University of Pisa

Department of Economics and Management
Master of Science in Economics

Looking for the roots of economic fluctuations

A FORMAL EXPLORATION OF CLASSIC AND KEYNESIAN ENDOGENOUS BUSINESS CYCLES

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Abstract

Can we still consider real crises and downturns as the effect of exogenous shocks? We are strongly persuaded that, in order to have a better understanding of the behaviour of the economy as a whole, the idea of capitalism as a self-sustained balanced system should be abandoned.

This work addresses two main tasks. First, it attempts to retrace the seminal contributions on endogenous business cycle theory, devoting particular attention to Keynesian and Classical/Marxian models. Second, we develop a model able to encompass, at the same time, Keynesian and Marxian drivers of fluctuations. What we obtain is an important interpretive puzzle. It emerges from the combined interaction of a *demand effect*, which resembles a rudimentary first approximation of an accelerator, and of a *hysteresis effect* in wage formation. The interesting result provided by our model is the possibility to describe the business cycle movements either by means of persistent harmonic oscillations, or of chaotic motions. These two different paths are useful in order to grasp the behaviour of the system when it is *profit-led* and when *wage-led*.

To

Cecilia, Rossana, Sara and Mattia.

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Acknowledgments

I would like to take this occasion to express all my gratitude and consideration to my mentor, Prof. Giovanni Dosi who during the last months taught me what Economic Science is and how it does work. Its critical approach in analyzing and interpreting reality routed me in a better understanding of the professional, social and political role of the Economist. His insightful hints gave me huge inspiration in elaborating my master thesis.
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Thanks to Jacopo who tolerated, understood and backed me every single day during the last months. Without his closeness these years would had not been so intense and exciting as they were.

1 Introduction

What is the most suitable representation of the economic system? Is it a constant steady growth path, or rather a cyclical path, that provides a more appropriate description of the economic mechanism? What are the sources of fluctuations and how business cycle and economic growth are influenced by income distribution? These are the main questions we are going to address in this work. The simplistic description of the economic behaviour provided by the mainstream *Real Business Cycle* theory, according to which a steady growing output trend is disturbed by exogenous supply-side shocks, is not able to take into account complexity, interdependency and endogenous evolution as drivers of economic system. The Real Business Cycle theory describes fluctuations as *exogenous equilibrium* phenomena. Moving away from the neoclassical approach, in order to give some explanations to the above questions, we are going to begin from theories that consider fluctuations as an *endogenous disequilibrium* phenomenon. We devote particular attention to theories influenced by Marxian and Keynesian approaches. Both authors have an intrinsic attempt to describe booms, stagnations and downturns as direct consequences of the nature of capitalism. Even though they find completely different sources in business cycle, both highlight the instability of the system. The idea that the economy is inherently unstable joins the theoretical approach of these two economists. Along with Marx, the Schumpeterian theory is meant to account for a process of growth systematically characterized by economic fluctuations, driven by technical change and innovation, and pushed by competition. This approach reveals a Marxian influence in describing competition as an inner struggle among capitalist firms without however abandoning the notion that the system eventually converges back to a Walrasian full-employment state. Conversely, the Keynesian approach emphasizes the disequilibrium nature of capitalistic dynamics, driven by endogenous fluctuations in demand.

A long tradition of economic theory has addressed economic phenomena as equilibrium ones. The famous article by Frisch (1933) was taken as a revolution for classical economics, since it shed some light on the dynamic properties of economic systems. Notwithstanding, the static idea is still present: neoclassical theory studies economic growth as a moving equilibrium, comparing different optimal growth paths that are due to different parameters values. Again an equilibrium approach is used, even though in a temporal framework, with attention to the law of motions of the system. A relaxation of the equilibrium approach allows for the emergence of fluctuations. These latter arise as the system, which reflects agent decisions, takes time in order to approach the equilibrium point. If a closed solution is admitted, it describes the history of variables, starting from a given initial condition. Here we stand with those who propose to go further, and explicitly study a full dynamic of the economic system. Dynamic analysis is directed at analyzing the stability, regularity and amplitude of oscillations. The study of equilibrium stability properties is a sort of byproduct. Quoting Dosi (2012):

> *Such methodological imperative (Dynamic first) demands that the explanation for why something exists, or why a variable takes the value it does, ought to rest on a process account of how it became what it is. Loosely speaking, that amounts to the theoretical imperative: provide the process story either by formally writing down some dynamical systems, or telling a good qualitative historical reconstruction (or, when possible, both). Putting it in terms of negative prescriptions: be extremely wary of any interpretation of what is observed that runs just in terms of ex-post equilibrium rationalizations ("it has to be like that, given rationality").*

Along with dynamics, nonlinearity is the other important element that characterizes some endogenous fluctuations models. The limitation of linear analysis, often used, is due to its inadequacy in representing persisting fluctuations, that are not explosive nor damped. The idea of nonlinearity stems from the awareness that economic system is a *'complex evolving system'* (Arthur et al., 1997; Kirman, 2011). In order to give a description of the behaviour of aggregate variables, nonlinear dynamical systems allow for more realistic outcomes.

The Goodwin's, Samuelson's, Kaldor's, Kalecki's models belong to the family of explicitly dynamic models. We present continuous and discrete time versions of the aforementioned contributions. Then we first elaborate a discrete time version of the Goodwin's model. Its structural instability will push us toward a generalised version, where we introduce, along with the 'classical' elements present in the class-struggle model, a demand effect. It incorporates a Keynesian perspective in explaining output growth rate. The model we provide bears some intuitions about the behaviour of a system where a Classical and a Keynesian engine are compared. In particular, we analyze what conditions determine a quasi-periodic structure and what determine the emergence of a chaotic behaviour.

2 Real business cycle theory

Before analyzing the theories that give an endogenous explanation to the business cycle, we want to briefly summarize the typical ingredients and conclusions of the neoclassical approach, that has been the dominant and leading theory in explaining business cycle sources. Basically it describes output fluctuations as the outcome of exogenous supply-side shocks. In this section we are going to describe in more details, the historical background, the main features and the criticisms of the *Real Business Cycle* theory, following the survey elaborated by Stadler (1994) and by King et al. (1999).

2.1 Historical background and main features

According to Kaldor (1940) and Lucas (1977), although every cycle is significantly different from each other, some general regularities arise (the so called *'macro stylized facts'*): relevant coherence of output variations in different economic sectors (multi-sector homogeneity in output movements), higher investment volatility than output, consumption less volatile than output, less variability of capital stock than output, pro-cyclical employment, anti-cyclical unemployment. The emergence of these stylized facts pushed economist to believe that output fluctuations do not depend on idiosyncratic shocks or on institutional factors. To explain fluctuations, given the persistence of the mentioned stylized facts, a sort of unified story could be constructed. In its seminal article, Frisch (1933) claimed stochastic shocks are the impulse mechanism responsible for variables deviation from the mean. Depending on their propagation, shocks can be more or less persistent in causing oscillations. Productivity shocks are considered the most relevant impulse mechanism. Preferences and policy shocks have been considered having a minor relevance in producing cycles. Solow (1956) exerted a big influence on tracing productivity drifts as the main source of output behaviour, becoming the reference point of the neoclassical growth theory. Using a *Cobb-Douglas* aggregate production function, therefore assuming constant return to scale and perfect market competition, the output growth rate is the sum of labour and capital growth rate (each one weighted by the factor contribution coefficient) plus the so called *Solow residual*:

$$g_y = \alpha g_k + (1 - \alpha) g_l + z \tag{1}$$

where z represents the total factor productivity, say, the exogenous productivity component that determines output growth. Later on, the Real Business Cycle (RBC) model assumes that the drift is instead an exogenous stochastic process. The dynamic evolution of the Solow residual is described by Prescott (1986) as a random walk:

$$z_t = z_{t-1} + \varepsilon_t \tag{2}$$

where ε is an independent, identically and normally distributed error. Describing the Solow's residual as a random walk implies the productivity shock to have a random behaviour which produces continuous oscillations. The *neoclassical growth theory* becomes a *theory of business cycle* as soon as the productivity shock is a random walk. The random path of the shock is responsible for the random path of output growth rate. Presuming to gain a scientific status, mainstream economic theory attempted to micro-found the RBC: aggregate output fluctuations are the result of the interactions among perfect rational maximizing agents. Being each individual perfectly rational, characterized by a quasi-linear utility function that respects the *Gorman's form*, the aggregation process becomes quite trivial (being the influence of income distribution avoided by the assumption of quasi-linear preferences): aggregate utility function is the simple sum of individual functions. The outcome of the final aggregation process is a sort of *'big individual'*, endowed with a *'big income'* that shows the average characteristics (linear combination) of other individuals. The basic assumptions used to micro-found macroeconomic models are:

i a representative agent framework in terms of consumption (households) and production (firms) units;

ii maximization of an objective function (production or utility function) under a resource constraint (available technology or available income);

iii rational expectations and market clearing conditions; no problems of asymmetric information are taken into account, at least in the initial versions;

iv propagation mechanisms for the productivity shocks that take several forms: a rise in output, being the aim of agents smoothing consumption, could result in a higher investment and capital stock; lags in investments can transfer current shocks in the future; a rise in productivity, having a positive correlation with wages, will determine an increase in wages, then, an increase in labour supply; inventories can be used by firms in order to absorb changes in demand.

2.2 Critical aspects of RBC

The basic assumptions of the model constitute also its main drawbacks: the nature of the shocks, the propagation mechanism, the possibility of accounting for recessions and finally, the representative agent framework are the points we are going to further discuss not as a strength but as a weakness of the model.

Nature of the shocks: the most widespread interpretation for the nature of the shocks regards them as common shocks. They are assumed to propagate their effects in a multi-sector, horizontal way, influencing all the production factors at the same intensity, independently from capital utilisation or labour skills. Normally, a productivity shock hits just a few sectors and not the entire economy. Multi-sector models have not been explored, firstly because they would undermine the hypothesis of representative firm. Secondly, modeling a multi-sector RBC framework is more difficult than assuming a total factor productivity effect. Allowing for a disaggregate model, where just some sectors are hit by productivity improvement, implies a higher variability at a sectoral level in order to reproduce output fluctuations. This happens because the variance of aggregate output across *n* sectors is lower than the variance within each sector. An other possible interpretation for the nature of the shocks is that they affect just the marginal efficiency of investment: only new capital goods are hit by productivity improvement. This kind of explanation could be considered more close to a Keynesian, demand-led shock. Empirical evidence shows that, even though investment in new capital goods is just 7% of GNP on average, it accounts for 20% of cyclical output volatility (Greenwood et al., 1992). Actually, investment specific shocks are very relevant in explaining output fluctuations, but the difficulty in managing models with vintage capital have been an obstacle in carrying on this research line. The typical example of productivity shock provided by RBC theorists is the oil price peak, reached during 1970s. Nonetheless, a variation in input prices cannot produce an upward or downward shift of a presumed production function, but just a movement along the existing frontier. Empirical evidence shows that energy price variations account for a range between 8% and 18% of output variations (Kim et al., 1992). The fictitious and *ad hoc* nature of productivity shocks creates ambiguity and misinterpretation in the determination of the effective sources of output fluctuations.

Propagation mechanism: the aim that RBC should satisfy is reproducing output variability. In order to obtain the serial output correlation, the model adds a random walk error, responsible for output volatility. But, without the random walk behaviour, a temporary productivity shock will produce non-autocorrelated variations. The RBC is unable to reproduce a serially correlated output with an uncorrelated shock: the output fluctuations entirely reflect the shock fluctuations, that are artificially introduced into the model. Moreover, even though RBC models produce cycles, they are quite different from the output trend registered in reality: output trend is strongly positively correlated in the short run, but it is weakly negatively correlated in the long run. Also the cyclical trend component seems to be mean-reverted: this path is completely different from the one described by a random walk.

Recession: as booms are considered the effect of positive productivity shocks, recessions are considered the effect of negative productivity shocks: this interpretation is quite economically questionable. The institutional framework (legal directive for environment protection or for labour safety) is responsible for slow output growth, changing the production possibility set (Hansen et al., 1993). Recessions can be seen as periods in which no new technology innovation occurs due to the depressive impact of bureaucracy and legislation on output. Recessions could also be considered as the effect of a mismatch between ex-ante resources allocations in investment projects and ex-post profit realisations.

Representative agent and aggregation process: RBC models are constructed under the hypothesis of perfectly rational agents. Consumption and production activities are described by means of utility and production functions. The hypothesis of perfect rationality at the individual level allows for continuous functions at the aggregate level: if each individual has a continuous function, the aggregate function must be continuous too. Notwithstanding, Kirman (2011) has strongly pointed out as a continuous aggregate demand/supply function could be the outcome of the interaction process among discontinuous individual demand functions. Moreover, the result of the representative agent is possible only if a perfect aggregation process is possible. Indeed, the aggregation process performed as a convex combination is allowed only if a very restrictive hypothesis on preferences is over-imposed: each agent should possess a preference relation characterized by a Gorman's form. Suppose the fictitious perfect aggregation has been performed, the Debreu-Mantel-Sonnenschein theorem demonstrates that, at an aggregate level, only three properties of the individual demand functions are maintained: continuity, Walras's Law, homogeneity of degree zero. Nothing guarantees the uniqueness and stability of the equilibrium point reached by the excess demand function. In addition, from the original general equilibrium theory, no information is provided regarding the adjustment market mechanism that allows to reach the equilibrium. How do agents change their demanded and supplied bundles according to price changes? The addition of the *tatonnement* mechanism tries to clarify what should be the behaviour of the aggregate excess demand function: prices of those commodities that are supplied in excess, should decrease; prices of those commodities that are demanded in excess should increase. If nothing can be said about equilibrium stability, actually there is no need to study what would be this equilibrium state. Quoting Kirman (2011):

> *Yet, as we know from the well known result of Sonnenschein, Mantel and Debreu, even with the typical rigorous restrictions on preferences, the equilibria of economies are not necessarily stable under this adjustment process. This is unfortunate since the tatonnement process requires little more information than the Walrasian mechanism at equilibrium. Yet the lack of stability is of great importance. If equilibria are not stable under plausible adjustment process, then their intrinsic interest becomes very limited indeed. If we cannot be sure that an economy will arrive at an equilibrium, why should we devote so much time to studying these states?*

The intertemporal substitution mechanism between labour and leisure is the trick used in order to justify that unemployment is a voluntary phenomenon and the economy is always on the labour supply curve (Mankiw, 1989). Individuals should decrease their labour supply according to decreases in real wages or in interest rate. Anyway, empirical evidence shows that elasticity between labour and relative prices is not so high: people slightly react to changes in productivity shocks. Agents do not decide how much to work according to their expectations on future changes in real wages. What Lucas considered the strength of this new model, the *micro-foundation procedure*, revealed to be the main weakness: no empirical evidence is recorded about the perfectly rational behaviour of agents (completeness and transitivity), hence no scientific merit can be recognised to this kind of micro-foundation process. On the other hand, even economies that display rules of thumb or bounded rationality can have well-behaved aggregate demand/supply functions. The hypothesis of perfect rationality, which is instrumental to operate a maximization procedure and to aggregate individual preferences, is the artifact that was introduced to give a scientific status to macroeconomics. Indeed, it is exactly this fundamental assumption, being absolutely over-imposed and with no empirical track record, that makes non scientific mainstream macroeconomic theory. The evolution of the RBC is the Dynamic Stochastic General Equilibrium model, where along with the New Classical elements before mentioned, some New Keynesian ingredients are added, such as imperfect information, sticky prices and market incompleteness. DSGE models inherit all the unrealistic assumptions and methodologies used in RBC models: exogenous stochastic real shocks, perfect maximizing agents, aggregate production/utility functions. Everything is modeled in a Walrasian framework, where the so called *Keynesian* ingredients are market frictions that, at the end, push the system toward a *second best* equilibrium. In this work we take an alternative route in explaining business cycle movements and explore instead endogenous factors driving fluctuations.

3 Samuelson's multiplier-accelerator model: the role of investment in output fluctuations

In this section we are going to discuss the multiplier-accelerator model proposed by Samuelson (1939a). This work is one of the seminal paper in economic literature that tries to describe how output volatility can be the result of investment variability, embodying a strong Keynesian interpretation of business cycle. It models the possibility to get output fluctuations in a linear framework. We analyse this contribution because it is able to give a very simple but formalized representation of endogenous business cycle fluctuations. Simplicity due to linearity is also its main limitation: it makes impossible to produce persistent output oscillations.

3.1 The model

Firstly, Samuelson clarifies how the concept of the multiplier is not simply related to the effect of government spending on output: the multiplier provides the ratio of increased income over investment (private and public). He underlines how the effect upon private investment was often disregarded. In order to fulfill this theoretical lack, he combines the accelerator and the multiplier effect.

Income is the sum of three different components:

i governmental deficit spending;

ii private consumption expenditure, induced by previous public expenditure;

iii induced private investment, assumed to be, according to the acceleration principle, proportional to lagged increase in consumption.

An initial public expenditure of one dollar, a propensity to consume equal to one-half and an accelerator-relation factor equal unity are assumed. Consumption in current period is related to income increase in the previous period: at t_1, aggregate income increases by just one dollar. In the second period, t_2, consumption increases by one half the income in the previous period. Since Samuelson's assumption is that investment has a unity proportional relation to variation in consumption, investment increases of fifty cents. So basically, a public expenditure injection of 1 dollar in t_1 will result in 1 dollar increase in t_2 plus the new government expenditure of 1 dollar in t_2, reaching two dollars. By changing the value of the propensity to consume and of the accelerator, the outcome of the model will be very different. The more relevant aspect of this model is that, according to the values assigned to the propensity to consume and to the accelerator coefficient, income can have an oscillatory behaviour. The model can be analytically represented as:

$$Y_t = g_t + C_t + I_t \tag{3}$$

$$C_t = \alpha Y_{t-1} \tag{4}$$

$$I_t = \beta(C_t - C_{t-1}) = \alpha\beta(Y_{t-1} - Y_{t-2}) \tag{5}$$

$$g_t = 1 \tag{6}$$

so that aggregate income will be:

$$Y_t = 1 + \alpha(1+\beta)Y_{t-1} - \alpha\beta Y_{t-2} \tag{7}$$

This equation means: if we have both $t-1$ and $t-2$ income we can obtain the level of current income as a weighted sum of the two. Obviously, the result will depend upon the weight of the two parameters. It is a non homogeneous linear second order difference equation whose solution is:

$$Y_t = \frac{1}{1-\alpha} + a_1 x_1^t + a_2 x_2^t \tag{8}$$

where x_1 and x_2 are roots of the quadratic equation:

$$x^2 - \alpha(1+\beta)x + \alpha\beta = 0 \tag{9}$$

and a_1 and a_2 are constant coefficients.
The fixed point, the point reached when $Y_t = Y_{t-1} = Y_{t-2}$ is:

$$Y^* = \frac{1}{1-\alpha} \tag{10}$$

In order the fixed point to be stable, three conditions must be simultaneously satisfied (see Gandolfo (1996)):

$$1 - \alpha(1+\beta) + \alpha\beta = 1 - \alpha > 0 \tag{11}$$

$$1 - \alpha\beta > 0 \tag{12}$$

$$1 + \alpha(1+\beta) + \alpha\beta > 0 \tag{13}$$

The first equation is always satisfied since, by assumption, propensity to consume is less than one. Also the third one is satisfied, being the sum of three positive components. The stability condition is established by the second inequality, so the system will be stable if:

$$\alpha\beta < 1 \tag{14}$$

or

$$\alpha < \frac{1}{\beta} \tag{15}$$

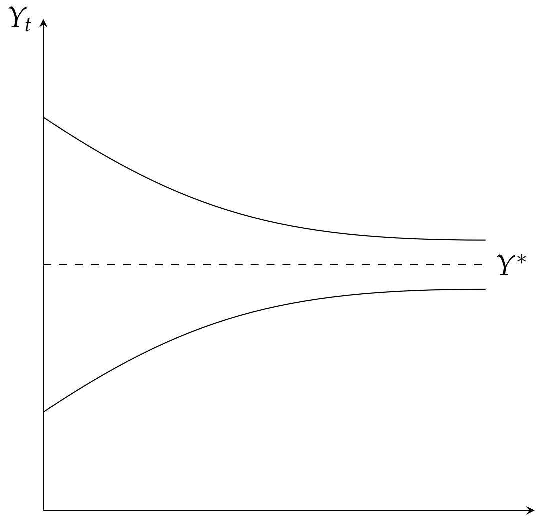


Figure 1: **Scenario A: Monotonic damping**.
Source: Lorenz (1993)

Computing the discriminant we get:

$$\Delta = \alpha^2(1+\beta)^2 - 4\alpha\beta \tag{16}$$

so that $\Delta \lesseqgtr 0$ if $\alpha^2(1+\beta)^2 - 4\alpha\beta \lesseqgtr 0$, hence:

$$\alpha \lesseqgtr \frac{4\beta}{(1+\beta)^2} \tag{17}$$

The output bahaviour can be classified according to the values of the roots of the characteristic equation (if they are real or complex) and if their moduli is strictly less than one or not. From the combination of the nature of the roots and stability properties, four different cases arise.

Scenario A: any point in this region lies below the function $\alpha = \frac{1}{\beta}$ and above the function $\alpha = \frac{4\beta}{(1+\beta)^2}$. A constant level of government expenditure will result in a constant convergence toward the equilibrium. A periodic injection of public expenditure will result in a monotonic output behaviour. Two real roots, whose moduli is less than one, determine a globally asymptotically stable behaviour of the output, that tends to converge towards the fixed point.

Scenario B: any point in this region satisfies the inequalities $\alpha < \frac{1}{\beta}$, $\alpha < \frac{4\beta}{(1+\beta)^2}$. A constant level of government expenditure such as an oscillatory one, both result in

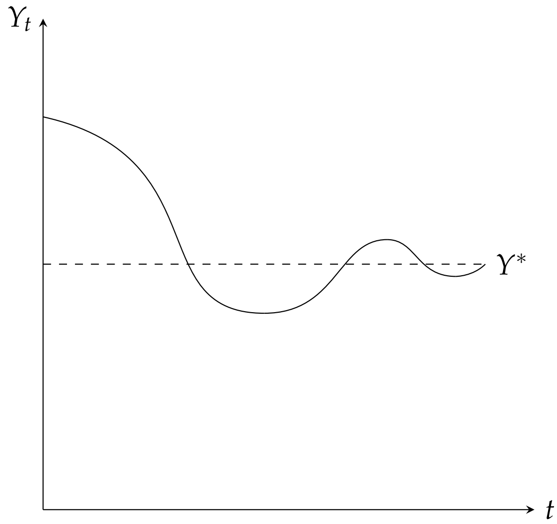


Figure 2: **Scenario B: Damped oscillations**.
Source: Lorenz (1993)

damped output fluctuations that converge toward the fixed point. Two complex roots, whose moduli is still less than one, generate this output movement.

Scenario C: any point in this region satisfies $\alpha > \frac{1}{\beta}$, $\alpha < \frac{4\beta}{(1+\beta)^2}$. A constant level of output expenditure will result in explosive oscillations around the fixed points. Two complex roots with moduli greater than one cause this divergent path.

Scenario D: any point in this region satisfies $\alpha > \frac{1}{\beta}$, $\alpha > \frac{4\beta}{(1+\beta)^2}$. A constant level of government expenditure will result in ever increasing output. A single impulse on net investment will increase output at infinite rate. On the other hand, a minimum disinvestment impulse will increasingly put downward pressure on output. This behaviour is generated by two real roots with moduli greater than one.

Samuelson underlines how the model is constructed under the assumption that the marginal propensity to consume and the accelerator are two exogenous parameters, even though in reality they endogenously depend on income. Notwithstanding, the model provides a first clear interpretation of output oscillation.

3.2 The relationship between the accelerator and the multiplier

In a second article published in the same year, Samuelson (1939b) analyses the reciprocal relationship between the accelerator and the multiplier. He starts from the

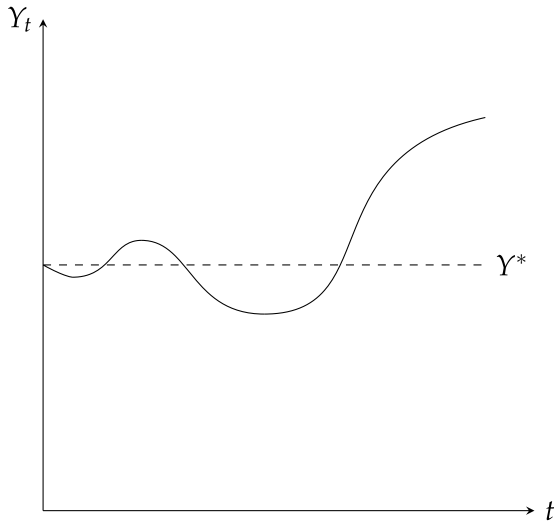


Figure 3: **Scenario C: Exploding oscillations**.
Source: Lorenz (1993)

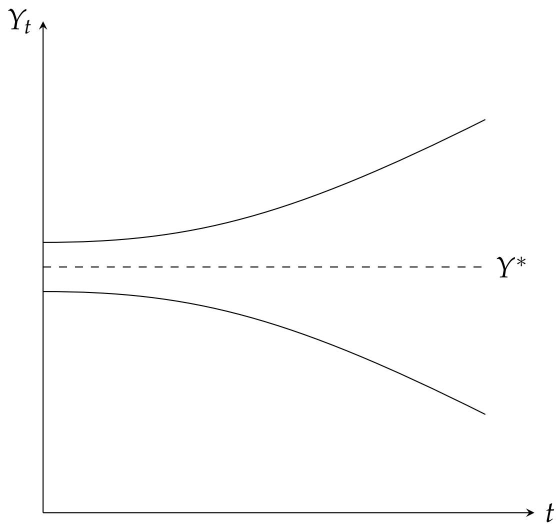


Figure 4: **Scenario D: Monotonic explosion**.
Source: Lorenz (1993)

consideration that some authors, as Clark and Frisch, stressed the role of the accelerator, interpreting consumption as the primary source of investment oscillation. On the other hand, Keynes looked at the role of investment in fueling income and consumption, hence at the multiplier effect, which is not only related to the positive effect of public investment, but also to the effect that private investment exerts on income.

The question that he poses is: will income expansion be continuous or will it start to decrease at some point? Assuming zero capital depreciation rate, in the fixed point, where consumption is constant over time, the propensity to consume should be equal to one: at this point net investment is equal to zero. But in normal circumstances, a noninduced component of investment should be present. The existence of the autonomous investment will increase the output level. The acceleration principle thus can determine the nature of oscillations but not the absolute levels of output and income. Note also that the effect of the accelerator is higher in societies with low level of autonomous investment and income, rather than in high income-investment ones. What in the previous model was the role of public expenditure, in this model becomes autonomous investment:

$$Y_{t+2} = A + \beta(C_{t+2} - C_{t+1}) + C_{t+2} \tag{18}$$

where β is the relation. According to the Keynesian multiplier, output in one period will be consumption in the following one, so that there exists a relation between consumption and income that is supposed to be linear:

$$C_{t+3} = f(Y_{t+2}) = f(A + (1+\beta)C_{t+2} - C_{t+1}) \tag{19}$$

More generally, we can easily compute consumption in one period if we know the relation and the consumption values in the two preceding periods:

$$C_{t+1} = f(A + (1+\beta)C_t - \beta C_{t-1}) \tag{20}$$

The results of this linear second order difference equation are exactly the same of the previously exposed ones. For certain values of the propensity to consume and of the relation, an expansion will always come to an end. For all values of α and β that lies in scenario B and C a depression movement always occurs: the demarcation line between B and C represents the passage from stable to unstable oscillations. Also, being the propensity to consume always less than unity, a positive propensity to save does exist. Notwithstanding, a positive propensity to save is not sufficient to bring the system toward the end of the cumulative process (this is true for those values of α and β such that the system is in the explosive oscillatory range). For any given value of the propensity to consume, low values of the relation will not generate a cyclical behaviour (Scenario A). For slightly larger values of the relation, the system shows convergent oscillations (Scenario B). Grater values of

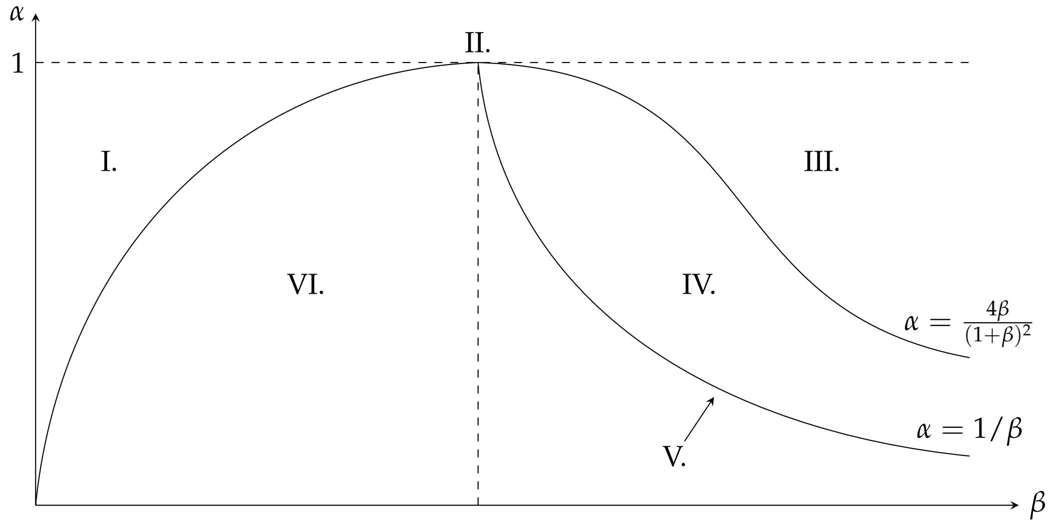


Figure 5: **Parameters regimes**.
Source: Lorenz (1993)

the relation will determine no more convergent, but divergent oscillations around the fixed points (Scenario C). Finally, for very high values of the accelerator, the system has a cumulative explosive path (Scenario D). For a better understanding see fig. 5. All paths are described maintaining a constant propensity to consume: if it endogenously depends on income, what can happen will be an infinite upward movement of the system. Economies with large investment and small propensity to consume are less hit by the destabilizing acceleration mechanism.

3.3 Variations and extensions

Samuelson built a model based on the two assumptions of multiplier and accelerator. According to the first, consumers spend a constant fraction of their income. Given any variation in disposable income, due to public or private investment, changes in spending are generated. These changes multiply the initial variation over the reciprocal of the saved income fraction (geometric series). According to the second, the proportion of capital over output is constant. Any variation in output determines a proportional variation in capital stock, that by definition is investment. The model is inherently Keynesian because it is demand-led: investment follows output variations. Capital is accumulated to keep production capacity constant. The system resembles the harmonic oscillations which can be explosive or damped. In order to obtain sustained and not damped oscillations, Hicks (1950) proposed to introduce a ceiling and a floor to contain the otherwise explosive behaviour of the system. Following the linear relation, during depressions, firms not only disinvest; they could also operate an active capital destruction if the disinvestment rate is higher than the depreciation rate. In order to prevent this unrealistic feature, a lower bound could be meaningful. On the other way around, when investment becomes very high, a natural upper bound due to resource constraint could be reasonable. These are the interpretations provided by Hicks to justify ceilings and floors needed to obtain persistent oscillations. The floor can be represented by the following equation, that substitutes the original $I_t = \alpha(Y_{t-1} - Y_{t-2})$:

$$I_t = \max[\alpha(Y_{t-1} - Y_{t-2}), -I^f] \tag{21}$$

where I^f is the absolute value of the floor disinvestment. The ceiling can instead be due to a fixed proportion production function:

$$Y_t = \min\left(\frac{K_t}{a}, \frac{L_t}{b}\right) \tag{22}$$

where the investment ceiling is the maximum amount of available labour force multiplied by the utilisation coefficients:

$$I^c = \frac{a}{b} L_t \tag{23}$$

Since Hicks never wrote a function with a ceiling is not clear whether it should be introduced within the investment function:

$$I_t = \min[I^c, \max[\alpha(Y_{t-1} - Y_{t-2}), -I^f]] \tag{24}$$

or adding it as a constraint in the aggregate production function:

$$Y_t = \min(C_t + I_t, I^c) \tag{25}$$

Anyway, the introduction of ceilings and floors is an expedient that allows to contain the explosive motion resulting from the Samuelson's model.

4 Kalecki's business cycle model: the role of time in investment decisions

In this section we are going to carry on the presentation of the business cycle models that have a Keynesian root. In particular we analyze the Kalecki's contribution on business cycle that is *in between* the multiplier-accelerator model of Samuelson and the Kaldor's trade cycle that we will discuss in the next section. The Keynesian trait present in Kalecki, which can be envisaged in the role played by expectations, and in particular by future expected profits in determining current investment, can be better understood reading its article (Kalecki, 1935):

> *Investments (together with capitalist consumption) determine profits and hence also the savings that they require, and not the reverse.*

In this proposition is clearly stated how investment determines profit accumulation, contradicting the usual Classical assumption, according to which, profits generate investment. Differently from Keynes, the investment process is not determined by the gap between the marginal efficiency of capital and the rate of interest. Kalecki considers that investment decisions are driven by the difference between the prospective rate of profits and the interest rate. The determinant of the prospective rate of profits are the long term expectations on returns and on price of investment goods. But expectations on future profits depend upon the current *"state of the art"* of the economy. Hence it is the short period equilibrium that determines prospective rate of profits.

An interesting link between Kalecki and the Marxian tradition is its view on the capitalist system as consisting of social classes, unlike the individualistic approach emphazised by the marginalist school. The national income is the sum of capitalists' and workers' consumption. In addition, unlike Keynes, even though he supports the idea of full-employment, he conceives this objective practically impossible to achieve within a capitalist organization of society. Capitalists need a reserve army to make the working class more disciplined. Thus unemployment is an intrinsic feature of capitalism (Kalecki, 1990):

> *The reserve of capital equipment and the reserve army of unemployed are typical features of capitalist economy, at least throughout a considerable part of the cycle.*

4.1 The linear model

As discussed in Gabisch et al. (1989), Kalecki published (see Kalecki, 1935) one of the first business cycle model, re-elaborated afterwards in other versions (Kalecki, 1937), (Kalecki, 1943). The initial versions were linear while the later ones were non-linear, even if the basic economic ideas were the same. Differently from Samuelson that introduced in his multiplier-accelerator model time lags by means of assumptions on consumption and investment decisions, the Kalecki model incorporates technical restrictions in the investment process. The time lag exists between the investment decision and the installation of new capital goods. He assumes the investment decision to occur at time t and $I^D(t)$ as the corresponding amount of investment. The production of capital goods requires a time interval θ, then the capital stock will be modified at $t + \theta$:

$$\dot{K} = I^D(t - \theta) \tag{26}$$

Being an equilibrium model, the production has to be financed such that I is introduced as an advanced payment. The value of the investment goods is:

$$W(t) = \int_{t-\theta}^{t} I^D(\tau) d\tau \tag{27}$$

and the average production of investment goods per unit of time is $A = W/\theta$:

$$A(t) = \frac{1}{\theta} \int_{t-\theta}^{t} I^D(\tau) d\tau = I(t) \tag{28}$$

Substituting for:

$$\dot{K}(t + \theta) = I^D(t) \tag{29}$$

we get:

$$I(t) = \frac{1}{\theta} \int_{t-\theta}^{t} \frac{dK(\tau + \theta)}{d\tau} \, d\tau \tag{30}$$

$$= \frac{1}{\theta}[K(t + \theta) - K(t)] \tag{31}$$

The investment decisions depend upon the level of income and the level of capital. In Kalecki (1935) a linear relation is used:

$$I^D(t) = asY(t) - kK(t) \tag{32}$$

Equating the value of investment goods with the determinant of the investment process, $I(t) = I^D(t)$ and being $Y(t) = I(t)/s$, we get:

$$\dot{K}(t) = \frac{a}{\theta} K(t) - (k + \frac{a}{\theta}) K(t - \theta)) \tag{33}$$

This is a mixed difference-differential equation. With $\theta = 1$ a solution of

$$\dot{K}(t) = aK(t) - (k + a)K(t - 1)) \tag{34}$$

is $K(t) = K_0 e^{\rho t}$ that is very similar to a second order difference equation. According to Gabisch, the analysis of this equation leads exactly to the same behaviour of the multiplier-accelerator model: because explosive oscillations should be excluded and because steady oscillations occur only for exactly one numerical value of k, the typical dynamic is characterized by damped oscillations in the complex roots case. In order to obtain permanent oscillations exogenous shocks are necessary.

4.2 Nonlinear case

In 1937 Kalecki substitutes the original linear relation with a non linear one, where the investment decision process is summarized by the relation:

$$I^D(t) = \phi(Y(t), K(t)) \tag{35}$$

I^D is assumed to be S-shaped and negatively affected by $K(t)$. Let:

$$Y(t) = f(I(t)) \tag{36}$$

be the short run equilibria of the economy depending on the amount of investment. Let's assume that

$$Y(t) = f(I^D(t-1)) \tag{37}$$

is a linear function. From the intersection between the S-shaped investment decision curve and the linear short run equilibria, he obtained the long run equilibrium Y^* (see fig. 6). The long run equilibrium is a stable point, since for levels of income lower than the equilibrium point, capital decreases due to the low investment, so there will be an upward shift of the investment decision curve. The opposite (a downward shift due to an increase in capital) will happen, for level of income higher than the equilibrium level. The existence of a cycle (see fig. 7) in the Kalecki nonlinear model has to be ensured by a strong shift of the investment decision curve, otherwise a monotonic return toward the steady state will occur. In particular, necessary conditions in order to obtain a cyclical movement from the introduction of the time-lag between the investment decision and the generated income are: (i) the effect of current investment on total equipment should be very large, affecting the rate of profits, hence the investment decision; (ii) the angle enclosed between the locus of points of short run equilibria and the S-shaped investment function should be small, hence the equilibrium point has a low degree of stability.

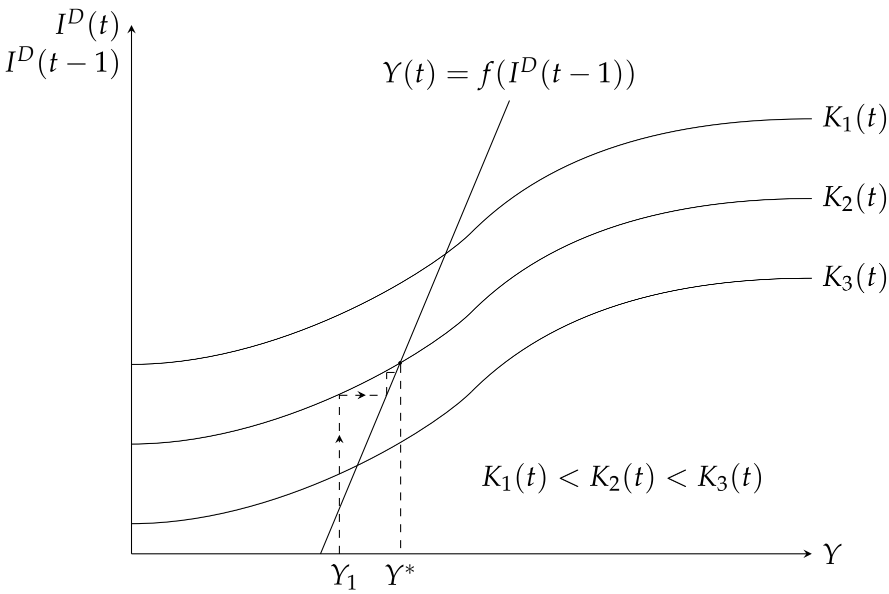


Figure 6: **The non linear Kalecki model**.
Source: Gabisch et al. (1989)

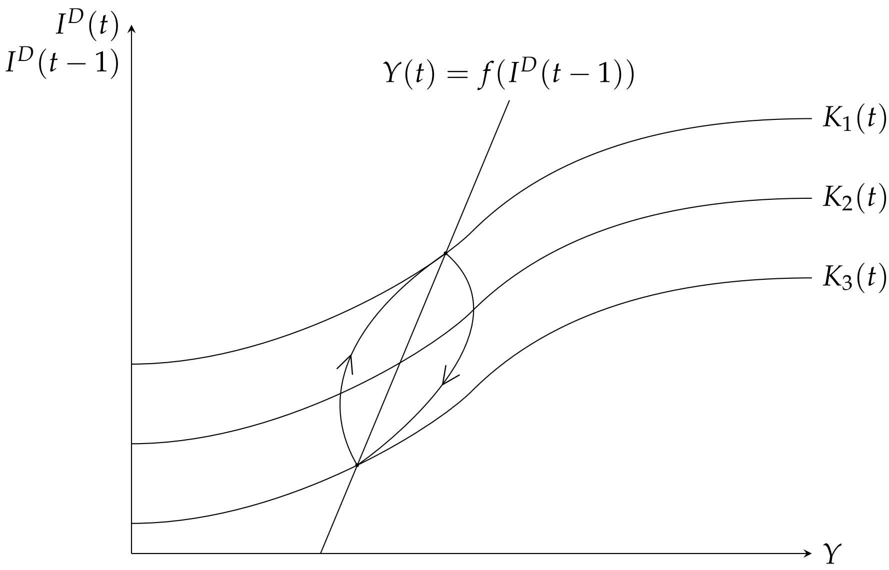


Figure 7: **Cycles in the non linear Kalecki Model**.
Source: Gabisch et al. (1989)

5 Kaldor trade cycle model: the S-shaped investment function

In 1940 Nicholas Kaldor (see Kaldor, 1940) published *'A model of trade cycle'* where he discussed how the main sources of business cycle can be envisaged in the combined interaction between the multiplier and the investment demand function. He focused on the necessary and sufficient conditions under which the combined effect of these two forces generate a cycle. The usual equilibrium condition of equality between savings and investment has to be discussed making a distinction between ex-ante and ex-post equilibrium. We quote from the original article:

> *Investment ex-ante is the value of the designed increments of stocks of all kinds (i.e., the value of the net addition to stocks plus the value of the aggregate output of fixed equipment), which differs from Investment ex-post by the value of the undesigned accretion (or de-cumulation) of stocks. Savings ex-ante is the amount people intend to save, i.e., the amount they actually would save if they correctly forecast their incomes. Hence ex-ante and ex-post Saving can differ only in so far as there is an unexpected change in the amount of incorne earned.*

If ex-ante Investment exceeds ex-ante Saving an increase in the level of activity is generated. This occurs or because ex-post Investment will be less than ex-ante Investment or either ex-post Saving will be higher than ex-ante Saving. A decrease in the level of activity is the result of the opposite discrepancy between ex-ante Investment and ex-ante Saving. This is so, either for the reduction in consumer expenditure due to the reduction of ex-post Saving as compared with ex-ante Saving, or for the excess of ex-post Investment with respect to ex-ante Investment.

5.1 An analytical description

Both the influence of Investment and Saving on the level of activity are functions themselves of income so if we denote x as the level of income, we characterized dS/dx and dI/dx as both positive derivatives. The first expression is nothing other than a consequence of the Keynesian consumption multiplier (being the reciprocal of Keynes' investment multiplier). The second means that demand for capital goods will increase with the level of activity. Following Kaldor, we are going to analyze the result of the interaction between these two effects in three different cases: (i) when both relations are linear, (ii) when only one of the two is non linear, (iii) when both are non-linear.

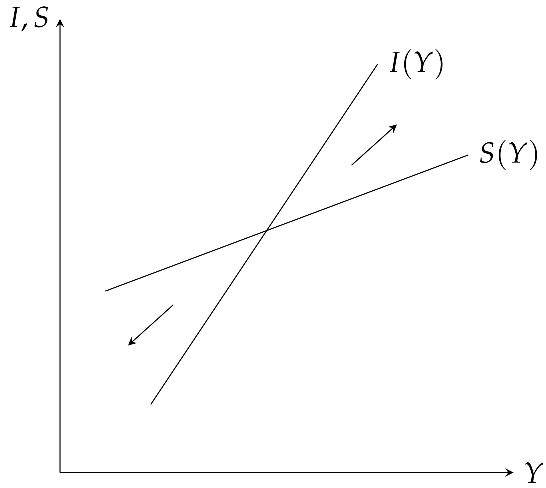


Figure 8: **Kaldor linear model: unstable equilibrium**.
Source: Gabisch et al. (1989)

Linear case

1. The first possible linear case is when the magnitude of the Investment response to changes in income is greater than the one of Saving:
$$\frac{dI}{dx} > \frac{dS}{dx} \tag{38}$$
In this case, only one unstable equilibrium point, as shown in the graph (see fig. 8), is achieved. Below the equilibrium point, $S > I$, leading to a contraction in the level of activity. Above the equilibrium point, $I > S$ leading to an expansion. The interaction of the combined effects will result either in a full-employment/hyper-inflationary state, or in a zero level of employment. Since both these two states are not so very common, this first case can be neglected.

2. The second one is when the Saving response to income changes is greater than the Investment response:
$$\frac{dS}{dx} > \frac{dI}{dx} \tag{39}$$
In this case there will be a single stable equilibrium point, so any discrepancy that can occur between the level of investment and saving is automatically stabilized (see fig. 9). The inherent stability of this scenario is so strong that Kaldor conceived it to be unrealistic such as the previous unstable case. Moreover, according to the principle of the accelerator, the magnitude of investment could be greater than the magnitude of saving for any level of activity,

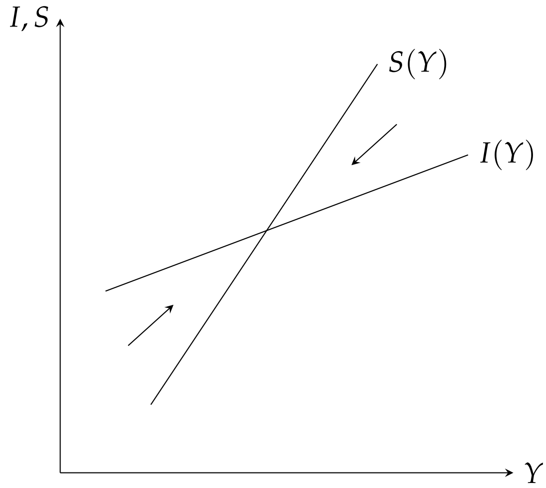


Figure 9: **Kaldor linear model: stable equilibrium**.
Source: Gabisch et al. (1989)

contradicting equation (39). The unrealistic outcomes of the linear case push the attention on the non-linear case.

Nonlinear case

1. First case: linear saving function and non linear investment function. The non linearity of the investment function can be justified by the fact that the level of investment would be small for both low and high level of activity. It will be small for low level of activity because when there is a surplus capacity, profits are not able to generate investment. Anyway, the level of investment can not be zero because of the long period investment, independent from the level of activity. Analogously, for very high level of activity, the increase in the cost of investment will stop it, decreasing the magnitude of the investment derivative.

2. Second case: linear investment function and non linear saving function. In this case the effect of extreme low and high level of activity on saving will be the opposite: for very low income level, saving could also be negative (borrowing activity), while for high income level, saving will be very high. These relations are reinforced by the interactions at the aggregate level: for low level of income, an increasing proportion of workers are paid out of capital funds; for high level of income profits will increase relatively to wages, leading to an increase in the propensity to save.

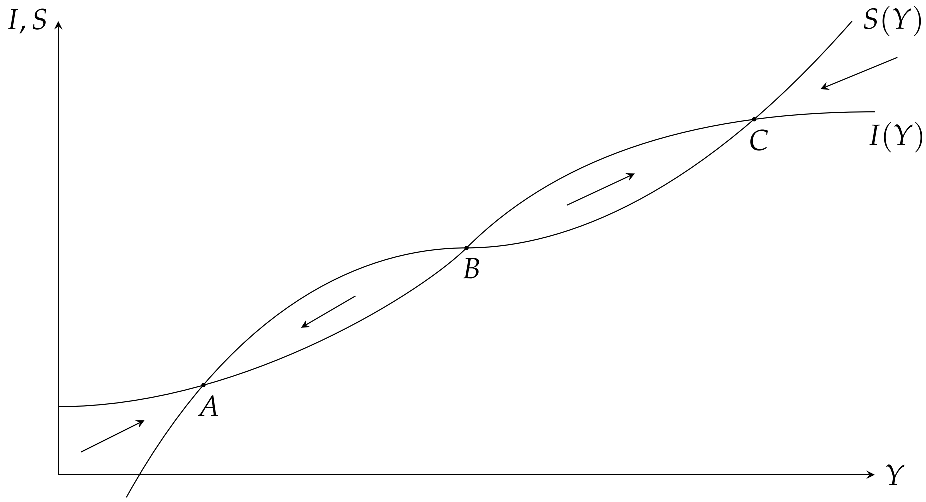


Figure 10: **Kaldor non linear model**.
Source: Gabisch et al. (1989)

Consider the case in which both functions are non-linear: in this scenario, from the intersection of the curves three equilibrium points are determined (see fig. 10). The two extreme ones A and C are stable, since below the equilibrium points, $I > S$ and the level of activity tends to increase, while above $S > I$ and the level of activity tends to decrease. The internal equilibrium B is the unstable one since for levels of activity above the equilibrium point there will be an expansion that will stop in C. For levels of activity lower than B there will be a contraction that will stop in A. Introducing non linear saving and investment relations, the economy could reach a stable point either at a high level of activity (C) or at a low level of activity (A). Anyway, Kaldor emphasizes how this stability is ensured only in the short run. We can now rewrite the necessary and sufficient assumptions needed to obtain a cyclical path of the level of activity:

- The normal value of dI/dx must be greater than the corresponding value of dS/dx.
- The extreme values of dI/dx, for very high level or very low level of income, must be smaller than the corresponding value of dS/dx.
- The level of investment in C, the upper equilibrium point, must be sufficient large for the $I(x)$ to fall in time relatively to $S(x)$. Conversely, in A, the lower equilibrium point, it must be sufficiently small for $I(x)$ to rise in time relatively to $S(x)$.

The amplitude of the cycle depends upon the shapes of the I and S curves which determines the distance between A and C. The amplitude will be the smaller, the shorter the range of activity over which the normal values of dI/dx and dS/dx are operative.

5.2 The cyclical behaviour of the model

As Appendix of the above quoted article, Kaldor introduces a comparison with Kalecki's model using the same kind of diagram present in Kalecki (see fig. 11). This model has been used as the prototype model for non linear business cycle models, being able to reproduce endogenous limit cycles. Let's see how this dynamic is generated. Investment is a function both of the real income and of the stock of capital in each point in time:

$$I = I(Y, K), \quad I_Y > 0, \quad I_K < 0 \tag{40}$$

and there exists a Y_1 such that $I_{YY} > 0$ $[< 0]$ if $Y < Y_1$ $[Y > Y_1]$. Let's assume the case in which the saving function is linear, $0 < S_Y < 1$ and $S_K > 0$. The low of motions that describe the behaviour of the system are:

$$\dot{Y} = \alpha(I(Y, K) - S(Y, K)) \tag{41}$$

and

$$\dot{K} = I(Y, K) - \delta K \tag{42}$$

with δ as the constant capital depreciation rate and α as an adjustment coefficient, both strictly positive. The Jacobian matrix of the system is:

$$\mathcal{J} = \begin{pmatrix} \alpha(I_Y - S_Y) & \alpha(I_K - S_K) \\ I_Y & I_K - \delta \end{pmatrix}$$

with determinant:

$$\det(\mathcal{J}) = \alpha(I_Y - S_Y)(I_K - \delta) - \alpha I_Y(I_K - S_K) \tag{43}$$

and trace:

$$\mathrm{Tr}(\mathcal{J}) = \alpha(I_Y - S_Y) + (I_K - \delta) \tag{44}$$

The eigenvalues of the Jacobian matrix are:

$$\lambda_{1,2} = \frac{\mathrm{Tr}(\mathcal{J}) \pm \sqrt{(\mathrm{Tr}(\mathcal{J})^2) - 4\det(\mathcal{J})}}{2} \tag{45}$$

so, in order to be the stability conditions for the continuous time dynamical systems fulfilled, the real parts of the eigenvalues must be negative, that is $\mathrm{Tr}(\mathcal{J}) < 0$ which implies $\alpha(I_Y - S_Y) + (I_K - \delta) < 0$. When $\alpha(I_Y - S_Y) + (I_K - \delta) > 0$ the fixed point

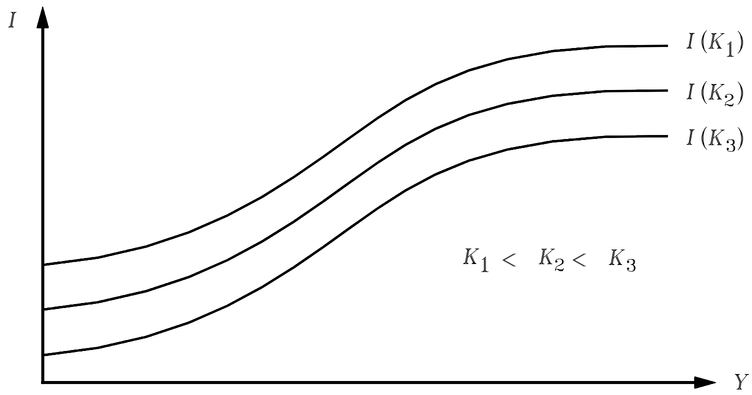


Figure 11: Kaldor's model.
Source: Lorenz (1993)

loses stability. Lorenz (see Lorenz, 1993) demonstrates how the Kaldor's model satisfies the criterion needed to obtain a limit cycle, according to the Poincaré-Bendixon theorem (see theorem 1). The possibility of the existence of a closed orbit depends upon the magnitude of the term $I_K - \delta$. Lorenz performs a graphical solution of the problem looking at the phase portrait (see fig. 12). Considering the locus of points for which:

$$\dot{K} = 0 = I(K, Y) - \delta K \tag{46}$$

in order to understand the slope of this curve is necessary to analyze the sign of the total derivative:

$$\frac{dK}{dY} = -\frac{I_Y}{I_K - \delta} > 0. \tag{47}$$

Hence this locus of points is an upward sloping curve. For all K above the curve $\dot{K} = 0$ investment decreases because of $I_K - \delta < 0$ hence $\dot{K} < 0$. For all K below the curve, $\dot{K} > 0$. Proceeding in the same way, the locus of points for which $\dot{Y} = 0$ is:

$$\dot{Y} = 0 = I(Y, K) - S(Y, K) \tag{48}$$

so that the total derivative is:

$$\frac{dK}{dY} = \frac{S_Y - I_Y}{I_K - S_K} \gtreqless 0. \tag{49}$$

The sign of the equation depends upon the effect of S_Y and of I_Y. The numerator is positive for very low and very high income levels, and is negative for levels around the equilibrium point. Thus, the locus of points obtained when $\dot{Y} = 0$ is negatively sloped for low and high income levels, and positively sloped in a neighborhood of the equilibrium point. The shaded area in the graph (see fig. 12) is a compact set that contains the unstable equilibrium points and the vector field points inwards

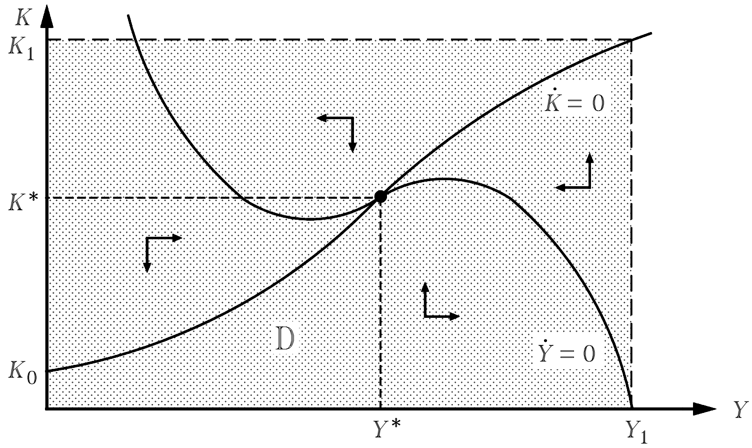


Figure 12: Phase portrait.
Source: Lorenz (1993)

the set, hence the conditions necessary for the existence of a limit cycle are satisfied. Summarizing, these conditions are:

- Locate a fixed point of the dynamical system and examine its stability properties.
- If the fixed point is unstable, search for an invariant set D enclosing the fixed point. When a closed orbit does not coincide with the boundary of D, the vector field described by the function f and g must point into the interior of D.

6 Goodwin's growth cycle model: fluctuations as result of the class-struggle

In 1967 Goodwin published its seminal work (see Goodwin, 1967) on growth cycle: it represents the first formalization of the distributive conflict between profits and wages stemming from the Marxian class struggle theory. The Goodwin's model can be considered a 'classical' model as based on a Say's Law, opposed to a Keynesian, demand-driven perspective. The main assumption of the model is related to the symbiotic but conflicting coexistence between capitalists and workers: capitalists save and immediately reinvest all their profits, without any concern about over-accumulation. Workers spend all income they receive. Crises can occur for two different reasons related to capital accumulation: overproduction crises or class-struggle crises. Overproduction crises occur because of the high rate of profit accumulation that determines an excess of supply, for a given wage level. In this case, capitalists should suffer some losses because they are not able to sell their production to workers, the only class that consumes. Class-struggle crises happen according to the following process: capital accumulation determines an increase in labour demand, leading to an employment increase, so strengthening workers contractual power. The profit accumulation has as a counter effect, the drop of profit rate. Again, less capital accumulation drives to lower production, lower employment, lower wage rate and higher profit rate. This cyclical and opposite movement of wages and profits is the same idea present in Smith and Ricardo, deepened by Marx who developed the idea of workers as an *industrial reserve army* that could eventually lead to the end of capitalism. What is important to stress, for the endogenous business cycle approach, is that labour market does not define the absolute value of real wages but just oscillations around an equilibrium value: it is not an equilibrium model, but a fluctuations model.
Within the classical view of capitalist system is possible to develop a model of economic fluctuations due to the combined interrelations between profits and wages. This is the kernel of Goodwin Growth Cycle model. In analyzing it, we are going to follow Medio (1979).

6.1 The model

The main features and assumptions are:

i Two economic forces called *employment effect* and *profits effect*: the former is a positive relation between employment rate and wage rate variation; the latter is a positive relation between the profit share and the output/production growth rate.

ii A constant capital/output ratio.

iii Wages are entirely spent, profits are entirely saved and reinvested.

iv Output growth rate equals profit rate.

v The equilibrium growth rate will be equal to a *natural* growth rate given by the sum of population and productivity growth rate. Technical progress is assumed to be Harrod neutral.

vi All quantities are real.

vii Disequilibrium dynamic: when employment rate is above the equilibrium rate, wages increase and their growth rate, being higher then productivity rate, erodes profit rate. Decreasing profit rate leads to a decrease in output growth rate, reducing employment rate. This implies a reduction in wages allowing for a profit rate expansion, restarting the cycle.

Although it is particularly meaningful from an economic point of view, this model presents a peculiar drawback: it is a *structurally unstable* model. Structural instability means that every minimal modification of the equations will destroy his fundamental characteristic, that is the possibility to depict persistent fluctuations of variables. The model is built on the *predator-prey* model elaborated by the two mathematicians Lotka (1925) and Volterra (1931) in order to study the relationship between preys and predators, in a biological context. The system is expressed in continuous time; maintaining the original notation used by Goodwin, variables are the following:

q is output;
k is capital;
w is wage rate;
$a = a_0 e^{\alpha t}$ is labour productivity, α is constant;
σ is capital-output ratio, the inverse of capital productivity;
$w/a = u$ is workers share of product, $(1-u)$ is capitalists share of product;
surplus=profit=savings=investment=$(1-u)q = k$;
profit rate $= \dot{k}/k = \dot{q}/q = (1-u)q/\sigma$;
$n = n_0 e^{\beta t}$ is labour supply, β is constant;
$l = q/a$ is employment;
$v = l/n$ is labour demand;
writing $(\dot{q}/\dot{l}) = d/dt(q/l)$ we get the output per capita growth rate:

$$(\dot{q}/\dot{l})/(q/l) = \dot{q}/q - \dot{l}/l = \alpha \tag{50}$$

Labour demand growth rate is:

$$(\dot{l}/l) = (1-u)\sigma - \alpha \tag{51}$$

Employment growth rate is:

$$(\dot{v}/v) = (1-u)\sigma - (\alpha+\beta) \tag{52}$$

The positive relation between real wages and employment is expressed by means of a linearized Philip Curve:

$$(\dot{w}/w) = -\gamma + \rho v \tag{53}$$

Hence:

$$(\dot{u}/u) = (\dot{w}/w) - \alpha = \rho v - (\alpha+\gamma) \tag{54}$$

So finally, we get the nonlinear continuous system expressed in term of u and v variation rates:

$$\dot{v} = [(1/\sigma - (\alpha+\beta)) - 1/\sigma u]v \tag{55}$$

$$\dot{u} = [-(\alpha+\gamma) + \rho v]u \tag{56}$$

The fixed points are the trivial one:

$$v^* = 0, u^* = 0 \tag{57}$$

and a non trivial one:

$$v^* = \frac{\alpha+\gamma}{\rho}, \quad u^* = 1 - \sigma(\alpha+n) \tag{58}$$

Goodwin states in his article (1967):

> *It has long seemed to me that Volterra's problem of the symbiosis of two populations, partly complementary, partly hostile, is helpful in the understanding of the dynamical contradictions of capitalism, especially when stated in a more or less Marxian form.*

We can simplify the notation of the model reducing it to four coefficients:

$a = b - (\alpha+\beta)$;
$b = 1/\sigma$;
$c = \alpha+\gamma$;
$d = \rho$;

We rewrite the system in the following way:

$$\dot{v} = (a - bu)v \tag{59}$$

$$\dot{u} = (-c + dv)u \tag{60}$$

The two fixed points now become the usual trivial equilibrium point $(u = 0, v = 0)$ and the non trivial one $(v = c/d, u = a/b)$. Linearizing the non linear system around the fixed points in order to study its qualitative behaviour, we get the Jacobian matrix. The partial derivatives of the Jacobian matrix evaluated at the trivial fixed point are:

$$\mathcal{J} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$$

The characteristic equation is:

$$\lambda^2 - \mathrm{Tr}(\mathcal{J})\lambda + \det(\mathcal{J}) = 0 \tag{61}$$

$\mathrm{Tr}(\mathcal{J}) = a - c, \det(\mathcal{J}) = -ac < 0$. In this case, which implies $\Delta > 0$ independently of the sign of the trace of $\mathcal{J}$, one eigenvalue is positive, the other is negative (say, $\lambda_1 > 0 > \lambda_2$). There is, then, a one-dimensional stable and a one-dimensional unstable eigenspace and the trivial equilibrium point is known as a **saddle point**. All orbits starting off-equilibrium eventually diverge from equilibrium except those originating in points on the stable eigenspace which converge to equilibrium.
The concept of *stability* can be distinguished in *Lyapunov stability* and *asymptotic stability*. We introduce the following definitions:

Definition 1. *The fixed point $\bar{x}$ is said to be* ***Lyapunov stable*** *(or simply stable) if for any $\varepsilon > 0$ there exist a number $\delta(\varepsilon) > 0$ such that if $\| x_0 - \bar{x} \| < \delta$ then $\| x(t) - \bar{x} \| < \varepsilon$ for all t>0.*

Definition 2. *The fixed point $\bar{x}$ is said to be* ***asymptotically stable*** *if a) it is stable and if b) there exists an $\eta > 0$ such that whenever $\| x_0 - \bar{x} \| < \eta$*

$$\lim_{t \to \infty} \| x(t) - \bar{x} \| = 0$$

Definition 3. *Let $\bar{x}$ be an asymptotically stable fixed point, then the set:*
$B(\bar{x}) = \{x \in \mathbb{R}^m \; s.t. \; \lim_{t\to\infty} \| x(t) - \bar{x} \| = 0\}$ is the ***domain*** *or* ***basin of attraction*** *of $\bar{x}$. If $B(\bar{x}) = \mathbb{R}^m$ (or, at any rate, $\bar{x}$ if it coincides with the state space) then $\bar{x}$ is said to be* ***globally asymptotically stable****. If stability only holds in a neighbourhood of $\bar{x}$ it is said to be* ***locally stable****.*

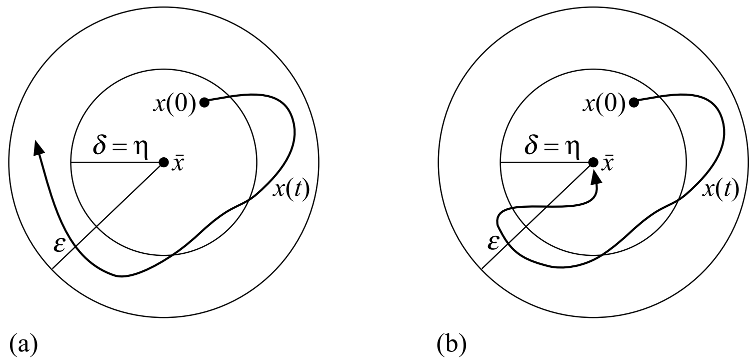


Figure 13: a) Lyapunov stability; b) Asymptotic stability.
Source: Medio et al. (2003)

Broadly speaking, when defined in terms of the union of properties (a) and (b), asymptotic stability implies that if orbits start near equilibrium they stay near it and eventually converge to it. Property (b) (convergence to equilibrium) does not imply property (a) (stability). That is, if (b) holds but (a) does not, we could have solutions that, before converging to $\bar{x}$, wander arbitrarily far from it. Systems for which property (a) holds, but property (b) does not, are called **weakly stable**. We are going to see an example: the case of centre.

The partial derivatives of the Jacobian matrix evaluated at the non trivial fixed point are:

$$\mathcal{J} = \begin{pmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix}$$

$\mathrm{Tr}(\mathcal{J}) = 0$, $\det(\mathcal{J}) = ac > 0$, $\Delta < 0$. In this special case we have a pair of purely imaginary eigenvalues. Orbits neither converge to, nor diverge from, the equilibrium point, but they oscillate regularly around it with a constant amplitude that depends only on initial conditions and a frequency equal to $\det(A)/2\pi$. The eigenspace coincides with the state space and the equilibrium point is called a **centre**.

Representing the system on the phase space we can observe how the trivial fixed point is **unstable**: every trajectory starting from a point in the positive quadrant will go away from the trivial point sooner or later but remaining in the quadrant. The trajectories will be attracted by the other stationary point, the centre, depicting closed orbits around it (see fig. 14).

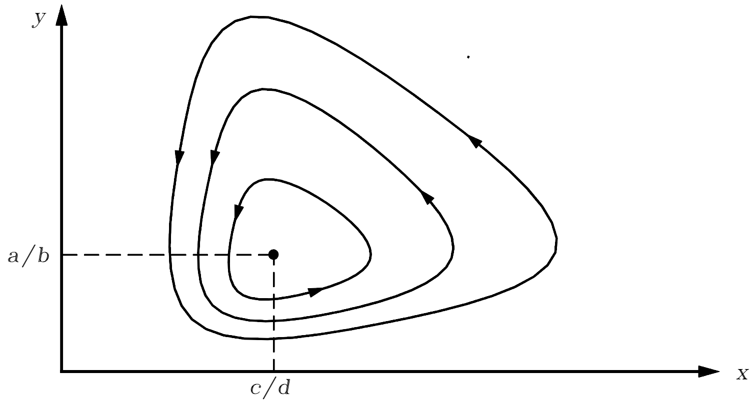


Figure 14: Closed orbits in a Predator Prey system.
Source: Lorenz (1993)

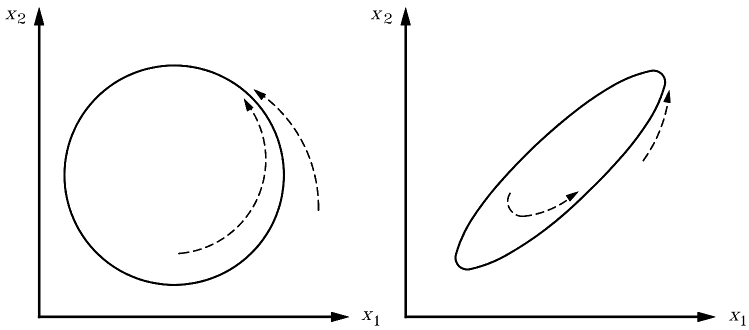


Figure 15: Limit cycles in topological equivalent dynamical systems.
Source: Lorenz (1993)

The equations (59) and (60) describe a dynamical system where every deviation from the stationary point is followed by an oscillatory movement of u, the output share of wages (so implicitly of profits), and of v, the employment rate.

6.1.1 Structural instability

In order to understand why the main drawback of the Goodwin model is its structural instability, we are going to define what is a structural stable system:

Definition 4. *Two dynamical systems are topologically equivalent if there exists a homeomorphism from the phase space of the first system to the phase space of the second system that transforms the phase flow of the first system to the phase flow of the second system.*

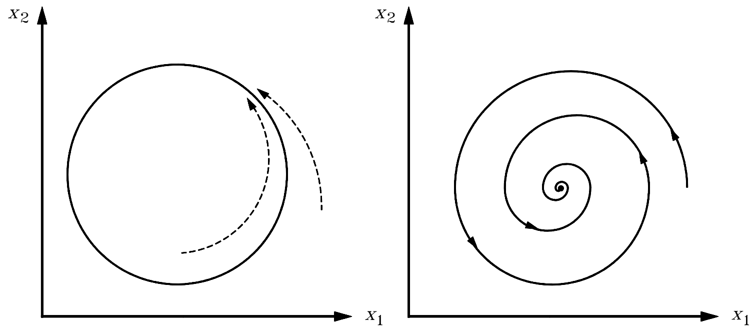


Figure 16: Topologically non equivalent dynamical systems.
Source: Lorenz (1993)

Fig. 15 illustrates the meaning of topological equivalence. It depicts an attracting circle, i.e., a limit cycle. The two elliptic attracting orbits differ in a geometric sense, but the property of a limit cycle persisted under the transformation. The homeomorphism that transforms one cycle in the other can be understood as a coordinate transformation.

Definition 5. *A dynamical system is structurally stable if for every sufficiently small perturbation of the vector field the perturbed system is topologically equivalent to the original system.*

The term "small perturbation" is usually interpreted in terms of the $C1$ norm:

Definition 6. *Two dynamical systems are close at a point x if the associated images, e.g. $f(x)$ and $g(x)$, and the first derivatives, $f(x)'$ and $g(x)'$, are close together.*

Considering these definitions, it is easily to verify that the Goodwin's model is structurally unstable: if direct partial derivatives evaluated at the non-trivial fixed point were not anymore the same, because of a small perturbation of the system, the trace of the Jacobian matrix would be different from zero.
Being $\mathrm{Tr}(\mathcal{J}) = 0$ a *necessary* condition to have a *centre*, any small perturbation will transform it into a *focus*, stable or unstable according to the trace's sign (see fig. 16).
If a small perturbation is performed, the fundamental characteristic of this model (generating persistent oscillations) will be lost. Trajectories will become spirals converging toward the fixed point or, diverging from it leading to the explosion of the system.
The direct consequence is that if we try to generalize the original Goodwin's system with more flexible hypotheses, the model will lose not only its elegant formalisation but also the possibility to give a good representation of the economic cycle. The growth cycle model presents an other drawback, again due to the nature of the stationary point: the amplitude of oscillations is entirely due to initial conditions.

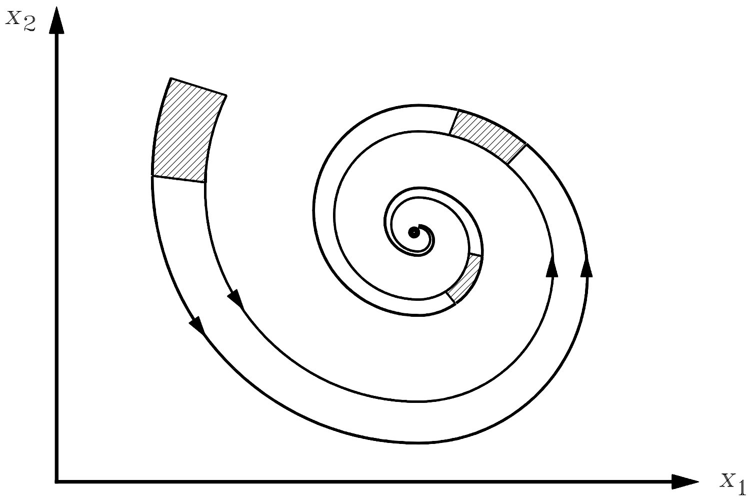


Figure 17: Dissipative systems: area contraction.
Source: Lorenz (1993)

Trajectories that start from points near to the centre have a limited amplitude. Viceversa, trajectories starting from points far from it have violent and explosive oscillations. This is the further confirmation that the Goodwin's model is not stable: nothing ensures that a trajectory starting from acceptable values in the phase space (u, v) will remain in the same region.

6.1.2 Dissipative and Conservative systems

The most common dynamical economic systems present in the literature are the so called *dissipative systems*. The term stems from the analysis of the physical systems characterized by a permanent input of energy which dissipates over time. If the energy input is interrupted, the system collapses to its equilibrium state. Dissipation in continuous time dynamical systems can be formally characterised by the property that the *divergence* (or *Lie derivative*) is lower than zero:

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} < 0 \tag{62}$$

A contraction of the phase-space volume occurs over time. These systems are not *area/volume preserving* (see fig. 17).
A classical example in economics is the Kaldor (1940) model previously discussed: the equilibrium point of this system is unstable so there is a tendency away from the equilibrium point; a spiraling flow emerges without closed orbits. This behaviour

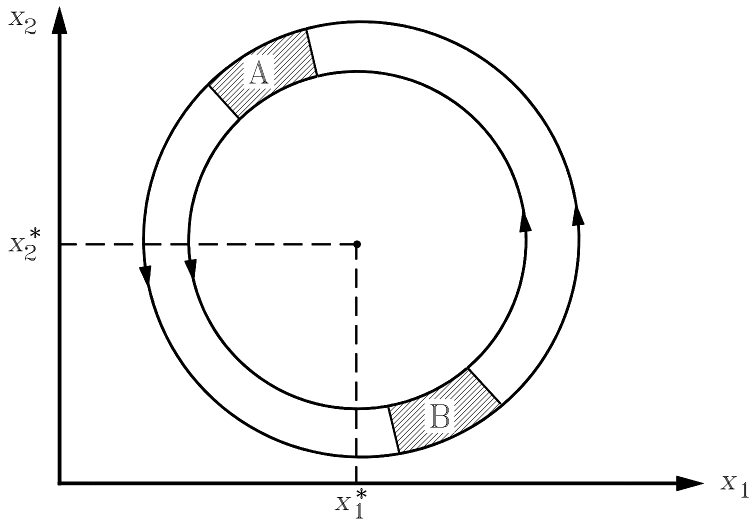


Figure 18: Conservative systems: area preservation.
Source: Lorenz (1993)

is determined by the trace's sign that is positive. Notwithstanding, the trace's sign is reversed on the phase space: a negative trace corresponds to a positive friction, such that the exploding fluctuations will be dampened for points sufficiently far away from the equilibrium point. A closed orbit arises when exploding and imploding forces collide, so that the trace will be zero.
A limit cycle, according to the Bendixon theorem, can exist if the trace's sign is not the same for the entire space:

Theorem 1 (Bendixon's theorem)**.** *Assume the functions f and g of a bi-dimensional dynamical system have continuous first order derivatives in S. If the trace has the same sign throughout S, then there is no periodic solution of the system lying entirely in S.*

Another typology of dynamical systems are the *conservative systems* where no frictions exist because neither inputs nor loss of energy emerge. According to the previous characterization, in conservative systems the trace/*Lie derivative* always equals zero for all points in the phase space. They are *area-volume preserving* systems (see fig. 18):

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = 0 \tag{63}$$

The zero trace implies the fixed points are centres or saddles. In physical systems, the pendulum motion is the classical example.
Regarding the Lotka-Volterra system, which is the underlined mathematical structure of the Goodwin's model, the trace is zero for all the space.

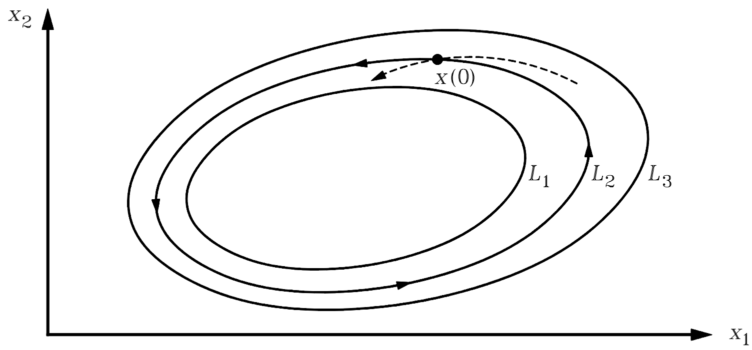


Figure 19: Conservative systems: the dashed line is impossible.
Source: Lorenz (1993)

According to the Hirsch/Smale theorem:

Theorem 2 (Hirsch-Smale's theorem)**.** *Every trajectory of the Lotka/Volterra equations is a closed orbit (except the fixed point (v^*, u^*) and the coordinate axes).*

The theorems stated mean that trajectories in conservative systems cannot cross the closed orbits (see fig. 19). The initial conditions determine which of the infinitely closed orbits describes the behaviour of the system.

6.2 From conservative to dissipative systems

As we saw, the main drawback of the Goodwin's model is its structural instability, that is the high sensitivity that the system shows when it is perturbed by any small perturbation. In the literature we find many attempts that try to overcome this negative aspect: Desay (1974); Flaschel (1984); Wolfstetter (1987); Velupillai (1978); Pohjola (1981). Except for the Pohjola's work, all the other contributions go in the direction of extending the labour share equation of the Goodwin's model. Here we are going to consider the contribution of Flaschel (1984), that is based on the same assumptions of Desai, Wolfstetter and Velupillai. The extension consists of abandoning the real framework of Goodwin and introducing a price equation that determines inflation. Prices are constructed using the mark-up theory, so that workers target, established by the original linear Phillips-curve $f(v) = \dot{w}/w$, now is augmented/diminished by the presence of a mark-up η. The modified Phillips curve has the following form:

$$f(v) = \hat{w} + \eta\pi \tag{64}$$

where $\eta\pi$ is the money illusion that now accounts for wages determination. The price equation has the following form:

$$\pi = g[(1+r)u - 1], \quad g' > 0, \quad g(0) = 0 \tag{65}$$

so that the extended Goodwin's model becomes:

$$\hat{u} = \rho v - \gamma - \alpha + \eta g[(1+r)u - 1] \tag{66}$$

$$\hat{v} = [1/\sigma - (\alpha + \beta)) - 1/\sigma u] \tag{67}$$

The slightly modification of the system can be summarized by the following general expression:

$$\begin{cases} \dot{v}/v = h(u) \\ \dot{u}/u = f(v) - g(u) \end{cases}$$

If we know evaluate the Jacobian matrix at the non trivial fixed point, we do not anymore obtain the usual trace equals zero that determines the conservative structure, but we get always a trace different from zero:

$$\mathcal{J} = \begin{pmatrix} 0 & [g(u^*) - (\alpha + \gamma)]/\sigma\rho \\ \rho[1 - \sigma(\alpha + n)] & -g'(u^*)[1 - \sigma(\alpha + n)] \end{pmatrix}$$

where $u^* = 1 - \sigma(\alpha + n)$ is the equilibrium share of workers in the Goodwin's model.

Theorem 3 (Olech's theorem)**.** *Assume that the Jacobian of the system fulfills:* $\mathrm{Tr}(\mathcal{J}) < 0$, $\det(\mathcal{J}) > 0$ *and* $\mathcal{J}_{12,21} \neq 0$ *everywhere in* $\mathbb{R}^2_+$*. Then, the equilibrium* u^*, v^* *of the system is asymptotically stable in the large, i.e., each trajectory which starts in* $\mathbb{R}^2_+$ *will approach the equilibrium point* (u^*, v^*) *without hitting the boundary of* $\mathbb{R}^2_+$ *.*

The interaction of u and v leads the flow to or away the steady-state depending on the sign of η. For $\eta < 0$ we get an unstable focus (node). For $\eta > 0$ we get a stable focus (node), for $\eta = 0$ we get a centre (bifurcation point), coming back to the Goodwin's original model. The extension proposed by Flaschel suggests how the system cannot be structurally stable since the topological properties are not preserved if the system is perturbed by the money illusion term $\eta\pi$. The money illusion is the term which allows to create a dependency of wages share growth rate not only on the level of employment, but on the level of wages as well. The conservative dynamic is destroyed and the dissipative structure emerges. The system is not anymore able to reproduce permanent oscillations: depending on the values of the parameters we end up in converging or diverging oscillations.

6.3 Discrete time versions

Up to know we have analyzed the Goodwin's model and its extensions in continuous time. Continuous time systems imply that decisions undertaken by agents produce instantaneous changes.
In the Goodwin's framework, two classes of individuals have to take decisions. First, capitalists decide how to invest. Their investment rule is given by the Say's Law, the reason for which the model is sometimes (and rather inappropriately) defined as a *'classical model'*. Second, workers have to decide how to set their wages demand. Modeling both sets of decisions in continuous time sounds quite unrealistic.
Investment is a time consuming process: equipments have to be purchased, stocked, introduced in production and so forth. Entrepreneurs usually make investment plans deciding today how much to invest tomorrow. A time interval that takes at least months exists between investment decision and capital production/utilisation. Investment and disinvestment activities cannot happen in an instantaneous way.
Wage bargaining is a process that takes time as well: labour contracts cannot be instantaneously modified. Workers decide today how much to receive for the labour activity they will offer tomorrow. The laws of motions of the dynamical systems are the functions that describe how individuals behave, without any sound of microfoundation. From our point of view, being investment and wage bargaining decisions characterised by an inherent lumpiness in time, discrete models are more appropriate to represent human decisions processes and physical constraints.
In what follows we are going to discuss two different modifications of the class struggle model, both in discrete time. One is the article of Pohjola (1981), that reduces the original two dimensional system into a one dimensional logistic equation. This contribution seems quite interesting because it allows to get chaotic behaviour. The other one is a contribution by Canry (2005) that is a linking attempt between the classical and the Keynesian specification of endogenous fluctuations.

6.3.1 A model of class-struggle with a chaotic dynamics

The interesting feature present in Pohjola's article is the attempt to obtain a chaotic behaviour from the predator-prey model. First of all, he rephrased the model in discrete time since a chaotic dynamic in continuous time needs at least a third dimension. Further, he substitutes the original Phillips curve equation with the Kuh (1967)'s specification. According to this specification, not the *wage rate* but the *level of wage* depends positively upon employment. This apparent slightly change allows to get a nonlinear first order, difference equation, the well known logistic-equation, that spans, for changes in the parameter value, from a stable equilibrium point, into stable cycles, finally into a chaotic dynamic. Even if they are generated by a deterministic process, chaotic solutions seem indistinguishable from the behaviour generated by stochastic processes. To describe the model we are going to follow the notation used by Sordi (1999).

Labour supply is taken to grow at a constant rate $\beta \geq 0$:

$$n_{t+1} = n_t(1+\beta), \quad \forall t \tag{68}$$

techinical progress is labour augmenting at a constant rate $\alpha \geq 0$

$$\frac{q_{t+1}}{l_{t+1}} = \frac{q_t}{l_t}(1+\alpha), \quad \forall t \tag{69}$$

where q is output and l employment. The capital-output ratio is constant:

$$\frac{k_t}{q_t} = \sigma, \quad \forall t \tag{70}$$

and the savings function is of the classical type (Say's Law), being investments equal profits:

$$k_{t+1} - k_t = (1-u_t)q_t \tag{71}$$

where $u_t = (w_t l_t)/q_t$ is the workers share of output. Writing the employment rate $v_t = l_t/n_t$, we obtain:

$$\frac{v_{t+1}}{v_t} = 1 + \frac{1-\sigma(\alpha+\beta+\alpha\beta-u_t)}{\sigma(1+\alpha)(\beta)} = 1 + \frac{1-\sigma g - u_t}{\sigma(1+g)} \tag{72}$$

where $g = \alpha + \beta + \alpha\beta$.

The only equation modified is the wage bargaining equation: as we said, is not the wage rate, but the wage level that depends upon employment rate and upon labour productivity a_t:

$$w_t = h(v_t)\frac{q_t}{l_t} \cong (-\gamma + \rho v_t)\frac{q_t}{l_t} \tag{73}$$

Thus we get:

$$u_t = \frac{w_t l_t}{q_t} = -\gamma + \rho v_t \tag{74}$$

Finally, substituting equation (74) into equation (72), we get a logistic equation:

$$x_{t+1} = (1+r)x_t(1-x_t) \tag{75}$$

where

$$r = \frac{1-\sigma g + \gamma}{\sigma(1+g)} \tag{76}$$

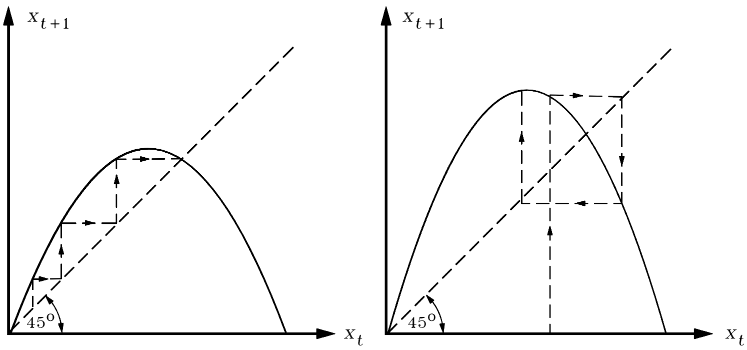


Figure 20: Loss of stability of logistic equation: map for $r < 1$ and $r > 1$.
Source: Lorenz (1993)

$$x_t = \frac{r\rho v_t}{(1+r)(1-\sigma g+\gamma)} \tag{77}$$

The behaviour of the logistic equation:

$$x_{t+1} = F(x_t, r), \quad x_t \in [0,1], \quad r \in [0,3]^{1} \tag{78}$$

is a prototypical example of chaotic dynamic (May, 1976): it is a map in which a so called *route to chaos* occurs by means of *period doubling bifurcation*. A bifurcation is a non linear phenomenon and describes a qualitative change in the orbits structure of a dynamical system when one ore more parameters are changed. The fixed points are determined by the intersection with the bisector. We report the standard values of the parameter r that determines the different behaviours of the map.

If $r < 1$ the phase curve will lie entirely below the $x_{t+1} = x_t$ line in the positive quadrant and $\bar{x} = 0$ is the only fixed point (in fact $\bar{x} = 0$ is an equilibrium $\forall r$). As r increases beyond 1, $\bar{x} = 0$ loses stability, but a new (positive) fixed point, $\bar{x} = 1 - 1/(1+r)$, appears at the intersection of the $x_{t+1} = x_t$ line and the phase curve (see fig. 20). This is locally attracting if the slope of $|F'(\bar{x}, r)| < 1$. The stability condition is $0 < r < 2$. The equilibrium is approached monotonically for $0 < r < 1$ and in an oscillatory fashion for $1 < r < 2$. But what happens if $r > 2$? The fixed point is now repelling but, on the other hand, we know that the solution trajectories are bounded. In order to understand what happens we consider the second iterative of the logistic equation:

$$F^2: \quad x_{t+2} = F(x_{t+1}) = F(F(x_t)) = F^2(x_t) \tag{79}$$

[1]Usually the parameter set is [0,4] with the standard formulation of the logistic equation: $x_{t+1} = r(1 - x_t)$

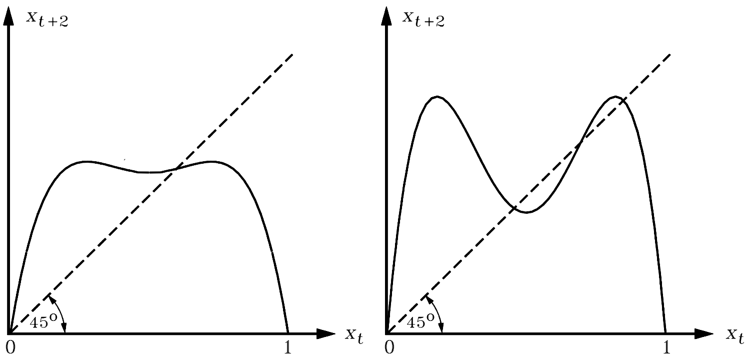


Figure 21: The second iterative in the Logistic map.
Source: Lorenz (1993)

which means that the mapping has to be applied twice:

$$x_{t+2} = (1+r)\left((1+r)(x_t - x_t^2) - (1-r)^2(x_t - x_t^2)^2\right) \tag{80}$$

Fig. 21 illustrates changes in the map F^2 when the value of the parameter is less (first graph) or greater (second graph) than unity. In the first graph, the fixed point of the second iterative is stable since $|F^{2'}(\bar{x}, r)| < 1$. This fixed point is also a fixed point of the first iterative. When the parameter increases, three points of intersection between the map and the bisector emerge. The former stable fixed point will lose its stability because the slope of F^2 is greater than one. At the two newly fixed points the slope of F^2 is smaller than unity. These two new fixed points are stable and they are called *fixed points of order 2*. The meaning of this stability is that a fixed point is reached every second period. But this is true also for the other stable fixed point. Trajectories jump between the two points of intersection from period to period. The stable equilibrium constellation consists in a permanent switching between two values. This situation is called a *period-2 cycle*. A bifurcation called *flip* occurs (see fig. 22).

If the parameter is increased even further, the slope of the second iterative will be larger than 1. The cycle will then become unstable. Each of the 2-period fixed points bifurcates into two new stable fixed points and an unstable one. The four stable fixed points form a period-4 cycle: they are fixed points of the fourth iterative. The period of the cycle has doubled and the afore mentioned bifurcation is called *period doubling* (see fig. 23). After n periods a stable cycle of period 2^n exists (see fig. 24 showing a numerical simulation of ours of the model).
We summarise the results regarding the behaviour of the logistic equation in table (1):

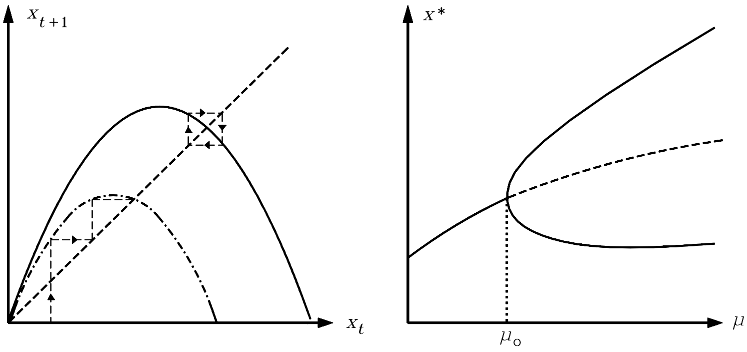


Figure 22: The supercritical flip Bifurcation.
Source: Lorenz (1993)

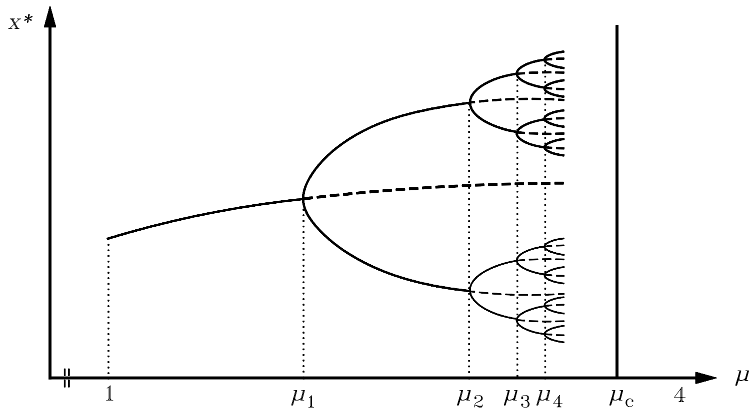


Figure 23: Period doubling bifurcation.
Source: Lorenz (1993)

Dynamic behaviour	**Value of** r
Stable equilibrium point	$0 < r < 2$
Monotonic convergence	$0 < r < l$
oscillations	$1 < r < 2$
Stable cycles of period 2	$2 < r < 2.570$
2-period cycle	$2 < r < 2.449$
4-period cycle	$2.449 < r < 2.544$
8-period cycle	$2.544 < r < 2.564$
Chaotic behaviour	$2.570 < r < 3$

Table 1: Region of parameter in the logistic equation.

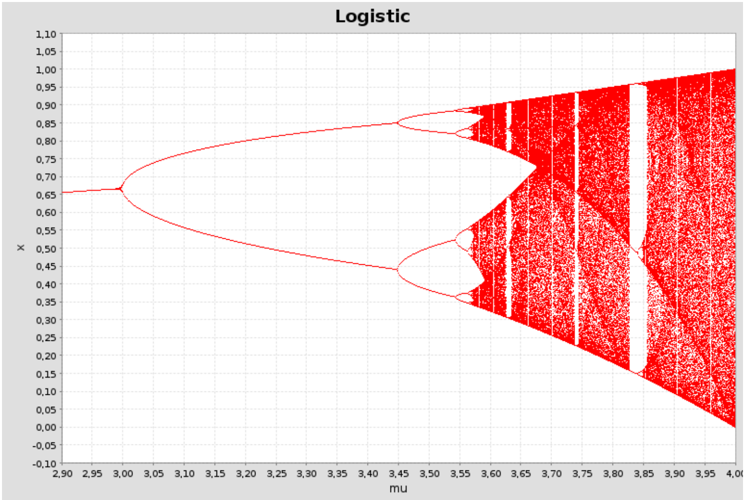
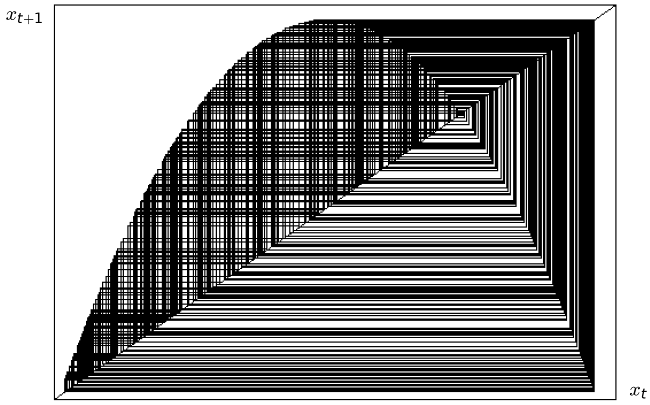


Figure 24: Period doubling bifurcation.
Source: our simulation

Pohjola highlights some implications due to the emergence of such a chaotic behaviour (see fig. 25): it has relevance for business cycle theory and for economic modeling in general. Firstly, the external random shocks used by RBC theorists are not necessary to reproduce fluctuations: they are the result of the combination of nonlinearity and discrete time adjustments, in deterministic equations. Secondly, if the business cycle displays such unpredictable behaviour, measuring the effects of policy intervention becomes a very difficult exercise. Thirdly, it has relevant implications for rational expectations theory: in a chaotic regime, even if agents know how economy functions today, they are not anymore able to predict its behavior tomorrow.
Notwithstanding, the route to chaos determined by the *period doubling bifurcation*, is called *deterministic chaos*: successive branches of the bifurcations tend to an accumulation point, so the chaotic region can be confined.

6.3.2 A model of wage-led vs. profit-led dynamics

The other model we are going to discuss is the one proposed by Canry (2005): this model tries to combine the traditional investment equation (Say's Law) with a demand-effect, that resembles the Keynesian tradition. In particular, the investment function does not simply depend upon savings that equal profits, but also on demand. In a Goodwin's model, recessions periods generate their own recovery thanks to the investment take off boosted by higher profits. In a Keynesian framework, recovery might not happen either because the economy is *wage-led* (see



Ergodic Behavior in the Logistic Equation; $\mu = 3.99$

Figure 25: Chaotic behaviour.
Source: Lorenz (1993)

Bhaduri et al., 1990), so investments growth less than savings when profits increase, or because investments are affected by the negative influence of a shortcoming in aggregate demand. Investment evolution depends on two antagonist effects: the positive impact of profits accumulation (classical approach) and the negative impact of economic activity decrease (Keynesian accelerator).

Output is determined by the aggregate demand that depends upon wage share. There is a threshold w^* that separates profits and wages led economies. Below w^*, increases in wage share outweigh decreases in profits (wage-led economy), so the accelerator effect is higher than the classical effect in determining investment. Above w^*, wages increases depress investment (profit-led economy) more than they stimulate demand: the classical effect prevails on the accelerator effect in determining investments. The interesting feature is that the model is able to reproduce a Goodwinian type equilibrium (the centre) and a Keynesian type disequilibrium.

The firm production function is:

$$Y_t = a_t L_t \tag{81}$$

The rates of growth of labour productivity a_t and labour force n_t are respectively α and β.
Differently from the Goodwin's model, aggregate activity depends upon aggregate demand and not upon the current stock of capital. At each period t consumption and investment determine production:

$$Y_t = C_t + I_t \tag{82}$$

Aggregate production equals aggregate income that is split into wages and profits. Capitalists invest (save) all their income (profits); workers consume all their wages. Consumption tomorrow depends upon current wages:

$$C_{t+1} = w_t L_t \tag{83}$$

Capitalists investment's rule depends upon profitability and upon demand they face. Profitability is measured by the profit share of the previous period. Demand is captured by the current level of consumption, so the author proposes a multiplicative form between profits share and aggregate consumption:

$$I_{t+1} = \eta C_{t+1}(1 - u_t) \quad \eta > 1 \tag{84}$$

where $(1 - u_t)$ is the profit share and η is an accelerator coefficient. Substituting equation (84) and equation (83) into equation (82) we get:

$$Y_{t+1} = Y_t[1 - (1 - u_t)(1 - \eta u_t)] \tag{85}$$

This low of motion determines the output dynamic that has an inverted U shaped form. Defining the employment rate as usual:

$$v_t = \frac{L_t}{N_t} = \frac{Y_t}{a_t N_t} \tag{86}$$

we obtain:

$$\frac{v_{t+1}}{v_t} = \frac{Y_{t+1} a_t N_t}{Y_t a_{t+1} N_{t+1}} = \frac{[1 - (1 - u_t)(1 - \eta u_t)]}{(1 + \alpha)(1 + \beta)} \tag{87}$$

Regarding labour market, Canry uses the usual Phillips curve in discrete time, but inserting, at first glance, a slightly modification, that determines a relevant change in the structure of the original Goodwin's model. The wage rate in Canry's formulation depends upon current employment level and not upon past employment level:

$$\frac{w_{t+1} - w_t}{w_t} = \rho v_{t+1} - \gamma, \quad \rho, \gamma > 0 \tag{88}$$

instead of the more meaningful expression:

$$\frac{w_{t+1} - w_t}{w_t} = \rho v_t - \gamma, \quad \rho, \gamma > 0 \tag{89}$$

This reversed interpretation comes from a (economically questionable) discretization of the continuous time version of the Phillips curve:

$$\frac{\dot{w}}{w} = \rho v - \gamma, \quad \rho, \gamma > 0 \tag{90}$$

Apart from the questionable economic meaning, the Phillips curve introduced by Canry determines a huge effect in terms of the structure of the system. It allows to transform the original Goodwin system:

$$\begin{cases} (u_{t+1} - u_t)/u_t = f(v_t) \\ (v_{t+1} - v_t)/v_t = g(u_t) \end{cases}$$

into:

$$\begin{cases} (u_{t+1} - u_t)/u_t = f(v_t, u_t) \\ (v_{t+1} - v_t)/v_t = g(u_t) \end{cases}$$

In fact from the modified Phillips curve we get:

$$w_{t+1} = \left(\frac{\rho}{1+\alpha} v_{t+1} + \frac{1-\gamma}{1+\alpha} \right) w_t \tag{91}$$

The analytical form of the system is:

$$\begin{cases} v_{t+1} = \left(\frac{[1-(1-u_t)(1-\eta u_t)]}{(1+\alpha)(1+\beta)} \right) v_t \\ u_{t+1} = \frac{1}{1+\alpha} \left(\frac{[1-(1-u_t)(1-\eta u_t)]}{(1+\alpha)(1+\beta)} \rho v_t + (1-\gamma) \right) w_t \end{cases}$$

It has three equilibria: the trivial one that is a locally stable point, a saddle point and a centre (see fig. 26)
The passage from the centre to the trivial equilibrium happens by means of a saddle (see fig. 27).
Varying the value of the parameter ρ the system bifurcates: when $\rho < 0.6$ the system oscillates around the centre. When $0,6 < \rho < 0,7$ the system bifurcates: the orbits enter in the basin of attraction of the saddle point. When $\rho > 0,7$ the system collapses into the trivial equilibrium point. It is remarkable the constant periodicity of employment rate and wage share trajectories along time (see figg. 28, 29) that gives origin to the invariant orbit around the centre.
Increasing values of ρ determine the exit from the Goodwin/profit led-region of the system: cyclical dynamic is interrupted because wages have been too much squeezed during a recession. As a result, consumption brings down investment in its fall, in spite of profit share recovery. As soon as investment increases when consumption drops, cycles are maintained, although the model is demand constrained. However, if consumption becomes too weak, due to very low wages, it may offset the positive effect of profit recovery on investment.

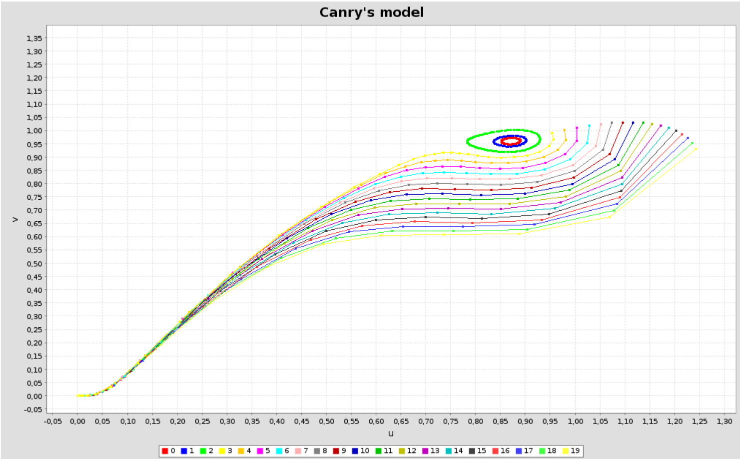


Figure 26: Dynamic of the system: different trajectories related to different initial conditions. In the neighborhood of the centre an infinite number of closed orbits arises. When the initial points are below the stable manifold of the saddle point, trajectories converge to the trivial equilibrium.
Source: our simulation

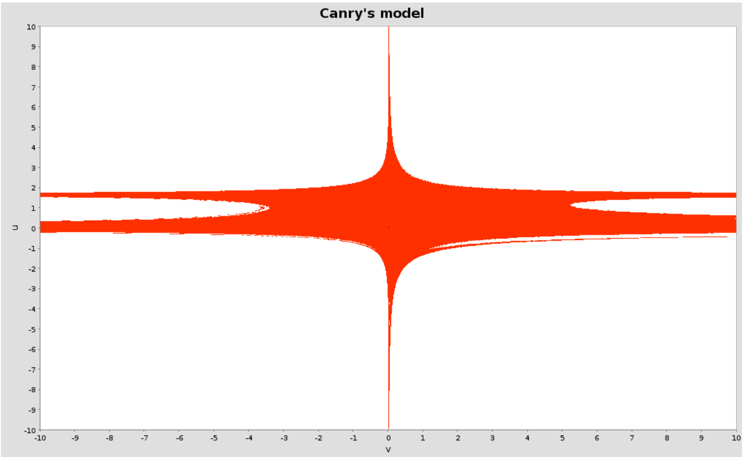


Figure 27: Basin of attraction: the red region depicts the basin of attraction of the trivial equilibrium point. In the white region between the two red branches of the basin, we observe closed curves.
Source: our simulation

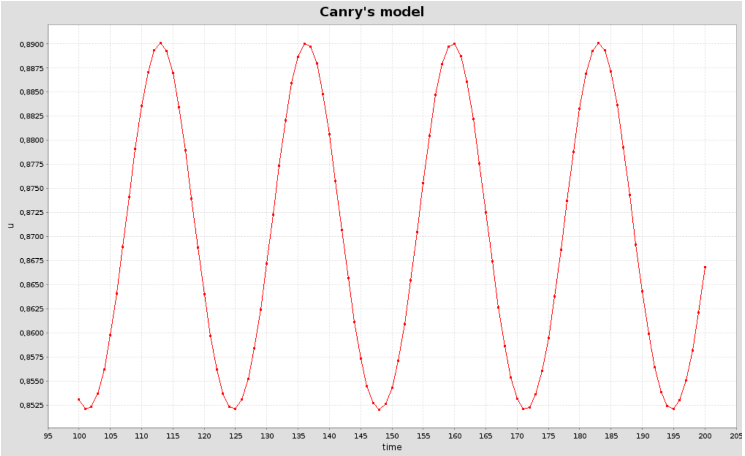


Figure 28: Wage share time series.
Source: our simulations

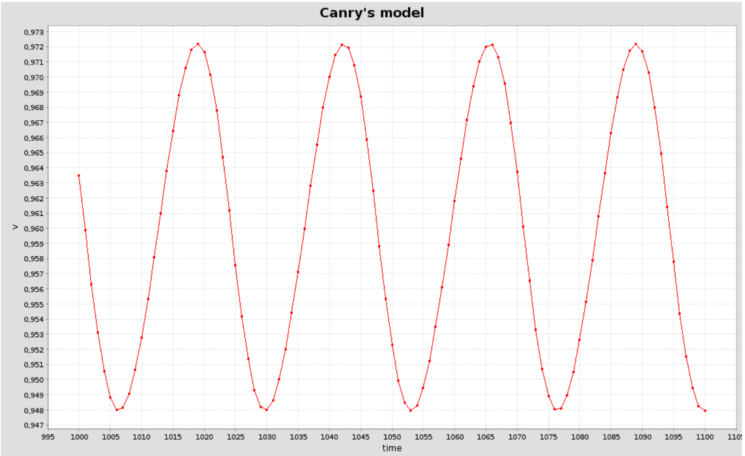


Figure 29: Employment rate time series.
Source: our simulation

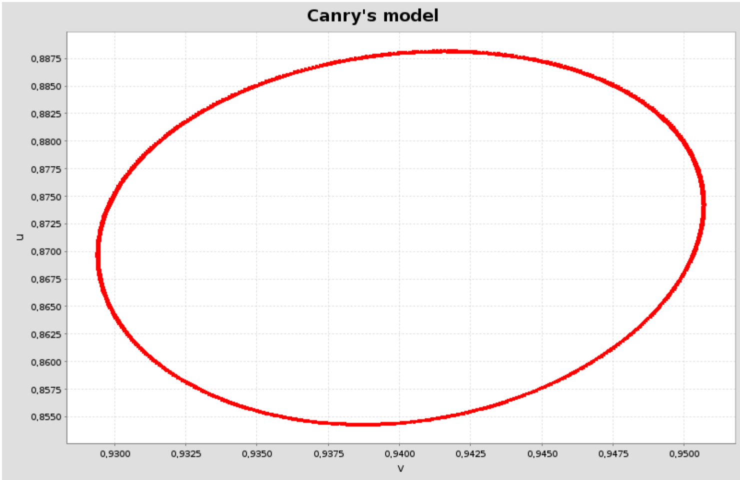


Figure 30: The state space shows the emergence of an invariant orbit.
Source: our simulation

Consumption and investment may thus fall together inducing a cumulative slump. Cycles disappear because neither consumption nor investment can restore growth. Slowdown exacerbates wage-share fall, whereas the economy is in the wage-led area: wages should increase to boost growth.

7 Exploring extensions of the Marx-Goodwin's model

Let us try to move some steps toward the construction of a model where both Keynesian and Marxian features live together. An important aspect is the time formalization: in modeling the system, we use discrete time which we consider more appropriate compared to continuous time in representing economic decisions (see Section 6.3 for further explanations). The first step we undertake is reformulating the Goodwin's model in a discrete time version. Studying the model analytically and via simulations, we identify a generic explosive behaviour of the system. Such structural instability makes impossible to get the same results obtained in the continuous version of the model. Next, we introduce a generalized extension of the system, that recovers a richer dynamic. The generalized version is built on a *coupled dynamic model*. We obtain both a limit cycle and a chaotic behaviour that occur by means of a Sacker and a period-doubling bifurcations. Indeed in one of the model formulations we get all the results above, related to the class-struggle model and its extension.

7.1 A discrete time version of the Goodwin's growth cycle model

The model, as we said, is a discrete time reformulation of the original one, but it presents some slightly modifications,

$$Y_t = AK_t \tag{92}$$

meaning that the output-capital ratio (Y/K) is constant and equals $A > 0$, such as in the Goodwin's original model; the dynamic of capital is similar to the one considered by Pasinetti (1960):

$$K_t = (1-\delta)K_{t-1} + I_t \tag{93}$$

where $0 < \delta < 1$ is the constant rate of capital depreciation.

$$L_t = \frac{Y_t}{a_t} \tag{94}$$

Labour demand L equals total output over labour productivity a_t. Labour productivity grows at a constant, exogenous rate $\alpha > 0$:

$$a_t = a_{t-1}(1+\alpha) \tag{95}$$

The current wage rate depends on lagged wages plus a correction factor constisting in the difference between the past employment rate and the "equilibrium" value, the zero wage-inflation rate of employment;

$$w_t = w_{t-1}(1 + \lambda(v_{t-1} - \bar{v})) \tag{96}$$

λ parametrizes the strength of workers reaction to "dis-equilibrium". Differently from the linearized Phillips curve [2] present in the growth cycle, in our equation, the coefficient multiplies the deviation from equilibrium. We assume that $\lambda < 1$. The employment rate is defined as the ratio of total labour demand over labour supply:

$$v_t = \frac{L_t}{N_t} = \frac{Y_t}{N_t a_t} = \frac{AK_t}{N_t a_t} \tag{97}$$

Population growth rate is assumed to grow at a constant rate $\beta > 0$:

$$N_t = N_{t-1}(1+\beta) \tag{98}$$

Finally, investments are function of the share of profits gained in the previous period, where $0 < s < 1$ is the capitalists' propensity to save:

$$I_t = s\pi_{t-1} = sY_{t-1}(1-u_{t-1}) \tag{99}$$

where:

$$u_t = \frac{w_t}{a_t} \tag{100}$$

is the workers output share. Making the necessary substitutions, we get a system of two equations in two variables, with six parameters:

$$\begin{cases} K_t = (1-\delta)K_{t-1} + sAK_{t-1}(1-u_{t-1}) \\ w_t = w_{t-1}\big(1+\lambda(v_{t-1}-\bar{v})\big) \end{cases}$$

Rewriting the system in terms of v_t and u_t we get:

$$\frac{v_t}{v_{t-1}} = \frac{K_t N_{t-1} a_{t-1}}{K_{t-1} N_t a_t} = \frac{1-\delta+sA(1-u_{t-1})}{(1+\alpha)(1+\beta)} \tag{101}$$

$$\frac{u_t}{u_{t-1}} = \frac{L_t a_{t-1}}{L_{t-1} a_t} = \frac{(1+\lambda(v_{t-1}-\bar{v}))}{1+\alpha} \tag{102}$$

From this expression, we rephrase the original Goodwin's model in the following form:

[2] Indeed, the labour market equation of the Growth-cycle, being expressed in real terms, is not exactly a Phillips curve which is a negative relation between changes in the money wage rate and the unemployment rate. It lies in between the P.C. and the so called Wage Curve. The last one is a real relation between the *levels* of the wage rate and the unemployment rate (see Blanchflower et al., 1994)

$$\begin{cases} v_t = v_{t-1}\big(1 - \delta + sA(1 - u_{t-1})\big)/(1+\alpha)(1+\beta) \\ u_t = u_{t-1}\big(1 + \lambda(v_{t-1} - \bar{v})\big)/(1+\alpha) \end{cases}$$

Such as in the predator-prey relation, the employment rate is the *'prey'* (because the wage rate has a positive relation with employment), while the share of wages is the *'predator'* (since increases in wage share depress the profit share that positively affects the level of activity). In order to study the property of the system, we perform the usual analysis of the fixed points. The system presents two fixed points: a trivial and a non trivial one.

$$(v^* = 0, u^* = 0) \quad \left(v^* = \frac{\alpha + \lambda\bar{v}}{\lambda}, u^* = 1 - \frac{\delta + g}{sA}\right) \tag{103}$$

where $g = \alpha + \beta + \alpha\beta$. The trivial one is a saddle node, the other one is an unstable focus. Unfortunately, the nature of centre that distinguishes the continuous version is not anymore obtainable. Performing simulations, we found that variables manifest an explosive behavior (see fig. 31, 32, 33). What we conclude is that the structural instability that characterises the Goodwin's system in continuous time determines the skip from a centre to an unstable focus. The Goodwin's cycle is not robust to the discretization process. The same results obtained by our simulations are discussed in Sordi (1999):

> *As was to be expected, given the centre character of the non-trivial fixed point of the model framed in differential equations and its structural instability [Medio (1979, pp. 39-40), Velupillai (1979)], the change in the time concept results in a qualitatively different behaviour of the solution, even in the case in which all other assumptions are kept unchanged.*

7.2 Overcoming the structural instability: a generalised version of the Goodwinian growth cycle with a Keynesian component

After having performed simulations of our discrete time version, a first remark has to be pointed out: while the discrete-time version presented by Canry reproduces, at least for some parameters values, the invariant closed orbits found by Goodwin, the results of our discretization are completely different: as we showed, we obtain an unstable focus. The different results are due to the different specification of the two models. The crucial point in the Canry's one is not a theoretical meaningful insight, such as for example, the introduction of the Keynesian framework; it is the specification of the Phillips Curve in terms of the current employment level. As we saw, this kind of specification allows to express the wage share rate variation both

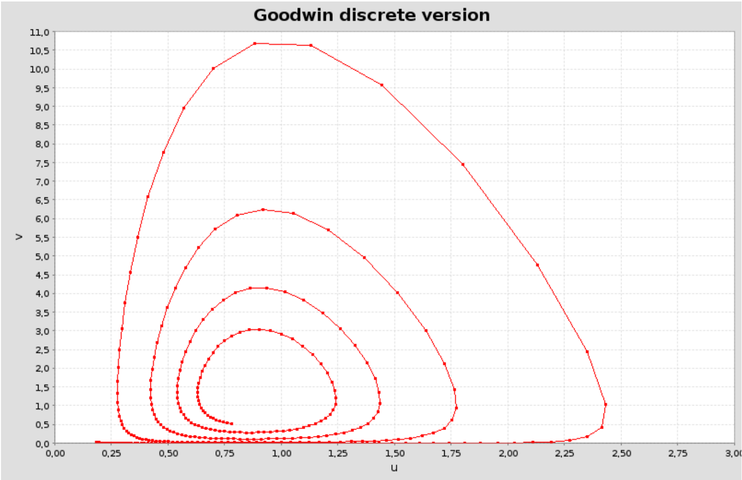


Figure 31: Explosive behaviour. State space.
Source: our simulation

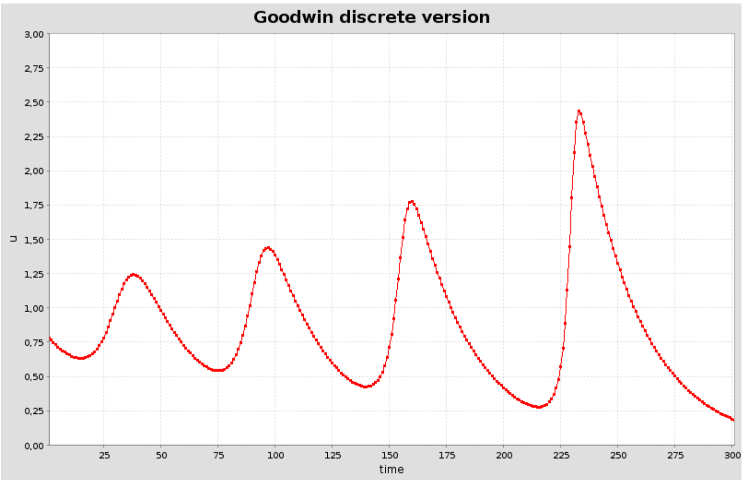


Figure 32: Wage share time series.
Source: our simulation

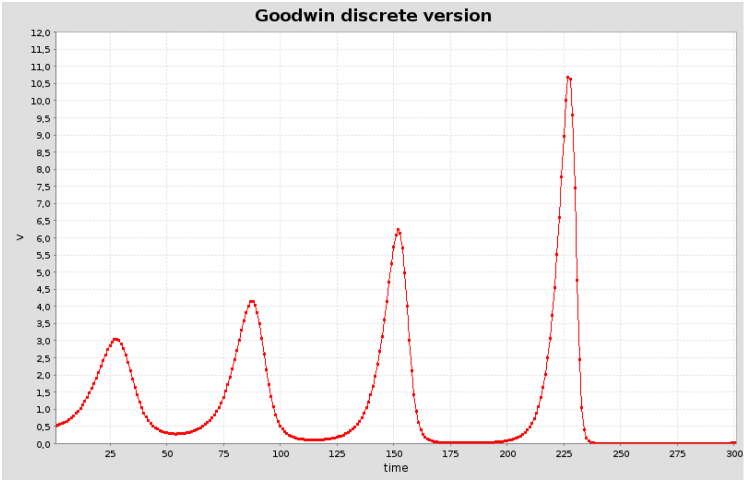


Figure 33: Employment rate time series.
Source: our simulation

in terms of the employment level and the wage share level. The question we are going to address is how the structural instability can be overcome: in the following section, we are going to analyse a generalized discrete time formulation of the original Lotka-Volterra system, in order to understand what kind of dynamics the system exhibits. As we observed previously, the original Goodwin's system can be expressed by the following discrete formalization[3]:

$$\begin{cases} (x_{t+1} - x_t)/x_t = e + by_t = f(y_t) \\ (y_{t+1} - y_t)/y_t = f + dx_t = g(x_t) \end{cases}$$

The discretization we used is the simplest one, the method of finite difference:

$$\dot{x} \simeq \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \tag{104}$$

We can approximate $\dot{x} \simeq x_{t+1} - x_t$

$$\frac{\dot{x}}{x} \simeq \frac{x_{t+1} - x_t}{x_t} \tag{105}$$

The same kind of discretization process is used by Goodwin (1989). In the Lotka-Volterra's framework, $d > 0$ and $b < 0$. In this specification, the variation rate of

[3]In the current section we are going to replace the usual notation of u and v with y and x respectively. Moreover we are not anymore constrained by the necessity to obtain less than unity outcomes since we do not treat shares or relative variables.

each variable depends only upon the other variable. The tracked way in continuous time to make the system structurally stable has been creating a dependence of at least one rate of variation not only on the level of other variable but on itself level. What we are going to analyse is the effect of a reciprocal interdependency of both variation rates on the levels of both variables. This is a generalized version of the Goodwin's model, where $a, c = 0$:

$$\begin{cases} (x_{t+1} - x_t)/x_t = ax_t + by_t + e = f(x_t, y_t) \\ (y_{t+1} - y_t)/y_t = cy_t + dx_t + f = g(x_t, y_t) \end{cases}$$

The first equation describes output growth rate's dynamic, the second equation wages growth rates dynamic. As said before, both the *output rate variation* and the *wage rate variation* depend upon the *level of wages and level of activity* of the previous period. Maintaining the same assumptions of the original model, we continue to assume that $b < 0$ and $d > 0$.
Following Medio (1979), we discuss the possible economic interpretation of the partial derivatives in this framework.

1. $\partial f(x_t, y_t)/\partial y_t < 0$: it represents the so called *'profits effect'* meaning that the higher the level of wages, the lower the output growth rate. A reduction in the profit margin will decrease resources available for investment activity. For any given output-capital ratio, high wage boosts will reduce the profit rate leading to the detriment of investment activity. In our formulation the parameter b embodies the profit effect and it is defined, recalling the original Goodwin's model, as: $0 < b \simeq 1/\sigma(1+\alpha)(1+\beta) < 1$. This parameter enters with a negative specification in the model.

2. $\partial f(x_t, y_t)/\partial x_t \gtreqless 0$ if $x_t \lesseqgtr 0$: it represents the *'demand effect'* meaning that the higher the *level* of activity, the higher the output growth rate up to the point it will be equal to x_0, where the last term is the so called *'normal'* level of employment. Above this point the output growth rate will increase only if the profit margin increases, implying an expansion in the productive capacity. This term embodies a Keynesian effect creating a dependence of output growth rate on demand. After having reached the maximum available productive capacity, in order to get positive income growth rate, a positive investment activity is necessary. Entrepreneurs acquire new capital inventories driven by the high consumption activity, that gives them the insurance to make future profits.

3. $\partial g(x_t, y_t)/\partial x_t > 0$: it represents the *employment effect* expressed by the Phillips curve, meaning that the higher the *level* of activity, the higher the wage rate

variations. It embodies the assumption that workers contractual power positively depends upon the level of employment. The parameter's definition is $0 < d \simeq \rho/(1+\alpha) < 1$. It enters in the model with a positive specification

4. $\partial g(x_t, y_t)/\partial y_t < 0$: it represents the so called *mark-up* effect, assuming that firms operate in a imperfect competition framework. When monetary wages growth rate is higher than labour productivity growth rate, capitalists increase prices. Since the increase in prices is not immediately compensated by an increase in monetary wages of the same magnitude, the wage growth rate is eroded. The prices growth rate depends upon the level of wages weighted by a mark up factor.

5. The two constants are present in the original Goodwin's model. In particular $0 < e \simeq (1+\sigma)/\sigma(1+\alpha)(1+\beta) < 1$ and $f \simeq (1-\gamma)/(1+\alpha) \lesseqgtr 0$.

The theoretical strength of this reformulation is the consideration of a *path dependence* in the rate of growths that each variable presents, creating a *coupled dynamic* interaction. Path dependence means that history and initial conditions matter in order to explain results.

7.2.1 Analysis of the system and simulations results

The fixed points of the system are fours:

$$(x^* = 0, y^* = 0), \quad (x^* = 0, y^* = -\frac{f}{c}), \quad (x^* = -\frac{e}{a}, y^* = 0) \tag{106}$$

and

$$(x^* = \frac{bf - ec}{ca - bd}, y^* = -\frac{-de + fa}{ca - bd}), \tag{107}$$

The partial derivatives of the system are:

$$\mathcal{J}_{11} = \frac{\partial f(x_t, y_t)}{x_t} = 2ax + by + e + 1, \quad \mathcal{J}_{12} = \frac{\partial f(x_t, y_t)}{y_t} = xb \tag{108}$$

$$\mathcal{J}_{21} = \frac{\partial g(x_t, y_t)}{x_t} = yd, \quad \mathcal{J}_{12} = \frac{\partial g(x_t, y_t)}{y_t} = 2cy + dx + f + 1 \tag{109}$$

The Jacobian matrix is:

$$\mathcal{J} = \begin{pmatrix} 2ax + by + e + 1 & xb \\ yd & 2cy + dx + f + 1 \end{pmatrix}$$

According to our hypothesis the signs and the magnitude of the parameters are the following:

$$-1 < b < 0, \quad -1 < c < 0, \tag{110}$$

$$0 < d < 1, \quad 0 < e < 1, \tag{111}$$

The only two parameters that can both be positive or negative are a, f; moroever we assume the magnitude of f can be greater than unity:

$$-1 < a < 1, \quad -2 < f < 2, \tag{112}$$

The stability conditions for a two dimensional map follow the usual characterization: a fixed point $\bar{x}$ is (locally) asymptotically stable if the eigenvalues λ_1 and λ_2 of the Jacobian matrix, calculated at the fixed point, are less than one in modulus. The necessary and sufficient conditions ensuring that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ are:

$$1 + \mathrm{Tr}(\mathcal{J}_{\bar{x}}) + \det(\mathcal{J}_{\bar{x}}) > 0 \tag{113}$$

$$1 - \mathrm{Tr}(\mathcal{J}_{\bar{x}}) + \det(\mathcal{J}_{\bar{x}}) > 0 \tag{114}$$

$$1 - \det(\mathcal{J}_{\bar{x}}) > 0 \tag{115}$$

The system shows the emergence of two different typologies of bifurcations. Each of the two generic bifurcations results from the loss of stability through the violation of one of these conditions:

i The Neimark–Sacker bifurcation occurs when the modulus of a pair of complex, conjugate eigenvalues is equal to one. Since the modulus of complex eigenvalues in $\mathbb{R}^2$ is simply the determinant of $\mathcal{J}$, this occurs at $\det(\mathcal{J}_{\bar{x}}) = 1$. If, moreover, conditions (113) and (114) are simultaneously satisfied (i.e., $\mathrm{Tr}(\mathcal{J}_{\bar{x}}) \in [-2, 2]$), there may be a Neimark bifurcation.

ii The flip bifurcation occurs when a single eigenvalue becomes equal to -1 that is, $1 + \mathrm{Tr}(\mathcal{J}_{\bar{x}}) + \det(\mathcal{J}_{\bar{x}}) = 0$, with $\mathrm{Tr}(\mathcal{J}_{\bar{x}}) \in [0, -2]$, $\det(\mathcal{J}_{\bar{x}}) \in [-1, 1]$ (i.e., conditions (114) and (115) are simultaneously satisfied).

The reformulation of the Hoph theorem in discrete time ensures the existence of the limit cycle showed in fig. 38.

Theorem 4. *Let the mapping $x_{t+1} = F(x_t, \mu)$, $x_t \in \mathbb{R}^2$, $\mu \in \mathbb{R}$, have a smooth family of fixed points $x^*(\mu)$ at which the eigenvalues are complex conjugate. If there is a μ_0 such that:*

$$|\lambda(\mu_0)| = 1, \quad \lambda^n(\mu_0) = \pm 1, \quad n = 1, 2, 3, 4 \tag{116}$$

and

$$\frac{d|\lambda(\mu_0)|}{d\mu} > 0 \tag{117}$$

then there is an invariant closed curve bifurcating from $\mu = \mu_0$.

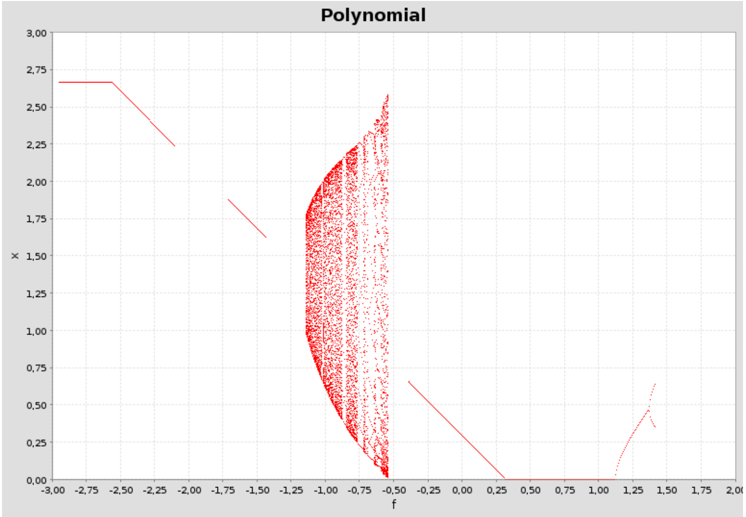


Figure 34: Neimark-Sacker bifurcation on x. From the stationary state, the x variable undergoes a Neimark-Sacker bifurcation. Then the stationary state becomes again stable.
Source: our simulation

The requirement for the Neimark-Sacker bifurcation is that the complex conjugate eigenvalues cross the unit circle, i.e., that $|\lambda| = 1$ at the bifurcation point $\mu = \mu_0$. Furthermore, it is required that the roots do not become real when they are iterated on the unit circle: the first four iterations λ^n must also be complex conjugate. Finally, the eigenvalues must cross the unit circle with nonzero speed for varying μ at μ_0.

Performing simulations we find a very rich dynamic of this generalised model. We found the emergence of both the Neimark-Sacker bifurcation (that allow to get the invariant orbit), so replicating results obtained by Goodwin, and chaotic dynamics. The parameters we analyze in order to observe if bifurcations of the system occur are $f = (1-\gamma)/(1+\alpha)$ and a. The observed Neimark-Sacker (see fig. 34) bifurcation occurs when the parameter f is in the range [-1.2,-0.5] and $a < 0$. The signs of all the other parameters is the one expressed before: $b < 0, c < 0, d > 0, e > 0$. Fig. 36 and fig. 37 show the effect of the different magnitude of the parameter b in determining the formation of the limit cycle. Fig. 38 is the state space obtained with the same parameters values that generate the Neimark-Sacker. A further confirmation of the existence of the invariant orbits is given by the periodic oscillations of the two variables presented in fig. 39 and in fig. 40. Finally the basin of attraction (see fig. 41) that shows how the behaviour of the fixed points (black dots) leads to the closed curve.

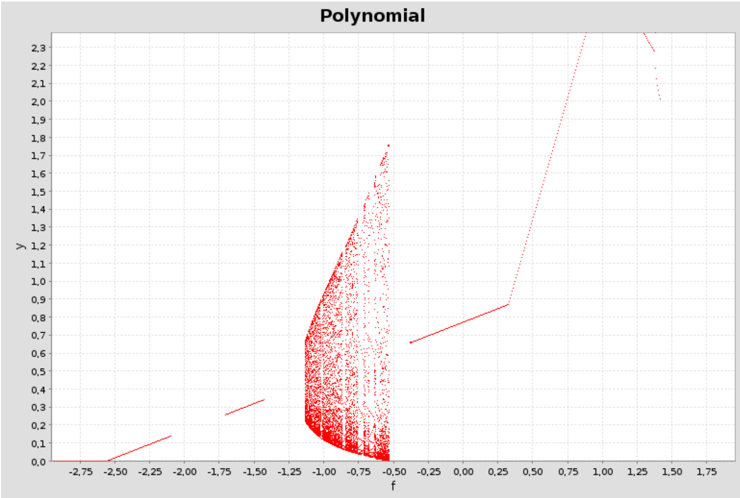


Figure 35: Neimark-Sacker bifurcation on y.
Source: our simulation

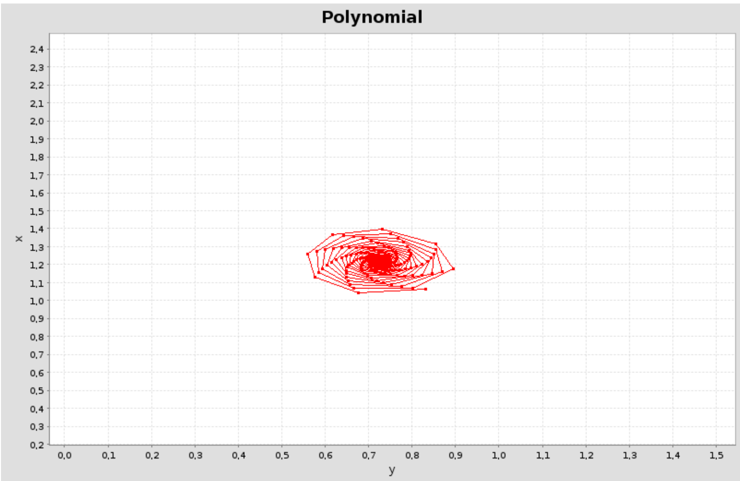


Figure 36: Limit cycle formation: increasing magnitude of the profits effect.
Source: our simulation

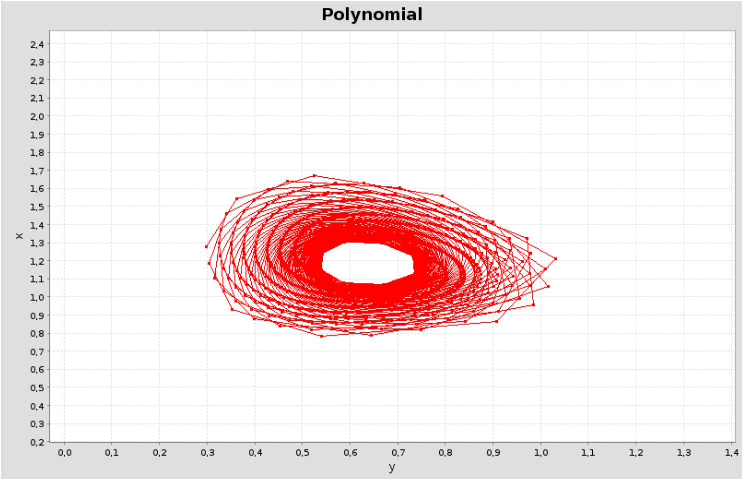


Figure 37: Limit cycle formation: increasing magnitude of the profits effect.
Source: our simulation

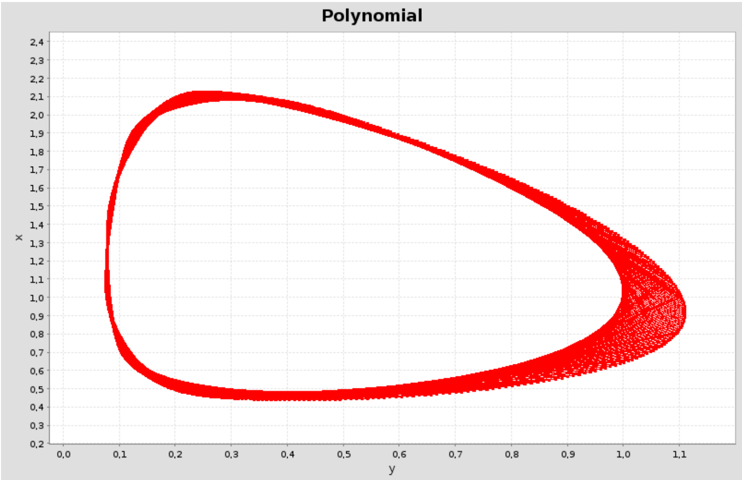


Figure 38:
The limit cycle relative to the Neimark-Sacker bifurcation. The solid part is given by the high number of iterations converging to the limit cycle.
Source: our simulation

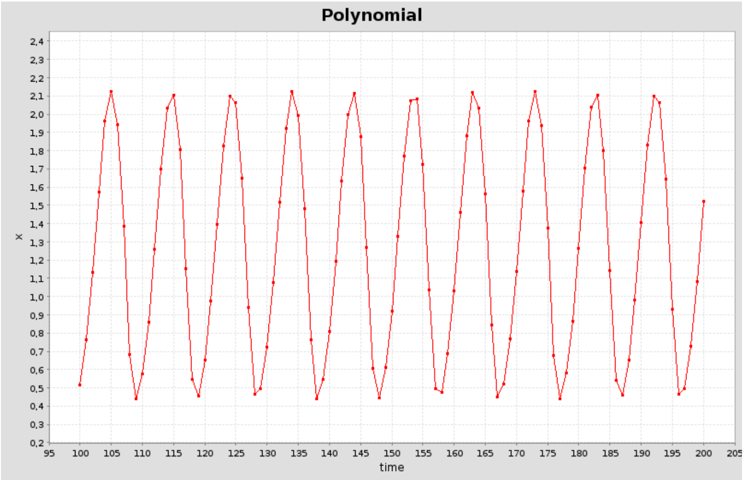


Figure 39: x-time trajectory.
Source: our simulation

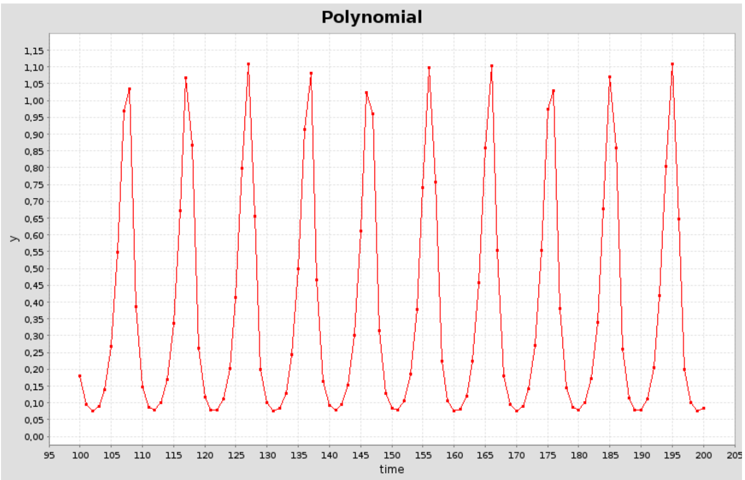


Figure 40: y-time trajectory.
Source: our simulation

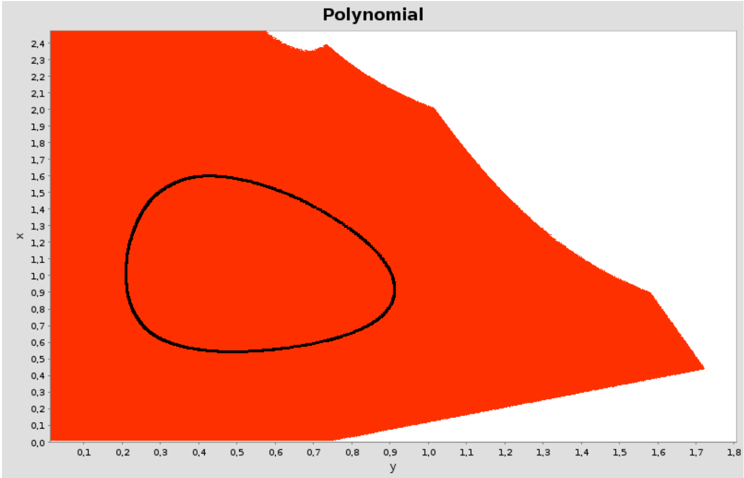


Figure 41: Basin of attraction represented by the red area. The white area is the region of unfeasible trajectories.
Source: our simulation

After having observed how the formation of the limit cycle occurs, we are going to present the results regarding the flip bifurcation. In figures 43 and 44 we show the emergence of the flip bifurcation for both variables with the corresponding basin of attraction in fig. 48 and 49, where are respectively illustrated the period 2-cycle and the period 4-cycle. The discontinuity presents in the flip bifurcation is due to the coexistence of multiple attractors (see fig. 45). The existence of multiple attractors emphasizes the role of initial conditions: the system shows a quite apparent path dependency, since depending on the initial conditions, it will end up in different *"states of the world"*. When the negative magnitude of a increases the system shows chaotic behaviour as presented in fig. 46 and in fig. 47.
Invariant, attracting sets and attractors with a structure more complicated than that of periodic or quasi-periodic sets are called chaotic. A discrete or continuous time dynamical system is chaotic if its typical orbits are aperiodic, bounded and such that nearby orbits separate fast in time. Chaotic orbits never converge to a stable fixed or periodic point, but exhibit sustained instability, while remaining forever in a bounded region of the state space. They are, as it were, trapped unstable orbits.

The chaotic attractors presented in fig. 50 and in fig. 51 illustrates the passage from the period-4 cycle that merges in two pieces of chaotic attractors toward a chaotic regime. The form of the attractor resembles the Henon attractor (see fig. 42). The sudden merging of two (or more) chaotic attractors or two (or more) separate pieces of a chaotic attractor, taking place when they simultaneously collide with an unstable fixed or periodic point (or its stable manifold) is called *crisis*. Crises of this type have been numerically observed, for example, in the Henon map (see Medio

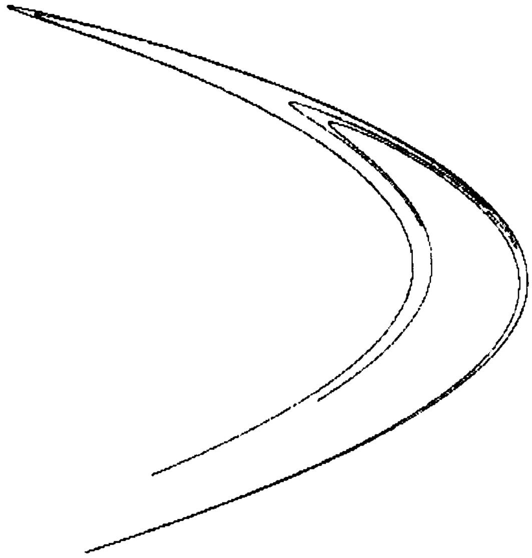


Figure 42: Henon attractor.
Source: Medio et al. (2003)

et al., 2003). Finally, in fig. 52 and in fig. 53 the time trajectories of variables give proof of the chaotic behaviour. In order to grasp the combined effect of the two parameters a and f, we realized a two parameters diagram bifurcation showed in fig. 54. The red area shows the combination of the parameters values a, f where the system is stable; the white area represents the regions of the parameters where occur the passage from stability to instability (possible emergence of the Neimark-Sacker bifurcation and of chaotic dynamic); the blue area the regions of period-2 cycle; the yellow area the parameters regions of period-4 cycles. Finally the black one shows the combination of a and f that determines divergent, unfeasible oscillations.

7.2.2 Interpretative notes

One aim of this work was realizing a discrete time version of the class struggle model discussed by Goodwin. The inherent structural instability was the obstacle to obtain similar results to the continuous case. We already discussed how the property of the Growth Cycle is not anymore valid in discrete time (persistent harmonic oscillations turn into explosive oscillations). On the other hand, Goodwin's model has a main theoretical limitation, from our point of view, that is the implicit Say's Law that drives the economy. Indeed, it is a supply-side model. A Keynesian investment function where past income variations influence current decisions is not take into consideration. The extension presented in the previous section is both an attempt to overcome the structural instability in a discrete framework, but

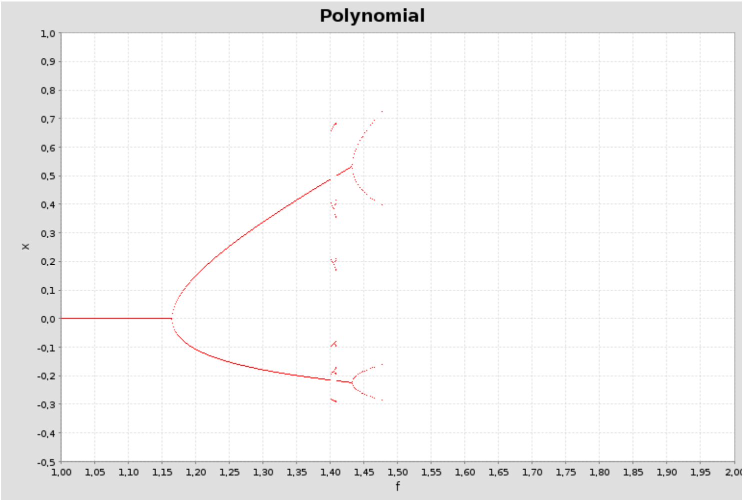


Figure 43: x: flip bifurcation.
Source: our simulation

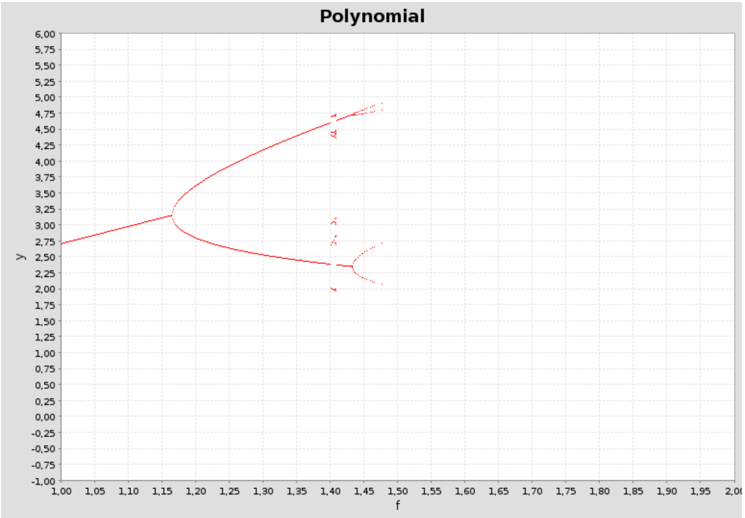


Figure 44: y: flip bifurcation.
Source: our simulation

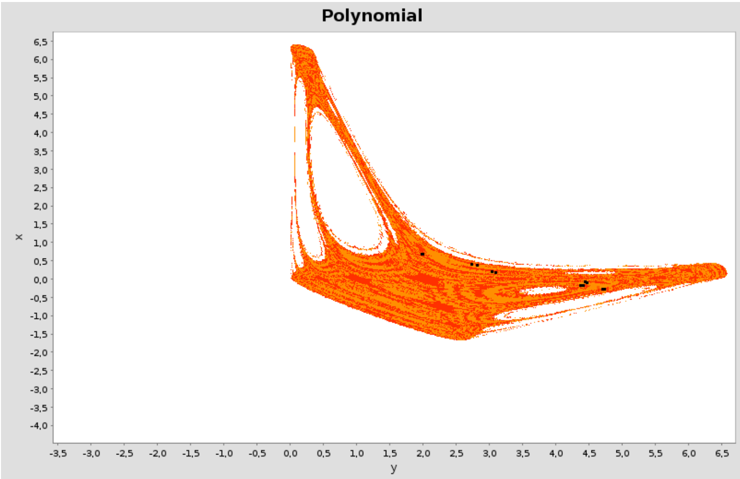


Figure 45: Multiple attractors.
Source: our simulation

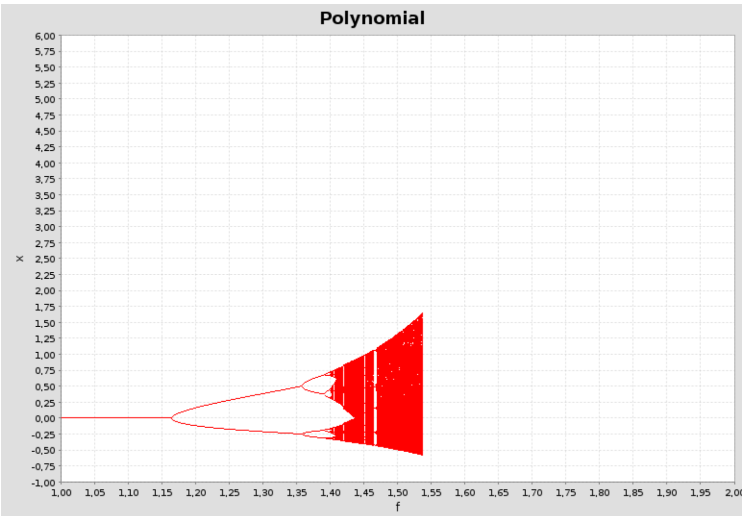


Figure 46: x: Routes to chaos.
Source: our simulation

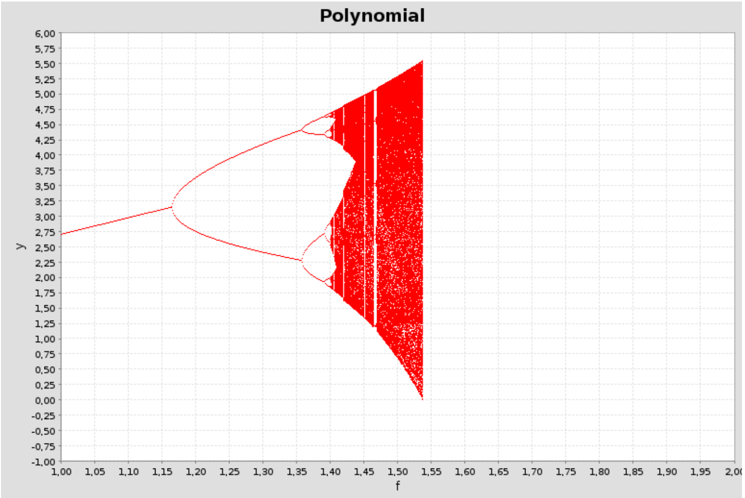


Figure 47: y: Routes to chaos.
Source: our simulation

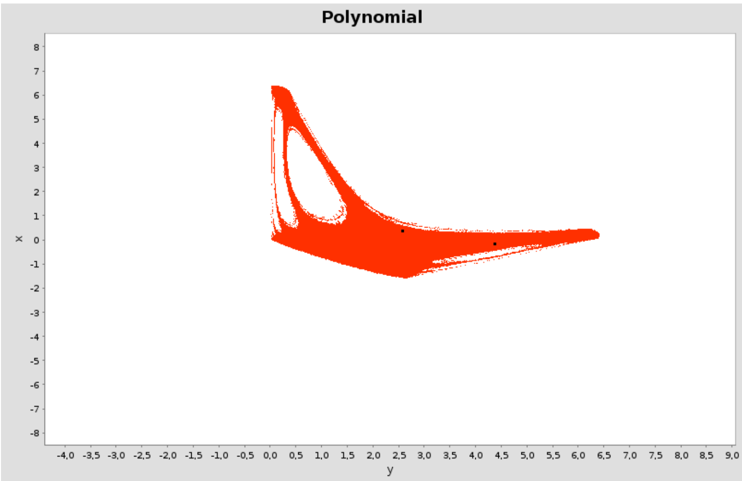


Figure 48: Period-2 cycle.
Source: our simulation

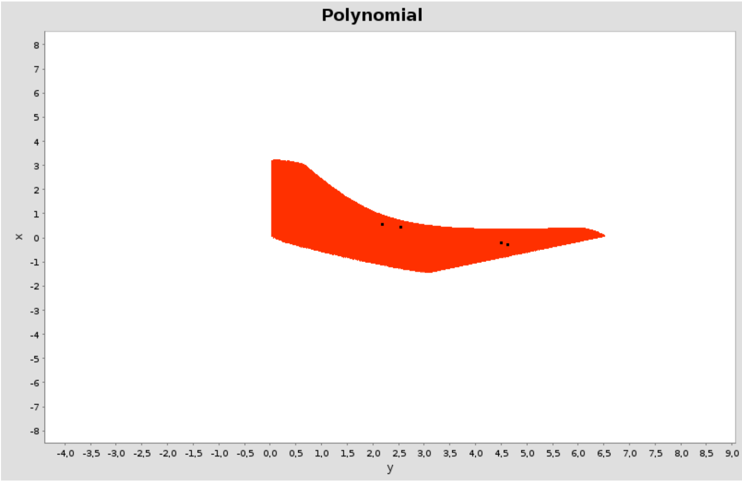


Figure 49: Period-4 cycle.
Source: our simulation

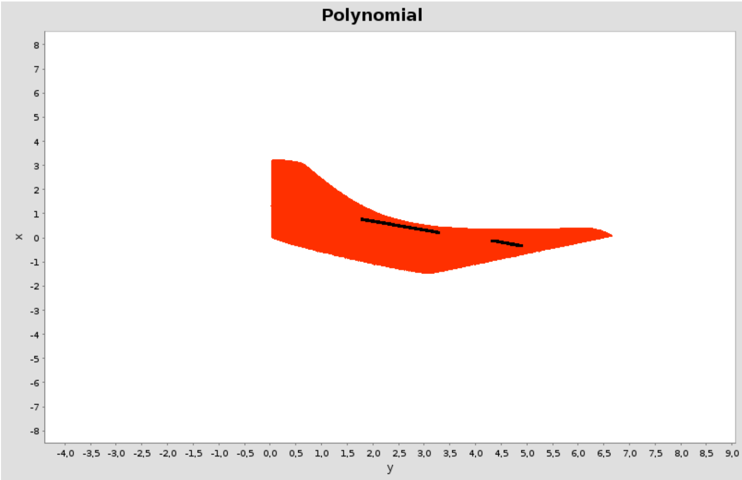


Figure 50: Formation of the Henon-like attractor.
Source: our simulation

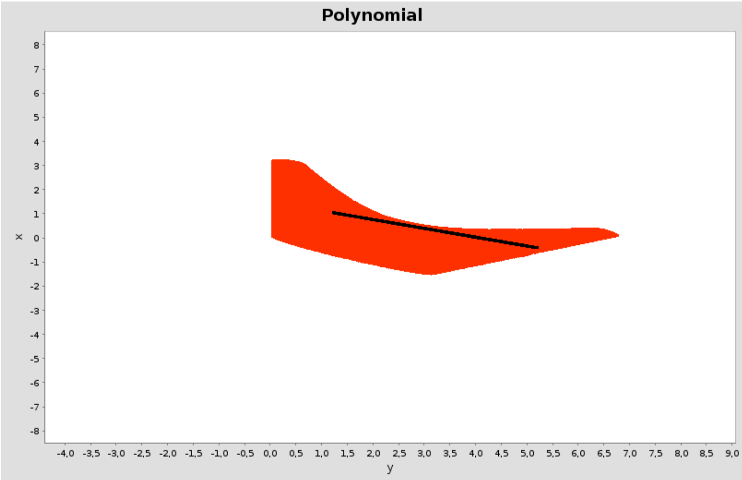


Figure 51: Henon-like attractor. Chaotic regime.
Source: our simulation

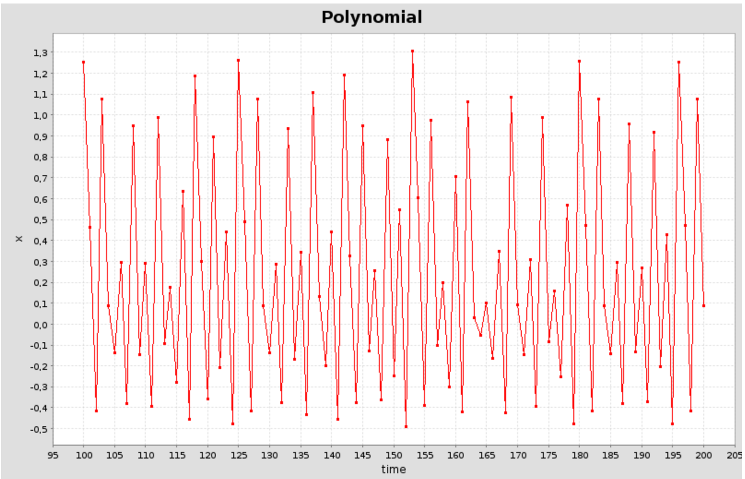


Figure 52: x chaotic oscillations.
Source: our simulation

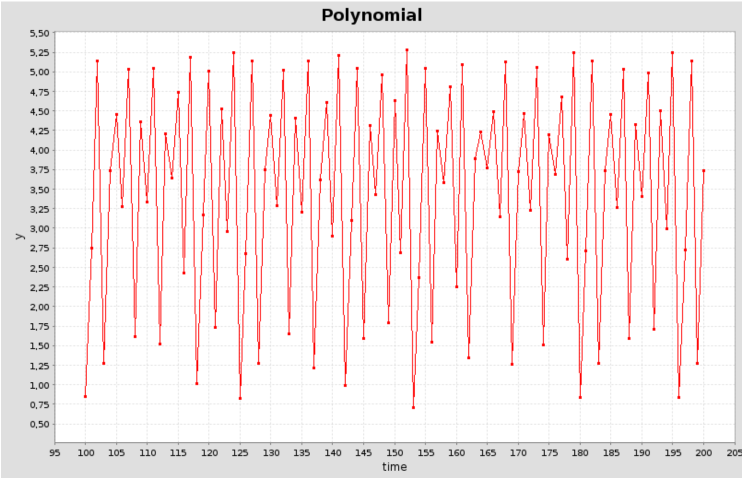


Figure 53: y chaotic oscillations.
Source: our simulation

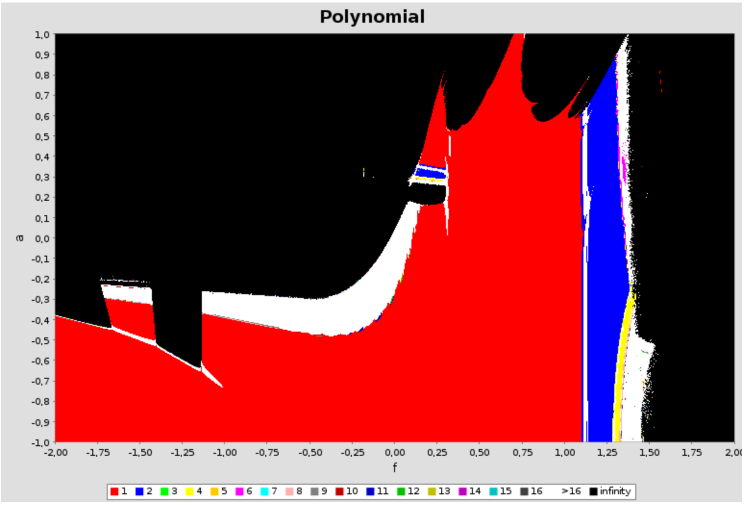


Figure 54: 2 parameters bifurcation diagram.
Source: our simulation

also the possibility to give a more complex and interesting explanation to the behaviour of the economy. In particular, the extension consists in introducing a demand effect in explaining output growth rate and a mark-up effect in explaining wages growth rate. While the mark-up effect has a defined negative influence on monetary wages growth rate (being excluded any instantaneous adjustment mechanism that reflects prices increases in wages increases), the demand effect is not unequivocally determined. The last figure (54) gives a rough idea of what happens in the selected parameters space. In particular, the white area describes both the periodic oscillations of income and wages already present in the predator-prey model and the emergence of a chaotic dynamic. The meaning of the economic cycle, which emerges when the Neimark-Sacker bifurcation occurs, is the following: for low level of employment, income and output growth rate are lower than the productivity and population growth rate. Wages grow less than productivity hence the incentive of capital accumulation becomes very high and greater than the disincentive to invest, due to the low level of consumption. Output growth rate starts to increase, pushed by the high capital accumulation, at a rate higher than productivity and population. Capitalists need to hire labour, so employment increases, but monetary wages carry on to decrease, up to the point they reach a minimum threshold. From this point on, workers contractual power increases, obtaining nominal wages higher than productivity and than prices growth rate: the share of wages over income increases. The reverse ordering takes place: capital accumulation is discouraged by low profits rate, so output growth rate starts to decrease. The level of employment goes down, workers contractual power diminishes and wages growth rate is lower than productivity and prices growth rate. In the white region where the profits effect is higher than the demand effect, the economy is still driven by a classical engine. The interpretation we provide is that, in the white region where chaotic oscillations replace quasi-periodic orbits (see the right hand side of fig. 54), the demand effect is higher than the profits effect: capital accumulation is higher sensitive to changes in consumption, but also the *animal spirits* take a crucial role in determining investment decisions. The interesting feature is that as soon as we introduce a Keynesian effect, embodied by the influence of past income on current output growth rate, resembling a rudimentary, first approximation form of an accelerator, the chaotic motion occurs. The so called *demand effect* lays actually in between an accelerator and a hysteresis effect in determining aggregate demand. The chaotic dynamic refers to the occurrence of crises and unpredictable events generated by an inherent instability: no cyclical constant fluctuations that could be predictable by agents occur. Notwithstanding, the route to chaos that happens in our model is determined by period doubling bifurcations, the so called safe boundary bifurcations. This kind of chaotic behaviour can be signaled by the parameters continuous changes.

Chaotic dynamics emerge in one region: for positive, high level of f. What is the meaning of the parameter f? As stated before: the magnitude and the sign of $f = 1 - \gamma / \ 1 + \alpha$ basically depends upon γ, assuming that productivity growth rate $\alpha \simeq 0.03$ is constant. γ is the vertical intercept of the linearized Phillips curve. The region in which a cycle emerges (the white area in the left hand side of fig. 54), where a form of stability (even though fluctuating) exists, is the region where $\gamma > 1$, being f negative. Since the parameter enters the linearized Phillips curve with a negative specification, it basically means that the monetary wage growth rate will become inflationary after having passed the NAIRU threshold. We are exactly in the area where the main properties of the *profit-led* economy described by Goodwin are valid. The economy is *profit-led* and wages negatively affect output growth. The system enters into a period-doubling bifurcation when f is higher than unity: for the parameter's range [1,1.30] the value of γ is lower than one. The parameter f is able to capture a form of hysteresis in determining the wage level, representing the effect of past wage on current wage level. Hysteresis means that the natural rate of unemployment endogenously changes. In this region we capture a very broad and rudimentary *wage-led* economy that is characterized by unstable conditions with eventually lead to chaos. According to the Post Keynesian approach, in absence of policy interventions, the NAIRU is unstable. Quoting Stockhammer (2008):

> *The inverse real balance effect and a wage-led demand regime do have an important consequence: the equilibrium will become unstable. If wages increase growth, growth increases employment and higher employment improves the bargaining position of labor, then a deviation from equilibrium will be self-sustaining.*

The chaotic dynamic obstacles the possibility to get any sort of prediction about the future behaviour of the system.

Concluding, as we said before, analysing one model, we recover Goodwin's cyclicality and chaotic dynamics as well. Differently from one would expect, introducing a demand component has a partly destabilizing role.

8 Conclusions

In this work we analysed some theories on endogenous business cycle, with particular attention to the Samuelson's, Kalecki's and Kaldor's models characterized by a Keynesian root, and to the Goodwin's model characterized by a classical root. A research study we implicitly followed is the investigation of the causality direction of the well-known Kaleckian *Investment=Profit* equation: is the profit accumulation that pushes output growth or are expected future profits that determine current investment? We devoted particular attention to the Goodwin's model and to its extensions, developing in the final section a generalized discrete time version of the class struggle model. The aim of the formulated model is twofold: from the one hand, overcoming the structural instability of the Goodwin's one (being the property of the Lotka-Volterra system not robust in discrete time); from the other hand, comparing Keynesian and Classical roots in leading to output fluctuations. Performing simulations we got the same results of the original Goodwinian one, for some parameter ranges. Additionally, our model is able to generate endogenous chaotic dynamics. The rich dynamics comes from the introduction of a coupled interaction of both variables in determining the output and the wage growth rate. It introduces the relevance of the *path dependence* in explaining variables movements. In particular, our reformulation takes into account two effects: a *demand effect*, as a first rudimental approximation of an accelerator, in influencing output growth rate, and a *hysteresis effect* in wage formation. This reformulation, which expresses the relevance of history in explaining current results, allows to compare the behaviour of the system when investments are wage-led (Keynesian approach) and when they are profit-led (Marxian approach). The interesting puzzle is that, for some parametrization, a profit-led economy has a more stable path (harmonic predictable oscillations) than a wage-led economy (erratic oscillations). Unlike the role of demand is usually considered having a stabilizing effect, according to the Post-Keynesian tradition its combined effect with hysteresis could create self-sustaining disequilibrium paths, in absence of any policy intervention. The main question that remains open is how to interpret empirical results on the ground of economic intuitions provided by this model. It is a challenging conjecture that demands corroboration both on the modeling side and on the econometric side.

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