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Exogenous impact and conditional quantile functions

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ABSTRACT. An exogenous impact function is defined as the derivative of a structural function with respect to an endogenous variable, other variables, including unobservable variables held fixed. Unobservable variables are fixed at specific quantiles of their marginal distributions.

Exogenous impact functions reveal the impact of an exogenous shift in a variable perhaps determined endogenously in the data generating process. They provide information about the variation in exogenous impacts across quantiles of the distributions of the unobservable variables that appear in the structural model. This paper considers nonparametric identification of exogenous impact functions under quantile independence conditions.

It is shown that, when valid instrumental variables are present, exogenous impact functions can be identified as functionals of conditional quantile functions that involve only observable random variables. This suggests parametric, semiparametric and nonparametric strategies for estimating exogenous impact functions.

1. INTRODUCTION

This paper proposes a quantile based approach to identification and estimation of a functional of a structural model which is informative about the impact of an exogenous change in a variable that, in the data generating process, is potentially endogenous.

The functional considered here is the derivative of a response (Y_1) with respect to a potentially endogenous variable (Y_2) with covariates and instrumental variables held fixed and with unobservable variables set equal to specified quantiles of their marginal distributions. This functional, defined formally below, applied to a structural model, produces what is termed the *exogenous impact function* for Y_2.

The approach relies on the existence of instrumental variables that satisfy local *quantile independence* conditions rather than the global mean independence conditions frequently used. A quantile independence condition requires that a conditional quantile, for example a conditional median, or percentile, of a random variable does not depend on the values of the conditioning variables[1].

It is shown that, under particular local quantile independence assumptions and some assumptions concerning the monotonicity of the structural relationships, the exogenous impact function can be identified with a functional of conditional quantile functions associated with the distributions of observable random variables. This leads directly to an

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[1] See Chapter 6 of Manski (1988) for a discussion of quantile independence. Recent uses of quantile independence conditions as the basis for developing estimators can be found in Newey and Powell (1990), Chaudurhi, Doksum and Samarov (1997) and Kahn (2001).

analog estimator of an identifiable exogenous impact function, constructed as the same functional of estimated conditional quantile functions.

A quantile based approach is attractive because, subject to identifiability conditions:

1. there is a simple and direct link between the exogenous impact function and conditional quantile functions,
2. it allows the extraction of information about the distribution of the effects of exogenous shifts in variables that are endogenous in the data generating process,
3. it does not require the existence of moments of any order.

In contrast an analysis in which identification and inference is based on *mean* independence assumptions does not in general directly generate distributional information and requires existence of at least first order moments.

A quantile based approach is well suited to the analysis of nonlinear structural models in which there are monotonicity restrictions because quantiles obey monotonic nonlinear transformations in the sense that the τ-quantile of $h(Y)$ is just the result of applying the function $h(\cdot)$ to the τ-quantile ($(1-\tau)$-quantile) of Y when $h(\cdot)$ is monotonic increasing (decreasing). It is this which allows us to forge the simple link between exogenous impact functions and conditional quantile functions.

The monotonicity assumptions made to secure the results of this paper are no more restrictive than those typically made in an instrumental variable based analysis of structural models based on mean independence conditions[2].

This first Section of the paper proceeds to consider a simple example of the sort of structural model studied here. The exogenous impact function is defined for this example, its relationship to conditional quantile functions is given with a summary of the conditions under which there is the stated correspondence, and there follows a brief discussion of estimation of exogenous impact functions. The Section concludes with the plan of the remainder of the paper.

1.1. A simple example. Here is a simple example of the sort of system considered here.

$$\begin{aligned} Y_1 &= h_1(Y_2, X, \varepsilon, \nu) \\ Y_2 &= h_2(X, Z, \nu) \end{aligned}$$

Y_1 and Y_2 are scalar random variables, X is a list of covariates and Z is a list of instrumental variables.

The following quantile independence conditions, somewhat stronger than are necessary, are assumed to hold: for some τ_ε and τ_ν

1. the unobservable continuously distributed scalar ε has conditional τ_ε-quantile given ν, X and Z independent of ν, X and Z.
2. the unobservable continuously distributed scalar ν has conditional τ_ν-quantile given X and Z independent of X and Z.

The functions h_1 and h_2 are not parametrically specified. h_1 is assumed to be differentiable with respect to scalar[3] Y_2 and ν. h_2 is assumed to be differentiable with respect to Z and ν. h_1 is monotonic increasing[4] in ε and h_2 is monotonic increasing in ν. There

[2] See the discussion in Section 2.5.3 of Manski (1988).

[3] The case in which Y_2 and ν are vectors is considered in Section 3.

[4] Assuming that h_1 is monotonic increasing in ε rather than decreasing is an innocuous normalisation.

are no assumptions concerning the existence of moments of any order. This alone makes the problem an interesting one to study.

Example 1: Returns to schooling. *To fix ideas it is helpful to have a concrete example. A system of the sort just described can be used to represent a prototypical model employed to study returns to schooling, with Y_1 measuring a labour market outcome, for example wages, Y_2 measuring schooling, ε capturing labour market induced heterogeneity and ν measuring ability.*

In the context of this example, and others, it can be useful to have information about the derivative of h_1 with respect to Y_2, the variables X, ε, ν, and therefore Y_2, being held fixed at interesting values. This is the exogenous impact function studied here.

Formally, the exogenous impact function is $\pi(\tau_\varepsilon, \tau_\nu, x, z)$, defined as follows.

$$\pi(\tau_\varepsilon, \tau_\nu, x, z) = \nabla_{y_2} h_1(Q_{Y_2|XZ}(\tau_\nu, x, z), x, Q_\varepsilon(\tau_\varepsilon), Q_\nu(\tau_\nu))$$

Here $\nabla_{y_2} h_1(\cdot,\cdot,\cdot,\cdot)$ is the partial derivative of h_1 with respect to its first argument, $Q_\varepsilon(\tau_\varepsilon)$ and $Q_\nu(\tau_\nu)$ are the τ_ε- and τ_ν-quantiles of the distributions of respectively ε and ν and $Q_{Y_2|XZ}(\tau_\nu, x, z)$ is the τ_ν-quantile of the conditional distribution of Y_2 given $X = x$ and $Z = z$, which, note, is equal to $h_2(x, z, Q_\nu(\tau_\nu))$ under the monotonicity assumption assumed to hold for the function h_2.

The exogenous impact function, $\pi(\tau_\varepsilon, \tau_\nu, x, z)$, is the rate at which Y_1 changes as the value of Y_2 is increased when $X = x$, $Z = z$, and when ε and ν have values equal to the specified quantiles of their respective distributions.

Example 1 continued. *In the context of the returns to schooling example the exogenous impact function is the rate at which wages increase as schooling is "exogenously" increased for a person with characteristics x and z and with a value of ν equal to the τ_ν-quantile of the distribution of ability and a value of ε equal to the τ_ε-quantile of the distribution of labour market heterogeneity. Of course this exogenous impact function only has this interpretation if the exogenous change in schooling leaves all other elements of the system undisturbed.*

The exogenous impact function is a rich source of information, revealing the variation in the impact of an exogenous shift in Y_2 as ε and ν vary across quantiles of their marginal distributions. This may be of particular interest when the exogenous shift in Y_2 is the consequence of a policy intervention and the distributional consequences of the intervention are of interest.

1.2. The exogenous impact function. Under certain conditions, set out in detail in Section 2 and summarised shortly, the exogenous impact function can be identified with a functional of conditional quantile functions associated with the distributions of observable random variables.

In this example involving just one endogenous variable, Y_2, when the identifiability conditions hold, the exogenous impact function can be expressed as follows.

$$\begin{aligned}\pi(\tau_\varepsilon, \tau_\nu, x, z) &= \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z) \\ &\quad + \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)}\end{aligned} \tag{1}$$

Here $Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2, x, z)$ is the conditional τ_ε-quantile of the distribution of Y_1 given $Y_2 = y_2$, $X = x$ and $Z = z$. The conditional quantile $Q_{Y_2|XZ}(\tau_\nu, x, z)$ was defined earlier. ∇_{y_2} indicates differentiation with respect to y_2 and ∇_{z_i} indicates differentiation with respect to the ith element of z. If the conditions required for (1) to be valid do hold then estimates of the conditional quantile functions, $Q_{Y_1|Y_2XZ}$ and $Q_{Y_2|XZ}$, lead directly to estimates of the exogenous impact function.

1.3. Identification. The required conditions constitute identifiability conditions in the sense that, when they apply, the exogenous impact function can be identified with a well defined functional of conditional quantile functions about which data are in principle informative.

The conditions include analogues of the order and rank conditions familiar in a mean independence based analysis. Each variable, Z_i, is an *instrumental variable*. For (1) to apply it is required that there exist at least one instrumental variable Z_i which possesses the quantile independence properties, namely that the τ_ε- and τ_ν-quantiles of ε and ν are independent of Z_i, and such that:

1. Z_i is excluded[5] *a priori* from the function h_1, which relates Y_1 to the endogenous variable Y_2,
2. Z_i does have a role in determining Y_2 via the auxiliary function h_2 in the sense that $\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z) \neq 0$ at the chosen values of x and z.

If there is no such instrumental variable then the exogenous impact function cannot be identified at the chosen values of x, z, τ_ε and τ_ν using (1). If there is more than one such instrumental variable[6] then (1) applies for each choice of Z_i and the exogenous impact function is in principle overidentified at the chosen values of x and z.

In this nonparametric, analysis, the conditions under which (1) applies are "local" in two senses. First, quantile independence conditions are required to hold only at the values τ_ε and τ_ν that appear in the exogenous impact function. Second, the condition $\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z) \neq 0$ is only required to hold at the chosen values of x, z and τ_ν.

If the separate impact of x and z on the exogenous impact function is to be identified then their separate impact on the conditional quantile functions must be identifiable at the chosen values of τ_ε and τ_ν. This rules out exact functional dependencies among X and Z.

1.4. Estimation. With suitably informative realisations of Y_1, Y_2, X and Z, the conditional quantile functions that appear on the right-hand side of (1), and their derivatives, can be estimated, and then evaluated, as required, at $y_2 = \hat{Q}_{Y_2|XZ}(\tau_\nu, x, z)$, producing an estimate of the exogenous impact function.

Estimation can be done using a parametric specification for the conditional quantile functions, or a semiparametric specification, for example deploying single index restrictions, or, if the curse of dimensionality can be kept at bay, using a fully nonparametric specification[7]. There may be efficiency gains from joint estimation of the two conditional quantile functions.

When the exogenous impact function is overidentfied one may wish to efficiently combine the estimates got by using different instrumental variables and there is scope for testing hypotheses concerning the validity of instruments.

This paper focuses on the identification of exogenous impact functions *via* their correspondence with conditional quantile functions. The interesting, but separate, estimation and inference issues that this analysis throws up are the subject of continuing research.

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[5] This is a slightly stronger condition than is required. In fact all that is needed is that Z_i is excluded from the function h_1 at the τ_ε quantile under consideration which could occur if for example

$$h_1(Y_2, X, Z, \nu) = \theta Y_2 + X'\beta + \left(Z'\delta\right)\varepsilon + \lambda\nu$$

and $\tau_\varepsilon = 0.5$ and the median of ε is zero.

[6] In this example there is a single endogenous variable appearing in the function h_1. With G endogenous variables appearing in h_1, at least G instrumental variables will be required in order to identify the G structural impact functions.

[7] For parametric estimation, see e.g., Koenker and Bassett (1978), Koenker and d'Orey (1984, 1994), for semiparametric estimation see e.g., Chaudhuri, Doksum and Samarov (1997) and Kahn (2001), for nonparametric estimation, see e.g., Chaudhuri (1991), Nahm (2001).

1.5. Related work. Rather little attention has been paid to endogeneity in the context of quantile regressions. Amemiya (1982) and Powell (1983) consider estimation in parametric models in which a conditional median is linear in endogenous and exogenous variables and independent of instrumental variables. A recent application can be found in Levin (2001).

The exogenous impact function studied here gives information about the variation across a population in responses to an exogenous change in a variable potentially endogenous in the data generating process. There is recent work aimed at understanding this sort of heterogeneity, which has come to be realised to be important in the evaluation of policy. Heckman, Smith and Clements (1997) explore non-quantile based approaches in a programme evaluation setting. Abadie, Angrist and Imbens (2001) propose a Quantile Treatment Effect estimator[8] in a study of the impact of subsidised training on the distribution of earnings.

The estimation procedure suggested by the results presented here is rather different to that proposed by Amemiya (1982) on which much subsequent work has been based. Rather than using the instrumental variables at the first step of estimation, one estimates quantile regressions which capture the dependency of quantiles of (a) Y_1 on Y_2, X and Z and of (b) Y_2 on X and Z, and then uses the instruments in a second step to retrieve an estimate of the exogenous impact function, in a sense applying a "bias correction".

The connection between exogenous impact functions and conditional quantile functions forged in this paper does not seem to have been noted, although the discussion[9] in Manski (1988) points to the path which leads to the result presented here.

1.6. Plan of the remainder of the paper. Section 2 provides a full set of conditions under which the exogenous impact function can be identified with a functional of conditional quantile functions, and outlines the method of proof. The proof is given in the Appendix. Section 3 sketches the extension to the case when there is more than one endogenous (Y_2) variable. Section 4 concludes.

2. EXOGENOUS IMPACT FUNCTIONS WITH ONE ENDOGENOUS VARIABLE

This Section considers the case in which there is a single endogenous variable, Y_2, and potentially many instrumental variables, Z_i, $i = 1, \ldots, M$.

A theorem giving conditions under which the exogenous impact function can be identified with a functional of conditional quantile functions is stated. Some brief remarks and an outline of the method of proof follow. The proof is given in the Appendix.

Theorem

Scalar Y_1 and Y_2, M-vector $Z = \{Z_i\}_{i=1}^{M}$, $M \geq 1$, and K-vector $X = \{X_i\}_{i=1}^{K}$, $K \geq 0$ satisfy the following equations.

$$\begin{aligned} Y_1 &= h_1(Y_2, X_1, \ldots, X_K, \varepsilon, \nu) \quad (2) \\ Y_2 &= h_2(X_1, \ldots, X_K, Z_1, \ldots, Z_M, \nu) \quad (3) \end{aligned}$$

These functions are written below in the abbreviated notation $h_1(Y_2, X, \varepsilon, \nu)$, $h_2(X, Z, \nu)$.

Let $Q_A(\tau)$ denote the τ-quantile of a random variable A. Let $Q_{A|B}(\tau, b)$ denote the conditional τ-quantile of A given $B = b$, and so forth.

Consider values x and z of respectively X and Z, $\tau_\varepsilon, \tau_\nu \in (0, 1)$ and a particular element, Z_i of Z.

[8] Their estimator is a weighted quantile regression estimator based on a parametric model, using a binary treatment indicator and a binary instrumental variable.

[9] In Section 6.2.6 of Manski (1988) - "...median independence combines nicely with real-valued response functions that are monotone in a scalar [unobservable] u..." which is the key to the analysis of this paper.

Assume the following.

1. ε and ν are continuously distributed random variables with independent support.
2. At $X = x$, $Z = z$, the conditional τ_ε-quantile $Q_{\varepsilon|\nu XZ}(\tau_\varepsilon, v, x, z)$ is equal to $Q_\varepsilon(\tau_\varepsilon)$ for all v.
3. At $X = x$, $Z = z$, the conditional τ_ν-quantile $Q_{\nu|XZ}(\tau_\nu, x, z)$ is equal to $Q_\nu(\tau_\nu)$.
4. At $X = x$, $Z = z$, $Y_2 = Q_{Y_2|XZ}(\tau_\nu, x, z)$, $\nu = Q_\nu(\tau_\nu)$, h_1 is a continuous function of Y_2, ε and ν, and monotonic increasing in ε and the partial derivatives of h_1 with respect to Y_2, $\nabla_{y_2} h_1$, and with respect to ν, $\nabla_\nu h_1$, exist and are finite.
5. At $X = x$, $Z = z$, $\nu = Q_\nu(\tau_\nu)$, h_2 is a continuous function of Z_i and ν, and monotonic increasing in ν, and the partial derivative of h_2 with respect to Z_i, $\nabla_{z_i} h_2$, exists and is non-zero.

Define the function

$$\rho(y_2, x, \varepsilon, \nu) = \nabla_{y_2} h_1(y_2, x, \varepsilon, \nu)$$

and the exogenous impact function

$$\pi(\tau_\varepsilon, \tau_\nu, x, z) = \rho(Q_{Y_2|XZ}(\tau_\nu, x, z), x, Q_\varepsilon(\tau_\varepsilon), Q_\nu(\tau_\nu))$$

Then

$$\begin{aligned} \pi(\tau_\varepsilon, \tau_\nu, x, z) &= \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z) \\ &\quad + \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)}. \end{aligned} \tag{4}$$

2.1. Remarks. The instrumental variables, $Z_1, \ldots, Z_M$, are excluded from the function, h_1, determining Y_1. Note that there is no requirement that moments of any order exist, nor any assumption concerning the location (e.g. the median) of the distributions of ε and ν.

The identification result is local in the sense that it tells us when the exogenous impact function can be identified as a functional of conditional quantile functions at particular quantiles of the distributions of ε and ν and particular values of X and Z.

Assumption 1 and the continuity conditions in Assumptions 4 and 5 lead to unambiguous definitions of quantiles. Assumptions 2 and 3 define values of τ_ε and τ_ν at which there is *quantile independence* at the chosen values of X and Z. Of course full independence of ε, ν, X and Z will ensure full quantile independence but this is a much stronger condition than is required.

The monotonicity requirements in Assumptions 4 and 5 are standard in instrumental variables based attacks on identification[10]. In fact the conditions are stronger than is required because all that is needed is that at the chosen values of τ_ε, τ_ν, x and z, conditional quantiles obey the following conditions

$$\begin{aligned} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2^*, x, z) &= h_1(y_2^*, x, Q_\varepsilon(\tau_\varepsilon), Q_\nu(\tau_\nu)) \\ Q_{Y_2|XZ}(\tau_\nu, x, z) &= h_2(x, z, Q_\nu(\tau_\nu)) \end{aligned}$$

[10] In a mean independence based attack monotonicity implies that the model is separable in ε and ν. See Section 2.5.3 of Manski (1988).

where $y_2^* = Q_{Y_2|XZ}(\tau_\nu, x, z)$. These conditions may hold even when h_1 is not monotonic in ε for all ε and h_2 is not monotonic in ν for all ν. The requirement that h_1 and h_2 be increasing in respectively ε and ν (rather than decreasing) is an innocuous normalisation.

If there is no Z_i satisfying the rank type condition $\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z) \neq 0$ at the chosen values of x and z then, at these values and at the chosen quantiles, the exogenous impact function cannot be identified as the functional (4) of conditional quantile functions. If there is more than one Z_i satisfying the rank type condition at the chosen values of x and z, then at these values and at the chosen quantiles, the exogenous impact function is overidentified.

If the separate impact of each element of X and Z on the exogenous impact function is to be identifiable then the conditional quantile functions, $Q_{Y_1|Y_2XZ}$ and $Q_{Y_2|XZ}$ must reveal the separate impact of x and z on conditional quantiles (at the chosen values of τ_ε and τ_ν) which rules out exact functional dependencies between X and Z at least in the region of values of X and Z and at the τ_ε- and τ_ν-quantiles that are of interest.

2.2. Outline of the method of proof of the Theorem. The monotonicity of h_1 with respect to ε implies that the conditional τ-quantile of Y_1 given $Y_2 = y_2$, $X = x$, $Z = z$, $Q_{Y_1|Y_2XZ}$, is related to y_2, x and z via the function h_1 with ε replaced by the conditional τ-quantile of the distribution of ε given ν, which, at $\tau = \tau_\varepsilon$, is equal to the marginal τ-quantile of ε under quantile independence. This need only be the case in a neighbourhood of the quantiles of interest.

The monotonicity assumption on h_2 ensures the existence of an inverse function, $\nu = g_2(X, Z, Y_2)$ which is substituted for ν in h_1. In fact this inverse function need only exist in a neighbourhood of $\nu = Q_\nu(\tau_\nu)$.

The derivatives of $Q_{Y_1|Y_2XZ}$ with respect to y_2 is then considered. This is evaluated at $y_2 = Q_{Y_2|XZ}(\tau_\nu, x, z)$. The term in $\nabla_{y_2} Q_{Y_1|Y_2XZ}$ arising from the direct (first) entry of y_2 in h_1 is, after evaluation at appropriate values of its arguments, the object to be identified. The problem now is to identify the second term in $\nabla_{y_2} Q_{Y_1|Y_2XZ}$ that arises from the appearance of y_2 in $g_2(x, z, y_2)$ which has been substituted for ν.

To accomplish this we consider the derivative of the conditional τ_ε-quantile of Y_1 given $Y_2 = Q_{Y_2|XZ}(\tau_\nu, x, z)$, $X = x$, $Z = z$ with respect to an instrumental variable z_i, $\nabla_{z_i} Q_{Y_1|Y_2XZ}$, and the derivative of the conditional τ_ν-quantile of Y_2 given $X = x$, $Z = z$ with respect to z_i, $\nabla_{z_i} Q_{Y_2|XZ}$.

We show that the term to be identified is just the ratio of $\nabla_{z_i} Q_{Y_1|Y_2XZ}$ to $\nabla_{z_i} Q_{Y_2|XZ}$. For this to be true, at the τ-quantiles considered, the instrumental variable Z_i must appear in the h_1 equation only through its influence on ν as captured by the inverse function $g_2(X, Z, Y_2)$ and it must be that $\nabla_{z_i} Q_{Y_2|XZ} \neq 0$.

Finally, upon rearranging terms, the exogenous impact function is identified as the functional of conditional quantile functions (1). A full proof is given in the Appendix.

2.3. The exogenous impact function for some specific models. This Section concludes with two examples in parametric models giving the exogenous impact function and the components of the functional of conditional quantile functions to which, under suitable conditions, it corresponds.

The first example is a linear simultaneous equations model, a simple example, but note that this analysis does not require the existence of moments which is required in conventional mean independence based analysis of this model. The second example is similar but a Box-Cox transformation is applied to Y_1.

Linear model. First consider the linear model

$$\begin{aligned} Y_1 &= \theta Y_2 + X'\beta_1 + \varepsilon + \lambda\nu \\ Y_2 &= X'\beta_2 + Z'\delta + \nu \end{aligned} \tag{5}$$

and suppose that the quantile independence conditions are satisfied for the τ_ε- and τ_ν-quantiles of respectively ε and ν.

The exogenous impact function is simply

$$\pi(\tau_\varepsilon, \tau_\nu, x, z) = \theta.$$

The monotonicity assumptions apply, and so,

$$\begin{aligned} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2, x, z) &= \theta y_2 + x'\beta_1 + Q_\varepsilon(\tau_\varepsilon) + \lambda(y_2 - x'\beta_2 - z'\delta) \\ &= (\theta+\lambda)y_2 + x'(\beta_1 - \lambda\beta_2) - z'\lambda\delta + Q_\varepsilon(\tau_\varepsilon) \\ Q_{Y_2|XZ}(\tau_\nu, x, z) &= x'\beta_2 + z'\delta + Q_\nu(\tau_\nu). \end{aligned}$$

We have

$$\begin{aligned} \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z) &= \theta + \lambda \\ \nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z) &= -\lambda\delta_i \\ \nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z) &= \delta_i \end{aligned}$$

and so, as long as the rank condition, $\delta_i \neq 0$ is satisfied,

$$\nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z) + \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)} = \theta$$

which is the desired exogenous impact function.

A Box-Cox model with endogeneity. Now consider the same model but with Y_1 subject to a Box-Cox transformation:

$$\begin{aligned} \frac{Y_1^\alpha - 1}{\alpha} &= \theta Y_2 + X'\beta_1 + \varepsilon + \lambda\nu \\ Y_2 &= X'\beta_2 + Z'\delta + \nu. \end{aligned}$$

Write the first equation as

$$Y_1 = (1 + \alpha(\theta Y_2 + X'\beta_1 + \varepsilon + \lambda\nu))^{1/\alpha}$$

The exogenous impact function is

$$\pi(\tau_\varepsilon, \tau_\nu, x, z) = \theta(1 + \alpha(\theta(x'\beta_2 + z'\delta + Q_\nu(\tau_\nu)) + x'\beta_1 + Q_\varepsilon(\tau_\varepsilon) + \lambda Q_\nu(\tau_\nu)))^{1/\alpha - 1}.$$

The monotonicity assumptions still apply, so

$$\begin{aligned} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2, x, z) &= (1 + \alpha(\theta y_2 + x'\beta_1 + Q_\varepsilon(\tau_\varepsilon) + \lambda(y_2 - x'\beta_2 - z'\delta)))^{1/\alpha} \\ &= A(\tau_\varepsilon, y_2, x, z)^{1/\alpha}, \text{ say,} \end{aligned}$$

$$\begin{aligned} \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2, x, z) &= \left(\theta + \frac{\lambda}{\alpha}\right) A(\tau_\varepsilon, y_2, x, z)^{1/\alpha - 1} \\ \nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2, x, z) &= -\frac{\lambda\delta_i}{\alpha} A(\tau_\varepsilon, y_2, x, z)^{1/\alpha - 1} \\ \nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z) &= \delta_i \end{aligned}$$

and on evaluating at $y_2 = Q_{Y_2|XZ}(\tau_\nu, x, z)$, the exogenous impact function is recovered.

3. Exogenous impact functions with many endogenous variables

3.1. Identification. Now consider the case in which there are $G \geq 1$ endogenous variables in the equation determining Y_1. The structural equations are then as follows.

$$\begin{aligned} Y_1 &= h_1(Y_{21}, \ldots, Y_{2G}, X, \varepsilon, \nu_1, \ldots, \nu_G) \\ Y_{2j} &= h_{2j}(X, Z, \nu_j) \quad j \in \{1, \ldots, G\} \end{aligned}$$

The exogenous impact functions are now, for $j \in \{1, \ldots, G\}$,

$$\pi_j(\tau_\varepsilon, \tau_{\nu_1}, \ldots, \tau_{\nu_G}, x, z) = \rho_j(y_{21}^*, \ldots, y_{2G}^*, x, Q_\varepsilon(\tau_\varepsilon), \nu_1^*, \ldots, \nu_G^*)$$

where

$$\rho_j(y_{21}, \ldots, y_{2G}, x, \varepsilon, \nu_1, \ldots, \nu_G) = \nabla_{y_{2j}} h_1(y_{21}, \ldots, y_{2G}, x, \varepsilon, \nu_1, \ldots, \nu_G).$$

and

$$\left.\begin{aligned} y_{2i}^* &= Q_{Y_{2i}|XZ}(\tau_{\nu_i}, x, z) \\ \nu_i^* &= Q_{\nu_i}(\tau_{\nu_i}) \end{aligned}\right\} \quad i = 1, \ldots, G. \tag{6}$$

The jth exogenous impact function gives the rate of change of Y_1 under exogenous changes in Y_{2j}, x and z fixed, with ε set equal to the τ_ε-quantile of its distribution and each ν_i set equal to the τ_{ν_i}-quantile of its distribution. Note that these settings for ε, x, z and the ν_i's fix the Y_{2i}'s at the values shown in equation (6).

Each "reduced form" equation for the Y_{2j}'s is specified as a monotonic function of a single unobservable ν_j, but this is not very restrictive as there is no requirement that the ν_j's be independently distributed.

Under conditions similar to those of the Theorem in Section 2, the jth exogenous impact function can be expressed in terms of conditional quantile functions as

$$\pi_j(\tau_\varepsilon, \tau_{\nu_1}, \ldots, \tau_{\nu_G}, x, z) = \nabla_{y_{2j}} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_{21}^*, \ldots, y_{2G}^*, x, z) - b_j$$

where the y_{2i}^*'s and ν_i^*'s are as in (6), b_j is the jth element of $b = [b_1 \ldots b_G]'$ and b is any solution to

$$c = -Ab \tag{7}$$

where M element vector c is

$$c = \begin{bmatrix} \nabla_{z_1} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_{21}^*, \ldots, y_{2G}^*, x, z) \\ \vdots \\ \nabla_{z_M} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_{21}^*, \ldots, y_{2G}^*, x, z) \end{bmatrix}$$

and $M \times G$ matrix A is

$$A = \begin{bmatrix} \nabla_{z_1} Q_{Y_{21}|XZ}(\tau_{\nu_1}, x, z) & \ldots & \nabla_{z_1} Q_{Y_{2G}|XZ}(\tau_{\nu_G}, x, z) \\ \vdots & \ddots & \vdots \\ \nabla_{z_M} Q_{Y_{21}|XZ}(\tau_{\nu_1}, x, z) & \ldots & \nabla_{z_M} Q_{Y_{2G}|XZ}(\tau_{\nu_G}, x, z) \end{bmatrix}.$$

When $M > G$ there is potentially a multiplicity of representations of each of the G exogenous impact functions which is the manifestation of over identification in this setting.

If the matrix A has rank less than G at the selected values of x and z and the selected quantiles of the distributions of ε and the ν's then there is no representation of the exogenous impact functions in terms of conditional quantile functions and the exogenous impact function is not identifiable by this means at the selected values of its arguments.

The conditions under which these conditional quantile based representations of the exogenous impact functions are valid are similar to those in the single endogenous variable case treated in the Theorem of Section 2. In particular it is required that, in addition to the matrix A having full column rank at the selected values of the τ_{ν_i}'s, x and z,

1. h_1 is monotonic increasing in ε and has finite partial derivatives with respect to each Y_{2j} and each ν_j,
2. each function h_{2j} is monotonic increasing in ν_j and has partial derivatives with respect to each z_i and ν_j,
3. the τ_ε-quantile of ε given x, z and $\nu_1, \dots, \nu_G$ is independent of x, z and $\nu_1, \dots, \nu_G$,
4. for each $i \in \{1, , G\}$, the τ_{ν_i}-quantile of ν_i given x and z is independent of x and z.

As in the Theorem of Section 2, many of these conditions need only be satisfied locally, at, or in a neighbourhood of, x, z, τ_ε and τ_ν.

3.2. Outline of the method of proof. The conditional quantile function representation given in Section 3.1 is obtained as follows.

Monotonicity of h_1 with respect to ε, and τ_ε-quantile independence leads to the following.

$$Q_{Y_1|Y_2XZ}(\tau_\varepsilon) = h_1(y_{21}, \dots, y_{2G}, x, Q_\varepsilon(\tau_\varepsilon), g_{21}(x, z, y_{21}), \dots, g_{2G}(x, z, y_{2G}))$$

The derivatives of $Q_{Y_1|Y_2XZ}$ are, now for brevity suppressing all arguments to h_1 and the g_{2j}'s, as follows: with respect to y_{2j},

$$\begin{aligned} \nabla_{y_{2j}} Q_{Y_1|Y_2XZ} &= \nabla_{y_{2j}} h_1 + \nabla_{\nu_j} h_1 \nabla_{y_{2j}} g_{2j} \\ &= \nabla_{y_{2j}} h_1 + b_j \end{aligned}$$

where in the second line $b_j = \nabla_{\nu_j} h_1 \nabla_{y_{2j}} g_{2j}$, and: with respect to z_i:

$$\nabla_{z_i} Q_{Y_1|Y_2XZ} = \sum_{j=1}^{G} \nabla_{\nu_j} h_1 \nabla_{z_i} g_{2j}. \tag{8}$$

Note that, without the exclusion of the Z_i's from the function h_1 at the chosen value of τ_ε, there would be additional terms in this derivative and the claimed result would not follow.

The monotonicity conditions placed on the h_{2j} functions ensure that there exist inverse functions as follows.

$$\nu_j = g_{2j}(X, Z, Y_{2j})$$

The argument in the proof in the Appendix leads to (10),

$$\nabla_{z_i} g_{2j}(X, Z, Y_{2j}) = -\nabla_{y_{2j}} g_{2j}(X, Z, Y_{2j}) \nabla_{z_i} h_{2j}(X, Z, \nu_j) \tag{9}$$

and, substituting in (8) gives the following.

$$\nabla_{z_i} Q_{Y_1|Y_2XZ} = -\sum_{j=1}^{G} \nabla_{\nu_j} h_1 \nabla_{y_{2j}} g_{2j} \nabla_{z_i} h_{2j}$$

Evaluating $\nabla_{z_i} h_{2j}$ at $\nu_j = Q_{\nu_j}(\tau_{\nu_j})$ we have $\nabla_{z_i} h_{2j} = \nabla_{z_i} Q_{Y_{2j}|XZ}$, and so, for $i \in \{1, \dots, M\}$

$$\begin{aligned} \nabla_{z_i} Q_{Y_1|Y_2XZ} &= -\sum_{j=1}^{G} \nabla_{\nu_j} h_1 \nabla_{y_{2j}} g_{2j} \nabla_{z_i} Q_{Y_{2j}|XZ} \\ &= -\sum_{j=1}^{G} b_j \nabla_{z_i} Q_{Y_{2j}|XZ}. \end{aligned}$$

The final equation can be written in matrix form as in equation (7) above and evaluation at x, z and

$$\nu_j = \nu_j^* = Q_{\nu_j}(\tau_{\nu_j}), \quad j = 1, \ldots, G$$

ensures that

$$y_{2j} = y_{2j}^* = Q_{Y_{2j}|XZ}(\tau_{\nu_j}, x, z), \quad j = 1, \ldots, G.$$

4. Concluding remarks

This paper has defined an exogenous impact function. This function gives the rate of change of a response to exogenous changes in variables which are potentially endogenous in a data generating process. The function can be evaluated at chosen quantiles of distributions of the unobservable random variables that drive a model, and at chosen values of covariates and instrumental variables.

This paper has shown that, under certain monotonicity conditions and local quantile independence conditions placed on instrumental variables, the exogenous impact function can be identified as a functional of conditional quantile functions pertaining to only observable random variables.

The result is interesting for a number of reasons.

1. Identification is nonparametric and may be achievable at some quantiles and covariate and instrumental variable values, but not at others.
2. With parametric models for conditional quantile functions the result can be used to explore the possibility of parametric identification.
3. The result offers the possibility of extracting information about the distribution of exogenous impacts across different quantiles of the marginal distributions of the unobservable variables that drive the structural model.
4. The result suggests parametric, semiparametric and nonparametric analog estimators of the exogenous impact function using various types of quantile regression function estimators.
5. The exogenous impact function can be defined and identified in contexts (e.g. financial markets) in which it is attractive to construct models with nonexistent low order moments.

Appendix: Proof of the Theorem of Section 2

Throughout consider particular values, x and z, of X and Z, and values τ_ε and τ_ν, at which the assumptions of the Theorem are satisfied.

Monotonicity of h_1 with respect to ε and continuity (Assumption 4) imply that

$$Q_{Y_1|\nu XZ}(\tau_\varepsilon, \nu, x, z) = h_1(h_2(x, z, \nu), x, Q_{\varepsilon|\nu XZ}(\tau_\varepsilon, \nu, x, z), \nu) \qquad \text{(A1)}$$

and since the conditional τ_ε-quantile of ε is independent of ν, X and Z (Assumption 2),

$$Q_{Y_1|\nu XZ}(\tau_\varepsilon, \nu, x, z) = h_1(h_2(x, z, \nu), x, Q_\varepsilon(\tau_\varepsilon), \nu).$$

Assumption 5 implies the existence of the inverse function

$$\nu = g_2(X_1, \ldots, X_K, Z_1, \ldots, Z_M, Y_2) \qquad \text{(A2)}$$

which is written, using an abbreviated notation, as $g_2(X, Z, Y_2)$.

Substituting for ν in (A1) using (A2) and noting that conditioning on ν, X and Z is the same as conditioning on Y_2, X and Z, gives the following.

$$Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2, x, z) = h_1(y_2, x, Q_\varepsilon(\tau_\varepsilon), g_2(x, z, y_2)) \tag{A3}$$

Let $y_2^* = Q_{Y_2|XZ}(\tau_\nu, x, z)$. The differentiability assured by Assumptions 4 and 5 implies that there are the following derivatives of the conditional quantile function (A3),

$$\begin{aligned}\nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau, y_2^*, x, z) &= \nabla_{y_2} h_1(y_2^*, x, Q_\varepsilon(\tau), g_2(x, z, y_2^*)) \\ &\quad + \nabla_{y_2} g_2(x, z, y_2^*) \nabla_\nu h_1(y_2^*, x, Q_\varepsilon(\tau), g_2(x, z, y_2^*)) \quad \text{(A4)} \\ \nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau, y_2^*, x, z) &= \nabla_{z_i} g_2(x, z, y_2^*) \nabla_\nu h_1(y_2^*, x, Q_\varepsilon(\tau), g_2(x, z, y_2^*)). \quad \text{(A5)}\end{aligned}$$

Consider equation (A2). We have, using Assumption 5,

$$0 = \nabla_{z_i} g_2(x, z, y_2^*) dz_i + \nabla_{y_2} g_2(x, z, y_2^*) dy_2 \tag{A6}$$

$$dy_2 = \nabla_{z_i} h_2(x, z, Q_\nu(\tau_\nu)) dz_i \tag{A7}$$

The monotonicity of h_2 with respect to ν (Assumption 5) implies that

$$Q_{Y_2|XZ}(\tau_\nu, x, z) = h_2(x, z, Q_\nu(\tau_\nu))$$

and since $\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z) \neq 0$ (Assumption 5), from (A7)

$$dz_i = \frac{dy_2}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)}$$

and so on substituting in (A6)

$$\nabla_{y_2} g_2(z, y_2^*) = -\frac{\nabla_{z_i} g_2(x, z, y_2^*)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)}. \tag{A8}$$

Substituting (A8) into (A4) gives

$$\begin{aligned}\nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau, y_2^*, x, z) &= \nabla_{y_2} h_1(y_2^*, x, Q_\varepsilon(\tau), g_2(x, z, y_2^*)) \\ &\quad - \frac{\nabla_{z_i} g_2(x, z, y_2^*)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)} \nabla_\nu h_1(y_2^*, x, Q_\varepsilon(\tau), g_2(x, z, y_2^*))\end{aligned}$$

and on using equation (A5),

$$\begin{aligned}\nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2^*, x, z) &= \nabla_{y_2} h_1(y_2^*, x, Q_\varepsilon(\tau_\varepsilon), g_2(x, z, y_2^*)) \\ &\quad - \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2^*, x, z)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)}\end{aligned}$$

Finally, noting that $y_2^* = Q_{Y_2|XZ}(\tau_\nu, x, z)$ and $g_2(x, z, y_2^*) = Q_\nu(\tau_\nu)$ and that the exogenous impact function is

$$\pi(\tau_\varepsilon, \tau_\nu, x, z) = \nabla_{y_2} h_1(Q_{Y_2|XZ}(\tau_\nu, x, z), x, Q_\varepsilon(\tau_\varepsilon), Q_\nu(\tau_\nu))$$

there is the result (4) of Theorem, as follows.

$$\begin{aligned}\pi(\tau_\varepsilon, \tau_\nu, x, z) &= \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z) \\ &\quad + \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)}\end{aligned}$$

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