J. Korean Math. Soc. **35** (1998), No. 1, pp. 135–148

A COLLISION AVOIDANCE CONTROL PROBLEM FOR MOVING OBJECTS AND A ROBOT ARM

JUNHONG HA AND JITO VANUALAILAI

ABSTRACT. We propose the new controls constructed via the second or direct method of Liapunov to solve the collision avoidance control problems for moving objects and a robot arm in the plane. We also explicate the controlling effect by the simulations.

1. Introduction

It is well known that the *second or direct method of Liapunov* is applicable to many works related to qualitative theory of differential games and collision avoidance control strategies. We refer to [1, 2, 3, 5] as the related references.

In a collision avoidance control strategy, Stonier[4] first proposed the use of the Liapunov method for the collision avoidance between two point masses or objects moving to designated areas or targets located in the horizontal plane. Via *Liapunov-like functions*, he determined analytic forms of control laws for the planar movement of the two point objects. In his approach, he used the so-called "right-of-way assumption" which allows one object to register the position of the other as a constant in a sufficiently small time interval before making a move. However, the assumption has two shortcomings. The first is the difficulty one may have in justifying the use of the components of the position vector at time t of the system trajectory as constants in the Liapunov-like function. The second is the difficulty in the use of the assumption in a multi-point system where the assumption poses the problem of deciding which object or objects should be held in a given time interval. These problems were overcome in VNH [6] where a single Liapunov-like function for the entire system, instead of a Liapunov-like function for each point mass, is

Received June 7, 1997.
1991 Mathematics Subject Classification: 65L07, 93-04, 73B52.
Key words and phrases: Liapunov function, feedback control, equilibrium state.

constructed. The two papers (Stonier [4] and VNH [6]) however have a common drawback. The Liapunov-like functions defined in them do not satisfy the Liapunov stability condition that they should be precisely zero at stable equilibrium points of the system. The attempt to satisfy this condition in both papers saw some restrictions placed on one of the two types of parameters associated with the findpath problem.

The two parameters are the *control and convergence parameters*, and the restriction is in the requirement that the control parameters be sufficiently small so that the existence of a stable equilibrium state of the system in a *neighborhood* of the center of a target could be guaranteed. In other words, with this restriction, the objects cease motion very close to but not at the center of their targets.

In this paper, the problem in the two papers is solved once and for all by the use of a function that satisfies the sufficient conditions of Liapunov's stability theorem. This *Liapunov function* can be easily constructed for a multi-point system, and it requires the control parameters only for the purpose of controlling the direction of the trajectory.

Moreover, the proposed function need not be generalized, as was the case in VNH [6], in order to address the important collision avoidance issue of a safe and smooth trajectory.

As an application of the proposed method, we consider the problem of controlling a planar robot arm to its target.

In the next section, a control strategy for moving objects by the Liapunov technique is considered. In Section 3, the single planar robot arm is considered, followed by simulations showing the robot movement.

2. Control plan for two moving objects by Liapunov technique

Consider the autonomous nonlinear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad t > t_0 \tag{2.1}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{2.2}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be smooth enough to guarantee existence, uniqueness. By $\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0)$ denote the solution of (2.1) and (2.2) passing $(t_0, \mathbf{x}_0)$. If a constant function $\mathbf{x} \equiv \mathbf{e}$ satisfy $f(\mathbf{e}) = \mathbf{0}$, then we call it an equilibrium state of (2.1).

The direct method of Liapunov states that this equilibrium state $\mathbf{x} \equiv \mathbf{e}$ is stable if, in a neighborhood $\mathcal{D}$ of $\mathbf{e}$, there exists a real scalar function $\mathcal{L}(\mathbf{x})$ such that

(i) $\mathcal{L}(\mathbf{e}) = 0$,
(ii) $\mathcal{L}(\mathbf{x}) > 0$ for all $\mathbf{e} \not\equiv \mathbf{x} \in \mathcal{D}$,
(iii) $\left.\dfrac{d\mathcal{L}(\mathbf{x})}{dt}\right|_{(2.1)} = \dfrac{\partial \mathcal{L}(\mathbf{x})}{\partial x} \cdot \mathbf{f}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathcal{D}$,

where $\left.\frac{d\mathcal{L}(\mathbf{x})}{dt}\right|_{(2.1)}$ denotes the time derivative of $\mathcal{L}(\mathbf{x})$ along a solution $\mathbf{x}$ of (2.1). When $\mathcal{L}(\mathbf{x})$ successfully meets the above conditions, it is called the Liapunov function for system (2.1).

Now let us consider the problem where there are two moving objects, each having a fixed antitarget in a plane workspace, such that each moving object has to be controlled to reach its target without colliding with the other moving object and its fixed target. We often call this problem the *collision avoidance control problem.* Precisely, the collision avoidance control problem is the problem of controlling the movement of the i-th moving object so that it reaches the the center of the i-th target, while ensuring the i-th moving object avoids the j-th target and the j-th moving object, which can be regarded as an antitarget of i-th object. (Here, $i, j \in \{1, 2\}$.)

For $i, j = 1, 2$, we will denote by A_i the i-th moving object and by T_j the j-th target. Let us regard the centers of the moving objects A_i as the points (x_i, y_i) on the plane. When each moving object A_i moves continuously depending on $t \in \mathbb{R}^+ = [0, \infty)$, we can regard (x_i, y_i) as a continuous function of $t \geq 0$. In this paper, as studied in Stonier [4] and VNN [6], we suppose that the dynamics of two point objects $(x_i, y_i), i = 1, 2$, are described by the system of ordinary differential equations,

$$(2.3) \qquad \begin{cases} \dot{x}_i = z_i \\ \dot{z}_i = u_i \\ \dot{y}_i = w_i \\ \dot{w}_i = v_i, \quad i = 1, 2. \end{cases}$$

Here, $(z_i, w_i) = (\dot{x}_i, \dot{y}_i)$ denotes the instantaneous velocity of the i-th point object and (u_i, v_i) denotes its instantaneous acceleration. By the

Liapunov technique, $(u_i, v_i), i, j = 1, 2$, are considered as *feedback controllers* to be obtained from a Liapunov function associated with the system (2.3).

Now, let us define the target set TS_i of the i-th target T_i with center (p_{i1}, p_{i2}) and radius rp_i and the moving object set AS_j of the j-th moving object A_j with center (x_j, y_j) and length rap_j of the j-th moving object A_j as follows:

$$TS_i = \{(x,y) : (x - p_{i1})^2 + (y - p_{i2})^2 \le rp_i^2\}, \quad i = 1, 2,$$

$$AS_j = \{(x,y) : (x - x_j)^2 + (y - y_j)^2 \le rap_j^2\}, \quad j = 1, 2.$$

In order to determine the controllers for the collision avoidance control problem, we need to define the Liapunov function such that object A_i approaches its targets while avoiding its antitargets. Accordingly, let us define the following the functions for approaching the target and avoiding antitargets:

the V_i function to make moving object A_i approach target T_i;

$$V_i = \frac{1}{2}\{(x_i - p_{i1})^2 + (y_i - p_{i2})^2 + z_i^2 + w_i^2\}, \quad i, j = 1, 2,$$

the W_{ij} function to make the moving object A_i avoid target $T_j, i \neq j$;

$$W_{ij} = \frac{1}{2}\{(x_i - p_{j1})^2 + (y_i - p_{j2})^2 - rp_j^2\}, \quad i, j = 1, 2,$$

the V_{12} function to make A_1 avoid A_2;

$$V_{12} = \frac{1}{2}\{(x_1 - x_2)^2 + (y_1 - y_2)^2 - \max\{rap_1^2, rap_2^2\}\},$$

and the G_i function which denotes the distance between centers of moving object A_i and target T_i;

$$G_i = \frac{1}{2}\{(x_i - p_{i1})^2 + (y_i - p_{i2})^2\}, \quad i = 1, 2.$$

Using the above functions V_i, W_{ij}, V_{12} and G_i, we can now define a Liapunov function $\mathcal{L}$ on $\mathcal{D}(\mathcal{L}) = \{\mathbf{x} \in \mathbb{R}^8 : V_{12}(\mathbf{x}) > 0, W_{ij}(\mathbf{x}) > 0, i, j = 1, 2\}$ into $\mathbb{R}^+$ for the system (2.3) as follows:

$$\begin{aligned} \mathcal{L}(\mathbf{x}) &= V_1(x_1, y_1, z_1, w_1) + V_2(x_2, y_2, z_2, w_2) \\ &\quad + \frac{\alpha_{12} G_1(x_1, y_1)}{W_{12}(x_1, y_1)} + \frac{\alpha_{21} G_2(x_2, y_2)}{W_{21}(x_2, y_2)} + \frac{\beta G_1(x_1, y_1) G_2(x_2, y_2)}{V_{12}(x_1, y_1, x_2, y_2)}, \end{aligned} \tag{2.4}$$

where $\mathbf{x} = (x_1, z_1, y_1, w_1, x_2, z_2, y_2, w_2) \in \mathbf{R}^8$ and α_{12}, α_{21} and β are positive constants, called the *control parameters*. The roles of the numerators

G_1, G_2 and G_1G_2 in $\mathcal{L}$ are to improve the trajectories leading to the targets and to reduce the effect of W_{12}, W_{21} and V_{12} when A_1 approached to T_1 or A_2 approached to T_2. Note that these functions did not appear in [4, 6].

To determine the control pairs $(u_i, v_i), i = 1, 2$, let us differentiate $\mathcal{L}(\mathbf{x})$ with respect to t, where $\mathbf{x}$ is a solution of (2.3). Indeed, by straightforward calculations we have

$$\begin{aligned}
&\left.\frac{d\mathcal{L}(\mathbf{x})}{dt}\right|_{(2.3)} = \\
&\left[(1+\frac{\alpha_{12}}{W_{12}}+\frac{\beta G_2}{V_{12}})(x_1-p_{11})-\frac{\alpha_{12}G_1}{W_{12}^2}(x_1-p_{21})-\frac{\beta G_1G_2}{V_{12}^2}(x_1-x_2)\right]z_1\\
+&\left[(1+\frac{\alpha_{12}}{W_{12}}+\frac{\beta G_2}{V_{12}})(y_1-p_{12})-\frac{\alpha_{12}G_1}{W_{12}^2}(y_1-p_{22})-\frac{\beta G_1G_2}{V_{12}^2}(y_1-y_2)\right]w_1\\
+&\left[(1+\frac{\alpha_{21}}{W_{21}}+\frac{\beta G_1}{V_{12}})(x_2-p_{21})-\frac{\alpha_{21}G_2}{W_{21}^2}(x_2-p_{11})-\frac{\beta G_1G_2}{V_{12}^2}(x_2-x_1)\right]z_2\\
+&\left[(1+\frac{\alpha_{21}}{W_{21}}+\frac{\beta G_1}{V_{12}})(y_2-p_2c_2)-\frac{\alpha_{21}G_2}{W_{21}^2}(y_2-p_1c_2)-\frac{\beta G_1G_2}{V_{12}^2}(y_2-y_1)\right]w_2.
\end{aligned}$$

Therefore the time derivative $\frac{d\mathcal{L}(\mathbf{x})}{dt}$ along the equation (2.3) is given by

$$\left.\frac{d\mathcal{L}(\mathbf{x})}{dt}\right|_{(2.3)} = -(\gamma_1 z_1^2 + \mu_1 w_1^2 + \gamma_2 z_2^2 + \mu_2 w_2^2) \tag{2.5}$$

provided that the feedback control variables $(u_i, v_i), i = 1, 2$ are given by

$$\begin{cases}
u_1 = -A(x_1-p_{11}) + \frac{\alpha_{12}G_1}{W_{12}^2}(x_1-p_{21}) + \frac{\beta G_1G_2}{V_{12}^2}(x_1-x_2) - \gamma_1 z_1\\
v_1 = -A(y_1-p_{12}) + \frac{\alpha_{12}G_1}{W_{12}^2}(y_1-p_{22}) + \frac{\beta G_1G_2}{V_{12}^2}(y_1-y_2) - \mu_1 w_1\\
u_2 = -B(x_2-p_{21}) + \frac{\alpha_{21}G_2}{W_{21}^2}(x_2-p_{11}) + \frac{\beta G_1G_2}{V_{12}^2}(x_2-x_1) - \gamma_2 z_2\\
v_2 = -B(y_2-p_{22}) + \frac{\alpha_{21}G_2}{W_{21}^2}(y_2-p_{12}) + \frac{\beta G_1G_2}{V_{12}^2}(y_2-y_1) - \mu_2 w_2,
\end{cases} \tag{2.6}$$

where $A = 1 + \frac{\alpha_{12}}{W_{12}} + \frac{\beta G_2}{V_{12}}$ and $B = 1 + \frac{\alpha_{21}}{W_{21}} + \frac{\beta G_1}{V_{12}}$. We call $\gamma_i, \mu_i, i = 1, 2$, *convergence parameters.*

In [6], it is shown that (1) the smaller α_{12}(resp. α_{21}) is, the shorter the distance between moving object A_1(resp. A_2) and target T_2(resp. T_1) can become, (2) when β is increase, the distances between moving objects get large and (3) the smaller γ_i, μ_i are, the faster the speed of the moving objects arriving at their targets become. (See Example 2.1 for the sake of verification via simulations.)

When we adopt the control pairs $(u_i, v_i), i = 1, 2$, defined by (2.6), it is easily verified that $\mathbf{e} = (p_{11}, 0, p_{12}, 0, p_{21}, 0, p_{22}, 0)$ is an equilibrium state of (2.3).

Summarizing the above observations, we have the following theorem:

THEOREM 2.1. *The equilibrium state* $\mathbf{e}$ *of* (2.3), *where* u_i *and* v_i, $i = 1, 2$ *are as in* (2.6), *is stable if we choose the parameters* $\gamma_i, \mu_i > 0, i = 1, 2$.

Proof. It is clear that $\mathcal{L}(\mathbf{e}) = 0$ owing to the factors, G_1 and G_2, being included in the numerators of $\mathcal{L}$. Since $\gamma_i, \mu_i > 0, i = 1, 2$, $\left.\frac{d\mathcal{L}(\mathbf{x})}{dt}\right|_{(2.3)} \leq 0$ by the equality (2.5). Hence $\mathbf{e}$ is stable. □

REMARK. Since the asymptotic stability of the system (2.3) with (2.6) is not expected in general, there exists a possibility that $E = \{\mathbf{x} \in \mathbb{R}^8 : \left.\frac{d\mathcal{L}(\mathbf{x})}{dt}\right|_{(2.3)} = 0, \ \mathbf{x} \neq \mathbf{e}\}$ is not empty. Indeed, if we consider the initial conditions $y_1(0) = y_2(0) = \dot{y}_1(0) = \dot{y}_2(0) = 0$ and $p_{12} = p_{22} = 0$ (other conditions are arbitrary), then $\mathbf{x}$ with $y_1 = y_2 = \dot{y}_1 = \dot{y}_2 = 0$ is a unique solution of (2.3), because the control pairs $(u_i, v_i), i = 1, 2$, are differentiable on $\mathcal{D}(\mathcal{L})$. Hence the moving objects never can escape from the x-axis. If both the initial positions are on the $x-$axis at the opposite from their targets, then $\mathbf{e}$ is not asymptotically stable.

We shall next give some examples to explain the above Remark, and simultaneously to explain how to obtain the asymptotical stability of $\mathbf{e}$ by simulations. In many realistic cases, we can find the cases where the antitargets are in front of the moving objects. Hence we focus on these ones.

The following data will be used as the initial data for Example 2.1 and Example 2.2:

Note that in the fourth order Runge-Kutta method is used in the simulations.

Time interval : $[0, 50]$
Target centers : $(p_{11}, p_{12}) = (5, 0), \quad (p_{21}, p_{22}) = (-5, 0)$
Target/moving object radii : $rp_1 = rp_2 = 2, \quad rap_1 = rap_2 = 1$
Initial states : $(x_1, z_1, y_1, w_1) = (-10, 1, 0, 1)$,
$(x_2, z_2, y_2, w_2) = (10, -1, 0, 1)$
Control parameters : $\alpha_{12} = 1, \quad \alpha_{21} = 1, \quad \beta = 1$

Convergence parameters : $\gamma_1 = \mu_1 = \gamma_2 = \mu_2 = 5$

EXAMPLE 2.1. For the given data, we see that two moving objects can never arrive to their targets.

To verify the roles of the each parameter as mentioned in the above remark, we consider two cases:

(1) $\alpha_{12} = \alpha_{21} = 0.1, \quad 1, \quad 5;$

(2) $\beta = 0.1, \quad 1, \quad 5.$

This example shows that the smaller α_{12}(resp. α_{21}) is, the shorter the distance between moving object A_1(resp. A_2) and target T_2(resp. T_1) can become for case (1) and when β is increase, the distances between moving objects get large for case (2).

The results of simulations are given in Fig. 2.1 and Fig 2.2.

EXAMPLE 2.2. This example shows that all trajectories go to their targets when the initial data in Example 2.1 are changed slightly as follows:

(1) $\alpha_{12} = 1.5.$

(2) $\gamma_1 = 4$, which means the speed of A_1 is faster than that of A_2.

(3) $rp_1 = 1.5$, which means the sizes of targets are different.

The results of simulations appear in Fig. 2.3.

3. A single planer arm

We consider a very simple robot arm which has a translational joint and a rotational joint in the horizontal plane. This arm consists of two links made up uniforms slender rods; the revolute first link with fixed length, and the prismatic second link which carries the payload at the gripper. It is assumed that sliding motion of the second link relative to the first link is due to a linear torque (there is no rotation of the second link relative to the first). It is also assumed that the rotation of the arm is caused solely by an applied actuator torque and is parallel to the earth's surface so that gravity is not a factor. Our objective is to move the gripper from an initial position to its target in the workspace, the accepted path being smooth. For a schematic representation of the robot arm in the horizontal xy-plane, we assume that the first link has a fixed length R, the arm has length $r(t)$ at time t, the arm has angular position $\theta(t)$ at time t, the arm has mass M located at which is the center of mass, the payload of mass m is located at the gripper, the linear torque is $f_r(t)$ at time t and the actuator torque is $\tau_\theta(t)$ at time t.

Using Lagrange' equations, it is easy to show that the equations of motion of the arm are given by

$$(3.1)\qquad \begin{cases} [MR^2 + mr^2(t)]\ddot{\theta}(t) + 2mr(t)\dot{r}(t)\dot{\theta}(t) = \tau_\theta(t) \\ m\ddot{r}(t) - mr(t)\dot{\theta}^2(t) = f_r(t). \end{cases}$$

The Liapunov method requires a state-space description of the equations of motion. Accordingly, let

$$\begin{aligned} x_1 &= \text{the angular position, } \theta(t), \text{ of the arm} \\ x_2 &= \text{the angular speed, } \dot{\theta}(t), \text{ of the arm} \\ x_3 &= \text{the translational position, } r(t), \text{ of the mass } m \\ x_4 &= \text{the translational speed, } \dot{r}(t), \text{ of the mass} m \\ u &= \text{the actuator torque, } \tau_\theta(t) \\ v &= \text{the linear torque, } f_r(t). \end{aligned}$$

These yield

$$(3.2)\qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{u - 2mx_2x_3x_4}{MR^2 + m{x_3}^2} \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{v + m{x_2}^2x_3}{m}. \end{cases}$$

Let us use the notation $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ as a n-vector. Let us give three variables x_2, x_3, x_4 in (3.2) the constraints such as

$$\begin{aligned} -a < {} & x_2 < a, \quad a > 0, \\ 0 < R < {} & x_3 < \text{Rmax}, \\ -b < {} & x_4 < b, \quad b > 0. \end{aligned}$$

The constraints may be expressed as the set

$$CS = \{\mathbf{x} \in \mathbb{R}^4 : -\infty < x_1 < \infty, \quad -a < x_2 < a, \quad R < x_3 < \text{Rmax}, -b < x_4 < b\}.$$

We shall next construct a Liapunov function that determines control pair (u, v) which makes the gripper $(x, y) = (o_x + x_3 \cos x_1, o_y + x_3 \sin x_1)$ access its target (p, q) in polar coordinates and $\mathbf{x} \in CS$, where (o_x, o_y) is the center of the robot in the xy-plane. In this section, we assume that $(o_x, o_y) = (0, 0)$ without loss of generality. Now we define the functions

V, W, Z, Q, G as follows:
V is the function that makes the gripper approach the target;

$$V = \frac{1}{2}\{\alpha_1(x_1 - p)^2 + \alpha_2(x_3 - q)^2 + x_2^2 + x_4^2\},$$

W is the function that adjusts the angular speed of the gripper;

$$W = (a + x_2)(a - x_2),$$

Z is that function to controls the length of the robot arm;

$$Z = (-R + x_3)(\text{Rmax} - x_3),$$

Q is the function that controls the translational speed of the robot arm;

$$Q = (b + x_4)(b - x_4),$$

G is the function which denotes the distance between gripper and its target;

$$G = \frac{1}{2}\{(x_1 - p)^2 + (x_3 - q)^2\}.$$

Using the above functions V, W, Z, Q and G, we can now define a total Liapunov function $\mathcal{L}$ on the domain $\mathcal{D}(\mathcal{L}) = CS$ for the system (3.2) by

$$\mathcal{L}(\mathbf{x}) = V + G\left(\frac{\beta}{W} + \frac{\mu}{Z} + \frac{\eta}{Q}\right), \tag{3.3}$$

where β, μ and η are positive constants, which are control parameters. It is clear that $\mathcal{L}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in CS$.

We remark that our Liapunov function is more simpler than the Liapunov-like function considered by Stonier [4].

Now, to determine the control pair (u, v) we differentiate the Liapunov function $L(\mathbf{x})$ with respect to t, where $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is the solution of (3.2). By straightforward calculations we have

$$\begin{aligned} \left.\frac{dL(\mathbf{x})}{dt}\right|_{(3.2)} &= \left[(x_1 - p)\left(\alpha_1 + \frac{\beta}{W} + \frac{\mu}{Z} + \frac{\eta}{Q}\right) + \left(1 + \frac{\beta G}{W^2}\right)\dot{x}_2\right] x_2 \\ &+ \left[(x_3 - q)\left(\alpha_2 + \frac{\beta}{W} + \frac{\mu}{Z} + \frac{\eta}{Q}\right) - \frac{\mu G(R + R_{\max} - x_3)}{Z^2}\right. \\ &+ \left.\left(1 + \frac{\mu G}{Q^2}\right)\dot{x}_4\right] x_4. \end{aligned}$$

If we substitute $(\dot{x}_2, \dot{x}_4)$ of (3.2) into the above equality and we take the control pair (u, v) as

$$u = (MR^2 + mx_3^2)\left(1 + \frac{\beta G}{W^2}\right)^{-1}\Big[(p - x_1)A_1 - \gamma_1 x_2\Big] + 2mx_2x_3x_4 \tag{3.4}$$

and

(3.5)

$$v = m\left(1 + \frac{\mu G}{Q^2}\right)^{-1}\left[(q - x_3)A_2 + \frac{\mu G(R + R_{\max} - x_3)}{Z^2} - \gamma_2 x_4\right] - mx_2^2 x_3,$$

then we have

$$\left.\frac{dL(\mathbf{x}(t))}{dt}\right|_{(3.2)} = -\gamma_1 x_2^2 - \gamma_2 x_4^2,$$

where $A_i = \alpha_i + \dfrac{\beta}{W} + \dfrac{\mu}{Z} + \dfrac{\eta}{Q}$ and γ_1, γ_2 are positive constants. We also call γ_1 and γ_2 convergence parameters, and their roles are similar to the cases of moving objects. We note that $\mathbf{e} = (p, 0, q, 0)$ is an equilibrium state of (3.2) with (3.4) and (3.5). For α_1, α_2, it is verified in simulations that the variation of x_1(or x_3) is faster than the one of x_2(or x_1). (See Example 3.1 for verification.)

Therefore we have proved the following theorem:

THEOREM 3.1. *The equilibrium state* $\mathbf{e}$ *of the system given by*

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \dfrac{W^2}{\beta G + W^2}\Big[A(p - x_1) - \gamma_1 x_2\Big] \\ \dot{x}_3 &= x_3 \\ \dot{x}_4 &= \dfrac{Q^2}{\eta G + Q^2}\left[B(q - x_2) + \dfrac{\mu G(R + R\max - 2x_3)}{Z^2} - \gamma_2 x_4\right] \end{cases} \tag{3.6}$$

is stable if we take $\gamma_i > 0, i = 1, 2$. *Moreover* $\mathbf{e}$ *of (3.6) is asymptotically stable.*

Proof. It is easily verified that $L(\mathbf{e}) = 0$ and $\left.\frac{dL(\mathbf{x})}{dt}\right|_{(3.2)} \leq 0$. Therefore The equilibrium state $\mathbf{e}$ is stable. The asymptotic stability of $\mathbf{e}$ follows from the fact $E = \{\mathbf{x} \in \mathbb{R}^4 : \left.\frac{dL(\mathbf{x})}{dt}\right|_{(3.2)} = 0\} = \{\mathbf{e}\}$. □

REMARK. In Stonier [4], he remarked that all parameters had to be selected carefully, because the solution orbits may oscillate if the parameters were slightly changed. However, in our case, we can see in the simulations that changing the parameters does not affect the stability of solution orbits at all, because the system (3.6) is stable.

EXAMPLE 3.1. The initial settings are given as follows.

Time Interval : $[0, 60]$
RK4 Step Size : 0.01
Target Centers: $(p, q) = (\pi/2, 7)$
Length of the first link : $R = 3$
Length of the total link : $R_{\max} = 8$
Initial States : $(x_1, x_2, x_3, x_4) = (0, 0.1, 4, 0.1)$
Control Parameters : $\beta = \mu = \eta = 1$
Convergence Parameters : $\gamma_1 = \gamma_2 = 20$

In this example, we give some simulation results of the effect of parameters α_1 and α_2. Consider the following cases where (1) $\alpha_1 = 0.1,\ \ 1,\ \ 10$ and $\alpha_2 = 1$, and (2) $\alpha_2 = 0.1,\ \ 1,\ \ 10$ and $\alpha_1 = 1$.

If $\alpha_1 > \alpha_2$, Fig. 3.1 shows that the angular speed is larger than the translational speed. If $\alpha_1 < \alpha_2$, Fig. 3.2 shows that the translational speed is larger than the angular speed.

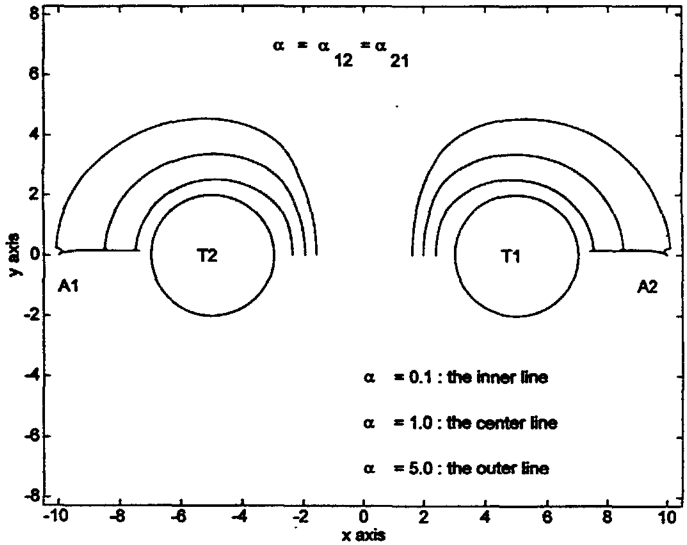

Fig. 2.1. Nature of parameters α_{12} and α_{21}

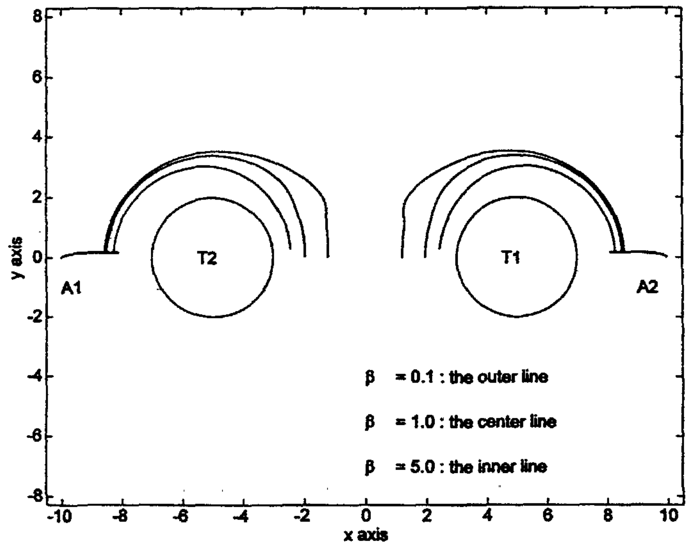

Fig. 2.2. Nature of parameter β

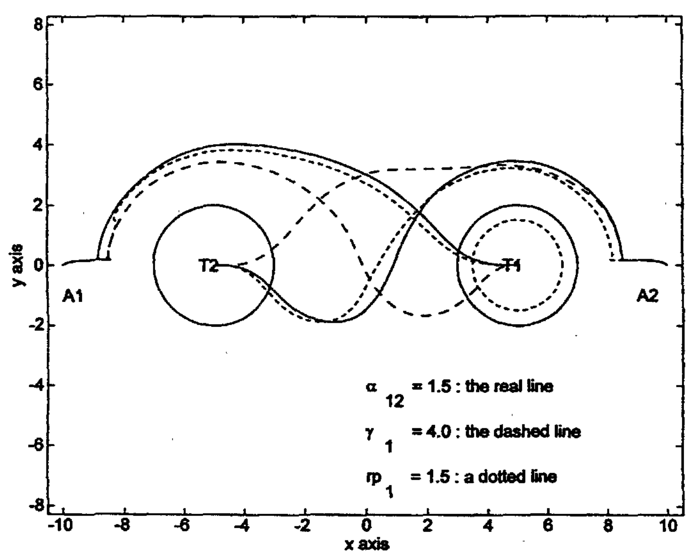

Fig. 2.3. Role of parameters α, β and radius rp

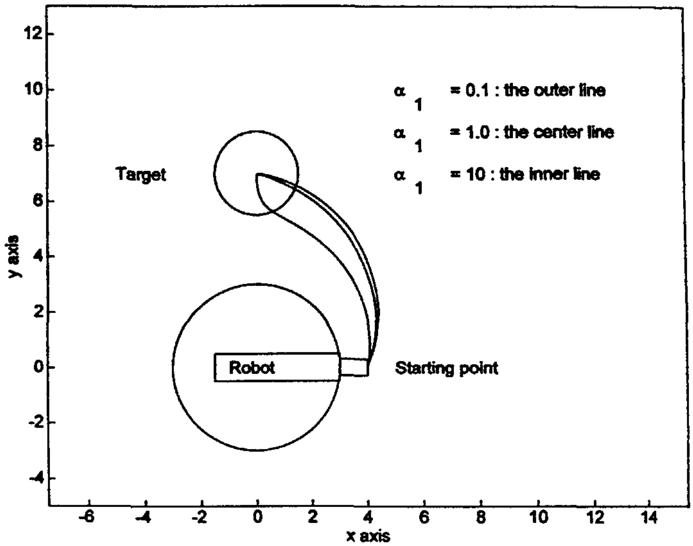

Fig. 3.1. Nature of parameters α_1

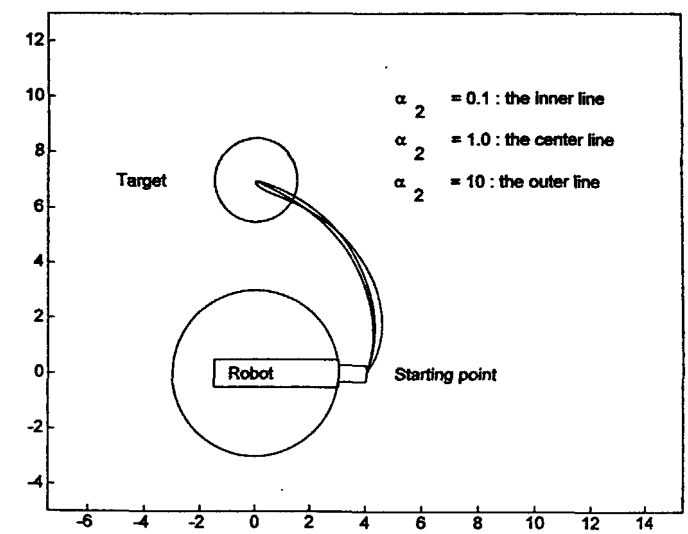

Fig. 3.2. Nature of parameters α_2

References

[1] G. Leitmann, *Guaranteed avoidance strategies*, J. of Opimization Theory and Applications **33** (1980), 569-577.

[2] J. M. Skowronski and R. J. Stonier, *The barrier in a pursuit-evasion game with two targets* Comput. Math. Appl. **13** (1987), 37-45.

[3] R. J. Stonier, *On qualitative differential games with two targets* J. of Opimization Theory and Applications **4** (1983), 587-599.

[4] ——, *Use of Liapunov techniques for collision-avoidance of robort arms* Control and Dynamic Systems **35** (1990), 185-214.

[5] T. L. Vincent and J. M. Skowronski, *Controllability with capture*, J. of Opimization Theory and Applications **29** (1979), 77-86.

[6] J. Vanualailai, Shin-ichi Nakagiri and Junhong Ha, *Liapunov function-based control functions for a collision avoidance control of two mass-points moving in a plane* Mathematics and Computers in Simulation **39** (1995), 125-141.

Junhong Ha
Basic Science Research Institute
Pusan National University
Pusan Korea
E-mail: hjh@math.ulsan.ac.kr

Jito Vanualailai
Department of Mathematics and Computing Science
University of the South Pacific
Fiji
E-mail: vanualailai@usp.ac.fj