**Instituto Superior de Ciências do Trabalho e da Empresa**

**Faculdade de Ciências da Universidade de Lisboa**

Departamento de Finanças do ISCTE

Departamento de Matemática da FCUL

# ALTERNATIVE STRUCTURAL MODELS TO APPROXIMATE MOODY'S KMV DISTANCE TO DEFAULT

**Guilherme Ferreira da Costa**

A Dissertation presented in partial fulfillment of the Requirements for the Degree of

**Master in Financial Mathematics**

October 2011

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# Resumo

Esta tese compara o uso de diferentes modelos estruturais para estimação dos activos de uma empresa e da da volatilidade dos mesmos, de modo a calcular o correspondente valor da *Distance to Default*, tal como definido pela Moody's KMV. A abordagem utilizada consiste em implementar a estimação de métodos baseados no modelo de Black-Scholes e no modelo CEV. Estes métodos são seguidamente utilizados para determinar os valores da *Distance to Default* para uma amostra de empresas, de modo a encontrar o método que melhor aproxima os valores da Moody's KMV. Alguns dos resultados obtidos foram melhores que os do modelo padrão, o modelo Black-Scholes. O uso de um modelo baseado em interpretar o valor de mercado da empresa como uma opção do tipo *Down and Out Call* sobre os activos da empresa, os quais se determinou seguirem o modelo CEV Square Root, e em postular directamente uma relação funcional entre a volatilidade dos activos da empresa e a volatilidade do valor da empresa em mercado, mostrou resultados substancialmente melhores que os restantes modelos.

**Palavras-Chave:** Moody's KMV, Distance to Default, Modelo Black-Sholes, Modelo CEV.

# Abstract

This thesis compares the use of different structural models in order to estimate a firm's assets and asset's volatility values and compute the corresponding Distance to Default value as defined by Moody's KMV. The approach used consists in implementing estimation methods based on the Black-Scholes Model and the Constant Elasticity of Variance Model. These methods are then applied to determine the Distance to Default values from a sample of firms, in search for the method that better approximates Moody's KMV values. Some of the results obtained were better than the benchmark model, the Black-Scholes model. The use of a model based on interpreting equity as one-year Down and Out Call Option on the firm's assets who were determined to follow a CEV Square Model and on postulating directly a functional relation between the firm's asset's volatility and equity volatility, showed substantially better results than the remaining models.

**Key-words:** Moody's KMV, Distance to Default, Black-Sholes Model, CEV Model.

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# Introduction

Structural models have been in use for years as a basis for credit risk and credit pricing models [1].

Moody's KMV (MKVM) implements a modified structural model called the Vasicek-Kealhofer model, which allows for the estimation of the assets and asset's volatility of a publicly traded firm. These estimates are used by MKVM to compute a "distance to default" (DD) value for which it determines an empirical distribution to generate the Expected Default Frequency (EDF) measure [2].

Due to commercial reasons, MKMV method is not fully disclosed. Moreover, for many investors in need of estimating default probabilities or merely comparing the risk of investing in different firms [3], replicating MKVM methodology would be an impractically complex and time-consuming task and would require the use of MKVM proprietary defaults database.

This thesis addresses the problem of estimating MKMV DD using structural models that result in methods different from the MKMV methodology. It proposes alternative structural models and analyses them in terms of: (a) their capacity for approximating DD values as computed by MKVM; (b) their computation time; (c) their ability to approximate MKVM EDF values.

The thesis is organized as follows. Chapter 1 describes MKMV general framework for estimating default probabilities. It also contains MKVM DD definition, how to compute it and how to calculate its corresponding EDF measure.

Chapter 2 exposes the proposed structural models from a theoretical point

[1] See, for example, the description of the Merton and Vasicek-Kealhofer models present in Arora et al. [2005].

[2] This is a commercially available estimate of a firm's one-year default probability.

[3] Note that it is theoretically possible to compare firms in terms of their default risk without knowing their respective default probabilities.

of view. It describes in detail each of the model's assumptions and motivations and derives the corresponding formulas that allow us to estimate the assets and asset's volatility values needed to compute a firm's DD.

In Chapter 3 we test the models introduced in the previous chapter, using a sample obtained from MKMV of DD and EDF measures for different firms. We measure the difference between MKVM DD and EDF values and our models values for each of the firms in the sample. We then compare our models based on these results and on their computation time.

Chapter 4 presents the thesis main conclusions.

# Chapter 1

# Moody's KMV Expected Default Frequency and Distance to Default

Moody's KMV EDF (Expected Default Frequency) credit risk measures are forward-looking default probabilities estimates for public (and private) firms. Each EDF measure is determined in three steps:

1. Estimate the firm's assets value and volatility;
2. Calculate the Distance to Default;
3. Map the Distance to Default to the probability of default.

## 1.1 Estimating the firm's assets value and volatility

In this step, for a publicly traded firm, asset value ($A$) and asset volatility ($\sigma_A$) are estimated from the market value of equity and book value of liabilities. In general, this can be achieved by choosing a structural model and using an options pricing based approach, which recognizes equity as a call option on the underlying assets of the firm [1]. For this purpose, MKVM uses a version of the Vasicek-Kealhofer model, whose original formulation can be found in Vasicek [1984].

[1] See Merton [1974] for the seminal example.

## 1.2 Calculating the distance to default

Having estimated $A$ and $\sigma_A$, MKVM computes the Distance to Default trough the following formula [2]:

$$DD = \frac{A - dp}{A\sigma_A},$$

where $dp$ denotes the default point.

The default point is an estimate of the value for which if the firm's assets value falls bellow, the firm will default.

It is important to stress that this is an estimate and that it does not coincide with the value of the firm's liabilities. As stated by MVMV [Crosbie and Bohn, 2003, p. 7], "in general firms do not default when their asset value reaches the book value of their total liabilities. (...) The asset value the firm will default, generally lies somewhere between total liabilities and current, or short-term liabilities".

## 1.3 Mapping the Distance to Default to the probability of default

Usually, the estimates described in Section 1.2 would allow us to compute a model implied default probability, either analytically or by using Monte Carlo Methods.

However, in the case of Moody's KMV use of the Vasiceck-Kealhofer model, default probabilities calculated in this manner "provide little discriminatory power" [Crosbie and Bohn, 2003, p. 14-18].

For this reason, EDF measures are computed using the DD empirical distribution. MKVM obtains a relationship between DD and default probability from data on historical default: [3] for each DD value, the company queries the default history for the proportion of firms with this DD value that defaulted over the following year.

[2] To be precise, MKVM DD values depend on the time horizon we are considering: there is a DD value for each time horizon; the formula presented here is a one year DD. See the example from page 32 in Dwyer and Qu [2007].

[3] MKMV uses a database including over 250,000 company-years of data and over 4,700 incidents of default or bankruptcy; see [Crosbie and Bohn, 2003, p. 14] and [Dwyer and Qu, 2007, p. 22].

# Chapter 2

# Theoretical Models

## 2.1 General Considerations

In each section of this chapter, we will follow the methodology described in Section 1.1:

i. Describe equity as a call option on the underlying assets of the firm;

ii. Choose a structural model that posits a stochastic process for the value of the firm's assets and from which it is possible to derive a valuation formula for the option specified in i.;

iii. Write an equation using the option valuation formula from ii. that relates $A$ and $\sigma_A$ with the firm's equity ($E$), the firm's equity volatility ($\sigma_E$) and the risk-free interest rate ($r$); apply Ito's lemma to the process from ii. or specify a functional relation between $\sigma_A$ and $\sigma_E$ in order to obtain the second equation needed to estimate $A$ and $\sigma_A$.

The approach followed was to choose as structural models for step ii. the Black-Scholes model, for being the simplest of the most widely used models in option valuation, and the Constant Elasticity of Variance (CEV) model [1].

The resulting empirical distribution of default probabilities as computed by MKMV has much wider tails than the Normal distribution resulting from the use of the Black-Scholes model [2]. The CEV Model posits a distribution

[1] To be precise, we used a particular case of this model called the Constant Elasticity of Variance Square-Root Model.

[2] [Crosbie and Bohn, 2003, p. 18]

for the time $t$ value of the assets that possess wider tails than the corresponding Lognormal distribution from the Black-Schols model. For this reason, the CEV model is a natural alternative to the Black-Scholes model. Moreover, the fact that it exhibits the so called leverage effect on volatility (the instantaneous variance of stock returns being inversely related to the asset price) led us to formulate the hypothesis that this is a more powerful model for the estimation of default probabilities [3].

## 2.2 Black-Scholes Model

The Black-Scholes model posits that the market value of a firm's underlying assets ($A$) follows the following stochastic process:

$$dA = \mu_A A dt + \sigma_A A dz, \tag{2.1}$$

where $\sigma_A$ denotes the annualized volatility of the asset and $dz$ is a standard Brownian motion.

### 2.2.1 Black-Scholes Model - 1 year - without default

In this section we assume that:

i. The firm's assets follow the process described by Equation (2.1);

ii. All the firm's debt is short-term debt with maturity equal to one year;

iii. There is no possibility of default occurring during the forthcoming year;

iv. In a year from now, it will be indifferent to an investor holding all the firm's equity or receiving the value of the firm's assets minus the short-term debt that must be paid.

These assumptions imply that holding equity today is equivalent to holding a call option on the assets of the firm with strike price equal to the short-term debt and maturity equal to one year. Using the Black-Scholes option pricing formula [4], we can therefore write

$$E = AN(d_1) - e^{-r\tau} std N(d_2), \tag{2.2}$$

[3] The existence of this leverage effect on volatility is supported by several empirical studies: see, for instance, Beckers [1980] and Christie [1982].

[4] See Black and Scholes [1973].

where $E$ denotes the equity value observed today in the market, $std$ is the amount of short-term debt the firm owes, $r$ is the risk-free interest rate, $N(.)$ is the standard normal cumulative distribution function, $\tau = 1$ is the time to maturity,

$$d_1 = \frac{\ln\left(\frac{A}{std}\right) + \left(r + \frac{\sigma_A^2}{2}\right)}{\sigma_A}$$

and

$$d_2 = d_1 - \sigma_A.$$

Since $E$ is a function of $A$ and $t$, we can use Ito's lemma to conclude that

$$dE = \Big[\frac{\partial E}{\partial t} + \mu_A A \frac{\partial E}{A} + \frac{1}{2}(\sigma_A A)^2 \frac{\partial^2 E}{\partial A^2}\Big] dt + \sigma_A A \frac{\partial E}{\partial A} dz.$$

This implies that

$$\frac{dE}{E} = \Big[\frac{1}{E}\frac{\partial E}{\partial t} + \mu_A \frac{A}{E}\frac{\partial E}{A} + \frac{1}{2}\sigma_A^2 \frac{A^2}{E}\frac{\partial^2 E}{\partial A^2}\Big] dt + \sigma_A \frac{A}{E}\frac{\partial E}{\partial A} dz$$

and so we obtain the relation

$$\sigma_E = \frac{A}{E}\frac{\partial E}{\partial A}\sigma_A = \frac{A}{E} N(d_1) {\sigma_A}^5. \tag{2.3}$$

Equity's volatility $\sigma_E$ can be estimated from equity's historical market values. Therefore, we can approximate $A$ and $\sigma_A$ by solving the system composed of equations (2.2) and (2.3)

$$\boxed{\begin{cases} E = AN(d_1) - e^{-r\tau} std N(d_2) \\ \sigma_E = \frac{A}{E} N(d_1)\sigma_A \end{cases}} \tag{2.4}$$

### 2.2.2 Black-Scholes Model - 1 year - with default

The assumptions from the previous section can be made more realistic by replacing assumption iii. with:

iii. Default can occur at anytime.

[5] See appendix A for the derivation of the equality $\frac{\partial E}{\partial A} = N(d_1)$.

Again, equity can be valued (approximately) as a call option with underlying equal to $A$ and strike equal to $std$, maturing in a year. Assumption iii. implies that the referred option should be a Down and Out Call with lower barrier equal to the $dp$ and zero rebate, accounting for the fact that the firm can default in case $A_t < dp$. Using the Black-Scholes option pricing formula for a Down and Out Call [6] , and noting that $\max(std, dp) = dp$, we get

$$\begin{aligned} E &= C(A, std, \tau) - \left(\frac{dp}{A}\right)^2 C\left(\frac{(dp)^2}{A}, std, \tau\right) \\ &= AN(d_1) - e^{-\tau r} stdN(d_2) - \left(\frac{dp}{A}\right)^2 \left[\frac{dp^2}{A} N(d_3) - e^{-\tau r} stdN(d_4)\right], \end{aligned} \tag{2.5}$$

where $\tau = 1$, $C(A, X, \tau)$ denotes the Black Scholes model valuation of a standard call with underling A, strike X and maturity $\tau$,

$$d_1 = \frac{\ln\left(\frac{A}{std}\right) + \tau(r + \frac{\sigma_A^2}{2})}{3\sigma_A},$$

$$d_2 = d_1 - \sigma_A,$$

$$d_3 = \frac{\ln\left[\frac{dp^2}{A\ std}\right] + \tau(r + \frac{\sigma_A^2}{2})}{3\sigma_A}$$

and

$$d_4 = d_3 - \sigma_A.$$

Using Ito's lemma as in the previous section, we can approximate $A$ and $\sigma_A$ by solving the system [7]

$$\boxed{\begin{cases} E = AN(d_1) - e^{-\tau r} stdN(d_2) - (\frac{dp}{A})^2 [\frac{(dp)^2}{A} N(d_3) - e^{-\tau r} stdN(d_4)] \\ \sigma_E = \frac{A}{E}\frac{\partial E}{\partial A}\sigma_A \end{cases}}$$

[6] See Merton [1973].

[7] Note that $\frac{\partial E}{\partial A}$ can be approximated using standard numerical techniques.

## 2.3 CEV Square Root Model

The Constant Elasticity of Variance (CEV) Square Root model posits that the market value of a firm's underlying assets follows the following stochastic process:

$$dA = \mu_A A dt + \delta_A \sqrt{A} dz. \tag{2.6}$$

It follows immediately from the above equation that

$$\sigma_A = \frac{\delta_A}{\sqrt{A}}. \tag{2.7}$$

### 2.3.1 CEV Model - 1 year - without default

The assumptions made in this section are the same as the ones described in Section 2.2.1, except for statement i., which we now replace with

i. The firm's assets follow the process described in (2.6).

Again, these assumptions imply that holding equity today is equivalent to holding a call option on the assets of the firm with strike price equal to the short-term debt and maturity equal to one year. Using the CEV model option pricing formula [8], we can therefore write

$$E = AQ_{\chi^2(4,2x)}(2k\ std) - std\ e^{-r\tau}[1 - Q_{\chi^2(2,2kX)}(2x)], \tag{2.8}$$

where $Q_\chi^2(a,b)$ denotes the complementary distribution function of a random variable that follows a non-central Chi-squared law with $a$ degrees of freedom and no-centrality parameter $b$, $\tau = 1$ is the time to maturity of the option,

$$k := \frac{2r}{\delta_A^2(e^{r\tau} - 1)},$$

and

$$x := kAe^{r\tau}.$$

Applying Ito's Lemma we can conclude that:

$$dE = \Big[\frac{\partial E}{\partial t} + \mu_A A \frac{\partial E}{A} + \frac{1}{2}(\delta_A \sqrt{A})^2 \frac{\partial^2 E}{\partial A^2}\Big] dt + \delta_A \sqrt{A} \frac{\partial E}{\partial A} dz.$$

[8] See Cox [1975].

This implies that

$$\frac{dE}{E} = \Big[\frac{1}{E}\frac{\partial E}{\partial t} + \mu_A \frac{A}{E}\frac{\partial E}{A} + \frac{1}{2}\delta_A^2 \frac{A}{E}\frac{\partial^2 E}{\partial A^2}\Big] dt + \delta_A \frac{\sqrt{A}}{E}\frac{\partial E}{\partial A} dz$$

and so we obtain the relation

$$\sigma_E = \frac{\sqrt{A}}{E}\frac{\partial E}{\partial A}\delta_A. \tag{2.9}$$

As in the previous chapter, we can approximate $A$ and $\delta_A$ by solving the system composed of equations (2.8) and (2.9)

$$\boxed{\begin{cases} E = AQ_{\chi^2(4,2x)}(2k\ std) - std\ e^{-r\tau}[1 - Q_{\chi^2(2,2kX)}(2x)] \\ \sigma_E = \frac{\sqrt{A}}{E}\frac{\partial E}{\partial A}\delta_A \end{cases}} \tag{2.10}$$

After approximating $A$ and $\delta_A$, an estimate of $\sigma_A$ can be obtained trough Equation (2.7).

### 2.3.2 CEV Model - 1 year - with default

In this section it is assumed that:

i. The firm's assets follow the process described by Equation (2.6);

ii. All the firm's debt is short-term debt with maturity equal to one year;

iii. Default can occur at anytime.

iv. In a year from now, it will be indifferent to an investor holding all the firm's equity or receiving the value of the firm's assets minus the short-term debt that must be paid.

Henceforth, $E_{CEV}(A, X, dp, r, \tau)$ denotes the computed value of a European Down and Out Call option with an underlying $A$ following the process described by Equation (2.6), strike $X$, lower barrier $dp$, maturity $\tau = 1$, zero rebate and risk-free rate $r$. This value is computed using a Monte Carlo method, detailed in appendix C.

The use of Monte Carlo Methods for the valuation of $E$ implies that the time required to compute $\frac{\partial E}{\partial A}$ numerically makes it unfeasible the use of Equation (2.9). Consequently, the approach followed was to postulate directly the functional relation supposed to exist between $\sigma_A$ and $\sigma_E$.

In order to do this, it is necessary to add to our list of assumptions:

vi. Equity also follows a CEV Square Root Model of the type described by Equation (2.8);

vii. $\delta_A \approx \delta_E$.

Since assumption vi. implies that $\sigma_E = \frac{\delta_E}{\sqrt{E}}$ and consequently that $\delta_E = \sigma_E\sqrt{E}$, we can approximate $\delta_A$ using assumption vii:

$$\delta_A \approx \delta_E = \sigma_E\sqrt{E}. \tag{2.11}$$

Regarding assumption vii., it is well worth noting that it is well known that in general $\sigma_A < \sigma_E$ [9]. What we are assuming, recurring here to Equation (2.9), is that $\sigma_A < \sigma_E$ due to the fact that we usually have $A > E$. We see no reason to believe the way $\sigma_A$ changes with $A$, is in any aspect, different from the way $\sigma_E$ varies with $E$.

Since equity can be seen as a Down and Out Call Option with zero rebate, as was done in Section 2.2.2, and using Equation (2.11), we can write

$$\boxed{\begin{cases} E = E_{CEV}(A, std, dp, r, \tau) \\ \delta_A = \sigma_E\sqrt{E} \end{cases}} \tag{2.12}$$

### 2.3.3 CEV Model - 10 years - with default

In this section we consider a more realistic picture of a firm's debt structure by defining two classes of debt: short-term debt and long-term debt.

The assumptions made are the following:

i. The firm's assets follow the process described by Equation (2.6);

ii. The firm's debt consists of short-term debt (*std*), maturing in one year, and long-term debt (*ltd*), which we assume will mature in ten years;

iii. Default can occur at anytime;

iv. In ten years from now, it will be indifferent to an investor holding all the firm's equity or receiving the value of the firm's assets minus *ltd*;

[9] See for instance Crosbie and Bohn [2003]

v. Equity follows a CEV Square Root Model of the type described by Equation (2.8);

vi. $\delta_A \approx \delta_E$.

It is worth noting that the above assumptions imply the following:

a) If somewhere between now and a year from now we have $A_t < dp$, our firms defaults;

b) In a year from now, our firm will have to pay the short-term debt $std$, implying that $A_1$ will be replaced by $A_1 - std$ and $dp$ by $dp - std$;

c) After reaching year one and paying $std$, we can approximate equity's value by that of a European Down and Out Call option with underlying $(A - dp)$, strike $ltd$, lower barrier $(dp - std)$, maturity $\tau = 9$, zero rebate and risk-free rate $r$.

From statements a), b) and c), it is possible to value equity's present value using Monte Carlo methods, as described in Appendix C .

Representing these valuation by $E = E_{CEV_{10}}(A, std, ltd, dp, r, \tau)$ we can then solve, as in the previous section, the following equations:

$$\boxed{\begin{cases} E = E_{CEV_{10}}(A, std, ltd, dp, r, \tau) \\ \delta_A = \sigma_E \sqrt{E} \end{cases}} \tag{2.13}$$

As stated before, after estimating $\delta_A$, $\sigma_A$ can be approximated trough Equation (2.7).

# Chapter 3

# Numerical Results

## 3.1 Sample and Moody's KMV results

The models described in Chapter one were applied to a sample of 20 firms, whose relevant financial data, as supplied by Moody's KMV, is summarized below in Table 3.1. $E$ denotes the firm's equity, $\sigma_E$, its equity volatility, $std$ the short-term debt, $dp$ the firm's default-point and $r$ its risk-free rate.

Each firm is identified with an abbreviated name. For the firm's complete name, as well as the date to which the presented data refers to, see appendix B.

All default probabilities presented were computed using Monte Carlo methods, generating for each computation $n = 1000$ sample paths, each with 120 equally spaced points.

Table 3.1: Sample Data

| | $E$ | $\sigma_E$ | *total debt* | *std* | *dp* | $r$ |
|---|---|---|---|---|---|---|
| NRTLQ | 6599.295 | 47.74% | 14170 | 6481 | 10325 | 3.41% |
| FRP | 634.72 | 47.72% | 2758.782 | 308.638 | 1931.147 | 3.07% |
| SPCB | 245.357 | 90.39% | 3146.126 | 606.156 | 2202.288 | 2.69% |
| LEAR | 1990.578 | 50.92% | 6924.567 | 3921.7 | 5423.133 | 3.84% |
| TROXA | 335.68 | 51.40% | 1238.3 | 408.7 | 866.81 | 1.93% |
| PGPDQ | 1976.713 | 33.44% | 2746.79 | 948.269 | 1922.753 | 4.53% |
| VSUNQ | 1057.599 | 44.03% | 755.771 | 97.428 | 529.04 | 2.45% |
| FNM | 63780.203 | 27.01% | 802294 | 335618 | 568956 | 4.90% |
| 8868 | 429.522 | 45.32% | 338.687 | 212.573 | 275.63 | 5.01% |
| ANS | 109.87 | 65.04% | 1110.947 | 83.949 | 777.663 | 3.08% |
| SIX | 584.243 | 49.22% | 2463.2 | 291.421 | 1724.24 | 5.50% |
| PRTL | 79.32 | 124.41% | 862.423 | 253.411 | 603.696 | 5.38% |
| VRSO | 0.57 | 2636.93% | 35.699 | 30.466 | 33.083 | 2.48% |
| IREP | 0.34 | 8470.44% | 135.488 | 124.998 | 130.243 | 2.50% |
| TMBAF | 37.67 | 226.67% | 1965 | 501 | 1375.5 | 2.72% |
| TOUS | 9.24 | 1226.35% | 2243.4 | 519 | 1570.38 | 2.69% |
| WOLV | 3.95 | 143.93% | 7.49 | 6.422 | 6.956 | 5.45% |
| BFTH | 26.92 | 168.19% | 1845.17 | 437.781 | 1291.619 | 5.32% |
| NEWC | 92.64 | 190.52% | 22995.418 | 12646.253 | 17820.836 | 5.26% |
| TRINQ | 1.57 | 1107.18% | 49.856 | 49.426 | 18.471 | 5.37% |

Moody's KMV Model results for this given sample are presented bellow.

Table 3.2: Moody's KMV

| | $A$ | $\sigma_A$ | $DD$ | $EDF$ |
|---|---|---|---|---|
| NRTLQ | 19515.295 | 19.23% | 2.45 | 0.0128 |
| FRP | 2824.308 | 17.16% | 1.84 | 0.0445 |
| SPCB | 2486.701 | 18.25% | 0.63 | 0.2479 |
| LEAR | 8383.416 | 14.95% | 2.36 | 0.0185 |
| TROXA | 1380.905 | 17.55% | 2.12 | 0.0269 |
| PGPDQ | 4711.857 | 15.41% | 3.84 | 0.0019 |
| VSUNQ | 1753.849 | 28.18% | 2.48 | 0.0057 |
| FNM | 872775.125 | 4.55% | 7.65 | 0.005 |
| 8868 | 735.261 | 27.56% | 2.27 | 0.0103 |
| ANS | 998.29 | 15.43% | 1.43 | 0.0861 |
| SIX | 3023.065 | 16.90% | 2.54 | 0.0143 |
| PRTL | 704.127 | 23.38% | 0.61 | 0.2563 |
| VRSO | 33.669 | 45.93% | 0.04 | 0.35 |
| IREP | 130.6 | 23.13% | 0.01 | 0.35 |
| TMBAF | 1341.097 | 16.98% | 0.00 | 0.35 |
| TOUS | 1405.017 | 20.53% | 0.00 | 0.35 |
| WOLV | 11.807 | 55.81% | 0.74 | 0.2 |
| BFTH | 1300.774 | 16.10% | 0.04 | 0.2 |
| NEWC | 18708.627 | 7.47% | 0.64 | 0.2 |
| TRINQ | 51.218 | 34.09% | 1.88 | 0.2 |

## 3.2 Black-Scholes Model

### 3.2.1 Black-Scholes Model - 1 year - without default

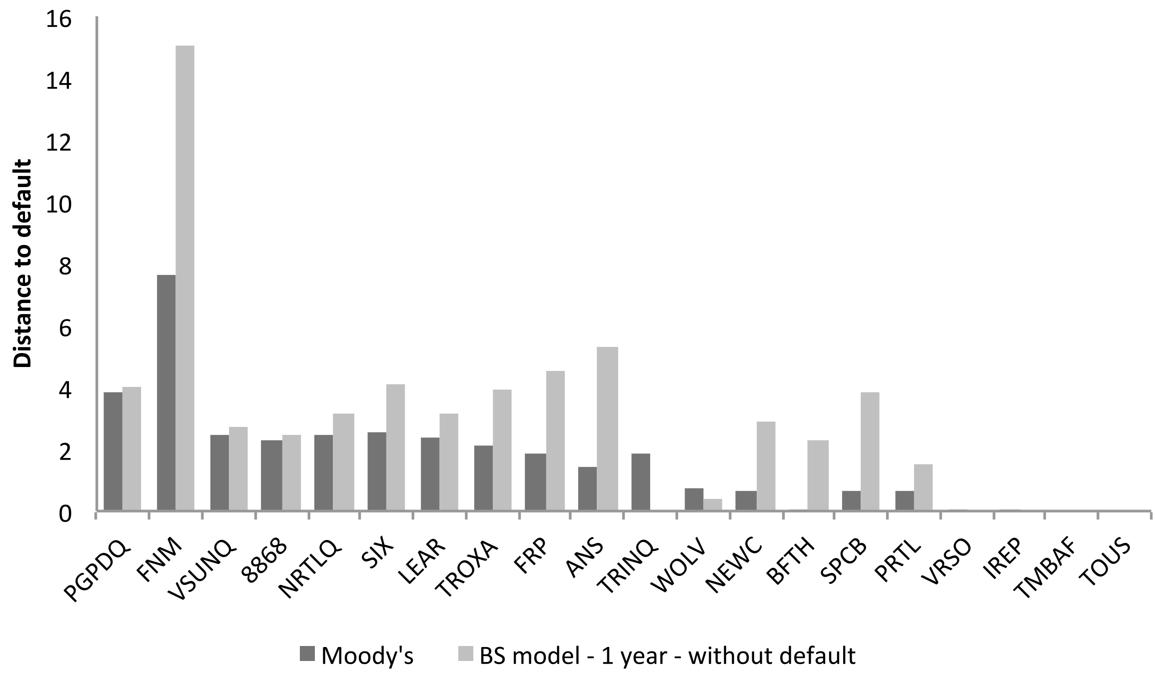
Table 3.3 shows the results obtained for this model.

Table 3.3: Black-Scholes - 1 year - without default

| | $A$ | $\sigma_A$ | $DD$ | $p$ |
|---|---|---|---|---|
| NRTLQ | 20289.3504 | 0.1560 | 3.1479 | 0 |
| FRP | 3309.0899 | 0.0924 | 4.5063 | 0 |
| SPCB | 3275.0752 | 0.0849 | 3.8569 | 0 |
| LEAR | 8649.8835 | 0.1186 | 3.1454 | 0 |
| TROXA | 1549.4566 | 0.1129 | 3.9033 | 0 |
| PGPDQ | 4601.8443 | 0.1436 | 4.0529 | 0 |
| VSUNQ | 1795.0556 | 0.2595 | 2.7182 | 0 |
| FNM | 827709.1852 | 0.0208 | 15.0193 | 0 |
| 8868 | 751.6408 | 0.2591 | 2.4445 | 0 |
| ANS | 1184.8575 | 0.0646 | 5.3226 | 0 |
| SIX | 2914.4842 | 0.0998 | 4.0931 | 0 |
| PRTL | 847.6201 | 0.1899 | 1.5152 | 0.0461 |
| VRSO | 0.5700 | 26.3693 | 0.0000 | 1 |
| IREP | 0.3400 | 84.7044 | 0.0000 | 1 |
| TMBAF | 1125.0819 | 0.4488 | 0.0000 | 1 |
| TOUS | 9.2400 | 12.2635 | 0.0000 | 1 |
| WOLV | 9.7109 | 0.7444 | 0.3811 | 0.7053 |
| BFTH | 1650.0290 | 0.0958 | 2.2678 | 0.002 |
| NEWC | 20806.3050 | 0.0491 | 2.9253 | 0 |
| TRINQ | 1.5700 | 11.0718 | 0.0000 | 1 |

Figure 3.2.1 compares this section model DD results with Moody's data. Firms are ordered in descending order of DD (as computed by MKMV).

Figure 3.1: Moody's vs BS-1Y-without default - distance to default




### 3.2.2 Black-Scholes Model - 1 year - with default

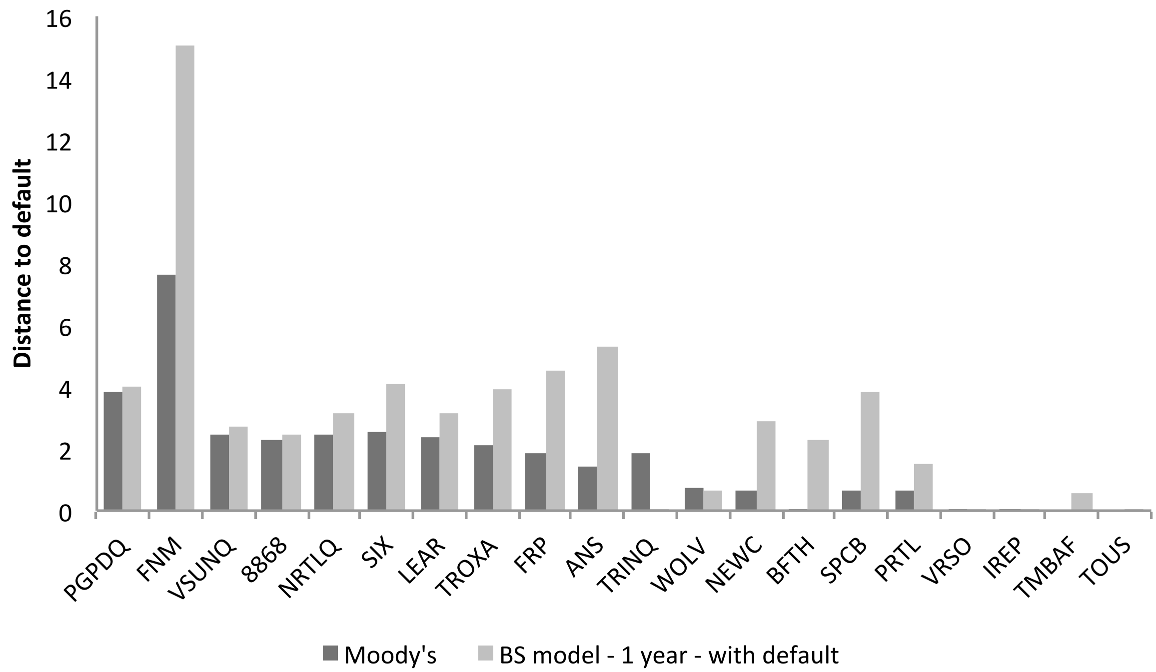
Table 3.4 shows the results obtained for this model.

Table 3.4: Black-Scholes - 1 year - with default

| | $A$ | $\sigma_A$ | $DD$ | $p$ |
|---|---|---|---|---|
| NRTLQ | 20289.350 | 0.156 | 3.148 | 0.000 |
| FRP | 3309.090 | 0.092 | 4.506 | 0.000 |
| SPCB | 3275.075 | 0.085 | 3.857 | 0.000 |
| LEAR | 8649.884 | 0.119 | 3.145 | 0.000 |
| TROXA | 1549.457 | 0.113 | 3.903 | 0.000 |
| PGPDQ | 4601.844 | 0.144 | 4.053 | 0.000 |
| VSUNQ | 1795.056 | 0.259 | 2.718 | 0.000 |
| FNM | 827709.185 | 0.021 | 15.019 | 0.000 |
| 8868 | 751.641 | 0.259 | 2.445 | 0.000 |
| ANS | 1184.858 | 0.065 | 5.323 | 0.000 |
| SIX | 2914.484 | 0.100 | 4.093 | 0.000 |
| PRTL | 847.693 | 0.190 | 1.516 | 0.046 |
| VRSO | 33.671 | 0.462 | 0.038 | 0.944 |
| IREP | 130.571 | 0.213 | 0.012 | 0.946 |
| TMBAF | 1566.434 | 0.226 | 0.540 | 0.531 |
| TOUS | 1605.235 | 0.267 | 0.081 | 0.899 |
| WOLV | 10.900 | 0.532 | 0.680 | 0.428 |
| BFTH | 1650.029 | 0.096 | 2.268 | 0.002 |
| NEWC | 20806.305 | 0.049 | 2.925 | 0.000 |
| TRINQ | 20.797 | 1.240 | 0.090 | 0.945 |

Figure 3.2.2 compares this section model DD results with Moody's data. Firms are ordered in descending order of DD (as computed by MKMV).

Figure 3.2: Moody's vs BS-1Y-with default - distance to default




## 3.3 CEV Model

### 3.3.1 CEV Model - 1 year - without default

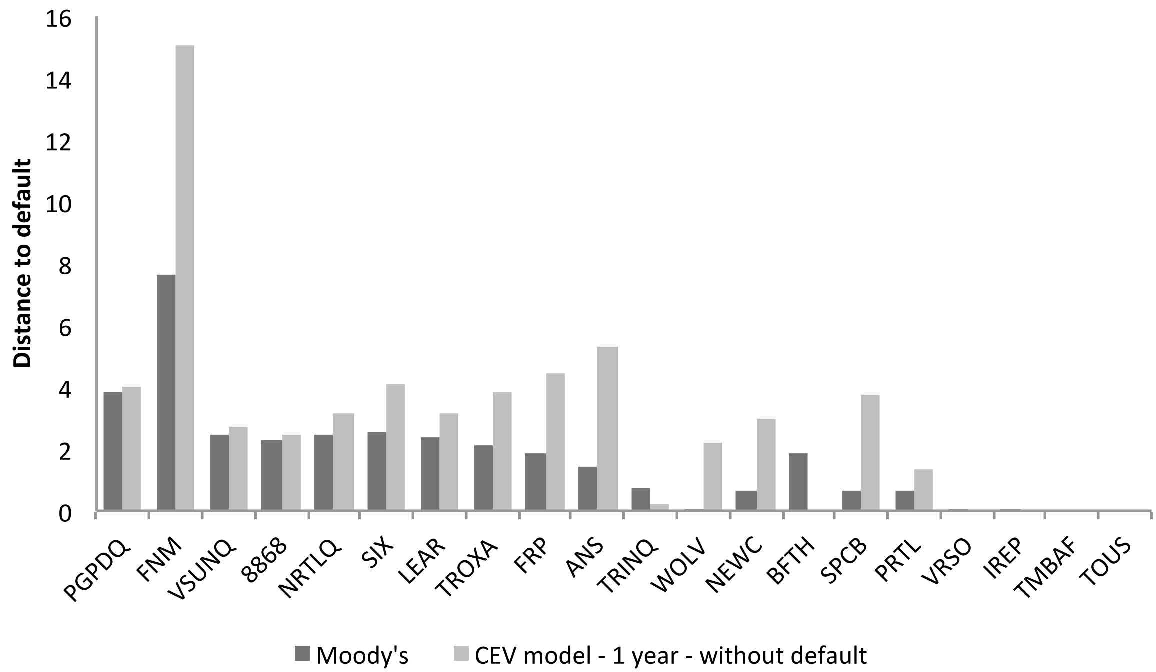
The results for this model are presented in Table 3.5.

Table 3.5: CEV - 1 year - without default

| | $A$ | $\sigma_A$ | $DD$ | $p$ |
|---|---|---|---|---|
| NRTLQ | 20283.581 | 0.157 | 3.130 | 0.000 |
| FRP | 3308.638 | 0.093 | 4.487 | 0.000 |
| SPCB | 3271.884 | 0.087 | 3.768 | 0.000 |
| LEAR | 8647.503 | 0.119 | 3.124 | 0.000 |
| TROXA | 1549.000 | 0.114 | 3.875 | 0.000 |
| PGPDQ | 4601.808 | 0.144 | 4.052 | 0.000 |
| VSUNQ | 1794.777 | 0.260 | 2.712 | 0.000 |
| FNM | 827709.152 | 0.021 | 15.019 | 0.000 |
| 8868 | 751.490 | 0.260 | 2.438 | 0.002 |
| ANS | 1184.595 | 0.065 | 5.278 | 0.000 |
| SIX | 2914.029 | 0.100 | 4.074 | 0.000 |
| PRTL | 840.710 | 0.204 | 1.382 | 0.087 |
| VRSO | 0.664 | 26.363 | 0.000 | 1.000 |
| IREP | 0.376 | 84.703 | 0.000 | 1.000 |
| TMBAF | 980.964 | 0.644 | 0.000 | 1.000 |
| TOUS | 27.344 | 12.174 | 0.000 | 1.000 |
| WOLV | 8.831 | 0.940 | 0.226 | 0.803 |
| BFTH | 1649.349 | 0.099 | 2.194 | 0.005 |
| NEWC | 20854.120 | 0.049 | 2.989 | 0.000 |
| TRINQ | 2.235 | 11.039 | 0.000 | 1.000 |

Figure 3.3.1 compares this section model DD results with Moody's data. Firms are ordered in descending order of DD (as computed by MKMV).

Figure 3.3: Moody's vs CEV-1Y-without default - distance to default




## 3.3.2 CEV Model - 1 year - with default

The results for this model are presented in Table 3.6.

Table 3.6: CEV - 1 year - with default

| | $A$ | $\sigma_A$ | $DD$ | $p$ |
|---|---|---|---|---|
| NRTLQ | 19878.210 | 0.275 | 1.747 | 0.034 |
| FRP | 3217.283 | 0.212 | 1.886 | 0.026 |
| SPCB | 2921.250 | 0.262 | 0.940 | 0.285 |
| LEAR | 8405.500 | 0.248 | 1.413 | 0.092 |
| TROXA | 1504.376 | 0.243 | 1.746 | 0.043 |
| PGPDQ | 4563.202 | 0.220 | 2.629 | 0.001 |
| VSUNQ | 1773.355 | 0.340 | 2.064 | 0.007 |
| FNM | 820250.907 | 0.075 | 4.068 | 0.000 |
| 8868 | 740.446 | 0.345 | 1.819 | 0.023 |
| ANS | 1110.323 | 0.205 | 1.464 | 0.088 |
| SIX | 2830.448 | 0.224 | 1.748 | 0.032 |
| PRTL | 738.620 | 0.408 | 0.448 | 0.592 |
| VRSO | N/A | N/A | 0.000 | 1.000 |
| IREP | N/A | N/A | 0.000 | 1.000 |
| TMBAF | 1387.079 | 0.374 | 0.022 | 0.929 |
| TOUS | N/A | N/A | 0.000 | 1.000 |
| WOLV | 9.805 | 0.914 | 0.318 | 0.737 |
| BFTH | 1401.547 | 0.233 | 0.336 | 0.650 |
| NEWC | 18260.200 | 0.136 | 0.177 | 0.746 |
| TRINQ | N/A | N/A | 0.000 | 1.000 |

The equations used in this model weren't able to provide estimates for the values of $A$ and $\sigma_A$ of the firms VRSO, IREP, TMBAF and TRINQ. This particularity of the model occurs for some firms with high EDF measures, for which equity is small relatively to total debt $dp$: when trying to solve the first equation of the System (2.12), we are faced with:

1. A positive drift for A;

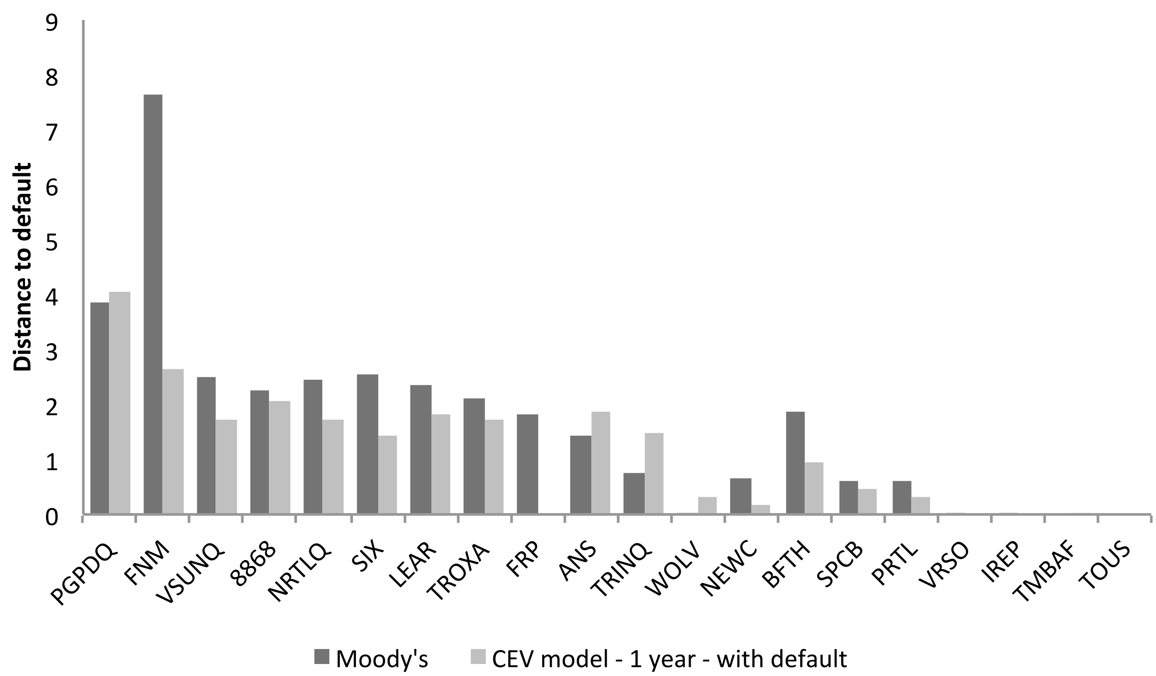
2. A lower barrier for $A$, $dp$; if $A$ is lesser than $dp$, $E_{CEV}(A, std, dp, r, \tau)$ is automatically valued at 0;

3. An upper barrier for $\sigma_A$; since $\delta_A$ is uniquely determined by $E$ and $\sqrt{E}$ trough the second equation of System (2.12), it is easy to see that $\sigma_A$ cannot be larger than $\frac{\delta_A}{\sqrt{dp}}$.

For some of the these firms, making $A$ minimum (equal to $dp$) results in a value for $\sigma_A$ for which $E_{CEV}(A, std, dp, r, \tau)$ is minimum but larger than $E$.

This behavior must not be considered a drawback of the model in question. We can interpret the results obtained by considering that, when no solution is found, the market values observed imply that the firm is in fact already in default. As such, for these cases, we set $DD = 0$ and assign the firm a probability of default equal to one. Figure 3.3.2 compares this section model DD results with Moody's data. As before, firms are ordered in descending order of DD (as computed by MKMV).

Figure 3.4: Moody's vs CEV-1Y-with default - distance to default



### 3.3.3 CEV Model - 10 year - with default

The results for this model are presented in Table 3.7.

Table 3.7: CEV - 10 year - with default

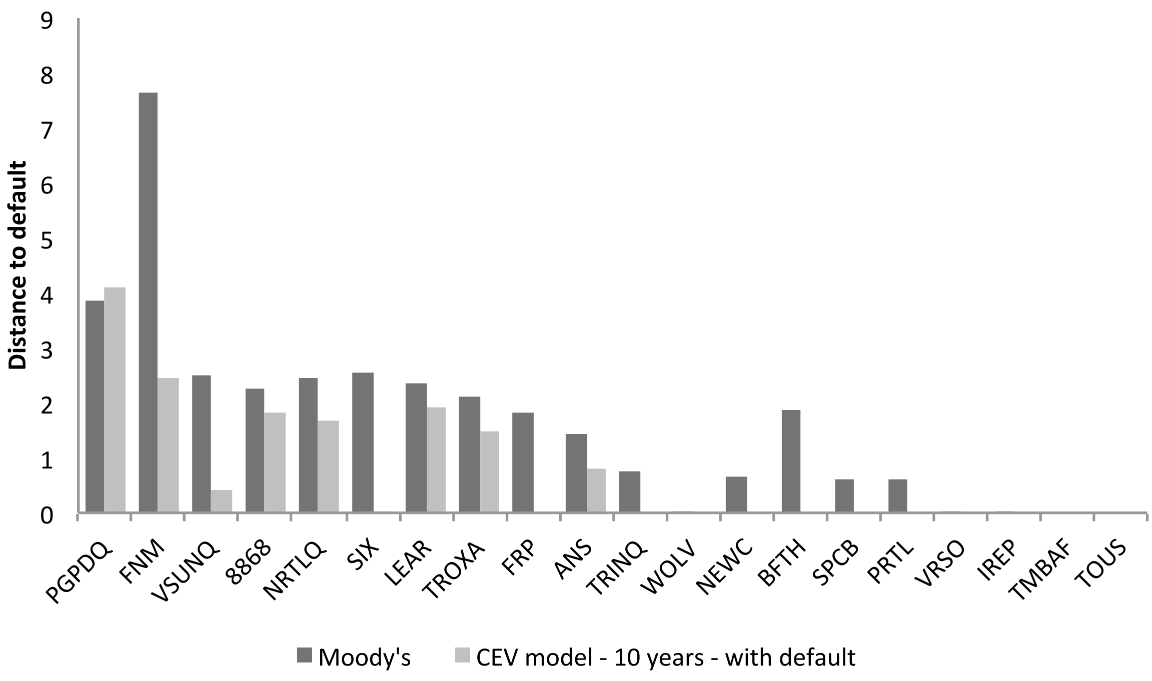
| | $A$ | $\sigma_A$ | $DD$ | $p$ |
|---|---|---|---|---|
| NRTLQ | 19534.545 | 0.277 | 1.699 | 0.042 |
| FRP | 2394.050 | 0.246 | 0.787 | 0.356 |
| SPCB | N/A | N/A | 0.000 | 1.000 |
| LEAR | 8642.239 | 0.375 | 0.022 | 0.073 |
| TROXA | 1386.160 | 0.253 | 1.481 | 0.090 |
| PGPDQ | 4331.129 | 0.226 | 2.461 | 0.002 |
| VSUNQ | 1550.290 | 0.364 | 1.811 | 0.024 |
| FNM | 824165.934 | 0.075 | 4.121 | 0.000 |
| 8868 | 774.317 | 0.338 | 1.908 | 0.013 |
| ANS | N/A | N/A | 0.000 | 1.000 |
| SIX | 1951.903 | 0.269 | 0.433 | 0.572 |
| PRTL | N/A | N/A | 0.000 | 1.000 |
| VRSO | N/A | N/A | 0.000 | 1.000 |
| IREP | N/A | N/A | 0.000 | 1.000 |
| TMBAF | N/A | N/A | 0.000 | 1.000 |
| TOUS | N/A | N/A | 0.000 | 1.000 |
| WOLV | N/A | N/A | 0.000 | 1.000 |
| BFTH | N/A | N/A | 0.000 | 1.000 |
| NEWC | N/A | N/A | 0.000 | 1.000 |
| TRINQ | N/A | N/A | 0.000 | 1.000 |

Again, it is important to note that there were firms in the sample for which the equations used weren't able to provide estimates for the values of $A$ and $\sigma_A$. The results indicate that this model is clearly unsatisfactory in its ability to distinguish default risk between firms with high $EDF$: all the firms in the sample with $EDF$ higher than 0.25 were classified as being in default. Such a small threshold implies that the model is not fit to use for practical purposes.

One possible explanation for this relies in the fact that we are using a constant risk-free rate (and consequently a constant drift) for such a large period of time.

Figure 3.3.3 compares this section model DD results with Moody's data. As before, firms are ordered in descending order of DD (as computed by MKMV).

Figure 3.5: Moody's vs CEV-10Y-with default - distance to default



## 3.4 Comparison of the Models

Table 3.8 compares the results obtained in terms of:

1. Sample mean of the difference between MKVM DD values and each of the other method's DD values ($\overline{\mathbf{DD}}$), computed by the formula:

$$\overline{\mathbf{DD}} = \frac{\sum_{i=1}^{20} |DDM_i - DDm_i|}{20},$$

   where $DDM_i$ and $DDm_i$ denote, respectively, the DD value supplied by MKVM for firm $i$ of the sample and the DD value obtained for firm $i$ from the model to which we are applying the formula.

2. Sample mean of the difference between MKVM default probability values and each of the other method's default probabilities ($\overline{\mathbf{p}}$), which is obtained trough the use of the expression:

$$\overline{\mathbf{p}} = \frac{\sum_{i=1}^{20} |pM_i - pm_i|}{20},$$

   where $pM_i$ denotes the one-year default probability estimated by MKVM for firm $i$ of the sample and $pm_i$ denotes the the same default probability for firm $i$ but obtained from the model to which we are applying the formula [1].

Table 3.8: Models Comparisons

| | $\overline{DD}$ | $\overline{p}$ |
|---|---|---|
| BS - 1Y - no default | 1.5148 | 0.0691 |
| BS - 1Y - default | 1.5239 | 0.0691 |
| CEV - 1Y - no default | 1.5027 | 0.0668 |
| CEV - 1Y - default | 0.6073 | 0.0450 |
| CEV - 10Y- default | 0.4920 | 0.0939 |

In general, the models proposed seem to provide poor results.

[1] MKVM truncates its EDF measures at 35%, so for each default probabilities estimated by our models that are larger than 0.35, we replace them with 0.35 in these computations.

Both $\overline{\mathbf{DD}}$ and $\overline{\mathbf{p}}$ are estimators of, respectively, the absolute difference between our model's DD values and MKMV values and the absolute difference between our model's default probabilities and MKMV EDF values.

Therefore, a value for $\overline{\mathbf{DD}}$ close to 1.5, as was obtained in three of the proposed models, implies that we should expect, on average, to commit an an absolute error of 1.5. This seems substantially high when compared to the DD values in our sample (note that changing a DD value by 1.5 in any of the firms would immediately lead us to consider her substantially riskier/safer than many other firms in the sample). Similarly, an error of approximately 600 basis points (or even 450), makes it implausible to use this models for, say, CDS pricing.

Surprisingly, the model described in section 3.3.2 provides much better results than the remaining ones and seems to be the best alternative to estimate Moody's KMV Distance to Default. It's $\overline{\mathbf{DD}}$ is half of the value obtained with the models from the first three sections. It possesses the second smallest $\overline{\mathbf{DD}}$ and smallest $\overline{\mathbf{p}}$.

# Chapter 4

# Conclusion

The approach followed in this thesis resulted in models that exhibited a too much poor performance for them to be used in practice as methods for the estimation of Moody's KMV Distance to Default.

Nevertheless, one of the proposed methods showed substantially better results than the remaining ones, and in particular showed better results than the methods based on the Black-Scholes model. This implies that it should be in no way excluded the possibility of obtaining different structural models better fit to estimate MKVM DD.

The model who exhibited better results was based on interpreting equity as one-year Down and Out Call Option on the firm's assets who were determined to follow a CEV Square Model and on postulating directly a functional relation between the firm's asset's volatility and equity volatility. This involved the use of Monte Carlo methods in order to evaluate Down and Out Call Options.

# Appendix A

# Auxiliary Results

## A.1 Derivation of Black-Scholes $\delta$

Applying the Chain Rule to the Black-Scholes formula for the pricing of a standard Call Option, results in:

$$\frac{\partial E}{\partial A} = N(d_1) + \frac{\partial N(d_1)}{\partial A} A - std\ e^{-r\tau} N(d_2)$$

Expanding the second parcel from the above expression, we have

$$\frac{\partial N(d_1)}{\partial A} = \frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{A} = f(d_1)\frac{\partial \frac{\ln\left(\frac{A}{std}\right)+\left(r+\frac{\sigma_A^2}{2}\right)}{\sigma_A}}{\partial A} = \frac{f(d_1)}{A\sigma_A\sqrt{\tau}},$$

with $f$ denoting the probability density function of the Normal distribution. Since

$$\frac{\partial N(d_2)}{\partial A} = \frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{A} = f(d_2)\frac{\partial(d_1 - \sigma_A\sqrt{\tau})}{\partial A} = \frac{f(d_2)}{A\sigma_A\sqrt{\tau}}$$

and

$$f(d_2) = \frac{\exp\left(-\frac{\left(d1-\sigma_a\sqrt{\tau}\right)^2}{2}\right)}{\sqrt{2\pi}} = \frac{\exp\left(-\frac{d_1^2}{2} + \ln\frac{A}{std} + r\tau\right)}{\sqrt{2\pi}} = \frac{f(d_1)Ae^{r\tau}}{std},$$

we can therefore conclude that

$$\begin{aligned}\frac{\partial E}{\partial A} &= N(d_1) + \frac{f(d_1)}{\sigma_A\sqrt{\tau}}A - std\ e^{-r\tau}\frac{e^{r\tau}f(d_1)}{std\ \sigma_A\sqrt{\tau}} = N(d_1) + \frac{f(d_1)}{\sigma_A\sqrt{T}} - \frac{f(d_1)}{\sigma_A\sqrt{T}} \\ &= N(d_1)\end{aligned}$$

# Appendix B

# Auxiliary Tables

## B.1 Firms used in the sample

Table B.1: Firm / date the financial data in the sample refers to

| | Full name | Date |
|---|---|---|
| NRTLQ | Nortel Networks INC | 14-Jan-08 |
| FRP | FairPoint Communications Inc. | 30-Jul-08 |
| SPCB | Spectrum Brands Inc. | 3-Feb-08 |
| LEAR | Lear Corp | 1-Jan-08 |
| TROXA | Tronox Inc. | 3-Dec-08 |
| PGPDQ | Pilgrim's Pride Corp | 3-Nov-07 |
| VSUNQ | Verasun Energy Group | 31-Oct-08 |
| FNM | Fannie Mae | 7-Sep-07 |
| 8868 | Urban Corporation | 13-Sep-07 |
| ANS | Ainsworth Lumber Co Ltd | 28-Jul-08 |
| SIX | Six Flags Inc | 16-Jun-07 |
| PRTL | Primus Telecomm Group Inc | 22-May-07 |
| VRSO | Verso Technologies Inc | 13-Apr-08 |
| IREP | Interep National Radio Sales Inc | 7-Mar-08 |
| TMBAF | Tembec Inc | 27-Feb-08 |
| TOUS | Tousa Inc | 5-Feb-08 |
| WOLV | Netwolves Corp | 6-Jun-07 |
| BFTH | Bally Total Fitness Holding Corp | 6-May-07 |
| NEWC | New Century Financial Corp | 21-Mar-07 |
| TRINQ | Trinsic Inc | 15-Feb-07 |

# Appendix C

# Monte Carlo Methods

## C.1 $E_{CEV}(A, X, dp, r, \tau)$

The steps we follow to compute $E_{CEV}(A, X, dp, r, \tau)$ are:

1. Determine a number of sample paths *ite* and number of points per path $n$ to be used. In our case, we defined $ite = 1000$ and $n = 10$.

2. Generate *ite* sample paths with $n$ points, using the distribution specified for $A$. This is possible when, as is the case, the distribution of $A_t|_{A_0=A}$ is known, by following the steps:

   i. Generate a pseudo-random number ($rand$) between 0 and 1;

   ii. Solve the equation $F_{A_{\frac{1}{n}}|_{A_0=A}} = rand$ to generate the path's next point;

   iii. Repeat the previous two steps, replacing the initial value for $A$ with the last step's result;

   iv. Stop when $A_1$ is determined.

3. For each path generated, determine its minimum. If it is less than $dp$, replace the path's endpoint by $X + 1$.

4. Replace each path's endpoint ($ep$) with the value $\max(X - ep, 0)$.

5. Sum all the endpoints of all paths and divide the result by *ite*. The value obtained is an estimate for the terminal payoff of our call.

6. Multiply the previous result by $e^{-r}$ to determine its present value.

## C.2 $E_{CEV_{10}}(A, std, ltd, dp, r, \tau)$

The steps we follow to compute $E_{CEV_{10}}(A, std, ltd, dp, r, \tau)$ are:

1. Determine a number of sample paths *ite* and number of points per path $n$ to be used. In our case, we defined $ite = 1000$ and $n = 12$.

2. Generate *ite* sample paths with $n$ points, using the distribution specified for $A$. This is possible when, as is the case, the distribution of $A_t|_{A_0=A}$ is known, by following the steps:

   i. Generate a pseudo-random number ($rand$) between 0 and 1;

   ii. Solve the equation $F_{A_{\frac{1}{n}}|_{A_0=A}} = rand$ to generate the path's next point;

   iii. Repeat the previous two steps, replacing the initial value for $A$ with the last step's result;

   iv. Stop when $A_1$ is determined;

   v. If our path's current minimum point is less than $dp$, set the current path's endpoint ($ep$) equal to 0 and start generating a new path. Otherwise, set $A = (A - std)$ and $dp = (dp - std)$ and return to iii;

   vi. Stop when $A_{10}$ is determined.

3. For each path generated, determine its minimum. If it is less than $dp$, replace the path's endpoint by $X + 1$.

4. Replace each path's endpoint ($ep$) with the value $\max(X - ep, 0)$.

5. Sum all the endpoints of all paths and divide the result by *ite*. The value obtained is an estimate for the terminal payoff of our call.

6. Multiply the previous result by $e^{-r}$ to determine its present value.

# Bibliography


Navneet Arora, Jeffrey Bohn, and Fanlin Zhu. Reduced Form Vs. Structural Models of Credit Risk: a Case Study of Three Models. *Journal of Investment Management*, 3(4):43–67, 2005.

Stan Beckers. The Constant Elasticity of Variance Model and Its Implications for Option Pricing. *Journal of Finance*, 35:661–673, 1980.

Fischer Black and Myron Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

Andrew Christie. The Stochastic Behavior of Common Stock Variances. *Journal of Financial Economics*, 10:407–432, 1982.

John Cox. Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. Standford University, 1975.

Peter Crosbie and Jeff Bohn. Modeling Default Risk. Moody's KMV Company, 2003.

Douglas Dwyer and Shisheng Qu. EDF 8.0 Model Enhancements. Moody's KMV Company, 2007.

Robert Merton. Theory of Rational Option Pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183, 1973.

Robert Merton. On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance*, 28:449–470, 1974.

Oldrich Alfons Vasicek. Credit Valuation. KMV Company, 1984.