Hyperpaths in Network Based on Transit Schedules

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The concept of a hyperpath was introduced for handling passenger strategies in route choice behavior for public transit, especially in a frequency-based transit service environment. This model for handling route choice behavior has been widely used for planning transit services, and hyperpaths are now applied in areas beyond public transit. A hyperpath representing more specific passenger behaviors on a network based on transit schedules is proposed. A link-based time-expanded (LBTE) network for transit schedules is introduced; in the network each link represents a scheduled vehicle trip (or trip segment) with departure time and travel time (or arrival time) between two consecutive stops. The proposed LBTE network reduces the effort to build a network based on transit schedules because the network is expanded with scheduled links. A link-based representation of a hypergraph with existing hyperpath model properties that is directly integrated with the LBTE network is also proposed. Transit passenger behavior was incorporated for transfers in the link-based hyperpath. The efficiency of the proposed hyperpath model was demonstrated. The proposed models were applied on a test network and a real transit network represented by the general specification of Google's transit feed.

The use of hyperpaths in public transportation was conceptualized by Nguyen et al. (*1*), Nguyen and Pallottino (*2*), and Spiess and Florian (*3*). Extensions have since been studied, including *k*-shortest hyperpaths (*4, 5*) and the one-to-one hyperpath (*6*). For a transit schedule–based network (*7–13*), Nguyen et al. proposed a hyperpath that uses the logit model (*14*), extending the hyperpath concept beyond the traditional application to frequency-based transit assignment. Gallo et al. introduced the graphical hyperlink with multiple node ends (*15*). This hyperpath concept can be extended further into transit schedule–based networks, by considering a less-complex network representation and several transit passenger behaviors (path choice) related to this improved network representation.

This paper proposes that a transit schedule can be represented by a vehicle run that serves as a schedule link. On this premise, the hyperpath on a transit schedule network can be defined and represented in a specific link-based scheme (*16*); this approach was tested by Ziliaskopoulos and Wardell on a multimodal time-dependent network (*17*). In this construct, a hyperlink represents a connection between two scheduled service links (e.g., a transfer between trips or a continuing trip on the same route). This hyperlink is represented with a link-to-link connector that captures both the feasibility and the cost of making a connection between the scheduled links. In this way, separate links for boarding and alighting behavior are not necessary (as might be needed in a node-based representation). Instead, the hyperlink includes the cost for each transfer between two consecutive links and can have a separate cost for each transfer movement, which differs from when a hyperlink cost is on the existing hyperpath.

A transit schedule network represented in a link-based scheme gives the same result as node-based models. Each scheduled vehicle trip between two consecutive stops is represented as a single transit schedule link, with a route and a mode. This is called a link-based time-expanded (LBTE) network. Because a basic search unit is along a link and in the link-to-link connections, it is not necessary to expand a physical stop to multiple stops, representing the same stop at different points in time [a diachronic graph (*8*)]. Thus for network representation, the benefit of the LBTE is twofold: passenger boarding and alighting behavior is represented on a hyperlink, instead of through separate boarding and alighting hyperlinks, and the network size is much smaller than a node-based time-expanded transit schedule network.

Because of its efficiency in network representation and its flexibility in representing transit passenger behavior, a logit-based hyperpath choice model is proposed for use on an LBTE transit schedule network. It is assumed that passengers have a preferred arrival time (PAT) at the destination, and thus a backward hyperpath search model is introduced. Because of different perceptions of the generalized cost of travel for each passenger, each hyperlink is managed by a logit-type function for the choice set of schedule alternatives. The proposed hyperpath search model is expected to give more strategic alternatives for passengers on the transit schedule network.

This study defines the link-based hyperlink, its cost, and the resulting hyperpath structure. An LBTE transit schedule network compatible with this definition is proposed, along with a weighting function for both deterministic and stochastic assignment cases. A label-correcting algorithm is provided to solve for the bounded optimal assignment. These algorithms were applied to a transit test network.

REPRESENTATION OF NETWORK FOR LINK-BASED HYPERPATH

Definitions

Gallo et al. define a hyperlink by using $e = [t(e), h(e)]$, where $t(e)$ is the tail node subset of hyperlink e and $h(e)$ is the head node subset of hyperlink e (*15*). Each hyperlink is represented in the form (node-link-node). Instead, this paper introduces a (link-to-link)

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Transportation Research Record: Journal of the Transportation Research Board, No. 2284, Transportation Research Board of the National Academies, Washington, D.C., 2012, pp. 29–39.
DOI: 10.3141/2284-04

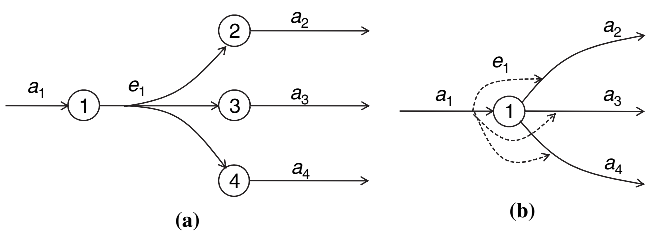

FIGURE 1 Diverging hyperlink: (*a*) existing node-based and (*b*) proposed link-based.

hyperlink $E = (\{e_a\}, \{e_b\})$ such that $\forall (e_a, e_b) \in M$ and $e_a, e_b \in L$, where M is a set of link-to-link connectors and L is the link set. The hypergraph is defined as $H(L, E)$. The difference between the proposed hypergraph and previous research is that the hyperlink connects two different link subsets, not two different node subsets. The forward and backward link sets are defined as $F^+_{e_a} = \{e_b \in L \backslash e_a | (e_a, e_b) \in M\}$ and $B^-_{e_b} = \{e_a \in L \backslash e_b | (e_a, e_b) \in M\}$. If $|F^+_{e_a}| > 1$ and $|B^-_{e_b}| = 1$, then hyperlink E is diverging (one link leading to more than one other link), and merging (many links leading to a single other link) can be defined if $|F^+_{e_a}| = 1$ and $|B^-_{e_b}| > 1$. Otherwise, a simple monotonically connected hyperlink (an elementary hyperlink) occurs when $|F^+_{e_a}| = 1$ and $|B^-_{e_b}| = 1$.

Figure 1 shows two possible representations of a hyperlink. Figure 1*a* represents a node-based hyperlink, and Figure 1*b* represents a link-based hyperlink.

In Figure 1*a*, e_1 is the hyperlink that uses a node-based representation, where e_1 connects Node 1 to a set of nodes {2, 3, 4}, which are subsequently connected to links a_2, a_3, and a_4, respectively. In Figure 1*b*, hyperlink e_1 connects a_1 to a_2, a_3, and a_4; that is, $(a_1, \{a_2, a_3, a_4\}) \in E$ such that $(a_1, a_2), (a_1, a_3), (a_1, a_4) \in M$. Also, the possible hyperlinks (dashed lines in Figure 1*b*) are provided by the combinations of a_1 and $\{a_2, a_3, a_4\}$.

Another assumption in Figure 1*b* is that the hyperlink can be represented as separate connections to each link. If the network contains link costs and a weight function for the separate connections for each hyperlink, a label-setting algorithm, which activates if all costs are finalized, can be used, or a label-correcting algorithm, which activates once each link cost is updated, can be used. For this reason, the hyperlink e in Figure 1*b* can be represented as F^+_e or $B^-_e\ \forall e \in L$.

For the elementary hyperlink, cost is updated with the link-to-link scheme introduced by Potts and Oliver, in which turn penalties in a transportation network are considered (*16*). In Figure 2, link e_a and link e_b have their own link costs c_{e_a} and c_{e_b}, respectively. Every cost label is updated at the end of each link (i.e., $\hat{c}_{e_a}$ and $\hat{c}_{e_b}$) following Bellman's optimality rule, $\hat{c}_{e_b} = \min\{\hat{c}_{e_b}, \hat{c}_{e_a} + c_{e_b} + c_{e_a e_b}\}$, where $c_{e_a e_b} \equiv c_{(e_a,\ e_b)}$ is the turn penalty cost. In the same manner, the hyperlink cost is updated with $\hat{c}_{e_b} = \min\{\hat{c}_{e_b}, c_{e_b} + w(\{\hat{c}_{e_a} + c_{e_a e_b}\}) | e_a \in B^-_{e_b}\}$, where $w(\cdot) \equiv \min_{\{e_a\} \subseteq B^-_{e_b}} f(\{c_{e_a} + c_{e_a e_b}\}), f(\cdot)$, is the weighting function for the hyperlink. Therefore, Bellman's optimality rule is satisfied because $w(\cdot)$ is the minimum value on the hyperlink.

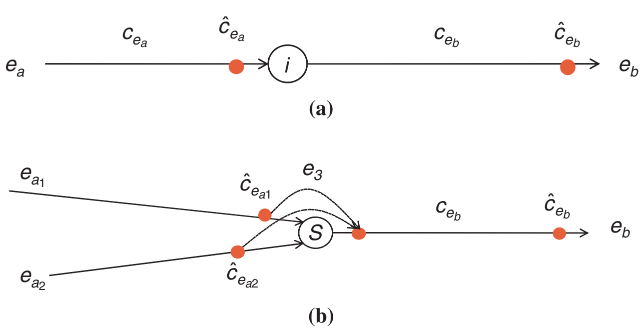

FIGURE 2 Cost update: (*a*) link-based and (*b*) hyperlink.

Proposed Hyperpath

The network conditions for a hyperpath were introduced by Nguyen et al. (*1*). These conditions include: (*a*) the hyperpath h_{rs} is an acyclic hyperpath with at least one link connecting the origin r to the destination s, and (*b*) at each node conditional probabilities for subsequent links (sum is equal to 1) exist. These conditions were defined again by Nielsen et al. (*4*) and Gallo et al. (*15*). The proposed hyperpath in the LBTE network from origin r to destination s is formed with a series of links and hyperlinks, $h_{rs} = (e_r, E_{m_r}, e_i, e_j, E_{m_i}, e_n, \ldots, E_{m_t}, e_s)$, which can be represented with a forward link set for the diverging hyperlink case with $h_{rs} = (e_r, e_i \in \bar{F}^+_{e_r}, e_j \in \bar{F}^+_{e_i}, \ldots, e_s \in \bar{F}^+_{e_k}, \ldots, e_s \in \bar{F}^+_{e_l})$, where $\bar{F}^+_e \subseteq F^+_e\ \forall e \in L$ and $r = h(e_r)$ and $s = t(e_s)$. Also, $\bar{F}^+_e$ is the subset of F^+_e. The subhypergraph $\bar{H} = (\bar{L}, \bar{E}) = (\bar{L}, \bar{M})$ assumes that a hyperlink can be separated into individual connections such that $\bar{L} \subseteq L$; $e_r, e_s \in \bar{L}$; and $|\bar{F}^+_e| \geq 1$ $\forall e \in \bar{L} \backslash \{e_s\}$ and $|\bar{B}^+_e| \geq 1\ \forall e \in \bar{L} \backslash \{e_r\}$. A bar over a variable indicates a subset of the variable's original set. This implicitly requires an acyclic network. In addition, the hypergraph can be represented by each origin link $e_r\ \bar{H}_{e_r} \equiv \{e_b | e_b \in \tilde{F}^+_{e_a}; \tilde{F}^+_{e_a} \subseteq \bar{F}^+_{e_a}, \forall a, b \in \bar{L}\} \forall e_r \in \bar{L}$ or $\bar{H}_{e_r} \equiv \{e_b | (e_a, e_b) \in \tilde{M}; \tilde{M} \subseteq \bar{M}, \forall a, b \in \bar{L}\}\ \forall e_r \in \bar{L}$ for satisfying the optimality conditions with minimum weight, consisting of $\tilde{F}^+_{e_a}$ for link e_a or $\tilde{M}$ for (e_a, e_b), since the hypergraph is the union of elementary paths with the connections M.

HYPERGRAPH ON LBTE TRANSIT SCHEDULE NETWORK

LBTE Transit Schedule Network

In a transit network, every stop is associated with (*a*) a sequence of points in time when a vehicle from a route will visit and (*b*) the travel time or the arrival time of the vehicle at the next available stop on the route. In one network representation, expansion of stops can be based on points in time, and the time points are connected and expanded spatially by each bus run (or route) (*8, 9, 18*). This is called an expanded node-based network, because the label is fundamentally updated through each node in a path search model. Instead of repeating the stop for each point in time, it is proposed that time points be assigned to each link connecting two stops by each run (or route). In this way, each link from a stop represents a run of each vehicle with departure time (previous stop departure time of the run, $t^{\text{dep}}_{e_a}$) and arrival time (next stop arrival time of the run, $t^{\text{arr}}_{e_a}$), as shown in Figure 3. The difference between the departure time and the arrival time at the next stop is the travel time $t^{\text{trv}}_{e_a}$; transfer cost including walking and waiting time is defined by $t^{\text{trsf}}_{e_a e_b}\ \forall e_b \in F^+_{e_a}$, and waiting time is defined by $t^{\text{wait}}_{e_a e_b}$. The proposed transit schedule network reduces the complexity of the time-expanded network, especially for creating transfer links among time points and runs (or routes).

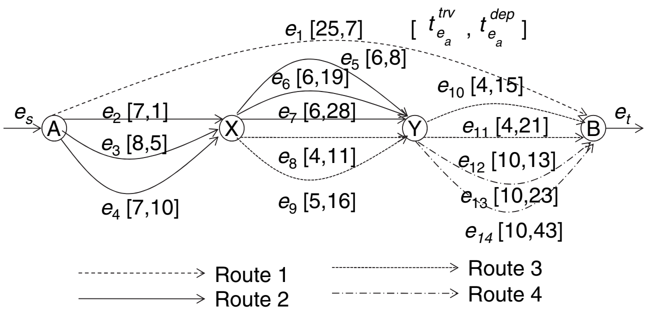

FIGURE 3 LBTE transit network.

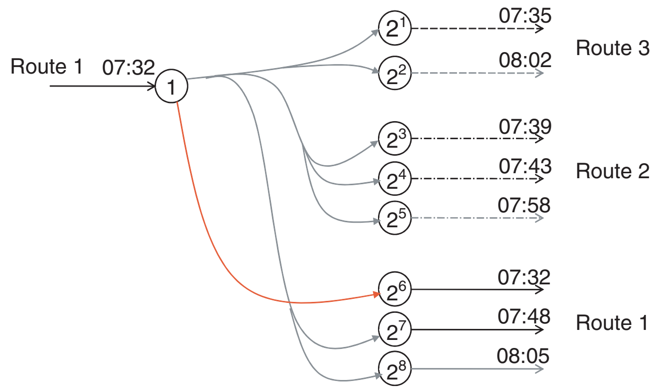

FIGURE 5 Hyperlink on LBTE network.

Hypergraph on LBTE Network

The hyperlink in a transit network fundamentally represents boarding and alighting passenger behaviors. Traditionally (*3*), the diverging hyperlink shows boarding behavior in terms of the passenger's choice of alternative routes and runs, and the merging hyperlink stands for alighting behavior from alternative routes and runs to a specific stop. In Figure 4*a*, X, S, and Y are the physical stops, and 1, 2, 3, 4, 5, and 6 are additional nodes for representing boarding and alighting behavior. In the proposed LBTE network shown in Figure 4*b*, alighting and boarding behavior is represented with a hyperlink. In Figure 4*a*, $e_1(\{3, 4\}, s)$ represents an alighting hyperlink and $e_2(s, \{3, 4\})$ is for boarding. In contrast, in Figure 4*b*, $e_3(a_1, \{a_2, a_4\})$ and $e_4(a_3, \{a_2, a_4\})$ represent alighting–boarding links as turn penalties in the link-based network representation (LBTE). When a hyperpath is generated by PAT in the LBTE network, hyperlinks e_3 and e_4 are merging hyperlinks. Representing this transit network with a node-based scheme requires nine nodes. But in Figure 4*b*, the link-based network representation requires the physical stops X, S, and Y for boarding–alighting behavior, but no additional nodes. The difference in network size between these two representations will be magnified in a transit schedule network because the number of nodes will increase with the time expansion on a node-based representation, and the number of links depends on the number of nodes. The proposed network representation requires more hyperlinks but will provide an easier representation of passenger behavior, specifically for the priority of movement according to arrival times in the transit schedule network. For example, for link a_4, the arrival time on link a_4 by passengers transferring or directly connecting from a_1 and a_3 will be estimated directly through the edges e_3 and e_4 in the proposed approach in Figure 4*b* but will be estimated through e_1 and e_2 sequentially in Figure 4*a*.

A hyperlink is associated with a route or run choice problem, typically associated with boarding or making a transfer at a stop. Especially in frequency-based transit assignment, a hyperlink is used to find an optimal strategy considering the compensation between waiting time improvements by combining frequencies of available routes and the travel time to destination (*1–3*). In a schedule-based network, the choice set is expanded to include temporal alternatives as well as the choice of route alternatives, as shown in Figure 5 (*7–13*). These temporal alternatives not only provide more detailed representation for arrivals and departures of vehicles and passengers but also provide the necessary acyclic property to the network representation, if it is assumed that the hyperlink does not allow a connection to an earlier point in time. All hyperlink connections to and from a schedule link should satisfy this condition as the network is created.

Cost and Weight Functions on Hyperlink

The weight function on an LBTE network was defined earlier as $f(\{c_{e_b} + c_{e_a e_b} | e_a \in B^-_{e_b}\})$ with the weighting in $w(\cdot) \equiv \min_{\{e_a\} \subseteq B^-_{e_b}} f(\{c_{e_a} + c_{e_a e_b}\})$. To prevent temporal violations and allow for a backward path search from the PAT at the destination, the weight function can

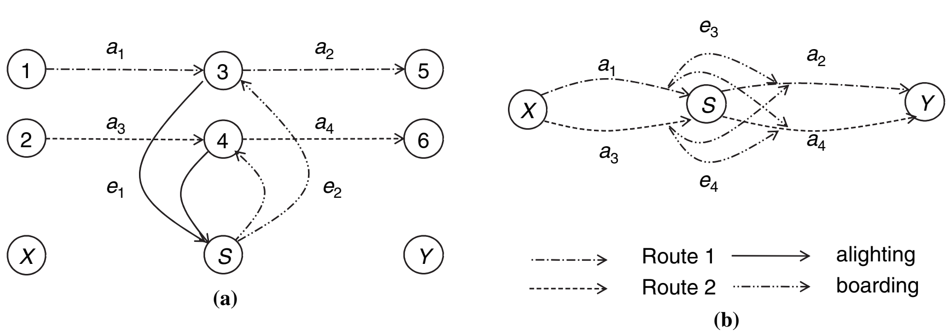

FIGURE 4 Hyperlink representation for alighting and boarding: (*a*) node-based and (*b*) link-based on LBTE network.

be more clearly defined as $f(\{c_{e_a} + c_{e_a e_b} | t_{e_a}^{\text{arr}} + t_{e_a e_b}^{\text{trsf}} < t_{e_b}^{\text{dep}} \text{ and } e_b \in F_{e_a}^+\})$ and $w_{e_a} = \min_{\{e_b\} \subseteq F_{e_a}^+} f(\{c_{e_a} + c_{e_a e_b} | t_{e_a}^{\text{arr}} + t_{e_a e_b}^{\text{trsf}} < t_{e_b}^{\text{dep}}; \forall e_b \in F_{e_a}^+\})$ or $w_{e_a} = \min_{\{e_b\} \subseteq F_{e_a}^+} f(\{c_{e_a} + c_{e_a e_b} | \forall e_b \in \bar{F}_{e_a}^+\})$, where $\bar{F}_{e_a}^+ = \{e_b \in L \backslash e_a | t_{e_a}^{\text{arr}} + t_{e_a e_b}^{\text{trsf}} < t_{e_b}^{\text{dep}}; (e_a, e_b) \in M\}$. Therefore, the cost at $h(e_a)$ from the destination link e_s is $\hat{c}_{e_a} = \min\{\hat{c}_{e_a}, c_{e_a} + w_{e_a} | e_{e_b} \in \bar{F}_{e_a}^+\}$. The weighting function may or may not satisfy the additive condition $w_{e_a} < c_{e_b} + c_{e_a e_b} \; \forall e_b \in \bar{F}_{e_a}^+$.

The weighting function can be defined by several forms according to traveler behavior, most notably in the relationship between path costs and path alternatives. The available functions are (*a*) an average model, in which the costs of all alternatives are averaged; (*b*) a modified version of the optimal strategy by Spiess and Florian considering the transit schedule (*3*); and (*c*) a log-sum model assuming stochastic user equilibrium behavior. The benefit of the average model is its simplicity, but it does not represent behavior of transit passengers well. When the average is used for the weighting function f, it is the same as a shortest path because the optimal alternative set always chooses the minimum cost alternative as $f(\{c_{e_a} + c_{e_a e_b}\}) \geq \min\{c_{e_a} + c_{e_a e_b}\}$. As an alternative, the optimal set can be configured by the relation between travel time and additional transfer and waiting time, similar to the optimal strategy suggested by Speiss and Florian as a deterministic weight function (*3*). The method brings the same sense of an optimal strategy; however, a deterministic choice might lead passengers to choose only the least-cost alternative. Third, assuming that the perception of cost is different for each passenger, a log-sum weight function will compensate for the increased costs of more alternatives with the availability of more alternatives. For this reason, this third option is better than the others when one considers the number of alternatives and a possible change in cost depending on the number of alternatives. In the third case, the weight function can be represented by the log-sum function shown in Equation 1:

$$w_{e_a} = \min_{\{e_b\} \subseteq \bar{F}_{e_a}^+} \frac{1}{\theta} \ln \sum_b \exp\left(\theta \cdot \hat{c}_{e_b}\right) \qquad \forall e_a \in E \tag{1}$$

where θ is the dispersion parameter for the logit model. The log sum plays a role in the choice model for the transit schedule alternatives. In the log-sum model, as more alternatives are added, no matter how high the cost, the overall cost will decrease (or the utility will increase). To manage this problem, the value of the dispersion parameter θ can be adjusted. It is also possible to reduce the number of alternatives by using a simple upper bound on the cost of alternatives in the alternative set. On the basis of the lowest-cost alternative, the upper bound is chosen from the number of likely alternatives. Alternatively, the logit probability of each alternative can be considered. The size of the set can be determined by allowing a certain minimum level of probability of a path, such as 0.0001.

In a transit schedule network, the link cost and weight could be generalized by including costs such as transfer time, waiting time, and number of transfers, as shown in Equations 2 and 3:

$$w_{e_a} = \min_{\{e_b\} \subseteq \bar{F}_{e_a}^+} \frac{1}{\theta} \ln \sum_b \exp\left\{\theta \begin{pmatrix} \hat{c}_{e_b} + \beta_{\text{trsfTime}} \cdot t_{e_a e_b}^{\text{trsf}} + \beta_{\text{waitTime}} \\ \cdot t_{e_a e_b}^{\text{wait}} + \beta_{\text{earlyDep}} \cdot t_{e_b}^{\text{earlyDep}} + \beta_{\text{trsf}} \end{pmatrix}\right\} \qquad \forall e_a \in E \tag{2}$$

$$\hat{c}_{e_a} = \beta_{\text{trvTime}} \cdot t_{e_a}^{\text{trv}} + w_{e_a} \tag{3}$$

where

$t_{e_a}^{\text{trv}}$ = (in-vehicle) travel time of link e_a,
$t_{e_a e_b}^{\text{trsf}}, t_{e_a e_b}^{\text{wait}}$ = transfer time and waiting time from link e_a to link e_b, respectively,
$t_{e_b}^{\text{earlyDep}}$ = relative departure time difference based on the latest departure time in the alternative set $\{e_b\}$, and
$\beta_{\text{trsfTime}}, \beta_{\text{waitTime}}, \beta_{\text{earlyDep}}, \beta_{\text{trsf}}, \beta_{\text{trvTime}}$ = parameters for transfer time, waiting time, relative departure time difference, transfer, and travel time, respectively.

Since w_{e_a} satisfies the bounded optimality condition in Equation 2, $\hat{c}_{e_a}$ satisfies the optimality in Equation 3.

SEARCH ALGORITHMS

Behavioral Assumptions

For the proposed backward hyperpath model, it is assumed that each passenger has his or her own PAT at the destination. The PAT will be set to be within a certain time window, such as 20 min earlier than the start time of work at 8:00 a.m., for each passenger. The defined PAT is used to create the hyperpath, and it is assumed that the path arriving within this time window will not incur any additional penalty. However, a penalty for early departure time is considered. When the PAT is given and the hyperpath is searched backward from the PAT, a set of alternatives may result that depart at different times from the passenger's origin. Among these departure times, it is accepted that the latest departure time is the preferred alternative.

For access to and egress from a transit stop, a specific Euclidean distance is assumed. However, a passenger is allowed to transfer to another stop when accessing his initial stop. Then at every stop, the passenger will board a transit vehicle among the alternatives that are defined in the path choice set. For alighting behavior, there is no specific consideration, although onboard congestion will cause longer alighting time. If an additional alighting time for onboard congestion is assumed, this is added to the cost of the transfer link. U-turns do not frequently happen in an uncongested transit network, because making a U-turn can generate a transfer burden and a much longer trip. However, if a congested transit schedule network is assumed, a U-turn may save waiting time and decrease the disutility created by congestion. A typical example was shown by Nuzzolo et al. (*8*). The LBTE network scheme will allow U-turn behavior without creating a cycle on consecutive links, allowing U-turns through consecutive nodes (*16*).

Label-Correcting Algorithm

A label-correcting algorithm is considered to be a search algorithm, not a label-setting algorithm. This is called the base hyperpath algorithm. A distinctive characteristic of a hyperlink is that its cost can be finalized only when it has final information on all the alternatives in the set. Also, the link-based scheme on a LBTE network allows a U-turn, so that an alternative for a link may not be finalized without the U-turn information, creating a recursive problem. For these reasons, a label-setting algorithm may not end at the optimal solution—there may be no finalized link costs to scan, although not all links have permanent labels. A label-correcting algorithm avoids these problems.

The proposed backward label-correcting on an LBTE schedule network algorithm is shown in Equation Box 1. The algorithm consists of one main function in Equation Box 1 and a subalgorithm for determining the optimal set of alternatives in Equation Box 2. Initially, the main algorithm in Equation Box 1 generates the adjacency list (connections) satisfying the temporal constraints, according to arrival and transfer times from any previous transit vehicle and the

EQUATION BOX 1 Overall Label-Correcting Hyperpath Algorithm

$\textbf{for}\ ((e_a, e_b) \in M)$ // generate adjacency list: $\boldsymbol{O}(L^2)$

$\quad \bar{F}^{+}_{e_a} = \{e_b \in L \setminus e_a \mid t^{\text{arr}}_{e_a} + t^{\text{trsf}}_{e_a e_b} < t^{\text{dep}}_{e_b}\};\ \bar{B}^{-}_{e_b} = \{e_a \in L \setminus e_b \mid t^{\text{arr}}_{e_a} + t^{\text{trsf}}_{e_a e_b} < t^{\text{dep}}_{e_b}\};\ \hat{c}_{e_a} := \infty;$

$\hat{c}_{e_s} := 0;\ Q := \varnothing;$

$\textbf{for}\ (e_a \in \bar{B}^{-}_{e_s})\ Q := Q \cup \{e_a\};$ // add egress links: $\boldsymbol{O}(K)$

$\textbf{while}\ (Q \neq \varnothing)$: $\boldsymbol{O}(L + R)$

$\quad$ Retrieve $e_a \in Q$ and $Q := Q - \{e_a\};$

$\quad \textbf{for}\ (e_c \in \bar{B}^{-}_{e_a})$ // add backward links: $\boldsymbol{O}(K)$

$\quad\quad \textbf{if}\ (\{e_c\} \cap Q = \varnothing)\ \textbf{then}\ Q := Q \cup \{e_c\}$: $\boldsymbol{O}(L)$

$\quad \textbf{for}\ (e_b \in \bar{F}^{+}_{e_a})$: $\boldsymbol{O}(K)$

$\quad\quad \hat{c}^{\text{new}}_{e_a} :=$ **set_optimal_set** (e_a, e_b); // cost update with optimally check: $\boldsymbol{O}(K \cdot L)$

$\quad\quad \textbf{if}\ (\hat{c}^{\text{new}}_{e_a} < \hat{c}_{e_a})\ \hat{c}_{e_a} := \hat{c}^{\text{new}}_{e_a};$

$\quad\quad \textbf{if}\ (\{e_a\} \cap Q = \varnothing)\ \textbf{then}\ Q := Q \cup \{e_a\}$: $\boldsymbol{O}(L)$

departure time of the next vehicle. Also, egress links are connected from the destination link and added to the search set Q. In the main loop, the labeling continues until Q is empty. The main loop has two subloops for building a hyperpath tree: processes (a) to add backward links from each processed link and (b) defining the optimal set of alternatives. For process a, since a hyperlink defines the (link-to-link) relation, search set Q is expanded by adding previous links. For each previous link, process b creates an optimal set and updates the link cost. Then, if the link cost satisfies the optimality condition, the link with the new cost is added to Q. A relative difference of the cost of one alternative relative to the minimum cost among all possible alternatives is used in consideration of alternatives to get the log-sum cost.

The label-correcting algorithm has a complexity of $\boldsymbol{O}(K^2L(L+R))$, where K is the maximum possible number of alternatives for a link, L is the total number of links, and R is the number of additional

EQUATION BOX 2 Subalgorithm Label-Correcting Hyperpath

// **set_optimal_set** (e_a, e_b): $\boldsymbol{O}(K \cdot L)$

$\textbf{if}\ (e_b = e_a)$ // if destination link

$\quad \textbf{if}\ (\bar{H}_{e_a} \cap \{e_b\} = \varnothing)\ \bar{H}_{e_a} := \bar{H}_{e_a} \cup \{e_b\}$, and update $\hat{c}_{e_a} := c_{e_b};$

else // if other links

$\quad \textbf{for}\ (e_b \in \bar{F}^{+}_{e_a})$ // search base link with min cost

$\quad\quad \textbf{if}\ (\hat{c}_{e_b} > \tilde{c}_{e_a}$ and $\hat{c}_{e_b} < \hat{c}_{\bar{e}_b})\ \bar{e}_b = e_b;\ \hat{c}_{\bar{e}_b} = \hat{c}_{e_b};$

$\quad \textbf{for}\ (e_b \in \bar{F}^{+}_{e_a})$ // logit probability

$\quad\quad \hat{c}^{\exp}_{e_b} = \hat{c}^{\exp}_{e_b} + \exp(\hat{c}_{e_b});$

$\quad\quad \textbf{if}\ (\hat{c}_{\bar{e}_b} - \ln(\hat{c}^{\exp}_{e_b}) \leq \beta)\ \hat{c}_{\bar{e}_b} = \ln(\hat{c}^{\exp}_{e_b})\ \tilde{F}^{+}_{e_a} := \tilde{F}^{+}_{e_a} \cup \{e_b\};$

$\quad w_{e_a} = \min_{\{e_b\} \subseteq \tilde{F}^{+}_{e_a}} f(\{e_b, (e_a, e_b)\});\ T_{e_a} = \arg\min_{\{e_b\}} f(\cdot);$ // weight function

$\quad \textbf{for}\ (e_b \in T_{e_a})$ // update hyperpath tree: $\boldsymbol{O}(K)$

$\quad\quad \textbf{if}\ (\bar{H}_{e_a} \cap \{e_b\} = \varnothing)\ \bar{H}_{e_a} := \bar{H}_{e_a} \cup \{e_b\};\ \boldsymbol{O}(L)$

$\quad \textbf{return}\ (w_{e_a} + c_{e_a});$

TABLE 1 Comparison of Hyperpath Model Complexity

Model	Outer Loop	Subalgorithm	Overall
SHT, Nguyen and Pallottino (*2*)	$\boldsymbol{O}(NK)$	$\boldsymbol{O}(K\log K)$	$\boldsymbol{O}(NK^2\log K)$
SBT, Gallo et al. (*15*)	$\boldsymbol{O}(K^2 \log N)$	$\boldsymbol{O}(F)$	$\boldsymbol{O}(FK^2\log N)$
SBT, Marcotte and Nguyen (*19*) and Nielsen (*20*)	$\boldsymbol{O}(NK^2)$	$\boldsymbol{O}(F)$	$\boldsymbol{O}(FNK^2)$
SBT-acyclic, Nielsen (*20*)	$\boldsymbol{O}(NK)$	$\boldsymbol{O}(F)$	$\boldsymbol{O}(FNK)$
Backward pass, Rochau et al. (*21*)	$\boldsymbol{O}(NK)$	$\boldsymbol{O}(F)$	$\boldsymbol{O}(FNK)$
Proposed model	$\boldsymbol{O}(K \log L)$	$\boldsymbol{O}(KL)$	$\boldsymbol{O}(K^2 l\log L)$

NOTE: SHT = shortest hypertree; SBT = shortest b-tree.

times one must revisit the same link in Q to correct the link's label. $(L + R)$ operations are taken within the while loop, and (K^2L) operations are taken in finding the set of alternatives, mainly dominated by the second for loop (Equation Box 2). Because the label-correcting algorithm is used, R is a critical determinant of the algorithm's performance. However, when one maximizes the acyclic property on the LBTE network by maintaining the links in Q in descending order of schedule time (05 in Equation Box 1) as transforming to a type of label-setting algorithm, the algorithm's complexity goes to $\boldsymbol{O}(K^2L^2)$ and $\boldsymbol{O}(K^2L \log L)$ when a heap is used in Q. This is shown in Table 1, which also shows a summary of the complexity of other hyperpath models. The complexity of each model, including the proposed model, is categorized as outer-loop, subalgorithm, and overall. Outer-loop is the algorithm in Equation Box 1 except Line 10, and Subalgorithm is Line 10 in Equation Box 1 or the algorithm of Equation Box 2. Overall is the combination of outer-loop with subalgorithm. For outer-loop, $\boldsymbol{O}(NK^2)$ is produced by Gallo et al. (*15*), Marcotte and Nguyen (*19*), and Nielsen (*20*), but Gallo et al. use heap sorting to obtain $\boldsymbol{O}(K^2 \log N)$ (*15*), and other models by Nguyen and Pallottino (*2*) and Nielsen (*20*) show a more simplified outer-loop with $\boldsymbol{O}(NK)$, typically as applied on an acyclic network. Considering the link-based approach on an acyclic network, including heap sorting, the proposed model shows the complexity of $\boldsymbol{O}(K \log L)$. Subalgorithm is generally a process for choosing an optimal alternative set. As shown in Table 1, other models except for that of Nguyen and Pallottino (*2*) simplify the process or assume that the set is given, where $\boldsymbol{O}(F)$ is the complexity of this simplified function or a given set. Because it is possible to generate a variety of subalgorithms, it is difficult to choose the best model among them. However, if it is assumed that the subalgorithm uses a logit-type function, the proposed model is sufficiently competitive with other models that use a link-based network representation.

APPLICATION

To test the algorithm, a test network is constructed, as shown in Figure 6. The network consists of five origin–destination nodes and nine intersections, including 12 bidirectional links (24 directional links) between each adjacent intersection pair. Thirty-six stops are located along the links, and four transit routes (encompassing eight directional routes, including northbound and southbound) serve the network. There are 41 trips among these routes during the period 7:14 to 9:24 a.m., as shown in Figure 7. On the schedule in Figure 7,

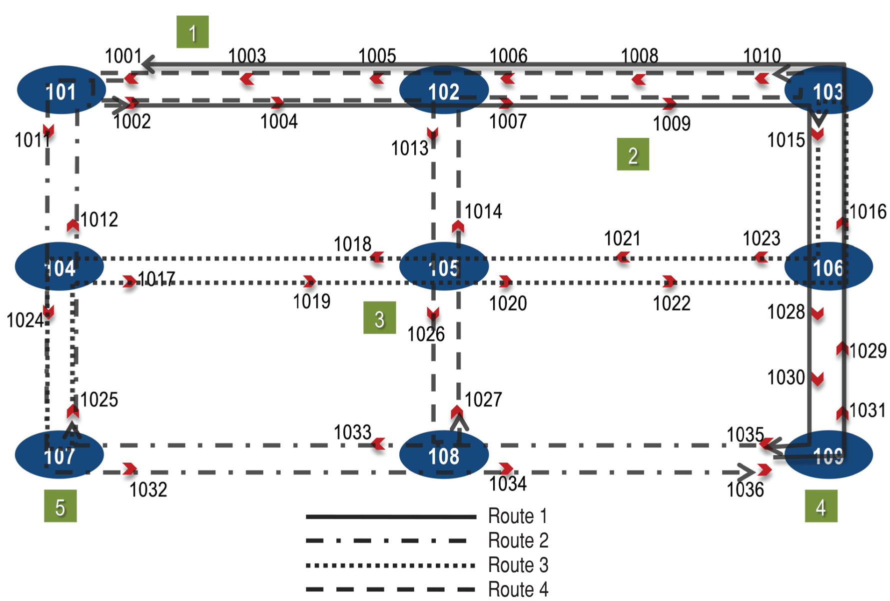

FIGURE 6 Sample test network.

1- South	111111	111221	111278	111321	111345	111478
1002	7:15:00	7:33:00	7:46:00	7:57:00	8:10:00	8:23:00
1004	7:17:00	7:35:00	7:48:00	7:59:00	8:12:00	8:25:00
1007	7:21:00	7:39:00	7:52:00	8:03:00	8:16:00	8:29:00
1009	7:25:00	7:43:00	7:56:00	8:07:00	8:20:00	8:33:00
1015	7:32:00	7:50:00	8:03:00	8:14:00	8:27:00	8:40:00
1028	7:42:00	8:00:00	8:13:00	8:24:00	8:37:00	8:50:00
1030	7:51:00	8:09:00	8:22:00	8:33:00	8:46:00	8:59:00
1035	7:59:00	8:17:00	8:30:00	8:41:00	8:54:00	9:07:00
1- North	111566	111697	111722	111833	111879	111923
1036	7:15:00	7:33:00	7:46:00	7:57:00	8:10:00	8:23:00
1031	7:23:00	7:41:00	7:54:00	8:05:00	8:18:00	8:31:00
1029	7:31:00	7:49:00	8:02:00	8:13:00	8:26:00	8:39:00
1016	7:37:00	7:55:00	8:08:00	8:19:00	8:32:00	8:45:00
1010	7:42:00	8:00:00	8:13:00	8:24:00	8:37:00	8:50:00
1008	7:46:00	8:04:00	8:17:00	8:28:00	8:41:00	8:54:00
1006	7:51:00	8:09:00	8:22:00	8:33:00	8:46:00	8:59:00
1005	7:54:00	8:12:00	8:25:00	8:36:00	8:49:00	9:02:00
1003	7:57:00	8:15:00	8:28:00	8:39:00	8:52:00	9:05:00
1001	8:00:00	8:18:00	8:31:00	8:42:00	8:55:00	9:08:00
2- South	222555	222623	222773	222837	222985	222995
1001	7:14:00	7:31:00	7:44:00	7:55:00	8:10:00	8:23:00
1011	7:16:00	7:33:00	7:46:00	7:57:00	8:12:00	8:25:00
1024	7:22:00	7:39:00	7:52:00	8:03:00	8:18:00	8:31:00
1032	7:30:00	7:47:00	8:00:00	8:11:00	8:26:00	8:39:00
1034	7:38:00	7:55:00	8:08:00	8:19:00	8:34:00	8:47:00
1036	7:46:00	8:03:00	8:16:00	8:27:00	8:42:00	8:55:00
2- North	222221	222322	222344	222411	222471	222499
1035	7:18:00	7:36:00	7:49:00	8:00:00	8:13:00	8:26:00
1033	7:28:00	7:46:00	7:59:00	8:10:00	8:23:00	8:36:00
1025	7:32:00	7:50:00	8:03:00	8:14:00	8:27:00	8:40:00
1012	7:40:00	7:58:00	8:11:00	8:22:00	8:35:00	8:48:00
1002	7:47:00	8:05:00	8:18:00	8:29:00	8:42:00	8:55:00
3-South	333623	333759	333866	333943		
1015	7:14:00	7:44:00	8:14:00	8:44:00		
1023	7:22:00	7:52:00	8:22:00	8:52:00		
1021	7:25:00	7:55:00	8:25:00	8:55:00		
1018	7:35:00	8:05:00	8:35:00	9:05:00		
1024	7:43:00	8:13:00	8:43:00	9:13:00		
1025	7:45:00	8:15:00	8:45:00	9:15:00		
3- North	333158	333258	333422	333574		
1025	7:14:00	7:44:00	8:14:00	8:44:00		
1017	7:16:00	7:46:00	8:16:00	8:46:00		
1019	7:22:00	7:52:00	8:22:00	8:52:00		
1020	7:30:00	8:00:00	8:30:00	9:00:00		
1022	7:38:00	8:08:00	8:38:00	9:08:00		
1016	7:46:00	8:16:00	8:46:00	9:16:00		
1015	7:54:00	8:24:00	8:54:00	9:24:00		
4- North	444123	444145	444213	444312	444432	
1027	7:15:00	7:33:00	7:46:00	8:03:00	8:18:00	
1014	7:23:00	7:41:00	7:54:00	8:07:00	8:22:00	
1007	7:31:00	7:49:00	8:02:00	8:14:00	8:29:00	
1009	7:37:00	7:55:00	8:08:00	8:24:00	8:39:00	
1010	7:42:00	8:00:00	8:13:00	8:33:00	8:48:00	
4- South	444611	444621	444631	444653		
1010	7:14:00	7:35:36	7:57:12	8:18:48		
1008	7:16:00	7:37:36	7:59:12	8:20:48		
1006	7:22:00	7:43:36	8:05:12	8:26:48		
1005	7:30:00	7:51:36	8:13:12	8:34:48		
1003	7:38:00	7:59:36	8:21:12	8:42:48		
1001	7:46:00	8:07:36	8:29:12	8:50:48		
1002	7:52:00	8:13:36	8:35:12	8:56:48		
1004	7:55:00	8:16:36	8:38:12	8:59:48		
1013	8:05:00	8:26:36	8:48:12	9:09:48		
1026	8:13:00	8:34:36	8:56:12	9:17:48		
1027	8:15:00	8:36:36	8:58:12	9:19:48		

FIGURE 7 Time schedule.

the six-digit number stands for the trip ID, such as 333623, in which the first three digits provide the route information (Route 3) and the other three digits represent a randomly assigned number for the trip on that route. The route and stops are in the far left column and trip ID with stop times in the right-hand columns. Thirty-two bidirectional transfers are provided on the network. To improve compatibility, all the transit-related input files follow Google's general transit feed specification (*22, 23*). Transfer information is given in Table 2 for transfers within 69 to 200 ft (21.03 to 60.96 m).

In this network, it is assumed that access and egress are made only by walking (walk-transit-walk) and within a maximum boundary of 0.23 mi (0.37 km). The speed of walking is assumed to be 3.1 mph (4.99 km/h). For the hyperpath search, the PAT time window is set from 8:40 to 9:00 a.m. and parameters of the path cost are assumed to be as follows: $\beta_{trvTime} = 1.0$, $\beta_{trsfTime} = 1.0$, $\beta_{waitTime} = 2.0$, $\beta_{trsf} = 0.5$, and $\beta_{earlyDep} = 2.0$. The resulting hyperpath is given in Figure 8 and Table 3. Table 3 gives the hyperpath list from Destination 1 to all origins (all-to-one), searching backward; an example hyperpath from each origin to Destination 1 is shown in Figure 8.

TABLE 2 Transfer Distance

From Stop	To Stop	Transfer Type	Transfer Distance (ft)
1001	1002	2	80
1003	1004	2	80
1005	1007	2	150
1006	1007	2	80
1008	1009	2	80
1001	1011	2	150
1002	1011	2	100
1005	1013	2	100
1006	1013	2	100
1007	1013	2	100
1010	1015	2	100
1012	1017	2	180
1014	1018	2	83
1014	1020	2	86
1018	1020	2	130
1021	1022	2	200
1016	1023	2	200
1012	1024	2	180
1017	1024	2	77
1014	1026	2	69
1018	1026	2	88
1020	1026	2	75
1016	1028	2	71
1023	1028	2	132
1029	1030	2	180
1025	1032	2	83
1027	1033	2	86
1027	1034	2	130
1033	1034	2	200
1031	1035	2	200
1031	1036	2	180
1035	1036	2	77

Figure 8 shows that hyperpaths from Origins 2 and 4 have a single elementary path to Destination 1, typically with single trips: 111879 boarding at 8:41 a.m. (Stop 1008) and alighting at 8:55 a.m. (Stop 1001), and 222499 boarding at 8:26 a.m. (Stop 1035) and alighting at 8:55 a.m. (Stop 1002), respectively.

The hyperpath from Origin 3 has three elementary paths with cost 32.33. For one resulting path from Origin 3 to Destination 1, there is direct access via Route 4, but this route makes a big detour by passing Intersection 103. The other paths use transfers at Intersections 102 and 104. The most preferred (highest probability) path is given by Elementary Path 3, taking a transfer between Stops 1024 and 1012 from Trip 333866 (Route 3) to Trip 222499 (Route 2). Elementary Paths 1 and 2 transfer at Intersection 102 from Trip 444432 (Route 4) to Trip 111833 (Route 1), alighting at Stop 1007 and boarding at Stop 1005 (Elementary Path 1) or Stop 1006 (Elementary Path 2). The main reason Elementary Path 3 is the preferred path is its later departure time from the origin. Trip 333866 arrives at Stop 1018 at 8:35 a.m., and Trip 222499 arrives at Stop 1002 at 8:55 a.m. Paths 2 and 3 depart at 8:22 a.m. at Stop 1014, arriving at Stop 1001 at 8:42 a.m., but this is attractive because taking Access Link 32 provides more alternatives than does Access Link 34. Also, for Elementary Paths 1 and 2, transferring to Stop 1006 is more preferred than transferring to 1005 because the transfer distance is 80 ft (24.38 m) versus 150 ft (45.72 m), respectively.

Finally, the hyperpath from Origin 5 gives two elementary paths but the more preferred path (Elementary Path 2) has a transfer between Trips 333574 and 222499, because the departure time parameter is larger than the parameters for transfer wait, walking time, and number of transfers.

CONCLUSION

A proposed hyperpath methodology was tested on an LBTE transit schedule network. The proposed LBTE network can be prepared by assigning the temporal elements of the transit schedule onto a link, which reduces the size of the expanded transit schedule network compared with existing approaches. Also introduced were a link-based hyperpath with a stochastic weight function and a label-correcting algorithm to solve for the assignment on the LBTE network. As well as reducing the effort to build a time-expanded transit schedule network, the proposed hyperpath can capture passengers' time-dependent stochastic behavior.

The proposed methodology can be applied to the broad area of schedule-based transit assignment and may also apply to the intermodal path choice environment described by Lozano and Storchi (*24*). With this link-based approach, it is possible to consider the priority of passenger boardings, because the turn penalty can be used to reflect a capacity constraint that would limit boarding if a scheduled vehicle trip were already full. In addition, to improve the performance of the proposed algorithm, it is possible to use a hierarchical hyperpath algorithm, exploiting vehicle trip-level schedules to improve the path search.

ACKNOWLEDGMENTS

This study was funded by the Explanatory Advanced Research Program through FHWA and by SHRP 2 Project C-10B. The authors thank the other members of the University of Arizona Transit Research Unit for their ideas and support.

<table>
<tr><td>origin 2 (22.4535) (Elem. Path 1)</td></tr>
<tr><td>22(access|2,1008|----|1)</td></tr>
<tr><td>145(transit|1008,1006|111879|1)</td></tr>
<tr><td>146(transit|1006,1005|111879|1)</td></tr>
<tr><td>147(transit|1005,1003|111879|1)</td></tr>
<tr><td>148(transit|1003,1001|111879|1)</td></tr>
<tr><td>15(egress|1001,1|----|1)</td></tr>
</table>

<table>
<tr><td>origin 3 (32.3296) (Elem. Path 1)</td><td>origin 3 (32.3296) (Elem. Path 2)</td><td>origin 3 (32.3296) (Elem. Path 3)</td></tr>
<tr><td>32(access|3,1014|----|0.0378888)</td><td>32(access|3,1014|----|0.0378888)</td><td>34(access|3,1018|----|0.962111)</td></tr>
<tr><td>273(transit|1014,1007|444432|1)</td><td>273(transit|1014,1007|444432|1)</td><td>249(transit|1018,1024|333866|1)</td></tr>
<tr><td>138(transit|1005,1003|111833|0.0604604)</td><td>137(transit|1006,1005|111833|0.93954)</td><td>181(transit|1012,1002|222499|1)</td></tr>
<tr><td>139(transit|1003,1001|111833|1)</td><td>138(transit|1005,1003|111833|0.0604604)</td><td>17(egress|1002,1|----|1)</td></tr>
<tr><td>15(egress|1001,1|----|1)</td><td>139(transit|1003,1001|111833|1)</td><td>1(destination|1,-2|----|1)</td></tr>
<tr><td>1(destination|1,-2|----|1)</td><td>15(egress|1001,1|----|1)</td><td></td></tr>
<tr><td></td><td>1(destination|1,-2|----|1)</td><td></td></tr>
</table>

<table>
<tr><td>origin 4 (36.7124) (Elem. Path 1)</td></tr>
<tr><td>48(access|4,1035|----|1)</td></tr>
<tr><td>178(transit|1035,1033|222499|1)</td></tr>
<tr><td>179(transit|1033,1025|222499|1)</td></tr>
<tr><td>180(transit|1025,1012|222499|1)</td></tr>
<tr><td>181(transit|1012,1002|222499|1)</td></tr>
<tr><td>17(egress|1002,1|----|1)</td></tr>
</table>

<table>
<tr><td>origin 5 (19.6263) (Elem. Path 1)</td><td>origin 5 (19.6263) (Elem. Path 2)</td></tr>
<tr><td>58(access|5,1025|----|1)</td><td>58(access|5,1025|----|1)</td></tr>
<tr><td>180(transit|1025,1012|222499|3.86926e-005)</td><td>230(transit|1025,1017|333574|0.999961)</td></tr>
<tr><td>181(transit|1012,1002|222499|1)</td><td>181(transit|1012,1002|222499|1)</td></tr>
<tr><td>17(egress|1002,1|----|1)</td><td>17(egress|1002,1|----|1)</td></tr>
</table>

FIGURE 8 Searched hyperpaths from Origin Nodes 2, 3, 4, and 5 to Destination Node 1. Entries are arranged as follows: origin node ID (hyperpath cost); link ID (link type | from node, to node | trip ID | probability to be chosen) (elem. = elementary).

TABLE 3 Hyperpath List Created from Destination Node 1

Link ID	Link Type (cost)	Link ID (Link Type|from Node, to Node|Trip ID|Probability to Be chosen)
1	Destination	—
2	Origin (22.4535)	22(access|2,1008|——|1)
4	Origin (32.3296)	32(access|3,1014|——|.0378888), 34(access|3,1018|——|.962111)
14	access	311(transit|1001,1002|444653|1)
15	egress	1(destination|1,−2|——|1)
16	access	302(transit|1002,1004|444631|1)
17	egress	1(destination|1,−2|——|1)
22	access	145(transit|1008,1006|111879|1)
24	access	93(transit|1009,1015|111345|1)
32	access	273(transit|1014,1007|444432|1)
34	access	249(transit|1018,1024|333866|1)
36	access	215(transit|1020,1022|333158|1)
135	transit	136(transit|1008,1006|111833|1)
136	transit	137(transit|1006,1005|111833|1)
137	transit	138(transit|1005,1003|111833|1)
138	transit	139(transit|1003,1001|111833|1)
139	transit	15(egress|1001,1|——|1)
140	transit	141(transit|1031,1029|111879|1)
179	transit	180(transit|1025,1012|222499|1)
180	transit	181(transit|1012,1002|222499|1)
181	transit	17(egress|1002,1|——|1)
182	transit	15(egress|1001,1|——|1)
183	transit	184(transit|1024,1032|222555|1)
184	transit	160(transit|1025,1012|222221|1)
248	transit	249(transit|1018,1024|333866|1)
249	transit	181(transit|1012,1002|222499|1)
269	transit	129(transit|1005,1003|111722|.0604604), 128(transit|1006,1005|111722|.93954)
270	transit	136(transit|1008,1006|111833|1)
272	transit	273(transit|1014,1007|444432|1)
273	transit	138(transit|1005,1003|111833|.0604604), 137(transit|1006,1005|111833|.93954)
274	transit	145(transit|1008,1006|111879|1)

NOTE: — = not applicable.

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The Transportation Network Modeling Committee peer-reviewed this paper.