

Entanglement Routers Using Macroscopic Singlets

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We propose a mechanism where high entanglement between very distant boundary spins is generated by suddenly connecting two long Kondo spin chains. We show that this procedure provides an efficient way to route entanglement between multiple distant sites. We observe that the key features of the entanglement dynamics of the composite spin chain are well described by a simple model of two singlets, each formed by two spins. The proposed routing mechanism is a footprint of the emergence of a Kondo cloud in a Kondo system and can be engineered and observed in varied physical settings.

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Introduction.—A high entanglement between two well separated qubits is the central resource for quantum communication tasks. It also facilitates the preparation of multiparticle entangled states [1] for measurement based quantum computation. One could ask whether many-body systems can serve as mediums for entanglement between arbitrary distant qubits in a multisite network. Though this is the most important question from an “applied” perspective, the thriving field of entanglement in many-body systems [2] remains focused on the entanglement of blocks and proximal spins. In fact, long-range entanglement between individual spins is notoriously uncommon [3]. There are proposals exploiting weak couplings of distant spins to a spin chain [4,5], but these have limited thermal stability or a very long time scale of entanglement generation. Alternatively, a global quench [6] or specific time-dependent couplings [7] may generate entanglement, though this decays with the system size. Finally, there is a proposal [8] for distance-independent entanglement through a local quench which, however, lacks the versatility of routing entanglement between multiple sites. A few quantum routers have been recently proposed [9], but harnessing a canonical many-body phenomenon for routing still is an open question.

Kondo systems [10–12] are very distinctive in the context of entanglement for at least two reasons. Despite being “gapless,” they support the emergence of a length scale ξ —the so-called Kondo screening length [10,11]—which can be tuned by varying only one parameter [10] and reflects in the entanglement [8], making it markedly different from other conventional gapless models. Furthermore, in Kondo systems, the impurity spin is maximally entangled [12] with a block of spins whose spatial extent may be varied at will by tuning ξ .

In this Letter, we propose a dynamical mechanism by which long-range distance-independent entanglement may be generated by the switch on of a single coupling suddenly connecting two *macroscopic singlets*. We show that this mechanism provides an efficient way to route entanglement

between various distant parties. By a macroscopic singlet we mean an arbitrarily long spin chain which has been engineered to behave as a Kondo system of pertinent ξ and thereby as a two-spin singlet. Indeed we show that the key features of our mechanism are remarkably well described by a four-spin system made of two singlets.

Simple example.—Let us first consider two spin singlets each formed by only two spins interacting with a Heisenberg interaction of strength J'_1 and J'_2 , respectively. The ground state of the composite system is then given by $|gs\rangle = |\psi^-\rangle \otimes |\psi^-\rangle$ with $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. One may generate high entanglement between the boundary spins 1 and 4, by merely turning on an interaction J_m between the spins 2 and 3. After quenching, the evolution of the system is ruled by the Hamiltonian $H = J'_1 \vec{\sigma}_1 \cdot \vec{\sigma}_2 + J'_2 \vec{\sigma}_3 \cdot \vec{\sigma}_4 + J_m \vec{\sigma}_2 \cdot \vec{\sigma}_3$ and, since the initial state is a global singlet, time evolution allows for a nonzero overlap only with the singlet subspace of the spectrum of H , i.e.,

$$|\psi(t)\rangle = e^{-iE_{S_1}t}|S_1\rangle\langle S_1|gs\rangle + e^{-iE_{S_2}t}|S_2\rangle\langle S_2|gs\rangle, \quad (1)$$

where, $|S_1\rangle$ and $|S_2\rangle$ are two singlet eigenvectors of H with energy $E_{S_1} = -4J_m$ and $E_{S_2} = 0$. In order to get maximal entanglement between the boundary spins 1 and 4—after a certain time t^* —one has to choose $J_m = J'_1 + J'_2$. Once this condition is satisfied the state of the system at time t , up to a global phase, is given by

$$|\psi(t)\rangle = \frac{-i \sin(2J_m t)}{2} (|0011\rangle + |1100\rangle) - \frac{\cos(2J_m t)}{2} \times (|1001\rangle + |0110\rangle) + \frac{e^{i2J_m t}}{2} (|0101\rangle + |1010\rangle).$$

Surprisingly, $|\psi(t)\rangle$ depends only on J_m and, by tracing out the spins 2 and 3, one gets the density matrix $\rho_{14}(t)$ from which the concurrence [13] between spins 1 and 4 is computed as

TABLE I. Typical values of J'_k to generate Kondo clouds of size ξ_k as given in Eq. (5).

N_k	J'_k
4	0.300
6	0.280
8	0.260
10	0.250
12	0.240
14	0.230
16	0.220
18	0.215
20	0.210
22	0.205
24	0.202
26	0.198
28	0.195
30	0.190
32	0.187
34	0.184
36	0.180
38	0.175

$$E = \max\left\{0, \frac{1 - 3 \cos(4J_m t)}{4}\right\}. \quad (2)$$

Equation (3) shows that E oscillates with a period of $\frac{\pi}{2J_m}$ and that, at time $t^* = \frac{\pi}{4J_m}$, the spins 1 and 4 form a singlet. In this simple setting one sees that (i) the dynamics is determined only by two singlet eigenvectors of H , (ii) that maximal entanglement is achieved only when $J_m = J'_1 + J'_2$, and (iii) the dynamics is oscillatory with period $2t^*$, which is only a function of J_m and, thus, does not depend on J'_1 and J'_2 separately.

Many-body systems.—We now show that the above simple dynamics and the resulting high entanglement between the boundary spins, may be reproduced even with many-body systems—for *arbitrary length scales*—using pertinent spin chains. We consider two Kondo spin chains [10] in the Kondo regime, i.e., two chains of lengths N_k described by

$$H_k = J'_k(J_1 \vec{\sigma}_1^k \cdot \vec{\sigma}_2^k + J_2 \vec{\sigma}_1^k \cdot \vec{\sigma}_3^k) + J_1 \sum_{i=2}^{N_k} \vec{\sigma}_i^k \cdot \vec{\sigma}_{i+1}^k + J_2 \sum_{i=2}^{N_k-2} \vec{\sigma}_i^k \cdot \vec{\sigma}_{i+2}^k, \quad k = R, L \quad (3)$$

where, J_1 and J_2 are nearest and next-to-nearest neighbor couplings, $k = R$ ($k = L$) labels the right (left) chain, $\vec{\sigma}_i^k$ is the vector of three Pauli operators at site i for the chain k and J'_R (J'_L) is the right (left) impurity coupling. A Kondo spin chain supports a crossover from a gapless Kondo regime for $J_2 < J_2^c = 0.2412J_1$ to a gapped dimerized regime for $J_2 > J_2^c$. In the Kondo regime the Kondo length is uniquely determined by the impurity coupling [10,12] and, for large chains, the explicit dependence is given by

$\xi_k = e^{\alpha/\sqrt{J'_k}}$, where α is a constant; ξ_k sets the size of a block of spins forming a singlet with the impurity [12]. In the following, we shall fix the value of J'_R and J'_L so that

$$\xi_k = N_k - 1, \quad k = R, L. \quad (4)$$

We report in Table I the values of the impurity couplings—determined for chains of arbitrary lengths in Ref. [12]—as N_k is increased. Equation (5) allows us to build two macroscopic singlets (i.e., extended over a distance ξ_k tuned by J'_k). The composite spin system is depicted in Fig. 1(a); the two impurities are the boundary spins of the composite system while, due to Eq. (5), the two Kondo clouds are tuned to take over each chain separately. Note that not only is this $J'_k \sim 1/\log^2 N_k$ much stronger than the weak couplings in Refs. [4,5], but also the chain is *gapless*, so it cannot lead to perturbative end-to-end effective Hamiltonians.

Initially, the two chains are separated and initialized in their ground states [see Fig. 1(a)] and the initial state of the composite chain is given by $|\psi(0)\rangle = \prod_{k=R,L} |GS_k\rangle$ where $|GS_k\rangle$ is the ground state of the chain k . Then, we switch on $H_I = J_m(J_1 \vec{\sigma}_{N_L}^L \cdot \vec{\sigma}_{N_R}^R + J_2 \vec{\sigma}_{N_L-1}^L \cdot \vec{\sigma}_{N_R}^R + J_2 \vec{\sigma}_{N_L}^L \cdot \vec{\sigma}_{N_R-1}^R)$ between the two chains [see Fig. 1(b)]. The Hamiltonian of the composite system of length $N = N_L + N_R$ is given by $H = H_L + H_R + H_I$. Now the ground state evolves according to $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$. From knowing $|\psi(t)\rangle$ one obtains the reduced density matrix of the boundary spins at a generic time t by tracing out all other spins from the state $|\psi(t)\rangle$ and evaluate the concurrence $E(t, J_m)$ between the boundary spins. The dynamics is now not analytically solvable and one has to resort to numerical simulations which, for $N > 20$, use the time-dependent density matrix renormalization group introduced in [14] while, for $N < 20$, one may use exact diagonalization.

If the composite system should reproduce the remarkable features of the simple example discussed above one should expect that $E(t, J_m)$ oscillates with a period depending only on J_m and that maximal entanglement is achieved provided that

$$J_m = \Phi(N)(J'_L + J'_R). \quad (5)$$

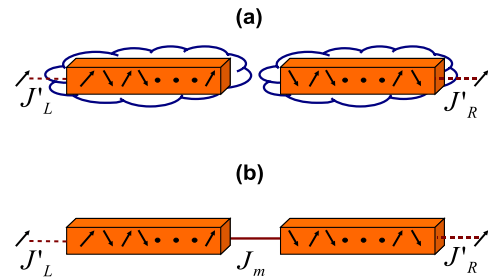


FIG. 1 (color online). (a) The composite system made of two separate Kondo spin chains initialized in their ground states in the Kondo regime. Equation (5) is satisfied by tuning J'_k . (b) To induce dynamics, one switches on the interaction between the two chains by the amount J_m .

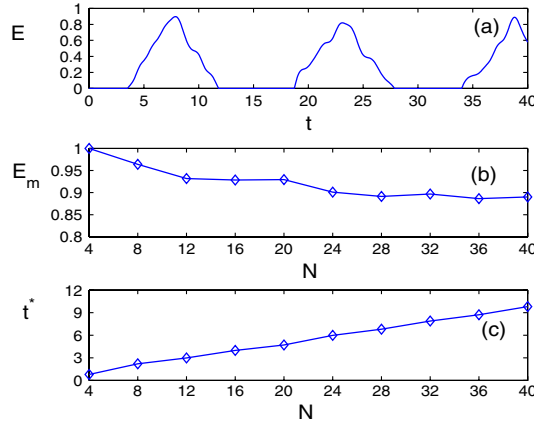


FIG. 2 (color online). (a) The entanglement dynamics vs time t ($J_2 = 0$) for a composite system of $N = 32$ when $N_L = N_R$. (b) The maximal entanglement E_m vs N . (c) t^* vs N .

$\Phi(N)$ accounts for the extended size of the Kondo singlets. Of course, for our dynamics to make sense at all one has to require that, as $N \rightarrow \infty$, J_m should take a nonzero and finite limiting value J (otherwise, one injects either zero or infinite energy). This condition alone suffices to determine $\Phi(N)$. Indeed, if $N_L \approx N_R \rightarrow \infty$, one has

$$\Phi(N) \sim \frac{J}{\alpha^2} \log^2\left(\frac{N}{2}\right), \quad (6)$$

since, in the Kondo regime, one has that $\xi_k = e^{\alpha/\sqrt{J_k}}$.

For the time being we will consider only chains for which $N_L = N_R$. In Fig. 2(a) we plot the evolution of the entanglement as a function of time for $J_m = 0.97J_1$ when $N = 32$ and $J_2 = 0$. We see that entanglement dynamics is oscillatory with a period $2t^*$. By restricting to the first period of oscillations only, one sees that there is an optimal value of J_m for which, at time t^* , the entanglement reaches its maximum E_m . In Fig. 2(b) we plot E_m as a function of N . Despite decreasing for short chains, entanglement remains very high and becomes *distance independent* for very long chains. It is remarkable that this distance-independent value seems to be 0.9 (e.g., for chains of length $N = 40$) whereas in Ref. [8] it was merely 0.7. To complete the picture of entanglement evolution in Fig. 2(c) we plot t^* as a function of N . We see that t^* increases linearly with N with a slope that is small enough to allow for fast dynamics. The linear dependence of t^* on N implies that t^* is related to J_m by $t^* \sim N \sim \xi_k \sim e^{\alpha\sqrt{[2\Phi(N)/J_m]}}$.

In Fig. 3(a) we have plotted the optimal value of J_m as a function of N . One sees that, as N increases, J_m goes to 1, thus confirming the assumption used in the derivation of $\Phi(N)$ [see Eq. (6)]. In Fig. 3(b) we plot $\Phi(N)$ versus $\log^2(N/2)$. The linearity of the plot provides an independent numerical confirmation of the result obtained in Eq. (6).

We have also investigated the situations for which N_L is different from N_R . In Fig. 4(a) we plot E_m versus N_L/N for a composite chain of length $N = 32$. Figure 4(a)

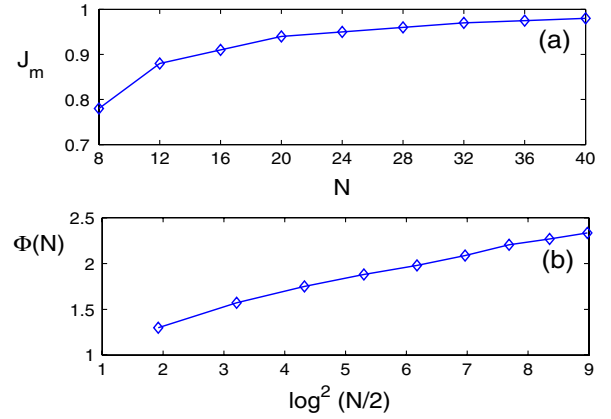


FIG. 3 (color online). (a) Optimal J_m vs N for $N_L = N_R$ in the Kondo regime. (b) The asymptotic behavior of $\Phi(N)$ vs $\log^2(N/2)$.

shows that the entanglement is maximal when $N_L = N_R$. In Fig. 4(b) we plot t^* as a function of N_L/N . Again one sees that the optimal time t^* is much shorter when $N_L \sim N_R$. Figures 4(a) and 4(b) lead us to conclude that efficient routing of entanglement is possible only if $N_L \sim N_R$.

The proposed mechanism for generating high entanglement relies heavily on Eq. (5) and, thus, on the fact that, for Kondo chains one can always tune the impurity couplings J'_L and J'_R so as to satisfy Eq. (5). As a result, entanglement generation between the boundary spins should vanish for $\xi_k < N_k/2$ as well as when the constituent Kondo chains are in the dimer regime (i.e., $J_2 > J_2^c$) where the cloud does not exist at all. We computed numerically E_m and t^* , for a chain composed of two Kondo spin chains in the dimer regime. The results are reported in Table II and compared with the results obtained for the same quantities when the two constituent chains are in the Kondo regime and Eq. (5) is satisfied. Table I shows that, as N increases, entanglement E_m (optimal time t^*) is very small (large); for instance, for $N = 40$, in the dimer regime, $E_m = 0.16$ and $t^* = 35.01$ while, in the Kondo regime, $E_m = 0.89$ and $t^* = 9.80$.

Entanglement router.—Our analysis allows us to engineer an efficient entanglement router dispatching

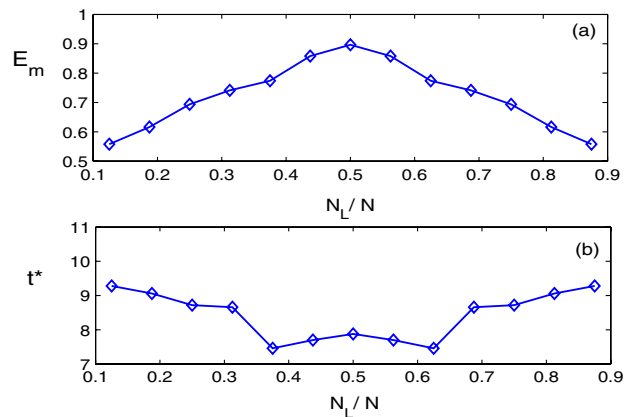


FIG. 4 (color online). Nonsymmetric case where $N = 32$ and N_L and N_R are varied: (a) E_m vs N_L/N . (b) t^* vs N_L/N .

TABLE II. Comparison between E_m and t^* for a Kondo spin chain in the Kondo ($J_2 = 0$) and dimer regimes ($J_2 = 0.42$). In the table K stands for Kondo and D for dimer.

N	8	12	16	20	24	28	32	36	40
E_m (K)	0.964	0.932	0.928	0.929	0.901	0.891	0.897	0.886	0.891
E_m (D)	0.957	0.903	0.841	0.783	0.696	0.581	0.468	0.330	0.160
t^* (K)	2.200	2.980	3.980	4.700	5.980	6.800	7.880	8.720	9.800
t^* (D)	3.780	7.290	10.32	13.41	16.89	20.43	24.51	27.12	35.01

entanglement between very distant qubits. A four-node entanglement router is sketched in Fig. 5. Each node, say A , B , C , and D , has a boundary spin whose coupling to its adjacent chain is tuned so as to generate a Kondo cloud reaching the dispatch center (Fig. 5). The dispatcher can entangle the spins of two arbitrarily chosen nodes, say A and B , by switching on a coupling J_m between the chains A and B and, thus, induce the quench dynamics previously analyzed. At $t = t^*$, the entanglement may be taken out of the boundary spins by a fast swap to any memory qubits in nodes A and B . Note that exclusive pairs of nodes, e.g., (A, B) and (C, D) , can be connected simultaneously.

Decoherence.—Spin chains with switchable or tunable couplings are realizable [15] with both superconducting qubits and spins in quantum dots. In the former, the effect of a reasonable dephasing of strength $0.005J_1$ [16] for $N = 12$ is about 10%. In the latter, a magnetic field in a random direction acts on each spin due to the dot nuclei [17]. For $N = 12$, a very strong magnetic field ($\sim 0.05J_1$) [17] decreases the entanglement by 5%.

Conclusions.—We proposed a mechanism for generating high entanglement between distant spins by switching on an appropriate interaction between two Kondo spin chains. In contrast to other recent networking schemes [9] it does not demand control of the intermediate spins or time-varying local fields. Our results hint that a Kondo spin chain satisfying Eq. (5) may be effectively described by an extended singlet formed by two spins since the key features of the entanglement dynamics can be easily understood using a simple model of a pair of two spin singlets.

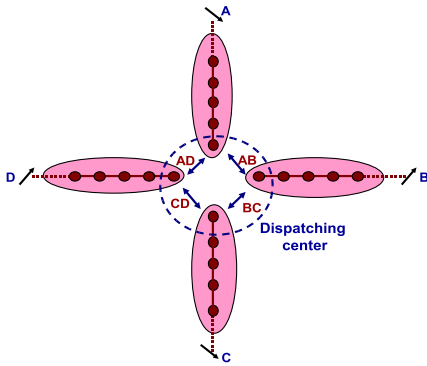


FIG. 5 (color online). A four-node router in which each user controls one boundary spin. A dispatcher connects two chains to induce dynamics in a channel composed of two spin chains.

In the absence of the Kondo cloud, entanglement is suppressed; thus, the remarkable dynamical behavior of the system is a new clear footprint of the emergence of the Kondo cloud in a Kondo system.

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