Irrational behavior in the Brown–von Neumann–Nash dynamics

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Abstract

We present a class of games with a pure strategy being strictly dominated by another pure strategy such that the former survives along solutions of the Brown–von Neumann–Nash dynamics from an open set of initial conditions.

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1 Introduction

Let $\pi : S = S_1 \times \cdots \times S_N \to \mathbb{R}^N$ be the payoff function of a finite strategic game, extended N–linearly to the polyhedron $\square(S) = \Delta(S_1) \times \cdots \times \Delta(S_N)$ of mixed strategy profiles $x = (x_i)_{i=1}^N = (x_i, x_{-i})$. Then the *Brown–von Neumann–Nash dynamics* is given by the system of differential equations on $\square(S)$

$$\dot{x}_{is} = k_i(s, x) - x_{is}\bar{k}_i(x), \quad i = 1, \ldots, N, \quad s \in S_i \tag{BNN}$$

with $k_i(s, x) = \max[0, \pi_i(s, x_{-i}) - \pi_i(x)]$ and $\bar{k}_i(x) = \sum_{\sigma \in S_i} k_i(\sigma, x)$. $k_i(s, x)$ is the excess payoff for strategy s of player i over his average payoff. For two person symmetric zero–sum games this differential equation was introduced by Brown and von Neumann (1950) as another numeric device (besides fictitious play) to compute optimal strategies. For general N person games it is the

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continuous time analog of the map that Nash (1951) used in his existence proof of equilibria.

Nash's map has been fundamental for the development of economic theory by inspiring a whole literature on existence of general equilibrium, starting from Arrow and Debreu (1954). A generalization of (BNN) has already been used by Nikaido (1959) as a price adjustment process. In game theory, however, (BNN) had to wait forty years for its comeback finally paved by Maynard Smith's (1982) creation of evolutionary game theory. It entered stage again as 'dynamics of rational deliberation' in Skyrms (1990), as 'canonical dynamics' in Swinkels (1993), and as the archetype of an 'innovative dynamics' in Weibull (1994b). Berger (1998) studied the stability of Nash equilibria under this dynamics, and Hofbauer (2000) showed global stability of completely mixed ESS. Sandholm (2001) uses it as prime example in his study of potential games, and Sandholm (2004) shows a fascinating relation of (BNN) to the regret-based decision rules of Hart and Mas-Colell (2001).

One can give a rough evolutionary interpretation of (BNN): Suppose there are N player populations (each of constant size) in which there is steady influx and outflux. New players joining the game use only strategies that are better than average, and better strategies are more likely to be adopted. More precisely, strategy $s \in S_i$ is adopted with probability proportional to the excess payoff $k_i(s, \cdot)$. On the other hand, randomly chosen players leave the game.

It is instructive to compare the BNN dynamics with two other, well-known dynamics. The *replicator dynamics* is the fundamental selection dynamics in evolutionary biology and in the social sciences. [1] Of similar importance in economic theory is the *best response dynamics*, which is (up to a rescaling of time) mathematically equivalent to the continuous time *fictitious play* learning process. [2] Both these dynamics are 'myopic adjustment dynamics' as defined by Swinkels (1993): Each player population moves towards a better reply against the current state of the other player populations.

From the viewpoint of classical game theory, however, both these dynamics have their drawbacks. The replicator dynamics is a *selection* dynamics (a once unused strategy will never occur), which in particular implies that every pure strategy profile is a stationary state. The best response dynamics, on the other hand, is an *innovative* dynamics (at least one of the (possibly unused) pure strategies with higher than average payoff grows in population share), implying

[1] See Maynard Smith (1982), Weibull (1995), Hofbauer and Sigmund (1998), or Fudenberg and Levine (1998).

[2] The best response dynamics was introduced by Gilboa and Matsui (1991) and Matsui (1992), and fictitious play is due to Brown (1951). For stability results see Gaunersdorfer and Hofbauer (1995), Hofbauer (1995, 2000), Fudenberg and Levine (1998), or Berger (2004).

that its constant solutions agree with the Nash equilibria, but in general it suffers from nonuniqueness of solutions.

What makes the BNN dynamics so interesting in evolutionary game theory is that it picks out the attractive features[3] of both the replicator and the best response dynamics:

- (BNN) is a myopic adjustment dynamics, as follows from its definition.
- Its stationary states are precisely the Nash equilibria, since it is an innovative dynamics.
- It has unique solutions by Lipschitz continuity.

2 (Ir)rationality

The present note deals with the following question. *Does the BNN dynamics asymptotically lead to rational behavior?*

Consider a game with a pure strategy q that is strictly dominated by a (possibly mixed) strategy p. A basic rationality postulate is that a player would never use q. It is easy to see that in the best response dynamics (or fictitious play), strategy q will eventually vanish from the population. Along interior orbits, q will also go extinct in the replicator dynamics, see Samuelson and Zhang (1992). More generally, Hofbauer and Weibull (1996) characterized the selection dynamics for which elimination of (even iteratively) strictly dominated strategies holds as 'convex monotone'. Basically, these dynamics are interpolations between the replicator and the best response dynamics. Their crucial property is that any pure strategy which is used and performs below average strictly decreases its frequency.[4] The BNN dynamics — though not a selection dynamics — obviously shares this property. This might suggest that (BNN), just like the replicator or the best response dynamics, asymptotically behaves 'as if' the population were a rational player (compare Weibull, 1994a). We show below that this is not the case.

Note that there are well known examples, e.g. Dekel and Scotchmer (1992) or Hofbauer and Weibull (1996), of strictly dominated strategies surviving in certain games under certain dynamics. However, in all these examples the dominating strategy p is mixed. For (BNN) the situation is worse: here the dominating strategy may itself be pure.

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[3] There are even arbitrarily smooth variants of (BNN), see Weibull (1994b), Hofbauer (2000), or Sandholm (2004), which share its crucial properties.

[4] A property called 'seeking the good' by Skyrms (1990) and 'weakly sign preserving' by Ritzberger and Weibull (1995).

We begin by studying the degenerate case of games with two equivalent strategies. For simplicity we consider symmetric two person games played within one population, with finite strategy set $S = S_1 = S_2$, payoff matrix $\pi : S \times S \to \mathbb{R}$, and $x \in \Delta(S)$ denoting a mixed population profile. Suppose there are two equivalent pure strategies, say $p, q \in S$, such that $\pi(p, x) = \pi(q, x)$ and hence $k(p, x) = k(q, x)$ for all x. Then $(x_p - x_q)^{\cdot} = -(x_p - x_q)\bar{k}(x)$. In particular, the subset $\{x \in \Delta(S) : x_p = x_q\}$ is invariant under (BNN). Consider a trajectory $x(t)$, $t \geq 0$, which stays away from the set of equilibria, so that $\bar{k}(x(t)) \geq \delta > 0$.[5] Then $x_p(t) - x_q(t) \to 0$ as $t \to \infty$.

Hence the BNN dynamics tends to *equalize* the proportions of equivalent strategies, at least along nonconvergent trajectories.[6] This equilibration property looks perfectly reasonable. It can be seen as an evolutionary version of the 'principle of insufficient reason': in case of indifference between two options, use each with equal probability. However, as we will show now, this property is incompatible with the elimination of strictly dominated strategies.

For this we need a game with nonconvergent trajectories. The obvious choice is a rock–scissors–paper game with $S = \{r, s, p\}$ and payoffs $0, 1, -3$ for a draw, win, or loss, respectively. This game has a unique equilibrium E at which players put weight $\frac{1}{3}$ on each option. A short calculation, compare Hofbauer (2000, section 4.3), shows that the nonnegative function $V(x) = \frac{1}{2}\sum_{\sigma \in S} k(\sigma, x)^2$ attains its unique minimum at E and satisfies

$$\dot{V} = \pi(\dot{x}, \dot{x}) - \bar{k}\pi(\dot{x}, x) = \sum_{\sigma \in S} \dot{x}_\sigma^2 - \bar{k} \sum_{\sigma \in S} k(\sigma, x)^2.$$

Near E the first term is of order $\bar{k}^2$ and the second term of order $\bar{k}^3$. This implies that $\dot{V} > 0$ near E. Hence V increases along trajectories near E, and E is a repellor for (BNN).[7] By the Poincaré–Bendixson theorem, orbits near E spiral outwards to some limit cycle. On the other hand, orbits starting on the boundary spiral inwards and approach some (possibly different) limit cycle. These two limit cycles bound an asymptotically stable annulus L.[8]

Consider now the augmented game with $S = \{r, s, p, q\}$ and payoff matrix

[5] Note that $\bar{k}(x)$ is nonnegative and vanishes precisely at equilibria, hence it is a measure for the distance of x from the set of Nash equilibria.

[6] This behaviour is quite different from that of the replicator dynamics, where the ratio x_p/x_q remains constant along solutions, or the best response dynamics, for which arbitrary drift between the equivalent strategies is possible.

[7] This holds more generally for payoffs a, b, c (instead of $0, 1, -3$) satisfying $c < a < b$ and $b + c < 2a$. For a different proof based on Poincaré maps see Berger (1998).

[8] Numerical simulations suggest that L is a unique limit cycle.

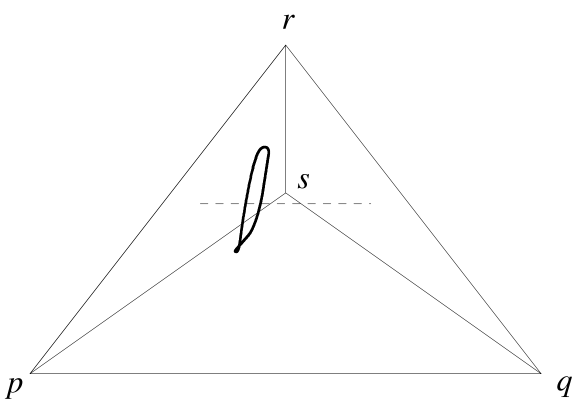

Fig. 1. The oscillating attractor for $\varepsilon = 0.2$.

given by

$$\begin{bmatrix} 0 & 1 & -3 & -3 \\ -3 & 0 & 1 & 1 \\ 1 & -3 & 0 & 0 \\ 1-\varepsilon & -3-\varepsilon & -\varepsilon & -\varepsilon \end{bmatrix}$$

with $\varepsilon \geq 0$. Strategies r, s, p play rock, scissors, paper, as before, while q is an ε-close variant of p which is strictly dominated by p for $\varepsilon > 0$. The oscillating attractor L for the $\{r, s, p\}$ game appears in the augmented game for $\varepsilon = 0$ on the invariant subset $\{x_p = x_q\}$. This asymptotically stable invariant set L attracts all trajectories in $\Delta(S)$, except the stationary solutions corresponding to the line of Nash equilibria $\frac{1}{3}(\lambda p + (1-\lambda)q + r + s)$, with $\lambda \in [0, 1]$. For $\varepsilon > 0$ this line is still invariant and consists of an orbit converging to the unique equilibrium $\frac{1}{3}(r + s + p)$. Since (BNN) depends continuously on the parameter ε, the oscillating attractor L continues for small ε to a nearby asymptotically stable invariant set L_ε (again a periodic orbit according to numerical simulations, see Figure 1) which still attracts all orbits except those starting in a small neighborhood of the invariant line.

For the perturbed game with $\varepsilon > 0$, strategy p strictly dominates q, but for small values of ε, x_p and x_q are almost equal along the attractor [9] L_ε, and hence the strictly dominated strategy q survives.

3 Conclusion

We showed by means of a simple 4×4 game that a pure strategy which is strictly dominated by another pure strategy may survive along a large set of

[9] According to numerical simulations, this limit cycle reaches the face $x_q = 0$ only for $\varepsilon \approx 0.47$.

solutions of the BNN dynamics. Indeed the construction works more generally for any game with an asymptotically stable attractor that is disjoint from the equilibrium set, and the argument also applies to other innovative dynamics. While (BNN) and its relatives are presently establishing themselves as a promising third type of evolutionary dynamics besides the replicator and the best response dynamics, they do not share one of the crucial properties of these two: 'as if' rationality.

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