EVOLUTIVE EQUILIBRIUM SELECTION I: SYMMETRIC TWO PLAYER BINARY CHOICE GAMES.

Richard Vaughan

Centre for Economic Learning and Social Evolution, University College London

Abstract The aim of the paper is the construction of a distributional model which enables the study of the evolutionary dynamics that arise for symmetric games, and the equilibrium selection mechanisms that originate from such processes. The evolution of probability distributions over the state variables is studied using the Fokker-Planck diffusion equation. Equilibrium selection using the ”basin of attraction” approach, and a selection process suggested by Pontryagin are contrasted. Examples are provided for all generic 2-person symmetric binary choice games. JEL Classification: C78

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1. INTRODUCTION

The aim of this paper is the construction of a distributional model which enables us to study the evolutionary dynamics that arise in the case of symmetric games, and the equilibrium selection mechanisms that originate from such processes. Recent approaches to equilibrium selection have focused on models employing stochastic difference or differential equations; equilibrium selection being determined by the "low-noise" limit of such processes. One branch of this literature employs continuous time continuous state representations of the relevant stochastic process, ; within this group we note in particular the papers by Foster and Young(1990), Fudenberg and Harris(1992), and Cabrales(1993). An alternative approach was taken by Kandori, Malaith and Rob(1993) and Young(1992), using z-tree techniques (see Freidlin and Wentzell(1983)). Subsequent papers by Kandori and Rob(1991), Samuelson(1993), Noldeke, Samuelson and Van Damme(1993), and Binmore and Samuelson(1993) use z-tree analysis in a wide variety of applications. The connection between these two approaches is not completely transparent. For example, the papers by Cabrales(1993), Binmore and Samuelson(1993), Binmore,Samuelson and Vaughan(1994) obtain equilibrium selection properties that do not coincide with those of Kandori, Malaith and Rob(1993). Study of these papers warn us that the outcome of the equilibrium selection process are sensitive; firstly to whether we deal with discrete or continuous models both in time and state space; secondly, as to the choice of the dynamic adjustment process, inclusive of the stochastic component; and thirdly, as to the nature of the boundary conditions that are imposed on the selection process.

The methodology of the present paper is distinct from that followed by the two approaches above, insofar as it is based on the evolution of probability distribution over the relevant state variables induced by the appropriate stochastic difference or differential equations. For the processes studied here we are led to a formulation employing the Fokker-Planck or diffusion equation. Such processes are well known in the statistical and physical sciences literature, including evolutionary biology, but have not appeared thus far in the game theoretic literature applied to equilibrium selection.

2. THE DISTRIBUTIONAL MODEL FOR THE SYMMETRIC 2x2 GAME

We begin with the analysis of the $2x2$ symmetric game. A population a time t of size $N(t)$ consists of two distinct phenotypes; each phenotype is associated with a fixed strategy, $s_i, i = 1, 2$; and the proportion of the population playing strategy s_i at time t is $p_i(t)$. When an individual of type i meets an individual of type j, the payoff matrix a_{ij} is defined by,

$$A = \begin{matrix} a, a & b, c \\ c, b & d, d \end{matrix} \tag{1}$$

where $a, b, c, d \in \mathbb{R}$. Generic types of $2x2$ games defined by the relative magnitudes of the payoffs will be considered further below.

The proportions playing each strategy are assumed to change over time. In the deterministic evolutionary case the relative rate of increase in the frequency of i is assumed to be governed by the replicator equation,

$$\frac{1}{p_i}\frac{dp_i}{dt} = (Ap(t))_i - p(t)'Ap(t) \tag{2}$$

where $A = [a_{ij}]$ and $p(t)' = [p_1(t), p_2(t)]$. Letting $p = p_1$ and $p_2 = 1 - p_1$, then (2) may be written as,

$$\frac{dp}{dt} = G(p, A) \tag{3}$$

where,

$$G(p) = ([a+d] - [b+c])p(1-p)(p - \frac{d-b}{(a+d)-(b+c)} \tag{4}$$

Associated with (3) we define the stochastic replicator dynamics by the introduction of an additional random term. Different dynamics are generated according to the choice of this term; one of the simplest which has been used in the existing literature is the introduction of a Wiener process, with the stochastic counterpart of (3) being defined by,

$$dp = G(p, A, N)pdt + \sigma(p, A, N)dz \tag{5}$$

where dz denotes a Wiener process of zero-mean and unit variance, and $\sigma^2(p, A, N) > 0$.

In formulating (5), we note the following,

(i) The drift term $G(p, A, N)$ may be formally dependent on the proportions in the population playing each strategy, the payoff matrix of the game, and the size of the total population. The appearance of N would rule out the replicator formulation (4). From the point of view of the present analysis the assumption of a replicator dynamic is not essential; indeed best response or other types of dynamic may be defined by the drift term.. In the present paper we shall assume N is fixed and of sufficient magnitude to allow us to view p as a continuous variable.

(ii) The diffusive term $\sigma(p, A, N)$ may also be dependent on the payoffs and size of total population, as well as the proportions playing any particular strategy. For sections 2-6, we shall assume that $\sigma^2(.)$ is strictly positive over the interval $[0, 1]$.

One method of approach to the solution of (5) is via the Ito stochastic calculus; in the present paper however we shall follow the approach of analysing directly the distribution function over p implied by an equation such as 4.

Let $f(p, t)$ denote the probability distribution function resulting from the stochastic dynamics implied by (5). As is well known, and shown in Appendix I, the evolution of $f(p, t)$ is governed by the Fokker-Planck equation,

$$\frac{\partial f(p,t)}{\partial t} = -\frac{\partial}{\partial p}(G(p)f(p,t)) + \frac{1}{2}\frac{\partial^2}{\partial p^2}(\sigma^2(p)f(p,t)) \tag{6}$$

subject to appropriate initial and boundary conditions.

In the present interpretation each separate p reflects a different "world" with a given proportion of the population playing strategy s_1; since p defines the proportion of the population following strategy s_1, then $0 \leq p \leq 1$, and boundary conditions at $p = 0, 1$ have to be imposed. If we look at the deterministic replicator dynamics equation (2) we see that any strategy not played by any individuals in the population, can never reappear in that population. In terms of population evolution this assumption reflects extinction of any phenotype, with no possible subsequent reappearance. In the present case however, even with $G(p)$ identical to (2), reappearance of a strategy may be possible due to the influence of the stochastic component $\sigma(.)$. Put simply, assume that the proportion of the population playing strategy s_1 is zero, and that the deterministic component of the stochastic differential equation is also zero at $p = 0$, i.e. $G(0) = 0$, then provided $\sigma^2(0) > 0$ then a stochastic perturbation could result in the reappearance of strategy s_1. A similar result exists if the proportion playing strategy s_2 becomes zero, a stochastic perturbation could reintroduce strategy s_2. As will be seen, the values that $G(.)$ and $\sigma^2(.)$ take at the boundaries are crucially important in determining the subsequent evolution of the system, as indeed are the accessibility or otherwise of the boundaries themselves.

Categorisation of boundaries is well known in the study of partial differential equations of the Fokker-Planck type. The first distinction is between accessible and inaccessible boundaries. If a boundary is accessible, we can further categorise it in terms of a reflecting or absorbing boundary.

(i) Inaccessible boundaries. In this case, if the process starts off from any interior point in (0,1), the boundary can never be reached. Formally, if $\lim_{p\to 0,1} \sigma^2(p) = 0$, and $G(p) > 0$, at $p = 0$, whilst $G(p) < 0$ at $p = 1$, then the boundaries are inaccessible. The intuition is clear, for any "world" with proportion p at a point close to either boundary, the drift coefficient $G(p)$ sets up a repulsive movement away from the boundary; against this must be set the stochastic component $\sigma^2(p)$. Such a term may act either in support of the drift coefficient, or act against it, driving the "world" at point p closer to the boundary. However if the stochastic component becomes progressively weaker as the boundary is approached, then the drift term becomes progressively more dominant, thus leading to the impossibility of a "world" ever reaching the boundary.

(ii) Absorbing Boundary. If we drop the assumption that $\lim_{p\to 0,1} \sigma^2(p)$= 0, then both boundaries become accessible, irrespective of the sign of $G(.)$ in the locality of the boundary. Once the boundary is reached it is perfectly reasonable to change the dynamics of the system, to reflect the requirements of the model. An absorbing boundary would reflect the perpetual extinction assumption; once a phenotype becomes extinct it forever remains thus. Stochastic perturbations at the boundary are thus ruled out.

(iii) Reflecting Boundary. The technical definition of "reflecting" in this case follows from the usage of this model in statistical physics. In the present context the terminology may be somewhat misleading since it seemingly implies that no "worlds" may be found at the boundary; this is not the case. "Worlds" may stick at the boundary and only depart if either a stochastic perturbation, in conjunction with the working of the drift coefficient, generates movement away from the boundary and back into the interior of $[0, 1]$

In order to proceed further a mathematical specification of these boundary conditions is required.

We find it convenient to introduce the notion of a probability flow in terms of the distribution $f(p,t)$.

Consider (5) defined over the entire real line, $-\infty < p < +\infty$. Integrate (6) from p to $+\infty$; then we have,

$$\frac{\partial F(p,t)}{\partial t} = -G(p)f(p,t) - \frac{1}{2}\frac{\partial}{\partial p}(\sigma^2(p)f(p,t)) + C \tag{7}$$

where C is an arbitrary constant of integration and,

$$F(p,t) = \int_p^\infty f(p,t)dp \tag{8}$$

$F(p,t)$ denotes the probability mass at and above p, therefore $\partial F(p,t)/\partial t$ denotes the change in this mass with respect to time; since this mass cannot be destroyed, i.e. no outflow of probability mass at or above p except back through p; $\partial F(p,t)/\partial t$ also defines the probability flow through p. Probability flow can only be induced via the stochastic equation (5), and therefore C in (6) must be zero. We thus have the definition of probability flow as,

$$\mathcal{F}(p,t) = G(p)f(p,t) - \frac{1}{2}\frac{\partial}{\partial p}(\sigma^2(p)f(p,t)) \tag{9}$$

Reflecting boundaries can now be easily expressed in terms of these flows, namely,

$$\mathcal{F}(0,t) = \mathcal{F}(1,t) = 0 \tag{10}$$

Whilst absorbing boundaries may be expressed in terms of the boundary conditions as,

$$f(0,t) = f(1,t) = 0 \tag{11}$$

The initial condition that we impose on the distribution $f(p,t)$ at $t = 0$ is,

$$f(p,0) = f^o(p) \tag{12}$$

and the normalization condition,

$$\int_0^1 f(p,t)dp = 1, \qquad t \geq 0 \tag{13}$$

The complete statement of the problem is therefore to find $f(p,t)$ for $t \geq 0$, which satisfies the Fokker-Planck equation (6), the initial condition (12), together with the appropriate boundary conditions, for example, either , (10), or (11).

3. THE ERGODIC DISTRIBUTION

In the present paper our main interest will centre on the limiting distribution,

$$\lim_{t\to\infty} f(p,t) = f^*(p) \tag{14}$$

The distribution is deemed to be ergodic if it is independent of the initial distribution $f(p,0)$. An ergodic distribution may not always exist, however sufficient conditions to guarantee ergodicity are the assumptions:

(A1) The boundary conditions are reflecting,

(A2) $G(p)$ is bounded on [0,1], and $\sigma^2(p) > 0$ on [0,1].

Such conditions are also sufficient to guarantee uniqueness of $f(p,t)$ on $[0,1], t \geq 0$. Existence of $f^*(p)$ is usually shown by construction; the conditions for uniqueness are well known in the literature (see e.g. Friedman(1964), Risken(1984)). For the remainder of this section we shall assume that (A1) and (A2) hold.

The determination of the functional form which satisfies the condition of time invariance is relatively easy to determine from the flow condition (9). If $f^*(p)$ is independent of t, then in equilibrium $h(p)$ must satisfy,

$$\mathcal{F}(p,t) = G(p)f^*(p) - \frac{1}{2}\frac{\partial}{\partial p}(\sigma^2(p)f^*(p)) = 0 \tag{15}$$

i.e. stationarity of the distribution must imply a zero probability flow not only at the boundaries of the interval $[0,1]$ but also at every point in its interior.

Solving (15) for $f^*(p)$ we thus have,

$$f^*(p) = \frac{C}{\sigma^2(p)} e^{\varphi(p)} \tag{16}$$

where,

$$\varphi(p) = 2\int_{p_o}^{p} \frac{G(x)}{\sigma^2(p)} dx \tag{17}$$

$p_o < p$ is an arbitrary constant, however upon integrating (17), and subst. into (16), this constant can be subsumed in the constant of integration C; which will be determined by the normalization condition (13) i.e.

$$C^{-1} = \int_0^1 \frac{e^{\varphi(x)}}{\sigma^2(x)} dx \tag{18}$$

The function $\varphi(p)$ has an important role in linking the stochastic dynamics to the equilibrium distribution eventually generated; in the statistical and physical sciences literature $\varphi(p)$ is called the potential function.

In the case where the variance $\sigma^2(p)$ is assumed to be constant over the interval of p, the relationship between the distribution $f^*(p)$ and the properties of the original deterministic dynamical system are particularly straightforward. Letting $\sigma^2(p) = V > 0$, and differentiating (16) successively with respect to p we have,

$$\frac{\partial f^*(p)}{\partial p} = f^*(p)\varphi'(p) = \frac{C}{V} e^{\varphi(p)} \varphi'(p) \tag{19}$$

$$\frac{\partial^2 f^*(p)}{\partial p^2} = f^*(p)[\varphi''(p) + (\varphi'(p))^2] = \frac{C}{V}[e^{\varphi(p)}\varphi''(p) + e^{\varphi(p)}(\varphi'(p))^2] \tag{20}$$

Now from (17) we have,

$$\varphi'(p) = 2G(p)/V \tag{21}$$

$$\varphi''(p) = 2G'(p)/V \tag{22}$$

When $G(p) = 0$ we have a rest point of the original deterministic system, and these rest points correspond to stationary points of $\varphi(p)$ and $f^*(p)$. Further, from (20) and (22) we note that at these stationary points of $f^*(p)$,

$$\frac{\partial^2 f^*(p)}{\partial p^2} = f^*(p)\varphi''(p) = f^*(p)\frac{2}{V}G'(p) \tag{23}$$

and for $f^*(p)$ strictly positive at the stationary points, unstable rest points of $G(p)$, $G'(p) > 0$, correspond to the minima of $f^*(p)$; with stable rest points, $G'(p) < 0$ corresponding to the maxima of $f^*(p)$.

As an example consider the replicator dynamics, with a payoff matrix given by equation (1); $G(p)$ is a polynomial of order 3, thus with a constant $\sigma^2(p) = V$, a typical relationship is represented by *Fig 1.*

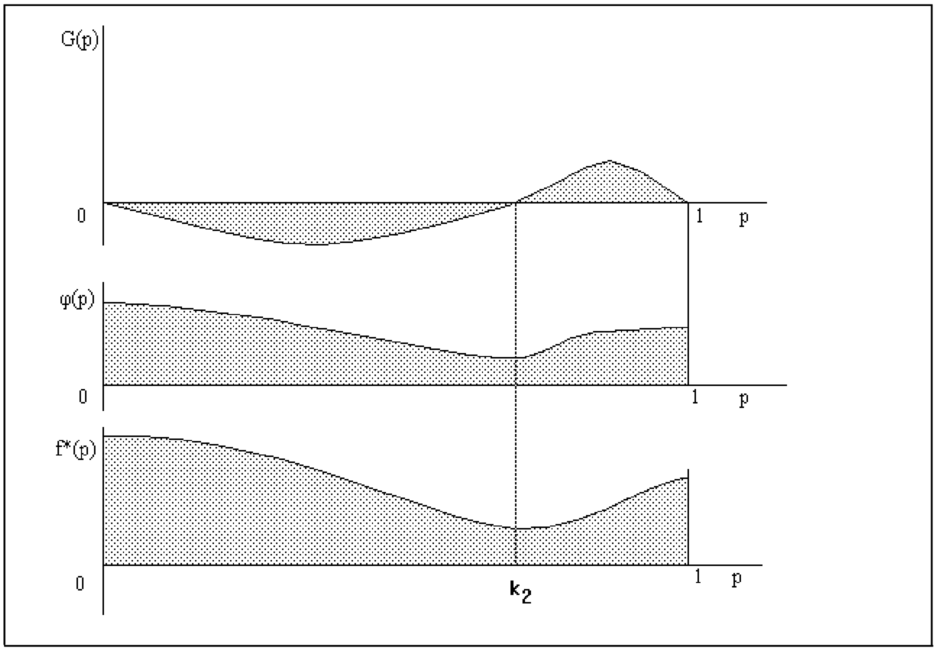


Fig.1

We may note that:

(a) The positions of the stationary points of $\varphi(p)$ and $f^*(p)$ are unaffected by the value of V; this follows from eq. (21).

(b) The ranking by value of $\varphi(p)$ and $f^*(p)$ at these stationary points are unaffected by the value of V. This can be seen by evaluating (16) defined at any two stationary points and then taking the quotient of these two values, remembering to evaluate (17) using the

constant variance.
(c) The absolute values of $\varphi(p)$ and $f^*(p)$ will vary with V. Apparent from equations (16) and (17).

The case of the non-constant variance is considered in Sections 4,7 and 8 below.

4. THE EQUILIBRIUM SELECTION PROCESS

In the case of the existence of an ergodic distribution, starting off from an initial distribution of worlds playing in different proportions the strategies s_1 and s_2, then we would expect to find after some period of time this distribution approaching $f^*(p)$. The time taken to approach this equilibrium will be considered elsewhere. In this section we shall be concerned with the process of equilibrium selection, or "low-noise" limit of the system (16)-(18).

Equilibrium selection means that the distribution function converges to a single point on the interval $[0, 1]$, leading to possible selection of mixed or single strategies. Provided the assumptions (A1) and (A2) hold, convergence to such a single point is not possible for the system (16)-(18). However, if we take the limit of the distribution $f^*(p)$ as the stochastic term $\sigma^2(p) \to 0$, then the distribution does collapse, either to a single or multiple points.

We return to the case where $\sigma^2(p)$ is allowed to vary over p, but still retain the assumption that $\sigma^2(p) > 0$. We propose to determine what happens to $f^*(p)$ as $\sigma^2(p)$ uniformly tends to zero. We therefore assume that $\sigma^2(p)$ can be represented in the form,

$$\sigma^2(p) = \beta V^2(p) \tag{24}$$

where $\beta > 0$ is some constant.

We are then interested in,

$$\lim_{\beta\to 0} f^*(p) = \lim_{\beta\to 0} \frac{C}{\beta V^2(p)} e^{\psi(p)/\beta} \tag{25}$$

where,

$$\beta\varphi(p) = \psi(p) = 2\int_0^p \frac{G(x)}{V^2(x)}dx \tag{26}$$

The basic theorem relating to (25) appears to have originated with Pontryagin, Andronov and Witt(1934).

Theorem 1. *Distributional Dominance*.

If $f^*(p)$ exists and if $\psi(p)$ attains a unique maximum at p^* in the closed interval $[0, 1]$ then,

$$\lim_{\beta\to 0} f^*(p) = f^*(p^*) \tag{27}$$

where f*(p^*) is the distribution centred on p^*, such that for any $\varepsilon > 0$, f*$(p) = 0$ for $p > p^* + \varepsilon$, and $p < p^* - \varepsilon$.

Proof: Appendix 2.

Corollary 1.

If $h^*(p)$ exists and $\psi(p)$ attains n maxima in the closed interval $[0,1]$ at points p_i^* such that $\psi(p_i^*) = \psi^*$ for $i = 1, \ldots, n$; then,

$$\lim_{\beta \to 0} f^*(p) = f^*(p) \tag{28}$$

where $f^*(p)$ is a function such that $f(p_i^*) = f(p_j^*)$ all $i, j = 1, .., n$; and for any $\varepsilon > 0$, $f^*(p) = 0$ for $p > p_i^* + \varepsilon$, and $p < p_i^* - \varepsilon$ for all $i = 1, \ldots, n$.

5. THE PRIMARY SELECTION MECHANISM: MAXIMUM POTENTIAL vs BASINS OF ATTRACTION

In the above section we have seen that the equilibrium selection mechanism, i.e. the low noise limit distribution is determined at that point at which $\varphi(p)$, is at its maximum, i.e. equivalent to the maximum of the equilibrium frequency distribution. In the paper by Kandori, Malaith and Rob (1993), an alternative criteria for equilibrium selection in terms of the "low-noise" limit is specified, that of the largest basin of attraction. In this section we consider the relationship between these two criteria, and determine whether one or the other should be the primary role of equilibrium selection.

We should of course note that the formulation of the KMR is quite different from the present case, being specified in terms of discrete space and discrete time; furthermore the equilibrium selection is formulated in terms of z-tree analysis, rather than in terms of the solution to the FPE subject to appropriate boundary conditions. Nevertheless, it may be argued that the formulation of a selection mechanism should be impervious to whether discrete or continuous time formulations are adopted. In order to pursue the matter, we shall consider the definition of basins of attraction in the present model, and establish their relationship, if any, with the distribution ultimately selected.

The "basin of attraction" is defined in relation to the deterministic dynamics of $G(p)$. Consider the set of n rest points of $G(p).0 \leq p_o < \ldots.. < p_n \leq 1$. Then for any two such points $(i < j)$ the "basin of attraction" is defined by $(p_i - p_j)$ with the attractor being p_i if $G(p) < 0$, and p_j if $G(p) > 0$. KMR show under their assumptions that the "low noise" limit distribution converges to the rest point p_m with the largest basin of attraction.

Let us apply this theory to the present model. For replicator dynamics we have at most three rest points of $G(p)$. In the case of two rest points, the basin of attraction is the entire unit interval, where the attractor is either of the end points depending on the payoff matrix. Quite clearly in this case the conditions for the limit distribution are identical whether specified in terms of the maximum potential or the largest basin of attraction.

In the case of three rest points, if the interior rest point is a stable point, then the entire unit interval is the basin of attraction, and so again we arrive at an identical prescription for the low noise limit in terms of the maximum potential or the largest basin of attraction.

The only case where possible differences could occur is therefore the three rest point case, with an unstable interior rest point. In this case if the "basin of attraction" approach could be correctly applied to the present model, then the low noise limit converges either to 0 or 1 depending on the relative magnitude of $p^*$and $(1 - p^*)$. Whereas the maximum potential approach requires knowledge of $\varphi(0)$ and $\varphi(1)$.

For the case of constant variance, the potential for the replicator dynamics is given by,

$$\varphi(p) = (2/V)\int^p G(p)dp = (2/V)([a+d]-[b+c])\int^p p(1-p)(p-p^*)dp \tag{29}$$

$$\text{where } p^* = \frac{d-b}{(a+d)-(b+c)} \text{ and } 0 < p^* < 1$$

Thus

$$\varphi(p) = (2/V)([a+d]-[b+c])\{\frac{1}{3}(1+p^*)p^3 - \frac{p^4}{4} - p^*\frac{p^2}{2}\} \tag{30}$$

and so,

$$\varphi(0) = 0; \varphi(1) = (2/V)([a+d]-[b+c])\{\frac{1}{3}(1+p^*) - \frac{1}{4} - p^*\frac{1}{2}\} \tag{31}$$

Thus $\varphi(0) > \varphi(1)$ if $\frac{1}{3}(1+p^*) - \frac{1}{4} - p^*\frac{1}{2} = (1/12)(1-2p^*) < 0$, i.e. if $p^* > 1/2$. But if $p^* > 1/2$ then $p = 0$ has the largest basin of attraction, and so both the maximum potential and basin of attraction approaches lead to the same low noise equilibrium selection. Indeed , once this equivalence has been shown there is no subsequent need to calculate the potential, knowledge of p^* alone is sufficient to determine the low noise limit distribution.

However, there still remains a problem with the basin of attraction result; we have only shown equivalence for the case of constant variance. In this case the deterministic dynamics are sufficient to determine the low noise limit. Once the variance is dependent on p then the potential becomes $\varphi(p) = 2\int[G(p)/\sigma^2(p)]dp$ and it becomes apparent that equivalence between the basin of attraction and the maximum potential approach no longer coincide. In this case using the basin of attraction approach would usually lead to the wrong prediction concerning the low noise limit; this point is considered further in Section 7.

6. EXAMPLES OF EQUILIBRIUM SELECTION

In this section we consider equilibrium selection, utilising Theorem 1, for the 2-person symmetric binary choice games; and then look at five particular examples; the "Stag-Hunt"game, the "Prisoners' Dilemma", "Hawk-Dove" also known as "Chicken", "Pure Co-ordination", and "Battle of the Sexes". Discussion of the nomenclature and raison d'etre for these games may be found in a variety of sources, e.g. Binmore( ),*),( ) and is not considered here.

The general payoff matrix is given by (1), and in this case,

$$G(p) = ([a+d]-[b+c])p(1-p)(p - \frac{d-b}{(a+d)-(b+c)}) \tag{32}$$

In this section we also assume a constant variance over the unit interval,

$$\sigma^2(p) = V \tag{33}$$

The equilibrium distribution is given by,

$$f^*(p) = \frac{C}{V}\exp\varphi(p) \tag{34}$$

$$\text{where } \varphi(p) = (2/V)\int_0^p G(x)dx$$

$$= (p^2/6V)\{(a+2(d-b)-c)4p - 3((a+d)-(b+c))p^2 - 6(d-b)\} \qquad (35)$$

There are a possible maximum of three stationary points of $G(p)$ on the interval $[0, 1]$. Provided, $0 < \frac{d-b}{(a+d)-(b+c)} < 1$, then Case 1 and Case2 exist:

Case 1

$$a > c, d > b; a \geq d$$

Stable rest points of $G(p)$ are to be found at $p = 0, p = 1$; and an unstable rest point at $p = \frac{d-b}{(a+d)-(b+c)}$. To apply Theorem 1 we therefore need to compare $\varphi(0)$ and $\varphi(1)$.

$$\varphi(0) = 0; \varphi(1) = (1/6V)[(a+2(d-b)-c)4 - 3((a+d)-(b+c)) - 6(d-b)]$$

$$= (1/6V)((a+b)-(c+d)) \qquad (36)$$

Thus, if $(a+b) > (c+d)$ then $\varphi(1) > \varphi(0)$ and $\lim_{V\to 0} f^*(p) = \delta(p-1)$, i.e. s_1 is selected.
and if $(a+b) < (c+d)$ then $\varphi(1) < \varphi(0)$ and $\lim_{V\to 0} f^*(p) = \delta(p)$, i.e. s_2 is selected.

Case 2

$$a < c, d < b$$

Unstable rest points of $G(p)$ are to be found at $p = 0, p = 1$; and a stable rest point at $p = \frac{d-b}{(a+d)-(b+c)}$. Since there exists only one stable rest point we can apply Theorem 1 and note that,

$$\lim_{V\to 0} f^*(p) = \delta\{p - \frac{d-b}{(a+d)-(b+c)}\} \qquad (37)$$

where $\delta\{.\}$ is the Dirac delta function; i.e.. the mixed strategy is selected.

If $\frac{d-b}{(a+d)-(b+c)}$ lies outside the interval $[0, 1]$ then there two stationary points at $p = 0$ *or* 1. and we thus have,
Case 3.

$$\varphi(0) = 0; \varphi(1) = (1/6V)[(a+2(d-b)-c)4 - 3((a+d)-(b+c)) - 6(d-b)] \qquad (38)$$

$$= (1/6V)((a+b)-(c+d)) \qquad (39)$$

Thus, as in Case 1 above,
if $(a+b) > (c+d)$ then $\varphi(1) > \varphi(0)$ and $\lim_{V\to 0} f^*(p) = \delta(p-1)$, i.e. s_1 is selected.
If $(a+b) < (c+d)$ then $\varphi(1) < \varphi(0)$ and $\lim_{V\to 0} f^*(p) = \delta(p)$, i.e. s_2 is selected.

Alternatively equilibrium selection may be expressed as follows. If,

$$sgn[(a+d)-(b+c)] = sgn[p - \frac{d-b}{(a+d)-(b+c)}] \qquad (40)$$

then the stable rest point is at $p = 1$, i.e. s_1 is chosen; otherwise the stable equilibrium is at $p = 0$.

Case 4

A knife-edge case exists when $(a+d) = (b+c)$. If $(a+d) = (b+c)$ and $d \neq b$ then stationary points of $G(p)$ are to be found at $p = 0, p = 1$;a third stationary point does not exist. If $(d-b) > 0$, then $p = 0$ is the stable equilibrium, whilst if $(d-b) < 0$, then $p = 1$ is the stable equilibrium. If $(a+d) = (b+c)$ and $d = b$, then $G(p) = 0$, and a continuum of stationary points p exist for $0 \leq p \leq 1$.

We can now note examples of the above solutions for a number of well-known games,

(i) The Stag-Hunt Game

The payoff matrix is given by,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & \alpha, \alpha & \alpha, 0 \\ s_2 & 0, \alpha & 1, 1 \end{array} \tag{41}$$

where $0 < \alpha < 1$. Thus s_1 is the ”safe strategy; irrespective of what the other player does, α is guaranteed. If s_2 is played it might yield the superior payoff of 1, but only if the other player also plays s_2; it only yields 0 if the other player plays s_1. Two strict Nash equilibria exist $[s_1, s_1]$ and $[s_2, s_2]$; which equilibrium should be chosen ?

Let us apply Th. 1; we have from (4),

$$G(p) = ([a+d] - [b+c])p(1-p)(p - \frac{d-b}{(a+d)-(b+c)}) \tag{42}$$

and thus with regard to (1), we have the identification,

$a = b = \alpha; c = 0$, and $d = 1$. Thus subst. in (42),

$$G(p) = [(\alpha+1) - \alpha]p[1-p][p - (1-\alpha)/((\alpha+1)-\alpha)] = p[1-p][p-(1-\alpha)] \tag{43}$$

We shall assume that the variance is constant over the interval of p, i.e. $\sigma^2(p) = V$.

Note first that since $\alpha < 1$, then we have three rest points in the interval $[0, 1]$. Since $a > c, d > b; a \geq d$, under the present identification, Case 1 above applies,

The equilibrium distribution is given by,

$$f^*(p) = \frac{C}{\sigma^2(p)} \exp \varphi(p) \tag{44}$$

where $\sigma^2(p) = V$ and,

$$\varphi(p) = 2 \int_{p_0}^{p} \frac{G(x)}{\sigma^2(x)} dx \tag{45}$$

$$= \frac{2p^2}{V}[(2-\alpha)\frac{p}{3} - \frac{p^2}{4} - \frac{(1-\alpha)}{2}] \tag{46}$$

Stable rest points of $G(p)$ are to be found at $p = 0$ and $p = 1$; and an unstable rest point at $p = (1-\alpha)$. To apply Th.1 we therefore need to compare, $\varphi(0)$ and $\varphi(1)$:

$$\varphi(0) = 0, \text{ and } \varphi(1) = (1/6V)(2\alpha - 1) \tag{47}$$

Thus if $\alpha < 1/2$ then $\varphi(1) < 0$; and if $\alpha > 1/2$ then $\varphi(1) > \varphi(0)$ and so if,

$$\alpha > 1/2 \text{ then } \lim_{V \to 0} f^*(p) = \delta(p-1) \tag{48}$$

and if

$$\alpha < 1/2 \text{ then } \lim_{V \to 0} f^*(p) = \delta(p) \tag{49}$$

where $\delta(.)$ is the Dirac delta function.

Thus s_1 is selected if $\alpha > 1/2$ and s_2 if $\alpha < 1/2$ according to this equilibrium selection process; i.e. we have the same solution as the equilibrium selection process suggested by Harsanyi and Selten(1988). The selection theories of Guth and Kalkofen(1989), and Carlsson and Van Damme(1993a) are known to coincide with Harsanyi and Selten for this case (see also Carlsson and Van Damme.(1993b)).

It should however be stressed that this selection is not only dependent on the specification of the payoff matrix, but also on the choice of the stochastic dynamic variables. In particular by changing the specification of $\sigma^2(p)$ we are able to switch the equilibrium choice at will between s_1 and s_2 as shown in Section 7. The replicator dynamic and the potential function for $V = 1$, and $\alpha = 0.6$ are shown in Fig.20.

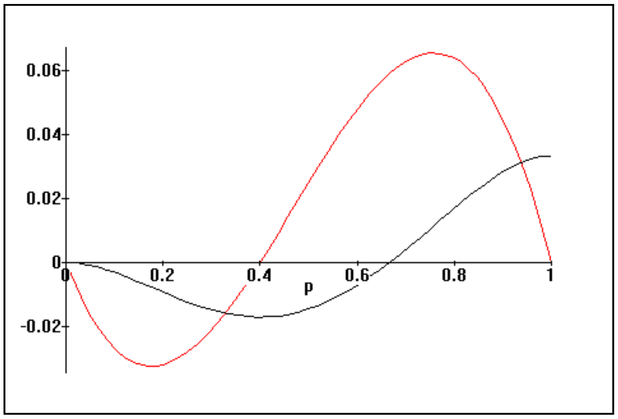


Fig.2. Replicator Dynamic and Potential for the Stag-Hunt game.

(ii) Hawk-Dove also known as Chicken

The payoff matrix for Hawk-Dove is given by,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & a,a & b,c \\ s_2 & c,b & 0,0 \end{array} \tag{50}$$

where $b < a < c$. Since $a < c, d < b$ and $(b/(b+c-a))$ lies within the unit interval, then Case (ii) above applies, and unstable rest points of $G(p)$ are to be found at $p = 0, p = 1$; and a stable rest point at $p = \frac{b}{b+c-a}$.Since there exists only one stable rest point we can apply Theorem 1 and note that,

$$\lim_{V \to 0} f^*(p) = \delta(p - \frac{b}{b+c-a}) \tag{51}$$

i.e.. the mixed strategy is selected.

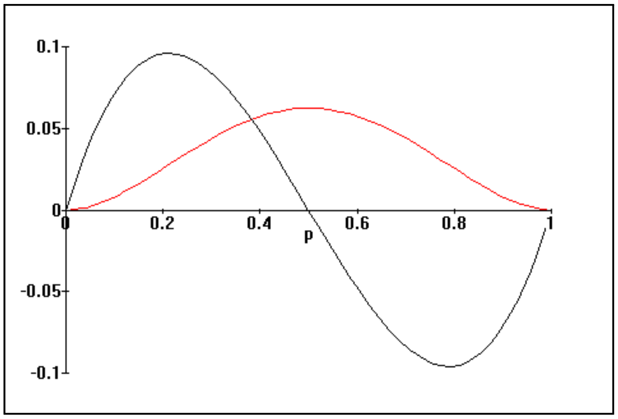


Fig.3. Replicator Dynamic and Potential for the Hawk-Dove Game.

The replicator dynamic and the potential function for V=1, and the payoff matrix

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 3,3 & 1,4 \\ s_2 & 4,1 & 0,0 \end{array} \tag{52}$$

are shown in Fig.3.

(iii) The Prisoner's Dilemma

The payoff matrix for the Prisoner's Dilemma is given by,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & a,a & b,c \\ s_2 & c,b & d,d \end{array} \tag{53}$$

where $c > a > d > b, (a+d) > (b+c)$, and $(d-b) > (a+d)-(b+c)$. Only two rest points exist in $[0,1]$, one of which is stable. Since

$$sgn[(a+d)-(b+c)] \neq sgn[p - \frac{d-b}{(a+d)-(b+c)}] \tag{54}$$

the only stable rest point occurs at $p = 0$, and since there is only one stable rest point we can apply Th.1 and note that strategy s_2 is selected.

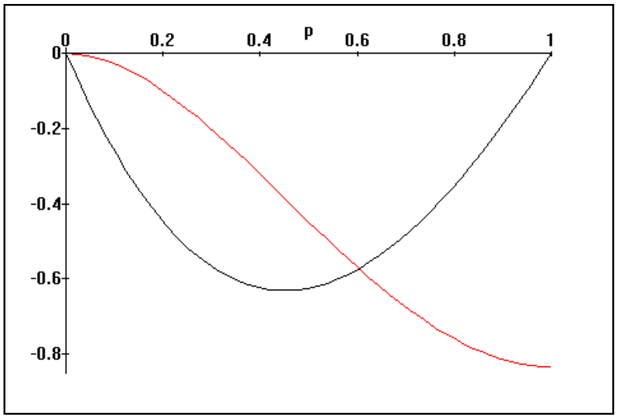


Fig.4. Replicator Dynamic and Potential for the Prisoner's Dilemma.

The replicator dynamic and the potential function for V=1, and the payoff matrix

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 5,5 & 0,7 \\ s_2 & 7,0 & 3,3 \end{array} \qquad (55)$$

are shown in Fig.4.

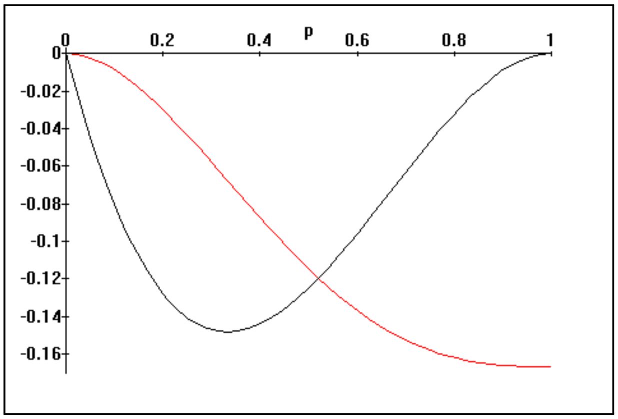


Fig.5. Knife-edge Prisoner's Dilemma

The replicator dynamic and the potential function for V=1, and the payoff matrix

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 3,3 & 0,4 \\ s_2 & 4,0 & 1,1 \end{array} \qquad (56)$$

for the knife-edge case are shown in Fig.5.

(iv) Pure Coordination

The payoff matrix for the pure coordination game is given by,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & a,a & 0,0 \\ s_2 & 0,0 & d,d \end{array} \tag{57}$$

where $a \gtreqless d$. In this case the potential function is,

$$\varphi(p) = (p^2/6V)\{(a+2d)4p - 3(a+d)p^2 - 6d\} \tag{58}$$

and so $\varphi(0) = 0$ and $\varphi(1) = (1/6V)(a-d)$, and hence if $a > d$, then s_1 is selected, and if $a < d$, then s_2 is selected.

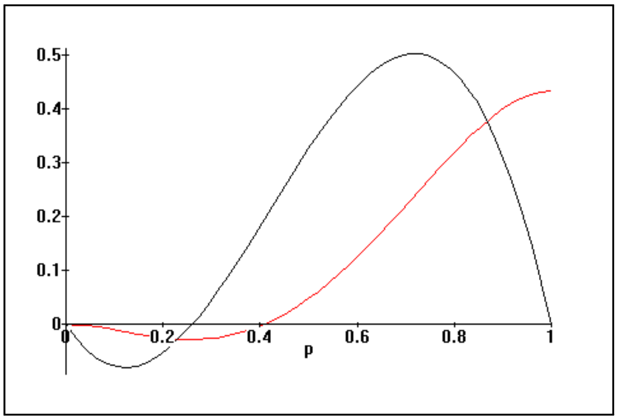


Fig.6. Replicator Dynamic and Potential for the Pure Co-ordination Game

The replicator dynamic and the potential function for V=1, and the payoff matrix

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 4,4 & 0,0 \\ s_2 & 0,0 & 2,2 \end{array} \tag{59}$$

are shown in Fig.6.

(v) Battle of the Sexes

The payoff matrix for the battle of the sexes game is given by,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 0,0 & b,c \\ s_2 & c,b & 0,0 \end{array} \tag{60}$$

where $b \lesseqgtr c$.

Case (ii) above applies, and unstable rest points of $G(p)$ are to be found at $p = 0, p = 1$; and a stable rest point at $p = \frac{b}{b+c}$.Since there exists only one stable rest point we can apply Theorem 1 and note that,

$$\lim_{V\to 0} f^*(p) = \delta(p - \frac{b}{b+c}) \tag{61}$$

i.e . the mixed strategy is selected.

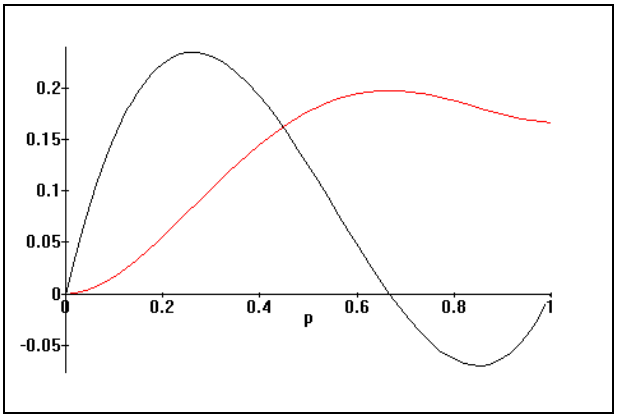


Fig.7 Replicator Dynamic and Potential for ”the Battle of the Sexes”.

The replicator dynamic and the potential function for V=1, and the payoff matrix

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 0,0 & 2,1 \\ s_2 & 1,2 & 0,0 \end{array} \tag{62}$$

are shown in Fig.7.

7. NASH EQUILIBRIA AND EQUILIBRIA OF STOCHASTIC REPLICATOR DYNAMICS

In the previous section we may note that the stationary points of the deterministic replicator dynamics are the Nash equilibria of the associated game. In the case where we have assumed a constant variance over the unit interval the set of maxima and minima of the potential function correspond to the stationary points of the deterministic replicator dynamics. By taking the ”low noise limit” of the stochastic replicator system, a point or points of the deterministic replicator, and thus a Nash equilibrium or equilibria are selected. We now turn to the case where the stochastic component is not a uniform function over the unit interval.

A number of initial questions may be posed?

(i) Are the stationary points of the deterministic evolutionary process identical to the maxima and minima of the potential function ?

(ii) What is now the relationship between the points selected by the ”low noise” limit (LNL) and the stationary points of the deterministic evolutionary process ?

(iii) Does the LNL still select one or more Nash equilibria ?

(iii) Does the nature of the stochastic term affect the Nash equilibrium that is chosen by the LNL ?

First we establish that in the case of a non-constant variance, the equivalence between rest points of the deterministic replicator system, and maxima and minima of the equilibrium distribution no longer holds. We assume that $\sigma^2(p) > 0$.over the unit interval for p.

We then have, differentiating equation (44),

$$\frac{\partial f^*(p)}{\partial p} = f^*(p)[\varphi'(p) - \frac{1}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p}] \tag{63}$$

and,

$$\frac{\partial^2 f^*(p)}{\partial p^2} = f^*(p)[\varphi'(p) - \frac{1}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p}]^2 + f^*(p)[\varphi''(p) - \frac{\partial}{\partial p}(\frac{1}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p})] \tag{64}$$

whereas from (17),

$$\varphi'(p) = 2G(p)/\sigma^2(p) \tag{65}$$

$$\varphi''(p) = [2/\sigma^2(p)][\frac{\partial G(p)}{\partial p} - \frac{G(p)}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p}] \tag{66}$$

Stationary points of $f^*(p)$ thus occur either when $f^*(p) = 0$ or when,

$$\varphi'(p) - \frac{1}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p} = 0 \tag{67}$$

i.e. subst. (65) into (67), when,

$$G(p) = (1/2)\frac{\partial \sigma^2(p)}{\partial p} \tag{68}$$

The second order condition for a maximum, $f^*(p) > 0$, requires, from (64) and (66),that,

$$[2/\sigma^2(p)][\frac{\partial G(p)}{\partial p} - \frac{G(p)}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p}] - \frac{\partial}{\partial p}[\frac{1}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p}] < 0 \tag{69}$$

Thus the maxima of $f^*(p)$ on the interval $[0, 1]$ cannot be determined solely in reference to the deterministic dynamics as reflected in $G(p)$ but also on $\sigma^2(p)$. Indeed, by an appropriate choice of $\sigma^2(p)$ the maxima of $f^*(p)$ can be shifted, a property which will be considered in Sections 6 below.

We now however have to consider the LNL. Again we assume that $\sigma^2(p)$ can be written in the form,

$$\sigma^2(p) = \beta V^2(p) \tag{70}$$

where letting $\beta \longrightarrow 0$ we take the variance uniformly to zero over the unit interval. From equation (68) we see the reappearance of the coincidence of the roots of G(p), the deterministic replicator dynamic with the maxima and minima of the potential function , provided $\beta\frac{\partial V^2(p)}{\partial p} \rightarrow 0$ as $\beta \rightarrow 0$.

We have thus established the equivalence of the Nash equilibria with the equilibria of the stochastic replicator dynamics for a positive population dependent variance. We have however not established that the equilibrium chosen is independent of this variance at the

LNL, the problem to which we now turn.

8. SWITCHING EQUILIBRIA

In this section we show that the equilibrium selected will be dependent on the functional form of the $\sigma^2(p)$ that is chosen. We give an example of such equilibrium switching for the Stag Hunt game.

We shall take as the definition of the variance the constant elasticity form,

$$\sigma^2(p) = V_0 e^{gp} \tag{71}$$

where g, V_0 are arbitrary constants, $V_0 > 0$.

From (16) the equilibrium distribution is defined by,

$$f^*(p) = \frac{C}{\sigma^2(p)} e^{\varphi(p,g)} = (C/V_0) e^{(\varphi(p,g) - gV_0 p)/V_0} \tag{72}$$

where for the stag hunt game we define,

$$\varphi(p, g) = 2((a + d) - (b + c)) \int p(1 - p)(p - p^*) e^{-gp} dp \tag{73}$$

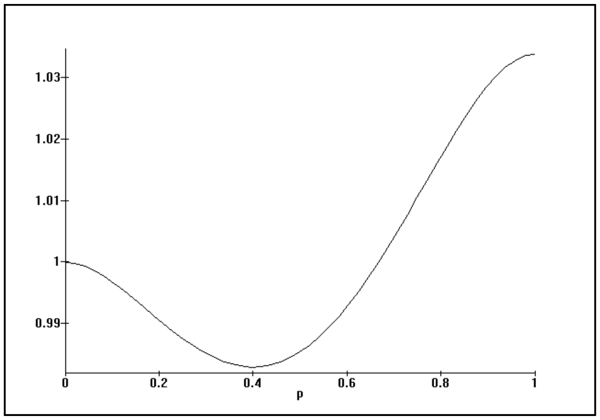
Interior rest points for a given V_0 will occur if there exist p such that $\varphi'(p, g) = gV_0$. Our particular interest however centres on the values of the potential function *after* the low noise limit has been taken, i.e. as $V_0 \to 0$. In this case the relevant values of the potential function are determined by those values of p for which $\varphi'(p, g) = 0$.

In the case where $g = 0$ then,

$$\varphi(0, 0) = 0 \text{ and } \varphi(1, 0) = (1/6V)(2\alpha - 1) \tag{74}$$

i.e. s_1 is selected if $\alpha > 1/2$ and s_2 if $\alpha < 1/2$, i.e. the risk dominant equilibrium is always selected by the distributional dominance theorem

The case of equilibrium switching can be easily demonstrated by comparing values of the potential $\varphi(p, g)$ for values of $p = 0$ and 1, for different values of g. Let us assume that $\alpha > 1/2$, so that s_1 is selected when $g = 0$; i.e. $\varphi(0, 0) < \varphi(1, 0)$. For positive g we have to compare the values of $\varphi(1, g)$ and $\varphi(0, g)$. Since $\varphi(0, g)$ remains constant as g varies but $\varphi(1, g)$ converges asymptotically to zero as g increases then there must exists some positive g such that $\varphi(1, g) < \varphi(0, g)$, i.e. for which the equilibrium s_2 is selected. How large must g in order for the switch to take effect ? This value of g depends upon the values of the payoff matrix. For example, letting $\alpha = 0.6$, a switch from s_1 to s_2 can be accomplished by letting $g \geq 2.06$.



When $g = 0$, the potential is greatest at $p = 1$, i.e. s_1 is selected as $V \rightarrow 0.$, the normalized distribution function for this case is as illustrated in Fig.8.

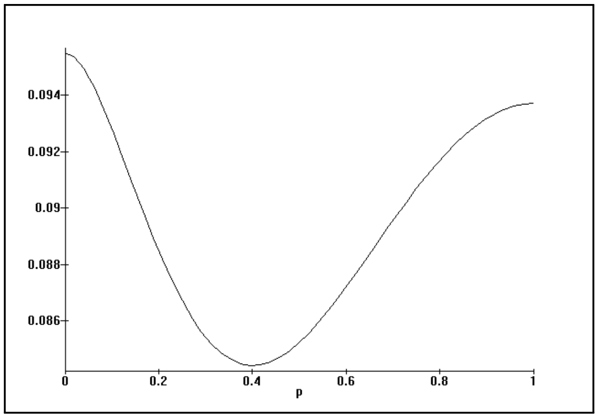


Fig.9. Normalized distribution function for the Stag-Hunt game with $\alpha = 0.6$ and $g = 2.4$.

For $g = 2.4$, the potential is greatest at $p = 0$, i.e. the strategy s_2 is selected as $V \rightarrow 0.$the normalized distribution function for this case is as illustrated in Fig.9.

The intuition is clear, the larger the variance in the largest basin of attraction compared to the smaller, the "easier" it is to get out of that basin, and hence the possibility that the largest basin is not the correct indicator for the equilibrium selection process. The importance of noise in the equilibrium selection process has been studied in the context of the Kandori, Mailath, Rob (1993) paper by Bergin and Lipman(1994) and Blume(1994)..

9. NON-ERGODICITY AND MUTATIONS

In all the models considered thus far we have assumed that the variance $\sigma^2(p)$ does not vanish on the interval $[0, 1]$. However, a number of existing models in the literature, e.g. Fudenberg and Harris(1993), Cabrales(1993) do have the property that this variance vanishes at the ends of the interval.

In this section we shall assume that the variance is represented by,

$$Vp(1-p) \tag{75}$$

where V is some positive constant. A number of possibilities exist for a solution in this case:

(i) In the first case we restrict the stochastic replicator dynamics to the interval $[0+\varepsilon, 1-\varepsilon]$, where ε is some arbitrary constant $0 \leq \varepsilon \leq 1/2$. Given appropriate boundary conditions, e.g. reflecting boundaries, then an equilibrium distribution of the following type is generated,

$$f^*(p) = \frac{C}{Vp(1-p)} e^{\varphi(p)} \tag{76}$$

where $\varphi(p)$ is the appropriate potential function determined with reference to the specification of $G(p)$ and $\sigma^2(p)$. The theory and equilibrium selection processes of the preceding sections therefore go through without alteration. Of course the drawback to such a solution is the motivation for the arbitrary imposition of the boundaries a small distance ε away from the end points of $[0, 1]$. For a discussion of such ε boundaries see Foster and Young(1990).

(ii) In the second case we have the non-ergodic solution. The distribution does converge to some limiting form, but this limit is dependent on initial conditions, i.e. the initial choice of p. In this case a solution of the form (76) does not exist. The non-ergodic solution has been discussed in this context by Fudenberg and Harris(1992). An intuitive explanation of the results can be made in terms of the deterministic dynamics, $G(p)$, allied with the term $Vp(1-p)$; assume that the deterministic dynamics are such, that dependent on the initial condition p, the world is either pushed towards 0 or 1; at it gets closer to the end points of the interval the stochastic term $Vp(1-p)$ becomes less and less significant, and thus a smaller and smaller chance of escaping from the deterministic pull. Applied to some initial distribution of worlds therefore, we have a build up of points at the ends of the unit interval.

(iii) In the third case we have the ergodic solution. The existence of a variance of type (75) need not preclude the generation of an ergodic distribution. Such a case exists if the deterministic dynamics are inward directed at the boundaries, i.e. towards some interior stable rest point. The distribution that results is given by (76), with the additional properties that,

$$\lim_{p\to 0} f^*(p) = \lim_{p\to 1} f^*(p) = 0 \tag{77}$$

However, if we propose to use these models for equilibrium selection, then (ii) and (iii) cause some problems. If (ii) applies then selection is determined by initial conditions, and stochastic replicator dynamics adds little to the deterministic solution. If (iii) applies then we only have a mixed strategy solution, and again the stochastic component appears to add little to the deterministic solution. The solution suggested in the literature is to modify the basic replicator dynamics by the addition of mutation rates between strategies. The addition of such mutations allows a change in the deterministic component, and an appropriate choice of functional form, allows the generation of an ergodic solution. The mixed strategy equilibrium is dependent on the rate of mutation between strategies, and allowing these mutations to tend to zero along with the stochastic term $\sigma^2(p)$ is an appropriate equilibrium selection process. Of course the order and relative magnitude of the limiting processes may be crucial to the selection process, and it is to these problems we now turn.

To the deterministic replicator term (4) we shall add two additional terms to reflect mutation rates as between the two strategies. We shall assume that $G(p)$ can be written as,

$$G(p) = ([a+d]-[b+c])p(1-p)(p-\frac{d-b}{(a+d)-(b+c)}) + \lambda(m_1(p)+m_2(p)) \tag{78}$$

where $\lambda > 0$ is some constant, and the mutation rates are functions which modify $G(p)$ such that,

$$\lim_{p\to 0} G(p) > 0 \text{ and } \lim_{p\to 1} G(p) < 0 \tag{79}$$

We shall assume that,

$$\lim_{p\to 0} \sigma^2(p) = 0 \text{ and } \lim_{p\to 1} \sigma^2(p) = 0 \tag{80}$$

and that the Gihman Skorohod (1972) conditions for a stable equilibrium distribution are satisfied. The above theory for the equilibrium selection process goes through unaltered. In this case there may be more than one interior stable rest point; so for given $\sigma^2(p), \varphi(p_i)$ at each of these rest points is calculated, and letting $\sigma^2(p)$ tend uniformly to zero we end up with a point mass centred on that p_i at which $\varphi(p_i)$ attains a maximum. However this particular p_i will be dependent on the mutation functions, and λ. Of particular interest is the additional selection process of letting λ tend to zero. We also have to consider the problem of the order in which the limits of $\sigma^2(p)$ and λ are taken.

As an example, consider the case which satisfies the conditions (79),

$$G(p) = Ap(1-p)(p-p^*) + \lambda(1-p) - \lambda p \tag{81}$$

where $A = (a+d)-(b+c)$ and $p^* = (d-b)/A$, and $\lambda > 0$. We take,

$$\sigma^2(p) = Vp(1-p), \qquad V > 0 \tag{82}$$

the resulting potential function which results is given by,

$$\begin{aligned}\varphi(p) &= 2\int \{[(Ap(1-p)(p-p^*) + \lambda(1-p) - \lambda p]/Vp(1-p)\}dp \\ &= (2/V)(Ap(\frac{p}{2}-p^*) + \lambda\log(p) + \lambda\log(1-p))\end{aligned} \tag{83}$$

and so the limiting distribution is,

$$\begin{aligned}f^*(p) &= (C/Vp(1-p))\exp\{(2/V)(Ap(\frac{p}{2}-p^*) + \lambda\log(p) + \lambda\log(1-p))\} \\ &= (C/V)\exp\{(2/V)(Ap(\frac{p}{2}-p^*) + (\lambda - \frac{V}{2})\log(p(1-p))\}\end{aligned} \tag{84}$$

where C is the normalization constant.

The Pontryagin theorem can now be applied by taking the limits of V andλ in the following strict order. Holding the mutation rates fixed, take the variance uniformly to zero;

the equilibria are then selected by the maxima of the function,

$$Ap(\frac{p}{2} - p^*) + \lambda \log(p(1-p)) \tag{85}$$

Then taking the mutation parameter λ to zero, we note the consequent change in the value of p that is selected. If the order of the limits were reversed, then taking λ to zero for fixed V, would result in the violation of the Gihman Skorohod (1972) conditions for a stable equilibrium distribution to exist.

As an example of the methodology, consider the case of the Stag-Hunt game, with the payoff matrix,,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 0.6, 0.6 & 0.6, 0 \\ s_2 & 0, 0.6 & 1, 1 \end{array} \tag{86}$$

Taking the Variance to zero, the function,

$$Ap(\frac{p}{2} - p^*) + \lambda \log(p(1-p)) \tag{87}$$

determines the value of p that is selected, Fig.1 provides the values for this function for $\lambda = 0.005$. As we take the mutation parameter λ closer to zero so we see that the proportion of the population playing strategy s_1 increases to unity.

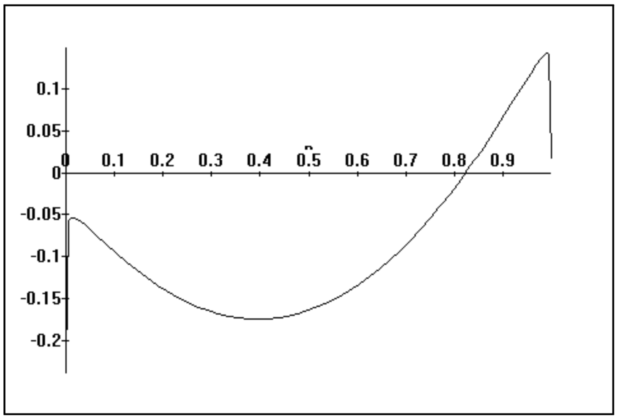


Fig.10. Mutation Potential Function for Stag-Hunt Game $\alpha = 0.6$

Changing the payoff $\alpha = 0.4$, we note the change in the potential as in Fig.11.

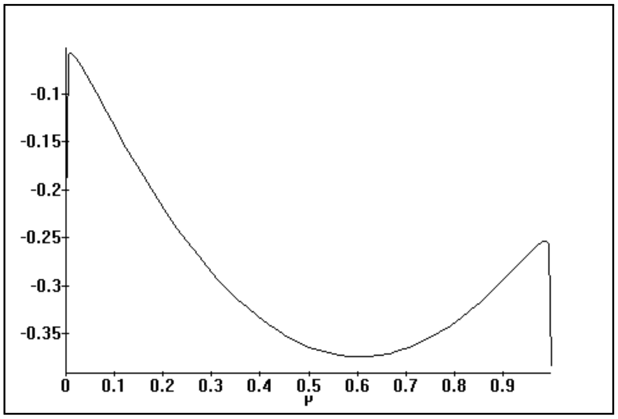


Fig.11. Mutation potential for alpha=0.4.

10. BEST RESPONSE AND REPLICATOR DYNAMICS

As was noted in Section 1, the use of the potential function and the Pontryagin low noise limit is not limited to the use of the Replicator Dynamic. However the use of a particular dynamic may be expected to strongly influence the potential function, and hence the equilibrium that is selected in the low noise limit.

We propose to compare the outcomes generated by the Replicator Dynamic and the Best Reply Dynamic, and use the Stag-Hunt game to exemplify the equilibrium selection as between the two dynamics.

The replicator equation is given by,

$$\frac{1}{p_i}\frac{dp_i}{dt} = (Ap(t))_i - p(t)'Ap(t) \tag{88}$$

where $A = [a_{ij}]$ and $p(t)' = [p_1(t), p_2(t)]$. Letting $p = p_1$ and $p_2 = 1 - p_1$, then (2) may be written as,

$$\frac{dp}{dt} = G(p, A) \tag{89}$$

where $G(p, A)$ is defined by eq.(3).

The best response dynamic is defined by,

$$\frac{1}{p_i}\frac{dp_i}{dt} = (Ap(t))_i - (Ap(t))_j \tag{90}$$

which in terms of the proportion p playing s_1 may be written as,

$$\frac{dp}{dt} = H(p, A) = \frac{G(p, A)}{(1-p)} \tag{91}$$

where $G(p, A)$ is as defined by eq.(3).

The potentual function for the replicator dynamic is, as we have seen, given by,

$$\varphi_{rep}(p) = 2\int_{p_o}^{p} \frac{G(x)}{\sigma^2(p)}dx \tag{92}$$

whereas the potential function for the best response dynamic is,

$$\varphi_{br}(p) = 2\int_{p_o}^{p} \frac{G(x)}{(\sigma^2(p))(1-p)}dx \tag{93}$$

Assuming the Gihman-Sokhorod conditions are satisfied, the equilibrium distribution is then given by eq.(16), with appropriate use of either (92) or ,(93).

A comparison of the potentials might suggest that the equilibrium selection for the Replicator and the Best response dynamic might not select the same equilibrium, and to this question we now turn using the Stag-Hunt game as an example. The replicator dynamic for the replicator stag hunt game is given by eq.(43) ,i.e.,

$$G(p) = p[1-p][p-(1-\alpha)] \tag{94}$$

and hence the potential is given by eq.( ) as,

$$\varphi_{rep}(p) = \frac{2p^2}{V}[(2-\alpha)\frac{p}{3} - \frac{p^2}{4} - \frac{(1-\alpha)}{2} \tag{95}$$

The best response dynamic for the stag-hunt game is given by,

$$H(p) = \frac{G(p)}{(1-p)} = p[p-(1-\alpha)] \tag{96}$$

and hence the potential is given by,

$$\varphi_{br}(p) = \frac{2}{V}[\frac{p^3}{3} - \frac{p^2}{2}(1-\alpha)] \tag{97}$$

For the case $\alpha = 0.4$ Fig.12 illustrates the two potentials,

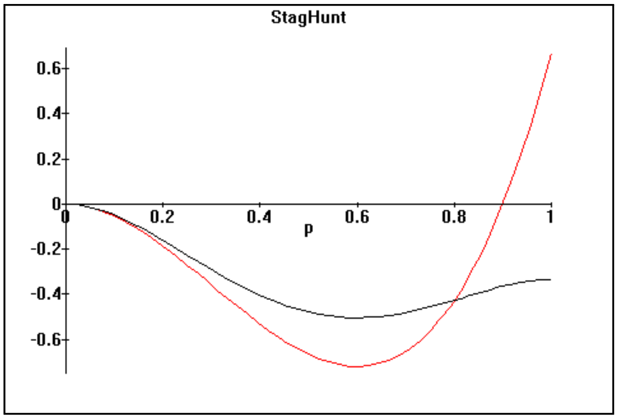


Fig.12. Stag Hunt Potentials, red BR;Black, Rep.alpha =0.4

As can be seen, equilibrium selection as between replicator and best response is not the same, replicator selects $s_2$whilst best response selects $s_{1.}$ The inadequacy of the basin of attraction approach as a method of equilibrium selection can be clearly seen by noting the replicator G(p) and best response function H(p) in Fig.13.

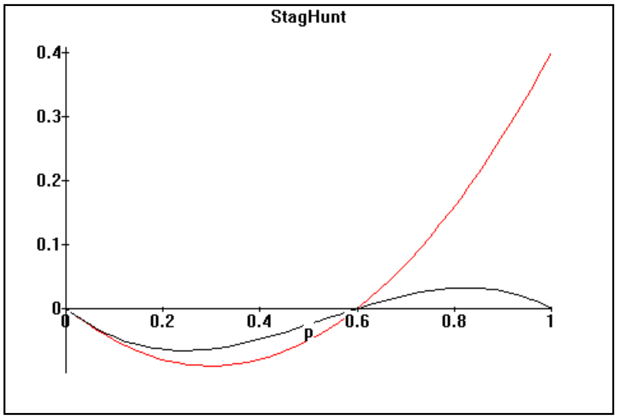


Fig.13. Replicator and Best Response G(p),H(p) functions

Both Replicator and Best Response have identical size basins of attraction for s_1 and s_2 , and thus since the basin of attraction for s_2 is greater than s_1 would predict that s_2 is selected. The potential approach predicts s_2 for the Replicator dynamic but s_1. We should note that in some instances the basin of attraction approach does not always lead to contradicitions as between BR and Replicator, thus for the case $\alpha = 0.6$.

11. TRAVERSAL TIMES

A well developed literature on the expected transition times between states exists for the diffusion system considered in the previous sections. Our main interest will centre on the expected transition time between multiple Nash equilibria (when such exist).We shall take as an example of multiple equilibria the Stag-Hunt game, and calculate the expected transition times between the Risk-Dominant and Payoff Dominant equilibria.

For the system (6), (10), the expected transition time from the equilibrium where the proportion playing strategy s_1 is zero to the equilibrium where all play strategy s_1, i.e. from $p = 0$ to $p = 1$, is given by,

$$T(0 \to 1) = \int_{p=0}^{p=1} \left\{ \int_{z=0}^{z=p} \frac{2}{V} exp(-\varphi(z))dz \right\} exp(\varphi(p))dp \tag{98}$$

whilst the expected transition time from the equilibrium $p = 1$ to $p = 0$ is given by,

$$T(1 \to 0) = \int_{p=0}^{p=1} \left\{ \int_{z=p}^{z=1} \frac{2}{V} exp(-\varphi(z))dz \right\} exp(\varphi(p))dp \tag{99}$$

The above formulae usually require numerical integration for solution. Approximations to the above equations which require less onerous computation are given by,

$$T(0 \rightarrow 1) = \left\{ \int_{p=0}^{p=p*} \frac{2}{V} exp(-\varphi(p))dp \right\} \left\{ \int_{p=0}^{p=1} exp(\varphi(p))dp \right\} \tag{100}$$

$$T(1 \rightarrow 0) = \left\{ \int_{p=p*}^{p=1} \frac{2}{V} exp(-\varphi(p))dp \right\} \left\{ \int_{p=0}^{p=1} exp(\varphi(p))dp \right\} \tag{101}$$

where p^* is the value of the unstable root of $G(p) = 0$, i.e. the boundary between the basins of attraction for the two equilibria. Further details of these approximations are given in Appendix 3.

Eq. (100) and (101) have a clear interpretation in terms of the equilibrium probability distribution (34). Subst. (34) in (100), (101), we have,

$$T(0 \rightarrow 1) = \frac{2}{V} \left\{ \int_{p=0}^{p=p*} f(p, V)dp \right\} \left\{ \int_{p=0}^{p=1} \frac{1}{f(p, V)} dp \right\} \tag{102}$$

$$T(1 \rightarrow 0) = \frac{2}{V} \left\{ \int_{p=p*}^{p=1} f(p, V)dp \right\} \left\{ \int_{p=0}^{p=1} \frac{1}{f(p, V)} dp \right\} \tag{103}$$

An example of the transition times between multiple equilibria, we note the case for the Stag-Hunt game with parameter $\alpha = 0.6$ in Table 1.

	$V = 1$	$V = 0.1$	$V = 0.01$	$V = 0.001$
$T(0 \rightarrow 1)(exact)$	0.9869177616	8.894299545	85.42426051	0.2850x10^9
$T(0 \rightarrow 1)(approx.)$	0.7929783316	7.416742	83.9263	0.2850x10^9
$T(1 \rightarrow 0)(exact)$	1.013590232	11.619484	1677.067376	0.6749x10^{23}
$T(1 \rightarrow 0)(approx.)$	1.207529680	13.09701	1678.565	0.6749x10^{23}

Table 1

Note the asymmetry in the transition times between the basins of attraction, fig 14 shows the relevant deterministic replicator potential function, and Fig.15 the potential function for the case of $V = 0.01$.

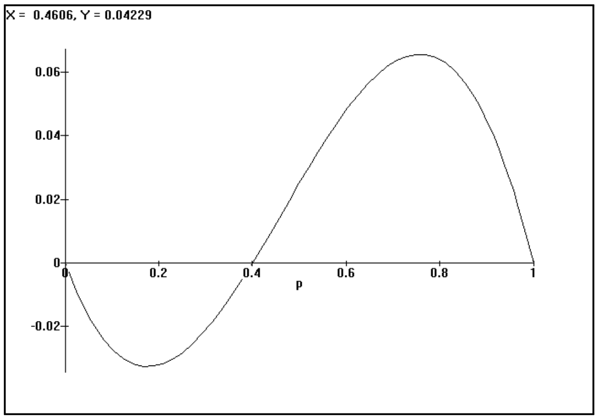


Figure 14. The deterministic replicator function for $\alpha = 0.6$

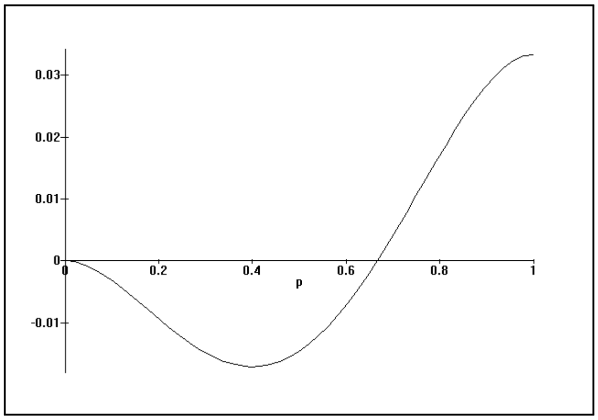


Figure 15. The potential function for the Stag-Hunt game with $\alpha = 0.6$

In addition to calculating the mean first passage time between the Nash equilibria, we may also derive the mean first passage time between any two proportions ,$p_{0,}$ $p_1$of the population playing strategy 1.

In this case,

$$T(p_0 \rightarrow p_1, p_1 > p_0) = \int_{p=p_0}^{p=p_1} \{ \int_{z=0}^{z=p} \frac{2}{V} exp(-\varphi(z))dz \} exp(\varphi(p))dp \qquad (104)$$

and in Fig 16. we show the transition time from $p = 0$ to $p = p^*$, for values of p^* from 0 to 1. $V = 0.001$ and $\alpha = 0.8$.

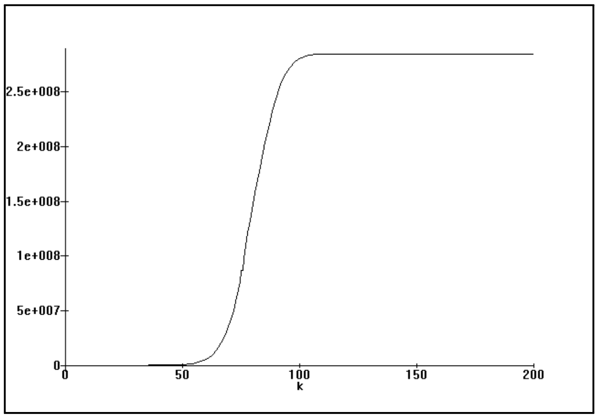


Figure 16.The transition time from $p = 0$ to p^*, for values of p^* from 0 to 1. The case of the Stag-Hunt game with $\alpha = 0.8$ and $V = 0.001$.

Appendix 1. Derivation of the Equation Governing the Evolution of the Probability Distribution Function $\mathbf{f(p, t)}$.

Let $f(p,t)$ denote the probability distribution function resulting from the stochastic dynamics implied by (4). In order to derive the equation governing the evolution of $f(p,t)$ we proceed as follows:

Let,

$$H(p,t) = \int_0^p f(p,t)dp \tag{1.1}$$

i.e. the probability mass at or below p at time t; i.e. the normal definition of the cumulative distribution function of p. We assume that within a given interval of time δt, changes in $f(p,t)$ are determined by the transition kernel, $\Gamma(p, p^*; \delta t; t)$; defining the proportion of probability mass at p^* at time t, which at the end of period δt ends up at or below p.

The equation governing the evolution of $H(p,t)$ is then given by,

$$H(p, t+\delta t) = \int_{-\infty}^{+\infty} h(p^*,t)\Gamma(p,p^*;\delta t;t)dp^* = \int_0^1 h(p^*,t)\Gamma(p,p^*;\delta t;t)dp^* \tag{1.2}$$

Note that values of p are restricted to the unit interval; this would ordinarily impose restrictions on the transition kernel $\Gamma(p,p^*;\delta t;t)$; e.g. $\Gamma(p,p^*;\delta t;t) = 0$, for p outside the unit interval. However, in keeping with the FPE literature, restrictions are imposed not on the $\Gamma(.)$ directly, but in terms of the boundary conditions defining the solution. Note further, that $p \gtreqless p^*$, i.e. that the transition kernel may induce zero, positive or negative jumps.

Equation (1.2) is the well known Chapman-Kolmogorov equation; the Fokker-Planck or diffusion equation may be derived by taking a Taylor series expansion of (1.2) around (p,t).

Expanding $H(p,t)$ in a Taylor series about the point (p,t), then differentiating with respect to p, we arrive at,

$$\frac{\partial f(p,t)}{\partial t}\delta t = -f(p,t) + \int f(p^*,t)\gamma(p,p^*;\delta t;t)dp^* + o(\delta t) \tag{1.3}$$

where $\gamma(p,p^*;\delta t;t) = \partial\Gamma(p,p^*;\delta t;t)/\partial p$; and $o(\delta t)$ represents a series of terms in δt such that$\lim_{\delta t \to 0} o(\delta t)/\delta t = 0$.

Equivalently, (1.3) may be written in terms of the size of the jump in p; letting $q = p - p^*$, we have,

$$\frac{\partial f(p,t)}{\partial t}\delta t = -f(p,t) + \int f(p-q,t)\gamma^*(p,p-q;\delta t;t)dq + o(\delta t) \tag{1.4}$$

Expanding $h(p-q,t)$ and $\gamma^*(p,p-q;\delta t;t)$ in Taylor series about the points (p,t) and (q,p) respectively, and letting,

$$\mu^j = \int_{-\infty}^{+\infty} q^j \gamma^*(q,p;\delta t;t)dq; j = 1, 2, \ldots. \tag{1.5}$$

i.e. the j th moment of the jump function, we may show,

$$\int_{-\infty}^{+\infty} f(p-q,t)\gamma^*(q,p-q;\delta t;t)dq = f(p,t) + \sum_{j=1}^{\infty}[\frac{(-1)^j}{j!}\frac{\partial^j}{\partial p^j}(\mu^j f(p,t))] \quad (1.6)$$

Hence substituting (1.6) in (1.4) we have,

$$\frac{\partial f(p,t)}{\partial t}\delta t = \sum_{j=1}^{\infty}[\frac{(-1)^j}{j!}\frac{\partial^j}{\partial p^j}(\mu^j f(p,t))] + o(\delta t) \quad (1.7)$$

The particular case of (1.7) we consider is where we assume $\gamma^*(q,p;\delta t;t)$ to be a normal distribution in the size of the jump, q, with mean and second moment defined respectively as,

$$\mu^1 = u(p,t)\delta t \,;\, \mu^2 = \sigma^2(p,t)\delta t \quad (1.8,1.9)$$

Since $\gamma^*(q,p;\delta t;t)$ is assumed normal, all odd moments higher than the first are zero; whilst for the even moments we have,

$$\mu^{2j} = \frac{(2j)!}{j!2^j}(\mu^2)^j; \quad (1.10)$$

$j = 1, 2, ...$

Hence all moments of $\gamma^*(q,p;\delta t;t)$ higher than the second are either identically zero, or of order $o(\delta t)$. Dividing through both sides of (1.7) by δt, and letting $\delta t \to 0$, we arrive at the Fokker-Planck equation (6) with appropriate definitions for the mean and variance.

APPENDIX 2

Theorem 1. *Distributional Dominance* .

If $f^*(p)$ as defined by (16) exists and if $\varphi(p)$, as defined by (17) attains a unique maximum at p^* in the closed interval $[0,1]$ then,

$$\lim_{\sigma^2(p)\to 0} f^*(p) = \delta(p-p^*) \tag{2.1}$$

where δ is the Dirac delta function, i.e. $\delta(p-p^*) = 1$ if $p = p^*$, and $\delta(p-p^*) = 0$ if $p \neq p^*$.

Proof

We have in equilibrium,

$$G(p)f(p,t) = \frac{1}{2}\frac{\partial}{\partial p}(\sigma^2(p)f(p,t)) \tag{2.2}$$

and thus the equilibrium distribution,

$$f^*(p) = \frac{C}{\sigma^2(p)}e^{\varphi(p)} \tag{2.3}$$

where C is the normalization constant, and,

$$\varphi(p) = 2\int_0^p \frac{G(x)}{\sigma^2(x)}dx \tag{2.4}$$

What happens to $f^*(p)$ as $\sigma^2(p)$ decreases uniformly ? Let $\sigma^2(p) = \beta V^2(p)$ where β is a parameter which we shall allow to tend to zero. Then,

$$f^*(p) = \frac{C}{\beta V^2(p)}e^{\psi(p)/\beta} \tag{2.5}$$

where,

$$\psi(p) = 2\int_0^p \frac{G(x)}{V^2(x)}dx \tag{2.6}$$

We also have the normalization condition,

$$\int_{-\infty}^{+\infty} f^*(x)dx = 1 \tag{2.7}$$

Thus subst. (2.5) in (2.7) we have,

$$\frac{\beta}{C} = \int_{-\infty}^{+\infty} \frac{1}{V^2(x)}e^{\psi(x)/\beta}dx \tag{2.8}$$

and so,

$$f^*(p) = e^{\psi(p)/\beta}/V^2(p)\int_{-\infty}^{+\infty} \frac{1}{V^2(x)}e^{\psi(x)/\beta}dx \tag{2.9}$$

Let $\psi(p)$ attain a unique maximum at p^*; and let $\psi(p^*) = J$; then we can define $\psi^*(p) = \psi(p) - J$, and note that,

$$f^*(p) = e^{\psi(p)/\beta}/V^2(p)\int_{-\infty}^{+\infty} \frac{1}{V^2(x)}e^{\psi(x)/\beta}dx$$

$$= e^{(\psi(p)-J)/\beta}/V^2(p)\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{(\psi(x)-J)/\beta}dx$$

$$= e^{\psi^*(p)/\beta}/V^2(p)\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{(\psi^*(x))/\beta}dx \tag{2.10}$$

and we thus ensure that at $p^*, \psi^*(p^*) = 0$. In order to continue we require *Lemma 1* ;

Lemma 1

$$\lim_{\beta\to 0}\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{(\psi^*(x))/\beta}dx = B(\beta) \tag{2.11}$$

where $\beta^{1/n}A(k''/M^2 \le B(\beta) \le \beta^{1/n}A(k')/M^2, M$ = Min $V^2(p)$ and $A(k''), A(k')$ are constants independent of β.

Proof

$$\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx = \int_{p^*-h}^{p^*+h}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx + \int_{-\infty}^{p^*-h}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx + \int_{p^*+h}^{+\infty}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx \tag{2.12}$$

where h is some small positive constant. The maximum of $\psi^*(p)$ occurs at p^* when $\psi^*(p^*) = 0$, thus the first integral on the R.H.S. of (2.12) contains a term which is independent of β. Outside the interval $[p^* + h, p^* - h]$ we have $\psi^*(p) < 0$, and thus each term in the second and third integrals on the R.H.S. of (2.12) tend to zero as β tends to zero; thus,

$$\lim_{\beta\to 0}\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx = \int_{p^*-h}^{p^*+h}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx \tag{2.13}$$

Now consider the function $\psi^*(p)$ in the interval $[p^* - h, p^* + h]$; the function is such that we can choose positive integers k', k”, and an even integer n, such that,

$$-k'(p-p^*)^n \ge \psi(p) \ge -k''(p-p^*)^n \tag{2.14}$$

where $k'' > k' > 0$. Thus,

$$\frac{1}{V^2(p)}e^{-k'(p-p^*)^n/\beta} \ge \frac{1}{V^2(p)}e^{\psi^*(p)/\beta} \ge \frac{1}{V^2(p)}e^{-k''(p-p^*)^n/\beta} \tag{2.15}$$

Now,

$$\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{-k'(p-p^*)^n/\beta}dx = \int_{p^*-h}^{p^*+h}\frac{1}{V^2(x)}e^{-k'(p-p^*)^n/\beta}dx + \int_{-\infty}^{p^*-h}\frac{1}{V^2(x)}e^{-k'(p-p^*)^n/\beta}dx + \int_{p^*+h}^{+\infty}\frac{1}{V^2(x)}e^{-k'(p-p^*)^n/\beta}dx \tag{2.16}$$

The last two integrals on the R.H.S. of (2.16) tend to zero as β tends to zero, and therefore,

$$\lim_{\beta\to 0}\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{-k'(p-p^*)^n/\beta}dx = \int_{p^*-h}^{p^*+h}\frac{1}{V^2(x)}e^{-k'(p-p^*)^n/\beta}dx \tag{2.17}$$

Let $M^2 = \text{Min } V^2(p) > 0$; then,

$$\int_{-\infty}^{+\infty} \frac{1}{V^2(x)} e^{-k'(p-p^*)^n/\beta} dx \leq \frac{1}{M^2} \int_{-\infty}^{+\infty} e^{-k'(p-p^*)^n/\beta} dx \tag{2.18}$$

Letting $z = (x - p^*)/\beta^{1/n}$ then,

$$\int_{-\infty}^{+\infty} e^{-k'(x-p^*)^n/\beta} dx = \beta^{1/n} \int_{-\infty}^{+\infty} e^{-k'z^n} dz = \beta^{1/n} A(k') \tag{2.19}$$

and so,

$$\begin{aligned} \int_{p^*-h}^{p^*+h} \frac{1}{V^2(x)} e^{-k'(p-p^*)^n/\beta} dx &= \int_{-\infty}^{+\infty} \frac{1}{V^2(x)} e^{-k'(p-p^*)^n/\beta} dx \\ &\leq \beta^{1/n} A(k')/M^2 \end{aligned} \tag{2.20}$$

We may similarly show that,

$$\int_{p^*-h}^{p^*+h} \frac{1}{V^2(x)} e^{-k'(p-p^*)^n/\beta} dx \geq \beta^{1/n} A(k'')/M^2 \tag{2.21}$$

and so,

$$\lim_{\beta \to 0} \int_{-\infty}^{+\infty} \frac{1}{V^2(x)} e^{\psi^*(x)/\beta} dx = B(\beta) \tag{2.22}$$

where $\beta^{1/n} A(k'')/M^2 \leq B(\beta) \leq \beta^{1/n} A(k')/M^2, M = \text{Min } V^2(p)$ and $A(k''), A(k')$ are constants independent of β. Thus,

$$f^*(p) = e^{\psi^*(p)/\beta} / V^2(p) B(\beta) \tag{2.23}$$

will tend to zero as $\beta \to 0$ for every value of p except $p = p^*$, when it will become infinitely large. From the normalization condition $\int f^*(p) = 1$; therefore we characterise,

$$\lim_{\sigma^2(p) \to 0} f^*(p) = \delta(p - p^*) \tag{2.24}$$

Appendix 3. Traversal Time

Let us define $T(p)$ as the expectation of the exiting time, the time needed for a world represented by some initial proportion p in the interval $[a, b]$, to leave the interval through one of its endpoints. It is well known that the equation governing $T(p)$ is,

$$-\frac{1}{2} b(p) \frac{\partial^2 T(p)}{\partial p^2} + G(p) \frac{\partial T(p)}{\partial p} + 1 = 0 \tag{3.1}$$

In the case where p must lie in the unit interval, the boundary conditions for the case of movement from a to b where $0 \leq a < b \leq 1$, are

$$\frac{\partial T(p)}{\partial p} = 0 \ for \ p = 0 \text{ and } T(p) = 0 \text{ at } b \tag{3.2}$$

and the boundary conditions for the case of movement from b to a are,

$$\frac{\partial T(p)}{\partial p} = 0 \, for \; p = 1 \text{ and } T(p) = 0 \text{ at } a \tag{3.3}$$

For the system (**??**), (105), the expected transition time from the equilibrium $p = a$ to $p = b$, is given by,

$$T(a \to b, b > a) = \int_{p=a}^{p=b} \{ \int_{z=0}^{z=p} \frac{2}{V} exp(-\varphi(z))dz \} exp(\varphi(p))dp \tag{3.4}$$

whilst the expected transition time from the equilibrium $p = b$ to $p = a$ is given by,

$$T(b \to a, b > a) = \int_{p=a}^{p=b} \{ \int_{z=p}^{z=1} \frac{2}{V} exp(-\varphi(z))dz \} exp(\varphi(p))dp \tag{3.5}$$

In the Stag-Hunt game we are interested in the transition times between the risk dominant and payoff dominant equilibria. The traversal time from 0 to 1, is given by,

$$T(0,1) = \int_{y=0}^{y=1} \{ \int_{z=0}^{z=y} \frac{2}{V} exp(-\varphi(z))dz \} exp(\varphi(y))dy \tag{3.6}$$

whilst the traversal time from 1 to 0, is given by,

$$T(1,0) = \int_{y=0}^{y=1} \{ \int_{z=y}^{z=1} \frac{2}{V} exp(-\varphi(z))dz \} exp(\varphi(y))dy \tag{3.7}$$

The calculation time for such integrals may be quite lengthy, we therefore propose to facilitate the calculation by using a well known approximation for small V.

For constant V, we note that,

$$\varphi(.) = (2/V)(\int_0^p G(x)dx = (2/V)U(p) \tag{3.8}$$

where,

$$U(p) = \int_0^p G(x)dx \tag{3.9}$$

Hence,

$$T(0,1) = \int_{y=0}^{y=1} \{ \int_{z=0}^{z=1} \frac{2}{V} exp(-2U(z)/V)dz \} exp(2U(y)/V)dy \tag{3.10}$$

Now consider Fig.1, in which the central stationary point is indicated at $k2$. If this central minimum of $-2U(p)$ is large and V is small, then $exp(-2U(z)/V)$ is sharply peaked at $p = k2$, while $exp(2U(z)/V)$ is very small near $x = k2$. Therefore,

$$\int_{z=0}^{z=y} \frac{2}{V} exp(-2U(z)/V)dz \tag{3.11}$$

is a very slowly varying function of y near $y = k2$. This means that the value of the integral will be approximately constant for those values of y which yield a value of $exp(U(y)/V)$ which is significantly different from zero. Hence in the inner integral we can set $y = k2$ and remove the resulting constant factor from inside the integral with respect to y. Thus we can approximate (105) by,

$$T(0,1)=\{C\int_{z=0}^{z=k2}\frac{2}{V}exp(-2U(z)/V)dz\}(1/C)\int_{y=0}^{y=1}exp(2U(y)/V)dy \qquad (3.12)$$

where C is the normalizing constant of the distribution. From the definition of the equilibrium distribution we can see that the first term in the integral (105),

$$C\int_{z=0}^{z=k2}\frac{2}{V}exp(-2U(z)/V)dz \qquad (3.13)$$

therefore defines the long run probability distribution to the left of the stationary $k2$.

Equivalently we can define the approximation to (105), as,

$$T(1,0)=\{C\int_{z=k2}^{z=0}\frac{2}{V}exp(-2U(z)/V)dz\}(1/C)\int_{y=0}^{y=1}exp(2U(y)/V)dy \qquad (3.14)$$

Since the latter integral is a common factor in both (105) and (105), we can see that the length of the traversal time varies directly with the mass of the probability distribution to the right and left of the point $k2$. Intuitively therefore, the greater the probability mass, the greater the deterministic forces implied with respect to that basin of attraction, and hence the greater the time taken to traverse from the attractor in that basin. Of course as the variance approaches zero, so the expected traversal time becomes infinite.

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