities in (2.2) become equalities, then l is called a *refined limit linear series*. Since refined limit series are in fact the ones playing the role of ordinary limit series on smooth curves, we shall usually drop the adjective "refined" unless it is necessary.

If $p \in X$ is a smooth point contained in a component Y, then we define the vanishing and ramification sequences of l at p to be those of l_Y at p. We have an analogue of the Plücker formula for limit linear series:

Proposition 2.1 ([EH86, Proposition 1.1])**.** *Let X be a genus g curve of compact type. If l is a crude limit g^r_d on X, then*

$$\sum_{p \text{ smooth point of } X} \left(\sum_{i=0}^{r} \alpha_i(l, p) \right) \leqslant (r+1)d + \binom{r+1}{2}(2g-2).$$

Equality holds if and only of l is a refined limit linear series.

Moreover, if X is a tree-like curve of genus g, with smooth points $p_1, \ldots, p_n$ and ramification sequences $\alpha^1, \ldots, \alpha^n$ of type (r, d), then we define

$$G^r_d(X, (p_1, \alpha^1), \ldots, (p_n, \alpha^n))$$

to be the scheme of all limit linear series with prescribed ramification satisfying

$$\alpha_i(l, p_j) \geqslant \alpha^j_i,$$

for all $i = 0, \ldots, r$ and $j = 1, \ldots, n$. This scheme has a determinantal structure and we have the following dimension estimate:

$$\dim G^r_d(X, (p_1, \alpha^1), \ldots, (p_n, \alpha^n)) \geqslant \rho(g, r, d) - \sum_{i=0}^{r} \sum_{j=1}^{n} \alpha^j_i.$$

In fact, one can prove that the estimate is correct for most tree-like curves:

Theorem 2.2 ([EH87, Theorem 1.1]). *Let X be a tree-like curve of genus g and suppose that each irreducible component Y, and the points $q_1, \dots, q_m \in Y$ where Y meets other components of X satisfy:*

1. *if $g(Y) = 1$, then $m = 1$;*
2. *if $g(Y) = 2$, then $m = 1$ and q_1 is not a Weierstrass point of Y;*
3. *if $g(Y) \geqslant 3$, then $(Y, q_1, \dots, q_m)$ is a general m-pointed curve.*

If $p_1, \dots, p_n \in X$ are general points, or arbitrary smooth points of X on smooth rational components, then, for any ramification sequences α^j,

$$\dim G^r_d(X, (p_1, \alpha^1), \dots, (p_n, \alpha^n)) = \rho(g, r, d) - \sum_{i=0}^{r} \sum_{j=1}^{n} \alpha^j_i.$$

Unfortunately, it is not always true that a limit linear series on a nodal curve $\mathscr{X}_0$ occurs as the limit of linear series on a family $\mathscr{X}$ of smooth curves specialising to $\mathscr{X}_0$, as is shown in Example 3.2 of [EH86]. Happily, one can check the smoothability of certain series, under some assumptions. To do so, we extend the above considerations to a family (satisfying some extra technical conditions) of curves with sections

$$\pi : \mathscr{X} \to B, \ p_1, \dots, p_n : B \to \mathscr{X},$$

where B is now an irreducible scheme, the fibres are curves of genus g of compact type, and the sections p_i are disjoint (with image in the smooth locus of π). One can show that there exists a scheme

$$\mathscr{G}^r_d(\mathscr{X}/B, (p_1, \alpha^1), \dots, (p_n, \alpha^n))$$

parametrising limit linear series on the fibres of the family with prescribed ramification α^i at the p_i. The key concept that appears in the proof of smoothability of limit linear series is that of *dimensional properness*. To understand it, consider an irreducible curve C of genus g with a linear series of type g^r_d with prescribed ramification $\alpha^1, \dots, \alpha^n$ at the smooth points $p_1, \dots, p_n \in C$. Then l is *dimensionally proper with respect to* $p_1, \dots, p_n$ if, letting

$$\pi : \widetilde{C} \to B$$
$$p_i : B \to \widetilde{C}$$

be the versal deformation of the pointed curve $(C; p_1, \dots, p_n)$, we have that the dimension of an irreducible component of $\mathscr{G}^r_d(\widetilde{C}/B, (p_1, \alpha^1), \dots, (p_n, \alpha^n))$ containing l is

$$\rho(g, r, d) - \sum_{i=0}^{r} \sum_{j=1}^{n} \alpha^j_i + \dim B.$$

Furthermore, a limit linear series with prescribed ramification at smooth points is dimensionally proper if each of its aspects is. The smoothing result is:

Theorem 2.3 ([EH86, Corollary 3.7]). *If X is a curve of compact type, and l a refined limit linear series which is dimensionally proper with respect to smooth points $p_1, \ldots, p_n \in X$ and their prescribed ramification $\alpha^1, \ldots, \alpha^n$, then l can be smoothed, maintaining the ramification conditions at the points p_j.*

An alternative, functorial formulation of the space of limit linear series is given in the work of Osserman [Oss06]. This proves to be useful when constructing the spaces of degenerations corresponding to our enumerative problems. We recall the main definitions by following the exposition in [Oss].

To begin with, let $\pi : \mathscr{X} \to B$ be a proper family of smooth curves of genus g with a section. Following Definition 4.2.1 in [Oss], the *functor $\mathscr{G}^r_d(\mathscr{X}/B)$ of linear series of type g^r_d* is defined by associating to each B-scheme T the set of equivalence classes of pairs $(\mathscr{L}, \mathscr{V})$, where $\mathscr{L}$ is now a line bundle on $\mathscr{X} \times_B T$ with degree d on all fibres, and $\mathscr{V} \subseteq \pi_{2*}\mathscr{L}$ is a subbundle of rank $r+1$, where π_2 denotes the second projection from the fibre product. For the precise definition of the equivalence relation, we refer the reader to [Oss]. This functor is represented by a scheme $G^r_d(\mathscr{X}/B)$ which is proper over B.

Assume now that the fibres of the family $\pi : \mathscr{X} \to B$ are nodal curves of genus g of compact type such that no nodes are smoothed. Hence all fibres have the same dual graph Γ. For each vertex v of Γ, let Y^v_t denote the irreducible component of $\mathscr{X}_t$ corresponding to v. Thus for each v we have a family $\mathscr{Y}^v$ of smooth curves over B with fibres given by Y^v_t. In this case the functor $\mathscr{G}^r_d(\mathscr{X}/B)$ of linear series of type g^r_d is defined as follows. Consider the product fibred over B

$$\prod_v \mathscr{G}^r_d(\mathscr{Y}^v/B).$$

Let T be a scheme over B. A T-valued point of the above product consists of tuples of pairs $(\mathscr{L}^v, \mathscr{V}^v)$, where $\mathscr{L}^v$ is a vector bundle of degree d on $\mathscr{Y}^v \times_B T$ and $\mathscr{V} \subseteq \pi_{2*}\mathscr{L}^v$ is a subbundle of rank $r+1$. Denote by $\mathscr{L}^{\vec{d}}$ the "canonical" line bundle of degree d and multidegree $\vec{d}$ on $\mathscr{X} \times_B T$ obtained as in 4.4.2 of [Oss]. Moreover, a line bundle has multidegree $\vec{d}^v$ if it has degree d on the component corresponding to the vertex v and degree zero on all the other components. Note also that for two distinct multidegrees $\vec{d}$ and $\vec{d}'$, there is a unique twist map $f_{\vec{d},\vec{d}'} : \mathscr{L}^{\vec{d}} \to \mathscr{L}^{\vec{d}'}$ obtained by performing the unique minimal number of line bundle twists. According to Definition 4.4.7 in loc.cit., a T-valued point of $\prod_v \mathscr{G}^r_d(\mathscr{Y}^v/B)$ is in $\mathscr{G}^r_d(\mathscr{X}/B)(T)$ if, for all multidegrees $\vec{d}$ of d, the map

$$\pi_{2*}\mathscr{L}^{\vec{d}} \to \bigoplus_v (\pi_{2*}\mathscr{L}^v)/\mathscr{V}^v$$

induced by the restriction to $\mathscr{Y}^v$ and $f_{\vec{d},\vec{d}^v}$ has its $(r+1)$st degeneracy locus equal to all of T. With this construction, $\mathscr{G}^r_d(\mathscr{X}/B)$ is also represented by a scheme $G^r_d(\mathscr{X}/B)$ proper over B.

Finally, if $\pi : \mathscr{X} \to B$ is a smoothing family (for details, see 4.5 of [Oss]), the irreducible components Y^v_t may not exist for certain $t \in B$ and it follows that

the dual graph of the fibres of the family is not constant. We assume from now on that there is a unique maximally degenerate fibre with dual graph Γ_0 (i.e. the family is locally smoothing). We fix a vertex $v_0 \in V(\Gamma_0)$ and set $\vec{d_0} := \vec{d}^{v_0}$. We then replace the tuples of pairs $(\mathscr{L}^v, \mathscr{V}^v)$ with tuples $(\mathscr{L}, (\mathscr{V}^v)_{v \in V(\Gamma_0)})$, where $\mathscr{L}$ is a line bundle of multidegree $\vec{d_0}$ on $\mathscr{X} \times_B T$, and for each $v \in V(\Gamma_0)$, the $\mathscr{V}^v$ are subbundles of rank $r+1$ of the twists $\pi_{2*}\mathscr{L}^{\vec{d}^v}$. Let $f : T \to B$ be a B-scheme. A T-valued point $(\mathscr{L}, (\mathscr{V}^v)_{v \in V(\Gamma_0)})$ is in $\mathscr{G}^r_d(\mathscr{X}/B)(T)$ if for an open cover $\{U_m\}_{m \in I}$ of B satisfying certain technical properties explained in 4.5.2 of [Oss], for all $m \in I$ and all multidegrees $\vec{d}$ of d, the map

$$\pi_{2*}\mathscr{L}^{\vec{d}}|_{(f \circ \pi_2)^{-1}(U_m)} \to \bigoplus_v \left(\pi_{2*}\mathscr{L}^{\vec{d}^v}|_{(f \circ \pi_2)^{-1}(U_m)}\right) / \mathscr{V}^v|_{f^{-1}(U_m)},$$

induced by the appropriate (local) twist maps, has its $(r+1)$st degeneracy locus equal to the whole of U_m.

Remark 2.4. The functor of linear series with points given by tuples

$$(\mathscr{L}, (\mathscr{V}^v)_{v \in V(\Gamma_0)})$$

is naturally isomorphic to the linear series functor with points given by tuples of pairs $(\mathscr{L}^v, \mathscr{V}^v)$ in the case of families where no nodes are smoothed (this is Proposition 4.5.5 in loc.cit.).

Remark 2.5. All the constructions can be shown to be independent of the choice of vertex v_0, twist maps, and open covers $\{U_m\}_{m \in I}$.

Note that all constructions are compatible with base change and moreover, the fibre over $t \in B$ is a limit linear series space when $\mathscr{X}_t$ is reducible, and a space of classical linear series when $\mathscr{X}_t$ is smooth. As a last remark, since working with (refined) limit linear series in the sense of Eisenbud and Harris is more convenient for practical purposes, we generally restrict to those (see Section 6 of [Oss06] for the connection between the two approaches).

2.2 Compactified Picard schemes

Recall that $\mathrm{Pic}(C)$ is the group of isomorphism classes of line bundles on a curve and it may furthermore be identified with the sheaf cohomology group $H^1(C, \mathcal{O}_C^*)$. If the curve is integral, $\mathrm{Pic}(C)$ is also isomorphic to the class group of Cartier divisors of C.

For a nonsingular projective curve C of genus g, one can show that $\mathrm{Pic}(C)$ has the structure of a variety and the following properties:

(i) There is a *Poincaré line bundle* $\widetilde{\mathscr{L}}$ on $\mathrm{Pic}(C) \times C$ such that for all points $[L] \in \mathrm{Pic}(C)$, the restriction $\widetilde{\mathscr{L}}|_{[L] \times C}$ is the line bundle L.

(ii) The variety $\mathrm{Pic}(C)$ has infinitely many components, one for each $d \in \mathbb{Z}$, denoted by $\mathrm{Pic}^d(C)$.

(iii) The Poincaré line bundle $\widetilde{\mathscr{L}}$ on $\mathrm{Pic}^d(C) \times C$ is such that for all points $[L] \in \mathrm{Pic}^d(C)$, the restriction $\widetilde{\mathscr{L}}|_{[L]\times C}$ is the line bundle L of degree d. This restriction induces a bijection between $\mathrm{Pic}^d(C)$ and the isomorphism classes of line bundles of degree d on C.

(iv) $\mathrm{Pic}^d(C)$ is nonsingular and projective of dimension g.

Of course we can extend these considerations to families of curves to obtain a *relative Picard variety of degree* d. More precisely, one can show (see for example Chapter XX1, Theorem 2.1 of [ACG11]) that for a family $\pi : \mathscr{X} \to B$ of smooth curves of genus $g > 1$ that admits a section, there exists a scheme $P_{d,g}$ over B and a Poincaré line bundle $\widetilde{\mathscr{L}}$ on $\mathscr{X} \times_B P_{d,g}$ which restricts to a degree d line bundle on each fibre of π and satisfies the following universal property: for every morphism $f : B' \to B$ and every line bundle $\mathscr{L}$ on $\mathscr{X} \times_B B'$ restricting to a degree d line bundle on each fibre of $h : \mathscr{X} \times_B B' \to B'$, there exists a unique lifting $\varphi : B' \to P_{d,g}$ of f such that

$$\mathscr{L} = (\mathrm{Id} \times \varphi)^* \widetilde{\mathscr{L}} \otimes h^* \mathscr{M},$$

for some line bundle $\mathscr{M}$ on B'.

Furthemore, for $g \geqslant 3$ and the universal family $f : \mathscr{C}_g \to \mathcal{M}_g$, we have the *universal Picard variety of degree* d:

$$\pi_d : P_{d,g} \to \mathcal{M}_g,$$

where for every $[C] \in \mathcal{M}_g$ with trivial automorphisms, we have $\pi_d^{-1}([C]) \simeq \mathrm{Pic}^d(C)$. In this case however, it was shown in [MR85] that a Poincaré bundle on $\mathscr{C}_g \times_{\mathcal{M}_g} P_{d,g}$ exists if and only if $(d - g + 1, 2g - 2) = 1$.

Since we ultimately want to construct degenerations of de Jonquières divisors, we shall concentrate on the universal Picard variety of degree d over $\mathcal{M}_{g,n}$ which we denote by $P_{d,g,n}$. Functorially, what we have said so far is expressed as follows: let $P_{d,g,n}$ be the universal Picard stack over $\mathcal{M}_{g,n}$. Sections of $P_{d,g,n}$ over a scheme B consist of flat and proper families $\pi : \mathscr{X} \to B$ of smooth curves of genus g, with n distinct sections $p_i : B \to \mathscr{X}$ and a line bundle $\mathscr{L}$ of relative degree d on $\mathscr{X}$. Morphisms between such objects are given by Cartesian diagrams

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\beta_2"] \arrow[d, "\pi"] & \mathscr{X}' \arrow[d, "\pi'"] \\
B \arrow[r, "\beta_1"'] \arrow[u, bend left, "p_i"] & B' \arrow[u, bend right, "p_i'"']
\end{tikzcd}$$

such that $p'_i \circ \beta_1 = \beta_2 \circ p_i$ for $i = 1, \ldots, n$, together with an isomorphism $\beta_3 : \mathscr{L} \to \beta_2^* \mathscr{L}'$.

So far we have only taken into consideration the case of smooth curves. Assume now that X is a nodal curve. We can describe how the Picard group of X is related to that of its normalisation $\widetilde{X}$ via the following exact sequence:

$$0 \to \bigoplus_{p \in X} \widetilde{\mathcal{O}}_p^* / \mathcal{O}_p^* \to \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X}) \to 0,$$

where for each $p \in X$, $\mathcal{O}_p$ is its local ring and $\mathcal{O}_p^*$ the integral closure of $\mathcal{O}_p$. It can be shown that $\mathrm{Pic}(X)$ has a scheme structure and, as before, we denote by $\mathrm{Pic}^d(X)$ its subscheme parametrising line bundles on X of degree d.

Unlike in the smooth case, one finds that while $\mathrm{Pic}^d(X)$ is always smooth of dimension g, it need not be projective, because it is not necessarily proper.

Example 2.6. If X is an irreducible nodal curve of genus g and if, moreover, X is rational, it follows that its normalisation $\widetilde{X}$ is isomorphic to $\mathbb{P}^1$. Then, from the exact sequence above we immediately get that $\mathrm{Pic}^d(X) \simeq (\mathbb{C}^*)^g$.

Let $Y_1, \ldots, Y_\gamma$ denote the irreducible components of X and let $L \in \mathrm{Pic}^d(X)$. We define the *multidegree* of L to be the vector of integers

$$\underline{d} = (\deg L_{Y_1}, \ldots, \deg L_{Y_\gamma}).$$

Moreover we set

$$\mathrm{Pic}^{\underline{d}}(X) := \{L \in \mathrm{Pic}^d(X) \mid L \text{ has multidegree } \underline{d}\}$$

and we have that

$$\mathrm{Pic}^d(X) = \prod_{|\underline{d}|=d} \mathrm{Pic}^{\underline{d}}(X), \tag{2.3}$$

where $|\underline{d}| = \deg L_{Y_1} + \ldots + \deg L_{Y_\gamma}$. Hence, unless X is irreducible, $\mathrm{Pic}^d(C)$ has infinitely many connected components, indexed by multidegree.

We would now like to extend our considerations to families of nodal curves. We encounter two major difficulties:

1. Let $\pi : \mathscr{X} \to B$ be a 1-parameter family of curves with smooth fibres for every $b \in B$ with $b \neq b_0$ and nodal fibre over b_0. Unfortunately, if the central fibre is not of compact type, or if the family $\mathscr{X}$ is not smooth, then line bundles over the complement $\mathscr{X} \setminus \mathscr{X}_{b_0}$ do not necessarily extend over $\mathscr{X}_{b_0}$.

2. Assume that the above family of curves $\pi : \mathscr{X} \to B$ is smooth. In this case line bundles over $\mathscr{X} \setminus \mathscr{X}_{b_0}$ always extend, so we have a scheme $P_{d,g} \to B$ which parametrises line bundles of degree d on the fibres of π. The fibre of $P_{d,g}$ over a point $b \in B$ is $\mathrm{Pic}^d(\mathscr{X}_b)$. It turns out that the scheme $P_{d,g}$ is smooth, but not separated, due to the fact that the special fibre has infinitely many connected components, as seen in (2.3).

Example 2.7. As a concrete example of the non-separatedness of $P_{d,g}$ (and expanding on our observation in (2.1)), suppose $\mathscr{X}_{b_0}$ has only two irreducible components Y_1, Y_2 connected at a single node. Then Y_1 and Y_2 are effective divisors on $\mathscr{X}$. Now consider the trivial bundle $\mathcal{O}_{\mathscr{X}}$ and the twisted bundle $\mathscr{L} = \mathcal{O}_{\mathscr{X}}(mY_1)$, for some integer m. Clearly $\mathscr{L}$ and $\mathcal{O}_{\mathscr{X}}$ are isomorphic on $\mathscr{X} \setminus \mathscr{X}_{b_0}$, but they are not isomorphic on $\mathscr{X}_{b_0}$ as the restriction of $\mathscr{L}$ to the special fibre is non-trivial. Moreover, $\mathcal{O}_{\mathscr{X}}$ has multidegree $(0,0)$, while $\mathscr{L}$ has multidegree $(-m, m)$ on the special fibre. Hence, twisting the bundle $\mathcal{O}_{\mathscr{X}}$ by the components of the special fibre yields infinitely many bundles that are isomorphic to $\mathcal{O}_{\mathscr{X}}$ on $\mathscr{X} \setminus \mathscr{X}_{b_0}$, but that belong to different connected components of $P_{d,g}$ on $\mathscr{X}_{b_0}$.

The same discussion applies also to the situation of n-pointed curves. Hence, to obtain a compactification of $P_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$, we need to construct a stack $\overline{P}_{d,g,n}$ with a map $\Psi_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$ with the following properties:

(i) $\Psi_{d,g,n}$ and $\overline{\mathcal{M}}_{g,n}$ fit in the following commutative diagram:

$$\begin{tikzcd} P_{d,g,n} \arrow[r, hook] \arrow[d] & \overline{P}_{d,g,n} \arrow[d, "\Psi_{d,g,n}"] \\ \mathcal{M}_{g,n} \arrow[r, hook] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

(ii) $\Psi_{d,g,n}$ is proper (or at least universally closed).

(iii) $\overline{P}_{d,g,n}$ has a geometrically meaningful description.

Unfortunately, in order to extend $P_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$, it is not enough to consider the stack of line bundles over families of n-pointed stable curves, as it is itself not complete. The solution is to enlarge the type of objects that we are working with, either by admitting more general sheaves than just line bundles, or a bigger class of curves. In this thesis we have opted for the latter, following the work of Caporaso [Cap94] and Melo [Mel11].

To give an illustration of the idea behind the choice of the new curves one introduces, consider the following example, where we ignore the markings.

Example 2.8. Let X be an irreducible nodal curve with one node p, its normalisation $\widetilde{X}$, and the two preimages of the node $p', p'' \in \widetilde{X}$. Take $L' \in \mathrm{Pic}^d(\widetilde{X})$. Therefore the set of line bundles on X pulling back to L' via the normalisation is a copy of $\mathbb{C}^*$, which we will now try to complete. Any line bundle $L \in \mathrm{Pic}^d(X)$ pulling back to L' is obtained by gluing the fibre $L'_{p'}$ with the fibre $L'_{p''}$. Such a gluing, after fixing a local trivialisation for L', is expressed as an isomorphism

$$\begin{aligned} L'_{p'} &\to L'_{p''} \\ 1 &\mapsto c \neq 0, \end{aligned}$$

where we used the fact that $L'_{p'} \simeq L'_{p''} \simeq \mathbb{C}$. Let L^c denote the line bundle on X corresponding to this gluing. Now if $c \to 0$, then the isomorphism above degenerates to the zero map. This means that the sections of L' compatible with this map are the ones that vanish at p''. In other words, L' "degenerates" to $L'(-p'')$, but by doing so it also decreases in degree. One way to fix this problem is to replace X by its blow-up at the node p, i.e. by the nodal curve Y that is constructed by adding to $\widetilde{X}$ a smooth rational component E connecting the points p' and p''. In this way, the limit of L^c as c approaches 0 is a line bundle $\widehat{L}$ such that

$$\begin{aligned} \widehat{L}|_{\widetilde{X}} &= L'(-p''), \\ \widehat{L}|_E &= \mathcal{O}_E(1). \end{aligned}$$

Hence $\deg \widehat{L} = d$ and furthermore, an easy argument shows that the limit $\widehat{L}$ is uniquely determined up to those automorphisms of Y that fix $\widetilde{X}$.

Similarly, if $c \mapsto \infty$, then the limit of L^c is a line bundle on Y that restricts to $L'(-p')$ on $\widetilde{X}$ and to $\mathcal{O}_E(1)$ on E.

The curve Y constructed above is an example of a *quasi-stable* curve, a type of curve that has very good properties for our purposes. To see this, consider the Hilbert scheme $H_{d,g,r}$ where we take $g - d + r = 0$. It has a natural action of $PGL(r+1)$ induced by coordinate change in $\mathbb{P}^r$ and we denote by $H^{ss}_{d,g,r}$ the locus of GIT-semi-stable curves for this action. For large d, the compactification of $P_{d,g}$ constructed by Caporaso in [Cap94] is given by:

$$\overline{P}_{d,g} = H^{ss}_{d,g,r}/PGL(r+1).$$

In fact, in [Gie82] and [Cap94] it is shown that the points of $H^{ss}_{d,g,r}$ correspond to *quasi-stable* curves equipped with *balanced* line bundles (we shall define these notions precisely below).

By construction, $\overline{P}_{d,g}$ is endowed with a proper morphism $\phi_d : \overline{P}_{d,g} \to \overline{\mathcal{M}}_g$, such that the preimage of the locus of automorphism-free curves under ϕ_d is isomorphic to $P_{d,g}$. Moreover, it turns out (see [Cap05]) that for large d and $(d-g+1, 2g-2) = 1$, the quotient stack associated to the GIT-quotient above is a smooth, irreducible Deligne-Mumford stack with a proper and strongly representable map onto $\overline{\mathcal{M}}_g$.

In [Mel11], this construction is extended to the moduli of n-pointed stable curves, which is our case of interest. We begin with the precise definitions of the notions of quasi-stable curves and balanced line bundles mentioned previously.

Let X be an n-pointed semi-stable curve of genus $g \geqslant 2$. For a subcurve $X' \subset X$, let $k_{X'} = \#\left(X' \cap \overline{X \setminus X'}\right)$. A *rational tail* C of X is a rational proper subcurve with $k_{X'} = 1$, whereas a *rational bridge* is a rational proper subcurve X' of X satisfying $k_{X'} = 2$. An *exceptional component* of X is a destabilising

component without marked points. Finally the semi-stable curve X is called *quasi-stable* if the following conditions are satisfied:

- all destabilising components are exceptional;
- rational tails do not contain any exceptional components;
- each rational bridge contains at most one exceptional component.

Definition 2.9. Let Y be a quasi-stable curve (obtained via semi-stable reduction) of the n-pointed stable curve X of genus $g \geqslant 2$ equipped with a line bundle L of degree d. The multidegree of L is *balanced* if

1. If $Y' \subset Y$ is an exceptional component, then $\deg_{Y'} L = 1$.
2. If Y' is a rational bridge, then $\deg_{Y'} L \in \{0, 1\}$.
3. If Y' is a rational tail, then $\deg_{Y'} L = -1$.
4. If Y' is a proper subcurve whose irreducible components are not contained in any rational tail or bridge, then $\deg_{Y'} L$ must satisfy the following inequality:

$$\left|\deg_{Y'} L - \frac{d(w_{Y'} - t_{Y'})}{2g-2} - t_{Y'}\right| \leqslant \frac{k_{Y'} - t_{Y'} - 2b^L_{Y'}}{2}, \tag{2.4}$$

where $w_{Y'} = 2(g_{Y'} - 2)$, $t_{Y'}$ is the number of rational tails meeting Y', and $b^L_{Y'}$ is the number of rational bridges where the degree of L vanishes and which meet Y' in two points.

Denote by $\overline{P}^X_d$ the set of all the pairs (Y, L) of quasi-stable curves Y of X equipped with a balanced line bundle L of degree d. Let $\overline{W}^X_{r,d} \subset \overline{P}^X_d$ denote all those pairs where the line bundles satisfy $h^0(Y, L) \geqslant r + 1$.

The compactification $\overline{P}_{d,g,n}$ of the Picard stack on the moduli stack of n-pointed stable curves is given by the line bundles with balanced multidegrees on quasi-stable curves.

Definition 2.10 ([Mel11, Definition 4.1])**.** For any integer d and $g \geqslant 3$, we denote by $\overline{P}_{d,g,n}$ the following category fibred in groupoids over the category of schemes over F: objects over a F-scheme B are families

$$(\pi : \mathscr{X} \to B, p_i : B \to \mathscr{X}, \mathscr{L})$$

of n-pointed quasi-stable curves over B and a balanced line bundle $\mathscr{L}$ on $\mathscr{X}$ of relative degree d. Morphisms between two such objects are given by Cartesian diagrams

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\beta_2"] \arrow[d, "\pi"] & \mathscr{X}' \arrow[d, "\pi'"] \\ B \arrow[r, "\beta_1"'] \arrow[u, bend left, "p_i"] & B' \arrow[u, bend right, "p'_i"'] \end{tikzcd}$$

such that $p'_i \circ \beta_1 = \beta_2 \circ p_i$ for $i = 1, \ldots, n$, together with an isomorphism $\beta_3 : \mathscr{L} \to \beta_2^* \mathscr{L}'$.

In Theorem 4.2 of [Mel11] it is shown that $\overline{P}_{d,g,n}$ is an Artin stack endowed with a (forgetful) universally closed morphism $\Psi_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$.

Remark 2.11. By construction, $\overline{P}_{d,g,n}$ contains $P_{d,g,n}$ for all $n \geqslant 0$.

Remark 2.12. The quotient stack $\overline{P}_{d,g}$ from above is the rigidification (in the sense of [ACV03]) of the stack $\overline{P}_{d,g,0}$. Unfortunately, the stacks $\overline{P}_{d,g,n}$ can never be Deligne-Mumford because there is an action of GL_n given by the scalar product on the line bundles which leaves the curves and the sections fixed. If moreover $(d - g + 1, 2g - 2) = 1$, the rigidification of $\overline{P}_{d,g,n}$ is a Deligne-Mumford stack and the morphism $\Psi_{d,g,n}$ is proper. For more details, see Section 7 of [Mel11].

Chapter 3

Enumerative study of de Jonquières divisors

In this chapter we study the dimension theory of de Jonquières divisors on algebraic curves. We begin by extracting as much information as possible about the geometry of the space $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ of de Jonquières divisors without using any degeneration techniques: in Section 3.1 we describe the space of de Jonquières divisors as a determinantal variety inside the symmetric product C_d and in Section 3.2 we write down its tangent space and the transversality condition which we are already able to verify in some special cases. In Section 3.3 we establish the existence result in the case of positive expected dimension by means of a class computation. We then proceed in Section 3.4 to construct degenerations of de Jonquières divisors using both limit linear series and compactified Picard schemes which will be used to prove Theorem 1.5 (dimension result) in Section 3.5, Theorem 1.8 (smoothness) in Section 3.6, and finally Theorem 1.9 (non-existence for non-complete series) in Section 3.7. We conclude in Section 3.8 with a discussion of de Jonquières divisors admitting negative coefficients.

3.1 The space of de Jonquières divisors as degeneracy locus

Fix an integer $k \leqslant d$ and two vectors of positive integers $\mu_1 = (a_1, \ldots, a_k)$ and $\mu_2 = (d_1, \ldots, d_k)$ such that $\sum_{i=1}^{k} a_i d_i = d$. The space $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ can be described as a degeneracy locus of vector bundles over C_d as follows: the idea is that the condition

$$a_1 D_1 + \ldots + a_k D_k \in \mathbb{P} V$$

is equivalent to the condition that the natural restriction map

$$V \to H^0(C, L/L(-a_1 D_1 - \ldots - a_k D_k))$$

has non-zero kernel. To reformulate this globally in terms of a morphism of two vector bundles over C_d, let the first bundle $\mathcal{E} = \mathcal{O}_{C_d} \otimes V$ be the trivial bundle. As for the second bundle, consider the diagram

$$\begin{tikzcd} & C \times C_d \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \supset \mathcal{U} \\ C & & C_d \end{tikzcd}$$

where σ and τ are the usual projections and $\mathcal{U}$ is the universal divisor defined as

$$\mathcal{U} = \{(p, D) \mid D \in C_d \text{ and } p \in D\} \subset C \times C_d.$$

Alternatively, identifying C_d with the Hilbert scheme $C^{[d]}$ of d points on C, one defines $\mathcal{U}$ as the universal family $\mathcal{U} \subset C \times C^{[d]}$. For the second bundle, consider the sheaf:

$$\mathcal{F}_d(L) = \tau_*(\sigma^* L \otimes \mathcal{O}_{\mathcal{U}}),$$

By cohomology and base change $\mathcal{F}_d(L)$ is indeed a vector bundle. The fibre of $\mathcal{F}_d(L)$ over any point $D \in C_d$ is given by the d-dimensional vector space $H^0(C, L/L(-D))$.

Finally, let $\Phi : \mathcal{E} \to \mathcal{F}_d(L)$ be the vector bundle morphism obtained by pushing down to C_d the restriction $\sigma^* L \to \sigma^* L \otimes \mathcal{O}_{\mathcal{U}}$. Moreover let

$$\Sigma_{k,N}(\mu_1, \mu_2) = \Big\{ E \in C_d \mid E = \sum_{i=1}^{k} a_i D_i \text{ for some } D_1 \in C_{d_1}, \dots, D_k \in C_{d_k} \Big\}.$$

The space $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ is defined as the r-*th degeneracy locus* of Φ, i.e. the locus in $\Sigma_{k,N}(\mu_1, \mu_2)$ where $\operatorname{rk} \Phi \leqslant r$.

Lemma 3.1. *For every point* $D \in DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$, *one has*

$$\dim_D DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l) \geqslant N - d + r.$$

Proof. From the description of $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ as a degeneracy locus, its codimension in $\Sigma_{k,N}(\mu_1, \mu_2)$ is at most

$$(\operatorname{rk} \mathcal{E} - r)(\operatorname{rk} \mathcal{F}_d(L) - r) = (r + 1 - r)(d - r) = d - r.$$

Since $\dim \Sigma_{k,N}(\mu_1, \mu_2) = N$, the dimension estimate follows. □

Finally, we record here an easy result that forms the base case for the induction argument in the proof of Theorem 1.9.

Lemma 3.2. *Let* C *be any smooth curve of genus* g *with a general linear series* $l \in G^r_d(C)$. *Fix an integer* $k \leqslant d$ *and two vectors of positive integers* $\mu_1 = (a_1, \dots, a_k)$ *and* $\mu_2 = (d_1, \dots, d_k)$ *such that* $\sum_{i=1}^{k} a_i d_i = d$ *and* $N - d + r < 0$.

1. *If* $g - d + r = 0$, *then* $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l) = \emptyset$.

2. *If* $g - d + r < 0$ *and* $N < g$, *then* $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l) = \emptyset$.

Proof. 1. Consider the following restriction of the Abel-Jacobi map:

$$\begin{aligned} u : \Sigma_{k,N}(\mu_1, \mu_2) &\to \mathrm{Pic}^d(C) \\ a_1 D_1 + \ldots + a_k D_k &\mapsto \mathcal{O}_C(a_1 D_1 + \ldots + a_k D_k). \end{aligned}$$

In the non-special regime $\mathrm{Pic}^d(C) = W^r_d(C)$. Moreover the image of φ is closed and $\dim \operatorname{im} u \leqslant N < g = \dim \mathrm{Pic}^d(C)$. Thus a general line bundle $L \in \mathrm{Pic}^d(C)$ is not contained in the image of u whence we conclude that the divisor $a_1 D_1 + \ldots + a_k D_k$ is not contained in a general linear series l of type g^r_d, i.e. $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l) = \emptyset$. This is Theorem 1.9 for non-special linear series.
As a consequence, for all $r' < r$, a general linear series l' in $G^{r'}_d(C)$ also has $DJ^{r',d}_{k,N}(\mu_1, \mu_2, C, l') = \emptyset$. To see this, let $c : G^{r'}_d(C) \to W^{r'}_d \subset W^r_d(C)$ be the forgetful map $(L, V) \mapsto L$. Note that a line bundle $L \in W^r_d(C) \setminus \operatorname{im} u$ does not admit de Jonquières divisors of length N. Now, since c is continuous (as the projection morphism from a Grassmann bundle), $c^{-1}(W^{r'}_d(C) \setminus \operatorname{im} u)$ is also open in $G^{r'}_d$ and nonempty. Hence no $l' \in c^{-1}(W^{r'}_d(C) \setminus \operatorname{im} u)$ admits a de Jonquières divisor of length N and our claim is proved.

2. Set $r_1 = d - g$ so that $g - d + r_1 = 0$ and $r < r_1$. We conclude from the discussion above that if $N < g$, then $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l) = \emptyset$ for a general linear series l. The non-existence for $N \geqslant g$ for $DJ^{r,d}_n(\mu, C, l)$ follows by an induction argument explained in Section 3.7.

□

3.2 Preliminary discussion of transversality

We deal here only with the case of complete linear series $l \in G^r_d(C)$ such that $|D| = l$ for some $D \in C_d$. Consider the alternative description of the space $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ as the intersection

$$DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l) = \Sigma_{k,N}(\mu_1, \mu_2) \cap |D|.$$

The condition for transversality of intersection is:

$$T_D(C_d) = T_D(\Sigma_{k,N}(\mu_1, \mu_2)) + T_D(|D|), \tag{3.1}$$

for a divisor $D = \sum_{i=1}^k a_i D_i$, such that $D_i \in C_{d_i}$ and the fixed vectors of strictly positive integers $\mu_1 = (a_1, \ldots, a_k)$ and $\mu_2 = (d_1, \ldots, d_k)$ satisfy $\sum_{i=1}^k a_i d_i = d$ and $N = \sum_{i=1}^k d_i$.

Recall that

$$T_D(C_d) = H^0(C, \mathcal{O}_D(D)),$$

as shown for instance in [ACGH85] chapter IV, §1. Moreover its dual is

$$T_D^\vee(C_d) = H^0(K_C/K_C(-D))$$

and the pairing between the tangent and cotangent space is given by the residue.

To compute $T_D(\Sigma_{k,N}(\mu_1,\mu_2))$, let $\mathcal{D}_i$ denote the diagonal in the a_i-th product $C_{d_i} \times \ldots \times C_{d_i}$ so that $\Sigma_{k,N}(\mu_1,\mu_2) = \mathcal{D}_1 \times \ldots \times \mathcal{D}_k/S_d$. Hence

$$T_D(\Sigma_{k,N}(\mu_1,\mu_2)) = T_{a_1 D_1}\mathcal{D}_1 \oplus \ldots \oplus T_{a_k D_k}\mathcal{D}_k.$$

Since $T\mathcal{D}_i = TC_{d_i}$ for all $i = 1,\ldots,k$,

$$\begin{aligned} T_D(\Sigma_{k,N}(\mu_1,\mu_2)) &= T_{D_1}C_{d_1} \oplus \ldots \oplus T_{D_k}C_{d_k} \\ &\simeq T_{(D_1,\ldots,D_k)}C_{d_1} \times \ldots \times C_{d_k} \\ &\simeq T_{D_1+\ldots+D_k}C_N \\ &= H^0(C, \mathcal{O}_C(D_1+\ldots+D_k)/\mathcal{O}_C), \end{aligned}$$

and is isomorphic to $H^0(K_C(-D_1-\ldots-D_k)/K_C(-D))^0$, where the superscript ^{0} denotes the annihilator of a vector space.

To determine $T_D|D|$, consider the following restriction of the Abel-Jacobi map:

$$u : C_d^r \to W_d^r(C)$$

with differential given by

$$\delta : \mathrm{im}(\alpha\mu_0)^0 \to \mathrm{im}(\mu_0)^0,$$

where δ denotes the restriction of the coboundary map

$$H^0(C, \mathcal{O}_D(D)) \to H^1(C, \mathcal{O}_C)$$

of the Mittag-Leffler sequence to $T_D C_d^r = \mathrm{im}(\alpha\mu_0)^0$, while

$$\alpha : H^0(C, K_C) \to H^0(C, K_C \otimes \mathcal{O}_D)$$

is the restriction mapping and

$$\mu_0 : H^0(C, K_C - D) \otimes H^0(C, \mathcal{O}_C(D)) \to H^0(C, K_C)$$

the cup-product mapping (see Chapter IV of [ACGH85] for details).

Let $D \in C_d^r$. Then $|D| \subset C_d^r$ and $u(D) \in W_d^r(C)$ with $u^{-1}(u(D)) = |D|$. Since δ is surjective by definition,

$$T_D|D| = T_D(u^{-1}(u(D))) = \ker\delta = \mathrm{im}(\delta^\vee)^0,$$

where the dual map $\delta^\vee$ is the restriction of α to $(\mathrm{im}(\mu_0)^0)^\vee = \mathrm{coker}(\mu_0)$.

The transversality condition (3.1) translates to

$$T_D C_d = H^0\Big(C, K_C\Big(-\sum_{i=1}^{k} D_i\Big)/K_C(-D)\Big)^0 + \mathrm{im}(\delta^\vee)^0$$

which is equivalent to:

$$H^0\Big(C, K_C\Big(-\sum_{i=1}^{k} D_i\Big)/K_C(-D)\Big) \cap \mathrm{im}(\delta^\vee) = 0.$$

If the Brill-Noether number $\rho(g, r, d) = 0$, then μ_0 is an isomorphism and this means that there are two possibilities for a differential in $H^0(C, K_C)$:

1. the differential is of the form $1 \otimes \omega$ with $\omega \in H^0(C, K_C - D)$. Then clearly $\delta^\vee(\omega) = 0$.

2. the differential is of the form $f \otimes \omega$, where f is a meromorphic function with divisor of poles D. Assuming that $f \otimes \omega \in H^0(C, K_C - D_1 - \ldots - D_k)$, we notice that it vanishes on $D_1 + \ldots + D_k$ if and only if

$$\omega \in H^0(C, K_C - D - D_1 - \ldots - D_k).$$

We conclude that the transversality condition (3.1) can be reformulated as:

$$H^0(C, K_C - D - D_1 - \ldots - D_k) = 0. \tag{3.2}$$

If the Brill-Noether number $\rho(g, r, d) \geqslant 1$, then the transversality condition becomes

$$H^0\Big(C, K_C - \sum_{i=1}^{k} D_i\Big) \cap \mathrm{coker}(\mu_0) = 0.$$

Since $H^0(C, K_C) = \mathrm{im}(\mu_0) \oplus \mathrm{coker}(\mu_0)$, the transversality condition becomes

$$H^0\Big(C, K_C - \sum_{i=1}^{k} D_i\Big) \subseteq \mathrm{im}(\mu_0).$$

Using the same argument as in the previous case, we obtain the same transversality condition (3.2) as in the case $\rho(g, r, d) = 0$.

Note that the condition (3.2) is immediately satisfied by non-special (i.e. g^r_d-s with $g - d + r = 0$) and canonical linear series therefore proving Theorems 1.5 and 1.8 in these cases. There are actually a few more cases where transversality follows without using degenerations to nodal curves.

3.2.1 The case $r = 1$

The argument in this case is similar to the one in Section 5 of [HM82]. The idea is to consider the map

$$\pi : C \to \mathbb{P}^1$$

given by the de Jonquières divisor $D = \sum_{i=1}^{k} a_i D_i$ and its versal deformation space $\mathcal{V}$. Moreover, let $\mathcal{V}' \subset \mathcal{V}$ be the subvariety of maps given by divisors with the same coefficients as D. Then the tangent space to $\mathcal{V}$ at π is $T_\pi \mathcal{V} = H^0(C, \mathcal{N})$, where $\mathcal{N}$ is the normal sheaf of π defined by the exact sequence

$$0 \to \mathcal{T}_C \to \pi^* \mathcal{T}_{\mathbb{P}^1} \to \mathcal{N} \to 0,$$

where $\mathcal{T}_C$ is the tangent sheaf of C and $\mathcal{T}_{\mathbb{P}^1}$ the tangent sheaf of $\mathbb{P}^1$.

Consider also the forgetful map $\beta : \mathcal{V} \to \mathcal{M}_g$, with differential β_* given by the coboundary map

$$H^0(C, \mathcal{N}) \to H^1(C, \mathcal{T}_C)$$

of the exact sequence above. We now identify the tangent space to $\mathcal{V}'$ with the subspace of $H^0(C, \mathcal{N})$ of sections of $\mathcal{N}$ that vanish in a neighbourhood of the points in the support of $D_1 + \ldots + D_k$, i.e. the sections of the sheaf $\mathcal{N}'$ defined by the sequence

$$0 \to \mathcal{T}_C \to \pi^* \mathcal{T}_{\mathbb{P}^1}(-(a_1 - 1)D_1 - \ldots - (a_k - 1)D_k) \to \mathcal{N}' \to 0.$$

Since π is a point in the general fibre of $\beta|_{\mathcal{V}'}$, from Sard's theorem it follows that the differential β_* restricted to $\mathcal{T}_\pi(\mathcal{V})$ is surjective. This in turn means that the map β' below is surjective:

$$H^0(C, \mathcal{N}') \xrightarrow{\beta'} H^1(C, \mathcal{T}_C) \to H^1(C, \pi^* \mathcal{T}_{\mathbb{P}^1}(-(a_1 - 1)D_1 - \ldots - (a_k - 1)D_k)) \to 0.$$

Now, note that $\mathcal{T}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(2)$ and moreover $\pi^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_C(D)$. Therefore

$$\begin{aligned} 0 &= H^1(C, \mathcal{O}_C(2D - (a_1 - 1)D_1 - \ldots - (a_k - 1)D_k)) \\ &= H^0(C, K_C - D - D_1 - \ldots - D_k) \end{aligned}$$

as desired.

3.2.2 The case $r = 2$

Denote by $\mathcal{Z}^N$ the smooth $(N + 2)$-dimensional subvariety of the N-th symmetric product of $\mathbb{P}^2$ corresponding to collinear length N zero-cycles in $\mathbb{P}^2$. Further imposing on $\mathcal{Z}^N$ that the coefficients in these zero-cycles add up to d (i.e. imposing d independent conditions) yields indeed that

$$\dim DJ^{2,d}_{k,N}(\mu_1, \mu_2, C, l) = N - d + 2$$

for every linear series l.

3.2.3 The case $g - d + r = 1$

Let $D = \sum_{i=1}^{k} a_i D_i$ be a de Jonquières divisor such that $l = |D|$ is a g^r_d with $g - d + r = 1$ and, as usual, let L denote the corresponding line bundle. This means that the residual linear series $K_C - l$ is an isolated divisor $E \in C_{2g-2-d}$ such that $K_C = \mathcal{O}_C(D + E)$. Consider the subspace

$$\mathcal{P}_d = \{L \in \mathrm{Pic}^d(C) \mid h^1(C, L) = 1\} \subset \mathrm{Pic}^d(C),$$

which, by the previous observation, has dimension $2g - 2 - d$. Now consider the space

$$\mathcal{Q} = \{(E, D_1 + \ldots + D_k) \in C_{2g-d-2} \times C_N \mid \mathcal{O}_C(E + a_1 D_1 + \ldots + a_k D_k) = K_C\}.$$

Polishchuk [Pol06] shows that this space is smooth with

$$\dim \mathcal{Q} = N + g - d - 1.$$

Hence, for a general fixed isolated divisor $E \in C_{2g-d-2}$, the space

$$\mathcal{Q}' = \{D_1 + \ldots + D_k \in C_N \mid \mathcal{O}_C(a_1 D_1 + \ldots + a_k D_k) = \mathcal{O}_C(K_C - E)\}$$

is also smooth and of dimension

$$(N + g - d - 1) - (2g - d - 2) = N - g + 1 = N - d + r,$$

which immediately implies the same for the space $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ for a general linear series l with $g - d + r = 1$.

We can in fact do better than this and prove transversality for an arbitrary linear series l with $g - d + r = 1$. From Polishchuk's result we have that the intersection

$$\Sigma = \{E + D \in C_{2g-2} \mid D \in \Sigma_{k,N}(\mu_1, \mu_2)\} \cap |K_C|$$

is transverse, i.e.

$$T_{E+D}(\Sigma) + T_{E+D}|K_C| = T_{E+D}(C_{2g-2}).$$

Using the facts that

$$\begin{aligned} T_{E+D}(\Sigma) &= T_E(C_{2g-d-2}) \oplus T_D(\Sigma_{k,N}(\mu_1, \mu_2)) \\ T_{E+D}(C_{2g-2}) &= T_E(C_{2g-d-2}) \oplus T_D(C_d) \\ T_{E+D}|K_C| &= T_E|E| \oplus T_D|K_C - E| = T_D|L|, \end{aligned}$$

we obtain

$$T_D(\Sigma_{k,N}(\mu_1, \mu_2)) + T_D|L| = T_D(C_d),$$

which is the sought after transversality condition.

Therefore, in order to prove Theorem 1.8, it remains to check the transversality condition (3.2) for $r \geqslant 3$ and $g - d + r \geqslant 2$. We do this using degenerations in Section 3.6.

3.3 Existence of de Jonquières divisors

Luckily, the question of existence is easily answered in a manner similar to that of the first proofs of the existence part of the Brill-Noether theorem ([Kem71] and [KL72a]). The idea is to simply look at the class of $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$ and establish its positiveness. Consider the *diagonal mapping* for C_d:

$$\begin{aligned} \epsilon : C_{d_1} \times \ldots \times C_{d_k} &\to C_d \\ D_1 + \ldots + D_k &\mapsto a_1 D_1 + \ldots + a_k D_k. \end{aligned}$$

It is well-known (see for example chapter VIII §5 of [ACGH85]) that the image, via ϵ of the fundamental class of $\epsilon(C_{d_1} \times \ldots \times C_{d_k})$ is equal to the coefficient of the monomial $t_1^{d_1} \cdot \ldots \cdot t_k^{d_k}$ in

$$\sum_{a \geqslant b} \frac{(-1)^{a+b}}{b!(a-b)!} \left(1 + \sum_{i=1}^{k} a_i t_i\right)^{N-g+b} \left(1 + \sum_{i=1}^{k} a_i^2 t_i\right)^{g-b} x^{d-N-a} \theta^a,$$

where θ is the pullback of the fundamental class of the theta divisor to C_d and x the class of the divisor $q + C_{d-1} \subset C_d$. Evaluating this formula on a linear series l of degree d and dimension r, and using that $\theta \mid_l = 0$, we obtain the following expression for the class of $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$:

$$\left(1 + \sum_{i=1}^{k} a_i t_i\right)^{N-g} \left(1 + \sum_{i=1}^{k} a_i^2 t_i\right)^{g} x^{d-N}[l].$$

If $N - d + r \geqslant 0$, this class is clearly positive and yields the non-emptiness of $DJ^{r,d}_{k,N}(\mu_1, \mu_2, C, l)$.

3.4 Degenerations of de Jonquières divisors

In the case of nodal curves, the usual correspondence between divisors and line bundles breaks down. Most significantly for our problem, the Abel-Jacobi map

$$C_d \to \mathrm{Pic}^d(C)$$

does not make sense any more, even though the two spaces C_d and $\mathrm{Pic}^d(C)$ are still defined. As a simple example of this failure, the sheaf of functions with one pole at one of the nodes is not a line bundle, while the sheaf of functions with two poles at the node has degree 3. We therefore first need to make sense of the statement that a linear series on a nodal curve admits a de Jonquières divisor. We do this in a variational setting, by considering families of smooth curves degenerating to a nodal curve and analysing what happens on the central fibre to limits of line bundles admitting de Jonquières divisors. As mentioned in the Introduction, we approach this issue from two points of view: limit linear series for central fibres of compact type in Section 3.4.1 and compactified Picard schemes for stable central fibres in 3.4.2.

3.4.1 Limit linear series approach

In the framework of 2.1, fix $Y \subset \mathscr{X}_0$ an irreducible component of the central fibre. Let $\mathscr{D}^* = (\sigma) \in |\mathscr{L}^*|$ (and $\mathscr{D}^*_t = (\sigma)|_{\mathscr{X}^*_t}$) be a divisor on $\mathscr{X}^*$, where σ is a section of $\mathscr{L}^*$. To find the limit of $\mathscr{D}^*$ on $\mathscr{X}_0$, we multiply σ by the unique power of $t \in B^*$ so that it extends to a holomorphic section σ_Y of the extension $\mathscr{L}_Y$ on the whole of $\mathscr{X}$ and so that it does not vanish identically on $\mathscr{X}_0$. The limit of $\mathscr{D}^*$ is the divisor $(\sigma_Y|_Y)$.

Definition 3.3 (De Jonquières divisors on a nodal curve of compact type)**.** Let $\mathscr{X}_0$ be a nodal curve of compact type equipped with a smoothable limit linear series l of type g^r_d. Fix an integer $k \leqslant d$ and two vectors of positive integers $\mu_1 = (a_1, \dots, a_k)$ and $\mu_2 = (d_1, \dots, d_k)$ such that $\sum_{i=1}^k a_i d_i = d$. The divisor $\sum_{i=1}^k a_i D_i$ with $D_i \in C_{d_i}$ on $\mathscr{X}_0$ is a *de Jonquières divisor for* l if for each aspect l_Y corresponding to an irreducible component $Y \subset X$, there is a section $\sigma_Y|_Y$ as above vanishing on $\sum_{D_{i,Y} \in Y} a_i D_{i,Y}$, where $D_{i,Y}$ is the specialization of the divisor D_i on the component Y.

The section $\sigma_Y|_Y$ will also vanish at the nodes of $\mathscr{X}_0$ belonging to Y, and in such a way that (2.2) is satisfied (we assume the limit series l to be refined, so we have equality in (2.2)). Hence the series l_Y of type g^r_d on Y admits the de Jonquières divisor

$$\sum_{i=1}^k a_i D_{i,Y} + \sum_{q \in \mathrm{Sing}(\mathscr{X}_0), q \in Y} \left(d - \sum_{i=1}^k a_i d_{i,Y} \right) q,$$

where the sum is over the preimages $q \in Y$ of the nodes of $\mathscr{X}_0$, and $d_{i,Y} = \deg D_{i,Y}$. We therefore have a way to go from de Jonquières divisors on a nodal curve of compact type to de Jonquières divisors on its smooth components, where the coefficients of the nodes must of course satisfy the inequality (2.2).

In what follows we construct the space of de Jonquières divisors on families of nodal curves of compact type and endow it once more with the structure of a degeneracy locus.

Denote by ℓ a T-valued point of $\mathscr{G}^r_d(\mathscr{X}/B)(T)$. In what follows, we construct a functor $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$, represented by a scheme which is projective over B, and which parametrises de Jonquières divisors for a family $\mathscr{X} \to B$ of curves of genus g of compact type equipped with a linear series ℓ.

Proposition 3.4. *Fix a projective, flat family of curves $\mathscr{X} \to B$ over a scheme B equipped with a linear series ℓ of type g^r_d. Let $\mu_1 = (a_1, \dots, a_k)$ and $\mu_2 = (d_1 \dots, d_k)$ be vectors of positive integers such that $\sum_{i=1}^k a_i d_i = d$. As usual, let $N = \sum_{i=1}^k d_i$. Consider also the relative divisors $\mathscr{D}_i \subset \mathscr{X}^{d_i}$. There exists a scheme $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$ projective over B, compatible with base change, whose point over every $t \in B$ parametrises objects $[\mathscr{X}_t, \mathscr{D}_1(t), \dots, \mathscr{D}_k(t)]$ such that*

$\sum_{i=1}^{k} a_i \mathscr{D}_i(t)$ *is a de Jonquières divisor of* ℓ_t. *Furthermore, every irreducible component of* $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$ *has dimension at least* $\dim B - d + r$.

Proof. We construct the functor $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$ as a subfunctor of the functor of points of the fibre product $\mathscr{X}^N$ over B. We show that it is representable by a scheme that is projective over B and which we also denote by $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$.

Let $T \to B$ be a scheme over B. Suppose first that all the fibres of the family are nonsingular. In this case, from the discussion above, a g^r_d on $\mathscr{X}$ is given by a pair $(\mathscr{L}, \mathscr{V})$, where $\mathscr{V} \subseteq \pi_{2*}\mathscr{L}$ is a vector bundle of rank $r+1$ on B. Then the T-valued point $[\mathscr{X}, \mathscr{D}_1, \ldots, \mathscr{D}_k]$ belongs to $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)(T)$ if the r-th degeneracy locus of the map

$$\mathscr{V} \to \pi_{2*}\mathscr{L}|_{\sum_{i=1}^{k} a_i \mathscr{D}_i}$$

is the whole of T. By construction $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$ is compatible with base change, so it is a functor, and it has the structure of a closed subscheme, hence it is representable and the associated scheme is projective.

Alternatively, more explicitly, take the projective bundle $\mathbb{P}\mathscr{V}$ corresponding to $\mathscr{V}$ which has rank r, with elements in its fibres given by sections $\sigma \in H^0(\mathscr{L}|_{\mathscr{X}_t})$ up to equivalence with respect to scalar multiplication. Consider the subscheme $\mathcal{DJ}'(\mathscr{X}, \mathscr{V})$ in $\mathbb{P}\mathscr{V}$ cut by the equations coming from the condition that the sections vanish on $\mathscr{D}_i$ with multiplicity at least a_i. This imposes in total $\sum_{i=1}^{k} a_i d_i = d$ conditions, so the dimension of every irreducible component of $\mathcal{DJ}'(\mathscr{X}, \mathscr{V})$ is at least $\dim B - d + r$. Collecting all irreducible components of $\mathcal{DJ}'(\mathscr{X}, \mathscr{V})$ such that the section σ does not vanish on the whole underlying curve, we obtain the desired $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$.

Now suppose that some of the fibres have nodes (that may or may not be smoothed by $\mathscr{X}$ - see Remark 2.4). From the discussion above, a g^r_d on $\mathscr{X}$ is a tuple $(\mathscr{L}, (\mathscr{V}^v)_{v \in V(\Gamma_0)})$. Let $v_j \in \Gamma_0$ be the vertex corresponding to an irreducible component. Denote by $\mathscr{D}_{i,j}$ the specialisation of $\mathscr{D}_i$ to v_j. Then the T-valued point $[\mathscr{X}, \mathscr{D}_1, \ldots, \mathscr{D}_k]$ belongs to $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)(T)$ if, for all vertices v_j, the r-th degeneracy locus of the map

$$\mathscr{V}^{v_j} \to \pi_{2*}\mathscr{L}^{d^{\vec{v}_j}}|_{a_i \mathscr{D}_{i,j}}$$

is the whole of T. Checking for compatibility with base change (and hence functoriality) is more delicate than in the previous case because the base change may change the graph Γ_0. However, arguing like in the proof of Proposition 4.5.6 in [Oss] yields the desired property. Representability and projectiveness then follow analogously.

Alternatively, if no nodes are smoothed in $\mathscr{X}$, for each vertex v of the dual graph Γ of the fibres, we have a family $\mathscr{Y}^v$ of smooth curves with the divisors $\mathscr{D}_i$ belonging to $\mathscr{Y}^v$ and additional sections q_j corresponding to the

preimages of the nodes. Consider now the space $\mathcal{DJ}'(\mathscr{Y}^\nu, \mathscr{V}^\nu)$ defined as in the case of families with smooth fibres by the vanishing at the $\mathscr{D}_i$. In addition, we cut $\mathcal{DJ}'(\mathscr{Y}^\nu, \mathscr{V}^\nu)$ with the equations corresponding to the vanishing of the sections at the points q_j, subject to the constraints explained in the discussion following Definition 3.3. We denote the space thus obtained by $\mathcal{DJ}'(\mathscr{Y}^\nu, \mathscr{V}^\nu)$ as well. Finally, the desired space $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$ is obtained by taking the fibre product over B of the $\mathcal{DJ}'(\mathscr{Y}^\nu, \mathscr{V}^\nu)$. The dimension estimate follows as in the case of smooth fibres. If there are smoothed nodes, for each $\nu \in V(\Gamma_0)$, consider the subscheme $\mathcal{DJ}'(\mathscr{X}, \mathscr{V}^\nu)$ in $\mathbb{P}\,\mathscr{V}^\nu$ cut by the vanishing conditions at the divisors $\mathscr{D}_i$ and at the nodes. Taking the fibre product over B yields the space $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$ and the dimension bound. □

Remark 3.5. Let $\phi : \mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell) \to \mathscr{X}$ be the forgetful map, which is projective by base change. Then the fibre of ϕ over a curve $\mathscr{X}_t$ is precisely $DJ^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$.

To conclude the study of the space $\mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}, \ell)$ of de Jonquières divisors for a family of curves, we investigate their smoothability.

Proposition 3.6. *Suppose the object* $[C, D_1, \ldots, D_k] \in B$ *is contained in an irreducible component* $U \subset \mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \mathscr{X}/B, \ell)$ *with* $\dim U = \dim B - d + r$. *Then the general point of* U *parametrises a de Jonquières divisor on a smooth curve.*

Proof. We essentially follow the argument in the proof of Theorem 3.4 of [EH86].

Let $\widetilde{\mathscr{X}} \to \widetilde{B}$ be the versal family of pointed curves around $[C, p_1, \ldots, p_n]$ and let $f : B \to \widetilde{B}$ be the map inducing $\pi : \mathscr{X} \to B$ with n sections corresponding to the marked points. Moreover, let $\widetilde{\mathscr{L}}$ be the corresponding linear series on $\widetilde{\mathscr{X}}$. Let $\widetilde{U} \subset \mathcal{DJ}^{r,d}_{k,N}(\mu_1, \mu_2, \widetilde{\mathscr{X}}/\widetilde{B}, \widetilde{\ell})$ be a component such that $U \subset f^*\widetilde{U}$ and denote by $\widetilde{C}$ the point of $\widetilde{U}$ corresponding to C. By Proposition 3.4, $\dim \widetilde{U} \geqslant \dim \widetilde{B} - d + r$. Hence, if $\widetilde{U}$ does not completely lie in the discriminant locus of $\widetilde{\mathscr{X}} \to \widetilde{B}$ which parametrises nodal curves, then a general point of $\widetilde{U}$ corresponds to a de Jonquières divisor on a smooth curve. On the other hand, if $\widetilde{U}$ lies over a component $\widetilde{B}'$ of the discriminant locus and $f(B) \subset \widetilde{B}'$, then

$$\dim \widetilde{U} \geqslant \dim \widetilde{B} - d + r > \dim \widetilde{B}' - d + r,$$

since $\widetilde{B}'$ is a smooth hypersurface in $\widetilde{B}$. Therefore every component of $f^*\widetilde{U}$ (hence also U) must have dimension strictly larger than $\dim B - d + r$ which contradicts the assumption on $\dim U$. Hence $\widetilde{U}$ cannot lie entirely in the discriminant locus, and we are done. □

3.4.2 Compactified Picard scheme approach

In this section we give a different approach to the degenerations of de Jonquières divisors, this time relying ourselves on the framework of compactified Picard stacks on the moduli stack of curves with marked points summarised in 2.2. We chose to work with this compactification (instead of using rank-1 torsion-free sheaves) because we want to use an induction procedure involving restrictions of line bundles on different irreducible components of the nodal curve. Rank-1 torsion-free sheaves would not allow this, since their restrictions to subcurves are not necessarily torsion-free themselves.

Consider a smooth 1-parameter family $\pi : \mathscr{X} \to B$ of curves of genus g over the smooth curve B such that the fibres over $B^* = B \setminus 0$ are smooth curves, while the special fibre is given by a stable nodal curve $\mathscr{X}_0$. Denote by $I(\mathscr{X}_0)$ the set of all irreducible components of the central fibre and by $N(\mathscr{X}_0)$ the set of nodes lying at the intersection of distinct irreducible components, together with their respective supports, i.e.

$$N(\mathscr{X}_0) = \{(q, C) \mid q \in C \cap C' \text{ where } C, C' \in I(\mathscr{X}_0)\}.$$

Suppose that $\mathscr{L}^*$ is a line bundle on $\mathscr{X}^*$ such that the restriction $\mathscr{L}_t$ to each fibre $\mathscr{X}_t$ is of degree d for all $t \in B^*$. Then, using Caporaso's approach [Cap94] we can extend $\mathscr{L}^*$ over the central fibre $0 \in B$ such that the fibre $\mathscr{L}_0$ is a limit line bundle on $\mathscr{X}_0$ (or possibly a quasi-stable curve of $\mathscr{X}_0$) of degree d. As observed before, this limit is not unique because, for any $m_C \in \mathbb{Z}$,

$$\mathscr{L} \otimes \mathcal{O}_{\mathscr{X}}\Big(\sum_{C \in I(\mathscr{X}_0)} m_C C\Big) \tag{3.3}$$

is also an extension of $\mathscr{L}^*$ to B. We call the new extension in (3.3) a *twisted line bundle*. Observe also the following "computation" rules

$$\begin{aligned} \mathcal{O}_{\mathscr{X}} &\simeq \mathcal{O}_{\mathscr{X}}\Big(\sum_{C \in I(\mathscr{X}_0)} C\Big) \\ \mathcal{O}_{\mathscr{X}}\Big(\sum_{C \in I(\mathscr{X}_0)} m_C C\Big)\Big|_{C'} &\simeq \mathcal{O}_{C'}\Big(\sum_{q \in C \cap C'} (m_C - m_{C'})q\Big). \end{aligned} \tag{3.4}$$

We encode this information in a *twist function*:

$$\begin{aligned} T : N(\mathscr{X}_0) &\to \mathbb{Z} \\ (q, C) &\mapsto m_{C'} - m_C \end{aligned}$$

and introduce the following

Definition 3.7. A *twist* of the line bundle $\mathscr{L}$ is a function $T : N(\mathscr{X}_0) \to \mathbb{Z}$ satisfying the following properties:

1. Given $C, C' \in I(\mathscr{X}_0)$ and $q \in C \cap C'$, then $T(q, C) = -T(q, C')$.
2. Given $C, C' \in I(\mathscr{X}_0)$ and $q_1, \ldots, q_n \in C \cap C'$, then
$$T(q_1, C) = \ldots = T(q_n, C) = -T(q_1, C') = \ldots = -T(q_n, C).$$
3. Given $C, C', \widehat{C}, \widehat{C'} \in I(\mathscr{X}_0)$, and points $q_C \in C \cap \widehat{C}$, $q_{C'} \in C' \cap \widehat{C'}$, $q \in C \cap C'$, and $\widehat{q} \in \widehat{C} \cap \widehat{C'}$, such that
$$T(q_C, C) = T(q_{C'}, C') = 0,$$
we have that
$$T(q, C) = T(\widehat{q}, \widehat{C}).$$

Remark 3.8. The definition for the twist T of a line bundle L on a single curve X is analogous.

Definition 3.9. Fix a quasi-stable curve Y of a stable curve X with n marked points $p_1, \ldots, p_n$. The line bundle L with balanced multidegree $\underline{d}$ on Y admits a de Jonquières divisor $\sum_{i=1}^{n} a_i p_i$ if there exists a twist T such that, for all $C \in I(Y)$,
$$L|_C = \mathcal{O}_C\Big(\sum_{p_i \in C} a_i p_i\Big) \otimes \mathcal{O}_C\Big(\sum_{q \in C} T(q, C) q\Big).$$
In other words, each restriction of L to the irreducible components C of Y admits the de Jonquières divisor
$$\sum_{p_i \in C} a_i p_i + \sum_{q \in C} T(q, C) q.$$

Remark 3.10. If C is an exceptional component, then the de Jonquières divisor has only the nodes q in the support.

Remark 3.11. If any of the coefficients in the divisor above are negative, we find ourselves in the situation described in Section 3.8.

Remark 3.12. Here our perspective on de Jonquières divisors on quasi-stable curves is naive in the sense that we ignore the precise vanishing or residue conditions at the nodes. In what follows we construct a space that not only contains the closure of the space of smooth curves with marked points and line bundles admitting de Jonquières divisors, but also some "virtual" components which we keep, in the same vein as the space of twisted canonical divisor of [FP18].

We now define the notion of de Jonquières divisors for a family of stable curves with n marked points. We work locally so that a Poincaré bundle exists (otherwise we would have to assume that $(d - g + 1, 2g - 2) = 1$).

Definition 3.13. Let $(\pi : \mathscr{X} \to B, p_i : B \to \mathscr{X}, \mathscr{L})$ be a flat, proper family of quasi-stable curves of genus g with n marked points equipped with a relative degree d balanced line bundle $\mathscr{L}$ such that $\mathscr{L}_t \in \overline{W}^{\mathscr{X}_t}_{r,d}$. For a fixed partition $\mu = (a_1, \ldots, a_n)$ of d we say that $\mathscr{L}$ admits the de Jonquières divisor $\sum_{i=1}^n a_i p_i$ if for all $t \in B$, $\mathscr{L}_t$ admits the de Jonquières divisor $\sum_{i=1}^n a_i p_i(t)$. Furthermore, define the locus $\mathcal{DJ}^{r,d}_{g,n,\mu}(B)$ of de Jonquières divisors in $\overline{P}_{d,g,n}$ by

$$\mathcal{DJ}^{r,d}_{g,n,\mu}(B) = \left\{ \left(\pi : \mathscr{X} \to B, p_i : B \to \mathscr{X}, \mathscr{L}\right) \mid \mathscr{L} \text{ admits divisor } \sum_{i=1}^n a_i p_i \right\}.$$

In what follows we also need the result below (for a proof, see [Ray70] Proposition 6.1.3).

Lemma 3.14. *Let B be a smooth curve and let $f : \mathscr{X} \to B$ be a flat and proper morphism. Fix a point $b_0 \in B$ and set $B^* = B \setminus b_0$. Let $\mathscr{L}$ and $\mathscr{M}$ be two line bundles on $\mathscr{X}$ such that $\mathscr{L}|_{f^{-1}(B^*)} \simeq \mathscr{M}_{f^{-1}(B^*)}$. Then*

$$\mathscr{L} = \mathscr{M} \otimes \mathcal{O}_{\mathscr{X}}(\mathcal{C}),$$

where $\mathcal{C}$ is a Cartier divisor on $\mathscr{X}$ supported on $f^{-1}(b_0)$.

The content of the following proposition is that, for a one-parameter family of quasi-stable curves, the limit of de Jonquières divisors is itself a de Jonquières divisor.

Proposition 3.15. *The locus $\mathcal{DJ}^{r,d}_{g,n,\mu}(B)$ is closed in $\overline{P}_{d,g,n}$.*

Proof. We use the valuative criterion. Take a map ι from B^* to $\mathcal{DJ}^{r,d}_{g,n,\mu}(B)$. We must show that there exists a lift $\bar{\iota}$ of ι from B, as shown in the commutative diagram below.

$$\begin{tikzcd} B^* \arrow[r, "\iota"] \arrow[d, hook] & \mathcal{DJ}^{r,d}_{g,n,\mu}(B) \\ B \arrow[ur, dashed, "\bar{\iota}"] & \end{tikzcd}$$

Since a map from B^* to $\mathcal{DJ}^{r,d}_{g,n,\mu}(B)$ is the same as a family

$$(\pi : \mathscr{X}^* \to B^*, p_i : B^* \to \mathscr{X}^*, \mathscr{L}^*),$$

we must show that we can extend this to a family

$$(\pi : \mathscr{X} \to B, p_i : B \to \mathscr{X}, \mathscr{L})$$

in $\mathcal{DJ}^{r,d}_{g,n,\mu}(B)$. In other words, we must show that if the general fibre

$$(\mathscr{X}_t, p_i(t), \mathscr{L}_t),$$

for $t \in B^*$, is such that $\mathscr{L}_t$ admits the de Jonquières divisor $\sum_{i=1}^{n} a_i p_i(t)$, then the central fibre $(\mathscr{X}_0, p_i(0), \mathscr{L}_0)$ is such that $\mathscr{L}_0$ also admits the de Jonquières divisor $\sum_{i=0}^{n} a_i p_i(0)$.

From Definition 3.9, the family admits de Jonquières divisors if there exists a twist T_t for each fibre $\mathscr{X}_t$, with $t \in B^*$, such that, for all components $C \in \mathscr{X}_t$ and all nodes $q \in C$,

$$\mathscr{L}_t|_C \simeq \mathcal{O}_C\Big(\sum_{p_i(t)\in C} a_i p_i(t)\Big) \otimes \mathcal{O}_C\Big(\sum_{q\in C} T_t(q, C)q\Big).$$

By shrinking B, and after possibly performing a base change, we may assume that the fibres of $\mathscr{X}$ are of constant topological type, the twist T_t is the same twist T over B^*, and there is no monodromy in the components of the fibres over B^*. We must now assign a twist T_0 to the central fibre $\mathscr{X}_0$ equipped with $\mathscr{L}_0$.

Recall that the twist T_0 is a function $T_0 : N(\mathscr{X}_0) \to \mathbb{Z}$. There are two types of elements in $N(\mathscr{X}_0)$:

- (q_0, C_0) where q_0 is a node not smoothed by the family $\mathscr{Y}$. Here

  $$T_0(q_0, C_0) = T(q_t, C_t),$$

  where q_t is the corresponding node in the component C_t in the generic fibre over $t \in B$.

- (q_0, C_0) where q_0 is smoothed by the family $\mathscr{X}$. Here the twist T_0 must be assigned "by hand".

To do so, note also that the component $C_0 \in I(\mathscr{X}_0)$ belongs to a connected subcurve X of $\mathscr{X}_0$ which consists of all components belonging to the same equivalence class with respect to twists at the non-smoothed nodes, i.e.

$$C_0, C_0' \in X \Leftrightarrow C_0 \sim C_0' \Leftrightarrow T(q, C_0) = T(q, C_0') = 0, \forall q \in C_0 \cap C_0'.$$

This yields a sub-family of $\mathscr{X} \to B$, which we call $\mathscr{X}'$, whose central fibre is X and whose generic fibre is given by the corresponding subcurves in $\mathscr{X}_t$. The markings p_i which lie on the fibres of $\mathscr{X}'$ give sections which we rename $p_i' : B \to \mathscr{X}'$, for $i = 1, \ldots, n'$, where $n' \leqslant n$. The nodes connecting $\mathscr{X}_t'$ to its complement in $\mathscr{X}_t$ also yield sections $q_j : B \to \mathscr{X}'$, for $q = 1, \ldots, m$, for some $m \geqslant 1$; we emphasize here that the $q_j(t)$ are smooth points of $\mathscr{X}'$. Since the twist T at the $q_j(t)$ is non-zero (by the definition of the equivalence classes), for $t \in B^*$ and for any component $C_t \in I(\mathscr{X}_t')$,

$$\mathscr{L}_t|_{C_t} \simeq \mathcal{O}_{C_t}\Big(\sum_{p_i'(t)\in C_t} a_i p_i'(t) + \sum_{q_j(t)\in C_t} T(q_j(t), C_t) q_j(t)\Big).$$

By our previous assumptions, $T(q_j(t), C_t)$ is constant for $t \in B$, so in what follows we omit the terms in the bracket. Hence the line bundles

$$\mathscr{L} \text{ and } \mathcal{O}_{\mathscr{X}'}\Big(\sum_{i=1}^{n'} a_i p_i + \sum_{j=1}^{m} T q_j\Big)$$

are isomorphic over B^* and they therefore differ by a Cartier divisor $\mathcal{C}$ on $\mathscr{X}'$, supported over $0 \in B$. This Cartier divisor is a sum of irreducible components of the fibre $\mathscr{X}_0' = X$, that is

$$\mathcal{C} = \sum_{C \in I(X)} m_{C_0} C_0, \text{ with } m_{C_0} \in \mathbb{Z}.$$

Since the non-smoothed nodes of the family $\mathscr{X}'$ all have zero twist, this Cartier divisor yields in fact the definition of the twist $T_0 : N(X) \to \mathbb{Z}$ for a node $q_0 \in C_0 \cap C_0'$ that is smoothed by $\mathscr{X}'$:

$$(q_0, C_0) \mapsto m_{C_0'} - m_{C_0}.$$

Putting everything together, we obtain a twist $T_0 : N(\mathscr{X}_0) \to \mathbb{Z}$ which by construction satisfies all the conditions of Definition 3.7. Moreover, for $t = 0$,

$$\mathscr{L}_0|_{C_0} = \mathcal{O}_{C_0}\Big(\sum_{p_i \in C_0} a_i p_i(0)\Big) \otimes \mathcal{O}_{C_0}\Big(\sum_{q_0 \in C_0} T_0(q_0, C_0) q\Big)$$

for each irreducible component C_0 of $\mathscr{X}_0$. By Definition 3.13,

$$(\mathscr{X} \to B, p_i : B \to \mathscr{X}, \mathscr{L}) \in \mathcal{DJ}^{r,d}_{g,n,\mu}(B).$$

We conclude that $\mathcal{DJ}^{r,d}_{g,n,\mu}(B)$ is closed. □

Remark 3.16. Arguing like in the proof of Lemma 6 of [FP18], one can show that the line bundle associated to a de Jonquières divisor on a quasi-stable curve can be smoothed to a line bundle on a nonsingular curve. More precisely, let $(\mathscr{X} \to B, p_i : B \to \mathscr{X}, \mathscr{L})$ be a smoothing of a quasi-stable curve with marked points $[X, p_1, \ldots, p_n]$ (so $\mathscr{X}_0 = X$). Suppose also that for some $(X, L) \in \overline{P}^X_d$,

$$\mathcal{O}_X\Big(\sum_{i=1}^{n} a_i p_i\Big) = L \in \overline{P}^X_d,$$

Then there exists a line bundle $\mathscr{L}' \to \mathscr{X}$ and an isomorphism $\mathscr{L}_0' \simeq L$, which is constructed by twisting $\mathscr{L}$.

For the next two results, assume that $(d - g + 1, 2g - 2) = 1$ so that the definitions of de Jonquières divisors hold not just locally, but also for families over any scheme B. We give a lower bound on the dimension of irreducible components of $\mathcal{DJ}^{r,d}_{g,n,\mu}(\overline{P}_{d,g,n})$.

Proposition 3.17. *Every irreducible component of* $\mathcal{DJ}^{r,d}_{g,n,\mu}(\overline{P}_{d,g,n})$ *has dimension at least* $3g-3+\rho(g,r,d)+n-d+r$.

Proof. The proof of this statement is the same as the one of Proposition 11 in [FP18]. The only difference is the dimension bound itself, which we explain below.

Let $[X, p_1, \dots, p_n, L] \in \mathcal{DJ}^{r,d}_{g,n,\mu}(\overline{P}_{d,g,n})$ and $L = \mathcal{O}_X\left(\sum_{i=1}^n a_i p_i\right)$ its associated twisted line bundle. We drop the markings p_i without contracting the unstable components that we obtain. We then add m new markings to X to get rid of the automorphisms of the unstable components (see loc. cit. for details) and we obtain a stable pointed curve $[X, q_1, \dots, q_m]$. Let $\mathcal{V}$ be its nonsingular versal deformation space. Hence

$$\dim \mathcal{V} = \dim \operatorname{Def}([X, q_1, \dots, q_m]) = 3g-3+m.$$

Let $\pi : \mathcal{C} \to \mathcal{V}$ be the universal curve and consider the relative moduli space $\epsilon : \mathcal{B} \to \mathcal{V}$ of line bundles of degree d on the fibres of π. Let $\mathcal{V}^* \subset \mathcal{V}$ be the locus of smooth curves and $\mathcal{B}^* \to \mathcal{V}^*$ the relative Picard scheme of degree d. Finally, let $\mathcal{W}^{r*}_d \subset \mathcal{B}^*$ be the codimension at most $(r+1)(g-d+r)$ locus of line bundles with dimension of the space of sections $r+1$. Let $\mathcal{W}^r_d$ be the closure of $\mathcal{W}^{r*}_d$ in $\mathcal{B}$. Then

$$\dim \mathcal{W}^r_d \geqslant \dim \mathcal{B} - (r+1)(g-d+r) + m = 3g-3+\rho(g,r,d)+m.$$

This then contributes to the lower bound in the same way as in loc. cit. □

Moreover, we also obtain an upper bound for the dimension of certain irreducible components of $\mathcal{DJ}^{r,d}_{g,n,\mu}(\overline{P}_{d,g,n})$ supported on the locus of marked quasi-stable curves with at least one node.

Proposition 3.18. *Every irreducible component of* $\mathcal{DJ}^{r,d}_{g,n,\mu}(\overline{P}_{d,g,n})$ *supported entirely on the locus of quasi-stable curves with* n *marked points and at least one node has dimension at most* $4g-4+n-d+r$.

Proof. Suppose $Z \subset \mathcal{DJ}^{r,d}_{g,n,\mu}(\overline{P}_{d,g,n})$ is an irreducible component supported entirely on the locus of quasi-stable curves with n marked points and at least one node. Let $(X, p_1 \dots, p_n, L) \in Z$ be a generic element and denote by Γ_Z the dual graph of the curve X. By the definition of Z, the set E of edges of Γ_Z has at least one element. Denote by v the vertices of Γ_Z and their set by V (with $|V| \geqslant 1$). By definition, each v corresponds to an irreducible component of X whose genus we denote by g_v. Recall the genus formula:

$$g-1 = \sum_{v \in V} (g_v - 1) + |E|. \tag{3.5}$$

The strategy in what follows is to bound the dimension of the space of

$$(X, p_1, \dots, p_n, L) \in Z$$

with graph exactly Γ_Z. Now X is equipped with a line bundle L of degree d with strictly balanced multidegree $\underline{d} = (d_v)_{v\in\Gamma_Z}$ and $h^0(X, L) = r + 1$. Denote by L_v the restriction of L to the irreducible component corresponding to the vertex v and by n_v the number of the marked points on it. Thus, for a fixed vertex v of Γ_v, and Assuming the result of Theorem 1.8, the dimension of the space of de Jonquières divisors of length n_v on the component corresponding to v is at most $3g_v - 3 + \rho(g_v, r_v, d_v) + n_v - d_v + r_v$, where $r_v := h^0(L_v) - 1$. The dimension bound is obtained by summing over the vertices

$$\begin{aligned}\dim Z &\leqslant \sum_{v\in V} (3g_v - 3 + \rho(g_v, r_v, d_v) + n_v - d_v + r_v)\\ &\leqslant 3\sum_{v\in V}(g_v - 1) + n + 2|E| - d + \sum_{v\in V} r_v + \sum_{v\in V}\rho(g_v, r_v, d_v),\end{aligned}$$

where we used the fact that $\sum_{v\in V} n_v \leqslant n + 2|E|$. The surplus of $2|E|$ comes from the preimages of the nodes on each component in case the twist from the definition of de Jonquières divisors is nonzero. From (3.5) we have

$$\dim Z \leqslant 3g - 3 + n - d - |E| + \sum_{v\in V} r_v + \sum_{v\in V}\rho(g_v, r_v, d_v).$$

To estimate $\sum_{v\in V} r_v$, let X_1 and X_2 be two connected subcurves of X intersecting each other at k nodes. From the Mayer-Vietoris sequence

$$0 \to H^0(X, L) \to H^0(X_1, L|_{X_1}) \oplus H^0(X_2, L|_{X_2}) \to \mathbb{C}^k$$

we obtain $h^0(X_1, L|_{X_1}) + h^0(X_2, L|_{X_2}) \leqslant r + 1 + k$. Consider in turn the same Mayer-Vietoris sequence for two connected subcurves of X_1 and of X_2, etc. , until we are left only with irreducible components. Working backwards and adding up the dimensions of the spaces of global sections for all irreducible components of X, we obtain

$$\begin{aligned}&\sum_{v\in V} h^0(L_v) = h^0(X, L) + |E| \Leftrightarrow\\ &\sum_{v\in V}(r_v + 1) = r + 1 + |E| \Leftrightarrow\\ &\sum_{v\in V} r_v = r + 1 + |E| - |V|.\end{aligned}$$

For the sum of Brill-Noether numbers, we use the bound $\sum_v \rho(g_v, r_v, d_v) \leqslant \sum_{v\in V} g_v$, which in turn yields, using (3.5), $\sum_{v\in V} g_v = g - 1 - |E| + |V|$. Hence $\dim Z \leqslant 4g - 3 + n - d + r - |E| \leqslant 4g - 4 + n - d + r$. □

3.5 The dimension theorem for complete linear series

We now give a proof of the dimension theorem (Theorem 1.5) for complete linear series (i.e. those with $s = g - d + r \geqslant 0$) that makes use of the framework of limit linear series as discussed in Section 3.4.1.

We construct a nodal curve $X = C_1 \cup_p C_2$ of genus g out of two general pointed curves (C_1, p) of genus g_1 and (C_2, p) of genus g_2, where $g_1 + g_2 = g$. Furthermore, we equip X with a limit linear series of type g^r_d which we construct from the corresponding aspects $g^r_{d_1}(b_1 p)$ on C_1 and $g^r_{d_2}(b_2 p)$ on C_2, where $b_1, b_2 \in \mathbb{Z}_{\geqslant 0}$. The genera g_j, the degrees d_j, and the multiplicities b_j are chosen in such a way as to allow for a convenient induction step, where the induction hypothesis is the dimension theorem for $g^r_{d_j}$ on C_j for $j = 1, 2$. We do this in two steps:

1. The proof for linear series with $s \geqslant 2$ and $\rho(g, r, d) = 0$ works by induction on s (while keeping $\rho(g, r, d) = 0$ fixed), with base case given by the canonical linear series on a general smooth curve (which has $s = 1$ and $\rho(g, r, d) = 0$). This is done in Section 3.5.1.

2. The proof for linear series with $\rho(g, r, d) > 0$ works by induction on $\rho(g, r, d)$ (and keeping s constant), with base case given by the linear series with $\rho(g, r, d) = 0$ from the previous step. This is done in Section 3.5.2.

In choosing the aspects $g^r_{d_j}(b_j p)$ on C_j (with $j = 1, 2$), one has to take the following restrictions into consideration, which ensure that the limit we constructed exists and is smoothable:

- as mentioned in the Introduction, a general pointed curve $(C_j, p) \in \mathcal{M}_{g_j,1}$ may carry a $g^r_{d_j}(b_j p)$ with ramification sequence at least $(\alpha_0, \ldots, \alpha_r)$ at the point p if and only if (cf. (1.2))

$$\sum_{i=0}^{r} (\alpha_i + g_j - d + r)_+ \leqslant g_j. \tag{3.6}$$

  In our case, the ramification sequence at p of $g^r_{d_j}(b_j p)$ is $(b_j, \ldots, b_j)$.

- the limit g^r_d on X must be refined in order to satisfy the hypotheses of the smoothability result of Eisenbud and Harris (Theorem 2.3). This means that the inequality in (2.2) must be in fact an equality, thus further constraining the choice of b_j.

Theorem 2.2 and Theorem 2.3 then ensure that the limit g^r_d on X that we chose is indeed smoothable. Assume that we are in the setting of Definition 3.3 with $\mathscr{X}_0 = X$. If the limit g^r_d on X admits a de Jonquières divisor $\sum_{i=1}^{n} a_i D_i$, then each aspect $g^r_{d_j}(b_j p)$ on C_j (with $j = 1, 2$) admits the de Jonquières divisor

$$\sum_{i=1}^{k} a_i D_{i,C_j} + \Big(d - \sum_{i=1}^{k} a_i d_{i,C_j}\Big) p,$$

where the following inequality must hold in order to preserve the chosen ramification at p:

$$d - \sum_{i=1}^{k} a_i d_{i,C_j} \geqslant b_j. \tag{3.7}$$

Removing the base point p from the series $g^r_{d_j}(b_j p)$, we are left with a general linear series $l_j := g^r_{d_j}$ on C_j (for $j = 1, 2$), with simple ramification at p and admitting a de Jonquières divisor

$$\sum_{i=1}^{k} a_i D_{i,C_j} + \Big(d_j - \sum_{i=1}^{k} a_i d_{i,C_j}\Big)p.$$

The strategy is to prove that

$$\dim DJ^{r,d}_{k,N}(\mu_1, \mu_2, X, l) \leqslant N - d + r$$

by using the dimension theorem for the spaces of de Jonquières divisors of the series l_j on C_j. By the upper semicontinuity of fibre dimension applied to the map ϕ from Remark 3.5 it follows that

$$\dim DJ^{r,d}_{k,N}(\mu_1, \mu_2, X_t, l_t) \leqslant N - d + r$$

for a smoothing of X to a general curve X_t equipped with a general linear series l_t of type g^r_d. Combining this with Lemma 3.1, we obtain the statement of the dimension theorem for a general curve with a general linear series.

3.5.1 Step 1: proof for $\rho(g, r, d) = 0$

Having fixed r and $s = g - d + r \geqslant 2$, the proof in this case works by induction on s. The base case is given by the dimension theorem for the canonical linear series, (the unique linear series with index of speciality $s = 1$ and vanishing Brill-Noether number), on a general smooth curve of any genus. This follows either from our discussion in Section 3.2 or from Theorem 1.1 a) of Polishchuk [Pol06] with $D = 0$. The induction step constructs a curve X of genus g with a limit linear series l of type g^r_d with index of speciality s and Brill-Noether number $\rho(g, r, d) = 0$ from two irreducible components: C_1 equipped with a linear series l_1 with index of speciality $s_1 = s - 1$ and Brill-Noether number $\rho(l_1) = 0$ and C_2 equipped with its canonical linear series (with index of speciality $s_2 = 1$). The induction hypothesis at each step is the dimension theorem for each of the components C_1 and C_2 equipped with their respective linear series l_1 and l_2.

We now show how to obtain the curve X. From the condition $\rho(g, r, d) = 0$, we get

$$g = s(r + 1),$$
$$d = g + r - s.$$

We start with a general curve C_1 of genus $(s-1)(r+1)$ equipped with a general linear series l_1 of type g^r_{g-s}. Hence the index of speciality of l_1 is

$$s_1 = (s-1)(r+1)-g+s+r = (s-1)(r+1)-(s-1)(r+1)-r+s-1+r = s-1$$

and its Brill-Noether number is

$$\rho((s-1)(r+1), r, g-s) = (s-1)(r+1) - (r+1)(s-1) = 0.$$

We choose a general point $p \in C_1$ to which we attach another general curve C_2 of genus $r+1$ equipped with its canonical linear series $l_2 = g^r_{2r}$. This series has index of speciality $s_2 = 1$ and Brill-Noether number

$$\rho(r+1, r, 2r) = 0.$$

Thus we obtained a curve $X = C_1 \cup_p C_2$ of genus g. We construct on X a refined limit linear series l of type g^r_d aspect by aspect using l_1 and l_2.

On C_1 we take the aspect to be the series $l_1(rp)$, which therefore has the following vanishing sequence on C_1:

$$(r, r+1, \ldots, 2r).$$

Since the limit is refined, the vanishing sequence on C_2 must be

$$(d-2r, \ldots, d-r),$$

so we take the aspect corresponding to C_2 to be the series $l_2((d-2r)p)$.

Finally, we check that the limit series

$$\{(C_1, l_1(rp)), (C_2, l_2((d-2r)p))\}$$

satisfies (1.2):

$$\text{on } C_1 : \sum_{i=0}^{r} (r + (s-1)(r+1) - d + r)_+ = (r+1)(s-1) \leqslant (r+1)(s-1),$$

$$\text{on } C_2 : \sum_{i=0}^{r} (d - 2r + r + 1 - d + r)_+ = r+1 \leqslant r+1.$$

Hence the limit linear series l on X is smoothable.

We now prove that

$$\dim DJ^{r,d}_{k,N}(\mu_1, \mu_2, X, l) \leqslant N - d + r.$$

For $j = 1, 2$, let $N_j = \sum_{i=1}^{k} d_{i,C_j}$ and therefore $N_1 + N_2 = N$. As seen above, $\sum_{i=1}^{k} a_i D_i \in DJ^{r,d}_{k,N}(\mu_1, \mu_2, X, l)$ if and only if

$$\sum_{i=1} a_i D_{i,C_j} + \Big(d - \sum_{i=1}^{k} a_i d_{i,C_j}\Big)p$$

is a de Jonquières divisor of length (at most) $N_j + 1$ of the aspect of l corresponding to C_j, where $j = 1, 2$.

We observe that if all points in the support of the D_i-s specialise on one of the C_j (with $j = 1, 2$), then

$$d - \sum_{i=1}^{k} a_i d_{i,C_j} = 0,$$

contradicting inequality (3.7). Hence we must have

$$d - 2r \leqslant \sum_{i=1}^{k} a_i d_{i,C_1} \leqslant d - r,$$
$$2r \geqslant \sum_{i=1}^{k} a_i d_{i,C_2} \geqslant r.$$

We now distinguish a few possibilities:

1. If $\sum_{i=1}^{k} a_i d_{i,C_1} = d - r$, then $\sum_{i=1}^{k} a_i d_{i,C_2} = r$ and moreover

$$\sum_{i=1}^{k} a_i D_{i,C_1} \in DJ^{r,d-r}_{k,N_1}(\mu_1', \mu_2', C_1, l_1)$$

and

$$\sum_{i=1}^{k} a_i D_{i,C_2} + rp \in DJ^{r,2r}_{k,N_2+1}(\mu_1'', \mu_2'', C_2, l_2),$$

where $\mu_1' = (a_i)_{D_{i,C_1}>0}$, $\mu_2' = (d_{i,C_1})$ are the strictly positive vectors corresponding to the component C_1, while $\mu_1'' = (a_i)_{D_{i,C_2}>0}$ and $\mu_2'' = (d_{i,C_2}, r)$ are the ones corresponding to C_2. By the induction hypothesis, the following inequalities must be satisfied

$$\dim DJ^{r,d-r}_{k,N_1}(\mu_1', \mu_2', C_1, l_1) = N_1 - d + 2r =: x \geqslant 0$$
$$\dim DJ^{r,2r}_{k,N_2+1}(\mu_1'', \mu_2'', C_2, l_2) = N_2 + 1 - r = (N - d + r) - x + 1 \geqslant 0,$$

where we used the fact that $N_1 + N_2 = N$. Furthermore, note that on C_2 we are actually only interested in the locus in $DJ^{r,2r}_{k,N_2+1}(\mu_1'', \mu_2'', C_2, l_2)$ consisting of divisors with p in their support. More precisely, consider the incidence correspondence

$$\Gamma = \{(D, p) \mid p \in D\} \subset DJ^{r,2r}_{k,N_2+1}(\mu_1'', \mu_2'', C_2, l_2) \times C$$

and let π_1, π_2 be the canonical projections. The locus we are after is $\pi_1(\pi_2^{-1}(p))$. By construction, π_2 is dominant and since p is general,

$$\dim \pi_1(\pi_2^{-1}(p)) = \dim DJ^{r,2r}_{k,N_2+1}(\mu_1'', \mu_2'', C_2, l_2) - 1.$$

Therefore the dimension estimate for $DJ^{r,d}_{k,N}(\mu_1, \mu_2, X, l)$ is

$$\begin{aligned}\dim DJ^{r,d}_{k,N}(\mu_1, \mu_2, X, l) \leqslant &\dim DJ^{r,d-r}_{k,N_1}(\mu_1', \mu_2', C_1, l_1) + \\ &+ DJ^{r,2r}_{k,N_2+1}(\mu_1'', \mu_2'', C_2, l_2) - 1 \\ =&N - d + r.\end{aligned}$$

2. If $d-2r < \sum_{i=1}^{k} a_i d_{i,C_1} < d-r$, then $2r > \sum_{i=1}^{k} a_i d_{i,C_2} > r$ and we obtain de Jonquières divisors of length $N_j + 1$ on the component C_j, for $j = 1, 2$. This yields

$$\begin{aligned}&\dim DJ^{r,d-r}_{k,N_1+1}(\mu_1', \mu_2', C_1, l_1) = N_1 + 1 - d + 2r =: x \geqslant 0 \\ &\dim DJ^{r,2r}_{k,N_2+1}(\mu_1'', \mu_2'', C_2, l_2) = N_2 + 1 - r = (N - d + r) - x + 2 \geqslant 0.\end{aligned}$$

Arguing as in the previous case (for both C_1 and C_2), we obtain the same upper bound for the dimension of $DJ^{r,d}_{k,N}(\mu_1, \mu_2, X, l)$:

$$\begin{aligned}\dim DJ^{r,d}_{k,N}(\mu_1, \mu_2, X, l) \leqslant &\dim DJ^{r,d-r}_{k,N_1}(\mu_1', \mu_2', C_1, l_1) - 1 + \\ &+ DJ^{r,2r}_{k,N_2+1}(\mu_1'', \mu_2'', C_2, l_2) - 1 \\ =&N - d + r.\end{aligned}$$

3. If $\sum_{i=1}^{k} a_i d_{i,C_1} = d - 2r$, then $\sum_{i=1}^{k} a_i d_{i,C_2} = 2r$ and we get de Jonquières divisors of length $N_1 + 1$ on C_1 and of length N_2 on C_2. This case is analogous to (1) and we again obtain the upper bound $N-d+r$ for $\dim DJ^{r,d}_{k,N}(\mu_1, \mu_2, X, l)$.

3.5.2 Step 2: proof for all $\rho(g, r, d) \geqslant 1$

Fix r, $s = g-d+r$, and $\rho(g, r, d) \geqslant 1$. We continue with the proof by induction on $\rho(g, r, d)$, where the base case is given by the dimension theorem for linear series with $\rho(g, r, d) = 0$ that we proved in Section 3.5.1. The induction step constructs a curve X of genus g with a linear series l of type g^r_d from two components: C_1 equipped with a linear series l_1 and C_2 equipped with l_2 such that $\rho(l) = \rho(g, r, d) = \rho(l_1) + 1$. As before, the induction hypothesis at each step is the dimension theorem for the components C_j and their corresponding linear series l_j, with $j = 1, 2$.

We start with a general curve C_1 of genus $g - 1$ equipped with a general linear series $l_1 = g^r_{d-1}$. We pick a general point $p \in C_1$ and attach to it an elliptic normal curve C_2 with its associated linear series $l_2 = g^r_{r+1}$. Note that the dimension theorem holds for the elliptic normal curve by virtue of the fact that l_2 is non-special (see the discussion in Section 3.2).

The resulting curve $X = C_1 \cup_p C_2$ has genus g and we construct on it a limit linear series l of type g^r_d aspect by aspect. On C_1 we take the aspect $g^r_{d-1}(p)$, hence p is a base point of the g^r_d on X with vanishing sequence on C_1 given by

$$(1, 2, \ldots, r + 1).$$

Since the limit g^r_d must be refined in order to be smoothable, the aspect on C_2 must have the following vanishing sequence at p

$$(d-r-1,\dots,d-1).$$

Thus the aspect on C_2 is given by the series $g^r_{r+1}((d-r-1)p)$.

We check that this limit g^r_d also satisfies (1.2):

$$\begin{aligned}&\text{on } C_1 : (r+1)(1+g-1-d+r) = (r+1)s \leqslant g-1,\\ &\text{on } C_2 : (r+1)(d-r-1+1-d+r) = 0 \leqslant 1,\end{aligned}$$

where in the first inequality we used the fact that $\rho(g,r,d) = g-(r+1)s \geqslant 1$. Hence l is a smoothable limit linear series on X. Moreover, its Brill-Noether number is

$$\rho(l) = \rho(g,r,d) = g-(r+1)s$$

while the linear series $l_1 = g^r_{d-1}$ on C_1 has Brill-Noether number

$$\rho(l_1) = \rho(g-1,r,d-1) = \rho(g,r,d)-1.$$

Finally, we observe here that the induction step leaves the indices of speciality unchanged since $s_1 = (g-1)-(d-1)+r = s$.

We now show that $\dim DJ^{r,d}_{k,N}(\mu_1,\mu_2,X,l) \leqslant N-d+r$. The argument is the same as in 3.5.1. For $j = 1,2$, denote by N_j the length of the divisor $\sum_{i=1}^k a_i D_{i,C_j}$. As for the $\rho(g,r,d) = 0$, there are a few possibilities:

1. If $\sum_{i=1}^k a_i d_{i,C_1} = d-1$, then $\sum_{i=1}^k a_i d_{i,C_2} = 1$ and moreover

$$\sum_{i=1}^k a_i D_{i,C_1} \in DJ^{r,d-1}_{k,N_1}(\mu'_1,\mu'_2,C_1,l_1)$$

and

$$\sum_{i=1}^k a_i D_{i,C_2} + rp \in DJ^{r,r+1}_{k,N_2+1}(\mu''_1,\mu''_2,C_2,l_2),$$

where the vectors $\mu'_1,\mu'_2,\mu''_1,\mu''_2$ are defined as in 3.5.1. By the induction hypothesis,

$$\begin{aligned}&\dim DJ^{r,d-1}_{k,N_1}(\mu'_1,\mu'_2,C_1,l_1) = N_1-d+1+r =: x \geqslant 0,\\ &\dim DJ^{r,r+1}_{k,N_2+1}(\mu''_1,\mu''_2,C_2,l_{C_2}) = N_2 = (N-d+r)+1-x.\end{aligned}$$

As discussed in 3.5.1, we have the bound

$$\begin{aligned}\dim DJ^{r,d}_{k,N}(\mu_1,\mu_2,X,l) \leqslant & DJ^{r,d-1}_{k,N_1}(\mu'_1,\mu'_2,C_1,l_1)+\\ &+ DJ^{r,r+1}_{k,N_2+1}(\mu''_1,\mu''_2,C_2,l_2)-1\\ =& N-d+r.\end{aligned}$$

2. If $d-r-1 < \sum_{i=1}^{k} a_i d_{i,C_1} < d-1$, then $r+1 > \sum_{i=1}^{k} a_{i,C_2} > 1$ and we get de Jonquières divisors of length N_1+1 on C_1 and length N_2+1 on C_2. Counting dimensions as before we obtain the upper bound $N-d+r$ for the dimension of $DJ^{r,d}_{k,N}(\mu_1,\mu_2,X,l)$.

3. If $\sum_{i=1}^{k} a_i d_{i,C_1} = d-r-1$, then $\sum_{i=1}^{k} a_i d_{i,C_2} = r+1$ and we have de Jonquières divisors of length N_1+1 on C_1 and length N_2 on C_2. We obtain the same upper bound $N-d+r$.

3.6 Smoothness

In this section we prove Theorem 1.8 which states that the space

$$DJ^{r,d}_{k,N}(\mu_1,\mu_2,C,l)$$

is smooth by showing that it arises as a transverse intersection of subvarieties of the symmetric product C_d. Recall that we already proved in Section 3.2 this result in the following cases:

- $l = K_C$
- $g-d+r \in \{0,1\}$
- $r \in \{1,2\}$.

This section is dedicated to the case $r \geqslant 3$ and $s \geqslant 2$. From the transversality condition (3.2), we have to show that $H^0(C, K_C - D - D_1 - \ldots - D_k) = 0$. To do this, we prove that

$$g-(d+N)+r' < 0,$$

where $r' = h^0(D+D_1+\ldots+D_k)-1 = r+n'$, for some integer $n' \geqslant 0$.

Suppose towards a contradiction that $n' \geqslant N-g+d-r$. Consider all flag curve degenerations $j : \overline{\mathcal{M}}_{0,g} \to \overline{\mathcal{M}}_g$ and let $\mathcal{Z} := \overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{C}}_g^N$, where $\overline{\mathcal{C}}_g = \overline{\mathcal{M}}_{g,1}$. Let $U \subset \mathcal{Z}$ be the closure of the divisors with $r' = r+n'$ and $n' \geqslant N-g+d-r$ on all curves from $\mathrm{im}(j) \subseteq \overline{\mathcal{M}}_g$. By assumption, the map $U \to \overline{\mathcal{M}}_{0,g}$ is dominant, hence $\dim U \geqslant g-3$. Applying Proposition 2.2 of [Far08], there exists a point $[\widetilde{R} := R \cup E_1 \cup \ldots \cup E_g, y_1, \ldots, y_N] \in U$, where R is a rational spine (not necessarily smooth), the E_i are elliptic tails and the y_i are the points in the supports of the divisors $D_1, \ldots, D_k$ such that either:

(i) the supports of the divisors $D_1, \ldots, D_k$ coalesce into one point, or else

(ii) the supports of the divisors $D_1, \ldots, D_k$ lie on a connected subcurve Y of $\widetilde{R}$ of arithmetic genus $p_a(Y) = N$ and $|Y \cap \overline{(\widetilde{R} \setminus Y)}| = 1$.

Denote by $q_1, \ldots, q_g$ the points of attachment of the elliptic tails to the rational spine.

A short computation using the Plücker formula allows us to immediately dismiss Case (i). We now deal with Case (ii). By assumption, there exists a proper flat morphism $\phi : \mathscr{X} \to B$ satisfying:

1. $\mathscr{X}$ is a smooth surface, B is a smooth affine curve with $0 \in B$ a point such that the fibre $\mathscr{X}_0$ is a curve stably equivalent to the curve $\tilde{R}$, and the fibre $\mathscr{X}_t$ is a smooth projective curve of genus g for $t \neq 0$. Furthermore, we have the relative divisors $\mathscr{D}_i \in \mathscr{X}^{d_i}$ with $\mathscr{D}_i(0) = D_i$, for $i = 1, \ldots, k$.

2. Let $\mathscr{X}^* = \mathscr{X} \setminus \mathscr{X}_0$. Then there exists a line bundle $\mathscr{L}^*$ on $\mathscr{X}^*$ of relative degree d and with $\dim H^0(\mathscr{X}_t, \mathscr{L}_t) = r + 1$ for $t \neq 0$. After (possibly) performing a base change and resolving the resulting singularities, the pair $(\mathscr{L}^*, \mathscr{V}^* := H^0(\mathscr{X}^*, \mathscr{L}^*))$ yields a refined limit linear series $\mathfrak{m} := g^r_d$ on $\tilde{R}$. The limit linear series $\mathfrak{m}$ has moreover the property that it admits the de Jonquières divisor $\sum_{i=1}^k a_i \mathscr{D}_i(0) = \sum_{i=1}^k a_i D_i$.

3. The line bundle $\mathscr{L}^*$ also has the following property: $\mathscr{N} := \mathscr{L}^*\big(\sum_{i=1}^k \mathscr{D}_i\big)$ is another line bundle on $\mathscr{X}^*$ of relative degree $d + N$ and with the property $h^0(\mathscr{X}_t, \mathscr{N}_t) = r + n' + 1$. The pair $(\mathscr{N}^*, \widetilde{\mathscr{V}}^* := H^0(\mathscr{X}^*, \mathscr{N}^*))$ also gives a limit linear series $\mathfrak{l} := g^{r+n'}_{d+N}$ on $\tilde{R}$. Furthermore, the limit linear series $\mathfrak{l}$ admits the de Jonquières divisor $\sum_{i=1}^k (a_i + 1) D_i$.

The situation can be reformulated as follows: for $t \neq 0$,

$$\dim H^0\Big(\mathscr{X}_t, \mathscr{N}_t\Big(-\sum_{i=1}^k \mathscr{D}_i(t)\Big)\Big) = r + 1.$$

Then $\mathscr{N}^* \otimes \mathcal{O}_{\mathscr{X}^*}(-\sum_{i=1}^k \mathscr{D}_i(B \setminus 0))$ induces the limit linear series g^r_d that we started with.

Now, for a component $C \subset \mathscr{X}_0$, let $(\mathscr{L}_C, \mathscr{V}_Z C) \in G^r_d(Z)$ be the C-aspect $\mathfrak{m}_C$ of the limit $\mathfrak{m} = g^r_d$. Then there exists a unique effective divisor $D_C \in C_N$ supported only at the points of $(C \cap \bigcup_{i=1}^k \mathscr{D}_i(B) \cap (C \cap \overline{\mathscr{X}_0 \setminus C})$ such that the C-aspect of $\mathfrak{m}$ has the property that the restriction map

$$\mathscr{V}_C \to \mathscr{V}_C|_{D_C}$$

has non-trivial kernel. For the C-aspect $\mathfrak{l}_C$ of the limit $\mathfrak{l}$ the situation is analogous, but now we have an effective divisor $D'_C \in C_{d+N}$ with $D'_C \geqslant D_C$. Moreover, the C-aspect of $\mathfrak{m}$ is of the form

$$\mathfrak{m}_C = (\mathscr{M}_C := \mathscr{N}_C \otimes \mathcal{O}_C(-D'_C + D_C), \mathscr{W}_C \subset \widetilde{\mathscr{V}}^*_C \cap H^0(\mathscr{M}_C))$$

Thus, the collection $\mathfrak{m}_Y := \{\mathfrak{m}_C\}_{C \subset Y}$ forms a limit g^r_d on Y, while the collection $\mathfrak{l}_Y := \{\mathfrak{l}_C\}_{C \subset Y}$ forms a limit $g^{r+n'}_{d+N}$ on Y.

Let $p = Y \cap \overline{(\widetilde{R} \setminus Y)}$ and $Z := \overline{\widetilde{R} \setminus Y}$. The vanishing sequence of the limit g^r_d at p is a subsequence of the vanishing sequence at p of the limit $g^{r+n'}_{d+N}$. The complement of this subsequence yields another limit linear series $g^{n'-1}_d$ on Y (see Lemma 2.1 of [Far08]). We distinguish two cases:

(I) $N < g$.
To begin with, we list two technical results that help us determine a lower bound for the ramification sequence at p of the limit linear series $g^{n'-1}_d$ in this case.

Lemma 3.19 (Corollary 1.6 of [EH83a]). *Let $C \simeq \mathbb{P}^1$ be an irreducible component of Z such that $q_j \in C$ for some $j = 1, \ldots, N$, where q_j is the point of attachment of the elliptic tail E_j to C. Let l be a limit linear series on Z and C' be another component of $\widetilde{R}$ and $q = C \cap C'$. If q' is another point on C, then for all but at most one value of i,*

$$a_i(l_C, q') < a_i(l_{C'}, q).$$

Lemma 3.20. *Let $\{\sigma_C \mid C \subseteq Y$ irreducible component$\}$ be the set of compatible sections corresponding to the divisor $D + D_1 + \ldots + D_k$. If $q \in C$, then $ord_q(\sigma_C) = 0$.*

Proof. The proof works by induction on the components of Y. By construction, the tree curve Y has at least two components, so the base case is $Y = C_1 \cup_{q'} C_2$. Denote by D_{C_1} and D_{C_2} the specialisations of the divisor $D + D_1 + \ldots + D_k$ on the two components C_1 and C_2. Then σ_{C_1} vanishes on D_{C_1} and

$$\mathrm{ord}_{q'}(\sigma_{C_1}) = d + N - \deg D_{C_1} = \deg D_{C_2}$$

and similarly σ_{C_2} must vanish on D_{C_2} and $\mathrm{ord}_{q'}(\sigma_{C_2}) = \deg D_{C_1}$. Hence, if $q \in C_1$ is a smooth point, then $\sigma_{C_1}(q) = 0$ and if $q \in C_2$ is a smooth point, then $\sigma_{C_2}(q) = 0$ and we are done.

Suppose now that Y has m irreducible components denoted $C_1, \ldots, C_m$ and let $D_{C_1}, \ldots, D_{C_m}$ be the specialisations of the divisor $D + D_1 + \ldots + D_k$ to each component. Let $C_m \cap C_{m-1} = q'$. Then

$$\mathrm{ord}_{q'}(\sigma_{C_{m-1}}) = d + N - \deg D_{C_1} - \ldots - \deg D_{C_{m-1}} = \deg D_{C_m}.$$

Furthermore, $\mathrm{ord}_{q'}(\sigma_{C_m}) = d + N - \deg C_m$. Thus, if $q \in Y$ is a smooth point belonging to C_m, then $\sigma_{C_m}(q) = 0$. If $q \in Y$ belongs to any of the components of the subcurve $C_1 \cup \ldots \cup C_{m-1}$, then $\sigma_{C_j}(q) = 0$ (with $j = 1, \ldots, m-1$), where we used the induction hypothesis and the fact that

$$D_{C_1} + \ldots + D_{C_{m-1}} + (\deg C_m) q'$$

is a divisor of degree $d + N$ on the subcurve $C_1 \cup \ldots \cup C_{m-1}$. □

Let $C \subset Z$ be the irreducible component meeting Y at p. Denote by C' the component of Y containing p. Suppose first that C contains at least one of the points q_j of attachment of the elliptic tails. Let $p' \in C$ be a general smooth point, which therefore has vanishing sequence

$$a_i((g^{r+n'}_{d+N})_C, p') = (0, 1, 2, 3, \ldots, r + n').$$

By Lemma 3.19 with $q = p$ and $q' = p'$, the vanishing sequence at p is

$$a_i((g^{r+n'}_{d+N})_{C'}, p) \geqslant (0, 2, 3, 4, \ldots, r + n' + 1).$$

By a similar argument,

$$a_i((g^r_d)_{C'}, p) \geqslant (0, 2, 3, 4, \ldots, r + 1).$$

Combining this with Lemma 3.20, we get the following ramification sequence for $g^{n'-1}_d$:

$$\alpha_i((g^{n'-1}_d)_{C'}, p) \geqslant (1, 1, \ldots, 1).$$

In fact we obtain a limit linear series $g^{n'-1}_d$ on Y with ramification

$$\alpha_i((g^{n'-1}_d)_Y, p) \geqslant (1, 1, \ldots, 1). \tag{3.8}$$

We check a necessary condition for such a limit series to exist (cf. Theorem 1.1 of [EH87]):

$$\sum_{i=0}^{n'-1} \widetilde{\alpha}_i + n'(N - d + n' - 1) \leqslant N. \tag{3.9}$$

Since we assumed $n' \geqslant N - g + d - r$ and using moreover the inequality (3.8) we obtain that

$$\sum_{i=0}^{n'-1} \widetilde{\alpha}_i + n'(N - d + n' - 1) \geqslant (N - g + d - r)(2N - g - r).$$

Denoting by $s := g - d + r$ and using $N > d - r$, we reformulate the necessary condition (3.9) as

$$(N - s)(N - s - r) < N$$

which is equivalent to the quadratic inequality

$$N^2 - (2s + r + 1)N + s(s + 1) < 0.$$

This implies that the solution N must be contained in the interval (N_1, N_2), where N_1 and N_2 are the solutions to the equation

$$N^2 - (2s + r + 1)N + s(s + 1) = 0.$$

This implies that

$$N < \frac{2s + r + 1 + \sqrt{(2s + r + 1)^2 - 4s(s + r)}}{2}.$$

We now show that for $s \geqslant 2$ and $r \geqslant 3$

$$\frac{2s + r + 1 + \sqrt{(2s + r + 1)^2 - 4s(s + r)}}{2} < d - r + 1, \tag{3.10}$$

contradicting thus the hypothesis $N - d + r \geqslant 1$. To do this, first note that a simple calculation yields

$$g \geqslant (r + 1)s \geqslant 2s + r + 1$$

for $s \geqslant 2$ and $r \geqslant 3$ which in turn yields

$$2s + r + 1 \leqslant g - ((r + 1)s - 2s + r + 1) = g - s(r - 1) + r + 1.$$

Another simple calculation gives, for $r \geqslant 3$ and $s \geqslant 2$:

$$\sqrt{(2s + r + 1)^2 - 4s(s + r)} \leqslant (2s + r + 1) - 4.$$

Putting it all together, we get a sufficient condition for the inequality (3.10) to be satisfied, namely:

$$\frac{2g - 2s(r - 1) + 2(r + 1) - 4}{2} < d - r + 1$$

which is equivalent to

$$(2 - r)(s - 1) < 0.$$

This is clearly satisfied for $r \geqslant 3$ and $s \geqslant 2$ which means (3.10) is also satisfied for these value ranges of r and s, contradicting thus the assumption $N - d + r \geqslant 1$.

(II) $N \geqslant g$.
In this case $Y = \widetilde{R}$ and we check the necessary condition for the existence of a linear series $g_d^{n'-1}$ on the tree curve Y with specified ramification at a point (also Theorem 1.1 of [EH87]):

$$n'(g - d + n' - 1) \leqslant g. \tag{3.11}$$

By our assumptions, $n' \geqslant N - g + d - r \geqslant d - r$ and we therefore have

$$\sum_{i=0}^{n'-1} \widetilde{\alpha}_i + n'(N - d + n' - 1) \geqslant (d - r)(g - r - 1).$$

Thus a necessary condition for (3.11) is that

$$(d-r)(g-r-1) \leqslant g,$$

which is equivalent to

$$g \leqslant \frac{(r+1)(d-r)}{d-r-1}.$$

However, we also know that $s = g - d + r \geqslant 2$ and $g \leqslant \frac{r+1}{r}(d-r)$, which immediately give $d \geqslant 3r$. This in turn yields

$$g \leqslant \frac{(r+1)(d-r)}{d-r-1} \leqslant \frac{r+1}{2r-1}(d-r) \leqslant d-r,$$

which contradicts the assumption that $s = g - d + r \geqslant 2$.

3.7 Non-existence for non-complete linear series

In this section we prove Theorem 1.9 which states that, for a general curve of genus g, if $n - d + r < 0$, the general linear series g^r_d with $g - d + r < 0$ does not admit de Jonquières divisors of length n of the type

$$a_1 p_1 + \ldots + a_n p_n,$$

where the points p_i in the support are distinct. Recall that in this case, we only need one partition $\mu = (a_1, \ldots, a_n)$ of d and we denote the space of de Jonquières divisors by $DJ^{r,d}_n(\mu, C, l)$. We proceed by induction. The base case is given by the non-existence statement in the case $n - d + r < 0$ and $n < g$ shown in Lemma 3.2. In the induction step we prove non-existence for $n \geqslant g$.

Consider the following quasi-stable curve Y of genus $g \geqslant 4$ with $n \geqslant g$ marked points consisting of a general curve C of genus $g-1$ and a rational bridge with $n+1$ rational components γ_j, for $j = 1, \ldots, n+1$. Since the curve is quasi-stable, at most one of the components of the rational chain is exceptional (i.e. it contains no marks). In our case, since we have n marks, there must be one such component which we denote by $\gamma_{j'}$, while each of the other rational components γ_j contains one of the marked points p_i. Let $C \cap \gamma_1 = q_1$, $C \cap \gamma_{n+1} = q_{n+2}$, and $\gamma_j \cap \gamma_{j+1} = q_{j+1}$ for $j = 1, \ldots, n+1$. The curve Y is equipped with a linear series $l = g^r_d = (L, V)$ with $g - d + r < 0$ corresponding to a line bundle L with $h^0(Y, L) > r + 1$. The bundle L has balanced multidegree $\underline{d}$, meaning that $\deg L_C = d - 1$ and $\deg L_{\gamma_j} = 0$ for all $j \neq j'$ and $\deg L_{\gamma_{j'}} = 1$. An easy Mayer-Vietoris sequence calculation yields that C is also equipped with a non-complete linear series $l_C = g^r_{d-1}$.

This configuration gives a de Jonquières divisor on Y corresponding to $l = (L, V)$ if there exists a twist T satisfying the following system of linear

equations:

$$T(q_1, C) + T(q_{n+2}, C) = d - 1$$
$$T(q_j, \gamma_j) + T(q_{j+1}, \gamma_j) + \sum_{p_i \in \gamma_j} a_i = 0 \text{ for all } j \neq j'$$
$$T(q_{j'}, \gamma_{j'}) + T(q_{j'+1}, \gamma_{j'}) = 1.$$

Note that at least one of the terms $T(q_1, C)$ and $T(q_{n+2}, C)$ must be non-zero. There are therefore two possibilities for solutions of this system:

1. Both $T(q_1, C)$ and $T(q_{n+2}, C)$ are non-zero. In this case we have a de Jonquières divisor $T(q_1, C)q_1 + T(q_{n+2}, C)q_{n+2}$ on C of length 2 corresponding to l_C. Note that since $2 < g - 1$ and
$$2 - (d - 1) + r = 3 - d + r < n - d + r < 0,$$
the induction hypothesis yields that l_C admits no such de Jonquières divisors.

2. Only one of the two terms is non-zero. We then have a de Jonquières divisor of length 1 corresponding to l_C. Since $1 < g - 1$ and
$$1 - (d - 1) + r < n - d + r < 0,$$
the induction hypothesis yields that l_C does not admit such de Jonquières divisors.

Hence l does not admit any de Jonquières divisors on Y of length $n \geqslant g$. We now explain how to conclude the non-existence statement for a general smooth curve with a general linear series of type g^r_d.

First note that Y is embedded in $\mathbb{P}^r$ by the linear series l and using the methods of Hartshorne-Hirschowitz and Sernesi [Ser84] (for the precise details, see for example Lemma 1.5 of [AFO17]) one shows that it is flatly smoothable to a general curve of genus g and degree d in $\mathbb{P}^r$. Thus we have a family $\pi : \mathscr{X} \to B$ of curves of genus g with central fibre $\mathscr{X}_0 = Y$. The family is equipped with a line bundle $\mathscr{L}$ of relative degree d and such that $h^0(\mathscr{X}_t, \mathscr{L}_t) > r + 1$ for all $t \in B$. Thus the family $(\pi : \mathscr{X} \to B, p_i : B \to \mathscr{X}, \mathscr{L}) \notin \mathcal{DJ}^{r,d}_{g,n,\mu}(B)$. Otherwise, if the smooth fibres of $\mathscr{X} \to B$ admitted de Jonquières divisors, then by Proposition 3.15, the central fibre would as well. However, we have just proven this not to be the case, which concludes the induction step.

Remark 3.21. Y is a quasi-stable curve obtained via semi-stable reduction from the stable curve X of genus g with no marked points and just one self-intersection node. Since X is d-general (see Definition 4.13 of [Cap05]), it follows that locally around X the forgetful morphism $\Psi_{d,g,0} : \overline{P}_{d,g,0} \to B$ (with $B \subset \overline{\mathcal{M}}_g$) is proper. Moreover, if $\Psi_{d,g,0}$ is proper, then so are $\Psi_{d,g,n} : \overline{P}_{d,g,n} \to B$ (with $B \in \overline{\mathcal{M}}_{g,n}$ - see for example the discussion in Sections 7 and 8 of [Mel11]) and $\mathcal{DJ}^{r,d}_{g,n,\mu}(B) \to B$.

3.8 De Jonquières divisors with negative terms

It is also worthwhile to study de Jonquières divisors whose partition μ of d contains negative terms. In fact, in Section 3.4.2 we saw that negative coefficients occur naturally when considering de Jonquières divisors on nodal stable curves, as the twists T may be negative. For simplicity of notation, we consider only de Jonquières divisors with distinct points in the support.

Definition 3.22. Fix a curve C equipped with a linear series $l \in G^r_d(C)$ and let

$$\mu = (a_1, \ldots, a_{n_1}, -b_1, \ldots, -b_{n_2})$$

be a partition of d of length n, where a_i, b_i are positive integers satisfying $\sum_{i=1}^{n_1} a_i - \sum_{i=1}^{n_2} b_i = d$ and n_1, n_2 are fixed positive integers with $n_1 + n_2 = n$. We define the space $DJ^{r,d}_{n_1,n_2}(\mu, C, l)$ of de Jonquières divisors with n_1 positive and n_2 negative terms corresponding to the linear series l on the curve C by the rule

$$\sum_{i=1}^{n_1} a_i p_i - \sum_{i=1}^{n_2} b_i q_i \in DJ^{r,d}_{n_1,n_2}(\mu, C, l)$$

if and only if

$$\sum_{i=1}^{n_1} a_i p_i \in DJ^{r',d'}_{n_1}(\mu', C, l'),$$

where $p_i, q_i \in C$, $\mu' = (a_1, \ldots, a_{n_1})$ is a positive partition of

$$d' = \sum_{i=1}^{n_1} a_i = d + \sum_{i=1}^{n_2} b_i,$$

and l' is the linear series of type $g^{r'}_{d'}$ given by $l' = l + \sum_{i=1}^{n_2} b_i q_i$.

Theorem 3.23. *Fix a general curve C of genus g equipped with a general linear series $l = (L, V) \in G^r_d(C)$, and let $\mu = (a_1, \ldots, a_{n_1}, -b_1, \ldots, -b_{n_2})$ be a partition of d of length n, where a_i, b_i are positive integers and $n = n_1 + n_2$. Assume that $g - d + r \geqslant \sum_{i=1}^{n_2} b_i$ (which ensures that l' is complete). If $n_1 - d' + r' \geqslant 0$, then the space $DJ^{r,d}_{n_1,n_2}(\mu, C, l)$ is of expected dimension $n - d' + r'$.*

Proof. Set $L' = L(\sum_{i=1}^{n_2} b_i q_i)$. We first show that

$$\dim DJ^{r',d'}_{n_1}(\mu', C, l') \geqslant n_1 - d' + r'.$$

We distinguish a few cases.

- If $d' = 2g - 2$ and $L' = K_C$, then $h^0(L') = g$.
- If $d' = 2g - 2$, but $L' \neq K_C$, then $h^0(L') = g - 1$.
- If $d' > 2g - 2$, then $h^0(C, L') = d' - g + 1$.

- If $d' < 2g-2$, then $h^0(C, L') > r$.

In all cases $h^0\big(C, L'|_{\sum_{i=1}^{n_1} a_i p_i}\big) = \sum_{i=1}^{n_1} a_i = d'$. With this in mind, we can describe the space $DJ_{n_1}^{r',d'}(\mu', C, l')$ as the locus in $C_{d'}$ where the vector bundle map Φ (constructed as in Section 3.1, but substituting L' for L) has rank at most $h^0(C, L')-1 = r'$. Hence the lower bound for the dimension of $DJ_{n_1}^{r',d'}(\mu', C, l')$ is given by

- $n_1 - (h^0(C, L') - r')(d' - r') = n_1 - d' + r' = n_1 - g + 1$ if $d' = 2g-2$ and $L' = K_C$,
- $n_1 - (h^0(C, L') - r')(d' - r') = n_1 - g$ if $d' \geqslant 2g-2$ and $L' \neq K_C$,
- $n_1 - (h^0(C, L') - r')(d' - r') = n_1 - d + r$ if $d' < 2g-2$.

The fact that

$$\dim DJ_{n_1}^{r',d'}(\mu', C, l') = n_1 - d' + r'$$

follows as in the case of effective de Jonquières divisors, by replacing the occurrences of L by L' in the proof of Theorem 1.5. Finally, including the points q_i in the dimension count, we get that the dimension of $DJ_{n_1,n_2}^{r,d}(\mu, C, l)$ is indeed $n - d' + r'$. $\square$

Chapter 4

Intersections of secant varieties

This chapter is dedicated to the study of intersections of incidence and secant varieties on algebraic curves. In Section 4.1 we establish some preliminary results on incidence and secant varieties before we prove Theorem 1.10 using a tangent space argument in Section 4.2. We construct degenerations of secant varieties for families of curves with nodal fibres of compact type using limit linear series in Section 4.3 and we use them to prove Theorem 1.11 in Section 4.4.

4.1 Preliminaries on incidence and secant varieties

As usual, let C be a general curve of genus g equipped with a linear series $l = (L, V)$ of type g^r_d. Let e and f be integers such that $0 \leqslant f < e \leqslant d$.

As mentioned in the Introduction, incidence varieties are special cases of secant varieties, namely $\Gamma_e(l) = V^r_e(l)$.

Secant (and therefore incidence) varieties $V^{e-f}_e(l)$ of effective divisors of degree e imposing at most $e - f$ conditions on l have a degeneracy locus structure inside the symmetric product C_e, obtained as follows: let $\mathcal{E} = \mathcal{O}_{C_e} \otimes V$ be the trivial vector bundle of rank $r + 1$ on C_e and $\mathcal{F}_e(L) := \tau_*(\sigma^* L \otimes \mathcal{O}_{\mathcal{U}})$ be the e-th secant bundle, where $\mathcal{U}$ is the universal divisor

$$\mathcal{U} = \{(p, D) \mid D \in C_e \text{ and } p \in D\} \subset C \times C_e,$$

and σ, τ are the usual projections:

$$\begin{tikzcd} & C \times C_e \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \supset \mathcal{U} \\ C & & C_e \end{tikzcd}$$

Let $\Phi : \mathcal{E} \to \mathcal{F}$ be the bundle morphism obtained by pushing down to C_e the restriction $\sigma^* L \to \sigma^* L \otimes \mathcal{O}_{\mathcal{U}}$. The space $V^{e-f}_e(l)$ is then the $(e - f)$-th

degeneracy locus of Φ, i.e. where $\operatorname{rk}\Phi \leqslant e-f$. To see that this is indeed the case, note that fibrewise, the morphism Φ is given by the restriction:

$$\Phi_D : H^0(C, L) \to H^0(C, L/L(-D)).$$

Now by definition, $D \in V_e^{e-f}(l)$ if and only if

$$\dim\ker\Phi_D = h^0(L-D) \geqslant r+1-e+f,$$

which is equivalent to the aforementioned condition $\operatorname{rk}\Phi \leqslant e-f$. The dimension estimate for $V_e^{e-f}(l)$ follows immediately from its degeneracy locus structure:

$$\dim V_e^{e-f}(l) \geqslant e-(r+1-e+f)(e-e+f) = e-f(r+1-e+f).$$

In particular,

$$\dim\Gamma_e(l) \geqslant r.$$

On the other hand, since $D \in \Gamma_e(l)$ is equivalent to there existing a divisor $E \in l$ such that $E-D \geqslant 0$, and since the dimension of the locus of such divisors E inside l is at most r, we immediately have that

$$\dim\Gamma_e(l) = r$$

for any linear series l of type g^r_d on C. Using the Porteus formula, one obtains (see for [ACGH85] Chapter VIII, Lemma 3.2) that the fundamental class of $\Gamma_e(l)$ is given by

$$\gamma_e(l) = \sum_{j=0}^{e-r} \binom{d-g-r}{j} \frac{x^k\theta^{e-r-j}}{(e-r-j)!},$$

where, as before, θ is the pullback of the fundamental class of the theta divisor to C_d and x is the class of the divisor $q + C_{d-1} \subset C_d$.

To obtain formula (1.4) giving the number (when expected to be finite) of divisors in the intersection

$$\Gamma_e(l_1) \cap \Gamma_e(l_2),$$

where $l_1 = g^{r_1}_{d_1}$ and $l_2 = g^{r_2}_{d_2}$, one may compute the product

$$\gamma_e(l_1)\gamma_e(l_2) \in H^{2e}(C_e, \mathbb{Z}) \simeq \mathbb{Z},$$

which, as shown in [ACGH85] Chapter VIII, Section §3, yields the desired count.

Unfortunately, the situation is not so simple in the general case of secant varieties with $r-e+f > 0$. Indeed, the fundamental class of $V_e^{e-f}(l)$ has been computed by MacDonald and its expression is very complicated and thus of

limited practical use, as can be seen in [ACGH85], Chapter VIII, §4. For a study of the dimension theory of secant varieties we refer the reader to [Far08].

In this thesis we are concerned instead with the study of intersections of incidence and secant varieties on a given general smooth curve and with the geometric interpretation of some unexpected enumerative results that arise in this context.

4.2 Intersections of incidence varieties

In this section we investigate the failure of transversality for intersections of incidence varieties in certain interesting cases. We begin in 4.2.1 by explaining why the enumerative formula (1.4) yields unexpected zero counts in some situations by making use of the dimension theorem for de Jonquières divisors. By studying the relevant tangent spaces we then prove Theorem 1.10 in 4.2.2.

4.2.1 Unexpected zero counts

Recall that for two linear series $l_1 = g^{r_1}_{d_1}$ and $l_2 = g^{r_2}_{d_2}$ on a general curve C and for the positive integer $e = r_1 + r_2$, we expect there to be a finite number of divisors in the intersection $\Gamma_e(l_1) \cap \Gamma_e(l_2)$ and this number is given by formula (1.4).

Consider the complete linear series $l_1 = g^{r_1}_{d_1}$, the pencil $l_2 = g^1_{d_2}$, and $e = r_1 + 1$. Formula (1.4) gives that the number of divisors $D \in C_{r_1+1}$ common to both l_1 and l_2 is

$$(d_1 - r_1)\binom{d_2 - 1}{r_1} - g\binom{d_2 - 2}{r_1 - 1}. \tag{4.1}$$

This number was first computed by Severi in the context of the theory of correspondences and coincidences on curves (see Section 74 of [SL21]).

From our point of view, this choice of parameters provides an interesting example of a zero count when $d_2 = r_1 + 2$ and $\rho(g, r_1, d_1) = 0$, because now

$$(d_1 - r_1)\binom{d_2 - 1}{r_1} - g\binom{d_2 - 2}{r_1 - 1} = \rho(g, r_1, d_1) = 0.$$

Thus we expect this intersection not to be well-behaved in the case of vanishing $\rho(g, r_1, d_1)$. Indeed, we have:

Proposition 4.1. *In the above setting, if $d_2 = r_1 + 2$ and $\rho(g, r_1, d_1) = 0$ there are three possibilities for the intersection $\Gamma_e(l_1) \cap \Gamma_e(l_2)$:*

(i) it is empty if $l_1 = K_C$ and l_2 is base point free;

(ii) it is strictly positive-dimensional if $l_1 = K_C$ and l_2 is not base point free;

(iii) it is empty if $l_1 \neq K_C$.

Proof. Let $D \in \Gamma_{r_1+1}(l_1) \cap \Gamma_{r_1+1}(l_2)$ and let $s_1 := g - d_1 + r_1$ be the index of speciality of the linear series l_1. Since $\rho(g, r_1, d_1) = 0$, it immediately follows that:

$$\begin{aligned} d_1 &= r_1(s_1 + 1) \\ g &= s_1(r_1 + 1). \end{aligned}$$

Since the curve C is general, the Brill-Noether number corresponding to the pencil l_2

$$\rho(g, 1, r_1 + 2) = s_1(r_1 + 1) - 2(s_1 - 1)(r_1 + 1) = (r_1 + 1)(2 - s_1)$$

must be non-negative. This is only possible if $s_1 = 1$ or $s_1 = 2$.

Assume first that $s_1 = 1$, so that $l_1 = K_C$. Then $K_C - D \geqslant 0$ for all $D \in C_{r_1+1} = C_g$ satisfying $g - (r_1 + 1) + \dim |D| = \dim |D| > 0$. Hence $D \in \Gamma_g(K_C)$ if and only $|D| = g^1_g$. If l_2 is base point free, then the intersection $\Gamma_g(K_C) \cap \Gamma_g(l_2)$ is empty. Otherwise, the intersection $\Gamma_g(K_C) \cap \Gamma_g(l_2)$ is at least 1-dimensional, hence not a finite, discrete set.

If $s_1 = 2$, then $l_1 = g^{r_1}_{3r_1}$ and $l_2 = g^1_{r_1+2}$. Note that in this case $l_1 = K_C - l_2$. By our assumption, there exists an effective divisor E_1 of degree $2r_1 - 1$ such that

$$|D + E_1| = l_1$$

and an effective divisor E_2 of degree 1 such that

$$|D + E_2| = l_2.$$

Therefore

$$K_C = |2D + E_1 + E_2|.$$

Since in this case $g = 2r_1 + 2$, we have $K_C = g^{2r_1+1}_{4r_1+2}$. Applying the dimension theorem for de Jonquières divisors (Theorem 1.5), we conclude that the locus of triples (D, E_1, E_2) inside $C_{r_1+1} \times C_{2r_1-1} \times C$ has dimension

$$(r_1 + 1 + 2r_1 - 1 + 1) - (4r_1 + 2) + (2r_1 + 1) = r_1.$$

Since l_1 has Brill-Noether number equal to zero, it follows that it is general and hence base point free (cf. Proposition 5.4 of [EH83b]). This implies further that

$$\dim(l_1 - D) = \dim(|E_1|) \leqslant r_1 - 1.$$

Moreover, E_2 is simply a point on C, which means that $\dim(|E_2|) = 0$. Putting everything together, we conclude that $\dim(|D|) \geqslant 1$. Since $D \in l_2$, we finally get that $|D| = g^1_{r_1+1}$. This is then equivalent to the statement

$$E_1 \in V^{r_1-1}_{2r_1-1}(l_1).$$

However, one easily checks that

$$\rho(g, r_1, 3r_1) + \dim V^{r_1-1}_{2r_1-1}(l_1) = -1,$$

from which we conclude, using Corollary 0.2 of [Far08], that $V^{r_1-1}_{2r_1-1}(l_1) = \emptyset$. Hence in this case, the intersection $\Gamma_{r_1+1}(l_1) \cap \Gamma_{r_1+1}(l_2)$ is empty. □

Using similar methods, we obtain a more general version of Proposition 4.1.

Proposition 4.2. *Let C be a general curve of genus g equipped with two complete linear series $l_1 = g^{r_1}_{d_1}$ and $l_2 = g^{r_2}_{d_2}$ such that*

$$r_1 > 1, \quad g - d_1 + r_1 > 0 \text{ and } l_2 = K_C - l_1.$$

Then if non-empty, the intersection $\Gamma_e(l_1) \cap \Gamma_e(l_2)$ is not transverse.

Proof. By assumption, $l_2 = g^{g-d_1+r_1-1}_{2g-2-d_1}$. Let $D \in \Gamma_e(l_1)$. Then there exists an effective divisor $E_1 \in C_{d_1-e}$ such that $|D + E_1| = l_1$. Moreover, it is easy to see that $\dim |2D + E_1| \geqslant d_1 + e - g$. If $\dim |2D + E_1| = d_1 + e - g$, then $|2D + E_1|$ is a non-special linear series of degree $d_1 + e$ and from the transversality of de Jonquières divisors, the dimension of the space of pairs (D, E_1) with this property is

$$d_1 - (d_1 + e) + (d_1 + e - g) = d_1 - g < r_1.$$

Therefore there is at most a $(r_1 - 1)$-dimensional family of divisors $D \in \Gamma_e(l_1)$ satisfying $|2D + E_1| = d_1 + e - g$ while the remainder of the divisors D in $\Gamma_e(l_1)$ are such that $\dim |2D + E_1| > d_1 + e - g$.

Now, if $D \in \Gamma_e(l_1)$ satisfies $\dim |2D + E_1| > d_1 + e - g$, then, by residuation, there exists an effective divisor E_2 such that

$$K_C = |2D + E_1 + E_2|.$$

Moreover, $l_2 = K_C - l_1 = |K_C - D - E_1| = |D + E_2|$, hence $\dim(l_2 - D) \geqslant 0$, i.e. $D \in \Gamma_e(l_2)$ for all $D \in \Gamma_e(l_1)$ with $\dim |2D + E_1| > d_1 + e - g$. Hence the intersection $\Gamma_e(l_1) \cap \Gamma_e(l_2)$ is not transverse. □

4.2.2 Proof of Theorem 1.10

Notice that we have almost proved Theorem 1.10, which states that the intersection

$$\Gamma_e(l_1) \cap \Gamma_e(K_C - l_1)$$

is never transverse for any complete linear series l_1. In order to extend the result of Proposition 4.2 and obtain Theorem 1.10 we change point of view to the tangent spaces of incidence varieties. We recall here the most important facts, some of which were already used in Section 3.2:

(i) The tangent space $T_D C_d = H^0(C, \mathcal{O}_D(D))$ and its dual is

$$T_D^\vee C_d = H^0(C, K_C/K_C - D),$$

with the pairing given by the residue.

(ii) The tangent space at a point of a linear series $|D| \subset C_d^r$ is $T_D|D| = \ker \delta$, where

$$\delta : \mathrm{im}(\alpha\mu_0)^0 \to \mathrm{im}(\mu_0)^0$$

is the differential of the Abel-Jacobi map $u : C_d^r \to W_d^r(C)$ while

$$\alpha : H^0(C, K_C) \to H^0(C, K_C \otimes \mathcal{O}_D)$$

is the restriction mapping and

$$\mu_0 : H^0(C, K_C - D) \otimes H^0(C, \mathcal{O}_C(D)) \to H^0(C, K_C)$$

the cup-product mapping.

(iii) Suppose D and D' are effective divisors of degree d and d' respectively, then the tangent spaces $T_{D+D'}C_{d+d'}$ and $T_D C_d$ are related via

$$H^0(C, \mathcal{O}_{D+D'}(D + D')) = H^0(C, \mathcal{O}_D(D)) \oplus H^0(C, \mathcal{O}_C(D + D')/\mathcal{O}_C(D)).$$

This follows from the exact sequence

$$0 \to \mathcal{O}_C(D)/\mathcal{O}_C \to \mathcal{O}_C(D + D')/\mathcal{O}_C \to \mathcal{O}_C(D + D')/\mathcal{O}_C(D) \to 0.$$

We remark here that the projection from the space $H^0(C, \mathcal{O}_{D+D'}(D+D'))$ onto $H^0(C, \mathcal{O}_D(D))$ is nothing but the truncation map for Laurent tails.

The transversality condition for the intersection of the incidence varieties $\Gamma_e(l_1)$ and $\Gamma_e(l_2)$ is:

$$T_D C_e = T_D\Gamma_e(l_1) + T_D\Gamma_e(l_2).$$

or equivalently, since $\dim \Gamma_e(l_1) + \dim \Gamma_e(l_2) = e$,

$$T_D\Gamma_e(l_1) \cap T_D\Gamma_e(l_2) = (0), \tag{4.2}$$

for some $D \in \Gamma_e(l_1) \cap \Gamma_e(l_2)$. Let $D_i \in l_i$, for $i = 1, 2$ such that $D \leqslant D_i$ and $E_i \in C_{d_i - d}$ such that $D + E_i = D_i$.

Let δ_i, α_i, $\mu_{0,i}$ denote the differential, restriction, and cup-product mapping corresponding to each of the divisors D_i, for $i = 1, 2$. With this notation, the tangent space to the incidence varieties is, for $i = 1, 2$:

$$T_D\Gamma_e(l_i) = \ker \delta_i \cap H^0(C, \mathcal{O}_D(D)).$$

Thus the transversality condition (4.2) becomes

$$\ker \delta_1 \cap \ker \delta_2 \cap H^0(C, \mathcal{O}_D(D)) = (0). \tag{4.3}$$

By construction, the restrictions of δ_1 and δ_2 to the space $H^0(C, \mathcal{O}_D(D))$ coincide and are both equal to the differential δ corresponding to D. Finally, recall that $\eta \in \ker \delta_i$ if and only if $\langle \delta_i \eta, \omega \rangle = 0$ for all $\omega \in \operatorname{coker} \mu_{0,i}$.

Returning now to the case of a linear series l_1 and its residual $l_2 = K_C - l_1$, we immediately see that $\mu_{0,1}$ and $\mu_{0,2}$ are the same multiplication map

$$\mu : H^0(C, D + E_1) \otimes H^0(C, D + E_2) \to H^0(C, 2D + E_1 + E_2).$$

Thus, $\eta \in \ker \delta_1 \cap \ker \delta_2 \cap H^0(C, \mathcal{O}_D(D))$ if and only if $\langle \delta\eta, \omega \rangle = 0$, for all $\omega \in \operatorname{coker} \mu$. But this condition is satisfied by any η in the kernels of both δ_1 and δ_2, so that the transversality condition (4.3) cannot be satisfied and this gives the proof of Theorem 1.10.

4.3 Degenerations of secant varieties

In this section we construct a space of degenerations of secant varieties for families of curves of compact type using limit linear series and the same idea of degeneracy loci.

Proposition 4.3. *Fix a proper, flat family of curves $\mathscr{X} \to B$ over a scheme B equipped with a linear series ℓ of type g^r_d. There exists a scheme $\mathcal{V}^e_{e-f}(\mathscr{X}, \ell)$ proper over B, compatible with base change, whose point over every $t \in B$ parametrises pairs $[\mathscr{X}_t, \mathscr{D}_t]$ of curves and divisors such that $\mathscr{D}_t$ is an $(e-f)$-th secant divisor of ℓ_t. Furthermore, every irreducible component of $\mathcal{V}^e_{e-f}(\mathscr{X}, \ell)$ has dimension at least* $\dim B - f(r + 1 - e + f)$.

Proof. We construct the functor $\mathcal{V}^e_{e-f}(\mathscr{X}, \ell)$ as a subfunctor of the functor of points of the fibre product $\mathscr{X}^e$ over B. We show that it is representable by a scheme that is proper over B and which we also denote by $\mathcal{V}^e_{e-f}(\mathscr{X}, \ell)$.

Let $T \to B$ be a scheme over B. Suppose first that all the fibres of the family are nonsingular. In this case, from Definition 4.2.1 of [Oss], $\ell = g^r_d$ on $\mathscr{X}/B$ is given by a pair $(\mathscr{L}, \mathscr{V})$, where $\mathscr{L}$ is a line bundle of degree d on $\mathscr{X} \times_B T$ and $\mathscr{V} \subseteq \pi_{2*}\mathscr{L}$ is a vector bundle of rank $r + 1$ on B, where π_2 is the second projection from the fibre product. Then the T-valued point $[\mathscr{X}, \mathscr{D}]$ belongs to $\mathcal{V}^e_{e-f}(\mathscr{X}, \ell)(T)$ if the $(e-f)$-th degeneracy locus of the map

$$\mathscr{V} \to \pi_{2*}\mathscr{L}|_{\mathscr{D}}$$

is the whole of T. By construction $\mathcal{V}^e_{e-f}(\mathscr{X}, \ell)$ is compatible with base change, so it is a functor, and it has the structure of a closed subscheme, hence it is representable and the associated scheme is proper.

Now suppose that some of the fibres have nodes (that may or may not be smoothed by $\mathscr{X}$). As we have seen already, a g^r_d on $\mathscr{X}$ is a tuple

$$(\mathscr{L}, (\mathscr{V}^v)_{v \in V(\Gamma_0)}),$$

where Γ_0 is the dual graph of the unique maximally degenerate fibre of the family, with a fixed vertex v_0, $\mathscr{L}$ is a line bundle of multidegree $\vec{d_0}$ (i.e. it has degree d on the component corresponding to v_0 and degree 0 otherwise) on $\mathscr{X} \times_B T$, and for each $v \in V(\Gamma_0)$, the $\mathscr{V}^v$ are subbundles of rank $r+1$ of the twists $\pi_{2*}\mathscr{L}^{\vec{d}^v}$. Let $v_i \in \Gamma_0$ be the vertex corresponding to the component containing a point p_i in the support of D. Then the T-valued point $[\mathscr{X}, D]$ belongs to $\mathcal{V}^e_{e-f}(\mathscr{X}, \ell)(T)$ if, for all i, the $(e-f)$-th degeneracy locus of the map

$$\mathscr{V}^{v_i} \to \pi_{2*}\mathscr{L}^{\vec{d}^{v_i}}|_{p_i}$$

is the whole of T. Checking for compatibility with base change (and hence functoriality) is more delicate than in the previous case because the base change may change the graph Γ_0. However, arguing like in the proof of Proposition 4.5.6 in loc.cit. yields the desired property. Representability and properness then follow analogously.

The dimension bound follows from the degeneracy locus construction of $\mathcal{V}^e_{e-f}(\mathscr{X}, \ell)$. □

For a linear series ℓ_1 of type $g^{r_1}_{d_1}$ on $\mathscr{X}$, denote by $\boldsymbol{\Gamma}_e(\mathscr{X}, \ell_1)$ the relative secant variety $\mathcal{V}^e_{r_1}(\mathscr{X}, \ell_1)$. Thus in this thesis we are interested in the intersection $\boldsymbol{\Gamma}_e(\mathscr{X}, \ell_1) \cap \mathcal{V}^e_{e-f}(\mathscr{X}, \ell_2)$, as we shall see explicitly in what follows.

4.4 Intersections of incidence and secant varieties

In this section we give a proof of Theorem 1.11. We recall the setup: consider a complete linear series $l_1 = g^{r_1}_{d_1}$ on a general curve of genus g with $g > d_1$. We study the intersection of $\Gamma_e(l_1)$ and $V^{e-f}_e(l_2)$, where $l_2 = g^{r_2}_{d_2} = K_C - l_1$ is the residual linear series to l_1 and in the case when

$$\dim \Gamma_e(l_1) + \dim V^{e-f}_e(l_2) \leqslant e - \rho(g, r_1, d_1) - 1. \tag{4.4}$$

We prove that the intersection is empty for an arbitrary linear series $l_1 \in G^{r_1}_{d_1}(C)$ when $f = 1$.

4.4.1 The case of minimal pencils

Before proving Theorem 1.11 in general we first focus on the case of minimal pencils. This will serve as a prototypical example of the strategy we develop in Section 4.4.2 to check the emptiness of the intersection of incidence and secant varieties

$$\Gamma_e(l_1) \cap V^{e-f}_e(K_C - l_1)$$

when condition (4.4) is satisfied.

Let $l_1 = g^1_{d_1}$ be a minimal pencil, i.e. such that the Brill-Noether number

$$\rho(g, 1, d_1) = 1.$$

It follows that

$$g = 2d_1 - 3. \tag{4.5}$$

Let $l_2 = g^{r_2}_{d_2} = K_C - l_1 = g^{d_1-3}_{3d_1-8}$. Then $\dim \Gamma_e(l_1) = 1$ and the expected dimension of $V^{e-f}_e(K_C - l_1)$, as mentioned in the Introduction, is:

$$e - f(r_2 + 1 - e + f).$$

Thus the non-existence condition (4.4) of Theorem 1.11 becomes

$$1 + e - f(r_2 + 1 - e + f) \leqslant e - 2.$$

To ease the computation and presentation, we deal here with the particular case

$$1 + e - f(r_2 + 1 - e + f) = e - 2. \tag{4.6}$$

We show that if (4.6) is satisfied, then the intersection

$$\Gamma_e(l_1) \cap V^{e-f}_e(l_2)$$

is empty. Condition (4.6) is equivalent to

$$f(r_2 + 1 - e + f) = 3$$

and we distinguish two possibilities:

I. If $f = 3$, then $r_2 - e + f = 0$ and $V^{e-f}_e(l_2) = \Gamma_e(l_2)$. Moreover,

$$e = r_2 + f = (d_1 - 3) + 3 = d_1. \tag{4.7}$$

Thus, as expected from the discussion in Section 4.2, we are in a degenerate situation and we are in fact looking at the inclusion of $l_1 = g^1_{d_1}$ inside the series $l_2 = K_C - l_1 = g^{d_1-3}_{3d_1-8}$. More precisely, suppose there exists a divisor $D \in C_e$ such that

$$D \in \Gamma_e(l_1) \cap \Gamma_e(l_2).$$

Thus, from (4.7) we have that $|D| = l_1$ and, as we have seen in the proof of Proposition 4.1, we have that

$$|2D + D'| = K_C$$

for some effective divisor D' of the correct degree. More precisely, the condition that $D \in \Gamma_e(l_2)$ is equivalent to

$$\dim(l_2 - D) = \dim(K_C - l_1) - D = \dim |D'| \geqslant 0. \tag{4.8}$$

Since the curve is general, the Petri map

$$\mu_0 : H^0(C, D) \otimes H^0(C, K_C - D) \to H^0(C, K_C)$$

is injective. Combining this with the base-point-free pencil trick, we get that

$$H^0(C, K_C - 2D) = H^0(C, D') = 0.$$

This then yields a contradiction with condition (4.8). Hence the intersection $\Gamma_e(l_1) \cap V^{e-f}_e(K_C - l_1)$ is empty in this case.

Remark 4.4. This actually provides an interesting example that contradicts the expectation of non-emptiness of secant varieties (see Theorem 0.5 in [Far08]). The inclusion of $l_1 = g^1_{d_1}$ in $l_2 = g^{r_2}_{d_2} = g^{d_1-3}_{3d_1-8}$ can be reformulated from the point of view of secant varieties as follows: there should exist an effective divisor $D' \in C_{2d_1-8}$ such that $g^1_{d_1} + D' = g^{d_1-3}_{3d_1-8}$. In other words, the secant variety $V^{e-f}_e(l_2)$, where $e = 2d_1 - 8$ and $f = d_1 - 4$ should be non-empty and this is indeed the expectation from dimensional considerations as:

$$e - (r_2 + 1 - e + f) = 0.$$

However, as we saw above, there are no such effective divisors D'.

II. If $f = 1$, then $e = d_1 - 4$ and $r_2 - e + f = 2$. Assume towards a contradiction that there exists a divisor

$$D \in \Gamma_e(l_1) \cap V^{e-f}_e(l_2).$$

Hence there exists an effective divisor $E \in C_4$ such that $D + E = l_1$. Moreover

$$l_2 - D = K_C - l_1 - D = g^{r_2-e+f}_{2d_1-4} = g^2_{2d_1-4}.$$

Taking the residue yields

$$l_1 + D = g^2_{2d_1-4}.$$

We have therefore obtained a "system of equations" for a pair of effective divisors $(D, E) \in C_{d_1-4} \times C_4$:

$$\begin{aligned} |D + E| &= g^1_{d_1} \\ |2D + E| &= g^2_{2d_1-4}. \end{aligned} \tag{4.9}$$

By our assumption, a solution for this system exists. Consider all flag curve degenerations $j : \overline{\mathcal{M}}_{0,g} \to \overline{\mathcal{M}}_g$ and let $\mathcal{Z} := \overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{C}}^{d_1}_g$, where $\overline{\mathcal{C}}_g = \overline{\mathcal{M}}_{g,1}$. Denote by $q_1, \ldots, q_g$ the points of attachment of the elliptic tails to the rational spine. Let $X \subset \mathcal{Z}$ be the closure of the divisors D and E satisfying (4.9) on all curves from $\mathrm{im}(j) \subseteq \overline{\mathcal{M}}_g$. Since, by assumption, X dominates $\overline{\mathcal{M}}_{0,g}$, then $\dim X \geqslant g - 3$. Applying Proposition 2.2 of [Far08], there exists a point $[\widetilde{R} := R \cup E_1 \cup \ldots \cup E_g, y_1, \ldots, y_{d_1}] \in X$, where R is a rational spine (not necessarily smooth), the E_i are elliptic tails, and the y_i are the points in the support of $D + E$ such that either:

(i) $y_1 = \ldots = y_{d_1}$, or else

(ii) $y_1, \ldots, y_{d_1}$ lie on a connected subcurve Y of $\widetilde{R}$ of arithmetic genus $p_a(Y) = d_1$ and $|Y \cap \overline{(\widetilde{R} \setminus Y)}| = 1$. Since $g = 2d_1 - 3$, it means that $g > d_1$ for $d \geqslant 2$ so that we may indeed find such a subcurve Y.

Case (i) is immediately dismissed via a short computation using the Plücker formula.

We focus on case (ii). By the assumption on $\widetilde{R}$, there exists a flat, proper morphism $\phi : \mathscr{X} \to B$ such that $\mathscr{X}$ is a smooth surface and B is a smooth affine curve. Let $0 \in B$ be a point such that the fibre $\mathscr{X}_0 := \phi^{-1}(0)$ is a curve stably equivalent to $\widetilde{R}$ and the other fibres $\mathscr{X}_t := \phi^{-1}(t)$ are smooth projective curves of genus g for $t \neq 0$. Moreover there are e sections $\sigma_i : B \to \mathscr{X}$ such that the $\sigma_i(0) = y_i$ are smooth points of $\mathscr{X}_0$ for all $1 \leqslant i \leqslant e$. As before, let $\mathscr{X}^* = \mathscr{X} \setminus \mathscr{X}_0$. There exists a line bundle $\mathscr{L}^*$ of degree $2d_1 - 4$ on $\mathscr{X}^*$ and a subbundle $\mathscr{V}^* \subset \phi_*\mathscr{L}^*$ of rank 2, such that for all $t \neq 0$,

$$\dim \mathscr{V}_t \cap H^0\left(\mathscr{X}_t, \mathscr{L}_t\left(-\sum_{j=1}^{e} \sigma_j(t)\right)\right) = 2.$$

Then, after possibly making a base change and resolving any resulting singularities, the pair $(\mathscr{L}^*, \mathscr{V}^*)$ induces a refined limit linear series of type $g^2_{2d_1-4}$ on $\widetilde{R}$, which we denote by $\tilde{l}$. Moreover, the vector bundle

$$\mathscr{V}^* \cap \phi_*\left(\mathscr{L}^* \otimes \mathcal{O}_{\mathscr{X}^*}\left(-\sum_{j=1}^{e} \sigma_j(B \setminus \{0\})\right)\right)$$

induces a limit linear series $l_1 = g^1_{d_1}$ on $\mathscr{X}_0$.

For a component X of $\mathscr{X}_0$, denote by $(\mathscr{L}_X, \mathscr{V}_X) \in G^2_{2d_1-4}(X)$ the X-aspect of $\tilde{l}$. There exists therefore a unique effective divisor D_X of degree e supported only at the points of $(X \cap \bigcup_{j=1}^{e} \sigma_j(B)) \cup (X \cap \overline{\mathscr{X}_0 \setminus X})$ such that the X-aspect of l_1 is of the form

$$l_{1,X} = (\mathscr{L}_X \otimes \mathcal{O}_X(-D_X), W_X \subset \mathscr{V}_X \cap H^0(X, \mathscr{L}_X \otimes \mathcal{O}_X(-D_X))) \in G^1_{d_1}(X).$$

The collection of aspects $\{l_{1,X}\}_{X \subset Y}$, which we will also denote by l_1, forms a limit $g^1_{d_1}$ on Y with a vanishing sequence that is a subsequence of the vanishing sequence of $\tilde{l}$. Moreover, the collection of aspects of l_1 on Z also yields a limit linear $g^1_{d_1}$ on Z whose vanishing sequence at p is a subsequence of the one of $\tilde{l}$.

Let $p = Y \cap \overline{(\widetilde{R} \setminus Y)}$ and let $Z := \overline{\widetilde{R} \setminus Y}$ and let R_Y, R_Z denote the rational spines corresponding to Y and Z, respectively. An easy argument shows that, without loss of generality, we may assume that all the points in the support of $D + E$ specialise on R_Y. Furthermore, arguing like above, we obtain limits l_1 and $\tilde{l}$ on both R_Y and R_Z.

To reach the desired contradiction, we obtain various bounds for the ramification sequences of the series l_1 and $\tilde{l}$ and show that they cannot be simultaneously satisfied.

Note that the points of attachment $q_1, \dots, q_g$ of the elliptic tails to the rational spine are all cusps, hence for $j = 1, \dots, g$,

$$\alpha((l_1)_{R_Y}, q_j) \geqslant (0,1) \text{ and } \alpha((l_1)_{R_Z}, q_j) \geqslant (0,1) \tag{4.10}$$

$$\alpha(\tilde{l}_{R_Y}, q_j) \geqslant (0,1,1) \text{ and } \alpha(\tilde{l}_{R_Z}, q_j) \geqslant (0,1,1). \tag{4.11}$$

Moreover, using the Plücker formula (2.1) on R_Y we have

$$\text{for } l_1 = g^1_{d_1} : \sum_{q \text{ smooth point}} \big(\alpha_0((l_1)_{R_Y}, q) + \alpha_1((l_1)_{R_Y}, q)\big) = 2d_1 - 2 \tag{4.12}$$

$$\text{for } \tilde{l} = g^2_{2d_1-4} : \sum_{q \text{ smooth point}} \big(\alpha_0(\tilde{l}_{R_Y}, q) + \alpha_1(\tilde{l}_{R_Y}, q)\big) = 6d_1 - 18. \tag{4.13}$$

Combining (4.10)-(4.13) we obtain that on R_Y the ramification at p is at most

$$\text{for } l_1 : \alpha_0((l_1)_{R_Y}, p) + \alpha_1((l_1)_{R_Y}, p) \leqslant d_1 - 2 \tag{4.14}$$

$$\text{for } \tilde{l} : \sum_{i=0}^{2} \alpha_i(\tilde{l}_{R_Y}, p) \leqslant 4d_1 - 18, \tag{4.15}$$

while on R_Z we have the upper bounds

$$\text{for } l_1 : \alpha_0((l_1)_{R_Z}, p) + \alpha_1((l_0)_{R_Z}, p) \leqslant d_1 + 1 \tag{4.16}$$

$$\text{for } \tilde{l} : \sum_{i=0}^{2} \alpha_i(\tilde{l}_{R_Z}, p) \leqslant 4d_1 - 12. \tag{4.17}$$

A further constraint for the ramification sequence at p is given by applying Lemma 3.20 to the current situation and we obtain the following

- If $\{\sigma_C \mid C \subseteq R_Y \text{ irreducible component}\}$ is the set of compatible sections corresponding to the divisor $D + E$ and if $q \in C$, then $\mathrm{ord}_q(\sigma_C) = 0$.
- Similarly, the compatible sections $\{\sigma_C \mid C \subseteq R_Y \text{ irreducible component}\}$ corresponding to the divisor $2D + E$ also have the property that, if $q \in C$, then $\mathrm{ord}_q(\sigma_C) = 0$.

The important observation in both cases is that the support of $D + E$ and of $2D+E$ are contained in Y and that $\deg(D+E) = d_1$ and $\deg(2D+E) = 2d_1-4$. Concretely, this means that both the vanishing sequence of $(l_1)_{R_Y}$ and that of $\tilde{l}_{R_Y}$ must have 0 as their first entry.

Combining this with the compatibility conditions for the vanishing of the sections (2.2) and the fact that the vanishing sequence at p of l_1 is a subsequence of the one of $\tilde{l}$ we see that the only possibility for the vanishing sequences at p of l_1 is

$$a((l_1)_{R_Y}, p) = (0, d_1 - 4) \text{ and } a((l_1)_{R_Z}, p) = (4, d_1)$$

while for the vanishing sequence of $\tilde{l}$ at p it is

$$a(\tilde{l}_{R_Y}, p) = (0, d_1 - 4, 2d_1 - 8) \text{ and } a(\tilde{l}_{R_Z}, p) = (4, d_1, 2d_1 - 4).$$

However the ramification sequence corresponding to the vanishing sequence

$$a((l_1)_{R_Z}, p) = (4, d_1)$$

is

$$\alpha((l_1)_{R_Z}, p) = (4, d_1 - 1)$$

which certainly breaks the upper bound in (4.16) and we have obtained the desired contradiction.

4.4.2 Proof of Theorem 1.11

This section is dedicated to proving Theorem 1.11, which states that for any linear series $l_1 = g^{r_1}_{d_1}$ on a general curve C there are no divisors $D \in C_e$ in the intersection

$$\Gamma_e(l_1) \cap V_e^{e-f}(K_C - l_1)$$

whenever $f = 1$, $g > d_1$, and

$$\dim \Gamma_e(l_1) + \dim V_e^{e-f}(l_2) \leqslant e - \rho(g, r_1, d_1) - 1.$$

In fact we give a general method to check this non-existence statement and apply it to the case $f = 1$ where the computations are most tractable.

For the linear series $l_1 = g^{r_1}_{d_1}$ on a general curve C of genus g, set

$$\rho := \rho(g, r_1, d_1).$$

Then we have an expression of the genus g in terms of ρ:

$$g = \frac{(r_1 + 1)d_1 - \rho}{r_1} - r_1 - 1. \tag{4.18}$$

Moreover, an easy computation shows that the residual linear series to l_1 is $l_2 = g^{r_2}_{d_2}$ where

$$r_2 = \frac{d_1 - \rho}{r_1} - 2 \tag{4.19}$$

$$d_2 = \frac{r_1 + d_1 - 2\rho}{r_1} - 2r_1 - 4. \tag{4.20}$$

The non-existence condition in the statement of the theorem is

$$r_1 + e - f(r_2 + 1 - e + f) \leqslant e - 1 - \rho,$$

or equivalently

$$f(r_2 + 1 - e + f) \geqslant r_1 + 1 + \rho. \tag{4.21}$$

Assume towards a contradiction that there exists a divisor $D \in C_e$ such that

$$D \in \Gamma_e(l_1) \cap V_e^{e-f}(l_2).$$

It follows that we also have a divisor $E = l_1 - D \in C_{d_1-e}$. Then

$$l_2 - D = K_C - l_1 - D$$

is a linear series of dimension

$$r_2 - e + f$$

and degree

$$\frac{r_1 + d_1 - 2\rho}{r_1} - 2r_1 - 4 - e.$$

By residuation we conclude that

$$l_1 + D = g_{d+e}^{r_1+f}. \tag{4.22}$$

We have therefore obtained a "system of equations" for two divisors $(D, E) \in C_e \times C_{d_1-e}$:

$$\begin{aligned} |D + E| &= g_{d_1}^{r_1} \\ |2D + E| &= g_{d_1+e}^{r_1+f} \end{aligned} \tag{4.23}$$

and by assumption a solution should exist.

Remark 4.5. We may view the condition $|2D + E| = g_{d_1+e}^{r_1+f}$ from the point of view of de Jonquières divisors: the dimension of the space of pairs (D, E) satisfying this is

$$d_1 - (d_1 + e) + (r_1 + f) = r_1 - e + f \geqslant 0.$$

Hence so far there is no reason to expect there not to be such a pair (D, E) satisfying the system (4.23).

By assumption, there exists therefore a pair of divisors $(D, E) \in C_e \times C_{d_1-e}$ satisfying the system (4.23). Assume furthermore that $g > d_1$ (we shall see later that in the case $f = 1$ this assumption does not lead to any loss of generality). We consider again all flag curve degenerations as in the case of minimal pencils. Applying Proposition 2.2 of [Far08], there exists a point $[\widetilde{R} := R \cup E_1 \cup \ldots \cup E_g, y_1, \ldots, y_{d_1}] \in X$, where R is a rational spine (not necessarily smooth), the E_i are elliptic tails, and the y_i are the points in the support of $D + E$ such that either:

(i) $y_1 = \ldots = y_{d_1}$, or else

(ii) $y_1, \ldots, y_{d_1}$ lie on a connected subcurve Y of $\widetilde{R}$ of arithmetic genus $p_a(Y) = d_1$ and $|Y \cap \overline{(\widetilde{R} \setminus Y)}| = 1$. This is possible since we have taken $g > d_1$.

Case (i) is again immediately dismissed via a short computation using the Plücker formula.

We focus on case (ii). Let $p = Y \cap \overline{(\widetilde{R} \setminus Y)}$ and let $Z := \overline{\widetilde{R} \setminus Y}$ and let R_Y, R_Z denote the rational spines corresponding to Y and Z, respectively. Just as in the case of minimal pencils, we have limits l_1 and $\tilde{l}$ on both R_Y and R_Z and we may assume that all points in the support of $D + E$ specialise on R_Y.

The strategy again is to constrain the vanishing (or, equivalently, ramification) sequence at p of the limit linear series $l_1 = g^{r_1}_{d_1}$ and $\tilde{l} := g^{r_1+f}_{d_1+e}$ on each of the components R_Y and R_Z. We make use of four important facts:

1. For refined limit linear series, the vanishing sequences at the point p must satisfy the following equalities:

$$\begin{aligned} &a_i((l_1)_{R_Y}, p) + a_{r_1-i}((l_1)_{R_Z}, p) = d_1 \text{ for } i = 0, \dots, r_1 \\ &a_i(\tilde{l}_{R_Y}, p) + a_{r_1+f-i}(\tilde{l}_{R_Z}, p) = d_1 + e \text{ for } i = 0, \dots, r_1 + f. \end{aligned} \tag{4.24}$$

2. The vanishing sequence at p of $l_1 = g^{r_1}_{d_1}$ is a subsequence of the one corresponding to $\tilde{l} = g^{r_1+f}_{d_1+e}$.

3. The Plücker formula (2.1) applied to both limit linear series on both components. The Plücker formula on R_Y yields:

$$\text{for } l_1 : \sum_{q \text{ smooth point of } R_Y} \left(\sum_{i=0}^{r_1} \alpha_i((l_1)_{R_Y}, q) \right) = (r_1 + 1)(d_1 - r_1) \tag{4.25}$$

$$\text{for } \tilde{l} : \sum_{q \text{ smooth point of } R_Y} \left(\sum_{i=0}^{r_1+f} \alpha_i(\tilde{l}_{R_Y}, q) \right) = (r_1 + f + 1)(d_1 + e - f - r_1). \tag{4.26}$$

The curve R_Y contains the points $q_1, \dots, q_{d_1}$ which are all cusps, and therefore have ramification sequences at least $(0, 1, \dots, 1)$. Combining this with (4.25) and (4.26) we obtain upper bounds for the ramification at p:

$$\text{for } l_1 : \sum_{i=0}^{r_1} \alpha_i((l_1)_{R_Y}, p) \leqslant (r_1 + 1)(d - r_1) - d_1 r_1 \tag{4.27}$$

$$\text{for } \tilde{l} : \sum_{i=0}^{r_1+f} \alpha_i(\tilde{l}_{R_Y}, p) \leqslant (r_1 + f + 1)(d_1 + e - f - r_1) - (f + r_1) d_1. \tag{4.28}$$

Using the same reasoning on R_Z we obtain the following bounds on the ramification at p:

$$\text{for } l_1 : \sum_{i=0}^{r_1} \alpha_i((l_1)_{R_Z}, p) \leqslant (r_1+1)(d-r_1) - (g-d_1)r_1 \tag{4.29}$$

$$\text{for } \tilde{l} : \sum_{i=0}^{r_1+f} \alpha_i(\tilde{l}_{R_Z}, p) \leqslant (r_1+f+1)(d_1+e-f-r_1) - (f+r_1)(g-d_1). \tag{4.30}$$

Since for a linear series l of type g^r_d,

$$\sum_{i=0}^{r} \alpha_i(l,p) = \sum_{i=0}^{r} a_i(l,p) - \frac{r(r+1)}{2}, \tag{4.31}$$

the upper bounds for the ramification give equivalently bounds for the vanishing at p.

4. The statement of Lemma 3.20 applied to the current situation, as in the case of the minimal pencils. We again obtain that both the vanishing sequence of $(l_1)_{R_Y}$ and that of $\tilde{l}_{R_Y}$ must have 0 as their first entry.

Putting everything together, the vanishing sequence at p corresponding to l_1 on R_Y is

$$a((l_1)_{R_Y}, p) = (0, x_1, \ldots, x_{r_1}),$$

for some strictly positive integers $x_1, \ldots, x_{r_1}$ smaller than d_1, while the sequence on R_Z is

$$a((l_1)_{R_Z}, p) = (d_1 - x_{r_1}, \ldots, d_1 - x_1, d_1).$$

On the other hand, the vanishing sequence at p corresponding to $\tilde{l}$ on R_Y is

$$a(\tilde{l}_{R_Y}, p) = (0, x_1, \ldots, x_{r_1}, x_{r_1+1}, \ldots, x_{r_1+f}),$$

where the strictly positive integers $x_{r_1+1}, \ldots, x_{r_1+f}$ are all smaller than $d_1 + e$ and exactly one of the x_i is equal to e. The sequence on R_Z is

$$a(\tilde{l}_{R_Z}, p) = (d_1 + e - x_{r_1+f}, \ldots, d_1, \ldots, d_1 + e),$$

which must also contain the terms $d_1 - x_{r_1}, \ldots, d_1 - x_1$.

Let $x = x_1 + \ldots + x_{r_1}$. Using (4.27), (4.29), and (4.31) and the fact that

$$g - d_1 = \frac{d_1 - \rho}{r_1} - r_1 - 1,$$

we have that

$$r_1\left(\frac{d_1}{r_1} - \frac{r_1+1}{2}\right) - \rho \leqslant x \leqslant r_1\left(\frac{d_1}{r_1} - \frac{r_1+1}{2}\right). \tag{4.32}$$

In order to prove the statement of Theorem 1.11, we find a contradiction to the inequality (4.32). If $f = 1$, then $\tilde{l} = g^{r_1+1}_{d_1+e}$ and

$$e \leqslant r_2 - r_1 - \rho + 1 = \frac{d_1 - (r_1+1)\rho}{r_1} - r_1 - 1. \tag{4.33}$$

Note that the above inequality also implies $e \leqslant g - d_1 - \rho$, hence the assumption $g > d_1$ does not lead to any loss of generality in the case $f = 1$.

Suppose first that none of the x_i with $i = 1, \ldots, r_1$ is equal to e. Thus the vanishing sequence at p corresponding to $\tilde{l}$ on R_Y is

$$a(\tilde{l}_{R_Y}, p) = (0, e, x_1, \ldots, x_{r_1}).$$

Combining (4.30) and (4.31) yields the inequality

$$\begin{aligned}(r_1+2)(d_1+e) - e - x - \frac{(r_1+1)(r_1+2)}{2} \leqslant &(r_1+2)(d_1+e-1-r_1) \\ &- (r_1+1)\left(\frac{d_1-\rho}{r_1} - r_1 - 1\right)\end{aligned}$$

which, after plugging in the expression (4.33) for e, reduces to

$$x \geqslant \frac{(r_1+1)(r_1+2)}{2} + (r_1+1)\left(\frac{d_1}{r_1} - r_1 - 1\right).$$

This contradicts the upper bound in (4.32). Hence this vanishing sequence cannot occur.

One the other hand, if e is one of the x_i with $i = 1, \ldots, r_1$, then the vanishing sequence at p corresponding to l_1 on R_Y is

$$a((l_1)_{R_Y}, p) = (0, e, x_1, \ldots, x_{r_1-1})$$

and on R_Z

$$a((l_1)_{R_Z}, p) = (d_1 - x_{r_1-1}, \ldots, d_1 - x_1, d_1 - e, d_1). \tag{4.34}$$

Moreover, the vanishing sequence at p corresponding to $\tilde{l}$ on R_Y is

$$(0, e, x_1, \ldots, x_{r_1-1}, y)$$

and the one on R_Z is

$$(d_1 + e - y, d_1 + e - x_{r_1-1}, \ldots, d_1 + e - x_1, d_1, d_1 + e). \tag{4.35}$$

Since the sequence (4.34) must be a subsequence of (4.35), we see that

$$d_1 + e - y = d_1 - x_i,$$

for some index i. In other words, $y = e + x_i$. Combining (4.30) and (4.31) again yields the inequality

$$(r_1+2)(d_1+e) - e - x_i - x - \frac{(r_1+1)(r_1+2)}{2} \leqslant (r_1+2)(d_1+e-1-r_1) \\ - (r_1+1)\left(\frac{d_1-\rho}{r_1} - r_1 - 1\right).$$

This leads to a contradiction with the upper bound in (4.32) in the same way as above.

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 17. September 2018 Mara Ungureanu