INCOME RISK AND CONSUMPTION INEQUALITY: A SIMULATION STUDY

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Income and Consumption Inequality: Decomposing Income Risk*

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Abstract

This paper assesses the accuracy of decomposing income risk into permanent and transitory components using income and consumption data. We develop a specific approximation to the optimal consumption growth rule and use Monte Carlo evidence to show that this approximation can provide a robust method for decomposing income risk. The availability of asset data enables the use of a more accurate approximation allowing for partial self-insurance against permanent shocks. We show that the use of data on median asset holdings corrects much of the error in the simple approximation which assumes no self-insurance against permanent shocks.

JEL: C30, D52, D91.

Keywords: income risk, inequality, approximation methods, consumption

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1 Introduction

Although the increase in income inequality over the last twenty five years in many modern economies has been well documented, attention has more recently focussed on the extent to which these increases were driven by the distribution of permanent or transitory shocks to individual income processes (see Burkhauser and Poupore, 1997; Buchinsky and Hunt, 1999; Moffitt and Gottschalk, 2002; Meghir and Pistaferri, 2004). Cross sectional income surveys alone cannot be used to make this distinction and even panel data cannot distinguish unexpected or uninsured events from predictable or insured ones. The combination of consumption and income data can reveal much more (see Blundell and Preston, 1998; Krueger and Perri, 2002). Under an assumption about intertemporal consumption choices such data can identify the distribution of uninsured transitory and permanent shocks to income.

This paper assesses the accuracy of identifying income risk using income and consumption data. It develops a specific approximation to the optimal consumption growth rule under CRRA preferences and shows that this approximation can provide an accurate method for decomposing income risk. More precisely, the approximation separates the variances of the permanent and transitory components of uninsured shocks to income. This result formalises the empirical approach adopted in Blundell and Preston (1998).

Several papers have looked at the variance of permanent shocks in the US in the 1980s (Moffitt and Gottschalk 1994, 2002, Meghir and Pistaferri 2004, Blundell, Pistaferri and Preston 2004). All conclude that permanent variances rose in the early 1980s. There is less agreement as to what happened subsequently though there is some evidence that it may have fallen back[1]. The Monte Carlo exercises in this paper are motivated by the sorts of numbers found in these papers.

The results in this paper show that a simple approximation to consumption behaviour can be used to identify the source of income risk using data on consumption inequality. The simplest approximation is based on individuals being unable to self-insure against permanent shocks but being able to insure fully against transitory shocks. This approximation implies that the cross-section variance of consumption will reflect only accumulated

[1]This view is supported in Meghir and Pistafferi (2002, p.10), Moffitt and Gottschalk (1994, p.12) and Blundell, Pistaferri and Preston (2004, p.22).

permanent shocks to income and further, the amount by which the cross-section variance of income exceeds the variance of consumption can be attributed to growth in the transitory variance. At a theoretical level, we show the order of the error of this approximation. Through simulation of individuals choosing consumption in an economy with permanent and transitory income shocks, we show that the approximation decomposes the income risk fairly accurately and correctly identifies changes in risk over time.

Error in the approximation arises through underpredicting self-insurance against permanent shocks and overpredicting self-insurance against transitory shocks. This error implies an underestimate of the actual risk to permanent income and an overestimate of the change in the variance of transitory shocks. The approximation of the consumption rule can be made more precise if more data is available to the econometrician. The availability of asset data enables the use of an approximation allowing for partial self-insurance against permanent shocks. The importance of using this additional information depends on the extent of self-insurance: if individuals are impatient, asset holdings are limited and the simple approximation is accurate. Further, we show that using data on median asset holdings corrects much of the error in the simple approximation and that using data on individual asset holdings does not add much improvement beyond this.

An alternative to the approximation we develop would be to estimate the income process structurally through dynamic programming. Such an approach requires estimates of preference parameters whereas although our approximation assumes a CRRA functional form, it does not require estimates of risk aversion or the discount rate. Further, the approximation does not have to specifiy the complete environment facing the individual.

In section 2 we derive the approximations which relate consumption inequality to income risk. Section 3 describes the environment we simulate and reports the results of our Monte Carlo experiments. Section 4 concludes.

2 The Evolution of Income and Consumption Variances

2.1 The income process

Consider an individual living for T periods. Until retirement at age R they work fixed hours to earn an income which evolves stochastically according to a process with a permanent-transitory decomposition. Specifically suppose log income in period t can be written

$$\ln y_t = \ln Y_t + u_t \qquad t = 1, \ldots, R$$

where Y_t represents the permanent component of income and u_t the transitory shock in period t. The final $T - R$ periods of life are spent in mandatory retirement with no labour income.

The permanent component is assumed to follow a random walk

$$\Delta \ln Y_t = \eta_t + v_t$$

where v_t is a permanent shock and η_t is a trend common to the members of the cohort. The process for income can therefore be written

$$\Delta \ln y_t = \eta_t + \Delta u_t + v_t. \tag{1}$$

We assume the shocks are orthogonal and unpredictable given prior information

$$\begin{aligned} E\left(u_t | Y_{t-1}, u_{t-1}, v_{t-1}\right) &= E\left(v_t | Y_{t-1}, u_{t-1}, v_{t-1}\right) \\ &= E\left(u_t v_t | Y_{t-1}, u_{t-1}, v_{t-1}\right) = 0. \end{aligned}$$

This is a popular specification compatible with an MA(1) process for changes in log income[2].

We let $\nu_t = (v_t, u_t)'$ denote the vector of shocks.

[2]See Macurdy 1982, Abowd and Card 1989, Moffitt and Gottschalk 2002, Meghir and Pistaferri 2004 for examples of papers modelling the time series properties of individual earnings using longitudinal data.

2.2 Consumption choice

Consumption and income are linked through the intertemporal budget constraint

$$\sum_{i=0}^{T-t} \frac{c_{t+i}}{(1+r)^i} + \frac{A_{T+1}}{(1+r)^{T-t}} = \sum_{i=0}^{R-t} \frac{y_{t+i}}{(1+r)^i} + A_t \tag{2}$$

where c_t denotes consumption in period t, A_t is assets at beginning of period t, r is a real interest rate, assumed for simplicity to be constant and T is length of lifetime. The terminal condition that $A_T = 0$ implies that individuals will not borrow more than the discounted sum of the minimum income they will receive in each remaining period.

Suppose the household plans at age t to maximise expected remaining lifetime utility

$$E_t \left[\sum_{i=0}^{T-t} \frac{U(c_{t+i})}{(1+\delta)^i} \right]$$

where δ is a subjective discount factor and $U : \mathbb{R} \to \mathbb{R}$ is a concave, three times continuously differentiable utility function.

The solution to the consumer problem requires expected constancy of discounted marginal utility λ_{t+k} across all future periods

$$\begin{aligned} U'(c_{t+i}) &= \lambda_{t+i} \\ E_t\lambda_{t+i} &= \left(\frac{1+\delta}{1+r}\right)^i \lambda_t, \qquad i = 0, 1, \ldots, T-t \end{aligned} \tag{3}$$

This is the familiar Euler condition for consumption over the life-cycle (see Hall 1978, Attanasio and Weber 1993, for example).

We show in the appendix that

$$\Delta \ln c_t = \varepsilon_t + \Gamma_t \ + \ \mathcal{O}(E_{t-1}{\varepsilon_t}^2)$$

where ε_t is an innovation term; Γ_t is the anticipated gradient to the consumption path reflecting precautionary saving and intertemporal substitution, common within a cohort if we assume CRRA preferences, and $\mathcal{O}(x)$ denotes a term with the property that

$$\lim_{x\to 0} \mathcal{O}(x)/x < \infty.$$

The innovation ε_t is tied to the idiosyncratic income shocks u_t and v_t through the lifetime budget constraint (2). We show below that we can approximate the relation between these innovations through a formula

$$\Delta \ln c_t \;=\; \pi_t \left[v_t + \alpha_t u_t\right] \;+\; \Gamma_t \;+\; \mathcal{O}(\|\nu_t\|^2) \;+\; \mathcal{O}(E_{t-1}\|\nu_t\|^2) \tag{4}$$

involving two additional parameters

- α_t: an annuitisation factor capturing the importance of transitory shocks to lifetime wealth relative to permanent shocks.
- π_t: a self-insurance factor capturing the significance of asset holdings as a component of current human and financial wealth.

To quantify the annuitisation factor, we need information on the time horizon and the interest rate. To quantify the self-insurance factor we need information on current asset holdings and on expected wage growth.

2.3 Variances

We assume that the variances of the shocks v_t and u_t are the same in any period for all individuals in any cohort but that these variances are not constant over time. The cross-sectional covariances of the shocks with previous periods' incomes are assumed to be zero. In this discussion we also assume that shocks are uncorrelated across individuals.

Define $V(u_t)$ to be the cross-section variance of transitory shocks in period t and $V(v_t)$ to be the corresponding variance of permanent shocks. Let $\bar{\pi}_t$ and $V(\pi_t)$ be the cross section mean and variance of π_t. Then the growth in the cross-section variance and covariances of income and consumption take the form

Theorem 1 *Assuming an income process (1) then*

$$\begin{aligned}
\Delta \mathrm{V}(\ln y_t) &= \mathrm{V}(v_t) + \Delta \mathrm{V}(u_t) \\
\Delta \mathrm{V}(\ln c_t) &= (\bar{\pi}_t{}^2 + \mathrm{V}(\pi_t))\mathrm{V}(v_t) + (\bar{\pi}_t{}^2 + \mathrm{V}(\pi_t))\alpha_t^2 \mathrm{V}(u_t) \\
&\qquad + \mathcal{O}(E_{t-1}\|\nu_t\|^3) \\
\Delta \mathrm{Cov}(\ln c_t, \ln y_t) &= \bar{\pi}_t \mathrm{V}(v_t) + \Delta[\bar{\pi}_t \alpha_t \mathrm{V}(u_t)] \\
&\qquad + \mathcal{O}(E_{t-1}\|\nu_t\|^3).
\end{aligned} \tag{5}$$

Proof: See appendix.

Permanent inequality ($\mathrm{V}(v_t)$) or growth in uncertainty ($\Delta\mathrm{V}(u_t)$) both result in growth of income inequality. Observing the cross-section distribution of income cannot, on its own, distinguish these. Taking income inequality together with consumption inequality and sufficient information on π_t and α_t we are able, however, to use the life-cycle model to separate the growth in permanent inequality from the growth in transitory uncertainty.

From these expressions we can approximately identify the growth in the transitory variance and the level of the permanent variances from the growth in consumption and income variances. The approximation used can take differing degrees of accuracy depending on the information available and assumptions made about π_t and α_t.

1. Particularly simple forms follow by allowing $\bar{\pi}_t \simeq 1$, $\mathrm{V}(\pi_t) \simeq 0$ and $\alpha_t \simeq 0$, implying no self-insurance and a long horizon. Such an approximation might be attractive if we lack information on assets. Specifically

$$\begin{aligned} \Delta\mathrm{V}(\ln c_t) &\simeq \mathrm{V}(v_t) \\ \Delta\mathrm{Cov}(\ln c_t, \ln y_t) &\simeq \mathrm{V}(v_t) \end{aligned} \tag{6}$$

   so that the within cohort growth in the variance of consumption identifies the variance of permanent shocks. This has the implication that the growth should always be positive, as noted, for example, by Deaton and Paxson (1994). The difference between the growth in the within cohort variances of income and consumption then identifies the growth in the variance of transitory shocks through the first equation in (5). The evolution of the covariance should follow that of the consumption variance and this provides one testable overidentifying restriction per period of the data.

2. If we have information on mean or median asset levels and mean or median incomes by age (but lack further information on the distribution) and are prepared to postulate an expected future path of increments to permanent income η_t then we might be prepared to approximate $\bar{\pi}_t$ by its value at mean (or median) income and assets,

say $\tilde{\pi}_t$, and take an approximation setting $\mathrm{V}(\pi_t) \simeq 0$ so that

$$\begin{aligned} \Delta \mathrm{V}(\ln c_t) &\simeq \tilde{\pi}_t{}^2 \mathrm{V}(v_t) + \tilde{\pi}_t{}^2 \alpha_t^2 \mathrm{V}(u_t) \\ \Delta \mathrm{Cov}(\ln c_t, \ln y_t) &\simeq \tilde{\pi}_t \mathrm{V}(v_t) + \Delta[\tilde{\pi}_t \alpha_t \mathrm{V}(u_t)] \end{aligned} \tag{7}$$

3. With information on the distribution of assets we might calculate $\bar{\pi}_t$ and $\mathrm{V}(\pi_t)$ and make full use of all terms in (5).

Cross section variances and covariances of log income and consumption can be estimated by corresponding sample moments with precision given by standard formulae. The underlying variances of the shocks can then be inferred by minimum distance estimation using (5) after choosing or estimating values for $\bar{\pi}_t$, $\mathrm{V}(\pi_t)$ and α_t, the minimised distance providing a χ^2 test of the overidentifying restrictions.

3 Monte-Carlo

3.1 Model and Calibration

Transitory and permanent shocks to log income are assumed log-normally distributed and truncated below.

$$\ln y_t = \ln Y_t + u_t \ , \quad u_t \sim N\left(0, \sigma_{u_t}^2\right) \tag{8}$$

$$\ln Y_t = \eta_t + \ln Y_{t-1} + v_t \ , \quad v_t \sim N\left(0, \sigma_{v_t}^2\right) \tag{9}$$

$$\eta_t = -\frac{\Delta \sigma_{u_t}^2 + \sigma_{v_t}^2}{2} \tag{10}$$

Transitory shocks are assumed to be *i.i.d.* within period with variance growing at a deterministic rate. In certain simulations permanent shocks are also *i.i.d.* within period with constant variance. We also consider adding stochastic volatility to the model. In such cases the permanent variance follows a two-state, first-order Markov process with the transition probability between alternative variances, $\sigma_{v,L}^2$ and $\sigma_{v,H}^2$, given by β.

$$\begin{array}{ccc} & \sigma_{v,L}^2 & \sigma_{v,H}^2 \\ \sigma_{v,L}^2 & 1-\beta & \beta \\ \sigma_{v,H}^2 & \beta & 1-\beta \end{array}$$

Utility is of the constant elasticity of substitution form,

$$U(c_t) = \frac{\gamma}{1+\gamma} c_t^{1+1/\gamma}. \tag{11}$$

Parameter values consistent across simulations are given in Table 1.

Table 1: **Parameter Values**

Parameter	Value	Parameter	Value	Parameter	Value
γ	-0.667	T	50	A_0	0
r	0.015	R	40	A_{T+1}	0

3.2 Results

The aim of the Monte Carlo exercise is to show the accuracy of estimation of the variances by approximation. In particular, we want to show the accuracy of estimates of the permanent variance and of changes in the transitory variance. We consider six sets of values for the variances, shown in Table 2. Experiments 1 and 2 have constant permanent variance. Experiments 3, 4, 5 and 6 have stochastic volatility. For these later four experiments, consumers believe that the permanent variance has an ex-ante probability β of changing in each t. In the simulations, the variance actually switches only once and this happens in period S, which we assume is common across all individuals. In other words, the distribution of idiosyncratic shocks is subject to a common shock.[3] The difference between experiments 3, 4, 5 and 6 is in the degree of impatience. In the base case subjective discount rate $\delta = 0.02$, whereas in the next two it takes higher and lower values of $\delta = 0.04$ and 0.01 and in the fourth we take a mixed population with half at 0.02 and a quarter each at 0.04 and 0.01.

For each experiment, we simulate consumption, earnings and asset paths for 50,000 individuals. To obtain estimates of the variance for each period, we draw random cross

[3] In solving the model for a particular individual, it is irrelevant whether a particular shock is idiosyncratic or common because the model is partial equilibrium.

sectional samples of 2000 individuals for each period from age 10 to 30. For the cases where there is a jump change in the variance at $S = 20$, this provides 9 years before the jump and 9 years after. We repeat this process 1000 times to provide information on the properties of the estimators.

As discussed above, we consider three approximations of differing subtlety. The simplest approximation, based on (6) and labelled *uncorrected*, would be accurate if it were not possible to insure at all against permanent shocks. In practice, individuals can use savings to partially insure against permanent shocks because individuals have finite horizons. We might therefore expect the accuracy of this simple approximation to depend on the cost of saving, which is explored by varying the discount rate. If we also have information on asset holdings, then the approximation can be corrected to take account of the amount of self-insurance through saving and we would not expect differences in the cost of saving to affect the accuracy of the corrected estimates.

The quality of the correction depends on the quality of information about assets. In one set of estimates, based on (7) and labelled *corrected at median*, we calculate π_t at sample median values of assets and incomes assuming known η_t and r. In another set, based on (5) and labelled *fully corrected*, we use the true means and variances of π, $\bar{\pi}_t$ and $V(\pi_t)$ as calculated from the full 50,0000 simulated cases.

In Figures 1 and 2 we compare means of the corrected and uncorrected estimates of the permanent variances for the six experiments. A three year moving average has been applied to smooth the time series variation in the estimates. In Table 3 we report average values across the nine year periods which come before and after jumps in the variance in the stochastic jump case.

Note firstly that the uncorrected estimates consistently underestimate the permanent variance and do so increasingly severely as age increases. This is because throughout these simulations $\bar{\pi}_t$ is consistently below unity and falling with age as can be seen for four of the cases in Figure 3. Furthermore the value of $\bar{\pi}_t$ is lower the more patient individuals are and therefore the more inclined to accumulate assets, hence the approximation is better if consumers are more impatient as is evident from comparisons within Figure 2[4].

[4]Carroll (1997) finds that households hold only small buffer stocks of saving until about age 50 (and then accumulate substantial retirement savings). This low level of asset holdings suggests individuals

Table 2: **Experiment Parameter Values**

Expt No	Description	δ	$\Delta\sigma^2_{u_t}$	$\sigma^2_{v_t}$	β	S
1	Constant low var	0.02	0.01	0.005	0.0	
2	Constant high var	0.02	0.01	0.015	0.0	
3	Stochastic volatility	0.02	0.01	$0.015 \rightarrow 0.005$	0.05	20
4	Impatience	0.04	0.01	$0.015 \rightarrow 0.005$	0.05	20
5	Patience	0.01	0.01	$0.015 \rightarrow 0.005$	0.05	20
6	Heterogeneity	mixed	0.01	$0.015 \rightarrow 0.005$	0.05	20

Table 3: **Estimated and True Permanent Variances**

Expt No	Period 10-19				Period 20-29			
	Uncorrected	Corrected at median	Corrected fully	Truth	Uncorrected	Corrected at median	Corrected fully	Truth
1	0.00404 (0.00032)	0.00480 (0.00047)	0.00489 (0.00048)	0.0050	0.00327 (0.00050)	0.00481 (0.000109)	0.00491 (0.00109)	0.0050
2	0.00950 (0.00084)	0.01390 (0.00132)	0.01422 (0.00135)	0.0150	0.00610 (0.00125)	0.01395 (0.00279)	0.01447 (0.00287)	0.0150
3	0.00987 (0.00086)	0.01385 (0.00134)	0.01413 (0.00137)	0.0150	0.00299 (0.00116)	0.00460 (0.00242)	0.00473 (0.00248)	0.0050
4	0.01270 (0.00101)	0.01423 (0.00126)	0.01442 (0.00129)	0.0150	0.00425 (0.00136)	0.00460 (0.00216)	0.00467 (0.00221)	0.0050
5	0.00871 (0.00080)	0.01374 (0.00139)	0.01403 (0.00141)	0.0150	0.00252 (0.00108)	0.00450 (0.00256)	0.00467 (0.00261)	0.0050
6	0.00941 (0.00091)	0.01376 (0.00139)	0.01361 (0.00135)	0.0150	0.00345 (0.00118)	0.00488 (0.00245)	0.00547 (0.00238)	0.0050

Table 4: **Tests of Overidentifying Restrictions**

	All tests χ^2_{19}		
Expt No	Uncorr	Corrected at median	Corrected fully
1	85.429 p=0.000	18.162 p=0.537	18.081 p=0.541
2	263.608 p=0.000	19.251 p=0.489	18.576 p=0.519
3	124.579 p=0.000	19.532 p=0.478	19.155 p=0.494
4	52.282 p=0.006	18.342 p=0.531	18.217 p=0.537
5	161.345 p=0.000	20.025 p=0.456	19.561 p=0.476
6	159.218 p=0.000	26.823 p=0.219	27.623 p=0.201

Nonetheless broad differences between simulations with different values of the variance $V(v_t)$ are clearly picked up. The drop in the variance in the cases with stochastic volatility is plainly identified on average, particularly when there is some aggregation over years either side of the jump, as can be seen in Table 3.

Correction for self-insurance possibilities, even only using sample median assets and incomes, secures a considerable improvement in estimates with the means across Monte Carlo replications very close to the true values in the simulations and no evident deterioration in quality with age.

Table 4 reports mean values across simulations of the χ^2 tests of overidentifying restrictions calculated with each set of estimates. While the equality of changes in variances and covariances implied by (6) is on average emphatically rejected, the approximate restrictions implied by (7) and (5) are contrastingly comfortably accepted.

are impatient and there is limited self-insurance against permanent shocks. Therefore, the closer the economy is to the buffer stock model, the closer our simple approximation will be to identifying the true decomposition of permanent and transitory shocks.

Figure 4 shows that the transitory variances are picked up with a high degree of accuracy in corrected and uncorrected estimates.

4 Conclusions

Increases in income inequality may reflect the variance of permanent shocks or increases in the variability of transitory shocks. The differing sources of risk have very different implications for welfare. In this paper, we show that simple approximations to consumption rules can be used to decompose income variability into its components. In assessing the accuracy of this decomposition we show that it is able to map accurately the evolution of transitory and permanent variances of income shocks.

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A.1 Appendix: Proof of Theorem 1

The approximation in section 2.3 uses the Euler equation to relate consumption growth to innovations. These innovations are related to income shocks through an approximation to the budget constraint. The validity of the approximation depends on the order of the error in approximations to the Euler equation and to the budget constraint. The aim of this appendix is firstly to show how the approximation relating consumption variance to income variance is derived and secondly to show the order of the error of this approximation.

A.1.1 Approximating the Euler Equation

We begin by calculating the error in approximating the Euler equation.

By (3)

$$E_t U'(c_{t+1}) = U'(c_t)\left(\frac{1+\delta}{1+r}\right) = U'(c_t + \Gamma_{t+1}) \tag{12}$$

for some Γ_t. If preferences are CRRA then Γ_{t+1} does not depend on c_t and is common to all households.

By exact Taylor expansion of marginal utility in $t+1$ around $c_t + \Gamma_{t+1}$, there exists a $\tilde{c}$ between $c_t + \Gamma_{t+1}$ and c_{t+1} such that

$$\begin{aligned} U'(c_{t+1}) \quad = \quad & U'(c_t + \Gamma_{t+1})\Bigg[1 + \frac{1}{\gamma(c_t + \Gamma_{t+1})}[\Delta \ln c_{t+1} - \Gamma_{t+1}] \\ & + \frac{1}{2}\beta(\tilde{c}, c_t + \Gamma_{t+1})[\Delta \ln c_{t+1} - \Gamma_{t+1}]^2\Bigg] \end{aligned} \tag{13}$$

where $\gamma(c) \equiv U'(c)/cU''(c) < 0$ and $\beta(\tilde{c}, c) \equiv \left[\tilde{c}^2 U'''(\tilde{c}) + \tilde{c}U''(\tilde{c})\right]/U'(c)$.

Taking expectations of (14)

$$\begin{aligned} E_t U'(c_{t+1}) \quad = \quad & U'(c_t + \Gamma_{t+1})\Bigg[1 + \frac{1}{\gamma(c_t + \Gamma_{t+1})}E_t[\Delta \ln c_{t+1} - \Gamma_{t+1}] \\ & + \frac{1}{2}E_t\left\{\beta(\tilde{c}, c_t + \Gamma_{t+1})[\Delta \ln c_{t+1} - \Gamma_{t+1}]^2\right\}\Bigg] \end{aligned} \tag{14}$$

Substituting for $E_t U'(c_{t+1})$ from (12),

$$\frac{1}{\gamma(c_t + \Gamma_{t+1})}E_t[\Delta \ln c_{t+1} - \Gamma_{t+1}] + \frac{1}{2}E_t\left\{\beta(\tilde{c}, c_t + \Gamma_{t+1})[\Delta \ln c_{t+1} - \Gamma_{t+1}]^2\right\} = 0$$

and thus

$$\Delta \ln c_{t+1} = \Gamma_{t+1} - \frac{\gamma(c_t + \Gamma_{t+1})}{2} E_t \left\{ \beta(\tilde{c}, c_t + \Gamma + t + 1)[\Delta \ln c_{t+1} - \Gamma_{t+1}]^2 \right\} + \varepsilon_{t+1} \quad (15)$$

where the consumption innovation ε_{t+1} satisfies $E_t \varepsilon_{t+1} = 0$. As $E_t \varepsilon_{t+1}^2 \to 0$, $\beta(\tilde{c}, c_t + \Gamma_{t+1})$ tends to a constant and therefore by Slutsky's theorem

$$\Delta \ln c_{t+1} = \varepsilon_{t+1} + \Gamma_{t+1} \ + \ \mathcal{O}(E_t |\varepsilon_{t+1}|^2) \quad (16)$$

The log of consumption therefore follows a martingale process with common drift.

A.1.2 Approximating the Lifetime Budget Constraint

The second step in the approximation is relating income risk to consumption variability. In order to make this link between the consumption innovation ε_{t+1} and the permanent and transitory shocks to the income process, we loglinearise the intertemporal budget constraint using a general Taylor series approximation (extending the idea in Campbell 1993).

Define a function $F : \mathbb{R}^N \to \mathbb{R}$ by $F(\xi) = \ln \sum_i \exp \xi_i$. By exact Taylor expansion around an arbitrary point $\xi^{\mathbf{0}} \in \mathbb{R}^N$

$$\begin{aligned} F(\xi) \ &= \ \ln \sum_{i=0}^{N} \exp \xi_i^0 \ + \ \sum_{i=0}^{N} \frac{\exp \xi_i^0}{\sum_{j=0}^{N} \exp \xi_j^0} (\xi_i - \xi_i^0) \\ &\quad + \frac{1}{2} \sum_{i=0}^{N} \sum_{j=0}^{N} \frac{\partial^2 F(\tilde{\xi})}{\partial \xi_i \partial \xi_j} (\xi_i - \xi_i^0)(\xi_j - \xi_j^0) \end{aligned} \quad (17)$$

where $\tilde{\xi}$ lies between ξ and $\xi^{\mathbf{0}}$. The use of $\tilde{\xi}$ is to make the expansion exact. The coefficients in the remainder term are given by

$$\frac{\partial^2 F(\tilde{\xi})}{\partial \xi_i \partial \xi_j} = \frac{\exp \tilde{\xi}_i}{\sum_j \exp \tilde{\xi}_j} (\delta_{ij} - \frac{\exp \tilde{\xi}_i}{\sum_j \exp \tilde{\xi}_j}),$$

where δ_{ij} denotes the Kronecker delta. These coefficients are bounded because $0 < \exp \tilde{\xi}_i / \sum_j \exp \tilde{\xi}_j < 1$.

Hence, taking expectations of (17) subject to information set $\mathcal{I}$

$$\begin{aligned} E_{\mathcal{I}}\left[F(\xi)\right] &= \ln \sum_{i=0}^{N} \exp \xi_i^0 + \sum_{i=0}^{N} \frac{\exp \xi_i^0}{\sum_{j=0}^{N} \exp \xi_j^0} (E_{\mathcal{I}} \xi_i - \xi_i^0) \\ &+ \frac{1}{2} \sum_{i=0}^{N} \sum_{j=0}^{N} E_{\mathcal{I}} \left(\frac{\partial^2 F(\tilde{\xi})}{\partial \xi_i \partial \xi_j} (\xi_i - \xi_i^0)(\xi_j - \xi_j^0) \right) \end{aligned} \tag{18}$$

We apply this expansion firstly to the expected present value of consumption, $\sum_{i=0}^{T-t} c_{t+i}(1+r)^{-i}$. Let $N = T - t$ and let

$$\begin{aligned} \xi_i &= \ln c_{t+i} - i \ln(1+r) \\ \xi_i^0 &= E_{t-1} \ln c_{t+i} - i \ln(1+r), \qquad i = 0, \ldots, N. \end{aligned} \tag{19}$$

Then, substituting equation (19) into equation (18) and noting only the order of magnitude for the remainder term,

$$\begin{aligned} E_{\mathcal{I}} \left[\ln \sum_{i=0}^{T-t} \frac{c_{t+i}}{(1+r)^i} \right] &= \ln \sum_{i=0}^{T-t} \exp[E_{t-1} \ln c_{t+i} - i \ln(1+r)] \\ &+ \sum_{i=0}^{T-t} \theta_{t+i} \left[E_{\mathcal{I}} \ln c_{t+i} - E_{t-1} \ln c_{t+i} \right] \\ &+ \mathcal{O}(E_{\mathcal{I}} \| \varepsilon_t^T \|^2) \end{aligned} \tag{20}$$

where

$$\theta_{t+i} = \frac{\exp \xi_i^0}{\sum_{j=0}^{N} \exp \xi_j^0} = \frac{\exp[E_{t-1} \ln c_{t+i} - i \ln(1+r)]}{\sum_{j=0}^{T-t} \exp[E_{t-1} \ln c_{t+j} - j \ln(1+r)]},$$

and ε_t^T denotes the vector of future consumption innovations $(\varepsilon_t, \varepsilon_{t+1}, \ldots, \varepsilon_T)'$.The term θ_{t+i} can be seen as an annuitisation factor for consumption.

We now apply the expansion (18) to the expected present value of resources, $\sum_{i=0}^{R-t}(1+r)^{-i} y_{t+i} + A_t - A_{T+1}(1+r)^{-(T-t)}$ Let $N = R + 1 - t$ and let

$$\begin{aligned} \xi_i &= \ln y_{t+i} - i \ln(1+r) \\ \xi_i^0 &= E_{t-1} \ln y_{t+i} - i \ln(1+r) \qquad i = 0, \ldots, N-1 \\ \xi_N &= \ln[A_t - A_{T+1}(1+r)^{-(T-t)}] \\ \xi_N^0 &= E_{t-1} \ln[A_t - A_{T+1}(1+r)^{-(T-t)}] \end{aligned} \tag{21}$$

Then, substituting equation (21) into equation (18), and again noting only the order of magnitude for the remainder term,

$$
\begin{aligned}
& E_{\mathcal{I}} \; \ln\left(\sum_{i=0}^{R-t} \frac{y_{t+i}}{(1+r)^i} + A_t - \frac{A_{T+1}}{(1+r)^{T-t}}\right) \\
&\quad = \; \ln\left[\sum_{i=0}^{R-t} \exp[E_{t-1}\ln y_{t+i} - i\ln(1+r)] + \exp E_{t-1}\ln[A_t - \frac{A_{T+1}}{(1+r)^{T-t}}]\right] \\
&\qquad + \pi_t \sum_{i=0}^{R-t} \alpha_{t+i}\left[E_{\mathcal{I}}\ln y_{t+i} - E_{t-1}\ln y_{t+i}\right] \\
&\qquad + (1-\pi_t)\left[E_{\mathcal{I}}\ln(A_t - \frac{A_{T+1}}{(1+r)^{T-t}}) - E_{t-1}\ln(A_t - \frac{A_{T+1}}{(1+r)^{T-t}})\right] \\
&\qquad + \mathcal{O}(E_{t-1}\|\nu_t^R\|^2) \qquad (22)
\end{aligned}
$$

where

$$
\alpha_{t+i} = \frac{\exp[E_{t-1}\ln y_{t+i} - i\ln(1+r)]}{\sum_{j=0}^{R-t}\exp[E_{t-1}\ln y_{t+j} - j\ln(1+r)]}
$$

can be seen as an annuitisation factor for income and

$$
\begin{aligned}
\pi_t &= 1 - \frac{\exp \xi_N^0}{\sum_{j=0}^{N}\exp \xi_j^0} \\
&= \frac{\sum_{i=0}^{R-t}\exp[E_{t-1}\ln y_{t+j} - j\ln(1+r)]}{\sum_{i=0}^{R-t}\exp[E_{t-1}\ln y_{t+j} - j\ln(1+r)] + \exp E_{t-1}\ln[A_t - A_{T+1}/(1+r)^{T-t}]}
\end{aligned}
$$

is (roughly) the share of expected future labor income in current human and financial wealth (net of terminal assets) and ν_t^R denotes the vector of future income shocks $(\nu_t', \nu_{t+1}', \ldots, \nu_R')'$.

We are able to equate equation (20) and (22) because the realised budget must balance, and so the expectation of the log budget constraint must also hold. We use (20) and (22), taking differences between expectations at the start of the period, before the shocks are realised, and at the end of the period, after the shocks are realised. This gives

$$
\varepsilon_t \; + \; \mathcal{O}(\varepsilon_t^2) \; + \; \mathcal{O}(E_{t-1}\varepsilon_t^2) = \pi_t(v_t + \alpha_t u_t) \; + \; \mathcal{O}(\|\nu_t\|^2) \; + \; \mathcal{O}(E_{t-1}\|\nu_t\|^2)
$$

where the left hand side is the innovation to the expected present value of consumption and the right hand side is the innovation to the expected present value of income. Squaring the two sides, equating expectations and eliminating terms which become negligible as $E_{t-1}\varepsilon_t^2 \to 0$ shows that terms which are $\mathcal{O}(E_{t-1}\varepsilon_t^2)$ are $\mathcal{O}(E_{t-1}\|\nu_t\|^2)$. Squaring again and equating then shows that terms which are $\mathcal{O}(\varepsilon_t^2)$ are $\mathcal{O}(\|\nu_t\|^2) \;+\; \mathcal{O}(E_{t-1}\|\nu_t\|^2)$. Thus

$$\varepsilon_t = \pi_t(v_t + \alpha_t u_t) \;+\; \mathcal{O}(\|\nu_t\|^2) \;+\; \mathcal{O}(E_{t-1}\|\nu_t\|^2)$$

and

$$\Delta \ln c_t = \Gamma_t \;+\; \pi_t(v_t + \alpha_t u_t) \;+\; \mathcal{O}(\|\nu_t\|^2) \;+\; \mathcal{O}(E_{t-1}\|\nu_t\|^2) \tag{23}$$

which is equation (4) in the text.

A.1.3 Cross Section Variances

We assume that the variances of the shocks v_t and u_t are the same in any period for all individuals in any cohort, that shocks are uncorrelated across individuals and that the cross-sectional covariances of the shocks with previous periods' incomes are zero.

Using equation (23) and the equation driving the income process (1) and noting that Γ_t is common within a cohort, the growth in the cross-section variance and covariances of income and consumption can now be seen to take the form[5]

$$\begin{aligned}
\Delta \mathrm{V}(\ln y_t) &= \mathrm{V}(v_t) + \Delta \mathrm{V}(u_t) \\
\Delta \mathrm{V}(\ln c_t) &= ({\bar{\pi}_t}^2 + \mathrm{V}(\pi_t))\mathrm{V}(v_t) + ({\bar{\pi}_t}^2 + \mathrm{V}(\pi_t))\alpha_t^2 \mathrm{V}(u_t) \\
&\qquad + \mathcal{O}(E_{t-1}\|\nu_t\|^3) \\
\Delta \mathrm{Cov}(\ln c_t, \ln y_t) &= \bar{\pi}_t \mathrm{V}(v_t) + \Delta[\bar{\pi}_t \alpha_t \mathrm{V}(u_t)] \\
&\qquad + \mathcal{O}(E_{t-1}\|\nu_t\|^3)
\end{aligned}$$

using the formula of Goodman (1960) for variance of a product of uncorrelated variables.

[5]Note that $\mathrm{Cov}(\ln y_{t-1}, u_{t-1}) = \mathrm{V}(u_{t-1})$ and $\mathrm{Cov}(\ln c_{t-1}, u_{t-1}) = \bar{\pi}_t \alpha_t \mathrm{V}(u_{t-1})$.

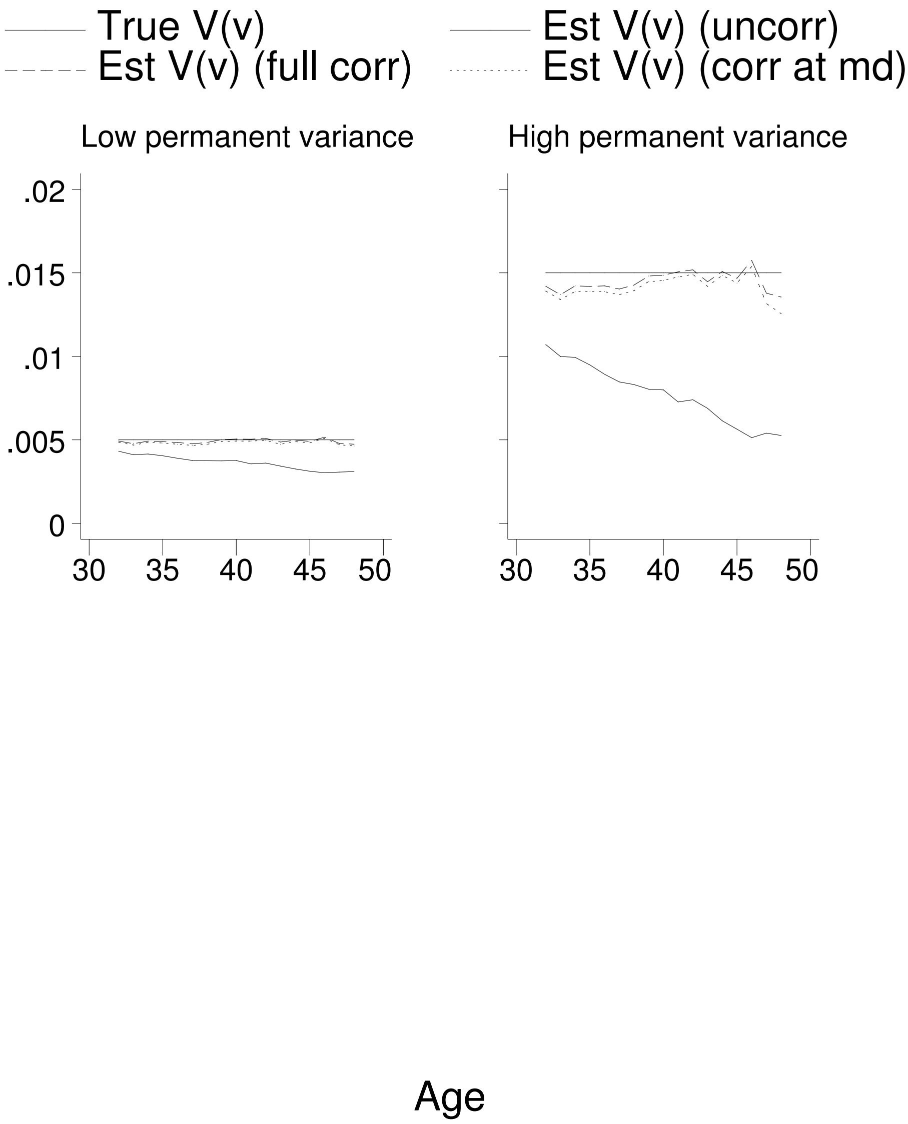

Figure 1: Permanent variance estimates

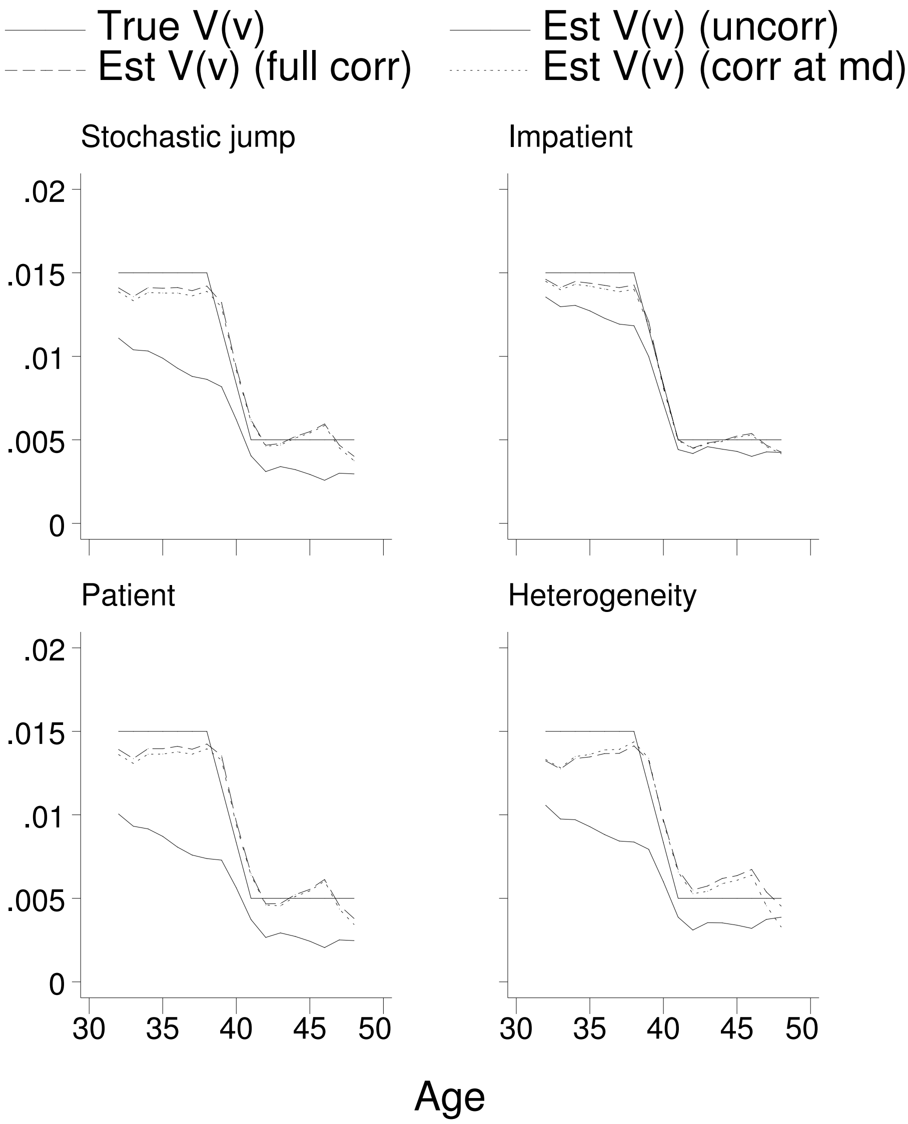

Figure 2: Permanent variance estimates

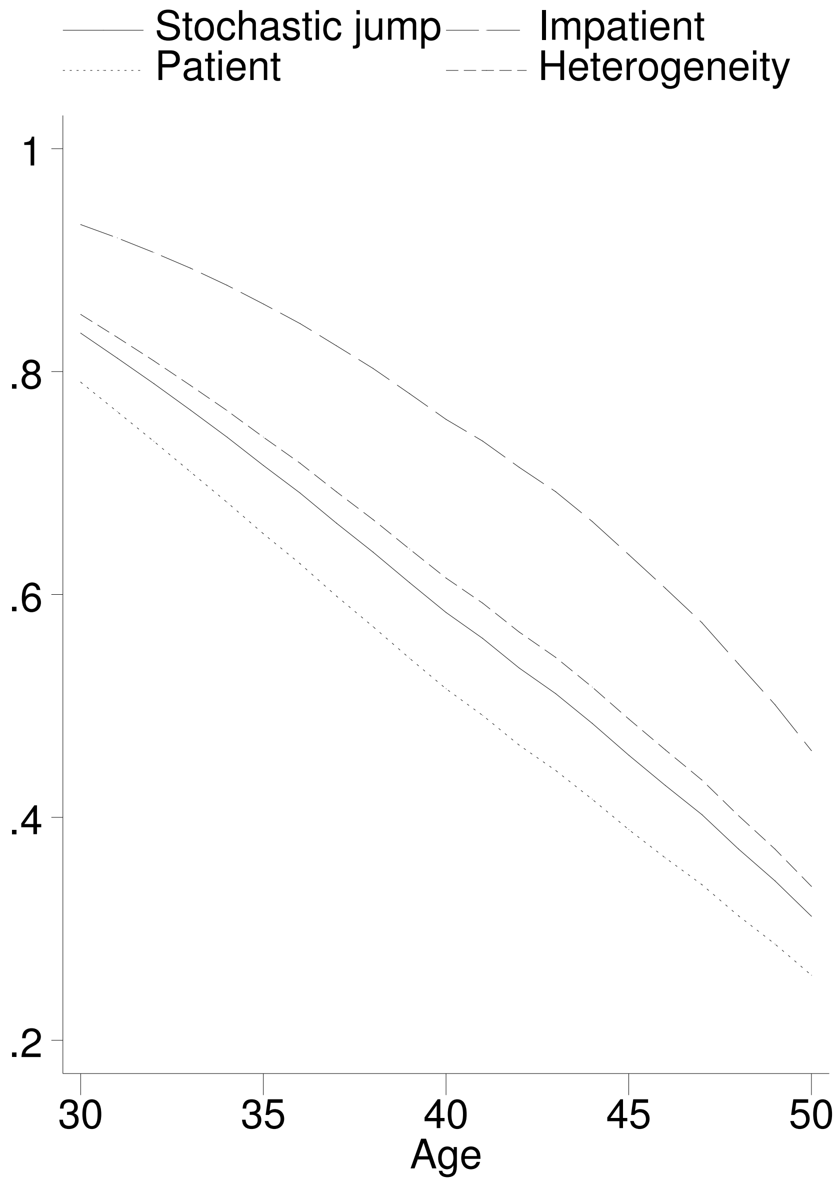

Figure 3: Variation of $\bar{\pi}_t{}^2$ with age

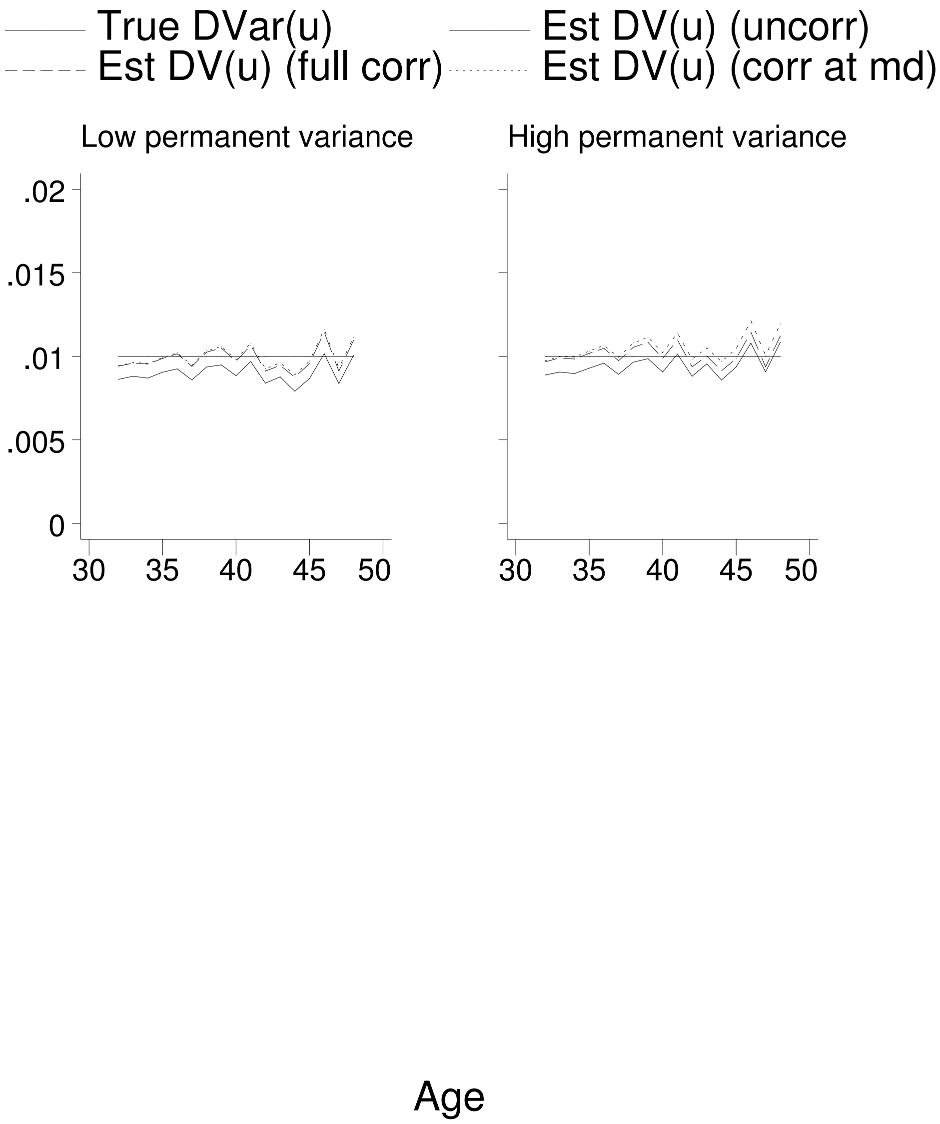

Figure 4: Transitory variance estimates