TRAFFIC MODELS FOR DYNAMIC SYSTEM OPTIMAL ASSIGNMENT

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Abstract

Most analyses on dynamic system optimal (DSO) assignment are done by using a control theory with an outflow traffic model. On the one hand, this control theoretical formulation provides some attractive mathematical properties for analysis. On the other hand, however, this kind of formulation often ignores the importance of ensuring proper flow propagation. Moreover, the outflow models have also been extensively criticized for their implausible traffic behaviour. This paper aims to provide another framework for analysing a DSO assignment problem based upon sound traffic models. The assignment problem we considered aims to minimize the total system cost in a network by seeking an optimal inflow profile within a fixed planning horizon. This paper first summarizes the requirements on a plausible traffic model and reviews three common traffic models. The necessary conditions for the optimization problem are then derived using a calculus of variations technique. Finally, a simple working example and concluding remarks are given.

1. Introduction

Dynamic traffic assignment (DTA) models are used to determine the network traffic pattern over time. A DTA model comprises two components: a traffic model and a travel choice model. The traffic model represents the propagation of traffic through the network. In general, traffic models can be classified into two different categories: outflow models and travel time models. A key difference between an outflow model and a travel time model is that the outflow model first determines the link outflow profile according to the given outflow function and the current traffic conditions, and then back calculates the corresponding link travel time. In contrast, the travel time model first determines the link travel time according to the given travel time function and the current traffic conditions, and then calculates the outflow profile.

The travel choice model determines the route choice and the departure time choice of each traveller in a road network. This is done based upon two principles: dynamic user equilibrium (DUE) assignment and dynamic system optimal (DSO) assignment. Under DUE assignment, travellers are assigned such that for each origin-destination (O-D) pair in the network, the total travel costs experienced by travellers, no matter which combination of travel routes and departure times they choose, are equal and minimal. DSO assignment assumes that travellers will cooperate in making their travel choices for the overall benefit of the whole system instead of their own individual benefits. Although the traffic pattern under DSO assignment may be regarded as unrealistic, it can provide a useful benchmark to evaluate various traffic management strategies.

Most DSO analyses are done by using the control theory with an outflow traffic model. On the one hand, this control theoretical formulation provides some attractive mathematical properties for analysis. On the other hand, however, this kind of formulation often ignores the importance of ensuring proper flow propagation. In addition, the outflow models have also been widely criticized for their implausible traffic behaviour. This paper aims to provide another framework for analysing a DSO problem, based upon sound traffic models. In the next section, we first summarize the requirements on a plausible traffic model. In section three, we give a brief review on three different traffic models: the outflow models; the deterministic queue model and the linear travel time model. In section four, we derive the necessary conditions for a DSO assignment problem. The assignment problem aims to minimize the total system cost in a network by seeking an optimal inflow profile within a fixed planning horizon. Different from the conventional control theoretical approach, we explicitly add a constraint to ensure proper flow propagation and adopt a calculus of variations technique to solve for the optimal solution. Finally a simple working example is illustrated in section five and concluding remarks are given in section six.

2. Desirable properties of traffic models

A traffic model is considered to be plausible if it satisfies the following five requirements.

1. Positivity - If we input a non-negative inflow profile, $e(s)$, into a traffic model, the corresponding state variables, include the outflow profile, $g(s)$, the amount of link traffic, $x(s)$, and hence the link travel time, given by this traffic model should also be non-negative.

2. First-in-first-out (FIFO) - If a traveller departs from the origin earlier, then he/she can expect he/she should be able to arrive at the destination earlier. In other words, the arrival time of a traveller should be always positively proportional to his/her corresponding departure time. This principle can be represented by $\dot{t}(s) > 0$, where $t(s)$ is the corresponding time of exit for a traveller enters the link at time s and the dot superscript refers to the derivative of a function with respect to time.

3. Flow Conservation - The rate of change of amount of link traffic, $x(s)$, at any time s should be always equal to the difference between the inflow rate and the outflow rate of the link at that time. This can be described by the differential equation $\dot{x}(s) = e(s) - g(s)$.

4. Flow Propagation - The cumulative traffic enters up to time s must has exited from the link by exactly time $t(s)$. This can also be expressed as $E(s) = G[t(s)]$, where $E(s)$ and $G[t(s)]$ correspond to the cumulative inflow by s and the cumulative outflow by $t(s)$ respectively. Differentiate both sides with respect to time s, we have $e(s) = g[t(s)]\dot{t}(s)$. The relationship shows the rate of flow along a vehicle trajectory should accord to $\dot{t}(s)$.

5. Causality - Causality states that traffic behaviour should be affected by local conditions and conditions downstream only, but not by conditions upstream. In other words, the exit time and corresponding instantaneous outflow for an inflow at time s, should only depend on the inflow at or before time s but not after.

3. Analysis of traffic models

3.1 Outflow models

Outflow models were first introduced by Merchant and Nemhauser (1978). Outflow rate from each link is considered to be a non-decreasing function of the amount of whole link traffic $x(s)$. The evolution of the state variable $x(s)$ is governed by the state equation $\dot{x}(s) = e(s) - g[x(s)]$ as flow conservation.

Outflow models, on the one hand, provide a linear and continuous state equation for analysis. On the other hand, they have also been extensively criticized for their implausible traffic propagation. For example, the models may lead to zero travel time for some travellers and infinitely long ones for the others (Astarita, 1996). Furthermore, Heydecker and Addison (1998) showed that the outflow models structurally violate causality. Given $x(s) = E(s) - G(s)$?and $E(s) = G[t(s)]$, it follows that $x[t(s)] = E[t(s)] - E(s)$. The instantaneous outflow $g[t(s)]$ depends on $x[t(s)]$ and hence on the inflow profile during $(s, t(s)]$. That is, the outflow $g[t(s)]$ depends on the inflow after the departure time s, and this is an obvious violation of causality.

The cell transmission model (CTM) proposed by Daganzo (1994) is also a outflow model. However, causality is observed in CTM. CTM discretizes each travel link into shorter segments or "cells". The outflow rate g_i is considered to be a function of the amount of traffic $x_i(s)$ in each cell i. This model differs from the other conventional outflow models by looking at g_i at one time step forward ($s+\Delta s$), rather than at the current time s, and it turns out that causality is satisfied. The problem of causality violation in outflow models may be bypassed using this discretization technique, while the details are out of the scope of the present paper.

3.2 Travel time models

In principle, a link travel time model can depend on any state variables such as link inflow rate, link outflow rate and amount of link traffic. However, Daganzo (1995) recognized that the travel time model should only depend on the amount of link traffic in order to ensure FIFO. Although Carey et al. (2003) have proposed another travel time model, which depends on the link inflow and outflow rates and is shown to be satisfying causality and FIFO, this paper will still focus on the former.

3.2.1 Deterministic queue model

The deterministic queue model, which is also known as the bottleneck model (Vickrey, 1969), satisfies all requirements in section two. This model corresponds to a freely flowing link with a flow-invariant travel time f together with a deterministic queue at its downstream end. The link capacity which refers to the maximum service rate of the queue is denoted as Q. The deterministic queue model states that when a queue exists, the link outflow is equal to the capacity and all travellers arrive before the queue dissipates will incur travel delay. Otherwise, when the queue length is zero, the outflow is taken as the inflow at the time of entry and the travellers are unimpeded. The outflow for each link thus can be expressed as

$$\tilde{g}(s) = \begin{cases} e(s) & (L(s) = 0,\ e(s) < Q) \\ Q & \text{otherwise} \end{cases} \tag{1}$$

where $\tilde{g}(s) = g(s+f)$. The state variable $L(s)$ refers to the amount of traffic that will be encountered in the queue by a traveller who enters the link at time s. The state variable is developed according to the state equation $\dot{L}(s) = e(s) - \tilde{g}(s)$. Using (1), the state equation can be also written as

$$\dot{L}(s) = \begin{cases} 0 & (L(s) = 0,\ e(s) < Q) \\ e(s) - Q & \text{otherwise} \end{cases} \tag{2}$$

Finally, the time of exit is given by $t(s) = s + f + L(s)/Q$. We should point out that this model can be difficult to analyse as $L(s)$ is not differentiable at $e(s)=Q$. This may also cause the optimisation problem become non-convex.

3.2.2 Linear Travel Time Model

Friesz et al. (1993) introduced another satisfactory travel time model that can be used in place of the deterministic queue model. The model considers the delay component of the link travel time to be a linear function of the amount of whole-link traffic $x(s)$ at the time of entry s to the link. In this case we have the state equation $\dot{x}(s) = e(s) - g(s)$. The functional form for the time of exit is given by $t(s) = s + f + x(s)/Q$ as before but with the present state variable. The outflow experienced by traffic that enters at time s can be established according to correct flow propagation (Heydecker and Addison, 1998) as

$$g[t(s)] = \frac{Qe(s)}{Q + e(s) - g(s)} \tag{3}$$

which depends on outflows at time s and hence on inflows at earlier times. The state equation can then be re-written as

$$\dot{x}(s) = e(s) - \frac{Q[s(s)]}{Q + e[s(s)] - g[s(s)]} \tag{4}$$

where $s(s)$ satisfies $s = s(s) + f + x[s(s)]/Q$, and it is regarded as an inverse function of $t(s)$. This travel model is more suitable for analysis as the state variable is smooth and continuously differentiable with time.

4. Analysis of dynamic system optimal

We seek an optimal inflow profile $e(s)$ that minimizes the total travel cost in the network within a fixed planning period T In this study, we consider three distinct components of the total travel cost associated with the chosen departure time. The first component is the travel time associated with amount of link traffic at the departure time s, which we denote as $y[x(s)] = f + x(s)/Q$. This travel time is determined by the travel time models specified in section three. Then, we add to this a time-specific cost associated with arrival time at the destination $t(s) = s + f + x(s)/Q$, which we denote as $f[t(s)]$. Finally, we further extend this by adding a time-specific cost associated with departure from the origin at time s, which we denote as $h(s)$. Possible choices of the departure and arrival time-specific cost functions are investigated by Heydecker and Addison (2004). In addition to their specifications, we further require that the sum of $h(s) + f[s+f]$ is a convex function of s, such that the overall objective function will be convex with respect to $e(s)$, as the travel time models considered in this paper are also convex. This construction means that any stationary point will be a minimum point of the objective function.

To enhance analytical tractability and facilitate understanding, we consider a simple network with only one route with one O-D pair. The objective function for the optimization problem is formulated as

$$\min_{e(s)} Z = \int_0^T \{h(s) + y[x(s)] + f[t(s)]\} e(s) ds \tag{5}$$

and is subject to the following set of constraints

$$\dot{x}(s) = e(s) - g(s) \tag{6}$$

$$g[t(s)]t'(s) = e(s) \tag{7}$$

$$\dot{E}(s) = e(s) \tag{8}$$

$$e(s) \geq 0 \tag{9}$$

$$g(s) \geq 0 \tag{10}$$

$$x(s) \geq 0 \tag{11}$$

$$g(s) \leq Q \tag{12}$$

$$E(T) = \overline{E} \tag{13}$$

Equations (6), (7) and (8) are the state equations for $x(s)$, $t(s)$ and $E(s)$ correspondingly. They actually also represent the flow conservation, the flow propagation and the evolution of cumulative inflow. Conditions (9) – (11) ensure the positivity of the traffic flow variables. The maximum outflow is captured by (12). Finally, (13) defines the total throughput for the whole study period. Note that we do not add an explicit constraint for FIFO as Carey (1992) showed that adding FIFO constraint will affect the structure of the formulation and cause the problem become non-convex. Indeed, our travel time models can satisfy FIFO structurally without any explicit constraint (Mun, 2001). The present analysis is developed with linear travel time model. However, the analysis can also be done with deterministic queue model. To do this, we can replace the flow conservation condition (6) by (2); replace the flow propagation condition (7) by (1); and change the notation $x(s)$ to $L(s)$.

The optimization problem involves finding a temporal inflow profile rather than a fixed value for the inflow. The problem thus is a dynamic optimization problem. To derive the optimality conditions, we first augment the objective function Z with the constraints to obtain

$$Z^* = \int_0^T \left\{ \begin{array}{l} \{h(s) + y[x(s)] + f[t(s)]\}e(s) + l(s)\{[e(s) - g(s)] - \dot{x}(s)\} \\ \quad + g(s)\{e(s) - g[t(s)]\dot{t}(s)\} \\ \quad + m(s)[e(s) - \dot{E}(s)] \\ \quad + n(T)[\overline{E} - E(T)] + h(s)[g(s) - Q] + r(s)e(s) + k(s)g(s) + z(s)x(s) \end{array} \right\} ds \tag{14}$$

where $l(s), g(s)$ and $m(s)$ are called the multipliers or costate variables for the state equations (6), (7) and (8). $r(s), k(s), z(s), h(s)$ and $n(T)$ are the Lagrange multipliers for constraints (9), (10), (11), (12) and (13). We then define H as the Hamiltonian function in which

$$\begin{aligned} H = {} & \{h(s) + y[x(s)] + f[t(s)]\}e(s) + l(s)[e(s) - g(s)] \\ & + g(s)e(s) + m(s)e(s) \\ & + n(T)[\overline{E} - E(T)] + r(s)e(s) + k(s)g(s) + z(s)x(s) \end{aligned} \tag{15}$$

and substitute H into Z^* to obtain

$$Z^* = \int_0^T \{H - l(s)\dot{x}(s) - g(s)g[t(s)]\dot{t}(s) - m(s)\dot{E}(s)\} ds \tag{16}$$

We now derive the total variation of Z^* with respect to all its arguments as

$$dZ^* = \int_0^T \left\{ \begin{array}{l} \dfrac{\partial H}{\partial e(s)} de(s) + \dfrac{\partial H}{\partial g(s)} dg(s) + \dfrac{\partial H}{\partial x(s)} dx(s) + \dfrac{\partial H}{\partial t(s)} dt(s) + \dfrac{\partial H}{\partial E(s)} dE(s) \\ \qquad - l(s)d\dot{x}(s) - g(s)g[t(s)]d\dot{t}(s) - m(s)d\dot{E}(s) \end{array} \right\} ds \tag{17}$$

Using integration by parts, the last three terms in the integrand can be re-written as

$$\int_0^T l(s)d\dot{x}(s)ds = \int_0^T l(s)ddx(s) = l(T)dx(T) - \int_0^T \dot{l}(s)dx(s)ds \tag{18}$$

$$\int_0^T g(s)g[t(s)]d\dot{t}(s)ds = \int_0^T g(s)g[t(s)]dd\,t(s)$$

$$= \{g(T)g[t(T)]d\,t(T) - g(0)g[t(0)]d\,t(0)\} - \int_0^T \{\dot{g}(s)g[t(s)] + g(s)g'[t(s)]\dot{t}(s)\}d\,t(s)ds \tag{19}$$

$$\int_0^T m(s)d\dot{E}(s)ds = \int_0^T m(s)ddE(s) = -\int_0^T \dot{m}(s)dE(s)ds \tag{20}$$

Note that $x(0)$ is fixed at zero and hence $dx(0) = 0$; $E(0)$ is fixed at zero and $E(T)$ is fixed at $\overline{E}$ so that $dE(0) = 0$ and $dE(T) = 0$. The prime superscript in the expressions denotes the derivate of a function with respect to its own argument. Finally, the expression for dZ^* can be re-expressed as

$$dZ^* = l(T)dx(T) + \{g(T)g[t(T)]d\,t(T) - g(0)g[t(0)]d\,t(0)\}$$
$$+ \int_0^T \left\{ \begin{array}{l} \frac{\partial H}{\partial e(s)}de(s) + \frac{\partial H}{\partial g(s)}dg(s) + \left(\frac{\partial H}{\partial x(s)} + \dot{l}(s)\right)dx(s) \\ + \left\{\frac{\partial H}{\partial t(s)} + \dot{g}(s)g[t(s)] + g(s)g'[t(s)]\dot{t}(s)\right\}d\,t(s) + \left(\frac{\partial H}{\partial E(s)} + \dot{m}(s)\right)dE(s) \end{array} \right\} ds \tag{21}$$

The optimality is achieved when Z^* is stationary, i.e. dZ^* equals zero for all variations in arguments of Z^*. This is ensured by setting the coefficients of all independent variations to zero, which leads to the following necessary conditions

$$\frac{\partial H}{\partial e(s)} = 0 \tag{22}$$

$$\frac{\partial H}{\partial g(s)} = 0 \tag{23}$$

$$l(T) = 0 \tag{24}$$

$$-\dot{l}(s) = \frac{\partial H}{\partial x(s)} \tag{25}$$

$$g(T)g[t(T)]d\,t(T) - g(0)g[t(0)]d\,t(0) = 0 \tag{26}$$

$$-\{\dot{g}(s)g[t(s)] + g(s)g'[t(s)]\dot{t}(s)\} = \frac{\partial H}{\partial t(s)} \tag{27}$$

$$-\dot{m}(s) = \frac{\partial H}{\partial E(s)} = 0 \tag{28}$$

Equations (22) and (23) are called the optimality conditions for the DSO assignment. Equations (25), (27) and (28) represent the costate equations for $l(s)$, $g(s)$ and $m(s)$ at optimality. Finally Equations (24) and (26) stand for the transversality or terminal conditions for $l(s)$ and $g(s)$. We can first deduce from (22) and (23) that

$$\{h(s) + y[x(s)] + f[t(s)]\} + l(s) + g(s) + m(s) + r(s) = 0 \tag{29}$$

$$-\,l(s) + h(s) + k(s) = 0 \tag{30}$$

Furthermore, $l(s)$ and $g(s)$ can be solved by the following costate equations deduced from (25) and (27) as

$$-\,\dot{l}(s) = y'[x(s)]e(s) + z(s) \tag{31}$$

$$-\{\dot{g}(s)g[t(s)] + g(s)g'[t(s)]\dot{t}(s)\} = \frac{\partial H}{\partial t(s)} = f'[t(s)]e(s) \tag{32}$$

Dividing both sides on (32) by $-g[t(s)]$ gets

$$\dot{g}(s) + \left\{\frac{g'[t(s)]}{g[t(s)]}\dot{t}(s)\right\}g(s) = -f'[t(s)]\dot{t}(s) \tag{33}$$

which is a first-order non-homogenous differential equation and it can be solved as

$$g(s) = \frac{-1}{g[t(s)]}\int_0^s g[t(t)]f'[t(t)]\dot{t}(t)dt \tag{34}$$

Finally, the following complementary slackness conditions hold

$$r(s)e(s) = 0; \quad r(s) \geq 0 \tag{35}$$

$$k(s)g(s) = 0; \quad k(s) \geq 0 \tag{36}$$

$$z(s)x(s) = 0; \quad z(s) \geq 0 \tag{37}$$

$$h(s)[g(s) - Q] = 0; \quad h(s) \geq 0 \tag{38}$$

For positive $e(s), g(s)$ and $x(s)$, $r(s), k(s)$ and $z(s)$ will all equal zero. Similarly, when $g(s)$ reaches the capacity Q, $h(s)$ will be zero. When these necessary conditions are solved simultaneously, we expect to obtain an optimal inflow profile that equates the marginal total costs for all departure time *s* at system optimal.

5. A simple example

To understand how the equations derived in section four work, we consider a simple example in which the deterministic queue model is adopted and the origin-specific cost is considered to be constant, i.e. $h'(s) = 0$. The study period $[0, T]$ is large enough so that all traffic can be cleared by the end of the time period, i.e. $\overline{E} < QT$. Heydecker and Addison (2004) showed that there will be three distinct intervals when this single bottleneck is at equilibrium: $[0, s_1]$, $[s_1, s_2]$ and $[s_2, T]$, which corresponds to when $e(s) = 0$, $e(s) > 0$ and $e(s) = 0$. The bottleneck will be congested throughout $[s_1, s_2]$ and hence *g*(s) = *Q* for $s \in [s_1, s_2]$. A constant outflow profile means $g'(s) = 0$ and (34) is thus reduced to $\dot{g}(s) = -f'[t(s)]\dot{t}(s)$. In addition, $h(s) = 0$, which can be deduced from the complementary condition (38). Furthermore, since the variation $dg(s)$ equals zero for all s, conditions (23) and hence (30) do not need to be considered.

Moreover, $r(s), k(s)$ and $z(s)$ are equal to zero since $e(s), g(s)$ and $x(s)$ are positive. Equation (31) thus becomes $-\dot{l}(s) = y'[x(s)]e(s)$. Finally, differentiate (29) with respect to time s, and substitute expressions for $\dot{l}(s)$, $\dot{g}(s)$ and $\dot{m}(s)$ leads to

$$\begin{aligned} &\{y'[x(s)]\dot{x}(s) + f'[t(s)]\dot{t}(s)\} + \dot{l}(s) + \dot{g}(s) + \dot{m}(s) = 0 \\ &\Rightarrow -y'[x(s)]Q = 0 \\ &\Rightarrow \quad y'[x(s)] = 0 \end{aligned} \tag{39}$$

It indicates that the travel time is constant with respect to the amount of link traffic at DSO condition, which implies that the queue length should be constant and hence $\dot{x}(s) = 0$. This also says $x(s)$ should remain at its initial value which is zero. As a result, the optimal inflow $e(s) = Q$ for all time s within $[s_1, s_2]$. Moreover, $y'[x(s)] = 0$ means $\dot{l}(s) = -y'[x(s)]e(s) = 0$ and thus $l(s)$ is constant. From the transversality condition (24) we can deduce that $l(s) = 0$ for all $s \in [s_1, s_2]$.

In addition, $\dot{g}(s) = -f'[t(s)]\dot{t}(s)$ implies that $g(s) = K - f[t(s)]$ for $s \in [s_1, s_2]$, where K is constant. Equation (28) shows that the costate variable $m(s)$ is constant throughout the planning period. The interpretation for this is that $-m(s)$ corresponds to the constant equilibrium cost C^* at equilibrium with departure time choice. Finally, replace the constant origin-specific cost by $\overline{h}$ and the constant travel time $y[x(s)]$ by the free flow travel time f, (29) can be re-expressed as

$$\{\overline{h} + f + f[t(s)]\} + \{K - f[t(s)]\} = C^* \tag{40}$$

for $s \in [s_1, s_2]$. We can immediately determine from (40) that $K = C^* - (\overline{h} + f)$. The costate variable $g(s) = C^* - (\overline{h} + f) - f[t(s)]$ can be interpreted as an external cost that transforms the system from equilibrium to a system optimal. For intervals $[0, s_1]$ and $[s_2, T]$, we have $g(s_1) = g(s_2) = 0$. Since $e(s) = 0$, hence $\dot{t}(s) = 0$ and thus $\dot{g}(s) = 0$ for all s within $[0, s_1]$ and $[s_2, T]$. Consequently, $g(s) = 0$ and thus no external cost should be added to the system in the two intervals.

6. Concluding remarks

In this paper we have provided an analysis framework for dynamic system optimal assignment with departure time choice based upon sound traffic models. First, we summarize all requirements on a traffic model and review three different traffic models. We particularly point out that the outflow models are widely used for analysis, however, these models do not give plausible traffic propagation and violate causality. Thus, outflow models should not be avoided. The travel time models are satisfactory. Nevertheless, we still have to note that the deterministic queue model can be difficult to analyse, as its state variable is not continuously differentiable with time.

We then derive the necessary conditions for a DSO assignment based upon the travel time models. The assignment problem aims to minimize the total system cost in a network by seeking an optimal inflow profile within a fixed planning horizon. Different from the conventional control theoretical approach, we have explicitly added a constraint to ensure proper flow propagation and adopted a calculus of variations technique to solve for the optimality conditions. It is then followed by a simple working example in which we consider a single bottleneck with constant origin-specific cost. The result agrees with traditional analysis on the same problem.

Traditional analysis tends to presume zero queuing at DSO. However, will it still be true if the origin-specific cost considered is time-varying instead of constant? Will it be possible that a traveller would rather encounter congestion in order to stay longer at origin for an overall net benefit? Furthermore, will it be better off if we tolerate congestion at some times to shorten the overall congested period? In particular, if we consider a oversaturated period, i.e. $\overline{E} > QT$, in which congestion must exist, the problem then becomes how we can manage this congestion rather to eliminate it. The answers to these questions are not straightforward. In fact, we expect congestion may exist at DSO condition under certain conditions, for example, when travellers with a highly negative origin-specific cost are considered; or a different traffic model such as the linear traffic model is adopted. The analysis work proposed here may facilitate us to understand these questions. Future work will also include extending the present work to multi-route and multi-commodity networks. Developing an efficient solution algorithm is also an important topic and is an area of our future research.

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