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VALUATION OF EUROPEAN-STYLE SWAPTIONS

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Mestrado em Matemática Financeira

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Abstract

The present work focuses on the pricing of European-style interest rate swaptions, using the Edgeworth expansion [Collin-Dufresne and Goldstein (2002)] and the Hyperplane approximations [Singleton and Umantsev (2002)], under multi-factor exponentially-affine models of the term structure. In a market without arbitrage opportunities, it is shown that an interest rate swaption can be priced as an option on a coupon-bearing bond. While the Edgeworth approximation suggests a cumulant expansion of the probability density function of the price of the underlying coupon-bearing bond, the Hyperplane approximation proposes a linearization of the exercise region, so that the same methods used when under one-factor models can be applied.

Both methods are analyzed in detailed, and then implemented considering a three-factor Gaussian model, and different maturities for the underlying interest rate swaps, as well as a range of strike prices for each swaption. While there are almost no differences between the results yielded by both approximations, the Edgeworth approximation proves to be significantly slower as the time-to-maturity of the underlying swap increases. Moreover, the Edgeworth approximation is less flexible, because it requires a closed-form solution for the moments of the distribution of the underlying asset (i.e. a coupon-bearing bond), which are not so readily available for non-affine term structure models.

Key words: Interest rate swaptions, coupon-bearing bonds, multi-factor exponentially-affine term structure models, Edgeworth expansion approximation, Hyperplane approximation.

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Contents

Chapter 1

Introduction

Over the past years, the pricing of swaptions has received great attention from researchers and practitioners. The increasing importance of swaps in financial markets[1], the connections to other financial instruments (it is a well known fact that a swaption can be priced as an option on a coupon-bearing bond), as well as the unavailability of closed-form pricing formulas when the interest rate dynamics are modeled using two or more factors, have fostered the need to develop simple, fast and accurate methods for swaption pricing.

Three of the approaches suggested in the literature to the problem of swaption pricing have gained superior notability: the Edgeworth expansion approximation [Collin-Dufresne and Goldstein (2002)], the Hyperplane approximation [Singleton and Umantsev (2002)] and the Stochastic Duration approximation [Munk (1999)].

The Edgeworth approximation, proposed by Collin-Dufresne and Goldstein (2002) suggests an application of the Edgeworth expansion technique to the characteristic function, in order to have an approximation —through the Fourier Inversion Theorem— of the probability density function of the price of the underlying coupon-bearing bond.

On the other hand, the Hyperplane approximation, introduced by Singleton and Umantsev (2002), proposes an approximation of the exercise probabilities of the swaption through a linearization of the exercise region, being the exercise boundary approximated by a hyperplane (a straight line, in the case of a two-factor interest rate model). The relevant exercise probabilities are then computed using the same numerical methods used for standard pure discount bonds.

In the third approach, Munk (1999) extends the results achieved by Wei (1997), and defines the concept of stochastic duration of a coupon-bearing bond as the time-to-maturity of a pure discount bond with the same instantaneous variance of relative price changes as the coupon-

[1] According to the Bank of International Settlements, the notional value in single currency interest-rate derivatives increased from 60 trillion dollars in 1999 to almost 450 trillion dollars in 2009, with the combined market of interest-rate swaps and options representing around 89% of the total notional value.

bearing bond. He then shows that an European-style option on a coupon-bearing bond (and therefore, a swaption) can be priced as a multiple of the price of a pure discount bond having the same stochastic duration of the underlying coupon-bearing bond (or interest rate swap).

As the majority of the literature concerning interest rate modelling, all the mentioned approaches assume that the interest rate dynamics are modeled by exponentially-affine term structure models. The affine framework became widely used among researchers and practitioners, given its analytical tractability, allowing the existence of closed-form solutions for interest rate derivatives [see, for example, Duffie and Kan (1996)], conserving at the same time the distinctive features that characterize term structure models (e.g. long-term mean reversion and, at least for a number of models, heterocedastic volatility). For instance, Dai and Singleton (2000) test the goodness-of-fit of multi-factor exponentially-affine term structure models to several time-series of interest rates, Cox et al. (1985) as well as Jamshidian (1989) show the existence of closed-form solutions for options on pure discount bonds, under one-factor square-root and Gaussian models, Longstaff and Schwartz (1992) extend the results of Cox et al. (1985) to a two-factor CIR model and Duffie et al. (2000) demonstrate that the entire family of exponentially-affine term structure models possesses closed-form pricing formulas for options on pure discount bonds.

The main purpose of this work is to analyze in detail some of the proposed method for swaption pricing, namely the Edgeworth expansion approximation [Collin-Dufresne and Goldstein (2002)] and the Hyperplane approximation [Singleton and Umantsev (2002)], under the generalistic assumption that the the underlying term structure dynamics are of the exponentially-affine form.

The remainder of this work is structured as follows. Chapter 2 describes the term structure framework that will be adopted, as well as the main features of the swaptions market. Chapters 3 and 4 analyze the Edgeworth expansion and the Hyperplane approximation approaches, and illustrate how to implement them under a three-factor Gaussian model of the term structure. Chapter 5 provides some numerical results, regarding the speed and accuracy of both approximations. Chapter 6 concludes.

Chapter 2

Preliminary results

Before moving towards the pricing of European-style swaptions, this Chapter describes the term structure framework that will be adopted, as well as the main features of the swaptions market.

2.1 Exponentially-affine term structure models

This section follows Björk (2004, section 22.3). Let $r(t)$ denote the short rate, which, in a M-factor model, is assumed to have the following dynamics:

$$r_t = \delta(t) + \sum_{k=1}^{M} x_k(t), \tag{2.1}$$

where $r(t) = r_0$ and $r_0 > 0$. Moreover, it is assumed that the processes $x_k(t)$ follow the following stochastic differential equation:

$$dx_k(t) = \mu_k[t, r(t)]\,dt + \sigma_k[t, r(t)]\,dW_k^{\mathbb{Q}}(t), \tag{2.2}$$

where $W_k^{\mathbb{Q}}(t)$ is a standard Brownian motion, defined in the risk-neutral probability measure $\mathbb{Q}$, with instantaneous correlation ρ_{kl} $(-1 \leq \rho_{kl} \leq 1)$, such that

$$dW_k^{\mathbb{Q}}(t)\,dW_l^{\mathbb{Q}}(t) = \rho_{kl}dt, \tag{2.3}$$

where $k, l = 1, ..., M$. Finally, let $\mathcal{F}_t$ denote the sigma-field generated by $\left\{W_k^{\mathbb{Q}}(t)\right\}_{k=1}^{M}$up to time t. An interest rate model is called an exponentially-affine term structure model, if the

functions $\mu_k[t, r(t)]$ and $\sigma_k[t, r(t)]$ satisfy certain conditions, namely

$$\mu_k[t, r(t)] = \alpha_k(t) r(t) + \beta_k(t) \tag{2.4}$$

and

$$\sigma_k[t, r(t)] = \sqrt{\gamma_k(t) r(t) + \eta_k(t)}, \tag{2.5}$$

such that the time-t value of a pure discount bond with maturity at time T (and unit face value) can be written as an exponentially-affine function of the short term interest rate [Dai and Singleton (2000)], that is if

$$\begin{aligned} P(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^{T_0} r_s ds\right)\middle|\mathcal{F}_t\right] &(2.6)\\ &= \exp\left[A(t,T) - \sum_{k=1}^{M} B_k(t,T) x_k(t)\right], &(2.7)\end{aligned}$$

where $A(t,T)$ and $B_k(t,T)$ are deterministic functions, satisfying the following equations:

$$\begin{cases} \frac{\partial A(t,T)}{\partial t} = \beta_k(t) B_k(t,T) - \frac{1}{2}\eta_k(t) B_k^2(t,T) \\ A(T,T) = 0 \end{cases} \tag{2.8}$$

and

$$\begin{cases} \frac{\partial B(t,T)}{\partial t} + \alpha_k(t) B_k(t,T) - \frac{1}{2}\gamma_k(t) B_k^2(t,T) + 1 = 0 \\ B(T,T) = 0 \end{cases}. \tag{2.9}$$

2.2 European-style interest rate swaptions

An interest rate swap —usually known as an *IRS*— is a financial contract by which one party exchanges a stream of fixed interest payments for the stream of floating-rate cash-flows of another party. This occurs because certain companies have a comparative advantage in fixed rate markets, while other companies have an advantage in floating rate markets. As so, a company may be borrowing fixed, when it wants floating, or borrowing floating when it is looking for fixed. This way, an interest rate swap transforms a fixed rate loan into a floating-rate loan or vice versa.

An European-syle interest rate swaption is a contract that gives its holder the right, but not the obligation, to enter an interest rate swap at a future date $T_0 > t$, with payments C on dates T_i, with $i = 1, ..., N$, that correspond to F payments per year. In financial markets, swaptions are quoted on the fixed component (generally known as *leg*) of the underlying swap.

There are two types of swaptions: the *receiver* swaption, which gives its holder the right to

enter a *receiver* swap (receive cash-flows at a fixed rate, and pay cash-flows at a floating-rate), and the *payer* swaption, which gives its holder the right to enter a *payer* swap (pay cash-flows at a fixed rate, and receive cash-flows at a floating-rate).

At time T_0, the exercise decision of the swaption is taken by considering the difference between the payment rate initially fixed for the swap (C) and the spot swap rate for the underlying swap, at time T_0, here denoted as $SR(T_0, T_N)$. Then, depending on the swaption under analysis, the exercise decision can be:

1. For receiver swaptions, the exercise happens only if $SR(T_0, T_N) < C$, as its holder is able to obtain an higher interest rate than the one quoted in the swap market. The terminal payoff of a receiver swaption will be then given by:

$$S[T_0, T_0, SR(T_0, T_N), C, N, F] = [C - SR(T_0, T_N)]^+ \times \frac{1}{F} \times \sum_{i=1}^{N} P(T_0, T_i); \qquad (2.10)$$

2. For payer swaptions, the exercise happens only if $SR(T_0, T_N) > C$, as its holder is able to obtain a lower interest rate than the one quoted in the swap market. The terminal payoff of a payer swaption will be then given by:

$$S[T_0, T_0, SR(T_0, T_N), C, N, F] = [SR(T_0, T_N) - C]^+ \times \frac{1}{F} \times \sum_{i=1}^{N} P(T_0, T_i). \qquad (2.11)$$

Moreover, it is a well-known fact that, at time-T_0, the value of the underlying interest rate swap is given by:

$$SR(T_0, T_N) = \frac{1 - P(T_0, T_N)}{\frac{1}{F}\sum_{i=1}^{N} P(T_0, T_i)}. \qquad (2.12)$$

As so, and considering the receiver swaption with terminal payoff given by equation (2.10),

$$\begin{aligned} S[T_0, T_0, SR(T_0, T_N), C, F, N] &= \left[C - \frac{1 - P(T_0, T_N)}{\frac{1}{F}\sum_{i=1}^{N} P(T_0, T_i)}\right]^+ \times \frac{1}{F} \times \sum_{i=1}^{N} P(T_0, T_i) \\ &= \left\{\left[P(T_0, T_N) + \frac{C}{F}\sum_{i=1}^{N} P(T_0, T_i)\right] - 1\right\}^+ \qquad (2.13) \end{aligned}$$

It is easily seen that

$$P(T_0, T_N) + \frac{C}{F}\sum_{i=1}^{N} P(T_0, T_i)$$

or, more simply (considering that $C_i = \frac{C}{F}$, for $i = 1, ..., N-1$, and $C_N = 1 + \frac{C}{F}$),

$$\sum_{i=1}^{N} C_i P\left(T_0, T_i\right) \equiv Y\left(T_0\right) \tag{2.14}$$

is in fact the value, at time T_0, of a coupon-bearing bond with cash-flow payments of C_i on dates T_i, with $i = 1, ..., N$. Therefore, in a market without arbitrage opportunities, a receiver swaption can be priced as a call option on the previously described coupon-bearing bond, with a strike equivalent to a monetary unit, and with the same time-to-maturity of the considered swaption. Similarly, a payer swaption can be priced as a put option on the same coupon-bearing bond, equally with a strike price of a monetary unit, and the same time-to-maturity of the considered swaption.

Taking this into account, the date-t price of an European-style call with maturity date at time T_0, strike price K and on a coupon-bearing bond with payments C_i on dates T_i (for $i = 1, \ldots, N$) is given by

$$\begin{aligned}
CB\left[t, T_0, K, \{C_i\}_{i=1}^{N}, \{T_i\}_{i=1}^{N}\right] &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_s ds} \max\left(\sum_{i=1}^{N} C_i P\left(T_0, T_i\right) - K, 0\right) \middle| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{N} C_i \mathbb{E}_{\mathbb{Q}}\left[\frac{P\left(T_0, T_i\right)}{e^{\int_t^{T_0} r_s ds}} \mathbb{I}_{\{Y(T_0)>K\}} \middle| \mathcal{F}_t\right] \\
&\quad -K \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_s ds} \mathbb{I}_{\{Y(T_0)>K\}} \middle| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{N} C_i P\left(t, T_i\right) \mathbb{E}_{\mathbb{Q}_{T_i}}\left[\mathbb{I}_{\{Y(T_0)>K\}} \middle| \mathcal{F}_t\right] \\
&\quad -K P\left(t, T_0\right) \mathbb{E}_{\mathbb{Q}_{T_0}}\left[\mathbb{I}_{\{Y(T_0)>K\}} \middle| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{N} C_i P\left(t, T_i\right) \Pi_{\mathbb{Q}_{T_i}}\left[\mathbb{I}_{\{Y(T_0)>K\}} \middle| \mathcal{F}_t\right] \\
&\quad -K P\left(t, T_0\right) \Pi_{\mathbb{Q}_{T_0}}\left[\mathbb{I}_{\{Y(T_0)>K\}} \middle| \mathcal{F}_t\right],
\end{aligned} \tag{2.15}$$

where, in going from the second to the third equality of the previous equation the risk-neutral measure $\mathbb{Q}$ was transformed into the well known T_i risk-neutral forward probability measure $\mathbb{Q}_{T_i}$ [El Karoui and Rochet (1989), Jamshidian (1991) and Geman et al. (1995)], with $i = 0, \ldots, N$, and where $Y\left(T_0\right)$ is defined in equation (2.14) as the date-T_0 price of the underlying coupon-bearing bond.

A closed-form solution for this problem has not yet been found for multi-factor exponentially-affine models, since the exercise boundary is a non-linear function of the state variables, and therefore the methodology proposed by Jamshidian (1989) for one-factor models cannot be

applied. As so, the pricing of European-style interest rate swaptions can only be performed through approximation schemes. In the next sections, some of the methodologies proposed in literature, namely the Edgeworth expansion approximation [Collin-Dufresne and Goldstein (2002)] and the Hyperplane approximation [Singleton and Umantsev (2002)], are discussed. Later, the results obtained from these approaches are compared, namely in terms of accuracy and speed.

Chapter 3

Edgeworth expansion approximation

Proposed by Collin-Dufresne and Goldstein (2002), the Edgeworth expansion for swaption pricing suggests an approximation of the exercise probabilities $\Pi_{Q_{T_i}}\left[Y\left(T_0\right) > K\right]$ through a cumulant expansion of the probability density function of the date-T_0 price of the underlying coupon-bearing bond. Here, the distribution cumulants are defined to be the coefficients $c^j_{Q_{T_i}}$ of a Taylor series expansion of the natural logarithm of the characteristic function $\varphi_{T_i}\left(\phi\right)$ of $Y\left(T_0\right)$, with $i = 0, ..., N$, i.e.

$$\ln\left[\varphi_{Q_{T_i}}\left(\phi\right)\right] = \sum_{j=1}^{\infty} c^j_{Q_{T_i}} \frac{(ik)^j}{j!}. \tag{3.1}$$

Moreover, under the Fourier inversion theorem, the relationship between the probability density function $\pi_{Q_{T_i}}$ and the characteristic function $\varphi_{Q_{T_i}}$ is given by

$$\pi_{Q_{T_i}}\left(y\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iky}\varphi_{Q_{T_i}}\left(k\right)dk. \tag{3.2}$$

Applying equations (3.1) and (3.2), and preserving cumulants only up to the third order, the previous expression can be rewritten as follows:

$$\begin{aligned}
\pi_{Q_{T_i}}\left(y\right) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[-iky + ikc^1_{Q_{T_i}} - \frac{k^2}{2}c^2_{Q_{T_i}} - i\frac{k^3}{6}c^3_{Q_{T_i}} + O\left(k^3\right)\right]dk \\
&\approx \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[-iky + ikc^1_{Q_{T_i}} - \frac{k^2}{2}c^2_{Q_{T_i}} - i\frac{k^3}{6}c^3_{Q_{T_i}}\right]dk,
\end{aligned} \tag{3.3}$$

where $c^j_{Q_{T_i}}$ represents the j-th distribution cumulant ($j = 1, ..., M$), under the forward probability measure Q_{T_i}.

Following Chu and Kwok [2007, equation (4.4)], integration procedures imply that

$$\pi_{Q_{T_i}}(y) \approx \left[\frac{1}{\sqrt{c^2_{Q_{T_i}}}} - \frac{c^3_{Q_{T_i}}\left(y - c^1_{Q_{T_i}}\right)}{2\left(c^2_{Q_{T_i}}\right)^{\frac{5}{2}}} + \frac{c^3_{Q_{T_i}}\left(y - c^1_{Q_{T_i}}\right)^3}{2\left(c^2_{Q_{T_i}}\right)^{\frac{7}{2}}}\right] n\left(\frac{y - c^1_{Q_{T_i}}}{\sqrt{c^2_{Q_{T_i}}}}\right), \tag{3.4}$$

where

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{3.5}$$

Using approximation (3.4), the expression for the exercise probability $\Pi_{Q_{T_i}}[Y(T_0) > K]$ follows immediately:

$$\begin{aligned}
\Pi_{Q_{T_i}}[Y(T_0) > K] &= \int_K^{+\infty} \pi_{Q_{T_i}}(y)\, dy \\
&\approx \int_K^{+\infty} \left[\frac{1}{\sqrt{c^2_{Q_{T_i}}}} - \frac{c^3_{Q_{T_i}}\left(y - c^1_{Q_{T_i}}\right)}{2\left(c^2_{Q_{T_i}}\right)^{\frac{5}{2}}} + \frac{c^3_{Q_{T_i}}\left(y - c^1_{Q_{T_i}}\right)^3}{6\left(c^2_{Q_{T_i}}\right)^{\frac{7}{2}}}\right] n\left(\frac{y - c^1_{Q_{T_i}}}{\sqrt{c^2_{Q_{T_i}}}}\right) dy \\
&= N(z) + \frac{c^3_{Q_{T_i}}}{6\left(c^2_{Q_{T_i}}\right)^{\frac{3}{2}}}\left(z^2 - 1\right) n(z), \qquad (3.6)
\end{aligned}$$

where

$$z = \frac{c^1_{Q_{T_i}} - K}{\sqrt{c^2_{Q_{T_i}}}}. \tag{3.7}$$

Given the approximation (3.6) for the exercise probabilities $\Pi_{Q_{T_i}}[Y(T_0) > K]$, the only step left in order to complete the algorithm is the computation of the moments and cumulants of the distribution of $Y(T_0)$, under the forward measure Q_{T_i} (with $i = 0, ..., N$). In other words, for each one of the $(N+1)$ forward measures, the algorithm determines the first j-th moments $(j = 1, ..., M)$ of $Y(T_0)$, defined by

$$m^j_{Q_{T_i}} = \mathbb{E}_{Q_{T_i}}\left[Y(T_0)^j\right]. \tag{3.8}$$

For any j, $Y(T_0)^j$ can be expressed as a simple sum of pure discount bond prices. On the other hand, since all bond prices are of the exponential-affine form, the previous equation is

once again rewritten as

$$\begin{aligned} m^j_{Q_{T_i}} &= \mathbb{E}_{Q_{T_i}}\left[\sum_{i_1,...,i_j=1}^{N}\left(C_{i_1}\times...\times C_{i_j}\right)\times\left(P\left(T_0,T_{i_1}\right)\times...\times P\left(T_0,T_{i_j}\right)\right)\right] \\ &= \mathbb{E}_{Q_{T_i}}\left[\sum_{i_1,...,i_j=1}^{N}\left(C_{i_1}\times...\times C_{i_j}\right)\times\left(e^{C\left(T_0,T_{i_j}\right)-\sum_{k=1}^{M}D_k\left(T_0,T_{i_j}\right)x_k(T_0)}\right)\right] \end{aligned} \tag{3.9}$$

where the functions $C\left(T_0,T_{i_j}\right)$ and $D_k\left(T_0,T_{i_j}\right)$ are the sums of functions $A\left(T_0,T_{i_j}\right)$ and $B_k\left(T_0,T_{i_j}\right)$, i.e.

$$C\left(T_0,T_{i_j}\right)\equiv\sum_{i_1,...,i_j=1}^{N}A\left(T_0,T_{i_j}\right) \tag{3.10}$$

and

$$D_k\left(T_0,T_{i_j}\right)\equiv\sum_{i_1,...,i_j=1}^{N}B_k\left(T_0,T_{i_j}\right), \tag{3.11}$$

for $j=1,...,M$. Finally, with the distribution moments, one can easily compute the corresponding cumulants, through the following formula [see, for instance Gardiner (1997), section 2.7]:

$$c^j_{Q_{T_i}}=m^j_{Q_{T_i}}-\sum_{i=1}^{j-1}\binom{j-1}{i}c^{j-1}_{Q_{T_i}}m^i_{Q_{T_i}} \tag{3.12}$$

Next section illustrates how to implement the Edgeworth expansion approximation, under a three-factor Gaussian model of the term structure. The Matlab algorithm for the selected example can be found in Appendix A3.

3.1 Implementation

In this work, a three-factor Gaussian model was considered for the purpose of illustrating the implementation of the Edgeworth expansion approximation. Following the description of the technique outlined in the previous section, the necessary steps for its implementation are:

1. Computation of the moments and cumulants of the distribution of $Y\left(T_0\right)$.

   Recalling equation (3.9), the moments of the distribution of $Y\left(T_0\right)$ can be computed through

$$m^j_{Q_{T_i}}=\sum_{i_1,...,i_j=1}^{N}\left(C_{i_1}\times...\times C_{i_j}\right)\times e^{C\left(T_0,T_{i_j}\right)}\times\mathbb{E}_{Q_{T_i}}\left[\left(e^{-\sum_{k=1}^{3}D_k\left(T_0,T_{i_j}\right)x_k(T_0)}\right)\right], \tag{3.13}$$

where $C\left(T_0, T_{i_j}\right)$ and $D_k\left(T_0, T_{i_j}\right)$ are defined by equations (3.10) and (3.11), considering functions $A\left(T_0, T_{i_j}\right)$ and $B_k\left(T_0, T_{i_j}\right)$ defined in Appendix A1. By equation (A.8), the model factors follow normal transition density functions. As so, since the linear combination of normally distributed variables is also normallly distributed, the expected value on the right-hand side of equation (3.13) can be calculated using the same idea as in Appendix A1: if Z is a normal random variable with mean μ and variance σ^2, then its moment-generating function will be given by:

$$\mathbb{E}\left[\exp\left(Z\right)\right] = \exp\left(\mu + \frac{1}{2}\sigma^2\right). \tag{3.14}$$

In this case, μ and σ^2 are defined to be the expected value and variance of the random variable $Z = \left[-\sum_{k=1}^{3} D_k\left(T_0, T_{i_j}\right) x_k\left(T_0\right)\right]$, under the forward measure Q_{T_i}, with $i = 0, ..., N$. Therefore:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{T_i}}\left[Z|\mathcal{F}_t\right] &= -D_1\left(T_0, T_i\right)\mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_1\left(T_0\right)|\mathcal{F}_t\right] - D_2\left(T_0, T_i\right)\mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_2\left(T_0\right)|\mathcal{F}_t\right] \\ &\quad -D_3\left(T_0, T_i\right)\mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_3\left(T_0\right)|\mathcal{F}_t\right],\end{aligned} \tag{3.15}$$

Using equation (A.35) yields:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{T_i}}\left[Z|\mathcal{F}_t\right] &= -D_1\left(T_0, T_i\right)\left[e^{-\kappa_1(T_0-t)}x_1\left(t\right) - M_{11} - M_{12} - M_{13}\right] \\ &\quad -D_2\left(T_0, T_i\right)\left[e^{-\kappa_2(T_0-t)}x_2\left(t\right) - M_{21} - M_{22} - M_{23}\right] \\ &\quad -D_3\left(T_0, T_i\right)\left[e^{-\kappa_3(T_0-t)}x_3\left(t\right) - M_{31} - M_{32} - M_{33}\right],\end{aligned} \tag{3.16}$$

where M_{kl} is defined by equation (A.34). Considering the conditional variance,

$$\begin{aligned}\sigma^2_{\mathbb{Q}_{T_i}}\left[Z|\mathcal{F}_t\right] &= D_1^2\left(T_0, T_i\right)\sigma^2_{\mathbb{Q}_{T_i}}\left[x_1\left(T_0\right)|\mathcal{F}_t\right] + D_2^2\left(T_0, T_i\right)\sigma^2_{\mathbb{Q}_{T_i}}\left[x_2\left(T_0\right)|\mathcal{F}_t\right] \\ &\quad +D_3^2\left(T_0, T_i\right)\sigma^2_{\mathbb{Q}_{T_i}}\left[x_3\left(T_0\right)|\mathcal{F}_t\right] \\ &\quad +2D_1\left(T_0, T_i\right)D_2\left(T_0, T_i\right)\rho_{12}\sigma_{\mathbb{Q}_{T_i}}\left[x_1\left(T_0\right)|\mathcal{F}_t\right]\sigma_{\mathbb{Q}_{T_i}}\left[x_2\left(T_0\right)|\mathcal{F}_t\right] \\ &\quad +2D_1\left(T_0, T_i\right)D_3\left(T_0, T_i\right)\rho_{13}\sigma_{\mathbb{Q}_{T_i}}\left[x_1\left(T_0\right)|\mathcal{F}_t\right]\sigma_{\mathbb{Q}_{T_i}}\left[x_3\left(T_0\right)|\mathcal{F}_t\right] \\ &\quad +2D_2\left(T_0, T_i\right)D_3\left(T_0, T_i\right)\rho_{23}\sigma_{\mathbb{Q}_{T_i}}\left[x_2\left(T_0\right)|\mathcal{F}_t\right]\sigma_{\mathbb{Q}_{T_i}}\left[x_3\left(T_0\right)|\mathcal{F}_t\right].\end{aligned} \tag{3.17}$$

Rearranging terms, equation (3.14) can be rewritten as[1]:

$$\mathbb{E}_{\mathbb{Q}_{T_i}}\left[\exp\left(Z\right)|\mathcal{F}_t\right] = \exp\left\{\sum_{k,l=1}^{3} D_k\left(T_0,T_i\right)M_{kl} + \frac{1}{2}\sigma^2_{\mathbb{Q}_{T_i}}\left[Z|\mathcal{F}_t\right] - \sum_{k=1}^{3} D_k\left(T_0,T_i\right)e^{-\kappa_k(T_0-t)}x_k\left(t\right)\right\}. \tag{3.18}$$

With the moments of the distribution, the necessary cumulants can be rapidly calculated using equation (3.12).

2. Calculation of the approximated exercise probabilities.

   After obtaining the cumulants of the distribution of $Y\left(T_0\right)$, the only step left to complete the implementation is the calculation of the approximated exercise probabilities, for all the considered forward probability measures. This can be easily performed through equations (3.6) and (3.7).

[1] Equation (3.18) corrects equations (28), (29) and (30) in Collin-Dufresne and Goldstein (2002).

Chapter 4

Hyperplane approximation

The Hyperplane approximation technique, introducted by Singleton and Umantsev (2002), suggests an approximation of the exercise probabilities $\Pi_{Q_{T_i}}\left[Y\left(T_0\right) > K\right]$, where the fundamental goal is the linearization of the exercise region, so that the same methods used when under one-factor models can be applied.

The first step of the algorithm, considering a three-factor interest rate model, is to compute $\left(x_{2,\alpha/2}, x_{2,1-\alpha/2}\right)$ and $\left(x_{3,\alpha/2}, x_{3,1-\alpha/2}\right)$ such that

$$\Pi_{Q_{T_i}}\left[x_{2,\alpha/2} < x_2\left(T_0\right) < x_{2,1-\alpha/2}\right] = 1 - \alpha \tag{4.1}$$

and

$$\Pi_{Q_{T_i}}\left[x_{3,\alpha/2} < x_3\left(T_0\right) < x_{3,1-\alpha/2}\right] = 1 - \alpha, \tag{4.2}$$

where α is a given level of significance (for instance, 1% or 5%). Both pairs can be easily calculated using the univariate transition density function followed by each factor under each one of the forward probability measures, depending on the chosen term structure model (e.g. the normal density function for Gaussian models, the non-central chi-square density function for models of the Cox-Ingersoll-Ross family, etc.).

After this, the next step is to find $x_{1,1}$, $x_{1,2}$, $x_{1,3}$ and $x_{1,4}$ so that the four computed points $\left(x_{1,1}; x_{2,\alpha/2}; x_{3,\alpha/2}\right)$, $\left(x_{1,2}; x_{2,1-\alpha/2}; x_{3,\alpha/2}\right)$, $\left(x_{1,3}; x_{2,\alpha/2}; x_{3,1-\alpha/2}\right)$ and $\left(x_{1,4}; x_{1-2,\alpha/2}; x_{3,1-\alpha/2}\right)$ fall on the exercise boundary $\left[Y\left(T_0\right) = K\right]$. Due to the non-linear nature of the exercise boundary, the corresponding equation must be solved using a numerical algorithm.

Armed with the coordinates for the four triplets, the next step of the algorithm involves the fitting of a hyperplane (which degenerates into a straight line, in the case of two-factor models)

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 1 \tag{4.3}$$

to the four points, which can be performed by using standard Ordinary Least Squares procedures, i.e.

$$\boldsymbol{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y} \tag{4.4}$$

where

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}, \tag{4.5}$$

$$\mathbf{X} = \begin{bmatrix} x_{1,1} & x_{2,\alpha/2} & x_{3,\alpha/2} \\ x_{1,2} & x_{2,1-\alpha/2} & x_{3,\alpha/2} \\ x_{1,3} & x_{2,\alpha/2} & x_{3,1-\alpha/2} \\ x_{1,4} & x_{1-2,\alpha/2} & x_{3,1-\alpha/2} \end{bmatrix} \tag{4.6}$$

and

$$\mathbf{Y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{4.7}$$

Finally, the exercise probabilities can be rewritten as:

$$\Pi_{Q_{T_i}}\left[Y\left(T_0\right) > K\right] \approx \Pi_{Q_{T_i}}\left(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 > K\right) \tag{4.8}$$

or

$$\Pi_{Q_{T_i}}\left[Y\left(T_0\right) > K\right] \approx \Pi_{Q_{T_i}}\left(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 < K\right), \tag{4.9}$$

which can be solved using the same techniques used for one-factor models. The choice between equation (4.8) or equation (4.9) depends on the location of the exercise region. A simple method to find the correct sign of the inequality is as follows: for each one of the exercise probabilities, corresponding to the $(N+1)$ forward measures, substitute the triplet (x_1, x_2, x_3) in the exercise boundary $[Y(T_0) = K]$ by $(0, 0, 0)$. Then, if $[Y(T_0) > K]$, the correct form of the exercise probability will be as in equation (4.8). Otherwise, if $[Y(T_0) < K]$, the correct exercise probability will be as in equation (4.9).

Next section illustrates how to implement the Hyperplane approximation under a three-factor Gaussian model of the term structure. Once again, the corresponding Matlab algorithm code for the selected example can be found in Appendix A3.

4.1 Implementation

As in the Edgeworth approximation, the three-factor Gaussian model is used to illustrate the implementation of the Hyperplane approximation. Therefore, following the structure outlined in the previous section, the necessary steps for the implementation of the algorithm are:

1. Computation of $\left(x_{2,\alpha/2}, x_{2,1-\alpha/2}\right)$ and $\left(x_{3,\alpha/2}, x_{3,1-\alpha/2}\right)$.

   In order to find both pairs, one needs the univariate transaction density function followed by x_2 and x_3. Following Appendix A1, and since the chosen term structure model belongs to the Gaussian family, all factors follow a normal transition density function, i.e.:

$$\Pi_{Q_{T_i}}\left[x_k\left(T_0\right) < X_k | \mathcal{F}_t\right] = \frac{1}{\sigma_k\sqrt{2\pi}} \int_{-\infty}^{X_k} e^{-\frac{\left(x_k-\mu_k\right)^2}{2\sigma_k^2}} dx \tag{4.10}$$

   where $k = 1, ..., 3$ and μ_k and σ_k^2 represent the expected value and the variance of x_i at time T_0, under the corresponding forward measure Q_{T_i}. Following Appendix A1, both parameters are given by

$$\mu_k \equiv \mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_k\left(T_0\right)|\mathcal{F}_t\right] = e^{-\kappa_k\left(T_0-t\right)} x_k\left(T\right) - M_{k1} - M_{k2} - M_{k3}, \tag{4.11}$$

   where M_{kl} is given by

$$M_{kl} = \frac{\sigma_k\sigma_l\rho_{kl}}{\kappa_l\kappa_k\left(\kappa_l+\kappa_k\right)}\left\{\left(\kappa_l+\kappa_k\right)\left(1-e^{-\kappa_k\left(T_0-t\right)}\right) - \kappa_k\left[e^{-\kappa_l\left(T_i-T_0\right)} - e^{-\kappa_k\left(T_0-t\right)-\kappa_l\left(T_i-t\right)}\right]\right\}, \tag{4.12}$$

   with $k, l = 1, .., 3$ and $i = 0, ..., N$, and

$$\sigma_k^2 \equiv \sigma_{\mathbb{Q}_{T_i}}^2\left[x_k\left(T_0\right)|\mathcal{F}_t\right] = \frac{\sigma_k^2}{2\kappa_k}\left[1-e^{-2\kappa_k\left(T_0-t\right)}\right]. \tag{4.13}$$

   As so, using equation (4.10), basic integration procedures yield that

$$\Pi_{Q_{T_i}}\left[x_k\left(T_0\right) > x_{k,\alpha/2}|\mathcal{F}_t\right] = 1 - \frac{\alpha}{2}, \tag{4.14}$$

   and, therefore,

$$x_{k,\alpha/2} = \Phi^1\left\{\frac{\alpha}{2}, \mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_k\left(T_0\right)|\mathcal{F}_t\right], \sigma_{\mathbb{Q}_{T_i}}^2\left[x_k\left(T_0\right)|\mathcal{F}_t\right]\right\}, \tag{4.15}$$

   for $k = 2, 3$. Similarly,

$$\Pi_{Q_{T_i}}\left[x_k\left(T_0\right) < x_{k,1-\alpha/2}|\mathcal{F}_t\right] = 1 - \frac{\alpha}{2} \tag{4.16}$$

implies that

$$x_{k,1-\alpha/2} = \Phi^1\left\{1-\frac{\alpha}{2}, \mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_k\left(T_0\right)|\mathcal{F}_t\right], \sigma^2_{\mathbb{Q}_{T_i}}\left[x_k\left(T_0\right)|\mathcal{F}_t\right]\right\} \tag{4.17}$$

where $\Phi^1\left(\theta,\mu,\sigma^2\right)$ denotes the inverse of the univariate normal cumulative distribution function, with expected value and variance given by μ and σ^2, respectively, and for the level of significance θ.

2. Computation of $x_{1,1}$, $x_{1,2}$, $x_{1,3}$ and $x_{1,4}$.

   After the previous step, one needs to compute $x_{1,1}$, $x_{1,2}$, $x_{1,3}$ and $x_{1,4}$, in order to obtain the triplets $\left(x_{1,1};x_{2,\alpha/2};x_{3,\alpha/2}\right)$, $\left(x_{1,2};x_{2,1-\alpha/2};x_{3,\alpha/2}\right)$, $\left(x_{1,3};x_{2,\alpha/2};x_{3,1-\alpha/2}\right)$ and $\left(x_{1,4};x_{1-2,\alpha/2};x_{3,1-\alpha/2}\right)$. This is done by solving four non-linear equations (one for each triplet) of the following form:

$$\sum_{i=1}^{N} C_i P\left(T_0,T_i\right) = K \tag{4.18}$$

   i.e.

$$\sum_{i=1}^{N} C_i \times \exp\left[A\left(T_0,T_i\right) - \sum_{k=1}^{3} B_k\left(T_0,T_i\right) x_k\left(T_0\right)\right] = K \tag{4.19}$$

   where $A\left(T_0,T_i\right)$ and $B_k\left(T_0,T_i\right)$ are the functions defined in Appendix A1.

3. Fitting of a hyperplane to the four computed points.

   As mentioned in the previous section, the fitting of a hyperplane (or a straight line, for two-factor models), can be easily performed using standard OLS procedures, i.e.

$$\begin{aligned}
\boldsymbol{\beta} &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y} \\
&= \left(\begin{bmatrix} x_{1,1} & x_{2,\alpha/2} & x_{3,\alpha/2} \\ x_{1,2} & x_{2,1-\alpha/2} & x_{3,\alpha/2} \\ x_{1,3} & x_{2,\alpha/2} & x_{3,1-\alpha/2} \\ x_{1,4} & x_{1-2,\alpha/2} & x_{3,1-\alpha/2} \end{bmatrix}' \cdot \begin{bmatrix} x_{1,1} & x_{2,\alpha/2} & x_{3,\alpha/2} \\ x_{1,2} & x_{2,1-\alpha/2} & x_{3,\alpha/2} \\ x_{1,3} & x_{2,\alpha/2} & x_{3,1-\alpha/2} \\ x_{1,4} & x_{1-2,\alpha/2} & x_{3,1-\alpha/2} \end{bmatrix}\right)^{-1} \\
&\quad \cdot \begin{bmatrix} x_{1,1} & x_{2,\alpha/2} & x_{3,\alpha/2} \\ x_{1,2} & x_{2,1-\alpha/2} & x_{3,\alpha/2} \\ x_{1,3} & x_{2,\alpha/2} & x_{3,1-\alpha/2} \\ x_{1,4} & x_{1-2,\alpha/2} & x_{3,1-\alpha/2} \end{bmatrix}' \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned} \tag{4.20}$$

4. Calculation of the approximated exercise probabilities.

With the computed $\boldsymbol{\beta}$, the approximated exercise probabilities can be rapidly computed using equations (4.8) or (4.9). As already mentioned, x_1, x_2, x_3 and x_4 follow normal transition density functions, with expected values and variances (in each one of the considered forward measures) given by equations (4.11) and (4.13), respectively. Hence, the distribution of

$$\boldsymbol{\beta}^T \mathbf{X}\left(T_0\right) = \beta_1 x_1\left(T_0\right) + \beta_2 x_2\left(T_0\right) + \beta_3 x_3\left(T_0\right), \tag{4.21}$$

under the forward measure Q_{T_i}, is a normal law characterized by having expected value and variance, respectively, given by

$$\mathbb{E}_{\mathbb{Q}_{T_i}}\left[\boldsymbol{\beta}^T \mathbf{X}\left(T_0\right)|\mathcal{F}_t\right] = \beta_1 \mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_1\left(T_0\right)|\mathcal{F}_t\right] + \beta_2 \mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_2\left(T_0\right)|\mathcal{F}_t\right] + \beta_3 \mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_3\left(T_0\right)|\mathcal{F}_t\right], \tag{4.22}$$

and

$$\begin{aligned}
\sigma^2_{\mathbb{Q}_{T_i}}\left[\boldsymbol{\beta}^T \mathbf{X}\left(T_0\right)|\mathcal{F}_t\right] &= \beta_1^2 \sigma^2_{\mathbb{Q}_{T_i}}\left[x_1\left(T_0\right)|\mathcal{F}_t\right] + \beta_2^2 \sigma^2_{\mathbb{Q}_{T_i}}\left[x_2\left(T_0\right)|\mathcal{F}_t\right] + \beta_3^2 \sigma^2_{\mathbb{Q}_{T_i}}\left[x_3\left(T_0\right)|\mathcal{F}_t\right] \\
&+2\beta_1\beta_2\rho_{12}\sigma_{\mathbb{Q}_{T_i}}\left[x_1\left(T_0\right)|\mathcal{F}_t\right]\sigma_{\mathbb{Q}_{T_i}}\left[x_2\left(T_0\right)|\mathcal{F}_t\right] \\
&+2\beta_1\beta_3\rho_{13}\sigma_{\mathbb{Q}_{T_i}}\left[x_1\left(T_0\right)|\mathcal{F}_t\right]\sigma_{\mathbb{Q}_{T_i}}\left[x_3\left(T_0\right)|\mathcal{F}_t\right] \\
&+2\beta_2\beta_3\rho_{23}\sigma_{\mathbb{Q}_{T_i}}\left[x_2\left(T_0\right)|\mathcal{F}_t\right]\sigma_{\mathbb{Q}_{T_i}}\left[x_3\left(T_0\right)|\mathcal{F}_t\right],
\end{aligned} \tag{4.23}$$

where $\mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_k\left(T_0\right)|\mathcal{F}_t\right]$ and $\sigma^2_{\mathbb{Q}_{T_i}}\left[x_k\left(T_0\right)|\mathcal{F}_t\right]$, with $k = 1, .., 3$, are given by equations (4.11) and (4.13), respectively.

Chapter 5

Numerical results

In this work, the three-factor Gaussian model of the term structure is considered, for the purpose of illustrating the implementation of the Edgeworth and the Hyperplane approximations, as well as testing their accuracy and speed. The purpose of this chapter is therefore to present the numerical results obtained from these analyses.

Three swaptions where considered in the analysis:

1. A 2-2 swaption (time-to-maturity of 2 years for the option, and 2 years for the underlying interest-rate swap, with semiannual payments);

2. A 2-5 swaption (time-to-maturity of 2 years for the option, and 5 years for the underlying interest-rate swap, with semiannual payments);

3. A 2-10 swaption (time-to-maturity of 2 years for the option, and 10 years for the underlying interest-rate swap, with semiannual payments).

All the three swaptions were considered to have strike rates ranging from 4% to 8% (at 0.1% increases), corresponding to a total of 41 contracts, for each swaption-swap maturity. The parameter values for the numerical implementations of the three-factor Gaussian model were obtained from Collin-Dufresne and Goldstein (2002, exhibit 1), and are as follows:

Table 5.1 - Three-factor Gaussian model parameters

$x_1(0)$	$x_2(0)$	$x_3(0)$	δ	κ_1	κ_2	κ_3	σ_1	σ_2	σ_3	ρ_{12}	ρ_{13}	ρ_{23}
0.01	0.005	−0.02	0.06	1.0	0.2	0.5	0.01	0.005	0.002	−0.2	−0.1	0.3

Moreover, in the Edgeworth approximation, the Taylor series expansion of the characteristic function was assumed only up to the third order. In the Hyperplane approximation, a level of

significance of 5% was considered. Both algorithms were run on an Intel Core Duo CPU, with 1.67 GHz and 2 Gb RAM.

Concerning the accuracy of both techniques, the absolute average price differences between the results obtained with the Edgeworth expansion and the Hyperplane approximations, for the 41×3 contracts analyzed, were as follows:

Table 5.2 - Absolute average price differences between the Edgeworth and the Hyperplane approximations

	Absolute average difference
Swaption 1 (2×2)	7.8×10^{-9}
Swaption 2 (2×5)	1.2×10^{-7}
Swaption 3 (2×10)	5.3×10^{-7}

Table 5.2 shows that there are almost no differences between the results obtained with the two approximations, even when considering underlying interest rate swaps with longer maturities.

In terms of the speed, running times for both approximations, for the considered swaption contracts, were as follows:

Table 5.3 - Running times (in seconds) for the Edgeworth and the Hyperplane approximations

	Edgeworth approximation	Hyperplane approximation
Swaption 1 (2×2)	0.571	1.774
Swaption 2 (2×5)	7.844	6.452
Swaption 3 (2×10)	1153.28	20.423

For the first swaption, the Edgeworth approximation was the fastest technique, mainly due to the fact that the underlying interest rate swap has a small number of payments, which decreases the associated computational cost. However, as the number of payments of the underlying swap increases, the computational costs associated with the Edgeworth approximation significantly increase, in comparison with the Hyperplane approximation. As a result, in the third swaption (and for the 41 considered contract strike prices), the running time for the Edgeworth approximation was around 19.2 minutes, and only 20.4 seconds for the Hyperplane approximation. This happens because each moment of the distribution of $Y(T_0)$ requires the

computation of a summation with N^M terms (for instance, the computation of the third moment of the distribution of $Y(T_0)$, for a swap with 20 payments, requires the computation of a sum with 8 000 terms), thus greatly increasing the running times necessary for the computation of higher order moments.

Chapter 6

Conclusions

In this work two approaches to the problem of swaption pricing were analyzed: the Edgeworth expansion and the Hyperplane approximations. With 3 swaptions, and a range of 41 strike prices for each swaption, no significant differences were found between the prices yielded by both approximations. However, in terms of speed, the Edgeworth expansion is significantly slower as the time-to-maturity of the underlying interest rate swap increases, due to the high computational costs associated with the computation of the moments of $Y(T_0)$, which have to be replicated for each one of the strike prices under analysis.

Moreover, the Edgeworth approximation seems to be less flexible, since it requires a closed-form solution for the moments of the distribution of $Y(T_0)$, which is not so readily derived for other term structure models, as it is for the considered three-factor Gaussian model. Therefore, an interesting extension of this analysis would be to test both approximations with other term structure models, neither Gaussian, nor of the exponentially-affine family. However, given space and time constraints, this topic is left for future work.

Appendix A

Auxiliary results - Three-factor Gaussian model

Under the risk-neutral probability measure $\mathbb{Q}$, the three-factor Gaussian model assumes that the short-term interest rate $r(t)$ has the following dynamics:

$$r(t) = \delta(t) + \sum_{k=1}^{3} x_k(t), \tag{A.1}$$

with $r(0) = r_0$ and $r_0 > 0$. Furthermore, the processes $x_k(t)$ satisfy the following stochastic differential equation:

$$dx_k(t) = -\kappa_k x_k(t)\, dt + \sigma_k dW_k^{\mathbb{Q}}(t), \tag{A.2}$$

where κ_k and σ_k are positive constants, and $\left(W_1^{\mathbb{Q}}, W_2^{\mathbb{Q}}, W_3^{\mathbb{Q}}\right)$ is a three-dimensional Brownian motion, with instantaneous correlation ρ_{kl} $(-1 \leq \rho_{kl} \leq 1)$, such that

$$dW_k^{\mathbb{Q}}(t)\, dW_l^{\mathbb{Q}}(t) = \rho_{kl} dt. \tag{A.3}$$

Finally, let $\mathcal{F}_t$ denote the sigma-field generated by the triplet (x_1, x_2, x_3) up to time t. Using the stochastic differential equation (A.2) and applying Itô's lemma to the process

$$y_k(t) = e^{\kappa_k t} x_k(t), \tag{A.4}$$

one obtains:

$$\begin{aligned} dy_k(t) &= \kappa_k e^{\kappa_k t} x_k(t)\, dt + e^{\kappa_k t} dx_k(t) \\ &= \kappa_k e^{\kappa_k t} x_k(t)\, dt + e^{\kappa_k t}\left[-\kappa_k x_k(t)\, dt + \sigma_k dW_k^{\mathbb{Q}}(t)\right] \\ &= e^{\kappa_k t}\sigma_k dW_k^{\mathbb{Q}}(t). \end{aligned} \tag{A.5}$$

Integrating both sides of the previous equation between s and t ($\geq s$) yields,

$$y_k(t) = y_k(s) + \sigma_k \int_s^t e^{\kappa_k u} dW_k^{\mathbb{Q}}(u). \tag{A.6}$$

Combining equations (A.4) and (A.6),

$$e^{\kappa_k t} x_k(t) = e^{\kappa_k s} x_k(s) + \sigma_k \int_s^t e^{\kappa_k u} dW_k^{\mathbb{Q}}(u), \tag{A.7}$$

i.e.

$$x_k(t) = e^{-\kappa_k (t-s)} x_k(s) + \sigma_k \int_s^t e^{-\kappa_k (t-u)} dW_k^{\mathbb{Q}}(u), \tag{A.8}$$

with $k = 1, ..., 3$. Recalling equation (A.1),

$$\begin{aligned} r(t) &= \delta(t) + e^{-\kappa_1(t-s)} x_1(s) + e^{-\kappa_2(t-s)} x_2(s) + e^{-\kappa_3(t-s)} x_3(s) + \sigma_1 \int_s^t e^{-\kappa_1(t-u)} dW_1^{\mathbb{Q}}(u) \\ &\quad + \sigma_2 \int_s^t e^{-\kappa_2(t-u)} dW_2^{\mathbb{Q}}(u) + \sigma_3 \int_s^t e^{-\kappa_3(t-u)} dW_3^{\mathbb{Q}}(u). \end{aligned} \tag{A.9}$$

In order to obtain the discount factor under the three-factor Gaussian model, one can make use of the following result: if Z is a normal random variable with mean μ and variance σ^2, then its moment-generating function will be given by:

$$\mathbb{E}[\exp(Z)] = \exp\left(\mu + \frac{1}{2}\sigma^2\right). \tag{A.10}$$

Taking this into account, one only needs to show that, for each (t, T) the random variable

$$I(t, T) := \int_t^T [x_1(s) + x_2(s) + x_3(s)]\, ds, \tag{A.11}$$

conditional to the sigma field $\mathcal{F}_t$ is in fact normally distributed. Basic integration by parts

yields

$$\begin{aligned}
\int_t^T x_k(s)\,ds &= Tx_k(T) - tx_k(t) - \int_t^T s dx_k(s) \\
&= \int_t^T (T-s)\,dx_k(s) + (T-t)\,x_k(t) \\
&= -\kappa_k \int_t^T (T-s)\,x_k(s)\,ds + \sigma_k \int_t^T (T-s)\,dW_k^{\mathbb{Q}}(s) + (T-t)\,x_k(t) \\
&= -\kappa_k x_k(t) \int_t^T (T-s)\,e^{-\kappa_k(s-t)}ds - \kappa_k\sigma_k \int_t^T (T-s) \int_t^s e^{-\kappa_k(s-u)} dW_k^{\mathbb{Q}}(u)\,ds \\
&\quad + \sigma_k \int_t^T (T-s)\,dW_k^{\mathbb{Q}}(s) + (T-t)\,x_k(t) \qquad \text{(A.12)}
\end{aligned}$$

Computing separately the previous expression,

$$\begin{aligned}
-\kappa_k x_k(t) \int_t^T (T-s)\,e^{-\kappa_k(s-t)}ds &= -x_k(t)\,T \int_t^T \kappa_k e^{-\kappa_k(s-t)}ds + x_k(t) \int_t^T \kappa_k s e^{-\kappa_k(s-t)}ds \\
&= x_k(t)\,T\left[e^{-\kappa_k(s-t)}\right]_t^T - x_k(t)\left[\frac{(\kappa_k s+1)\,e^{-\kappa_k(s-t)}}{\kappa_k}\right]_t^T \\
&= x_k(t)\left[Te^{-\kappa_k(T-t)} - T - \frac{(\kappa_k T+1)\,e^{-\kappa_k(T-t)} - (\kappa_k t+1)}{\kappa_k}\right] \\
&= x_k(t)\left[-(T-t) - \frac{e^{-\kappa_k(T-t)}-1}{\kappa_k}\right] \\
&= -x_k(t)\,(T-t) - \frac{e^{-\kappa_k(T-t)}-1}{\kappa_k} x_k(t)\,, \qquad \text{(A.13)}
\end{aligned}$$

and, again through integration by parts

$$\begin{aligned}
&-\kappa_k\sigma_k \int_t^T (T-s) \int_t^s e^{-\kappa_k(s-u)} dW_k^{\mathbb{Q}}(u)\,ds \\
&= -\kappa_k\sigma_k \int_t^T \left[\int_t^s e^{\kappa_k u} dW_k^{\mathbb{Q}}(u)\right] ds \left[\int_t^s (T-v)\,e^{-\kappa_k v} dv\right] \\
&= -\kappa_k\sigma_k \left[\int_t^T e^{\kappa_k u} dW_k^{\mathbb{Q}}(u)\right] \left[\int_t^T (T-v)\,e^{-\kappa_k v} dv\right] + \kappa_k\sigma_k \int_t^T \left[\int_t^s (T-v)\,e^{-\kappa_k v} dv\right] e^{\kappa_k s} dW_k^{\mathbb{Q}}(s) \\
&= -\kappa_k\sigma_k \int_t^T \left[\int_s^T (T-v)\,e^{-\kappa_k v} dv\right] e^{\kappa_k s} dW_k^{\mathbb{Q}}(s) \\
&= -\kappa_k\sigma_k \int_t^T \left[\frac{(T-s)\,e^{-\kappa_k s}}{\kappa_k} + \frac{e^{-\kappa_k T} - e^{-\kappa_k s}}{\kappa_k^2}\right] e^{\kappa_k s} dW_k^{\mathbb{Q}}(s) \\
&= -\sigma_k \int_t^T \left[(T-s) + \frac{e^{-\kappa_k(T-s)}-1}{\kappa_k}\right] dW_k^{\mathbb{Q}}(s)\,. \qquad \text{(A.14)}
\end{aligned}$$

Recalling equation (A.12) and adding up the previous terms, one obtains

$$\int_t^T x_k(s)\,ds = \frac{1-e^{-\kappa_k(T-t)}}{\kappa_k}x_k(t) + \frac{\sigma_k}{\kappa_k}\int_t^T \left[1-e^{-\kappa_k(T-s)}\right] dW_k^{\mathbb{Q}}(s). \tag{A.15}$$

Since any Itô's integral, with a deterministic integrand, possesses a normal distribution with zero mean and variance equal to its quadratic variation, then:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[I(t,T)\,|\mathcal{F}_t\right] &= \frac{1-e^{-\kappa_1(T-t)}}{\kappa_1}x_1(t) + \frac{\sigma_1}{\kappa_1}\times 0 + \frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}x_2(t) + \frac{\sigma_2}{\kappa_2}\times 0 + \\
&\quad + \frac{1-e^{-\kappa_3(T-t)}}{\kappa_3}x_3(t) + \frac{\sigma_3}{\kappa_3}\times 0 \\
&= \sum_{k=1}^{3} B_k(t,T)\,x_k(t), \qquad (A.16)
\end{aligned}$$

where

$$B_k(t,T) = \frac{1-e^{-\kappa_k(T-t)}}{\kappa_k}. \tag{A.17}$$

Concerning the computation of the conditional variance,

$$\begin{aligned}
\sigma^2\left[I(t,T)\,|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left\{\left[I(t,T) - \mathbb{E}_{\mathbb{Q}}\left(I(t,T)\,|\mathcal{F}_t\right)\right]^2 |\mathcal{F}_t\right\} \\
&= \mathbb{E}_{\mathbb{Q}}\left\{\frac{\sigma_1}{\kappa_1}\int_t^T \left[1-e^{-\kappa_1(T-s)}\right] dW_1^{\mathbb{Q}}(s) + \frac{\sigma_2}{\kappa_2}\int_t^T \left[1-e^{-\kappa_2(T-s)}\right] dW_2^{\mathbb{Q}}(s) + \right. \\
&\quad \left. + \frac{\sigma_3}{\kappa_3}\int_t^T \left[1-e^{-\kappa_3(T-s)}\right] dW_3^{\mathbb{Q}}(s)\right]^2 |\mathcal{F}_t\right\}.
\end{aligned}$$

Using Itô's isometry, simple integration procedures imply that:

$$\begin{aligned}
&\sigma^2\left[I(t,T)\,|\mathcal{F}_t\right] \\
&= \frac{\sigma_1^2}{\kappa_1^2}\int_t^T \left[1-e^{-\kappa_1(T-s)}\right]^2 ds + \frac{\sigma_2^2}{\kappa_2^2}\int_t^T \left[1-e^{-\kappa_2(T-s)}\right]^2 ds + \frac{\sigma_3^2}{\kappa_3^2}\int_t^T \left[1-e^{-\kappa_3(T-s)}\right]^2 ds \\
&\quad + 2\rho_{12}\frac{\sigma_1}{\kappa_1}\frac{\sigma_2}{\kappa_2}\int_t^T \left[1-e^{-\kappa_1(T-s)}\right]\left[1-e^{-\kappa_2(T-s)}\right] ds \\
&\quad + 2\rho_{13}\frac{\sigma_1}{\kappa_1}\frac{\sigma_3}{\kappa_3}\int_t^T \left[1-e^{-\kappa_1(T-s)}\right]\left[1-e^{-\kappa_3(T-s)}\right] ds \\
&\quad + 2\rho_{23}\frac{\sigma_2}{\kappa_2}\frac{\sigma_3}{\kappa_3}\int_t^T \left[1-e^{-\kappa_2(T-s)}\right]\left[1-e^{-\kappa_3(T-s)}\right] ds \qquad (A.18)
\end{aligned}$$

i.e.

$$
\begin{aligned}
&\sigma^2\left[I\left(t,T\right)|\mathcal{F}_t\right] \\
&= \frac{\sigma_1^2}{\kappa_1^2}\left[s-\frac{2e^{-\kappa_1(T-s)}}{\kappa_1}-\frac{e^{-2\kappa_1(T-s)}}{2\kappa_1}\right]_t^T+\frac{\sigma_2^2}{\kappa_2^2}\left[s-\frac{2e^{-\kappa_2(T-s)}}{\kappa_2}-\frac{e^{-2\kappa_2(T-s)}}{2\kappa_2}\right]_t^T \\
&\quad+\frac{\sigma_3^2}{\kappa_3^2}\left[s-\frac{2e^{-\kappa_3(T-s)}}{\kappa_3}-\frac{e^{-2\kappa_3(T-s)}}{2\kappa_3}\right]_t^T \\
&\quad+2\rho_{12}\frac{\sigma_1}{\kappa_1}\frac{\sigma_2}{\kappa_2}\left[s-\frac{e^{-\kappa_1(T-s)}}{\kappa_1}-\frac{e^{-\kappa_2(T-s)}}{\kappa_2}+\frac{e^{-(\kappa_1+\kappa_2)(T-s)}}{(\kappa_1+\kappa_2)}\right]_t^T \\
&\quad+2\rho_{13}\frac{\sigma_1}{\kappa_1}\frac{\sigma_3}{\kappa 3}\left[s-\frac{e^{-\kappa_1(T-s)}}{\kappa_1}-\frac{e^{-\kappa_3(T-s)}}{\kappa_3}+\frac{e^{-(\kappa_1+\kappa_3)(T-s)}}{(\kappa_1+k_3)}\right]_t^T \\
&\quad+2\rho_{23}\frac{\sigma_2}{\kappa_2}\frac{\sigma_3}{\kappa_3}\left[s-\frac{e^{-\kappa_2(T-s)}}{\kappa_2}-\frac{e^{-\kappa_3(T-s)}}{\kappa_3}+\frac{e^{-(\kappa_2+\kappa_3)(T-s)}}{(\kappa_3+\kappa_3)}\right]_t^T \\
&= \frac{\sigma_1^2}{\kappa_1^2}\left[T-t-\frac{2-2e^{-\kappa_1(T-t)}}{\kappa_1}-\frac{1-e^{-2\kappa_1(T-t)}}{2\kappa_1}\right]+\frac{\sigma_2^2}{\kappa_2^2}\left[T-t-\frac{2-2e^{-\kappa_2(T-t)}}{\kappa_2}-\frac{1-e^{-2\kappa_2(T-t)}}{2\kappa_2}\right] \\
&\quad+\frac{\sigma_3^2}{\kappa_3^2}\left[T-t-\frac{2-2e^{-\kappa_3(T-t)}}{\kappa_3}-\frac{1-e^{-2\kappa_3(T-t)}}{2\kappa_3}\right] \\
&\quad+2\rho_{12}\frac{\sigma_1}{\kappa_1}\frac{\sigma_2}{\kappa_2}\left[T-t-\frac{1-e^{-\kappa_1(T-t)}}{\kappa_1}-\frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}+\frac{1-e^{-(\kappa_1+\kappa_2)(T-t)}}{(\kappa_1+\kappa_2)}\right] \\
&\quad+2\rho_{13}\frac{\sigma_1}{\kappa_1}\frac{\sigma_3}{\kappa_3}\left[T-t-\frac{1-e^{-\kappa_1(T-t)}}{\kappa_1}-\frac{1-e^{-\kappa_3(T-t)}}{\kappa_3}+\frac{1-e^{-(\kappa_1+\kappa_3)(T-t)}}{(\kappa_1+\kappa_3)}\right] \\
&\quad+2\rho_{23}\frac{\sigma_2}{\kappa_2}\frac{\sigma_3}{\kappa_3}\left[T-t-\frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}-\frac{1-e^{-\kappa_3(T-t)}}{\kappa_3}+\frac{1-e^{-(\kappa_2+\kappa_3)(T-t)}}{(\kappa_2+\kappa_3)}\right] \qquad \text{(A.19)}
\end{aligned}
$$

Since $\rho_{kl}=1$, for any $k=l$, the previous equation can be rewritten as:

$$
\sigma^2\left[I\left(t,T\right)|\mathcal{F}_t\right]=\sum_{k,l=1}^{3}\rho_{kl}\frac{\sigma_k\sigma_l}{\kappa_{kl}}\left[T-t-B_k\left(t,T\right)-B_k\left(t,T\right)+B_{k+l}\left(t,T\right)\right]. \qquad \text{(A.20)}
$$

Therefore, combining equations (A.10), (A.11), (A.16), (A.17) and (A.20) the discount factor under the three-factor Gaussian model is given by:

$$
\begin{aligned}
P(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s d_s\right)|\mathcal{F}t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T\left[\delta\left(s\right)+x_1\left(s\right)+x_2\left(s\right)+x_3\left(s\right)\right]d_s\right)|\mathcal{F}t\right], \qquad \text{(A.21)}
\end{aligned}
$$

and, therefore,

$$\begin{aligned} P(t,T) &= \exp\left[-\int_t^T \delta(s)\,ds\right] \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T \left[x_1(s)+x_2(s)+x_3(s)\right] d_s\right)|\mathcal{F}t\right] \\ &= \exp\left[-\int_t^T \delta(s)\,ds\right] \exp\left\{-\mathbb{E}_{\mathbb{Q}}\left[I(t,T)\,|\mathcal{F}_t\right] + \frac{1}{2}\sigma^2\left[I(t,T)\,|\mathcal{F}_t\right]\right\} \\ &= \exp\left[A(t,T) - \sum_{k=1}^{3} B_k(t,T)\,x_k(t)\right] \end{aligned} \tag{A.22}$$

where[1]

$$A(t,T) = -\int_t^T \delta(s)\,ds + \frac{1}{2}\sum_{k,l=1}^{3} \rho_{kl}\frac{\sigma_k\sigma_l}{\kappa_k\kappa_l}\left[T-t-B_k(t,T)-B_l(t,T)+B_{k+l}(t,T)\right] \tag{A.23}$$

and $B_k(t,T)$ is defined as in equation (A.17).

Using a change of numeraire [El Karoui and Rochet (1989), Jamshidian (1991) and Geman et al. (1995)], it is easily shown that the state variables have the following dynamics in the forward measure $\mathbb{Q}_{T_i}$:

$$dx_k(t) = \left[-\kappa_k x_l(t) - \sum_{l=1}^{3} \sigma_k\sigma_l\rho_{kl}B_k(T_i-t)\right]dt + \sigma_k dW_k^{\mathbb{Q}_{T_i}}(t), \tag{A.24}$$

where

$$dW_k^{\mathbb{Q}_{T_i}}(t) = dW_k^{\mathbb{Q}}(t) + \sum_{l=1}^{3} \sigma_k\sigma_l\rho_{kl}B_k(T_i-t)\,dt, \tag{A.25}$$

with $i = 0, ..., N$ and $k = 1, ..., 3$.

Applying Itô's lemma to the process

$$y_k(t) = e^{\kappa_k t}x_k(t), \tag{A.26}$$

then:

$$\begin{aligned} dy_k(t) &= \kappa_k e^{\kappa_k t}x_k(t) + e^{\kappa_k t}dx_k(t) \\ &= \kappa_k e^{\kappa_k t}x_k(t) + e^{\kappa_k t}\left\{\left[-\kappa_k x_k(t) - \sum_{j=1}^{3}\sigma_k\sigma_j\rho_{kj}B_j(T_i-t)\right]dt + \sigma_k dW_k^{\mathbb{Q}_{T_i}}(t)\right\} \\ &= -e^{\kappa_k t}\left[\sum_{j=1}^{3}\sigma_k\sigma_j\rho_{kj}B_j(T_i-t)\right]dt + \sigma_k e^{\kappa_k t}dW_k^{\mathbb{Q}_{T_i}}(t). \end{aligned} \tag{A.27}$$

[1] Equation (A.23) corrects equation (26) in Collin-Dufresne and Goldstein (2002).

Integrating both sides of the previous equation between s and t ($\geq s$) yields

$$\begin{aligned} y_k(t) &= y_k(s) + \int_s^t -e^{\kappa_k u} \sum_{j=1}^{3} \sigma_k \sigma_j \rho_{kj} B_{\kappa_j}(T_i - u)\, du + \sigma_k \int_s^t e^{\kappa_k u} dW_k^{\mathbb{Q}_{T_i}}(u) \\ &= y_k(s) - \int_s^t e^{\kappa_k u} \sigma_k \sigma_1 \rho_{k1} B_{\kappa_1}(T_i - u)\, du - \int_s^t e^{\kappa_k u} \sigma_k \sigma_2 \rho_{k2} B_{\kappa_2}(T_i - u)\, du \\ &= \int_s^t e^{\kappa_k u} \sigma_k \sigma_3 \rho_{k3} B_{\kappa_3}(T_i - u)\, du + \sigma_k \int_s^t e^{\kappa_k u} dW_k^{\mathbb{Q}_{T_i}}(u) \end{aligned} \tag{A.28}$$

Through standard integration procedures, the first integral yields:

$$\begin{aligned} &\int_s^t e^{\kappa_k u} \sigma_k \sigma_1 \rho_{k1} B_{\kappa_1}(T_i - t)\, du \\ &= \sigma_k \sigma_1 \rho_{k1} \int_s^t e^{\kappa_k u} \left[\frac{1 - e^{-\kappa_1 (T_i - u)}}{\kappa_1} \right] du \\ &= \frac{\sigma_k \sigma_1 \rho_{k1}}{\kappa_1} \int_s^t e^{\kappa_k u} - e^{\kappa_k u - \kappa_1 (T_i - u)} du \\ &= \frac{\sigma_k \sigma_1 \rho_{k1}}{\kappa_1} \left[\frac{e^{\kappa_k u}}{\kappa_k} - \frac{e^{\kappa_k u - \kappa_1 (T_i - u)}}{(\kappa_1 + \kappa_k)} \right]_s^t \\ &= \frac{\sigma_k \sigma_1 \rho_{k1}}{\kappa_1 \kappa_k (\kappa_1 + \kappa_k)} \left[(\kappa_1 + \kappa_k) e^{\kappa_k u} - \kappa_k e^{\kappa_k u - \kappa_1 (T_i - u)} \right]_s^t \\ &= \frac{\sigma_k \sigma_1 \rho_{k1}}{\kappa_1 \kappa_k (\kappa_1 + \kappa_k)} \left[(\kappa_1 + \kappa_k) e^{\kappa_k t} - \kappa_k e^{\kappa_k t - \kappa_1 (T_i - t)} - (\kappa_1 + \kappa_k) e^{\kappa_k s} + \kappa_k e^{\kappa_k s - \kappa_1 (T_i - s)} \right] \\ &= \frac{\sigma_k \sigma_1 \rho_{k1}}{\kappa_1 \kappa_k (\kappa_1 + \kappa_k)} \left\{ (\kappa_1 + \kappa_k) \left(e^{\kappa_k t} - e^{\kappa_k s} \right) - \kappa_k \left[e^{\kappa_k t - \kappa_1 (T_i - t)} - e^{\kappa_k s - \kappa_1 (T_i - s)} \right] \right\}. \end{aligned} \tag{A.29}$$

Appying the same process to the remaining integrals,

$$\begin{aligned} &\int_s^t e^{\kappa_k u} \sigma_k \sigma_2 \rho_{k2} B_{\kappa_2}(T_i - t)\, du \\ &= \frac{\sigma_k \sigma_2 \rho_{k2}}{\kappa_2 \kappa_k (\kappa_2 + \kappa_k)} \left\{ (\kappa_2 + \kappa_k) \left(e^{\kappa_k t} - e^{\kappa_k s} \right) - \kappa_k \left[e^{\kappa_k t - \kappa_2 (T_i - t)} - e^{\kappa_k s - \kappa_2 (T_i - s)} \right] \right\}, \end{aligned} \tag{A.30}$$

and

$$\begin{aligned} &\int_s^t e^{\kappa_k u} \sigma_k \sigma_3 \rho_{k3} B_{\kappa_3}(T_i - t)\, du \\ &= \frac{\sigma_k \sigma_3 \rho_{k3}}{\kappa_3 \kappa_k (\kappa_3 + \kappa_k)} \left\{ (\kappa_3 + \kappa_k) \left(e^{\kappa_k t} - e^{\kappa_k s} \right) - \kappa_k \left[e^{\kappa_k t - \kappa_3 (T_i - t)} - e^{\kappa_k s - \kappa_3 (T_i - s)} \right] \right\}. \end{aligned} \tag{A.31}$$

Therefore, recalling equation (A.28),

$$\begin{aligned} y_k(t) &= y_k(s) - \frac{\sigma_k\sigma_1\rho_{k1}}{\kappa_1\kappa_k(\kappa_1+\kappa_k)}\left\{(\kappa_1+\kappa_k)\left(e^{\kappa_k t}-e^{\kappa_k s}\right)-\kappa_k\left[e^{\kappa_k t-\kappa_1(T_i-t)}-e^{\kappa_k s-\kappa_1(T_i-s)}\right]\right\} \\ &\quad -\frac{\sigma_k\sigma_2\rho_{k2}}{\kappa_2\kappa_k(\kappa_2+\kappa_k)}\left\{(\kappa_2+\kappa_k)\left(e^{\kappa_k t}-e^{\kappa_k s}\right)-\kappa_k\left[e^{\kappa_k t-\kappa_2(T_i-t)}-e^{\kappa_k s-\kappa_2(T_i-s)}\right]\right\} \\ &\quad -\frac{\sigma_k\sigma_3\rho_{k3}}{\kappa_3\kappa_k(\kappa_3+\kappa_k)}\left\{(\kappa_3+\kappa_k)\left(e^{\kappa_k t}-e^{\kappa_k s}\right)-\kappa_k\left[e^{\kappa_k t-\kappa_3(T_i-t)}-e^{\kappa_k s-\kappa_3(T_i-s)}\right]\right\} \\ &\quad +\sigma_k\int_s^t e^{\kappa_k u}dW_k^{\mathbb{Q}_{T_i}}(u). \end{aligned} \tag{A.32}$$

Combining equations (A.26) and (A.32),

$$\begin{aligned} e^{\kappa_k t}x_k(t) &= e^{\kappa_k s}x_k(s) - \frac{\sigma_k\sigma_1\rho_{k1}}{\kappa_1\kappa_k(\kappa_1+\kappa_k)}\left\{(\kappa_1+\kappa_k)\left(e^{\kappa_k t}-e^{\kappa_k s}\right)-\kappa_k\left[e^{\kappa_k t-\kappa_1(T_i-t)}-e^{\kappa_k s-\kappa_1(T_i-s)}\right]\right\} \\ &\quad -\frac{\sigma_k\sigma_2\rho_{k2}}{\kappa_2\kappa_k(\kappa_2+\kappa_k)}\left\{(\kappa_2+\kappa_k)\left(e^{\kappa_k t}-e^{\kappa_k s}\right)-\kappa_k\left[e^{\kappa_k t-\kappa_2(T_i-t)}-e^{\kappa_k s-\kappa_2(T_i-s)}\right]\right\} \\ &\quad -\frac{\sigma_k\sigma_3\rho_{k3}}{\kappa_3\kappa_k(\kappa_3+\kappa_k)}\left\{(\kappa_3+\kappa_k)\left(e^{\kappa_k t}-e^{\kappa_k s}\right)-\kappa_k\left[e^{\kappa_k t-\kappa_3(T_i-t)}-e^{\kappa_k s-\kappa_3(T_i-s)}\right]\right\} \\ &\quad +\sigma_k\int_s^t e^{\kappa_k u}dW_k^{\mathbb{Q}_{T_i}}(u), \end{aligned}$$

i.e.

$$\begin{aligned} x_k(t) &= e^{-\kappa_k(t-s)}x_k(s) \\ &\quad -\frac{\sigma_k\sigma_1\rho_{k1}}{\kappa_1\kappa_k(\kappa_1+\kappa_k)}\left\{(\kappa_1+\kappa_k)\left(1-e^{-\kappa_k(t-s)}\right)-\kappa_k\left[e^{-\kappa_1(T_i-t)}-e^{-\kappa_k(t-s)-\kappa_1(T_i-s)}\right]\right\} \\ &\quad -\frac{\sigma_k\sigma_2\rho_{k2}}{\kappa_2\kappa_k(\kappa_2+\kappa_k)}\left\{(\kappa_2+\kappa_k)\left(1-e^{-\kappa_k(t-s)}\right)-\kappa_k\left[e^{-\kappa_2(T_i-t)}-e^{-\kappa_k(t-s)-\kappa_2(T_i-s)}\right]\right\} \\ &\quad -\frac{\sigma_k\sigma_3\rho_{k3}}{\kappa_3\kappa_k(\kappa_3+\kappa_k)}\left\{(\kappa_3+\kappa_k)\left(1-e^{-\kappa_k(t-s)}\right)-\kappa_k\left[e^{-\kappa_3(T_i-t)}-e^{-\kappa_k(t-s)-\kappa_3(T_i-s)}\right]\right\} \\ &\quad +\sigma_k\int_s^t e^{-\kappa_k(t-u)}dW_k^{\mathbb{Q}_{T_i}}(u) \\ &= e^{-\kappa_k(t-s)}x_k(s) - M_{k1} - M_{k2} - M_{k3} + \sigma_k\int_s^t e^{-\kappa_k(t-u)}dW_k^{\mathbb{Q}_{T_i}}(u), \end{aligned} \tag{A.33}$$

where

$$M_{kl} = \frac{\sigma_k\sigma_l\rho_{kl}}{\kappa_l\kappa_k(\kappa_l+\kappa_k)}\left\{(\kappa_l+\kappa_k)\left(1-e^{-\kappa_k(t-s)}\right)-\kappa_k\left[e^{-\kappa_l(T_i-t)}-e^{-\kappa_k(t-s)-\kappa_l(T_i-s)}\right]\right\}. \tag{A.34}$$

Since any Itô's integral, with a deterministic integrand, possesses a normal distribution with

zero mean and variance equal to its quadratic variance, then:

$$\mathbb{E}_{\mathbb{Q}_{T_i}}\left[x_k\left(t\right)|\mathcal{F}_s\right] = e^{-\kappa_k(t-s)}x_k\left(s\right) - M_{k1} - M_{k2} - M_{k3}, \tag{A.35}$$

where M_{kl} is defined by equation (A.34), and

$$\begin{aligned}
\sigma^2_{\mathbb{Q}_{T_i}}\left[x_k\left(t\right)|\mathcal{F}_s\right] &= \mathbb{E}_{\mathbb{Q}_{T_i}}\left\{\left[x_k\left(t\right) - \mathbb{E}_{\mathbb{Q}_{T_i}}\left(x_k\left(t\right)|\mathcal{F}_s\right)\right]^2|\mathcal{F}_s\right\} \\
&= \sigma_k^2\mathbb{E}_{\mathbb{Q}_{T_i}}\left\{\left[\int_s^t e^{-\kappa_k(t-u)}dW_k^{T_i}\left(u\right)\right]^2|\mathcal{F}_s\right\} \\
&= \sigma_k^2\int_s^t e^{-2\kappa_k(t-u)}du \\
&= \sigma_k^2\left[\frac{e^{-2\kappa_k(t-u)}}{2\kappa_k}\right]_s^t \\
&= \frac{\sigma_k^2}{2\kappa_k}\left[1 - e^{-2\kappa_k(t-s)}\right],
\end{aligned} \tag{A.36}$$

with $k = 1, ..., 3$.

Appendix B

Matlab codes for the selected examples

The purpose of this Appendix is to present the Matlab algorithms used in the selected examples. Given space constraints, only the codes referring to the pricing of Swaption 1 defined in Chapter 5 are presented.

B.1 Edgeworth expansion approximation

```
%Given the parameters established for the three-factor Gaussian model, this
%routine prices swaptions using the Edgeworth expansion approximation
%[Collin-Dufresne and Goldstein (2002)].

clear; clc;
format long;

%Declaration of variables:
n = 4;                                      %Number of swap payments
tau = 2;                                    %Time-to-maturity of the swaption
fr = 2;                                     %Frequency of the swap payments
X = [0.01; 0.005; -0.02];                   %Vector of state variables at t=0
delta = 0.06;                               %Delta
K = [1; 0.2; 0.5];                          %Vector of Ks
S = [0.01; 0.005; 0.002];                   %Vector of Sigmas
R = [1 -0.2 -0.1; -0.2 1 0.3; -0.1 0.3 1];%Matrix of Rhos

f = length(X);                              %Number of model factors
```

```
tic;                                          %Timer initiation

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Computation of the discount factors - Appendix A1

c = 1;
for j = tau:(1/fr):tau+(n/fr)
    s = 0;
    for k = 1:1:f
    Bk(c,k) = (1-exp(-K(k,1)*j))/K(k,1);
        for l = 1:1:f
            a = (((S(k,1)*S(l,1)*R(k,l))/(K(k,1)*K(l,1)))
                *(j-((1-exp(-K(k,1)*j))/K(k,1))-((1-exp(-K(l,1)*j))/K(l,1))
                +((1-exp(-(K(k,1)+K(l,1))*j))/(K(k,1)+K(l,1))));
            s = s+a;
        end
    end
    A(c,1) = -delta*j+s/2;
    P(c,1) = exp(A(c,1)-Bk(c,:)*X);
    c = c+1;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Computation of the expected value and variance of Xk(T0), for k = 1,...,3,
%under the QTi forward measure - Appendix A1

c = 1;
for j = tau:(1/fr):tau+(n/fr)
    s = 0;
    for k = 1:1:f
        for l = 1:1:f
            s = s+((S(k,1)*S(l,1)*R(k,l))/(K(l,1)*K(k,1)*(K(l,1)+K(k,1))))
                *((K(l,1)+K(k,1))*(1-exp(-K(k,1)*(tau-0)))-K(k,1)
                *(exp(-K(l,1)*(j-tau))-exp(-K(k,1)*(tau-0)-K(l,1)*(j-0))));
```

```
        end
        Mij(c,k) = s;
        VAR(c,k) = ((S(k,1)^2)/(2*K(k,1)))*(1-exp(-2*K(k,1)*(tau-0)));
    end
    c = c+1;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Implementation of the Edgeworth expansion approximation

c1 = 1;
for c = 0.04:0.001:0.08                    %Swaption contract strikes
    C(1,1:n-1) = (c/fr);                   %Auxiliary payment vector
    C(1,n) = 1+(c/fr);                     %Auxiliary payment vector

    %Computation of the first moment of the distribution - Equations
    %(3.16) and (3.18)
    c2 = 1;
    for i1 = (1/fr):(1/fr):(n/fr)
        s = 0;
        for k = 1:1:f
            G(k,c2) = ((1-exp(-K(k,1)*(tau+i1-tau)))/K(k,1));
            N(k,c2) = G(k,c2)*exp(-K(k,1)*tau);
            for l = 1:1:f
                a1 = (((S(k,1)*S(l,1)*R(k,l))/(K(k,1)*K(l,1)))
                     *((tau+i1-tau)-((1-exp(-K(k,1)*(tau+i1-tau)))
                     /K(k,1))-((1-exp(-K(l,1)*(tau+i1-tau)))/K(l,1))
                     +((1-exp(-(K(k,1)+K(l,1))*(tau+i1-tau)))/(K(k,1)
                     +K(l,1)))));
                s = s+a1;
            end
        end
        A(1:n+1,c2) = -delta*(tau+i1-tau)+s/2;
        NX(1:n+1,c2) = N(:,c2)'*X;
        CAUX(1,c2) = C(1,i1*fr);
```

```
    c2 = c2+1;
end

c2 = 1;
for j = tau:(1/fr):tau+(n/fr)
    c3 = 1;
    for i1 = (1/fr):(1/fr):(n/fr)
        s1 = 0;
        s2 = 0;
        for k = 1:1:f
            s1 = s1+(G(k,c3)*Mij(c2,k));
            for l = 1:1:f
                s2 = s2+(G(k,c3)*G(l,c3)*sqrt(VAR(c2,k))
                     *sqrt(VAR(c2,l))*R(k,l));
            end
        end
        M(c2,c3) = s1+s2/2;
        c3 = c3+1;
    end
    c2 = c2+1;
end

LAUX = A+M-NX;
for j = 1:1:n+1
    MOMENTS(j,1)=exp(LAUX(j,:))*CAUX';
end

%Computation of the second moment of the distribution - Equations
%(3.16) and (3.18)
c2 = 1;
for i1 = (1/fr):(1/fr):(n/fr)
    for i2 = (1/fr):(1/fr):(n/fr)
        s = 0;
        for k = 1:1:f
            G(k,c2) = ((1-exp(-K(k,1)*(tau+i1-tau)))/K(k,1))+((1
                      -exp(-K(k,1)*(tau+i2-tau)))/K(k,1));
```

```
            N(k,c2) = G(k,c2)*exp(-K(k,1)*tau);
            for l = 1:1:f
                a1 = (((S(k,1)*S(l,1)*R(k,l))/(K(k,1)*K(l,1)))
                     *((tau+i1-tau)-((1-exp(-K(k,1)*(tau+i1-tau)))
                     /K(k,1))-((1-exp(-K(l,1)*(tau+i1-tau)))/K(l,1))
                     +((1-exp(-(K(k,1)+K(l,1))*(tau+i1-tau)))/(K(k,1)
                     +K(l,1)))));
                a2 = (((S(k,1)*S(l,1)*R(k,l))/(K(k,1)*K(l,1)))
                     *((tau+i2-tau)-((1-exp(-K(k,1)*(tau+i2-tau)))
                     /K(k,1))-((1-exp(-K(l,1)*(tau+i2-tau)))/K(l,1))
                     +((1-exp(-(K(k,1)+K(l,1))*(tau+i2-tau)))/(K(k,1)
                     +K(l,1)))));
                s = s+a1+a2;
            end
        end
        A(1:n+1,c2) = -delta*(tau+i1-tau)-delta*(tau+i2-tau)+s/2;
        NX(1:n+1,c2) = N(:,c2)'*X;
        CAUX(1,c2) = C(1,i1*fr)*C(1,i2*fr);
        c2 = c2+1;
    end
end

c2 = 1;
for j = tau:(1/fr):tau+(n/fr)
    c3 = 1;
    for i1 = (1/fr):(1/fr):(n/fr)
        for i2 = (1/fr):(1/fr):(n/fr)
            s1 = 0;
            s2 = 0;
            for k = 1:1:f
                s1 = s1+(G(k,c3)*Mij(c2,k));
                for l = 1:1:f
                    s2 = s2+(G(k,c3)*G(l,c3)*sqrt(VAR(c2,k))
                         *sqrt(VAR(c2,l))*R(k,l));
                end
            end
```

```
            M(c2,c3) = s1+s2/2;
            c3 = c3+1;
        end
    end
    c2 = c2+1;
end

LAUX = A+M-NX;
for j = 1:1:n+1
    MOMENTS(j,2)=exp(LAUX(j,:))*CAUX';
end

%Computation of the third moment of the distribution - Equations
%(3.16) and (3.18)
c2 = 1;
for i1 = (1/fr):(1/fr):(n/fr)
    for i2 = (1/fr):(1/fr):(n/fr)
        for i3 = (1/fr):(1/fr):(n/fr)
            s = 0;
            for k = 1:1:f
                G(k,c2) = ((1-exp(-K(k,1)*(tau+i1-tau)))/K(k,1))+((1
                          -exp(-K(k,1)*(tau+i2-tau)))/K(k,1))+((1
                          -exp(-K(k,1)*(tau+i3-tau)))/K(k,1));
                N(k,c2) = G(k,c2)*exp(-K(k,1)*tau);
                for l = 1:1:f
                    a1 = (((S(k,1)*S(l,1)*R(k,l))/(K(k,1)*K(l,1)))
                         *((tau+i1-tau)-((1-exp(-K(k,1)*(tau+i1-tau)))
                         /K(k,1))-((1-exp(-K(l,1)*(tau+i1-tau)))/K(l,1))
                         +((1-exp(-(K(k,1)+K(l,1))*(tau+i1-tau)))/(K(k,1)
                         +K(l,1)))));
                    a2 = (((S(k,1)*S(l,1)*R(k,l))/(K(k,1)*K(l,1)))
                         *((tau+i2-tau)-((1-exp(-K(k,1)*(tau+i2-tau)))
                         /K(k,1))-((1-exp(-K(l,1)*(tau+i2-tau)))/K(l,1))
                         +((1-exp(-(K(k,1)+K(l,1))*(tau+i2-tau)))/(K(k,1)
                         +K(l,1)))));
                    a3 = (((S(k,1)*S(l,1)*R(k,l))/(K(k,1)*K(l,1)))
```

```
                        *((tau+i3-tau)-((1-exp(-K(k,1)*(tau+i3-tau)))
                        /K(k,1))-((1-exp(-K(l,1)*(tau+i3-tau)))/K(l,1))
                        +((1-exp(-(K(k,1)+K(l,1))*(tau+i3-tau)))/(K(k,1)
                        +K(l,1)))));
                    s = s+a1+a2+a3;
                end
            end
            A(1:n+1,c2) = -delta*(tau+i1-tau)-delta*(tau+i2-tau)
                          -delta*(tau+i3-tau)+s/2;
            NX(1:n+1,c2) = N(:,c2)'*X;
            CAUX(1,c2) = C(1,i1*fr)*C(1,i2*fr)*C(1,i3*fr);
            c2 = c2+1;
        end
    end
end

c2 = 1;
for j = tau:(1/fr):tau+(n/fr)
    c3 = 1;
    for i1 = (1/fr):(1/fr):(n/fr)
        for i2 = (1/fr):(1/fr):(n/fr)
            for i3 = (1/fr):(1/fr):(n/fr)
                s1 = 0;
                s2 = 0;
                for k = 1:1:f
                    s1 = s1+(G(k,c3)*Mij(c2,k));
                    for l = 1:1:f
                        s2 = s2+(G(k,c3)*G(l,c3)*sqrt(VAR(c2,k))
                             *sqrt(VAR(c2,l))*R(k,l));
                    end
                end
                M(c2,c3) = s1+s2/2;
                c3 = c3+1;
            end
        end
    end
```

```
    c2 = c2+1;
end

LAUX = A+M-NX;
for j = 1:1:n+1
    MOMENTS(j,3)=exp(LAUX(j,:))*CAUX';
end

%Computation of the cumulants of the distribution - Equation (3.12)
CUMULANTS(:,1) = MOMENTS(:,1);
for j = 1:1:n+1
   for k = 2:1:3
       s = 0;
       for l = 1:1:(k-1)
           s = s+(nchoosek(k-1,l)*CUMULANTS(j,k-l)*MOMENTS(j,l));
       end
       CUMULANTS(j,k) = MOMENTS(j,k)-s;
   end
end

%Computation of the exercise probabilities [equations (3.6) and (3.7)]
%and pricing of the swaption contract under analysis
s = 0;
for i = 1:1:n
    for j = 1:1:n+1
        z = (CUMULANTS(j,1)-1)/sqrt(CUMULANTS(j,2));
        EP(j,1) = normcdf(z)+(CUMULANTS(j,3)/(6*CUMULANTS(j,2)^(3/2)))
                  *((z^2)-1)*((1/sqrt(2*pi))*exp((-z^2)/2));
    end
    s = s+(C(1,i)*P(i+1,1)*EP(i+1,1));
end
SWAPTION(c1,1) = s-(1*P(1,1)*EP(1,1));

%Clearing of variables
A = zeros(); M = zeros(); NX = zeros(); CAUX = zeros();
```

```
    c1 = c1+1;
end

SWAPTION
toc
```

B.2 Hyperplane approximation

```
%Given the parameters established for the three-factor Gaussian model, this
%routine prices swaptions using the Hyperplane approximation [Singleton and
%Umantsev (2002)].

clear; clc;
format long;

%Declaration of variables:
n = 4;                                      %Number of swap payments
tau = 2;                                    %Time-to-maturity of the swaption
fr = 2;                                     %Frequency of the swap payments
X = [0.01; 0.005; -0.02];                   %Vector of state variables at t=0
delta = 0.06;                               %Delta
K = [1; 0.2; 0.5];                          %Vector of Ks
S = [0.01; 0.005; 0.002];                   %Vector of Sigmas
R = [1 -0.2 -0.1; -0.2 1 0.3; -0.1 0.3 1];%Matrix of Rhos
alfa = 0.05;                                %Level of significance

f = length(X);                              %Number of model factors
Y = [1; 1; 1; 1];                           %Vector for the OLS procedures

tic;                                        %Timer initiation

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Computation of the discount factors - Appendix A1

c = 1;
```

```
for j = tau:(1/fr):tau+(n/fr)
    s = 0;
    for k = 1:1:f
    Bk(c,k) = (1-exp(-K(k,1)*j))/K(k,1);
        for l = 1:1:f
            a = (((S(k,1)*S(l,1)*R(k,l))/(K(k,1)*K(l,1)))*(j-((1
                -exp(-K(k,1)*j))/K(k,1))-((1-exp(-K(l,1)*j))/K(l,1))
                +((1-exp(-(K(k,1)+K(l,1))*j))/(K(k,1)+K(l,1)))));
            s = s+a;
        end
    end
    A(c,1) = -delta*j+s/2;
    P(c,1) = exp(A(c,1)-Bk(c,:)*X);
    c = c+1;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Implementation of the Hyperplane approximation

c1 = 1;
for c = (0.04):(0.001):(0.08)             %Swaption contract strikes
    C(1,1:n-1) = (c/fr);                  %Auxiliary payment vector
    C(1,n) = 1+(c/fr);                    %Auxiliary payment vector
    c2 = 1;
    for j = tau:(1/fr):tau+(n/fr)
        s = 0;
        %Expected value and variance of Xk(T0), for k = 1,...,3,
        %under the QTi forward measure - Equations (4.11) and (4.13)
        for k = 1:1:f
            for l = 1:1:f
                s = s+((S(k,1)*S(l,1)*R(k,l))/(K(l,1)*K(k,1)*(K(l,1)
                    +K(k,1))))*((K(l,1)+K(k,1))*(1-exp(-K(k,1)*(tau-0)))
                    -K(k,1)*(exp(-K(l,1)*(j-tau))-exp(-K(k,1)*(tau-0)
                    -K(l,1)*(j-0))));
            end
```

```
    EV(c2,k) = X(k,1)*exp(-K(k,1)*(tau-0))-s;
    VAR(c2,k) = ((S(k,1)^2)/(2*K(k,1)))*(1-exp(-2*K(k,1)*(tau-0)));
end

%Computation of functions A(.) and B(.) of the discount factors,
%taking into account the expected value and variance of Xk(T0),
%under the QTi forward measure - Appendix A1
c3 = 1;
for m = (1/fr):(1/fr):(n/fr)
    s = 0;
    for k = 1:1:f
    Bk2(c3,k) = (1-exp(-K(k,1)*m))/K(k,1);
        for l = 1:1:f
            a = (((sqrt(VAR(c2,k))*sqrt(VAR(c2,l))*R(k,l))/(K(k,1)
                *K(l,1)))*(m-((1-exp(-K(k,1)*m))/K(k,1))-((1
                -exp(-K(l,1)*m))/K(l,1))+((1-exp(-(K(k,1)+K(l,1))*m))
                /(K(k,1)+K(l,1))));
            s = s+a;
        end
    end
    A2(c3,1) = -delta*m+s/2;
    c3 = c3+1;
end

%Equations (4.15) and (4.17), for k = 2,3
X2(c2,1) = norminv(alfa/2,EV(c2,2),sqrt(VAR(c2,2)));
X2(c2,2) = norminv(1-alfa/2,EV(c2,2),sqrt(VAR(c2,2)));
X3(c2,1) = norminv(alfa/2,EV(c2,3),sqrt(VAR(c2,3)));
X3(c2,2) = norminv(1-alfa/2,EV(c2,3),sqrt(VAR(c2,3)));

%Computation of the second and third columns of X, to use in the
%OLS procedures
XAUX(1:2,3) = X3(c2,1);
XAUX(3:4,3) = X3(c2,2);
XAUX(1,2) = X2(c2,1);
XAUX(2,2) = X2(c2,2);
```

```
XAUX(3,2) = X2(c2,1);
XAUX(4,2) = X2(c2,2);

%Equation (4.19)
a = @(x1)((C(1,1)*exp(A2(1,1)-Bk2(1,1)*x1-Bk2(1,2)*XAUX(1,2)
    -Bk2(1,3)*XAUX(1,3))+C(1,2)*exp(A2(2,1)-Bk2(2,1)*x1-Bk2(2,2)
    *XAUX(1,2)-Bk2(2,3)*XAUX(1,3))+C(1,3)*exp(A2(3,1)-Bk2(3,1)*x1
    -Bk2(3,2)*XAUX(1,2)-Bk2(3,3)*XAUX(1,3))+C(1,4)*exp(A2(4,1)
    -Bk2(4,1)*x1-Bk2(4,2)*XAUX(1,2)-Bk2(4,3)*XAUX(1,3)))-1);
XAUX(1,1) = fzero(a,0);

b = @(x1)((C(1,1)*exp(A2(1,1)-Bk2(1,1)*x1-Bk2(1,2)*XAUX(2,2)
    -Bk2(1,3)*XAUX(2,3))+C(1,2)*exp(A2(2,1)-Bk2(2,1)*x1-Bk2(2,2)
    *XAUX(2,2)-Bk2(2,3)*XAUX(2,3))+C(1,3)*exp(A2(3,1)-Bk2(3,1)*x1
    -Bk2(3,2)*XAUX(2,2)-Bk2(3,3)*XAUX(2,3))+C(1,4)*exp(A2(4,1)
    -Bk2(4,1)*x1-Bk2(4,2)*XAUX(2,2)-Bk2(4,3)*XAUX(2,3)))-1);
XAUX(2,1) = fzero(b,0);

c = @(x1)((C(1,1)*exp(A2(1,1)-Bk2(1,1)*x1-Bk2(1,2)*XAUX(3,2)
    -Bk2(1,3)*XAUX(3,3))+C(1,2)*exp(A2(2,1)-Bk2(2,1)*x1-Bk2(2,2)
    *XAUX(3,2)-Bk2(2,3)*XAUX(3,3))+C(1,3)*exp(A2(3,1)-Bk2(3,1)*x1
    -Bk2(3,2)*XAUX(3,2)-Bk2(3,3)*XAUX(3,3))+C(1,4)*exp(A2(4,1)
    -Bk2(4,1)*x1-Bk2(4,2)*XAUX(3,2)-Bk2(4,3)*XAUX(3,3)))-1);
XAUX(3,1) = fzero(c,0);

d = @(x1)((C(1,1)*exp(A2(1,1)-Bk2(1,1)*x1-Bk2(1,2)*XAUX(4,2)
    -Bk2(1,3)*XAUX(4,3))+C(1,2)*exp(A2(2,1)-Bk2(2,1)*x1-Bk2(2,2)
    *XAUX(4,2)-Bk2(2,3)*XAUX(4,3))+C(1,3)*exp(A2(3,1)-Bk2(3,1)*x1
    -Bk2(3,2)*XAUX(4,2)-Bk2(3,3)*XAUX(4,3))+C(1,4)*exp(A2(4,1)
    -Bk2(4,1)*x1-Bk2(4,2)*XAUX(4,2)-Bk2(4,3)*XAUX(4,3)))-1);
XAUX(4,1) = fzero(d,0);

%Computation of OLS procedures - Equation (4.20)
BETA(c2,:) = (((XAUX'*XAUX)^-1)*XAUX'*Y)';

%Equations (4.8) or (4.9), (4.21), (4.22) and (4.23)
```

```
        if C*exp(A2)>1
            PR(c2,1) = normcdf(1,BETA(c2,:)*EV(c2,:)',sqrt(BETA(c2,1)^2
                       *VAR(c2,1)+BETA(c2,2)^2*VAR(c2,2)+BETA(c2,3)^2
                       *VAR(c2,3)+2*BETA(c2,1)*BETA(c2,2)*sqrt(VAR(c2,1))
                       *sqrt(VAR(c2,2))*R(1,2)+2*BETA(c2,1)*BETA(c2,3)
                       *sqrt(VAR(c2,1))*sqrt(VAR(c2,3))*R(1,3)+2*BETA(c2,2)
                       *BETA(c2,3)*sqrt(VAR(c2,2))*sqrt(VAR(c2,3))*R(2,3)));
        elseif C*exp(A2)<1
            PR(c2,1) = 1-normcdf(1,BETA(c2,:)*EV(c2,:)',sqrt(BETA(c2,1)^2
                       *VAR(c2,1)+BETA(c2,2)^2*VAR(c2,2)+BETA(c2,3)^2
                       *VAR(c2,3)+2*BETA(c2,1)*BETA(c2,2)*sqrt(VAR(c2,1))
                       *sqrt(VAR(c2,2))*R(1,2)+2*BETA(c2,1)*BETA(c2,3)
                       *sqrt(VAR(c2,1))*sqrt(VAR(c2,3))*R(1,3)+2*BETA(c2,2)
                       *BETA(c2,3)*sqrt(VAR(c2,2))*sqrt(VAR(c2,3))*R(2,3)));
        end

        c2 = c2+1;
    end

    %Pricing of the considered swaption contract
    SWAPTION(c1,1) = sum(C'.*P(2:size(P,1),1).*PR(2:size(PR,1),1))
                     -1*P(1,1)*PR(1,1);

    c1 = c1+1;
end

SWAPTION
toc                                          %Timer stoppage
```

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