A non-perturbative approach to quantum chromodynamics

Templeton, Stephen John

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Ph.D Thesis by Stephen John Templeton

Department of Mathematics, University of Durham, Durham U.K.

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Declaration

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This thesis has not been submitted for any other degree.

The following parts are claimed to be original unless otherwise acknowledged in the text. Also some parts have been done independantly by other groups who are also acknowledged in the text.

Most of sections 5 to 10 were done in collaboration with E. Corrigan, P. Goddard and D. B. Fairlie and published in Nuclear Physics B [1]. Sections 12 to 14 (including Appendix C) were done in collaboration with E. Corrigan and P. Goddard and published in Nuclear Physics B [2]. Part of sections 6 and 11 and most of sections 16 to 21 were done in collaboration with E. Corrigan, P Goddard and H. Osborn and will be published in Nuclear Physics B [3]. Also part of section 11 and all of section 15 are claimed to be original and are unpublished.

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Abstract

The approach of calculating non-perturbative effects in Quantum Chromodynamics by expanding about non-trivial classical solutions of the equations of motion is described. Some of the techniques required for this are developed in references [1,2,3] on which this thesis is predominantly based. The general self-dual solutions are discussed. With these as background fields the Green and massless Dirac functions are solved for arbitrary group representation. Then with the help of these the determinants and collective coordinate zero modes required for the first order quantum corrections are calculated.

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CONTENTS

CHAPTER 1 - INTRODUCTION

Section 1 - Preliminaries

It is generally believed that Quantum Chromodynamics (QCD) is the theory of strong interactions. A great deal of progress has been made in solving this theory perturbatively, that is expanding the fields about the trivial solutions of the equations of motion. However it is clear that perturbation theory cannot give us everything we want from QCD. There must also be non-perturbative effects as indicated by the existence of non-trivial solutions to the equations of motion - the instantons first discovered by Belavin et. al. [4]. One approach to studying this problem is the semi-classical method where instead of perturbing about the trivial solution one perturbs about the non-trivial instanton solutions. This was started by 't Hooft [5,6,7] (see also [8-13]) who calculated the first order ("one loop") quantum corrections about a one-instanton solution. Since then various people [14-21] have attempted to extend this to more general instanton solutions. Also Amati and Rouet [22] have shown in principle how to extend the one-instanton case to all orders.

This thesis is based on a series of papers [1,2,3] which calculate the first order quantum corrections about the most general self-dual multi-instanton solutions of Atiyah, Drinfeld, Hitchin and Manin [28-31]. The programme is not yet complete but the results obtained so far have since been extended a little by Osborn [23] and independently by Berg and Lüscher [24]. Their results are mentioned briefly in the Conclusion (chapter 5).

It is well known that the first order quantum corrections are given in terms of determinants of certain covariant Laplacian operators. In section 4 we give a derivation of this result based on Schwartz [17, 19] which in my view makes some problems connected with "gauge fixing" clearer. The simplest determinant, that of the fundamental scalar covariant Laplacian, is calculated in chapter 4. By using the techniques of chapter 3 this can be extended to the adjoint representation and by the results of [25,15] the spinor and vector determinants are simply related to the scalar - see section 4 for the vector case. Also relevant to the first quantum corrections are the instanton zero modes which are derived in section 11 and the required normalisation matrix is calculated in section 15.

The determinants of operators θ are calculated by using

$$\delta \ln(\det \theta) = \delta \operatorname{Tr}(\ln \theta) = \operatorname{Tr}(\delta\theta.\theta^{-1}) \tag{1}$$

where the expressions are suitably regulated and δ denotes the variation with respect to some parameters in θ. The inverse θ^{-1} of the operator is the corres-ponding Green function. In chapter 2, after explaining in detail the properties of the most general instanton solution, the Green function for the fundamental scalar Laplacian is derived. In chapter 3 the appropriate results of chapter 2 are extended to the adjoint representation. By a further trivial generalisation they are also extended to arbitrary tensor products of fundamental representations. Chapter 4 introduces the regularisation method used (zeta function regularisation) and goes some way towards calculating the determinant. The remainder of chapter 1 introduces many of the well known concepts required for the following chapters and serves to make the conventions explicit. Finally chapter 5, the conclusion, discusses what further steps have been done.

Appendix A explains some of the notations and conventions used in connection with the quaternions e_α. Appendix B studies the large $|x|$ behaviour of some of the functions occurring in the text. Appendix C sketches the long algebraic calculations required for section 14 on the tensor products.

All references are collected together at the end in order of appearance in the text and are referred to by square brackets, eg. [25].

Equations are referred to by section and equation number in round brackets, eg. (10.25). The section number is omitted if it refers to the current section.

Section 2 - Yang-Mills Theory and QCD. §2

The Euclidean Yang-Mills theory for gauge group G is given by the Lagrangian

$$\mathcal{L}_{Y.M.} = -\frac{1}{2g^2}\, tr\left(F_{\alpha\beta}F_{\alpha\beta}\right) \quad , \qquad 0 \leq \alpha,\beta \leq 3 \tag{1}$$

where g is the coupling constant and the field strength tensor $F_{\alpha\beta}$ is defined by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] \quad , \qquad \partial_\alpha \equiv \frac{\partial}{\partial x_\alpha} \; . \tag{2}$$

A_α is a matrix in group G space and tr is the trace of these matrices. The gauge potential is defined by

$$A_\alpha = A_\alpha^a t_a \tag{3}$$

where A_α^a are the (real) components of the potential and t_a are the fundamental representation matrices of the group G. They satisfy

$$[t_a, t_b] = f_{abc}\, t_c \quad ; \qquad t_a^{+} = -t^a \tag{4}$$

where f_{abc} are the (real, totally antisymmetric) structure constants of the group G, and they are normalised by

$$tr\left(t_a t_b\right) = -\tfrac{1}{2}\,\delta_{ab} \; . \tag{5}$$

Since the techniques for general instanton solutions apply equally well to all classical compact simple Lie groups we will treat the cases of $G = O(n)$, SU(n) and Sp(n) together. For each of these the generators t_a are independant antihermitian nxn matrices. For O(n) the entries are real (hence t_a are antisymmetric), for SU(n) they are complex and for Sp(n) they are quaternionic. In the hermitian conjugate for quaternionic matrices one takes the transpose of the n x n matrix and the quaternionic conjugate (see Appendix A) of each entry. For SU(n) only traceless t_a are allowed otherwise the group generated is U(n). For each group the number of independent n x n antihermitian matrices gives the dimension N of the group:

$$\begin{aligned} &O(n)\ :\ N = \tfrac{1}{2}n(n-1) \\ &SU(n)\ :\ N = n^2-1 \qquad U(n)\ :\ N = n^2 \\ &Sp(n)\ :\ N = 2n^2+n \end{aligned} \tag{6}$$

Thus the index on t_a runs from 1 to N and there are N independant components of the gauge potential A_α.

With the above definitions

$$F_{\alpha\beta} = F^a_{\alpha\beta}\, t_a \tag{7}$$

where the components

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A_\alpha + f_{abc} A^b_\alpha A^c_\beta \tag{8}$$

are real. Hence using the normalisation (5) we see that the Lagrangian (1) is positive

$$\mathcal{L}_{Y.M.} = +\frac{1}{4g^2} F^a_{\alpha\beta} F^a_{\alpha\beta}\ . \tag{9}$$

It is also invariant under the gauge transformation

$$A_\alpha \rightarrow A'_\alpha = g^+ A_\alpha g + g^+ \partial_\alpha g \tag{10}$$

where $g(x)$ is an arbitrary position dependant element of the group G defined by

$$g(x) = exp\left(g_a(x)\, t_a\right) \tag{11}$$

so that

$$g^+(x)\, g(x) = 1_n = g(x)\, g^+(x) \tag{12}$$

($g_a(x)$ are arbitrary real functions). The gauge invariance of (1) is most easily seen in terms of the gauge covariant derivative

$$D_\alpha(x) = \frac{\partial}{\partial x_\alpha} + A_\alpha(x) \tag{13}$$

which under the gauge transformation (10) becomes

$$D_\alpha \rightarrow D'_\alpha = g^+ D_\alpha\, g \tag{14}$$

(in all differential operator equations like this both sides are considered as acting on an arbitrary function to the right). From (2) we can rewrite the field strength as

$$F_{\alpha\beta} = \left[D_\alpha, D_\beta\right] \tag{15}$$

so using (12) it transforms as

$$F_{\alpha\beta} \rightarrow F'_{\alpha\beta} = g^+ F_{\alpha\beta}\, g\ . \tag{16}$$

Now we can immediately see that (1) is invariant because of the cyclic property of the trace.

The QCD Lagrangian is given by the Yang-Mills Lagrangian (1) with group $\mathcal{G} = SU(3)$ which is the kinetic energy and self interaction of the gluons (spin 1 gauge particles), along with a kinetic term for the quarks and an interaction between gluons and quarks. The quark fields ψ_f^i are four-component Dirac spinors with a colour index $i = 1,\cdots,n=3$ (it is a vector in group space) and a flavour index $f = u,d,s,c,b,\cdots$. The interaction is determined by exact SU(3) gauge invariance with the quarks transforming under the fundamental representation

$$\psi_f \rightarrow \psi_f' = g^+\psi_f \ . \tag{17}$$

The interaction term invariant under (10) and (17) which contains the Dirac kinetic energy is

$$\psi_f^\dagger \gamma_\alpha D_\alpha \psi_f \tag{18}$$

where γ_α are the Euclidean Dirac matrices (see Appendix A), D_α is the covariant derivative (13) and the conjugate † includes the spinor and group vector transposes. Combining (1), (18) and a mass term for the quarks gives the QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{2g^2} tr\left(F_{\alpha\beta} F_{\alpha\beta}\right) + \sum_f \left(\psi_f^\dagger \gamma_\alpha D_\alpha \psi_f + m_f \psi_f^\dagger \psi_f\right) \tag{19}$$

where m_f are the masses of the different quark flavours.

Section 3 - Quantisation of QCD. §3

The quantum theory of QCD is given in terms of the path integral

$$Z = \int d[A]d[\psi^\dagger]d[\psi] \exp(-S[A,\psi^\dagger,\psi]) \tag{1}$$

where $S = \int \mathcal{L}_{QCD} d^4x$ is the classical action and $d[A]$ etc. are the measures over the space of all functions $A_\alpha(x)$ with appropriate boundary conditions. In order to have Fermi statistics for the quarks ψ must be an anticommuting field. In particular the expectation value of some quantum operator f is given by the following path integral over the classical f

$$\langle f[A,\psi^\dagger,\psi]\rangle = \frac{1}{Z}\int d[A]d[\psi^\dagger]d[\psi] f[A,\psi^\dagger,\psi] \exp(-S) \ . \tag{2}$$

All these path integrals are calculated in Euclidean space as is everything in this thesis. Amplitudes of real processes in Minkowski space are obtained from the corresponding Euclidean ones by analytic continuation.

In order to calculate these path integrals we need some approximation scheme. The greatest contribution to (1) or (2) appears to come from the fields which are close to those that minimise the action as then the factor $exp(-S)$ is largest. This is the idea of ordinary perturbation theory, the classical action is a positive quantity and minimised to zero by

$$\psi^{+}=\psi=0 \quad , \quad A_\alpha = g^{+}\partial_\alpha g \ . \tag{3}$$

Such a gauge potential which gives zero field strength is a pure gauge.

However in addition to the contribution near the solutions (3) one would also expect large contributions near where the action is stationary. The requirement of stationary action gives the classical equations of motion $[D_\alpha, F_{\alpha\beta}]=0$ of which (3) is the trivial solution. Other solutions are identified by the following observations made by Belavin et. al. [4].

If one considers solutions where only A_α is non-trivial one can restrict attention to the Yang-Mills action

$$S = \int \mathcal{L}_{YM}\, d^4x = -\frac{1}{2g^2}\int tr\left(F_{\alpha\beta}F_{\alpha\beta}\right)d^4x \ . \tag{4}$$

We do not need to restrict the gauge group so G is arbitrary. The dual $\tilde{F}$ of F is defined by

$$\tilde{F}_{\alpha\beta} = \tfrac{1}{2}\,\varepsilon_{\alpha\beta\gamma\delta}F_{\gamma\delta} \tag{5}$$

so that $\tilde{\tilde{F}}=F$ where $\varepsilon_{\alpha\beta\gamma\delta}$ is totally antisymmetric and $\varepsilon_{0123}=1$. Then using the fact that

$$\tilde{F}_{\alpha\beta}\tilde{F}_{\alpha\beta} = F_{\alpha\beta}F_{\alpha\beta} \tag{6}$$

we obtain

$$S = -\frac{1}{2g^2}\int \tfrac{1}{2}\,tr\left[\left(F_{\alpha\beta}\pm\tilde{F}_{\alpha\beta}\right)\left(F_{\alpha\beta}\pm\tilde{F}_{\alpha\beta}\right)\right]d^4x \pm \frac{1}{2g^2}\int tr\left[F_{\alpha\beta}\tilde{F}_{\alpha\beta}\right] \ . \tag{7}$$

Since the first term is positive (cf. (2.9)) this gives the following bound for S

$$S \geq \frac{1}{2g^2}\left|\int tr\left(F_{\alpha\beta}\tilde{F}_{\alpha\beta}\right)d^4x\right| \tag{8}$$

or defining the topological quantum number q by

$$q = -\frac{1}{16\pi^2}\int tr\left(F_{\alpha\beta}\tilde{F}_{\alpha\beta}\right)d^4x \tag{9}$$

the inequality (8) becomes

$$S \geq \frac{8\pi^2}{g^2}|q| \quad . \tag{10}$$

From (7) we see that the bound is saturated by

$$\begin{matrix} F_{\alpha\beta} = +\tilde{F}_{\alpha\beta} \;, & q \text{ positive} \\ F_{\alpha\beta} = -\tilde{F}_{\alpha\beta} \;, & q \text{ negative} \end{matrix} \tag{11}$$

The number q in (9) is a topological invariant and always an integer provided $F_{\alpha\beta}$ vanishes sufficiently rapidly that the action S is finite (so e^{-S} is non-zero) and hence A_α tends to a pure gauge (3) at infinity. We also need the technical requirement thet the limiting pure gauge depends only on the direction of approach to infinity. This is equivalent to A_α being regular when mapped by stereographic projection on to the sphere S^4. The above result is shown by Belavin et. al. [4] from the following easily verified observation:

$$-\frac{1}{16\pi^2}tr\left(F_{\alpha\beta}\tilde{F}_{\alpha\beta}\right) = \partial_\mu Q_\mu \tag{12}$$

where

$$Q_\mu = -\frac{1}{8\pi^2}\varepsilon_{\mu\alpha\beta\gamma}\, tr\left(A_\alpha\partial_\beta A_\gamma + \tfrac{2}{3}A_\alpha A_\beta A_\gamma\right) . \tag{13}$$

The integration of (12) over all Euclidean space can be transformed into a surface integral assuming A_α is regular everywhere

$$q = \lim_{R\to\infty}\int_{|x|=R} dS_\mu Q_\mu \quad . \tag{14}$$

As R tends to infinity A_α tends to a pure gauge so Q_μ becomes

$$Q_\mu = \frac{1}{24\pi^2}\varepsilon_{\mu\alpha\beta\gamma}\, tr\left(g^+\partial_\alpha g . g^+\partial_\beta g . g^+\partial_\gamma g\right) \tag{15}$$

where we have used (2.12) repeatedly and the antisymmetry of ε to put the first term of Q_μ into the same form as the second. Then Belavin et. al. show that (15) is the group invariant measure and that the integral q of (14) counts the number of times the map $g(x)$ from the S^3 sphere $|x|=R$ to G covers an SU(2) subgroup of G.

Since for any gauge potential (with suitable boudary conditions) the number q (9) is always an integer, any smooth deformation of A_α leaves q unchanged (this is its topological invariance) so the gauge potentials can be classified into homotopy classes specified by q which can take all integer values. The path integrals (1) or (2) therefore split up into components where the integral over A in each component is restricted to have its topological charge q equal to some specified integer k.

$$Z = \sum_{k=-\infty}^{\infty} \int_{q=k} d[A]\, e^{-S[A]} \tag{16}$$

Within each component the action is bounded by (10) and minimised to the bound when $F_{\alpha\beta}$ satisfies (11). Each component can be approximated by expanding about the gauge potentials which satisfy (11). For the k = 0 component this is ordinary perturbation theory. To calculate the first order quantum correction for all k is the aim of this work. The solutions for positive k are called k-instantons and k-anti-instantons for k negative. They are all constructed in section 5.

Section 4 - First Order Quantum Corrections about k-instantons. §4

We want to approximate the k th component of the sum (3.16). To do this we expand the action about a particular solution of the self-duality equation (3.11)

$$F_{\alpha\beta} = \tilde{F}_{\alpha\beta} \ . \tag{1}$$

However there are many such solutions for given k. In the case of SU(n) there is a $4kn - (n^2 - 1)$ parameter family of such gauge inequivalent solutions - those that cannot be connected by a regular gauge transformation (2.10). These self-dual k-instanton solutions we will write as $A_\alpha(\lambda_r)$ where λ_r are $4kn - (n^2 - 1)$ real parameters. In addition all gauge transformations of these are solutions

$$A_\alpha(g(x), \lambda) = g^\dagger(x) A_\alpha(\lambda)\, g(x) + g^\dagger(x) \partial_\alpha g(x) \tag{2}$$

which is a further infinite dimensional parameter set (one set of gauge rotation parameters for each point in space-time). This large set of solutions which minimise the action lead to problems. The first order quantum corrections correspond to approximating the expansion of the action by the constant value at the classical solution and the quadratic part of the variations (the linear part vanishes by the equations of motion). The quadratic part gives rise to Gaussian integrations which one can evaluate in path integrals except in those directions corresponding to infinitesimal variations of the parameters in (2). In these directions the quadratic term vanishes so the Gaussian integration diverges. This problem is solved by first Gaussian integrating in only those directions orthogonal to the infinitesimal variations and then exact integration in the parallel directions. This resolution corresponds to the Fadeev and Popov procedure [26]. It is easiest to understand in the finite dimensional case and then assume the result generalises to the infinite dimensional case. The following is an adaption from the derivation by Schwartz [17,19] which is particularly suited to our formalism.

Consider the finite dimensional integral

$$\int d^n\left(\frac{\phi_i}{\sqrt{\pi\alpha}}\right) f(\phi)\exp\left(-\frac{1}{\alpha}S(\phi)\right) \tag{3}$$

where ϕ is an n-vector. Suppose $S(\phi)$ is a scalar such that

$$S(\phi) \geqslant \beta \tag{4}$$

for all ϕ and

$$S(\phi(\lambda)) = \beta \tag{5}$$

where λ_α runs over some r ($<n$) dimensional manifold. Asymptotically as $\alpha \to 0$ the integral (3) is approximated by steepest descent. To do this we expand quadratically about $\phi(\lambda)$

$$S(\phi(\lambda)+\phi) = S(\phi(\lambda)) + \phi^T M\phi + O(\phi^3) \tag{6}$$

where

$$M_{ij} = \left.\frac{\partial^2 S}{\partial\phi_i\partial\phi_j}\right|_{\phi(\lambda)} \tag{7}$$

($\partial S/\partial\phi_i|_{\phi(\lambda)} = 0$ as $\phi(\lambda)$ is a stationary point of S). Now M has r zero eigenvalues corresponding to variations of the parameter λ.

$$M\frac{\partial\phi(\lambda)}{\partial\lambda_a}=0\ , \quad 1\leq a\leq r\ . \tag{8}$$

It is these directions that we should treat seperately. Instead of integrating over each of the n components of ϕ we should integrate over the n independant variables λ_a $(1\leq a\leq r)$ and μ_b $(r+1\leq b\leq n)$. The μ variables are defined by

$$\phi^i=\phi^i(\lambda)+\xi^i_b(\lambda)\mu_b \tag{9}$$

where $\xi_b(\lambda)$ is a basis of $(n-r)$ orthogonal vectors to $\partial\phi/\partial\lambda_a$. We then make a change of variables from ϕ^i to (λ_a,μ_b) and integrate over the latter. Note that for given ϕ the coordinates (λ,μ) are not necessarily unique so there may be some multiple counting. This corresponds to the Gribov ambiguity [27] but can be neglected when we are considering only low order perturbation theory. Under this change of variables the integral (3) becomes

$$\int\frac{d^r\lambda}{(\sqrt{\pi\alpha})^r}\cdot\frac{d^{n-r}\mu}{(\sqrt{\pi\alpha})^{n-r}}\,\Delta f\left(\phi(\lambda)+\xi_b\mu_b\right)\exp\left(-\frac{\beta}{\alpha}-\frac{1}{\alpha}\mu_b\xi_b^T M\xi_{b'}\mu_{b'}+O(\mu^3,\lambda^3)\right) \tag{10}$$

where the Jacobean factor Δ is the determinant of the n x n transformation matrix following from (see (9))

$$d\phi^i=\left(\frac{\partial\phi^i(\lambda)}{\partial\lambda_a}+\frac{\partial\xi^i_b(\lambda)}{\partial\lambda_a}\mu_b\right)d\lambda_a+\xi^i_b\,d\mu_b \tag{11}$$

thus

$$\Delta=\det\left(\frac{\partial\phi^i(\lambda)}{\partial\lambda_a}+\frac{\partial\xi^i_b(\lambda)}{\partial\lambda_a}\mu_b\ ,\ \xi^i_b(\lambda)\right)\ . \tag{12}$$

In the $\alpha\to0$ limit it is reasonable to neglect the $O(\mu^3,\lambda^3)$ term in the exponential and the $O(\mu)$ term in Δ and f. Thus writing $\chi_a\equiv\dfrac{\partial\phi(\lambda)}{\partial\lambda_a}$

$$\begin{aligned}\Delta&=\det\left(\chi^i_a,\xi^i_b\right)\left(1+O(\mu)\right)\\&=\left[\det\left(\chi^i_a,\xi^i_b\right)^T\det\left(\chi^i_a,\xi^i_b\right)\right]^{1/2}(1+O(\mu))\\&\simeq\det{}^{1/2}\begin{pmatrix}\chi^i_a\chi^i_{a'}\ , & \chi^i_a\xi^i_{b'}\\ \xi^i_b\chi^i_{a'}\ , & \xi^i_b\xi^i_{b'}\end{pmatrix}\ .\end{aligned} \tag{13}$$

We can choose the $(n-r)$ vectors ξ_b to be not only orthogonal to χ_a but to be orthonormal to each other, in particular we may choose them to be eigenvectors of M when the orthogonality automatically holds. Then the Jacobean reduces to

$$\Delta=\det{}^{1/2}\begin{pmatrix}\chi^i_a\chi^i_{a'} & 0\\ 0 & 1_{n-r}\end{pmatrix}=\det{}^{1/2}\left(\chi^i_a\chi^i_{a'}\right)\ . \tag{14}$$

When ξ_b are the eigenvectors of M with (non-zero) eigenvalues e_b:

$$\mathcal{M}\xi_b = e_b\xi_b \tag{15}$$

we can easily carry out the integration in (10) which becomes

$$\int d^r\left(\frac{\lambda}{\sqrt{\pi\alpha}}\right) \det{}^{1/2}\left(\chi_a^T(\lambda)\chi_a(\lambda)\right) f(\phi(\lambda)) \prod_{b=r+1}^{n} \left(\frac{1}{e_b}\right)^{1/2} \cdot \exp\left(-\beta/\alpha\right). \tag{16}$$

Note that in the Gaussian integration over μ the terms linear in μ in (Δf) vanish and those quadratic in μ are a factor α smaller than (16) so the correction due to neglecting these terms comes in the higher order quantum corrections. The product over the eigenvalues in (16) would be just the determinant of $\mathcal{M}$ if it were not for the zero eigenvalues. Rewriting (16) we have

$$(\pi\alpha)^{-r/2} e^{-\beta/\alpha} \int d^r\lambda\, f(\phi(\lambda)) \det{}^{1/2}\left(\chi_a^T(\lambda)\chi_{a'}(\lambda)\right) / \overline{\det}{}^{1/2}(\mathcal{M}(\lambda)) \tag{17}$$

where $\overline{\det}$ means the product over only the non-zero eigenvalues or equivalently the determinant of $\mathcal{M}+\Pi$ where Π is the projector onto the zero eigenfunctions.

We generalise the above argument to the infinite dimensional case. This could be treated by putting space-time on a lattice in a finite volume and taking the limit as the lattice spacing goes to zero and the volume goes to infinity. The required path integral for QCD from (3.2) and (2.19) is

$$\int d[\psi^+] d[\psi] d[A] \exp\left\{-S_{YM}[A] - \psi^+(\gamma_\alpha D_\alpha + m)\psi\right\} f(A, \psi^+, \psi) \tag{18}$$

in which the Gaussian integral over ψ^+, ψ can be done giving

$$\int d[A] \exp\left(-S_{YM}[A]\right) F(A). \tag{19}$$

$F(A)$ will contain terms like $\det(\gamma\cdot D+m)$ and $(\gamma\cdot D+m)^{-1}$ which depend on A. (19) is then evaluated by expanding about the solution (2) and the result corresponding to (17) is

$$\pi^{-r/2} g^{-r} e^{-S[A]} \int d^r\lambda\, d^{n^2}[g(x)]\, F(A) \det{}^{1/2}\left(-\int \mathrm{tr}(Z_\alpha^s Z_\alpha^{s'})\right) \det{}^{1/2}(-\mathcal{D}^2) / \overline{\det}{}^{1/2}\left(-\mathcal{D}^2\delta_{\alpha\beta} + \mathcal{D}_\alpha\mathcal{D}_\beta - 2\mathcal{F}_{\alpha\beta}\right) \tag{20}$$

where $\mathcal{D}_\alpha$ the adjoint covariant derivative and $\mathcal{F}_{\alpha\beta}$ are defined on $n \times n$ matrices ϕ by

$$\mathcal{D}_\alpha\phi \equiv \partial_\alpha\phi + [A_\alpha, \phi] \quad ; \quad \mathcal{F}_{\alpha\beta}\phi \equiv [F_{\alpha\beta}, \phi]. \tag{21}$$

All the A_α in (20) are evaluated at the classical solution $A_\alpha(g(x), \lambda)$ of (2) with topological number k. $S[A]$ is the action given by (3.10) as $8\pi^2 k/g^2$ with g^2 the coupling constant corresponding to $\propto$ in (17). r is the number of zero eigenvalues of the quadratic part of the action

$$\Delta_{\alpha\beta} \equiv -\left[\mathcal{D}^2\delta_{\alpha\beta} - \mathcal{D}_\alpha\mathcal{D}_\beta + 2\mathcal{F}_{\alpha\beta}\right] \tag{22}$$

whose determinant (over non-zero eigenvalues only) corresponds to $\overline{\det}\, M$ in (17). These zero eigenvalues are made up of r_2 gauge zero modes $\mathcal{D}_\alpha\phi$ where $\phi(x)$ is an arbitrary normalisable antihermitian matrix function (r_2 is infinite in the continuum limit) and r_1 instanton zero modes Z^s_α which are the variations of the instanton solution orthogonal to the gauge zero modes. The functions ϕ_n are chosen to be a complete orthonormal set of eigenfunctions of $-\mathcal{D}^2$ and are the infinitesimal gauge transformations of the instanton solution.

The Z^s_α satisfy

$$\eta_{\alpha\beta}\mathcal{D}_\alpha Z_\beta = 0 \tag{23}$$

which ensures that an infinitesimal variation of A_α proportional to Z_α preserves self-duality because $\eta_{\alpha\beta}$ is anti-self-dual so

$$\eta_{\alpha\beta}\mathcal{D}_\alpha \delta A_\beta = \tfrac{1}{2}\eta_{\alpha\beta}\delta F_{\alpha\beta} \tag{24}$$

vanishes precisely when $F_{\alpha\beta}$ remains self-dual. Note the gauge zero modes $\mathcal{D}_\alpha\phi_n$ automatically satisfy (23). Further we require

$$\mathcal{D}_\alpha Z_\alpha = 0 \tag{25}$$

which ensures that Z^s_α is orthogonal to all the gauge zero modes $\mathcal{D}_\alpha\phi_n$ provided the Z^s_α are also normalisable. In section 11 we will construct all the normalisable solutions to (23) and (25) and hence show that $r_1 = 4kn$ (for SU(n)). They turn out to be $r_1 - (n^2 - 1)$ zero modes that are true variations of the instanton parameters and $(n^2 - 1)$ which are of the form $\mathcal{D}_\alpha\psi$ where ψ is a unnormalisable function. $\mathcal{D}_\alpha\psi$ correspond to the $(n^2 - 1)$ global gauge rotations but they are not quite such (see section 11).

The zero mode determinant $\det X^T X$ in (17) becomes

$$\det\left(-\int tr\left(Z^s_\alpha Z^{s'}_\alpha\right)\right)\det\left(-\mathcal{D}^2\right) . \tag{26}$$

It splits into the two factors due the orthogonality requirement (25) and the second factor arises from

$$\det\int tr\left[(\mathcal{D}_\alpha\phi_m)^\dagger(\mathcal{D}_\alpha\phi_n)\right] = -\det\int tr\left(\phi_m^\dagger\mathcal{D}^2\phi_n\right) = \det\left(-\mathcal{D}^2\right) \tag{27}$$

due to ϕ_n being the orthonormal eigenfunctions of $-\mathcal{D}^2$. The normalisation matrix of the instanton zero modes in (26)

$$N_{ss'} = -\int tr\left(Z_\alpha^s Z_\alpha^{s'}\right) \tag{28}$$

is calculated in section 15.

In addition we need the ratio of determinants

$$\det{}^{1/2}\Delta_0 \Big/ \overline{\det}^{1/2}\Delta_{\alpha\beta} \tag{29}$$

occurring in (21) where $\Delta_{\alpha\beta}$ is given by (22) and

$$\Delta_0 = -\mathcal{D}^2 \,. \tag{30}$$

The ratio (29) is not the familiar result of

$$\det\Delta_0 \Big/ \overline{\det}^{1/2}\Delta_1 \tag{31}$$

where

$$(\Delta_1)_{\alpha\beta} = -\left(\mathcal{D}^2\delta_{\alpha\beta} + 2\mathcal{F}_{\alpha\beta}\right) \tag{32}$$

is the quadratic part of the action with a "gauge fixing" term $-\mathcal{D}_\alpha\mathcal{D}_\beta$

$$(\Delta_1)_{\alpha\beta} = \Delta_{\alpha\beta} - \mathcal{D}_\alpha\mathcal{D}_\beta \,. \tag{33}$$

However since $\mathcal{D}_\alpha\phi$ are the zero modes of $\Delta_{\alpha\beta}$ we have

$$\Delta_{\alpha\beta}\mathcal{D}_\beta\mathcal{D}_\gamma = 0 \tag{34}$$

and as

$$\det\left(-\mathcal{D}_\alpha\mathcal{D}_\beta\right) = \det\Delta_0 \tag{35}$$

we obtain the result

$$\overline{\det}\,\Delta_1 = \overline{\det}\,\Delta_{\alpha\beta}\cdot\det\Delta_0 \,. \tag{36}$$

Thus the ratios of determinants (29) and (31) are equal. Further one can show [25] $\overline{\det}\,\Delta_1$ and $\det\Delta_0$ are simply related using the operator

$$T_\alpha = \begin{pmatrix} \delta_{\alpha\beta}\mathcal{D}_\beta \\ \eta^a_{\alpha\beta}\mathcal{D}_\beta \end{pmatrix} \,. \tag{37}$$

Then evaluating $T^\dagger T$ and $TT^\dagger$ gives

$$\begin{aligned} T_\alpha^\dagger T_\alpha &= -\begin{pmatrix} \delta_{\alpha\beta}\delta_{\alpha\gamma} & \delta_{\alpha\beta}\eta^b_{\alpha\gamma} \\ \eta^a_{\alpha\beta}\delta_{\alpha\gamma} & \eta^a_{\alpha\beta}\eta^b_{\alpha\gamma} \end{pmatrix}\mathcal{D}_\beta\mathcal{D}_\gamma \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \delta_{ab} \end{pmatrix}\Delta_0 \end{aligned} \tag{38}$$

and

$$\begin{aligned} T_\alpha T_\beta^\dagger &= -\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \eta^a_{\alpha\gamma}\eta^a_{\beta\delta}\right)\mathcal{D}_\gamma\mathcal{D}_\delta \\ &= -\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \bar{\eta}^a_{\alpha\beta}\bar{\eta}^a_{\gamma\delta}\right)\mathcal{D}_\gamma\mathcal{D}_\delta \\ &= -\left(\mathcal{D}^2\delta_{\alpha\beta} + 2\mathcal{F}_{\alpha\beta}\right) \equiv (\Delta_1)_{\alpha\beta} \end{aligned} \tag{39}$$

where we have used the results (A.8) for η and $\bar{\eta}$ and the self-duality of the potential through

$$\eta_{\alpha\beta}\mathcal{D}_\alpha\mathcal{D}_\beta = \tfrac{1}{2}\eta_{\alpha\beta}\mathcal{F}_{\alpha\beta} = 0\,. \tag{40}$$

Thus we have the relationship first proved for general self-dual fields by D'Adda and Di Vecchia [25]

$$\ln \overline{\det}\,\Delta_1 = \ln \overline{\det}\; T T^+ = \ln \det T^+ T = 4 \ln \det \Delta_0\,. \tag{41}$$

This can easily be seen to be true from (38) and (39) in the context of zeta function regularisation by the representation (16.4). Hence the required ratio (29) or (31) becomes

$$\left(\det \Delta_0\right)^{-2}. \tag{42}$$

Using the results of chapter 3 any result for the adjoint rep-resentation such as (42) can be obtained from the corresponding result for the fundamental representation. Thus all that we need to calculate is

$$\det\left(-D^2\right) \tag{43}$$

which is the aim of chapter 4.

Because it contains so many infinite factors the expression (20) is best normalised by the same in the case $k=0$ with $F(A)\equiv 1$ where the classical solution is given by (3.3). The answer is (for SU(n))

$$\pi^{-2kn}\, g^{-4kn}\, e^{-8\pi^2 k/g^2}\, \frac{\int d^{4kn}\lambda\; d^{r_2}[g(x)]\; F(A)\, \det^{1/2} N_{ss'} \left(\det \Delta_0\right)^{-2}\Big|_{A_k}}{\int d^{r_2}[g(x)]\, \left(\det\Delta_0\right)^{-2}\Big|_{A_0}}\,. \tag{44}$$

Since all the determinants in (44) are gauge invariant, if $F(A)$ is also gauge invariant then the integral over the gauge function g is trivial and cancels out between the numerator and denominator.

Notice that r_2 is the same for the case $k=0$. This is because we can choose the same set of gauge functions ϕ_n as before. Calling the adjoint covariant derivative for 0 and k instanton number $\mathcal{D}_0$ and $\mathcal{D}_k$ respectively, ϕ_n are the eigenfunctions of $-\mathcal{D}_k^2$. The determinant corresponding to (27) for $k=0$ is

$$\det\, \mathrm{tr}\int(\mathcal{D}_{0\alpha}\,\phi_m)(\mathcal{D}_{0\alpha}\,\phi_n) = -\det\left(a_{mr}\, .\, \mathrm{tr}\int \phi_r\, \mathcal{D}_0^2\, \phi_s\, .\, a_{ns}\right) \tag{45}$$

where

$$\phi_m = a_{mr}\psi_r \tag{46}$$

with ψ_r being the complete orthonormal set of eigenfunctions of $-\mathcal{D}_0^2$.

Thus the determinant (45) is

$$\det a \,.\, \det(-\mathcal{D}_0^2) \,.\, \det a^\dagger \;. \tag{47}$$

But from (47)

$$\begin{aligned}(aa^\dagger)_{mn} = a_{mr}a^*_{nr} &= \int tr(\phi_m^\dagger\psi_r)\int tr(\psi_r^\dagger\phi_n) \\ &= \int tr(\phi_m^\dagger\phi_n) \\ &= \delta_{mn}\end{aligned} \tag{48}$$

where we have used the completeness relation for ψ_r and the orthonormality of ϕ_n. Thus $\det a . \det a^\dagger = 1$ in (47) giving $\det(-\mathcal{D}_0^2)$ in (44) as expected. A similar argument shows that $\mathcal{D}_k^2$ arising in (26) independent of the choice of orthonormal functions ϕ_n.

Further r_1 is zero for $k = 0$. This fact is verified along with $r_1 = 4kn$, $(k \neq 0)$ in section 11.

CHAPTER 2 - THE GENERAL SELF-DUAL GREEN AND DIRAC FUNCTIONS

Section 5 - Construction of the General Self-dual Instantons.

Atiyah, Drinfeld, Hitchin and Manin [28-31] have shown how to construct the most general self-dual instanton solution of (4.1). The construction involves only linear algebra and works for arbitrary compact classical Lie group. Here the method is explained in elementary terms [1,32] in the same way for all the groups Sp(n), SU(n) (U(n)) and O(n) though some differences between these cases are pointed out.

The most general solution of (4.1) is

$$A_\alpha(x) = v^\dagger(x)\frac{\partial}{\partial x_\alpha}v(x) \tag{1}$$

where $v(x)$ is some non-square matrix whose dimensions depend on the group and the topological charge k. It is defined in terms of another matrix Δ specified below by

$$\Delta^\dagger(x)\,v(x) = 0 \tag{2}$$

and normalised by

$$v^\dagger(x)v(x) = 1_n \ . \tag{3}$$

The dimension and rank of the matrix $\Delta(x)$ is such that there are exactly n independent columns of v satisfying (2). The condition (3) then implies that the columns are chosen to be orthonormal. The elements of $v(x)$ must be respectively real, complex and quaternionic for the groups O(n), U(n) and Sp(n). With this requirement there is still an ambiguity in v defined by (2) and (3). It can be multiplied on the right by any element of the appropriate gauge group

$$v(x) \to v'(x) = v(x)g(x) \ . \tag{4}$$

Clearly v' also satisfies (2), (3) and still consists of the correct type of elements. The change (4) causes the gauge potential defined in (1) to transform as

$$A_\alpha(x) \to A_\alpha^+(x) = g^\dagger v^\dagger \partial_\alpha(vg) = g^\dagger A_\alpha g + g^\dagger\partial_\alpha g \tag{5}$$

where (3) has been used. Hence the transformation (4) is exactly a gauge transformation (2.10).

The matrix Δ defining v in (2) must be linear in x_α. It is given by

$$\Delta(x) = a + bx \tag{6}$$

where a, b are constant rectangular matrices and x is the quaternionic representation of the Euclidean coordinates x_α

$$x = x_\alpha e_\alpha . \tag{7}$$

(See Appendix A for the definition and properties of the quaternions e_α). In the cases of O(n) and Sp(n), a and b have quaternionic entries so the product bx is quaternionic. For U(n) the quaternions are represented by 2×2 matrices (see Appendix A) and a, b have complex entries and an extra two component index

$$\Delta_A = a_A + b_B x_{BA} \qquad 1 \leq A, B \leq 2 \ . \tag{8}$$

The dimensions of a and b are

$$\begin{array}{lll} (k+n) \times k & \text{quaternionic for } Sp(n) & \\ (2k+n) \times k & \text{complex for } U(n) & (\text{or } (2k+n) \times 2k \text{ including the} \\ & & \text{two component index in the right hand factor}) \\ (4k+n) \times k & \text{quaternionic for } O(n) & (\text{or } (4k+n) \times 4k, \text{both the quat-} \\ & & \text{ernionic indices in the right hand factor}). \end{array} \tag{9}$$

The matrices v have dimensions of respectively

$$\begin{array}{ll} (k+n) \times n & \text{real} \\ (2k+n) \times n & \text{complex} \\ (4k+n) \times n & \text{quaternionic .} \end{array} \tag{10}$$

For each case one can see that (2) comprises the correct number of equations to give v at least n independent columns of the appropriate type. But first we should check that it is possible to choose v real for O(n). We require that the equation (2) holds as a complex equation for each quaternion component of Δ (v has none) so we can multiply it on the left and right by ε which is the two dimensional alternating symbol acting on the two component quaternion indices (see Appendix A). Now for quaternionic Δ

$$\varepsilon \Delta^{+} \varepsilon^{-1} = \Delta^{T} \tag{11}$$

by (A.3) where T is both the matrix and quaternionic transpose (ie. $+ = T*$). Hence for O(n) (2) is equivalent to $\Delta^{T} v = 0$ or taking the complex conjugate

$$\Delta^{+}v^{*}=0 . \tag{12}$$

Thus both $v+v^{*}$ and $i(v-v^{*})$ are independent real solutions of (2) and all solutions may be chosen real.

The rank condition on Δ which ensures that v has exactly n independent columns is that it must have maximal rank of k, 2k and 4k for Sp(n), U(n) and O(n) respectively for each value of x.

So far equation (1) with (2), (3) and (6) define a gauge potential A_{α} since it is automatically antihermitian in the sense of section 2 by the condition (3). However for general a,b in Δ it is not self-dual. The most general self-dual solution is given by the above construction provided a,b satisfy the quadratic constraints

$$\Delta^{+}(x)\Delta(x) = f^{-1}(x)\ 1_2 . \tag{13}$$

This states that for each x, $\Delta^{+}\Delta$ must be proportional to the unit 2×2 matrix in the space of the left hand 2-component index of Δ^{+} and the right hand 2-component index of Δ. $f^{-1}(x)$ is the matrix function of proportionality and as implied by the notation the inverse exists because of the rank condition above. For Sp(n) the right hand 2-component of Δ^{+} and the left hand one of Δ are contracted in the quaternionic multiplication and so f is a $k\times k$ real symmetric matrix. For O(n) these central components are left uncontracted so f is $2k\times 2k$ real symmetric. For U(n) there are no central indices and f is $k\times k$ complex hermitian.

The constraint (13) being true for all x can be shown to be equivalent to

$$\begin{array}{lll} \text{(i)} & a^{+}a = \mu\, 1_2 & \\ \text{(ii)} & b^{+}b = \nu\, 1_2 & \\ \text{(iii)} & a^{+}b = \varepsilon (b^{+}a)^{t}\varepsilon^{-1} & (t \text{ is the } 2\times 2 \text{ transpose only}) \end{array} \tag{14}$$

using the properties of quaternions in Appendix A. μ and ν necessarily have the same properties as f above. Again one can show that (4iii) is equivalent to

$$\begin{aligned} a^+b &= \rho_\alpha e_\alpha^+ \\ b^+a &= \rho_\alpha e_\alpha \end{aligned} \tag{15}$$

for some ρ_α which again necessarily has the properties of f. Thus

$$\Delta^+\Delta = a^+a + a^+bx + x^+b^+a + b^+bx^2 = \left(\mu + 2\rho_\alpha x_\alpha + \nu x^2\right)1_2 \tag{16}$$

where we have used

$$e_\alpha^+ x + x^+ e_\alpha = tr\left(e_\alpha^+ x\right)1_2 = tr\left(e_\alpha x^+\right)1_2 = 2x_\alpha \tag{17}$$

which follows from (A.9).

We will prove in section 6 that when (13) holds the A_α constructed above is indeed self-dual with topological charge k. The latter requires that f is non-singular for all x including "x at infinity". More precisely we need also that b^+b or ν is non-singular.

As mentioned in section 2 the antihermitian potentials for U(n) are not automatically traceless (unlike Sp(n) and O(n)). If we want the most general SU(n) potential we take the general U(n) solution and perform a U(1) gauge transformation $g(x) = e^{i\theta(x)}$ (θ real) to make it traceless

$$A_\alpha \rightarrow A_\alpha' = e^{-i\theta} A_\alpha e^{i\theta} + e^{-i\theta}\partial_\alpha e^{i\theta} . \tag{18}$$

Thus we require

$$0 = tr A_\alpha' = tr A_\alpha + in\partial_\alpha\theta \qquad \left(tr 1 = n\right) . \tag{19}$$

(19) always has a real solution θ as the antihermiticity of A_α ensures that $tr A_\alpha$ is pure imaginery and the integrability condition

$$\partial_\beta tr A_\alpha = \partial_\alpha tr A_\beta \tag{20}$$

is easily verified by differentiating the definition (1) of A_α and using the cyclic property of the trace.

Section 6 - Self-duality and Topological Charge. §6

The requirement for self-duality (5.13) means that any quaternion multiplying $\Delta^+\Delta$ (on the outside 2-component indices) commutes with it. This property will be crucial in many of the proofs throughout this work. Before proving that the A_α constructed in section 5 is self-dual we introduce the projection matrix

$$\rho(x) = v(x)\,v^{+}(x)\ .\tag{1}$$

From (5.2) and (5.3) we see that it is indeed a projection and has other properties:

$$\begin{aligned} &\rho^2 = \rho \\ &\rho^+ = \rho \\ &\Delta^+\rho = \rho\Delta = 0 \\ &v^+\rho = v^+\ ,\ \rho v = v\ . \end{aligned}\tag{2}$$

These properties are shared by

$$\rho = 1 - \Delta(\Delta^+\Delta)^{-1}\Delta^+ = 1 - \Delta f \Delta^+\tag{3}$$

and hence by the completeness of the columns of Δ and v the equality implied in (3) does hold. In the $\Delta f \Delta^+$ term of (3) the right 2-component index of Δ is contracted with the left hand one of Δ^+ for all groups. For Sp(n) the outer indices are left uncontracted so ρ is a quaternionic $(k+n) \times (k+n)$ matrix. For O(n) the left hand 2-component index of Δ is contracted with the left hand one of f and similarly for the right hand index of Δ^+ and f. From this one can show that ρ is real $(4k+n) \times (4k+n)$. These conventions of contracting two component indices will be followed throughout and not explicitly mentioned again.

From the definition (2.2) we can calculate the field strength tensor

$$F_{\alpha\beta} = \partial_\alpha(v^+v_\beta) - \partial_\beta(v^+v_\alpha) + [v^+v_\alpha, v^+v_\beta]\tag{4}$$

where we have used the shorthand

$$v_\alpha \equiv \partial_\alpha v\ .\tag{5}$$

Evaluating (4) gives

$$F_{\alpha\beta} = v^+_\alpha v_\beta + v^+v_\alpha v^+v_\beta - (\alpha \leftrightarrow \beta)\tag{6}$$

where $-(\alpha\leftrightarrow\beta)$ means subtract the same term with α and β interchanged ie. antisymmetrise in them. Using (5.3) to switch the α derivative on to v^+ and then using (1) gives

$$F_{\alpha\beta} = v^+_\alpha(1-\rho)v_\beta - (\alpha\leftrightarrow\beta)\ .\tag{7}$$

From (3) the $1-\rho$ term is $\Delta f \Delta^+$ and by (5.2) we can transfer both the derivatives on to the Δ's:

$$F_{\alpha\beta} = v^+\Delta_\alpha f \Delta^+_\beta v - (\alpha\leftrightarrow\beta)\ .\tag{8}$$

Because of the linearity of Δ (5.6) its derivative has a simple form

$$\partial_\alpha \Delta \equiv \Delta_\alpha = b e_\alpha \tag{9}$$

where by (5.7)

$$\partial_\alpha x = e_\alpha \, . \tag{10}$$

Using the property mentioned at the beginning of this section that quaternions commute with f, (8) gives

$$F_{\alpha\beta} = v^+ b\left(e_\alpha e_\beta^+ - e_\beta e_\alpha^+\right) f b^+ v \, . \tag{11}$$

From the property of quaternions (A.9) we can immediately see that $F_{\alpha\beta}$ is self-dual as $\bar{\eta}_{\alpha\beta} = \sigma^a \bar{\eta}^a_{\alpha\beta}$ is:

$$F_{\alpha\beta} = 2i\, v^+ b\, \bar{\eta}_{\alpha\beta} f b^+ v \, . \tag{12}$$

The topological charge is given by (3.9). To evaluate it we use the result [3]

$$\mathrm{tr}\left(F_{\alpha\beta} F_{\alpha\beta}\right) = 2\partial^2 T \tag{13}$$

where

$$T = \mathrm{tr}\left(b^+ P b f + b^+ b f\right) \, . \tag{14}$$

Since $F_{\alpha\beta}$ is self-dual the left hand side of (13) is exactly what appears in the integrand of (3.9). The property (13) enables us to replace the four dimensional integral of (3.9) by an integral over the surface at infinity. Hence the topological charge is

$$q = -\frac{1}{8\pi^2} \lim_{R\to\infty} \int_{|x|=R} dS_\alpha \partial_\alpha T \, . \tag{15}$$

By expanding f in powers of $\frac{1}{|x|}$ one can check that $f = O\left(\frac{1}{x^2}\right)$ and $b^+ P b = O\left(\frac{1}{x^2}\right)$ as $|x| \to \infty$ (see Appendix B). Hence only the second term of T contributes on the surface at infinity in (15). Using

$$f(x) 1_2 = \left(b^+ b\right)^{-1} \frac{1}{x^2} + O\left(\frac{1}{|x|^3}\right) \tag{16}$$

since $b^+ b$ is required to be non-singular the integral (15) becomes

$$q = -\frac{1}{8\pi^2} \int dS_\alpha \left(\frac{-2x_\alpha}{x^4}\right) \mathrm{tr}\left(b^+ b \left(b^+ b\right)^{-1}\right) \, . \tag{17}$$

The integral $\int dS_\alpha x_\alpha / x^4$ is just the angular integral $\int d\Omega = 2\pi^2$. $b^+ b$ is a $2k \times 2k$ matrix (for $U(n)$ and $Sp(n)$) hence the trace term just gives $2k$ so the result is that

$$q = -\frac{1}{16\pi^2} \int \mathrm{tr}\left(F_{\alpha\beta} F_{\alpha\beta}\right) d^4 x = k \, . \tag{18}$$

The property stated in (13) and (14) can be verified directly by differentiating (14). However this is somewhat tedious and not repeated here. Nevertheless it is possible to derive the result indirectly as is done in two different ways in sections 15 and 21. These proofs use an intermediate result [3] which also occurs in the direct proof and is useful elsewhere

$$\partial^2 f = -4\,\mathrm{tr}_2\left(f b^+ P b f\right) \tag{19}$$

where tr_2 means we only take the trace over the 2-component indices (on the left of b^+ and the right of b). To verify this we use (5.13) and (9) giving

$$\partial_\alpha f . 1_2 = -f\left(\Delta^+ b e_\alpha + e_\alpha^+ b^+ \Delta\right) f \; . \tag{20}$$

Differentiating again gives

$$\partial^2 f . 1_2 = -f\left(2e_\alpha^+ b^+ b e_\alpha\right) f + 2 f\left(\Delta^+ b e_\alpha + e_\alpha^+ b^+ \Delta\right) f \left(\Delta^+ b e_\alpha + e_\alpha^+ b^+ \Delta\right) f \tag{21}$$

which we can simplify using the properties (A.11) for quaternions so

$$e_\alpha \Delta^+ b e_\alpha = -2\varepsilon\left(\Delta^+ b\right)^t \varepsilon^{-1} = -2b^+\Delta \tag{22}$$

where for the last step the constraints (5.14) are used. Then (21) becomes

$$\partial^2 f = -4 f\,\mathrm{tr}_2\left(b^+ b\right) f + 4 f\,\mathrm{tr}_2\left(b^+ \Delta f \Delta^+ b\right) f \tag{23}$$

which is identical to (19) by (3).

Using the result (19) and continuing in a similar fashion one can verify (13) and (14) directly. A further interesting relation is (see also Osborn [23])

$$T = \tfrac{1}{4}\partial^2\,\mathrm{tr}.\ln \Delta^+\Delta = -\tfrac{1}{2}\,\partial^2 \ln \det f \tag{24}$$

which can be verified in a similar manner. Combining this result with (13) gives

$$\mathrm{tr}\left(F_{\alpha\beta} F_{\alpha\beta}\right) = \partial^2 \partial^2 \ln \det f \tag{25}$$

which is a generalisation of an equivalent result for the 't Hooft solution given by Jackiw, Nohl and Rebbi (JNR. [33]). The equivalence can be seen when $\det f$ is calculated for the 't Hooft solution in section 10.

We have now completed the proof that the construction of section

5 gives a self-dual gauge potential of topological charge k. We have not proved that it the most general solution. This is is a much deeper statement and was proved by Atiyah et. al. [30]. However it is possible to check that the solution constructed has the correct number of parameters as required by the results of Bernard et. al. [34] for general gauge groups. This is the subject of the next section.

The corresponding results for anti-self-dual gauge potential are trivially obtained from all the foregoing by replacing x everywhere by $x^{\dagger}$.

Section 7 - Canonical Form of Δ and Parameter Count. §7

Not all the Δ satisfying the quadratic constraints (5.13) give different A_{α}. Under the transformation

$$\Delta \to \Delta' = K\Delta L \tag{1}$$

where K and L are square constant matrices, A_{α} is unchanged provided K and L are of particular form depending on the gauge group [30,1,32].

$$\begin{array}{llll} Sp(n): & K \in Sp(k+n), & L \in GL(\mathbb{R}, k) \\ U(n): & K \in U(2k+n), & L \in GL(\mathbb{C}, k) \\ O(n): & K \in O(4k+n), & L \in GL(\mathbb{H}, k). \end{array} \tag{2}$$

GL(Q , k) is the group of non-singular $k \times k$ matrices with elements in Q (real, complex or quaternion respectively).

Under transformation (1) $\Delta^{\dagger}\Delta$ becomes

$$\Delta'^{\dagger}\Delta' = L^{\dagger}\Delta^{\dagger}K^{\dagger}K\Delta L = L^{\dagger}\Delta^{\dagger}\Delta L \tag{3}$$

by (2) $K^{\dagger}K = 1$. The fact that L does not act on the outer two-component indices of $\Delta^{\dagger}\Delta$ means that $\Delta'^{\dagger}\Delta'$ is still proportional to the unit 2×2 matrix and

$$f' = L^{-1} f (L^{\dagger})^{-1}. \tag{4}$$

Hence Δ' is still linear in x as $\Delta' = a' + b'x$ with

$$\begin{aligned} a' &= KaL \\ b' &= KbL \end{aligned} \tag{5}$$

and still satisfies the quadratic costraints so defining a self-dual potential. If v was a solution of (5.2) then $v' = Kv$ is a solution of

$$\Delta'^{+} v' = 0 \tag{6}$$

by (2) and the entries are of the correct type. The resulting potential is

$$A'_\alpha = v^+ K^+ \partial_\alpha K v = v^+ \partial_\alpha v = A_\alpha \tag{7}$$

since K is constant and $K^+K = 1$.

One can see that (1) is the largest possible invariance in general that preserves A_α. Using these transformations we can bring Δ to a special form. Here we do it only for U(n), however the procedure is similar for the other groups (see [1,32]). Also in any other statement that depends on the type of group we only treat U(n) for simplicity.

Since $b^+b = \nu 1_2$ is hermitian $k \times k$ we can diagonalise it by a unitary L and then since the diagonal elements (the eigenvalues of ν) are real and positive (b^+b is non-singular) we can transform it to the unit matrix by a diagonal L. So there exists some L such that $b' = bL$ satisfies

$$b'^+ b' = 1_{2k}\,. \tag{8}$$

The 2k columns of b' are thus orthonormal and we can adjoin a further n columns orthonormal to the previous ones so constructing a unitary $(2k+n) \times (2k+n)$ matrix U

$$U = (u \,|\, b') \qquad \text{such that} \quad u^+ b' = 0. \tag{9}$$

We can transform b' using this unitary matrix as K

$$b' \to U^+ b' = \left(\frac{0}{1_{2k}}\right). \tag{10}$$

This is the canonical form of b. Under the above transformations a has also changed. We write the new matrices as

$$a = \begin{pmatrix} u_1 & u_2 \\ r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 \\ 1_k & 0 \\ 0 & 1_k \end{pmatrix} \tag{11}$$

where the $r_{\alpha\beta}$ blocks are $k \times k$ and the u_β blocks are $n \times k$ with β being the two component right hand index of a and b. The constraint (5.14iii) then takes the form

$$r_{22} = r_{11}^+ \quad , \quad r_{21} = -r_{12}^+ \,, \tag{12}$$

(5.14ii) is automatic (it was assumed in the construction of b in (11)) whereas (5.14i)

$$ {}_A a^\dagger a_B = (u_A)^\dagger u_B + (r_{cA})^\dagger r_{cB} = \mu\, \delta_{AB} \tag{13}$$

still has to be satisfied. We can further refine the canonical form by diagonalising (13) with a unitary L. The form of b in (11) can be preserved if simultaneously we transform by

$$ K = \begin{pmatrix} U & & \\ & L^\dagger & \\ & & L^\dagger \end{pmatrix} \tag{14}$$

where U is an arbitrary $n \times n$ unitary matrix so that K is unitary as $L^\dagger L = 1$. Thus in the canonical form of (11) the constraints take the form of (12) with

$$ {}_{iA}(a^\dagger a)_{jB} = \mu_i\, \delta_{ij}\, \delta_{AB} \qquad 1 \le i,j \le k \;\; ; \;\; 1 \le A,B \le 2 \tag{15}$$

for some real positive constants μ_i.

There are still some invariances of (11), (12) and (15). They are preserved by (14) with

$$ (L)_{jl} = e^{i\theta_j}\, \delta_{jl} \qquad 1 \le i,j \le k, \;\; \theta_j \text{ real.} \tag{16}$$

We may restrict U to be special unitary as the overall phase of the unitary matrix may be absorbed into L without changing (a, b) . Thus there are $k + n^2 - 1$ transformations (14), (16) which preserve the form of (11), (12) and (15) without changing A_α.

We can now count the number of parameters in the instanton solution. In a of (11) with the condition (12) there are $4kn + 4k^2$ real parameters. The constraints (15) for $A \neq B$ give $2k^2$ real equations, for $A = B$ but $i \neq j$ give $2k(k-1)$ equations and for $A = B$, $i = j$ give k equations or $4k^2 - k$ in total. Hence there are $4kn + k$ parameters remaining and when the residual invariances of (14), (16) are taken into account there are $4kn - (n^2 - 1)$ independent parameters of the instanton solution in general.

For $2k < n$ the residual freedom of (14), (16) is not quite right. Some of the matrices U do not change the u_B part of b in (11). We can use U to transform all but 2k of the n rows of u to zero. Then the remaining invariance preserving this is of a $2k \times 2k$ special unitary matrix removing $4k^2+k-1$ degrees of freedom altogether. Because of the zero rows of a there are now only $4k.2k+4k^2$ parameters initially in a with the same number of constraints thus the number of instanton parameters for $2k<n$ is $4k^2+1$. Hence the number of parameters for $U(n)$ and $SU(n)$ agree with [34] as they also do for $Sp(n)$ and $O(n)$ [1,32].

The count of parameters can also be made in a similar way directly from the initial form of the constraints (5.14) and agrees with the above results. For $2k \geq n$ there are $16k^2+8kn$ real parameters in (a,b) with $10k^2$ equations from (5.14) and a $(2k+n)^2-1+2k^2$ parameter invariance due to the transformations (1) with (2). For $2k<n$ we can use K to make all but 4k rows of (a,b) equal zero. Then $32k^2$ parameters are constrained by $10k^2$ with a $(4k)^2-1+2k^2$ parameter invarince.

Section 8 - The Fundamental Scalar Green Function. §8

We can now use the construction of the general self-dual gauge potential to solve some other equations of interest. In this section the spin 0 (scalar) Green function in the background of A_α is obtained for the fundamental representation. From this it is possible to obtain the spin $\frac{1}{2}$ (spinor) and spin 1 (vector) Green functions using the results of Brown et. al. [35,36]. The corresponding results for the adjoint representation are derived in chapter 3.

The fundamental scalar green function G is defined by

$$D^2 G(x,y) = 1_n \delta(x-y) \tag{1}$$

with D_α the covariant derivative given by (2.13). We want to solve (1) with A_α given by (5.1). In this representation the Green function transforms under gauge transformations (2.10) by

$$G(x,y) \rightarrow g^\dagger(x) G(x,y) g(y)\,. \tag{2}$$

To solve (1) we notice that as $x \rightarrow y$, G must have the same singularity as the ordinary scalar Green function in order to give the Dirac δ function on the right hand side. Thus

$$G(x,y) = -\frac{H(x,y)}{4\pi^2(x-y)^2} \tag{3}$$

for some H where $H(x,x)=1$ and it must have the same transformation property as (2). The simplest expression in terms of the self-dual construction of section 5 with these properties is $v^\dagger(x)v(y)$. In fact this gives the correct answer so [1,32]

$$G(x,y) = -\frac{v^\dagger(x)\,v(y)}{4\pi^2(x-y)^2} \tag{4}$$

though in principle there are many terms which vanish as $x \to y$ which could be added to $v^\dagger(x)v(y)$. In order to prove that (4) solves (1) we need to know how the covariant derivative acts on $v^\dagger$.

$$D_\alpha v^\dagger = \partial_\alpha v^\dagger + v^\dagger(\partial_\alpha v)v^\dagger = v^\dagger \partial_\alpha P \tag{5}$$

by (5.3) and (5.1). Manipulating (5) in the same way as the the calculation of $F_{\alpha\beta}$ in section 6 we obtian

$$D_\alpha v^\dagger = -v^\dagger b e_\alpha f \Delta^\dagger. \tag{6}$$

This result will be used extensively. Another useful result following from this and the derivative of f (6.20) is

$$D_\alpha(v^\dagger b e_\alpha f) = -v^\dagger b e_\alpha f \Delta^\dagger b e_\alpha f - v^\dagger b e_\alpha f(\Delta^\dagger b e_\alpha + e_\alpha^\dagger b^\dagger \Delta)f. \tag{7}$$

Then using (6.22) we see the $2e_\alpha \Delta^\dagger b e_\alpha$ cancels the $e_\alpha e_\alpha^\dagger b^\dagger \Delta$ term giving

$$D_\alpha(v^\dagger b e_\alpha f) = 0. \tag{8}$$

From (6) and (8) it is easy to evaluate the second covariant derivative of $v^\dagger$

$$D^2 v^\dagger = -v^\dagger b e_\alpha f\, \partial_\alpha \Delta^\dagger = -4 v^\dagger b f b^\dagger. \tag{9}$$

The Green function (4) can now be verified

$$D^2 G(x,y) = -\partial^2\left(\frac{1}{4\pi^2(x-y)^2}\right)v^\dagger(x)v(y) - 2\partial_\alpha\left(\frac{1}{4\pi^2(x-y)^2}\right)D_\alpha v^\dagger(x)v(y) - \left(\frac{1}{4\pi^2(x-y)^2}\right)D^2 v^\dagger(x)v(y). \tag{10}$$

The first term gives $+\delta(x-y)v^\dagger(x)v(y)$ which is equal to the right hand side of (1) as $v^\dagger(x)v(x) = 1_n$. The second term by (6) gives

$$-\frac{4(x-y)_\alpha}{4\pi^2(x-y)^4} v^\dagger b e_\alpha f \Delta^\dagger v(y) \tag{11}$$

(all matrices are evaluated at x unless otherwise explicitly indicated). Using (5.2) and the linearity of Δ (5.6)

$$\Delta^\dagger(x)v(y) = (\Delta^\dagger(x) - \Delta^\dagger(y))v(y) = (x-y)^\dagger b^\dagger v(y). \tag{12}$$

Then the $(x-y)_\alpha e_\alpha = (x-y)$ commutes with f to combine with $(x-y)^\dagger$ from

(12) to give $(x-y)^2$. Thus (11) is equal to

$$-\frac{4}{4\pi^2(x-y)^2}\, v^+ b f b^+ v(y) \tag{13}$$

which cancels the last term in (10) because of (9). Hence the result is verified generalising that of Brown et. al. [36] for the 't Hooft solutions.

Section 9 - The Fundamental Dirac Zero Modes. §9

The massless Dirac equation in the background field A_α for the fundamental representation is

$$\gamma_\alpha D_\alpha \psi = 0 \tag{1}$$

where γ_α are the Euclidean Dirac matrices defined in Appendix A and ψ is a four component spinor (cf. (2.18)). Following Grossmann[37] we write ψ as two 2-component spinors

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \tag{2}$$

and (1) becomes

$$e_\alpha^+ D_\alpha \psi_L = 0 \quad , \quad e_\alpha D_\alpha \psi_R = 0 \,. \tag{3}$$

For self-dual $F_{\alpha\beta}$ we have by (2.15)

$$\eta_{\alpha\beta} D_\alpha D_\beta = \tfrac{1}{2}\eta_{\alpha\beta} F_{\alpha\beta} = 0 \tag{4}$$

so that operating on the ψ_R part of (3) with $e_\alpha^+ D_\alpha$ gives

$$e_\alpha^+ D_\alpha e_\beta D_\beta = (\delta_{\alpha\beta} + i\eta_{\alpha\beta}) D_\alpha D_\beta = D^2 \tag{5}$$

by (A.4). Hence $D^2\psi_R = 0$ so ψ_R has no normalisable solutions as the inverse of D^2 exists (it is the Green function of section 8). Thus we only need to consider the first part of (3).

Defining χ by

$$_A(\psi_L) = \chi_B\, \varepsilon_{BA} \tag{6}$$

where A is the 2-component spinor index of ψ_L, the equation for ψ_L becomes

$$0 = (e_\alpha^+)_{CA}\, \varepsilon_{BA} D_\alpha \chi_B = D_\alpha (\chi e_\alpha \varepsilon)_C \tag{7}$$

by (A.3). The ε multiplying the right hand side of (7) is irrelevant so the equation we need to solve is $D_\alpha(\chi e_\alpha) = 0$. We see immediately from (8.8) that $v^+ bf$ solves this [1] hence there are k independant solutions for ${}_A(\psi)$ which are the k columns of

$$v^+ b \varepsilon_A f \,. \tag{8}$$

One can see that this is $O\left(\frac{1}{|x|^3}\right)$ as $|x| \to \infty$ (this follows from (6.16) and $v^+ b = O\left(\frac{1}{|x|}\right)$ - see Appendix B) and hence is normalisable. From general theorems (see eg. [38]) there are exactly k independant normalisable solutions of

$$e_\alpha^+ D_\alpha \psi_L = 0 \tag{9}$$

so (8) gives the complete set. This is also proven later in this section.

The ith solution is

$$_A\psi_i(x) = v^+ b \varepsilon_A f_i \qquad 1 \leq i \leq k \tag{10}$$

with normalisation matrix

$$N_{ij} = \int \psi_i^{+A}(x) \psi_j^A(x) \, d^4x \tag{11}$$

where the Dirac trace has been taken. From (10) the integrand of (11) is

$$tr_2\left(f \varepsilon^{-1} b^+ v v^+ b \varepsilon f\right) \tag{12}$$

which happens to be the right hand side of (6.19). Hence

$$N = -\frac{1}{4} \int \partial^2 f \, d^4x \tag{13}$$

which we convert to a surface integral and using (6.20) gives

$$N = -\frac{1}{4} \lim_{R \to \infty} \int_{|x|=R} dS_\alpha \partial_\alpha f = \frac{1}{4} \lim_{R \to \infty} \int_{|x|=R} dS_\alpha f \left(\Delta^+ b e_\alpha + e_\alpha^+ b^+ \Delta\right) f \,. \tag{14}$$

From the behaviour (6.16) of f the integrand is $O\left(\frac{1}{|x|^3}\right)$ and so the integral is

$$N = \frac{1}{4} \int_{|x|=R} dS_\alpha \frac{x_\beta}{x^4} (b^+ b)^{-1} \left(b^+ b . 2 \delta_{\alpha\beta}\right) (b^+ b)^{-1} = \pi^2 (b^+ b)^{-1} . \tag{15}$$

Thus the orthonormal solutions are

$$\psi_i^A = \frac{1}{\pi} v^+ b \varepsilon_A f \left(b^+ b\right)_i^{1/2} . \tag{16}$$

There is an alternative derivation of (16) which shows this to be the complete orthonormal set. This was first done by Osborn [39] using the derivation of the Dirac Green function from the scalar Green function by Brown et. al. [36]. The Dirac Green function is defined by

$$\gamma_\alpha D_\alpha S(x,y) = 1_4 \delta(x-y) 1_n - \sum_i \psi_i(x) \psi_i^+(y) \tag{17}$$

where $\psi_i(x)$ are the complete orthonormal set of solutions to (1). The solution of (17) is [36]

$$S(x,y) = \frac{1-\gamma_5}{2}\gamma_\alpha D_\alpha G(x,y) + \frac{1+\gamma_5}{2} G(x,y)\overleftarrow{D}_\alpha \gamma_\alpha \tag{18}$$

where G is the scalar Green function and

$$G(x,y)\overleftarrow{D}_\alpha \equiv -\frac{\partial}{\partial y_\alpha} G(x,y) + G(x,y)A_\alpha(y) . \tag{19}$$

Substituting (18) in (17) gives

$$\delta(x-y)1_4\otimes 1_n - \sum_i \psi_i(x)\psi_i^+(y) = \begin{pmatrix} e_\alpha e_\beta^+ \otimes D_\alpha G(x,y)\overleftarrow{D}_\beta & 0 \\ 0 & e_\alpha^+ e_\beta \otimes D_\alpha D_\beta G(x,y) \end{pmatrix} \tag{20}$$

where the right hand 4×4 Dirac matrix is written in 2×2 block form using the form of the Dirac matrices in Appendix A. From (5) the lower right hand block is $1_2\otimes D^2 G(x,y) = \delta(x-y)1_2\otimes 1_n$ so confirming that there are no ψ_R components of ψ. The upper left block can be calculated using the explicit form (8.4) for G using similar techniques to previous calculations. After some work (see Osborn [39] for details) it gives

$$_A(e_\alpha e_\beta^+)_B\, D_\alpha G(x,y)\overleftarrow{D}_\beta = \delta(x-y)\,\delta_{AB}\, 1_n - \frac{1}{\pi^2} v^+(x) b \varepsilon_A f(x) b^+ b f(y)_B \varepsilon^{-1} b^+ v(y) \tag{21}$$

so confirming that (16) is the complete orthonormal set.

Section 10 - Conformal Transformations and the 't Hooft Solutions. §10

So far we have only considered the general formalism in terms of the matrix Δ satisfying the quadratic constraints (5.13). These have not been solved in general but certain classes of solutions are known. When $n \geqslant 2k$ Drinfeld and Manin [31] have essentially solved it completely for the k instanton solution in U(n). Christ et. al. [32] have solved (5.13) for $k=3$ in Sp(n) (for $k=1$ [4] and $k=2$ [33] the solutions were known before the construction of Atiyah et. al.). In addition to describing the $n \geqslant 2k$ solutions we give here the solutions corresponding to the 't Hooft [40] multi-instantons for $SU(2) = Sp(1)$ as extended by Jackiw, Nohl and Rebbi (JNR [33]) using the conformal invariance of the classical theory.

For $n > 2k$ in U(n) we can always use the transformation (7.1) to make all but 4k rows of rows of (a,b) zero as mentioned in section 7. Thus the problem is reduced to considering the $n=2k$ case. The general gauge potential for $n>2k$ is then just that for $n=2k$ embedded in a larger matrix with zeroes filling the rest

$$A_\alpha^{U(n)} = \left(\begin{array}{c|c} A_\alpha^{U(2k)} & 0 \\ \hline 0 & O_{n-2k} \end{array}\right). \tag{1}$$

For $n=2k$, using the canonical form specified by (7.11), (7.12) and (7.15) we can choose r_{11} and r_{12} arbitrarily ($4k^2$ real parameters). Then r_{22} and r_{21} are determined by (7.12). Choose k real parameters μ_i

$$0<\mu_1 \leq \mu_2 \leq \cdots\cdots \leq \mu_k \tag{2}$$

so that the diagonal a^+a specified by (7.15) is positive definite as required by the non-singularity of f in (5.13). The parameters μ_i are still not arbitrary, they must be chosen to satisfy the inequalities

$$(a^+a - r^+r) \geq 0 \tag{3}$$

ie. the eigenvalues of the left hand side are all non-negative. Then since u is a $2k \times 2k$ matrix we can always solve for it in

$$u^+u = a^+a - r^+r \tag{4}$$

by for example taking the square root of the right hand side. Also μ_i must be chosen so that

$$\Delta^+\Delta = a^+a + r^+x + x^+r + x^2 \tag{5}$$

is non singular for all x.

The ambiguity in u as defined in (4) just corresponds to the unitary matrix U in the tranformation (7.14) so does not alter A_α. Hence the $4k^2+k$ parameters r_{11}, r_{12}, μ_i uniquely determine A_α. However not all such parameters give different A_α (though they would in the corresponding case for Sp(k)) due to the transformations of the form (7.14) ,(7.16) so that if r and r' are related by

$$(r'_{AB})_{ij} = e^{i(\theta_i-\theta_j)}(r_{AB})_{ij} \qquad 1\leq i,j\leq k \tag{6}$$

then with the same μ_i they give equivalent A_α. Thus $k-1$ parameters are spurious (since only the difference $\theta_i-\theta_j$ occurs) giving $4k^2+1$

true parameters as before. If the matrix (3) has some zero eigenvalues, ie. its rank is less than 2k, then one can solve (4) for an $n \times 2k$ matrix u where $n \geq \text{rank}(a^+a - r^+r)$. So these special cases give the U(n) potentials for $n < 2k$, the only problem is specifying when this occurs.

Before giving the JNR solutions we show how conformal transformations act on the self-dual solutions [1]. A conformal transformation in Euclidean space is given by $x \to x'$ with

$$x' = (\alpha x + \beta)(\gamma x + \phi)^{-1} \quad , \quad x = (\alpha - x'\gamma)^{-1}(x'\phi - \beta) \tag{7}$$

where $\alpha, \beta, \gamma, \phi$ are quaternions. The transformation is non-singular provided

$$\kappa^2 \equiv \left[\det\begin{pmatrix}\alpha & \beta \\ \gamma & \phi\end{pmatrix}\right] = \alpha^2\phi^2 + \beta^2\gamma^2 - \mathrm{tr}(\alpha^+\beta\phi^+\gamma) \neq 0 . \tag{8}$$

Under (7) the length element transforms as $(dx)^2 \to (dx')^2$

$$(dx')^2 = \Omega^2(x)(dx)^2 \quad , \quad \Omega(x) = \kappa \,/\, |\gamma x + \phi|^2 \tag{9}$$

or equivalently

$$(dx)^2 = \omega^2(x')(dx')^2 \quad , \quad \omega(x') = \kappa \,/\, |\alpha - x'\gamma|^2 . \tag{10}$$

The gauge potential $A_\alpha(x)$ transforms in the same way as the derivative $\frac{\partial}{\partial x_\alpha}$ giving

$$A'_\alpha(x) = \frac{\partial x'_\beta}{\partial x_\alpha} A_\beta(x') . \tag{11}$$

The conformal invariance of Yang-Mills theory means that if $A_\alpha(x)$ is a solution of the equations of motion then so is $A'_\alpha(x)$ given by (11). For self-dual solutions one can implement the conformal transformation (7) by the following transformation on the parameters of the instanton

$$\begin{aligned} a \to a' &= a\phi + b\beta \\ b \to b' &= b\alpha + a\gamma . \end{aligned} \tag{12}$$

Then Δ becomes

$$\begin{aligned} \Delta'(x) &= a(\gamma x + \phi) + b(\alpha x + \beta) \\ &= (a + bx')(\gamma x + \phi) = \Delta(x')(\gamma x + \phi) \end{aligned} \tag{13}$$

so

$$(\Delta'(x))^+ \Delta'(x) = (\gamma x + \phi)^+ \Delta^+(x')\Delta(x')(\gamma x + \phi) . \tag{14}$$

Now since $\Delta^+(x)\Delta(x)$ is proportional to the unit 2×2 matrix for all x the quaternion $(\gamma x + \phi)$ commutes with it giving

$$(\gamma x + \phi)^+(\gamma x + \phi) = |\gamma x + \phi|^2 1_2 \tag{15}$$

hence $\Delta'(x)$ also satisfies the constraints (5.13). Thus we obtain a new self-dual solution with $v'(x)$ defined from $\Delta'(x)$. Because of (13) $v(x')$ is such a solution so

$$A'_\alpha(x) = v^\dagger(x')\frac{\partial}{\partial x_\alpha}v(x') = \frac{\partial x'_\beta}{\partial x_\alpha}v^\dagger(x')\frac{\partial}{\partial x'_\beta}v(x') = \frac{\partial x'_\beta}{\partial x_\alpha}A_\beta(x') \tag{16}$$

as in (11).

Atiyah et.al. [30] give the parameters (a, b) for the 't Hooft solution. It is a simple matter to extend them to the JNR [33] solution by using (12). Thus for SU(2) = Sp(1) we get

$$\begin{matrix} a_{0i} = \alpha_0\mu_i \ , & a_{ij} = \alpha_i\delta_{ij} \\ b_{0i} = \beta_0\mu_i \ , & b_{ij} = \beta_i\delta_{ij} \end{matrix} \qquad 1 \le i,j \le k \tag{17}$$

where α_i, β_i are quaternions and μ_i real. These clearly satisfy the constraints (5.13) or (5.14) and are preserved in form by (12). By the transformations (7.1) on Δ it is possible to restrict (17) to be

$$\begin{matrix} a_{0i} = -y_0\lambda_i/\lambda_0 \ , & a_{ij} = y_i\delta_{ij} \\ b_{0i} = \lambda_i/\lambda_0 \ , & b_{ij} = -\delta_{ij} \end{matrix} \qquad 1 \le i,j \le k \tag{18}$$

where y_i are quaternions and λ_i are real. Hence there are 5k + 4 parameters in this (only the ratio λ_i/λ_0 occurs). This is not in canonical form but it is much more convenient as both a_{ij} and b_{ij} have a diagonal form. With the parameters (18)

$$\Delta(x) = \begin{pmatrix} -\frac{\lambda_i}{\lambda_0}(y_0 - x) \\ \delta_{ij}(y_i - x) \end{pmatrix} \tag{19}$$

where the top block is 2 x 2k and the bottom 2k x 2k complex for U(2) or equivalently 1 x k and k x k quaternionic for Sp(1). With (19), a solution of (5.2) is

$$v_i(x) = \phi(x)^{-1/2}\,\frac{\lambda_i(y_i - x)}{(y_i - x)^2} \qquad 0 \le i \le k \tag{20}$$

where the normalisation ϕ, chosen to give (5.3), is

$$\phi(x) = \sum_{i=0}^{k}\frac{\lambda_i^2}{(y_i - x)^2} \ . \tag{21}$$

This gives the familiar gauge potential of JNR [33]

$$A_\alpha(x) = \sum_i v_i^\dagger(x)\,\partial_\alpha v_i(x) = -\tfrac{i}{2}\,\eta_{\alpha\beta}\,\partial_\beta \ln\phi \ . \tag{22}$$

The choice of v in (20) is singular at $x = y_i$ but one can always choose v so that this does not happen. The original 't Hooft solution is recovered if one sets $y_0 = \lambda_0 1$, in (18) and takes the limit $\lambda_0 \to \infty$. All the foregoing and subsequent expressions are well defined in this limit giving the corresponding results for the 't Hooft solution.

The parameters λ_i, y_i in (18) are not completely arbitrary as $\Delta^+\Delta$ must be non-singular.

$$ {}_i\Delta^+\Delta_j = \left[\frac{\lambda_i\lambda_j}{\lambda_0^2}(y_0-x)^2 + \delta_{ij}(y_i-x)^2\right].1_2 \qquad 1\leq i,j\leq k. \tag{23}$$

We require that the determinant of this be non-zero for all x.

Consider the eigenvalue equation

$$\sum_j {}_i\Delta^+\Delta_j\,\psi_j = \mu\psi_i \tag{24}$$

which is solved by

$$\psi_i = \frac{\lambda_i(y_0-x)^2}{\lambda_0^2}\left(\sum_1^k\lambda_j\psi_j\right)\left(\mu-(y_i-x)^2\right)^{-1}. \tag{25}$$

Multiplying by λ_i, summing and assuming $\sum_j\lambda_j\psi_j$ is non-zero we obtain a compatability condition on μ

$$1 = \frac{(y_0-x)^2}{\lambda_0^2}\sum_{i=1}^k\frac{\lambda_i^2}{\mu-(y_i-x)^2}. \tag{26}$$

Writing (26) as a k th degree polynomial in μ we obtain the product of the k roots μ (the required determinant) as the constant term of the polynomial. This gives

$$\det\Delta^+\Delta = \left[\frac{1}{\lambda_0^2}\sum_{i=0}^k\lambda_i^2\prod_{\substack{j=0\\ \neq i}}^k(y_j-x)^2\right]^2 \tag{27}$$

(the power of 2 is due to the unit 2×2 matrix in (23)). To check that (27) is non-singular for all x we consider the critical points $x=y_j$ some j (other x clearly give non-zero determinant as all terms are positive definite). From these it is easy to see that $\det\Delta^+\Delta\neq 0$ if and only if

$$\begin{aligned}\lambda_i &\neq 0 \ , \quad 0\leq i\leq k\\ y_i&\neq y_j \ , \quad 0\leq i<j\leq k.\end{aligned} \tag{28}$$

Since only λ_i^2 appear in the potential we can restrict to $\lambda_i>0$.

The inverse of $\Delta^+\Delta$ can easily be calculated giving [39]

$$f_{ij} = \frac{\delta_{ij}}{(y_i-x)^2} - \frac{\lambda_i\lambda_j\,\phi^{-1}}{(y_i-x)^2(y_j-x)^2} \qquad 1\leq i,j\leq k \tag{29}$$

which despite appearances is non-singular as $x\to y_i$ provided (28) is satisfied. The projection matrix P is given by

$$P_{ij}(x) = v_i(x)v_j^+(x) = \phi^{-1}\frac{\lambda_i\lambda_j(y_i-x)(y_j-x)^+}{(y_i-x)^2(y_j-x)^2} \qquad 0\leq i,j\leq k . \tag{30}$$

Using the above formulae one can write down the Green function (Brown et. al. [36]) from (8.4)

$$G(x,z) = -\frac{1}{4\pi^2\phi^{1/2}(x)\phi^{1/2}(z)(x-z)^2}\cdot\sum_{i=0}^{k}\frac{\lambda_i^2(y_i-x)^\dagger(y_i-z)}{(y_i-x)^2(y_i-z)^2} \tag{31}$$

and the Dirac functions (Grossmann [37]) from (9.10)

$$\psi_i^A(x) = \phi^{-3/2}\frac{\lambda_i}{(y_i-x)^2}\sum_{j=0}^{k}\frac{\lambda_j^2(y_j-x)_A^\dagger}{(y_j-x)^4} - \phi^{-1/2}\frac{\lambda_i(y_i-x)_A^\dagger}{(y_i-x)^4} \qquad 1\leq i\leq k \tag{32}$$

for the JNR solutions. The normalisation of ψ in (32) is given by (9.15) as [39]

$$N_{ij} = \int \psi_i^{\dagger A}\psi_j^A d^4x = \pi^2\left(\delta_{ij}+\frac{\lambda_i\lambda_j}{\lambda_0^2}\right)^{-1} = \pi^2\left(\delta_{ij}-\frac{\lambda_i\lambda_j}{\sum_{l=0}^{k}\lambda_l^2}\right). \tag{33}$$

CHAPTER 3 - ADJOINT REPRESENTATION AND TENSOR PRODUCTS

Section 11 - The Adjoint Representation and Variations of Instantons

In addition to the fundamental representation functions treated in sections 8 and 9, physics requires us to understand the adjoint representation (see chapter 1). This and all other representations can be treated together by considering tensor products of the fundamental representation. In particular if ϕ_1 transforms under the fundamental representation of group G_1 and similarly for ϕ_2 under G_2

$$\phi_I(x) \to g_I^+(x)\phi_I(x) \quad , \; g_I \in G_I \quad I = 1,2 \tag{1}$$

then the tensor product is $\phi_1 \otimes \phi_2$ which transforms under the fundamental representation of $G_1 \otimes G_2$ to give

$$\phi_1 \otimes \phi_2 \to g_1^+ \phi_1 \otimes g_2^+ \phi_2 . \tag{2}$$

The covariant derivative is given by

$$\mathcal{D}_\alpha^{1\otimes 2}(\phi_1 \otimes \phi_2) = (D_\alpha^1 \phi_1) \otimes \phi_2 + \phi_1 \otimes (\mathcal{D}_\alpha^2 \phi_2) \tag{3}$$

where D_α^I is the covariant derivative and A_α^I the gauge potential of the group G_I. The adjoint representation is given by $G_1 = G_2^* = G$ with the potentials satisfying

$$A_\alpha^2 = (A_\alpha^1)^* \tag{4}$$

and one performs the reduction

$$\phi^a = t^a_{ji}(\phi_1^i \otimes \phi_2^j) \qquad 1 \leq i,j \leq n \tag{5}$$

where t^a are the fundamental $n \times n$ representation matrices of the group G (see section 2). To see this we apply the adjoint covariant derivative (4.21) to (5). The matrix form of the field ϕ^a is (cf. (2.3) and (2.7))

$$\phi = \phi^a t^a \tag{6}$$

so

$$\mathcal{D}_\alpha \phi \equiv \partial_\alpha \phi + [A_\alpha, \phi] = \left[\partial_\alpha \phi^c + A_\alpha^a \phi^b f_{abc}\right] t^c \tag{7}$$

by (2.4). Inserting the form (5) of ϕ^b and using (2.4) again

$$\begin{aligned} (\mathcal{D}_\alpha \phi)^c &= \left\{ t^c_{ji}\partial_\alpha + A_\alpha^a [t^c, t^a]_{ji} \right\} (\phi_1^i \otimes \phi_2^j) \\ &= t^c_{ji}\left[\partial_\alpha(\phi_1^i \otimes \phi_2^j) + A_\alpha^{ik}\phi_1^k \otimes \phi_2^j + \phi_1^i \otimes A_\alpha^{*\,jk}\phi_2^k\right] \end{aligned} \tag{8}$$

where in the last step we have used $-A_\alpha^{kj} = A_\alpha^{*\,jk}$. Thus we see that the right hand side of (8) is the same reduction (5) of the tensor product covariant derivative (3) when the relation (4) between the potentials holds. Henceforth we will not perform this explicit reduction from $G\otimes G^*$ to the adjoint representation as this tends to hide the elegance and unity of the various formulae.

Before treating the adjoint representation and tensor products in general we consider the variations of the instanton solutions. Since there is a connection between these zero mode variations and the adjoint Dirac solutions as discussed by Brown et. al. [38] this will give us some insights into the adjoint representation as well as being useful in their own right as discussed in section 4.

Under an infinitesimal variation δA_α of the gauge potential, the field $F_{\alpha\beta}$ changes by

$$\delta F_{\alpha\beta} = \delta[\mathcal{D}_\alpha, \mathcal{D}_\beta] = [\mathcal{D}_\alpha, \delta A_\beta] - [\mathcal{D}_\beta, \delta A_\alpha]. \tag{9}$$

With this variation $F_{\alpha\beta}$ remains self-dual provided the contraction with the anti-self-dual tensor $\eta_{\alpha\beta}$ vanishes. Thus by the anti-symmetry of $\eta_{\alpha\beta}$

$$\tfrac{1}{2}\eta_{\alpha\beta}\,\delta F_{\alpha\beta} = \eta_{\alpha\beta}\,\mathcal{D}_\alpha\,\delta A_\beta = 0 \tag{10}$$

which is the same as (4.23). But by (A.4)

$$i\eta_{\alpha\beta}\,\mathcal{D}_\alpha\,\delta A_\beta = \mathcal{D}_\alpha\,\delta A_\alpha - e_\alpha^+ e_\beta\,\mathcal{D}_\alpha\,\delta A_\beta \tag{11}$$

and if we impose the condition corresponding to (4.25) that δA_α is orthogonal to the gauge zero modes

$$\mathcal{D}_\alpha\,\delta A_\alpha = 0 \tag{12}$$

then (10) becomes

$$e_\alpha^+\mathcal{D}_\alpha\left(e_\beta\,\delta A_\beta\right) = 0\,. \tag{13}$$

This is exactly the Dirac equation for the ψ_L components (9.9) except that it is for the adjoint representation. Thus if ψ are the adjoint Dirac zero modes

$$\psi = e_\alpha\,\delta A_\alpha\, u \quad \text{or} \quad \delta A_\alpha = u'^+ e_\alpha^+\psi \tag{14}$$

where u and u' are arbitrary two component spinors.

It is possible to find all the zero modes δA_κ and hence the adjoint Dirac functions by considering variations of the parameters of the instanton solutions [3,20,24]. The parameters are the constant matrices (a,b) in Δ satisfying (5.13). The general variation of this is

$$\delta\Delta = \delta a + \delta b x \tag{15}$$

which in order to preserve the constraints must satisfy

$$\delta\Delta^+\Delta + \Delta^+\delta\Delta \propto 1_2 . \tag{16}$$

Many of the solutions of (16) will correspond to the algebraic transformations (7.1) so do not give variations of A_κ. These are identified later. The variation δv in v resulting from $\delta\Delta$ must satisfy the following equations resulting from (5.2)

$$\Delta^+\delta v + \delta\Delta^+ v = 0 \tag{17}$$

and from (5.3)

$$\delta v^+ v + v^+\delta v = 0 . \tag{18}$$

The general solution of (17) is

$$\delta v = -\Delta f\,\delta\Delta^+ v + v\,\delta u \tag{19}$$

for arbitrary $n \times n$ δu. The condition (18) that v must remain normalised then implies

$$\delta u + \delta u^+ = 0 . \tag{20}$$

Substituting (19) into the variation of the expression (5.1) for A_α

$$\begin{aligned} \delta A_\alpha &= v^+\partial_\alpha\delta v + \delta v^+\partial_\alpha v \\ &= -v^+\partial_\alpha(\Delta f\delta\Delta^+ v) - v^+\delta\Delta f\Delta^+\partial_\alpha v + \partial_\alpha\delta u + A_\alpha\delta u - \delta u A_\alpha \end{aligned} \tag{21}$$

where (5.1), (5.3) and (20) have been used for the last three terms. For the first two terms in (21) we use (5.2) that $v^+\Delta = \Delta^+ v = 0$ to give

$$\delta A_\alpha = v^+\left(\delta\Delta f e_\alpha^+ b^+ - b e_\alpha f \delta\Delta^+\right) v + \mathcal{D}_\alpha\delta u . \tag{22}$$

The infinitesimal variation (22) is the most general satisfying (10) by construction and the completeness of the self-dual solutions. The term $\mathcal{D}_\kappa\delta u$ corresponds to an infinitesimal gauge transformation since δu is antihermitian by (20) and gives the gauge zero modes $\mathcal{D}_\kappa\phi$ of section 4. Imposing (12) (or (4.25)) gives

$$0 = \mathcal{D}_\alpha \delta A_\alpha = \mathcal{D}^2 \delta u + (D_\alpha v^+ \delta\Delta) f e_\alpha^+ b^+ v - v^+ \delta\Delta (f e_\alpha^+ b^+ v \overleftarrow{D}_\alpha) - (D_\alpha v^+ b e_\alpha f) \delta\Delta^+ v + v^+ b e_\alpha f (\delta\Delta^+ v \overleftarrow{D}_\alpha) \tag{23}$$

since

$$\mathcal{D}_\alpha (BC) = \partial_\alpha (BC) + A_\alpha BC - BCA_\alpha = (D_\alpha B) C - B(C\overleftarrow{D}_\alpha) \tag{24}$$

by (4.21) and (9.19). Now the third and fourth terms of (23) vanish because of the equation for the fundamental Dirac solution (8.8) and its hermitian conjugate, hence (23) becomes

$$\mathcal{D}^2 \delta u + v^+ b e_\alpha f (\delta\Delta^+ \Delta - \Delta^+ \delta\Delta) f e_\alpha b^+ v + v^+ \delta b e_\alpha f e_\alpha^+ b^+ v - v^+ b e_\alpha f e_\alpha^+ \delta b^+ v = 0. \tag{25}$$

Since we can always choose b to be in the canonical form (7.11) we can choose $\delta b = 0$. Making this choice and using the properties of the quaternions (A.11), (25) becomes

$$\mathcal{D}^2 \delta u + 2 v^+ b f \, tr_2 (\delta\Delta^+ \Delta - \Delta^+ \delta\Delta) f b^+ v = 0. \tag{26}$$

As $\delta b = 0$ and one can show that (16) implies

$$tr_2 (\delta a^+ b x - x^+ b^+ \delta a) = 0 \tag{27}$$

the two dimensional trace in (26) reduces to

$$tr_2 (\delta a^+ a - a^+ \delta a). \tag{28}$$

With $\delta b = 0$ the conditions (16) comprise $7k^2$ real equations on the $4k(2k+n)$ parameters in δa giving $4kn + k^2$ independent variations δa. Of these $k^2 + n^2 - 1$ must be gauge transformations due to the parameter count of section 7 of the number of gauge inequivalent instantons. These will be the δa which correspond to the transformations (7.14) that preserve the canonical form of b. These are

$$\delta a = Ha - ah \tag{29}$$

where

$$H = \begin{pmatrix} h' & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix}, \qquad \begin{array}{ll} h'^+ = -h' & n \times n \\ h^+ = -h & k \times k \end{array} \tag{30}$$

is an arbitrary antihermitian infinitesimal matrix so that $1+H$ is unitary and of the form (7.14). There are $k^2 + n^2 - 1$ independent variations of this form as expected. One can then see that with δa given by (29) the δA_α of (22) with $\delta b = 0$, $\delta u = 0$ is

$$\delta A_\alpha = \mathcal{D}_\alpha \psi \quad ; \quad \psi = - v^+ H v \tag{31}$$

which is indeed a pure gauge transformation. Because of this extra freedom of δa we can impose further constraints in addition to (16). Since in (26) δa only occurs in the term (28) it is natural to choose the k^2 real conditions

$$tr_2\left(\delta a^+ a - a^+ \delta a\right) = 0\ . \tag{32}$$

Then (26) is

$$\mathcal{D}_\alpha \delta A_\alpha = \mathcal{D}^2 \delta u = 0 \tag{33}$$

so δA_α of (22) satisfies the condition (12) if $\delta u = 0$. When we combine condition (16) with (32) on δa they are equivalent to

$$\delta a^+ \Delta = \varepsilon\left(\Delta^+ \delta a\right)^t \varepsilon^{-1}. \tag{34}$$

To summarise, the variation of the potential given by

$$\delta A_\alpha = v^+ \delta a f e_\alpha^+ b^+ v - v^+ b e_\alpha f\, \delta a^+ v \tag{35}$$

satisfies the two conditions (10) and (12) provided δa satisfies (34). Thus (35) gives all the instanton zero modes of section 4 satisfying (4.23) and (4.25) when δa is considered as a finite matrix satisfying (34). Because of the counting after (28) and the extra conditions (32) (or more directly counting the number of independent solutions to (34)) there are exactly 4kn independent δa. This confirms the statement in section 4 that $r_1 = 4kn$.

There are still $n^2 - 1$ pure gauge transformations in (35) despite the fact that $\mathcal{D}_\alpha \delta A_\alpha = 0$. These occur when δa is given by (29) satisfying (32) so that δA_α is the pure gauge of (31). For the large $|x|$ behaviour of v given in Appendix B (31) is

$$\delta A_\alpha = \mathcal{D}_\alpha \psi \quad ; \quad \psi = -u^+(x) h' u(x) + O\left(\tfrac{1}{x^2}\right) \quad \text{as} \quad |x| \to \infty \tag{36}$$

where $u(x)$ is the unitary matrix with winding number k that occurs in the pure gauge form (3.3) of A_α as $|x| \to \infty$ (see Appendix B). If ψ was just h', δA_α would be a global gauge transformation, but this would not be normalisable so of no interest to section 4. All the δA_α of (35) are normalisable with normalisation calculated in section 15.

Note that in the case $k = 0$ solutions analogous to (36) do not occur. If they did they must solve $\mathcal{D}^2\psi=0$ with $A_\alpha=g^+\partial_\alpha g$ for unitary g. Then

$$\mathcal{D}^2\psi = g^+\partial^2(g\psi g^+)g=0 \tag{37}$$

and since g is invertible

$$g\psi g^+ = c \tag{38}$$

for some constant $n\times n$ matrix c. However with this $\delta A_\alpha=\mathcal{D}_\alpha\psi$ vanishes so there are no normalisable δA_α sitisfying the conditions (10), (12) for $k=0$.

There is another property because of the extra constraint (32). If A_α is initially an SU(n) potential - ie. it is traceless - then the variation (35) is also traceless so preserving A_α as an SU(n) potential if (32) holds. In fact this is also true of (22) ($\delta b\neq 0$) provided δu is traceless.

$$\mathrm{tr}\,\delta A_\alpha = \mathrm{tr}\left(e_\alpha{}^+b^+P\,\delta\Delta f - \delta\Delta^+Pbe_\alpha f\right) + \partial_\alpha\,\mathrm{tr}(\delta u). \tag{39}$$

Then using

$$\Delta^+\delta\Delta = \varepsilon\left(\delta\Delta^+\Delta\right)^t\varepsilon^{-1} \quad , \quad b^+\delta\Delta = \varepsilon\left(\delta\Delta^+b\right)^t\varepsilon^{-1} \tag{40}$$

(which follow from (16) and (32)) and properties of the quaternions one can show that (39) reduces to $\partial_\alpha\mathrm{tr}(\delta u)$. Note for Sp(n) and O(n) δA_α is automatically traceless since it is antihermitian.

Section 12 - The Adjoint and Tensor Product Dirac Function §12

Since the δA_α of (11.35) satisfies the conditions (11.10) and (11.12) we can construct the adjoint Dirac zero modes from (11.14)

$$\psi^A = {}_A e_\alpha u\,\delta A_\alpha = 2v^+\delta a\,u f_A b^+v + 2v^+b\varepsilon_A f u^t\varepsilon\,\delta a^+v. \tag{1}$$

Because there are 4kn δa and u is an arbitrary 2-component spinor there appear to be 8kn Dirac zero modes. However this is incorrect as if δa satisfies (11.35) then so does $\delta a.\underset{\sim}{q}$ for any quaternion $\underset{\sim}{q}$ which can then be absorbed into u by $u'=\underset{\sim}{q}u$. Thus there are only 2kn zero modes in (1) which is the correct result by general theorems (see for example [38]). It can be checked by a completeness relation analogous to (9.20) for the fundamental representation (see Appendix C and also the comment after (15.24)).

We can rewrite (1) in the form

$$\psi^A = v^+ b\varepsilon_A f c_2^T v - v^+ c_1 f_A b^+ v \tag{2}$$

where c_1 and c_2 are appropriate $(2k+n) \times k$ complex constant matrices. Instead of giving them in terms of the δa by (1) (however see section 15) we can find directly the relations they must satisfy so that the Dirac equation

$$(e_\alpha^+)_{AB} \mathcal{D}_\alpha \psi^B = 0 \tag{3}$$

is solved [2]. Substituting (2) in (3) and using (11.24) gives

$$\mathcal{D}_\alpha(v^+ b e_\alpha \varepsilon_A f)c_2^T v - v^+ b e_\alpha \varepsilon_A f (c_2^T v)\overleftarrow{\mathcal{D}}_\alpha - \mathcal{D}_\alpha(v^+ c_1) f_A e_\alpha^+ b^+ v + v^+ c_1 (f_A e_\alpha^+ b^+ v)\overleftarrow{\mathcal{D}}_\alpha = 0. \tag{4}$$

The first and last terms again vanish due to (8.8) so

$$v^+ b e_\alpha f \Delta^+ c_1 f_A e_\alpha^+ b^+ v - v^+ b e_\alpha \varepsilon_A f c_2^T \Delta f e_\alpha^+ b^+ v = 0. \tag{5}$$

By the properties of the quaternions (A.11) this becomes

$$2 v^+ b_B f ({}_A\Delta^+ c_1 - c_2^T \Delta \varepsilon_A) f_B b^+ v = 0 \tag{6}$$

which will vanish and hence ψ^A of (2) solve the Dirac equation if

$${}_A\Delta^+ c_1 = c_2^T \Delta \varepsilon_A. \tag{7}$$

Since this must be true for all x it is equivalent to the equations

$$\begin{aligned} {}_A a^+ c_1 &= c_2^T a\, \varepsilon_A \\ {}_A b^+ c_1 &= c_2^T b\, \varepsilon_A . \end{aligned} \tag{8}$$

These comprise $4k^2$ complex equations for the $2(2k+n)k$ unknowns in c_1, c_2 hence giving $2kn$ independent complex solutions (c_1, c_2) - arbitrary complex linear combinations of solutions still solve (8).

One can generalise (2) to an arbitrary tensor product as considered at the beginning of section 11. The solution is [2]

$$\psi_A^{ij} = (v_1^+ b_1 \varepsilon_A f_1)_{ir} (v_2^+ c_2)_{jr} + (v_1^+ c_1)_{is} (v_2^+ b_2 \varepsilon_A f_2)_{js} \qquad \begin{matrix} 1 \le i \le n_1\,;\ 1 \le r \le k_1 \\ 1 \le j \le n_2\,;\ 1 \le s \le k_2 \end{matrix} \tag{9}$$

where Δ_I, v_I $(I = 1,2)$ are the matrices defining $A_\alpha^I(x)$ for the group G_I. If G_I are $U(n_I)$ groups and $A_\alpha^I(x)$ have topological charge k_I with Δ_I, v_I having the correct dimensions then c_1 is $(2k_1 + n_1) \times k_2$ and c_2 is $(2k_2 + n_2) \times k_1$. With the covariant derivative acting as in (11.3) we see in the same way as for (2) that (9) solves the tensor product

Dirac equation provided

$$ {}_A\Delta_1^{\dagger} c_1 = c_2^{T} \Delta_{2A}^{*} , \qquad A = 1,2 \tag{10}$$

(cf. (7)). This has $\tilde{k} = k_1 n_2 + k_2 n_1$ independent solutions.

By the results of section 11 the adjoint representation corresponds to when

$$ \begin{gathered} n_1 = n_2 , \quad k_1 = k_2 \\ \Delta_2 = \Delta_1^{*} \varsigma , \quad v_2 = v_1^{*} \end{gathered} \tag{11}$$

so that $A_2 = A_1^{*}$ (11.4) holds. Then the condition (10) is the same as (7) and the tensor product Dirac function (9) becomes identical to (2).

Section 13 - The Tensor Product Green Function §13

Brown et.al.[36] constructed the adjoint Green function for the general 't Hooft solution. Using their methods Christ et. al. [32] extended it to the most general SU(2) solution. In a similar manner one can further extend it to the tensor product of any two groups and hence in particular to the adjoint representation of any group [2].

One would expect from the fundamental representation (8.4) that the tensor product Green function would contain a term

$$ H(x,y) = \begin{pmatrix} v^{+}(x) v(y) \\ v^{+}(x) v(y) \end{pmatrix} \cdot \frac{-1}{4\pi^2 (x-y)^2} . \tag{1}$$

Here and henceforth we write the two factors of the tensor product of (11.2) one above the other. The upper factor refers to G_1 with Δ_1, V_1 etc. and the lower factor to G_2. We have dropped the explicit $1,2$ labels on the $v(x)$ etc. in (1) as they are always determined by their position in the equations. With this convention the tensor product covariant derivative of (11.3) can be written

$$ \tilde{D}_\alpha \equiv D_\alpha^{1\otimes 2} = \begin{pmatrix} D_\alpha \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ D_\alpha \end{pmatrix} . \tag{2}$$

The Green function equation is then

$$ \tilde{D}_\alpha \tilde{D}_\alpha \tilde{G}(x,y) = \left(\begin{matrix} D^2 \\ 1 \end{matrix} + \begin{matrix} 1 \\ D^2 \end{matrix} + 2 \begin{matrix} D_\alpha \\ D_\alpha \end{matrix} \right) \tilde{G}(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \delta(x-y) . \tag{3}$$

The covariant Laplacian on the left hand side of (3) acts on (1) to give

$$(\tilde{\mathcal{D}})^2 H(x,y) = \begin{pmatrix} v^+(x)v(y) \\ v^+(x)v(y) \end{pmatrix} \delta(x-y) - \frac{2}{4\pi^2(x-y)^2} \begin{pmatrix} D_\alpha v^+(x)\, v(y) \\ D_\alpha v^+(x)\, v(y) \end{pmatrix}. \tag{4}$$

The first term gives the right hand side of (3) but the second term is non-zero so there must be an additional term in G. In deriving (4) we have used the Green function equation for the fundamental representation (8.1). The second term in (4) is evaluated as follows. Using (8.6) and (8.12) the upper and lower terms are

$$-v^+(x) b e_\alpha f(x)(x-y)^+ b^+ v(y). \tag{5}$$

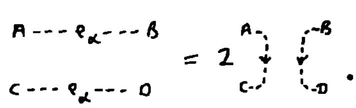
Then by (A.10) we get a factor $2(x-y)^2$ which cancels the $1/(x-y)^2$ term multiplying the second term of (4) to give

$$-\frac{1}{\pi^2} \begin{matrix} v^+(x)\, b\, f(x)\, b^+ v(y) \\ v^+(x)\, b\, f(x)\, b^+ v(y). \end{matrix} \tag{6}$$

Here we have introduced the further notation that broken lines denote contraction over the two-component indices of b,Δ. Later we will use solid lines to denote contraction over the k component indices of b, Δ and f. The arrow on the broken line means that the alternating 2×2 matrix ε_{AB} has been inserted. Thus

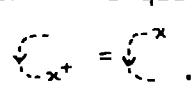
$$\begin{aligned} &A \text{-----} B \equiv \delta_{AB} \\ &A \text{--}\!\!>\!\!\text{---} B \equiv \varepsilon_{AB} \qquad 1 \le A,B \le 2 \;;\quad i \text{——} j \equiv \delta_{ij} \quad 1 \le i,j \le k \\ &A \text{---}\!\!<\!\!\text{---} B \equiv -\varepsilon_{AB} = \varepsilon^{-1}_{AB}\,. \end{aligned} \tag{7}$$

The identity for quaternions (A.10) used in obtaining (6) has the diagrammatic form



(8)

The $(x-y)^+$ in the top and bottom parts from (5) come together to give $(x-y)(x-y)^+ = (x-y)^2$ since for quaternions $\varepsilon x^+ = x^t \varepsilon$ by (A.3) so



(9)

The notation introduced above simplifies many of the expressions and calculations since otherwise they would contain many contracted and uncontracted indices on matrices which contain four or more indices.

In the Green function we need to add a term C to H in order to cancel the term (6). Analagously to Brown et. al. [36] we try the following form for C suggested by (6)

$$C = -\frac{1}{4\pi^2}\;\begin{matrix} v^+(x)\,b & & b^+v(y) \\ v^+(x)\,b & & b^+v(y)\end{matrix} \tag{10}$$

where

$$\begin{matrix} i & & k \\ j & & l \end{matrix} \equiv M_{ij,kl} \qquad \begin{matrix} 1 \le i,k \le k_1 \\ 1 \le j,l \le k_2 \end{matrix} \tag{11}$$

is a constant $k_1 k_2 \times k_1 k_2$ matrix. Evaluating the second covariant derivative of (10) gives

$$\frac{1}{\pi^2}\;\begin{matrix} v^+b\,f \\ v^+b'f \end{matrix} - \left[\begin{matrix}\Delta^+\Delta \\ b^+b\end{matrix} + \begin{matrix} b^+b \\ \Delta^+\Delta\end{matrix} - \begin{matrix}\Delta^+b \\ \Delta^+b'\end{matrix}\right] \begin{matrix} b^+v(y) \\ b^+v(y)\end{matrix} \tag{12}$$

(all matrices are functions of x unless explicitly shown otherwise). This will cancel (6) if M is the matrix inverse of the term in square brackets in (12). This term appears to depend on x so making M non-constant, but this is not the case. The x dependance cancels out due to relations like

$$\begin{matrix} x^+b^+b \\ a^+b' \end{matrix} = \begin{matrix} b^+b \\ a^+bx \end{matrix} \tag{13}$$

where the left hand side is part of the third term and the right hand side part of the second term in the square brackets of (12). (13) follows from the use of (9) and the fact that b^+b is proportional to the unit 2×2 matrix. Thus

$$M^{-1}_{ij,kl} = \begin{matrix} i \\ j \end{matrix}\left[\begin{matrix} a^+a \\ b^+b\end{matrix} + \begin{matrix} b^+b \\ a^+a\end{matrix} - \begin{matrix} a^+b \\ a^+b \end{matrix}\right]\begin{matrix} k \\ l \end{matrix} \tag{14}$$

and the Green function is given by $\tilde{G} = H + C$ with H defined in (1) and C in (10) with M as in (14).

We now examine some properties of M. Because the matrix in square brackets in (12) was independant of x it shows that the right hand side of (14) and hence M is invariant under

$$b \to b \;, \quad a \to a + bx \;. \tag{15}$$

It is also invariant under interchange of a and b

$$b \to a \;, \quad a \to b \;. \tag{16}$$

This is because of the constraint (5.14iii) so that

$$\begin{matrix} a^+b \end{matrix} = \begin{matrix} b^+a \end{matrix} \;. \tag{17}$$

Combining (15) and (16) shows that M is also invariant under

$$a \to a\ ,\quad b \to b + ay\ . \tag{18}$$

Under the transformation

$$a \to ax\ ,\quad b \to by \tag{19}$$

M only scales by $(|x||y|)^{-2}$. Combining the transformations (15), (18) and (19) we obtain the conformal transformation (10.12) on the parameters (a,b) under which M only scales by κ^{-2} given by (10.8).

In the construction of $\tilde{G}$ we have to assume M is non-singular. We have not proved this directly in general but it can be inferred from the existence of various quantities in the following section. Also in section 20, $\det M$ is evaluated for the JNR solutions of section 10 and shown to be non-singular precisely when $\Delta^{+}\Delta$ or f is non-singular for all x.

Section 14 - Self-dual Construction for Tensor Products §14

The self-dual construction for tensor products was first treated in [2]. The methods presented here are based on this reference. However since the work of [2] Drinfeld and Manin [41] have also treated tensor products on a more abstract basis. By making their work more concrete it is possible to simplify some of the algebraic proofs of this section [42]. However with the presentation here the connection with the earlier results is clearer. Since many of the calculations are rather long and tedious they are relegated to Appendix C where they are sketched.

The tensor product $G_1 \otimes G_2$ can be considered as a subgroup of some larger group $\tilde{G}$. For the case we are treating $SU(n_1) \otimes SU(n_2) \subset SU(n_1 n_2)$ so $\tilde{G} = SU(n_1 n_2)$. When the gauge potetials of G_1, G_2 are self-dual then the gauge potential for $G_1 \otimes G_2$

$$\tilde{A}_\alpha = \begin{pmatrix} A_\alpha & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & A_\alpha \end{pmatrix} \tag{1}$$

gives

$$\tilde{F}_{\alpha\beta} = \begin{pmatrix} F_{\alpha\beta} & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & F_{\alpha\beta} \end{pmatrix} \tag{2}$$

and hence is self-dual. $\tilde{A}_\alpha$ is also self-dual when considered as a potential of $\tilde{G}$. Calculating the topological charge of $\tilde{A}_\alpha$ from (3.9) gives

$$\tilde{k} = k_1 n_2 + k_2 n_1 \,. \tag{3}$$

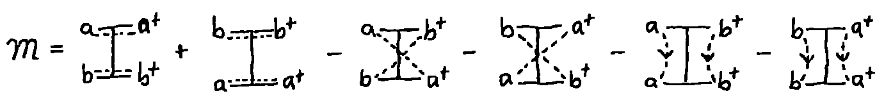
Hence $\tilde{A}_\alpha$ is a self-dual gauge potential in $U(\tilde{n})$

$$\tilde{n} = n_1 n_2 \tag{4}$$

with topological charge $\tilde{k}$ so must be given by some $(2\tilde{k}+\tilde{n}) \times 2\tilde{k}$ matrix $\tilde{\Delta}$ and $(2\tilde{k}+\tilde{n}) \times \tilde{n}$ matrix $\tilde{v}$ as in section 5. The obvious choice of the tensor product $v_1 \otimes v_2$ for $\tilde{v}$ is wrong as it has the incorrect dimension of $(2\tilde{k}+\tilde{n}+4k_1k_2) \times \tilde{n}$ and also if it was correct, the tensor product Green function

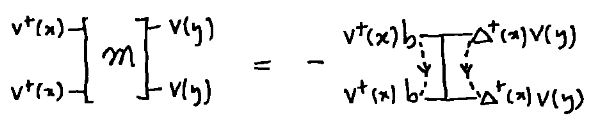
$$\tilde{G}(x,y) = -\frac{\tilde{v}^+(x)\tilde{v}(y)}{4\pi^2(x-y)^2} \tag{5}$$

would be given by $H(x,y)$ in (13.1) which is wrong. However we can write the Green function in terms of $v_1 \otimes v_2$ if we introduce the $(2k_1+n_1) \times (2k_2+n_2) = 2\tilde{k}+\tilde{n}+4k_1k_2$ dimensional square matrix $\mathcal{M}$ defined by



(6)

with I being M defined in (13.11) and (13.14). Under the conformal transformations (10.12) $\mathcal{M}$ is completely invariant. This is seen by using the fact that M of (13.14) only scales and then checking that $\mathcal{M}$ is invariant under (13.15), (13.16) and (13.19). In particular we can replace a by $\Delta(x)$ without changing $\mathcal{M}$. With this replacement



(7)

since only the last term of (6) fails to vanish by $v^+(x)\Delta(x) = 0$. From $\Delta^+(x)v(y) = (x-y)^+ b^+ v(y)$ (8.12) we see that the right hand side of (7) is $4\pi^2(x-y)^2 C(x,y)$ with C given by (13.10). Combining this with H in (13.1) gives the Green function as

$$\tilde{G}(x,y) = -\frac{1}{4\pi^2(x-y)^2}\{v_1(x)\otimes v_2(x)\}^+\{1-\mathcal{M}\}\{v_1(y)\otimes v_2(y)\}\,. \tag{8}$$

This is of the necessary form (5) if

$$\tilde{v}(x) = (1-\mathcal{M})^{1/2}\left(v_1(x)\otimes v_2(x)\right). \tag{9}$$

For $\tilde{v}$ to have the correct dimension, $1-\mathcal{M}$ must have rank $2\tilde{k}+\tilde{n}$ or exactly $4k_1k_2$ zero eigenvalues and the remaining eigenvalues must be positive so that (8) and (5) are equivalent. In Appendix C we construct the $4k_1k_2$ unit eigenvectors χ of $\mathcal{M}$. Also $\tilde{v}$ in (9) must satisfy (5.3), (5.2) for some $\tilde{\Delta}$ and give the correct A_α from (5.1). The first and last requirements hold as the right hand side of (7) is $4\pi^2(x-y)^2C(x,y)$ so

$$\tilde{v}^+(x)\,\tilde{v}(y) = \{v_1(x)\otimes v_2(x)\}^+\{v_1(y)\otimes v_2(y)\} + 4\pi^2(x-y)^2C(x,y). \tag{10}$$

Setting $x=y$ gives

$$\tilde{v}^+(x)\,\tilde{v}(x) = 1\otimes 1 \tag{11}$$

and differentiating (8) with respect to y and then setting $x=y$ gives

$$\tilde{A}_\alpha(x) = \tilde{v}^+(x)\,\partial_\alpha\tilde{v}(x) = A^1_\alpha(x)\otimes 1 + 1\otimes A^2_\alpha(x). \tag{12}$$

The final requirement is to construct a $\tilde{\Delta}(x)$ linear in x of the correct dimension satisfying the constraints (5.13) and

$$\tilde{v}^+(x)\,\tilde{\Delta}(x) = 0. \tag{13}$$

Clearly $\tilde{\Delta}$ cannot be the tensor product of Δ_1 and Δ_2 as this is quadratic in x. However the fact that Δ can be viewed as a mapping from the space of massless Dirac solutions [30] suggests that the right hand k-component index corresponds to the k independent pairs of matrices (c_1, c_2) used in the tensor product Dirac solution. In diagrammatic form (12.10) becomes

$$\left[\Delta^+c\right] = \left[\Delta^+c\right] \quad \text{or} \quad \left[a^+c\right] = \left[a^+c\right] \quad \text{and} \quad \left[b^+c\right] = \left[b^+c\right]. \tag{14}$$

(9) suggests that we define $\tilde{\Delta}$ in terms of a $(2\tilde{k}+\tilde{n}+4k_1k_2)\times\tilde{k}$ $\hat{\Delta}_R$ by

$$\tilde{\Delta}_R = (1-\mathcal{M})^{1/2}\,\hat{\Delta}_R \qquad R=1,2 \tag{15}$$

so that it is projected down to the correct dimension. (Note that $1-\mathcal{M}$ is not a projection as can be checked by any explicit numerical example). Because of the property (C.15) of $1-\mathcal{M}$ we can add any linear combination of the $4k_1k_2$ eigenvectors χ to $\hat{\Delta}$ leaving $\tilde{\Delta}$ unchanged. Using these facts suggests a form for $\hat{\Delta}$ is

$$\left[\hat{\Delta}_{rR}\right] = [\eta]\left[\Delta^+c_r \; k\cdots R\right] + \left[-\Delta L \; R \; -c_r\right] + \left[-c_r \; R \; -\Delta L\right] \qquad 1\le r\le\tilde{k},\; R=1,2 \tag{16}$$

where L_1 on the upper and L_2 on the lower rows are square matices of dimension k_1 and k_2 respectively and $\mathcal{N}$ is defined analogously to χ (C.1) to be

$$[\mathcal{N}] = {}^{-a}_{-a'}[N_1] + {}^{-b}_{-b'}[N_2] + {}^{a}_{be_\alpha'}[Q_\alpha] + {}^{be_\alpha}_{a'}[Q_\alpha] \qquad (17)$$

where N_1, N_2, Q_α are square $k_1.k_2$ dimensional matrices. In Appendix C we show that

$$\tilde{V}^+\tilde{\Delta} = (v_1 \otimes v_2)^+ (1-\mathcal{M})\hat{\Delta} = 0 \qquad (18)$$

provided the matrices N_1, N_2, Q_α satisfy

$$\begin{aligned} {}_{a^+a}[N_1] + {}_{b^+b}[N_2] + {}_{a^+be_\alpha}[Q_\alpha] &= -\ {}^{-L-}_{} \\ {}^{a^+a}[N_1] + {}^{b^+b}[N_2] + {}^{a^+be_\alpha}[Q_\alpha] &= +\ {}^{}_{-L-}\ . \end{aligned} \qquad (19)$$

It turns out that $(1-\mathcal{M})\hat{\Delta}$ and hence $\tilde{\Delta}$ depend only on L_1, L_2 (see Appendix C). (19) determines N_1, N_2, Q_α in terms of L_1, L_2 up to an arbitrariness due to addition of χ (C.1) which is then cancelled by $1-\mathcal{M}$ in $\tilde{\Delta}$. Thus $\tilde{\Delta}$ contains the $k_1^2 + k_2^2$ degrees of freedom of L_1, L_2.

In Appendix C we also calculate

$$\tilde{\Delta}^+\tilde{\Delta} = \hat{\Delta}(1-\mathcal{M})\hat{\Delta} \qquad (20)$$

which is also independent of N_1, N_2, Q_α and using some completeness relations for the c_1, c_2 matrices it simplifies to (see Appendix C)

$$(\tilde{\Delta}^+\tilde{\Delta})_{rR,sS} = (Z^+\Omega^{-1}Z)_{rs}\,\delta_{RS} \qquad (21)$$

where

$$Z_{rs} = \left[{}^{c_r^+c_s}_{L}\right] + \left[{}^{L}_{c_r^+c_s}\right] \qquad (22)$$

and

$$\Omega_{rs} = \left[{}^{c_r^+\rho c_s}_{f}\right] + \left[{}^{f}_{c_r^+c_s}\right]. \qquad (23)$$

Thus we see that $\tilde{\Delta}^+\tilde{\Delta}$ is indeed proportional to the unit 2×2 matrix. For suitable choice of L_1 and L_2, Z is non-singular and since $\rho = vv^+$ is a positive matrix and f is positive definite Ω must be non-singular. Hence $\tilde{\Delta}^+\tilde{\Delta}$ is also non-singular so satisfying both the conditions for a $\tilde{k}$-instanton $U(\tilde{n})$ solution thus verifying the construction for the tensor product.

One can now see that all the arbitrariness due to L_1, L_2 is contained in the constant $\tilde{k}\times\tilde{k}$ matrix Z which exactly corresponds to the $GL(\mathbb{C},\tilde{k})$ invariance of Δ in the original self-dual construction as given in (7.1) and (7.2).

Having constructed $\tilde{\Delta}$ and $\tilde{v}$ we can now rederive the functions of interest from the corresponding ones in the fundamental representation. The Green function is given by (5) which we have already shown is equal to that originally constructed in section 13. The tensor product Dirac function is given by

$$\tilde{\psi} = \tilde{v}^+\tilde{b}\,\varepsilon\,\tilde{f} = (v_1\otimes v_2)^+(1-m)\hat{b}\,\varepsilon\, Z^{-1}\Omega(Z^+)^{-1} \tag{24}$$

with normalisation

$$\int\tilde{\psi}^+\tilde{\psi} = \pi^2(\tilde{b}^+\tilde{b})^{-1} = \pi^2\left(\hat{b}^+(1-m)\hat{b}\right)^{-1} \tag{25}$$

where we have used the results (9), (15) and (21). Given the full expression for each of the matrices in (24) and (25), these are rather complicated expressions. However they simplify greatly as shown in Appendix C to

$$\tilde{\psi}^A_r = \left\{\begin{matrix} v^+bf \\ v^+c_s \end{matrix}\Bigg]^{A} + \begin{matrix} v^+c_s \\ v^+bf \end{matrix}\Bigg]^{A}\right\}\{(Z^+)^{-1}\}_{sr} \tag{26}$$

and normalisation

$$\int\tilde{\psi}^{+A}_r\tilde{\psi}^A_s = \pi^2\left(Z^{-1}\bar{\Omega}\,Z^{+^{-1}}\right)_{rs} \tag{27}$$

where $\bar{\Omega}$ is the same as Ω in (23) but with all the implied Δ's replaced by b's. (27) can easily be seen to be true from (21) as

$$\tilde{b}^+\tilde{b} = \lim_{|x|\to\infty}\frac{\tilde{\Delta}^+\tilde{\Delta}}{x^2} \tag{28}$$

since it is the quadratic term in $\tilde{\Delta}^+\tilde{\Delta}$.

Thus we have reconstructed the Dirac solutions of (12.9) which in diagrammatic form are

$$\psi^r_A = \begin{matrix} v^+b\,\varepsilon_A f \\ v^+c_r \end{matrix}\Bigg] + \begin{matrix} v^+c_r \\ v^+b\,\varepsilon_A f \end{matrix}\Bigg] \tag{29}$$

with normalisation following from (27) and (26) of

$$\int \psi_A^{+r} \psi_A^s \, d^4x = \pi^2 \bar{\Omega}_{rs} = \pi^2 \left\{ \left[\frac{c_r^+ (1 - b(b^+b)^{-1}b^+) c_s}{(b^+b)^{-1}} \right] + \left[\frac{(b^+b)^{-1}}{c_r^+ c_s} \right] \right\}. \tag{30}$$

The completeness relation for the solutions (29) is obtained by evaluating (9.20) for the tensor product Green function (see Appendix C). Since the latter does not contain (c_1, c_2) explicitly the relations give a completeness relation for the matrices (c_1, c_2) which is used in Appendix C to obtain some of the above results.

Section 15 - Normalisation of the Instanton Zero Modes §15

Using the normalisation of the Dirac equation in (14.29) and (14.30) we can obtain the corresponding normalisation of the instanton zero modes (11.35). Also we can derive the expression for $\mathrm{tr}(F_{\alpha\beta} F_{\alpha\beta})$ given by (6.13) and (6.14). First we do the latter.

For the adjoint representation there are always the following four independent solutions to the Dirac equation (12.3) (see [43,44])

$$\begin{aligned} \psi_A &= {}_A e_\alpha e_\beta^+ u . F_{\alpha\beta} \\ \psi_A &= {}_A e_\alpha e_\beta^+ x u . F_{\alpha\beta} \end{aligned} \tag{1}$$

where u is an arbitrary 2-spinor. These correspond to those variations of the instanton due to the conformal transformations (10.12) with $c = bu$ and $c = au$ respectively in the expression (14.29) for the adjoint representation. For example in the first case of (1) the two solutions are given by

$$(c_1)_R = (b_1)_R \quad , \quad (c_2)_R = (b_2)_R = b_1^* \varepsilon_R \qquad R = 1,2. \tag{2}$$

using the relation (12.11) for the adjoint representation. Then (14.29) becomes

$$\psi_A^R = v^+ b \varepsilon_A f_R \varepsilon^{-1} b^+ v - v^+ b_R f_A b^+ v \qquad R = 1,2 \tag{3}$$

which can be shown to be equal to

$$\psi_A^R = \tfrac{1}{8} (e_\alpha e_\beta^+)_{AR} \, F_{\alpha\beta} \tag{4}$$

with $F_{\alpha\beta}$ from (6.11), where the properties (A.10) of the quaternions are used. The integrand of the normalisation integral (14.30) is then

$$\begin{aligned} \mathrm{tr}\, \psi_A^{+R} \psi_A^S &= \tfrac{1}{64} (e_\alpha e_\beta^+)_{AR}^* (e_\gamma e_\delta^+)_{AS} \, \mathrm{tr}\, F_{\alpha\beta}^+ F_{\gamma\delta} \\ &= \tfrac{1}{64} (e_\alpha e_\beta^+ e_\gamma e_\delta^+)_{RS} \, \mathrm{tr}\, F_{\alpha\beta} F_{\gamma\delta} \end{aligned} \tag{5}$$

since $F_{\alpha\beta}$ is antisymmetric and antihermitian. Setting $R=S$ and summing gives the trace of the quaternions in (5) and using the results of Appendix A this gives

$$\mathrm{tr}\,\psi_A^{+R}\psi_A^R = -\frac{1}{32}\,\bar\eta^a_{\alpha\beta}\,\bar\eta^a_{\gamma\delta}\,\mathrm{tr}\left(F_{\alpha\beta}F_{\gamma\delta}\right). \qquad (6)$$

But $\frac{1}{4}\bar\eta^a_{\alpha\beta}\bar\eta^a_{\gamma\delta}$ is just the projection onto self-dual tensors hence

$$\mathrm{tr}\,\psi_A^{+R}\,\psi_A^R = -\frac{1}{8}\,\mathrm{tr}\,F_{\alpha\beta}F_{\alpha\beta}\,. \qquad (7)$$

The ψ on the left hand side is given by (14.29) and using the equality of (14.24) and (14.26) we have

$$\psi_A^r = \left(\tilde v^+\tilde b\,\varepsilon_A\tilde f\,Z^+\right)_r \qquad (8)$$

so

$$\psi_A^{+r}\psi_A^s = \left(Z\,\tilde f_A\varepsilon^{-1}\tilde b^+\tilde P\,\tilde b\;\;\varepsilon_A\tilde f\,Z^+\right)_{rs}. \qquad (9)$$

But this contains $f\,\mathrm{tr}_2\left(b^+Pb\right)f$ where all the terms are for the adjoint representation. The same relation (6.19) must hold for this as the proof goes through in exactly the same way. Thus (9) is equal to

$$-\frac{1}{4}\left(Z\,\partial^2\tilde f\,Z^+\right)_{rs} \qquad (10)$$

or using the result (14.21) gives

$$-\frac{1}{4}\,\partial^2\Omega_{rs}\,. \qquad (11)$$

Combining (11) with (7) gives the form of (6.13)

$$\mathrm{tr}\,F_{\alpha\beta}F_{\alpha\beta} = 2\,\partial^2\Omega_{RR} \qquad (12)$$

with Ω_{RS} defined in (14.23) and c_R given by (2). So we get

$$\Omega_{RR} = \begin{bmatrix} {}_a b_1^+ P b_{,R} \\ f \end{bmatrix} + \begin{bmatrix} f \\ {}_R b_1^+ b_{,R} \end{bmatrix} \qquad (13)$$

which from the relation (12.11) for the adjoint representation becomes

$$\Omega_{RR} = \mathrm{tr}\left(b^+Pbf + b^+bf\right). \qquad (14)$$

This is exactly T of (6.14) so (12) and (14) combine to give a proof of the result (6.13), (6.14).

We can normalise the instanton zero modes (11.35)

$$\delta A_\alpha = v^+\delta a\,f e_\alpha^+\,b^+v - v^+b e_\alpha f\,\delta a^+ v \qquad (15)$$

with δa satisfying (11.34)

$$\delta a^+\Delta = \varepsilon\left(\Delta^+\delta a\right)^t\varepsilon^{-1} \qquad (16)$$

using the normalisation (14.30) of the adjoint Dirac zero modes. But first we must make the connection between the δa in the instanton zero modes and the (c_1, c_2) in the Dirac zero modes more precise. In the adjoint case the conditions on (c_1, c_2) are (12.7)

$$_A\Delta^+ c_1 = c_2^T \Delta \varepsilon_A. \tag{17}$$

Taking the hermitian conjugate of (17) and multiplying by ε gives

$$c_1^+ \Delta \varepsilon_A = -_A\Delta^+ c_2^*. \tag{18}$$

Thus we see that if (c_1, c_2^T) is a solution of (17) so is $(-c_2^*, c_1^+)$ so we can choose the 2kn solutions (c_1, c_2) to have the form

$$\begin{aligned} (c_1^{r1}, c_2^{r1}) &= (d_1^r, d_2^{*r}) \\ (c_1^{r2}, c_2^{r2}) &= (-d_2^r, d_1^{r*}) \end{aligned} \qquad 1 \le r \le kn \tag{19}$$

for some d_1^r, d_2^r. Thus we see that

$$c_2^{rR} = c_1^{*rS} \varepsilon_{SR} \tag{20}$$

with the choice (19). Substituting (20) back in the equations (18) gives

$$(c_1^{rR})^+ \Delta \varepsilon_A = -_A\Delta^+ c_1^{rS} \varepsilon_{SR} \tag{21}$$

which can be rewritten

$$_R c_1^{+r} \Delta_B = \varepsilon_{RS}\, {}_A(\Delta^+ c_1^r)_S\, \varepsilon^{-1}_{AB}. \tag{22}$$

This is exactly the form of (16) so

$$\delta a_R^r = c_1^{rR} \qquad 1 \le r \le kn \tag{23}$$

solves (16). R is the right hand 2-component index of δa. (23) only gives kn of the δa, but as in the comment after (12.1) $\delta a . q$ (q a quaternion) also solves (16) and gives new independent δA_α so the 4kn δa are given by

$$\delta a_A^{r\mu} = c_1^{rR} (e_\mu)_{RA} \qquad 1 \le r \le kn \ , \ 0 \le \mu \le 3 . \tag{24}$$

(Note that the right hand side of (24) does not give new (c_1, c_2)'s as they are just linear combinations of the old). Using the δa of (24) in (15) gives

$$\delta A_\alpha^{r\mu} = v^+ c_1^{rR} f_R e_\mu e_\alpha^+ b^+ v - v^+ b e_\alpha e_\mu^+{}_R f (c_1^{rR})^+ v. \tag{25}$$

The relation (20) between c_1 and c_2 means that part of the second term of (25) is

$$\begin{aligned} {}_A e_\alpha e_\mu^+{}_R (c_1^{rR})^+ &= {}_A e_\alpha e_\mu^+ \varepsilon_R (c_2^{rR})^T \\ &= {}_R e_\mu e_\alpha^+{}_B\, \varepsilon_{AB} (c_2^{rR})^T \end{aligned} \tag{26}$$

where the property (A.3) has been used in the last step. Thus we see that δA_α of (25) is

$$ {}_R(e_\mu e_\alpha^+)_A \left[v^+ c_1^{\,rR} f_A b^+ v - v^+ b \,\varepsilon_A f (c_2^{\,rR})^T v \right] = -{}_R(e_\mu e_\alpha^+)_A \,\psi_A^{\,rR} \tag{27} $$

with ψ given by (14.29) for the adjoint representation which is equal to (12.2).

The required normalisation matrix for δA_α is

$$ N_{r\mu,s\nu} \equiv \int tr\left[(\delta A_\alpha^{r\mu})^+ \,\delta A_\alpha^{s\nu} \right] \tag{28} $$

so by (27) the integrand is

$$ {}_A(e_\alpha e_\mu^+)_R \; {}_S(e_\nu e_\alpha^+)_B \, tr\left(\psi_A^{+\,rR} \,\psi_B^{\,sS} \right) = 2\, {}_S(e_\nu e_\mu^+)_R \, tr\left(\psi_A^{+\,rR} \,\psi_A^{\,sS} \right) \tag{29} $$

using (A.10). Integrating (29) by the Dirac normalisation (14.30) gives us the required result

$$ N_{r\mu,s\nu} = 2\pi^2 \, {}_S(e_\nu e_\mu^+)_R \; \bar{\Omega}_{rR,sS} \tag{30} $$

where for the adjoint representation

$$ \bar{\Omega}_{rR,sS} = tr\left[(c_1^{\,rR})^+ (1 - b(b^+b)^{-1}b^+)\, c_1^{\,sS} (b^+b)^{-1} \right] + \varepsilon_{RR'} \,\varepsilon_{SS'} \, tr\left[(c_1^{\,sS'})^+ \, c_1^{\,rR'} \,(b^+b)^{-1} \right]. \tag{31} $$

The normalisation can be rewritten in terms of the $\delta a^{r\mu}$ by absorbing the e_ν, e_μ^+ into the $\bar{\Omega}$.

$$ N_{r\mu,s\nu} = 2\pi^2 \, tr\left[(\delta a^{r\mu})^+ (1 - b(b^+b)^{-1}b^+)\, \delta a^{s\nu} (b^+b)^{-1} + (\delta a^{s\nu})^+ \,\delta a^{r\mu} (b^+b)^{-1} \right]. \tag{32} $$

This is now true for any pair of δa's satisfying (16).

CHAPTER 4 - THE DETERMINANT OF D^2 AND THE AXIAL ANOMALY

Section 16 - Introduction and Regularisation of Determinants

When one tries to quantise Yang-Mills theory by expanding about instanton solutions, determinants of operators in the background field arise (see section 4). This chapter is devoted to calculating these. In particular we only need consider the covariant Laplacian for spin zero fields in the fundamental representation. By using the results of d'Adda and di Vecchia [25] and Brown and Creamer [15] we can express the spin $\frac{1}{2}$ and spin 1 determinants in terms of the spin 0 result. Further, using the results of section 14 [2] we can extend the fundamental representation to others, and in particular the adjoint. Thus in principle any determinant required for physics is calculable.

However we have so far been unable to calculate the simplest determinant completely. What has been obtained is the variation of the determinant under arbitrary variations of the background gauge potential, expressed as an integral of a particular expression (18.8)[3]. This is closely related to the result obtained by Brown and Creamer for the special case of the 't Hooft solution [15] (see also [20,23,24]). Also it is identical to the result independantly found by Belavin et. al. [20] and more recently by Berg and Lüscher [24] for the general self-dual solution. The latter group has also succeeded in integrating up the variation to write the determinant as a sum of 4 and 5-dimensional integrals. Even though the integrals for the full result have not been done in general, it is possible to find the part arising from conformal variations [3]. This is calculated in section 19 and the final result will contain in addition a conformally invariant factor. The known conformally non-invariant factor can be evaluated explicitly for the JNR solutions in section 20.

The determinant of the Laplacian is highly singular with both infrared and ultraviolet divergences. The former can be handled by calculating on a large sphere S^4 of radius a. Then the divergence in the limit $a\to\infty$ is cancelled by dividing out the corresponding determinant with zero background field. In section 4 we saw that this was a natural ratio to take. Once this is realised it is possible to do all the calculation in Euclidean space $\mathbb{R}^4$. The second divergence must be handled in the usual way of regularisation and renormalisation. The most elegant scheme for this purpose for our case is the zeta function regularisation of Ray and Singer [45-48]. This is introduced in the remainder of this section. However Berg and Lüscher [24] achieve their results by Pauli-Villars regulators which at least for the first order quantum corrections is equivalent. They also work on S^4 throughout and only at the end take the limit $a\to\infty$ and obtain equivalent results to us.

Using the zeta function regularisation and the procedures of this and the following sections we can rederive the result of the axial anomaly. In the self-dual case the gives us another derivation of the expression (6.13), (6.14) of $\mathrm{tr}\, F_{\alpha\beta} F_{\alpha\beta}$ - the one originally found [3] (see section 21).

We wish to evaluate the determinant of a positive definite self adjoint differential operator θ over a compact 4-dimensional manifold (eg. S^4). The (positive) eigenvalues $\lambda_1, \lambda_2, \cdots$ are discrete due to the compactness requirement. Define the corresponding generalised zeta function by

$$\zeta_\theta(s) = \sum_n \lambda_n^{-s} . \tag{1}$$

This sum can be proven to be convergent for $\mathrm{Re}(s) > 2$. If θ was just a finite dimensional matrix this sum would be finite and hence always convergent and clearly

$$\zeta_\theta(0) = \dim \theta \tag{2}$$

and

$$\zeta_\theta'(0) = -\ln \det \theta . \tag{3}$$

For general θ we can define the determinant on the right hand side of (3) to be given by these expressions using the analytic continuation of (1). The latter can be given by the integral representation of the zeta function

$$\zeta_\theta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, Tr\, \mathcal{G}(t)\, dt \tag{4}$$

where

$$\mathcal{G}(t) = exp(-t\theta) \tag{5}$$

and hence can be defined by

$$-\frac{\partial}{\partial t}\mathcal{G}(x,y,t) = \theta(x)\mathcal{G}(x,y,t) \tag{6}$$

with

$$\mathcal{G}(x,y,0) = \delta(x-y) \tag{7}$$

and the trace in (4) is the operator trace given by

$$Tr\,\mathcal{G}(t) = \int d^4x\; tr\big(\mathcal{G}(x,x,t)\big). \tag{8}$$

The tr trace in (8) is the internal (eg. group) index trace. The heat kernal $\mathcal{G}$ defined by (6), (7) is fully discussed by Gilkey [49] and also in [50,51]. From the definition (1) we see that

$$\zeta_{\theta/\mu}(s) = \mu^s \zeta_\theta(s) \tag{9}$$

for a scalar μ, thus

$$\zeta'_{\theta/\mu}(0) = ln\,\mu\,.\,\zeta_\theta(0) + \zeta'_\theta(0) \tag{10}$$

giving from the definition of the determinant (3)

$$det(\theta/\mu) = \mu^{-\zeta_\theta(0)}\,.\,det\,\theta\,. \tag{11}$$

So $\zeta_\theta(0)$ acts exactly as one would expect of a dimension so (2) is its generalised definition.

In Appendix A of [3] it is shown that the determinant of the operator $\theta=-D^2+\frac{1}{6}R$ on the sphere S^4 with radius a and scalar curvature R has the following variation under the change of radius by δa

$$-\delta\, ln\, det\,\theta = \delta\zeta'_\theta(0) = \delta\left[\frac{1}{96\pi^2}\int d^4x\left[ln\,\Omega(x).\, tr\left(F_{\alpha\beta}F_{\alpha\beta}\right)\right] - \frac{N}{45}\, ln\, a\right] \tag{12}$$

where $\Omega(x) = 2\left(1+\frac{x^2}{a^2}\right)^{-1}$ for coordinates on S^4 obtained by stereographic projection and $N = tr 1$ is the dimension of the gauge field representation (n for SU(n) and 2n for Sp(n)). The integral in (12) is convergent and vanishes in the flat space limit $a\to\infty$. The remaining term diverges but is independant of the classical solution so is cancelled if we calculate

$$\delta \ln\left(\det\theta/\det\theta_0\right) \tag{13}$$

where θ_0 is θ with $A_\alpha=0$. Since the divergent term is independant of the parameters of the instanton, the variation with respect to these is also convergent. Hence we can take the limit at the start and evaluate the variation of the determinant on flat space $\mathbb{R}^4$ which simplifies the calculation.

Section 17 - The Variation of the Determinant on Flat Space §17

We wish to calculate the flat space determinant of $\theta=-D^2$, the fundamental scalar Laplacian. The heat kernal defined in (16.6), (16.7) has the asymptotic expansion as $t\searrow 0$

$$\mathcal{G}(x,y,t)\sim\frac{1}{16\pi^2t^2}\exp\left[-\frac{1}{4t}(x-y)^2\right]\sum_{n=0}^{\infty}a_n(x,y)t^n. \tag{1}$$

By equating powers of t in the equation for $\mathcal{G}$ we obtain [52-57]

$$\begin{aligned}&(i)\quad (x-y)_\alpha D_\alpha a_0(x,y)=0\ , \quad a_0(x,x)=1\\ &(ii)\quad n\,a_n(x,y)+(x-y)_\alpha D_\alpha a_n(x,y)=D^2a_{n-1}(x,y)\qquad n\geq 1\end{aligned} \tag{2}$$

which can be solved iteratively for $a_n(x,y)$. From the integral expression (16.4) for $\zeta(s)$ we see that the value at $s=0$ and the residues of the poles at $s=1,2$ are determined by the small t behaviour of $\mathcal{G}$. Thus the asymptotic expansion (1) enables us to calculate these:

$$\begin{aligned}\mathrm{Res}_{s=2}\,\zeta(s)&=\frac{1}{16\pi^2}\int \mathrm{tr}\,a_0(x,x)\,d^4x\\ \mathrm{Res}_{s=1}\,\zeta(s)&=\frac{1}{16\pi^2}\int \mathrm{tr}\,a_1(x,x)\,d^4x\\ \zeta(0)&=\frac{1}{16\pi^2}\int \mathrm{tr}\,a_2(x,x)\,d^4x.\end{aligned} \tag{3}$$

$\zeta(s)$ is regular at $s=0$ because the $1/\Gamma(s)$ in (16.4) cancels the pole in the integral. The $a_n(x,x)$ can be easily evaluated by repeated covariant differentiation of (2) and then setting $x=y$

$$\begin{aligned}a_0(x,x)&=1\\ a_1(x,x)&=D^2a_0(x,y)\big|_{x=y}=0\\ a_2(x,x)&=\tfrac{1}{6}D^2D^2a_0(x,y)\big|_{x=y}=\tfrac{1}{12}F_{\alpha\beta}F_{\alpha\beta}.\end{aligned} \tag{4}$$

The last result uses $F_{\alpha\beta}=[D_\alpha,D_\beta]$ (2.15). Thus the residue at $s=2$ is infrared divergent, at $s=1$ it is zero and

$$\zeta(0) = \frac{1}{12.16\pi^2}\int d^4x \; tr\left(F_{\alpha\beta}F_{\alpha\beta}\right). \tag{5}$$

This is - k/12 for self-dual solutions with topological charge k defined in (3.9).

By subtracting out the pole at $s=0$ in the integral of (16.4) we can evaluate $\zeta'(0)$. It is

$$\zeta'(0) - \frac{\gamma}{16\pi^2}\int d^4x \; tr\; a_2(x,x) = \left[\int_0^\infty dt\, t^{s-1}\, Tr\left(e^{tD^2}\right) - \frac{1}{16\pi^2 s}\int d^4x \; tr\, a_2(x,x)\right]_{s=0} \tag{6}$$

where the right hand side is defined by analytic continuation to $s=0$ from sufficiently large s and

$$\gamma = \frac{d}{ds}\left(\frac{1}{s\Gamma(s)}\right)_{s=0} \tag{7}$$

is Euler's constant. e^{tD^2} is the operator form of the function $\mathcal{G}$ in (16.5). To evaluate (6) requires a complete knowledge of $\mathcal{G}(x,x,t)$ so because of this and the resolution of the infrared problems in the last section we take the variation of the parameters in D^2. We can vary (6) directly or equivalently first vary $\zeta(s)$ which gives us a better understanding of why it works.

Since $a_0(x,x)=1$ its variation is zero so the residue of $\delta\zeta(s)$ at $s=2$ vanishes by (3) so $\delta\zeta(s)$ is regular everywhere. Also since the action in (5) is stationary under arbitrary variations about a solution of the equations of motion, we have $\delta\zeta(0)=0$ from (5). Thus in

$$\delta\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s}\, Tr\left[e^{tD^2}\delta D^2\right] \tag{8}$$

the integral is regular at $s=0$. Hence the variation of the determinant given by differentiating (8) (or varying (6)) is given by

$$\delta\zeta'(0) = \int_0^\infty dt \;\; t^s \; Tr\left[e^{tD^2}\delta D^2\right]\Big|_{s=0} \tag{9}$$

where again we mean analytic continuation to $s=0$. However since $\delta\zeta(s)$ is regular we can replace this by the limit as $s\searrow 0$. With $s=0$ (9) can be integrated to give formally

$$\left[\; Tr\; e^{tD^2} G\, \delta D^2\right]_{t=0}^{t=\infty} \tag{10}$$

where the Green function G is the inverse of D^2 (8.1)

$$D^2 G(x,y) = \delta(x-y). \tag{11}$$

The upper limit $t=\infty$ vanishes as the heat kernal e^{tD^2} vanishes (at least on S^4). The lower limit of integration $t=0$ gives $-\mathrm{Tr}(G\delta D^2)$ which is the right hand side of (1.1) but it is ill defined. However we will show that when the integrations implied in (10) are done first then the limit $t\searrow 0$ exists so giving the regularised definition of (1.1). More rigourously one can integrate (9) by parts for large s and analytically continue:

$$\delta\zeta'(0) = -s\int_0^\infty dt\, t^{s-1}\,\mathrm{Tr}\left[e^{tD^2} G\,\delta D^2\right]\Big|_{s=0}. \tag{12}$$

Since

$$\delta D^2 = D_\alpha \delta A_\alpha + \delta A_\alpha D_\alpha \tag{13}$$

we can use the cyclic property of the Tr which also holds for differential operators due to integration by parts and the fact that the functions occurring vanish sufficiently rapidly at infinity. Thus we may rewrite (12) as

$$\delta\zeta'(0) = -s\int_0^\infty dt\, t^{s-1}\,\mathrm{Tr}\left[e^{tD^2}\delta A_\alpha \vec{D}_\alpha G + e^{tD^2} G\,\overleftarrow{D}_\alpha \delta A_\alpha\right]\Big|_{s=0} \tag{14}$$

where, as in (9.19)

$$\begin{aligned} \vec{D}_\alpha G(x,y) &= \frac{\partial}{\partial x_\alpha} G(x,y) + A_\alpha(x)\, G(x,y) \\ G(x,y)\overleftarrow{D}_\alpha &= -\frac{\partial}{\partial y_\alpha} G(x,y) + G(x,y)\, A_\alpha(y). \end{aligned} \tag{15}$$

We now see that the variation is much simpler to calculate as all that is required is the residue of the pole at $s=0$ of

$$-\int dt\, t^{s-1}\,\mathrm{Tr}\left[\delta A_\alpha\left(\vec{D}_\alpha G\, e^{tD^2} + e^{tD^2} G\,\overleftarrow{D}_\alpha\right)\right]. \tag{16}$$

This pole is determined solely by the constant term in the asymptotic expansion of the Tr part as $t\searrow 0$. In fact as stated after (11) the limit $t\to 0$ of this part exists since the coefficients of the terms $1/t^{3/2}$, $1/t$ and $1/t^{1/2}$ which could occur, turn out to vanish. So we wish to calculate

$$-\int d^4x\, d^4y\; \mathrm{tr}\left[\delta A_\alpha(x)\left(\vec{D}_\alpha G(x,y)\,\mathcal{G}(y,x,t) + \mathcal{G}(x,y,t)\, G(y,x)\overleftarrow{D}_\alpha\right)\right] \tag{17}$$

as $t\searrow 0$. To do this we notice that as $x\to y$

$$G(x,y) \sim -\frac{1}{4\pi^2(x-y)^2} \tag{18}$$

which is what gives rise to the potentially singular terms in (17). So we split the Green function into singular and non-singular parts

$$G(x,y) = -\frac{1}{4\pi^2}\left(\frac{\Phi(x,y)}{(x-y)^2} + R(x,y)\right) \tag{19}$$

where Φ and R are non-singular as $x\to y$ and each have the same gauge transformation property as G. For a special choice of Φ the contribution from the singular part of (19) in (17) vanishes. This choice is $a_0(x,y)$ - the lowest order term in the expansion of the heat kernal (1) and is defined by (2i). The solution is

$$\Phi(x,y) = a_0(x,y) = \mathcal{P}\exp\left(\int_x^y A_\alpha dx_\alpha\right) \tag{20}$$

where $\mathcal{P}$ denotes path ordering of the A_α matrices along the straight line joining x to y. One can check that with this Φ, R in (19) is indeed non-singular.

To show the singular part of (17) vanishes we evaluate

$$\int d^4y\ \vec{D}_\alpha\left(\frac{\Phi(x,y)}{(x-y)^2}\right)\mathcal{G}(y,x,t). \tag{21}$$

By considering an expansion of the integrand of (21) in powers of $(x-y)$ (it is clear that for small t only x close to y contributes due to the $\exp-(x-y)^2/4t$ in $\mathcal{G}$ of (1)) we see that the only integrals arising are of the form

$$\frac{1}{t^2}\int d^4z\, z_{\alpha_1} z_{\alpha_2}\cdots z_{\alpha_N}(z^2)^{-M}e^{-z^2/4t}. \tag{22}$$

For $N/2-M>-2$ this integral is convergent. For N odd it vanishes and for N even it is proportional to $t^{N/2-M}$. The lowest power of t arising in this way is $t^{-3/2}$ with $N=1$ and $M=2$ coming from the $\partial_\alpha 1/(x-y)^2$. Thus only the first two terms in the t expansion of $\mathcal{G}(x,y,t)$ will contribute as $t\searrow 0$. So we need only consider the

$$\frac{1}{16\pi^2t^2}\int d^4y\, D_\alpha\left[\frac{\Phi(x,y)}{(x-y)^2}\right]\left(\Phi(y,x) + t\,a_1(y,x)\right)e^{-(x-y)^2/4t} \tag{23}$$

part of (21). Again covariantly differentiating (2) and setting $x=y$ we see that

$$\begin{aligned} a_1(x,x) &= D^2\Phi\big|_{x=y} = 0 \\ D_\alpha a_1(x,y)\big|_{x=y} &= \tfrac{1}{2}D_\alpha D^2\Phi\big|_{x=y} = -\tfrac{1}{6}\left[D_\beta, F_{\beta\alpha}\right]. \end{aligned} \tag{24}$$

Provided the equations of motion are satisfied the latter vanishes and hence

$$\partial_\alpha a_1(x,y)\big|_{x=y} = \left[D_\alpha a_1(x,y) - A_\alpha \cdot a_1(x,y)\right]_{x=y} = 0 . \tag{25}$$

Thus $a_1(x,y) = O(x-y)^2$ as $x \to y$ so the lowest power of t it can contribute is $t^{1/2}$ which vanishes as $t \to 0$. We are left only with

$$\frac{1}{16\pi^2 t^2}\int d^4y\, e^{-(x-y)^2/4t}\left[\frac{1}{(x-y)^2}\, D_\alpha \Phi(x,y).\Phi(y,x) - \frac{2(x-y)_\alpha}{(x-y)^2}\,\Phi(x,y).\Phi(y,x)\right] \tag{26}$$

from (23). The second term vanishes as by (20)

$$\Phi(x,y)\Phi(y,x) = 1 \tag{27}$$

so the integral is of the form (22) with $N=1$ and $M=2$ so is convergent and is zero. The first term of (26) is expanded about $x=y$ using the fact that

$$\partial_\alpha (BC) = B\vec{D}_\alpha C - B\overleftarrow{D}_\alpha C \tag{28}$$

for any B, C. The zeroth order term of $D_\alpha\Phi(x,y).\Phi(y,x)$ vanishes as

$$D_\alpha \Phi(x,y)\big|_{x=y} = 0 . \tag{29}$$

The first order term, by (28) is

$$\frac{\partial}{\partial y_\alpha}\left[\vec{D}_\beta \Phi(x,y).\Phi(y,x)\right]_{x=y} = (\vec{D}_\beta \Phi)(\vec{D}_\alpha \Phi) - (\vec{D}_\beta \Phi \overleftarrow{D}_\alpha)\Phi\Big|_{x=y} = -\tfrac{1}{2}F_{\alpha\beta} . \tag{30}$$

The last step of (30) and also (29) were again obtained by differentiating (2) and setting $x=y$. The second order term is

$$\frac{\partial}{\partial y_\alpha}\frac{\partial}{\partial y_\beta}\left[\vec{D}_\gamma \Phi(x,y).\Phi(y,x)\right]_{x=y} = +\vec{D}_\gamma \Phi\,\overleftarrow{D}_\alpha \overleftarrow{D}_\beta \Phi\Big|_{x=y} \tag{31}$$

due to (28), (29) and by further differentiations of (2) becomes

$$\begin{aligned}\tfrac{1}{2}\left(D_\gamma D_\alpha D_\beta \Phi + D_\gamma D_\beta D_\alpha \Phi\right)_{x=y} &= -\tfrac{1}{6}\left(\left[D_\gamma, F_{\alpha\beta}\right] + \left[D_\alpha, F_{\gamma\beta}\right] + \left[D_\gamma, F_{\beta\alpha}\right] + \left[D_\beta, F_{\gamma\alpha}\right]\right) \\ &= -\tfrac{1}{6}\left[D_\alpha, F_{\gamma\beta}\right] - \tfrac{1}{6}\left[D_\beta, F_{\gamma\alpha}\right] .\end{aligned} \tag{32}$$

Then using these the Taylor expansion is

$$D_\alpha\Phi(x,y).\Phi(y,x) = \tfrac{1}{2}F_{\alpha\beta}(x)(x-y)_\beta - \tfrac{1}{6}\left[D_\beta, F_{\alpha\gamma}(x)\right](x-y)_\beta (x-y)_\gamma + O(|x-y|^3) . \tag{33}$$

The first term of (33) does not contribute as the integral over y_α in (26) vanishes. The second term becomes $[D_\beta, F_{\alpha\beta}]$ as

$$\frac{1}{16\pi^2 t^2}\int d^4z\, \frac{z_\beta z_\gamma}{z^2}\, e^{-z^2/4t} = \tfrac{1}{4}\delta_{\beta\gamma} \tag{34}$$

so it also vanishes due to the equations of motion. Higher order terms in (33) give contributions which vanish as $t \to 0$, so confirming that the singular part of (19) gives no contribution to (17).

We thus see that the only non-zero part of (17) comes from the non-singular $R(x,y)$ term of (19). Using

$$\frac{1}{16\pi^2 t^2}\int d^4z\, e^{-z^2/4t} = 1 \tag{35}$$

its contribution in (17) is

$$-\delta \ln \det(-D^2) = \delta \zeta'(0) = \int d^4x \; tr\left[\delta A_\alpha(x) \, J_\alpha(x)\right] \tag{36}$$

where

$$J_\alpha(x) = \frac{1}{4\pi^2}\left[\vec{D}_\alpha R(x,y) + R(x,y)\overleftarrow{D}_\alpha\right]_{x=y} . \tag{37}$$

This is exactly the result of Brown and Creamer [15] which they derived from (1.1) in a somewhat ad hoc manner. The result holds provided A_α satisfies the equations of motion, for arbitrary variations δA_α away from such a solution. In particular it holds for self-dual A_α and J_α is evaluated for these in the following section. The special variations δA_α for changes in the parameters of the self-dual solutions were given in section 11.

Section 18 - The Variation of the Self-dual Determinant §18

Since we know the Green function for the self-dual case (8.4) all that we need to calculate $R(x,y)$ which occurs in the current (37) is the path ordered exponential $\Phi(x,y)$. This does not have a simple closed form like the Green function G but all that is required for the calculation of J_α is the expansion of Φ as a power series in $(x-y)$ up to cubic order. The path ordered exponential can be evaluated directly from (17.20) by using

$$\Phi(x,y) = P\exp\int_x^y A_\alpha dx_\alpha = \lim_{\lambda_{i+1}-\lambda_i \to 0}\left[v^+(x_0)P(x_1)P(x_2)\cdots P(x_{n-1})v(x_n)\right]$$
$$x_i = \lambda_i x + (1-\lambda_i)y \quad , \quad 0=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n=1 \tag{1}$$

One can then show that it necessarily has the form

$$\Phi(x,y) = v^+(x)\left[1 + (x-y)^2 b H(x,y) b^+\right]v(y) \tag{2}$$

some $H(x,y)$. More simply [3] one can use (2) as an ansatz and solve the defining equation (17.2)

$$(x-y)_\alpha \vec{D}_\alpha \Phi(x,y) = 0 = \Phi(x,y)\overleftarrow{D}_\alpha (x-y)_\alpha \tag{3}$$

for $H(x,y)$ as a power series in $(x-y)$. The required function is then

$$R(x,y) = -v^+(x) \, b H(x,y) b^+ v(y). \tag{4}$$

Writing $h(x,y) = (x-y)^2 H(x,y)$ and substituting (2) in (3) we obtain

$$(x-y)_\alpha \frac{\partial}{\partial x_\alpha} h(x,y) = (x-y)^2 f(x) + (x-y) f(x) \Delta^+(x) b\, h(x,y)$$
$$(x-y)_\alpha \frac{\partial}{\partial y_\alpha} h(x,y) = -(x-y)^2 f(x) + h(x-y) b^+ \Delta(y) f(y) (x-y)^+. \tag{5}$$

Writing $\bar{x} = \frac{1}{2}(x+y)$ and $z = \frac{1}{2}(x-y)$ and taking the difference of the two equations in (5) gives

$$z_\alpha \frac{\partial}{\partial z_\alpha} h = 2z^2\left[f(\bar{x}+z) + f(\bar{x}-z)\right] + z f(\bar{x}+z)\Delta^+(\bar{x}+z) b h - h b^+ \Delta(\bar{x}-z) f(\bar{x}-z) z^+ \tag{6}$$

which can be solved iteratively for h as a Taylor series in z starting from $h(x,x) = 0$. The solution is

$$h = 2z^2\left[f(\bar{x}) + \tfrac{1}{3} f(\bar{x})\left\{z\Delta^+(\bar{x}) b - b^+\Delta(\bar{x}) z^+\right\} f(\bar{x})\right] + O(z^4) \tag{7}$$

which gives for R

$$R(x,y) = -\tfrac{1}{2} v^+(x) b\left[f(\bar{x}) + \tfrac{1}{6} f(\bar{x})\left((x-y)\Delta^+(\bar{x}) b - b^+\Delta(\bar{x})(x-y)^+\right) f(\bar{x})\right] b^+ v(y) + O(x-y)^2. \tag{8}$$

Using this in the expression for the current (17.37) it is easy to show that

$$J_\alpha = \frac{1}{12\pi^2} v^+ b f\left(e_\alpha \Delta^+ b - b^+\Delta e_\alpha^+\right) f b^+ v. \tag{9}$$

It is also straightforward to show that this is covariantly conserved as indeed one would expect [15]

$$\mathcal{D}_\alpha J_\alpha \equiv \left[D_\alpha, J_\alpha\right] = 0 . \tag{10}$$

In the expression (11.22) for the variation of the gauge potential under instanton parameter variations

$$\delta A_\alpha = v^+\left(\delta\Delta f e_\alpha^+ b^+ - b e_\alpha f \delta\Delta^+\right) v + \mathcal{D}_\alpha \delta u \tag{11}$$

the term δu that corresponds to a gauge transformation will not contribute in the determinant. Integrating by parts gives

$$\int d^4x\, \mathrm{tr}\left[(\mathcal{D}_\alpha \delta u) J_\alpha\right] = -\int d^4x\, \mathrm{tr}\left[\delta u\left(\mathcal{D}_\alpha J_\alpha\right)\right] = 0 \tag{12}$$

by (10) and $J_\alpha = O(1/|x|^5)$ as $|x|\to\infty$, hence the non-zero part comes from

$$\mathrm{tr}\left(\delta A_\alpha J_\alpha\right) = \frac{1}{12\pi^2} \mathrm{tr}\left[P b f\left(e_\alpha \Delta^+ b - b^+\Delta e_\alpha^+\right) f b^+ P\left(\delta\Delta f e_\alpha^+ b^+ - b e_\alpha f \delta\Delta^+\right)\right] . \tag{13}$$

It is possible to write this entirely in terms of P using the results from section 8 of

$$\begin{aligned} e_\alpha^+ b^+ P &= (\partial_\alpha \Delta^+) P = -\Delta^+ \partial_\alpha P \\ (\partial_\alpha P) P &= -\Delta e_\alpha^+ f b^+ P \\ P \delta P &= -P \delta\Delta f \Delta^+ \\ P \partial_\alpha (P \partial_\alpha P) &= -4 P b f b^+. \end{aligned} \tag{14}$$

This immediately gives

$$\frac{1}{48\pi^2} \mathrm{tr}\left\{P\left(\partial_\beta(P\partial_\beta P)\partial_\alpha P - \partial_\alpha P \partial_\beta(\partial_\beta P . P)\right) P\left[\delta P, \partial_\alpha P\right]\right\} \tag{15}$$

which can be simplified using some relations obtained by differentiating $P^2 = P$.

$$
\begin{aligned}
&P\delta P + \delta P.P = \delta P \quad , \quad P.\delta P.P = 0 \\
&P.\partial_\alpha P + \partial_\alpha P.P = \partial_\alpha P \quad , \quad P.\partial_\alpha P.P = 0 \\
&2\partial_\alpha P.\partial_\alpha P + \partial^2 P.P + P.\partial^2 P = \partial^2 P \\
&2P.\partial_\alpha P.\partial_\alpha P + P.\partial^2 P.P = 0 \ .
\end{aligned}
\tag{16}
$$

The terms in (15) where there is no second derivative of P vanish since $P.\partial_\beta P.\partial_\beta P.\partial_\alpha P.P = -\frac{1}{2}P.\partial^2 P.P.\partial_\alpha P.P$ is zero by (16). The remaining second derivative terms of (15) can be simplified using

$$
\begin{aligned}
\partial_\alpha P.P.\partial_\alpha P &= \partial_\alpha P \partial_\alpha P(1-P) \\
&= \tfrac{1}{2}(\partial^2 P - \partial^2 P.P - P\partial^2 P)(1-P) \\
&= \tfrac{1}{2}(1-P)\partial^2 P(1-P)
\end{aligned}
\tag{17}
$$

to give as a final result

$$
-\delta \ln \det(-D^2) = \zeta'(0) = -\frac{1}{96\pi^2}\int d^4x \ \mathrm{tr}\left[\delta P\left(\partial^2 P(1-P)\partial^2 P - 2\partial_\alpha P.\partial^2 P.\partial_\alpha P\right)\right] \tag{18}
$$

which is exactly as independantly obtained by Belavin et. al. [20].

The remaining problem is to integrate up the expression (18). Some attempts to do this by Osborn [23] and Berg and Lüscher [24] are mentioned in the conclusion. However in the case of those variations corresponding to conformal transformations it is possible to integrate up the variation of the determinant completely. This is done in the following section.

§19

Section 19 - The Determinant Variation under Conformal Transformations

We wish to calculate the variation of the determinant of the Laplacian where the gauge potential is replaced by its conformally transformed potential $A_\alpha(x) \to A'_\alpha(x)$. The new covariant derivative $D'_\alpha(x)$ can be related to the old using (10.11)

$$
D'_\alpha(x) \equiv \frac{\partial}{\partial x_\alpha} + A'_\alpha(x) = \frac{\partial x'_\beta}{\partial x_\alpha}\left(\frac{\partial}{\partial x'_\beta} + A_\beta(x')\right) = \frac{\partial x'_\beta}{\partial x_\alpha} D_\beta(x') \ . \tag{1}
$$

Thus the Laplacian becomes

$$
\begin{aligned}
(D'(x))^2 &= \frac{\partial x'_\beta}{\partial x_\alpha} D_\beta(x') \frac{\partial x'_\gamma}{\partial x_\alpha} D_\gamma(x') \\
&= \omega(x')^{-3} D^2(x')\omega(x')
\end{aligned}
\tag{2}
$$

where the latter result can easily be checked using the relationship between x and x' (10.7) and the expression for ω (10.10). Since in the determinant x and x' are dummy variables (2) gives

$$\det(-D'^2) = \det(-\omega^{-3}D^2\omega)\,. \tag{3}$$

For an infinitesimal conformal transformation (10.7) with

$$\alpha = 1+\delta\alpha\ , \quad \beta=\delta\beta\ , \quad \gamma=\delta\gamma\ , \quad \phi = 1+\delta\phi \tag{4}$$

we have from (10.10)

$$\omega = 1+\delta\omega\ , \quad \delta\omega = \tfrac{1}{2}\,tr(\delta\phi-\delta\alpha) + tr(x\delta\gamma) \equiv \sigma + \tau_\alpha x_\alpha\,. \tag{5}$$

Hence

$$\begin{aligned}\delta\zeta(s)\big|_{D'^2} &= \delta\zeta(s)\big|_{\omega^{-1}D^2\omega} = \delta\,\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,Tr\, e^{t\omega^{-3}D^2\omega} \\ &= \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s}\,Tr\left[e^{tD^2}(D^2\delta\omega - 3\delta\omega\, D^2)\right] = -\frac{2}{\Gamma(s)}\int_0^\infty dt\, t^s\,Tr\left[e^{tD^2}D^2\delta\omega\right] \\ &= \frac{2s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,Tr\left[e^{tD^2}\delta\omega\right]\end{aligned} \tag{6}$$

where in the final step we have integrated by parts for large s. Thus $\delta\zeta'(0)$ is given by the residue of the pole at $s=0$ of

$$2\int_0^\infty dt\, t^{s-1}\,tr\left[\mathcal{G}(x,x,t)\delta\omega(x)\right]. \tag{7}$$

Using the asymptotic expansion of $\mathcal{G}$ (17.1), as in section 17 only the $a_2(x,x)$ term appears and by the result (17.4) the residue gives [17, 18,19]

$$\delta\zeta'(0) = \frac{1}{96\pi^2}\int tr(F_{\alpha\beta}F_{\alpha\beta})\delta\omega(x)\, d^4x\,. \tag{8}$$

Since by (5) $\delta\omega$ is linear in x we can make use of the fact (6.13), (6.14) that $\frac{1}{2}tr\,F_{\alpha\beta}F_{\alpha\beta} = \partial^2 T$ to integrate (8) by parts twice.

$$\begin{aligned}\frac{1}{96\pi^2}\int tr(F_{\alpha\beta}F_{\alpha\beta})\delta\omega\, d^4x &= \frac{1}{48\pi^2}\int \partial^2 T\,(\sigma+\tau_\alpha x_\alpha) \\ &= \frac{1}{48\pi^2}\lim_{R\to\infty}\int_{|x|=R} dS_\beta\left[\partial_\beta T.(\sigma+\tau_\alpha x_\alpha) - T\tau_\beta\right].\end{aligned} \tag{9}$$

Since b^+Pbf in T (6.14) decreases like $|x|^{-4}$ at infinity it gives no contribution in the limit. The other term of T is asymptotically

$$b^+bf = 1\,|x|^{-2} - (x^+b^+a + a^+bx)(b^+b)^{-1}|x|^{-4} + O(|x|^{-4}) \tag{10}$$

which after integration in (9) gives

$$\begin{aligned}\delta\zeta'(0) &= \frac{1}{24}\,tr\left[(b^+a\tau + \tau^+a^+b)(b^+b)^{-1}\right] - \frac{k\sigma}{6} \\ &= \frac{1}{12}\,tr\left[(b^+a\delta\gamma + \delta\gamma^+a^+b)(b^+b)^{-1}\right] - \frac{k}{12}\,tr(\delta\phi-\delta\alpha) \\ &= \frac{1}{12}\,tr\left[\delta(b^+b)(b^+b)^{-1}\right] - \frac{k}{12}\,tr(\delta\phi+\delta\alpha)\,.\end{aligned} \tag{11}$$

The last step uses the conformal variation of b^+b coming from (10.12) with (4)

$$\begin{aligned}\delta(b^+b) &= (a\delta\gamma + b\delta\alpha)^+b + b^+(a\delta\gamma + b\delta\alpha) \\ &= b^+a\,\delta\gamma + \delta\gamma^+a^+b + (\delta\alpha^+ + \delta\alpha)\,b^+b.\end{aligned} \tag{12}$$

Now it is trivial to write (11) as a total variation by the fact that under the transformation (4) κ^2 given by (10.8) has the variation

$$\delta\kappa^2 = \delta(\alpha^2\phi^2) = tr(\delta\alpha + \delta\phi). \tag{13}$$

Hence

$$\delta\zeta'(0) = \tfrac{1}{12}\,\delta\, tr\ln b^+b - \tfrac{k}{12}\,\delta\kappa^2 . \tag{14}$$

The $tr\ln$ we can rewrite as $\ln\det$ and we can also absorb the κ term inside the $\ln$ as $\delta\ln\kappa^2 = \delta\kappa^2/\kappa^2 = \delta\kappa^2$ as $\kappa = 1$ initially.

$$\delta\zeta'(0) = \delta\,\tfrac{1}{12}\ln\left[\det(\nu)^2 . \kappa^{-2k}\right] \tag{15}$$

where $b^+b = \nu 1_2$ and ν is a $k\times k$ hermitian matrix.

The expression (15) tells us how the determinant varies under conformal transformations. However since the $A_\kappa(x)$ in the determinant is expressible entirely in terms of (a, b) and is invariant under the transformations (7.5) $(a,b)\to K(a,b)L$, one would expect the determinant from (15) to be so expressible. Thus

$$\zeta'(0) = \tfrac{1}{12}\ln\left[\det(\nu)^2\,\lambda(a,b)\right] \tag{16}$$

where $\lambda(a,b)$ is chosen so that $\zeta'(0)$ has the correct conformal transformation properties as given by (15) and makes it invariant under the algebraic transformations (7.5). Hence we need

$$\begin{aligned}\lambda(KaL, KbL) &= (\det K)^{-4}.\,\lambda(a,b)\\ \lambda(a', b') &= \kappa^{-2k}\,\lambda(a,b)\end{aligned} \tag{17}$$

with a', b' given by (10.12). To find such a function we notice that the matrix (13.14)

$$M^{-1}_{ij,kl} = (a^+a)_{ik}(b^+b)_{jl} + (b^+b)_{ik}(a^+a)_{jl} - tr\left[(a^+b)_{ik}(b^+a)_{lj}\right] \tag{18}$$

which was introduced for the construction of the adjoint Green function, has the transformation properties

$$M^{-1} \to (K^+\otimes K^+)\,M^{-1}(K\otimes K) \tag{19}$$

under (7.5) and

$$M^{-1} = \kappa^2 M^{-1} \tag{20}$$

under (10.12) (see result mentioned after (13.19)). Hence the determinant of M transforms as respectively

$$\begin{aligned}&\det M \to (\det K)^{-4k}\det M\\ \text{and}\quad &\det M \to \kappa^{-2k^2}\det M .\end{aligned} \tag{21}$$

So a possible choice for $\lambda(a,b)$ is $(\det M)^{1/k}$ giving

$$\delta\zeta'(0) = \frac{1}{12k}\,\delta \ln \det\left[(b^+b \otimes b^+b)M\right]. \tag{22}$$

(The matrices a^+a and b^+b in (18) to (22) are all considered as the k x k hermitian matrices μ and ν respectively - see (5.14)). Using the result (16.11) that

$$\det(-D^2/\mu^2) = \mu^{-2\zeta_{-D^2}(0)}.\det(-D^2) \tag{23}$$

with $\zeta_{-D^2}(0) = -\frac{1}{12}k$ from (17.5) we have

$$\delta \det(-D^2/\mu^2) = \delta \det\left[\mu^{-2}(b^+b \otimes b^+b)M\right]^{-1/12k} \tag{24}$$

or by the result quoted in section 16 that the ratio of determinants (16.13) is infrared finite

$$\frac{\det(-D_k^2)}{\det(-D_0^2)} = \det\left[(b^+b \otimes b^+b)M\right]^{-1/12k} C. \tag{25}$$

C is some unknown function of (a,b) which is invariant under both conformal transformations and the algebraic transformations (7.5). It must be determined so that the resulting $\zeta'(0)$ has the correct variations of (18.18) under general parameter variations. This we do not know how to do yet. C also contains a numerical factor (which could depend on k and the group) which is independant of the parameters of the instanton so is not calculable by these methods, but in the conclusion we mention how Berg and Lüscher [24] have calculated it by using the complete answer for the SU(2) one-instanton case [5].

Section 20 - Conformal Behaviour of Det($-D^2$) for the JNR Instanton §20

In the case of the 't Hooft or JNR solutions of section 10 we can explicitly evaluate the conformally non-invariant part (19.25) of $\det(-D^2)$ [3]. Since this will involve the evaluation of $\det M$ we will be able to show explicitly that M is non-singular precisely when $\Delta^+\Delta$ or f is non-singular for all x. Using the solutions for (a,b) given in (10.18) we have

$$(b^+b)_{ij} = \delta_{ij} + \frac{\lambda_i\lambda_j}{\lambda_0^2} \tag{1}$$

and

$$M^{-1}_{ij,rs} = \left(\delta_{ir} y_i^2 + \frac{\lambda_i\lambda_r}{\lambda_o^2} y_o^2\right)\left(\delta_{js} + \frac{\lambda_j\lambda_r}{\lambda_o^2}\right) + \left(\delta_{ir} + \frac{\lambda_i\lambda_r}{\lambda_o^2}\right)\left(\delta_{js} y_j^2 + \frac{\lambda_j\lambda_s}{\lambda_o^2} y_o^2\right)$$
$$- tr_2\left(\delta_{ir} y_i^+ + \frac{\lambda_i\lambda_r}{\lambda_o^2} y_o^+\right)\left(\delta_{js} y_j + \frac{\lambda_j\lambda_s}{\lambda_o^2} y_o\right) \qquad 1 \le i,j,r,s \le k \tag{2}$$
$$= \delta_{ir}\delta_{js}(y_i - y_j)^2 + \frac{\delta_{ir}\lambda_j\lambda_s}{\lambda_o^2}(y_o - y_i)^2 + \frac{\delta_{js}\lambda_i\lambda_r}{\lambda_o^2}(y_o - y_j)^2 .$$

Introducing

$$s_{ij} = (y_i - y_j)^2 \lambda_i^{-2} \lambda_j^{-2} \quad , \; 0 \le i,j \le k , \tag{3}$$

(2) can be written

$$M^{-1}_{ij,rs} = \lambda_i\lambda_j\lambda_r\lambda_s\left[s_{oi}\delta_{ir} + s_{oj}\delta_{js} + s_{ij}\delta_{ir}\delta_{js}\right]. \tag{4}$$

From (19.25) we wish to calculate $\det(b^+b)^{2k}\det M$. Since in the gauge potential $\lambda_o,\ldots,\lambda_k$ and $y_o,\ldots,y_k$ occur symmetrically (see (10.21) and (10.22)) they must also do so in this expression. Though not immediately apparent one can check this is so by performing appropriate transformations $(a,b)\to(a,b)L$. This symmetry is useful in determining the form of the result. The first factor is easily shown to be (cf. the coefficient of the highest power of x in (10.27))

$$\det(b^+b)^{2k} = \lambda_o^{-4k}\left(\sum_{i=0}^{k}\lambda_i^2\right)^{2k}. \tag{5}$$

$\operatorname{Det} M$ is rather more difficult. Defining

$$S_{ij,rs} = s_{oi}\delta_{ir} + s_{oj}\delta_{js} + s_{ij}\delta_{ir}\delta_{js} \tag{6}$$

then

$$\det M^{-1} = \lambda_o^{-4k}\left(\prod_{i=0}^{k}\lambda_i^2\right)^{2k}\det S \tag{7}$$

where it is written so the above mentioned symmetry is apparent. To evaluate $\det S$ we use

$$\delta \ln\det S = tr\left(S^{-1}\delta S\right). \tag{8}$$

To calculate S^{-1} we use the methods of Brown et. al. [36] where they effectively calculated M for the 't Hooft solution. The result is

$$S^{-1}_{ij,rs} = t_{ij}\delta_{ir}\delta_{js} - \tfrac{1}{2}t_{ij}\delta_{rs}\left(\delta_{jr}+\delta_{ir}\right) - \tfrac{1}{2}t_{rs}\delta_{ij}\left(\delta_{is}+\delta_{ir}\right)$$
$$+ \tfrac{1}{2}\delta_{ij}\delta_{rs}t_{ir} + \tfrac{1}{2}\delta_{ij}\delta_{rs}\delta_{ir}\sum_{l=0}^{k}t_{il} \tag{9}$$
$$- \tfrac{1}{2}t_{ij}t_{rs}\left(m_{ir} + m_{js} - m_{is} - m_{jr}\right)$$

where

$$t_{ij} = \begin{cases}(s_{ij})^{-1} & , \; i \ne j \\ 0 & , \; i = j\end{cases} \qquad 0 \le i,j \le k \tag{10}$$

and

$$(m^{-1})_{ij} = p_{ij} \equiv \delta_{ij}\sum_{l=0}^{k}t_{il} - t_{ij} \quad , \; 1 \le i,j \le k \tag{11}$$

(ie. m is the matrix inverse of ρ). It can be verified that the product of the matrices in (6) and (9) give unity. Using the expression (9) for S^{-1} and the variation of (6) we obtain

$$tr(S^{-1}\delta S) = \sum_{1\le i,j\le k} t_{ij}\,\delta s_{ij} + 2\sum_{1\le i\le k} t_{0i}\,\delta s_{0i} - \sum_i\Big(m_{ii}\sum_{l=0}^{k}(t_{il})^2\,\delta s_{il}\Big) + \sum_{i,j} m_{ij}\,(t_{ij})^2\,\delta s_{ij}\,. \tag{12}$$

By varying the expression (11) for ρ we see that (12) becomes

$$\delta \ln \det S = 2\sum_{0\le i<j\le k}(s_{ij})^{-1}\,\delta s_{ij} + \sum_{1\le i,j\le k} m_{ij}\,\delta\rho_{ij}\,. \tag{13}$$

Integrating up the variation gives

$$\det S = 2^k \det\rho \prod_{0\le i<j\le k}(s_{ij})^2 \tag{14}$$

where the integration constant of 2^k is found by comparing the term $\prod_{1\le i<j\le k}(s_{ij})^2\prod_{i=1}^{k}s_{0i}$ which only comes from the main diagonal of $\det S$ and $\det\rho$. From the result so far it is clear that $\det\rho$ must also have the permutation symmetry.

Determinants of the form of $\det\rho$ have been studied by Sylvester [58] and Borchard [59]. It can be shown that in $\det\rho$ there are $(k+1)^{k-1}$ terms each with coefficient plus 1 which are products of k t_{ij}'s

$$t_{i_1j_1}\cdot t_{i_2j_2}\cdots t_{i_kj_k} \tag{15}$$

such that each index from 0 to k occurs at least once and there is no factor which can be written in cyclic form

$$t_{i_1i_2}\cdot t_{i_2i_3}\cdots\cdots t_{i_ri_1}\,. \tag{16}$$

From (7), (14) and the construction of $\det\rho$ in which all the t_{ij} are non-negative we can see that $\det M^{-1}$ is non-singular if and only if all the $\lambda_i, 0\le i\le k$ and the $s_{ij}, 0\le i<j\le k$ are non-zero. This is exactly the condition (10.28) that $\Delta^+\Delta$ is non-singular. Finally the conformally non-invariant factor of $\det(-D^2)$ in (19.25) is

$$\left[2^k\Big(\sum_{i=0}^{k}\lambda_i^2\Big)^{-2k}\Big(\prod_{i=0}^{k}\lambda_i^2\Big)^{k}\prod_{0\le i<j\le k}(y_i-y_j)^4\cdot\det\rho\right]^{1/12k} \tag{17}$$

with $\det\rho$ given by the above construction.

Section 21 - The Axial Anomaly and the Index Theorem §21

In (6.13) and (6.14) we gave an expression for $tr(F_{\alpha\beta}F_{\alpha\beta})$ for self-dual solutions. We now present how it was originally found by considering the axial anomaly (see section 15 for an alternative derivation). In general the axial current is defined by suitably regularising

$$J^5_\alpha(x) = tr\left(\gamma^5\gamma_\alpha\, S(x,x)\right) \tag{1}$$

where $S(x,y)$ is the fundamental fermion Green function defined by

$$\gamma_\alpha D_\alpha\, S(x,y) = \delta(x-y) - \Pi(x,y). \tag{2}$$

Π is the projection operator onto the Dirac zero modes - $\Pi(x,y) = \sum_i \psi_i(x)\psi_i^\dagger(y)$ where ψ_i is a complete orthonormal set of solutions to $\gamma\cdot D\psi = 0$. Defining the Green function G_F by

$$-(\gamma\cdot D)^2\, G_F(x,y) = \delta(x-y) - \Pi(x,y) \tag{3}$$

we can express the fermion Green function as

$$S(x,y) = -\gamma\cdot D\, G_F(x,y). \tag{4}$$

To define the axial current from (1) we regulate G_F by

$$G_F(x,y) = \frac{1}{\Gamma(s)}\int_0^\infty dt\; t^{s-1}\left(e^{t(\gamma D)^2} - \Pi\right)\Big|_{s=1} \tag{5}$$

where as in section 17 we analytically continue from sufficiently large s. Also

$$e^{t(\gamma D)^2} - \Pi = \mathcal{G}_F(x,y,t) \tag{6}$$

solves the system

$$\left.\begin{aligned} (\gamma\cdot D)^2\,\mathcal{G}_F &= \frac{\partial}{\partial t}\mathcal{G}_F \\ \mathcal{G}_F(x,y,0) &= \delta(x-y) - \Pi(x,y) \end{aligned}\right\}. \tag{7}$$

That (5) is a sensible definition can be seen by expanding $\mathcal{G}_F$ in a complete set of eigenfunctions of $(\gamma\cdot D)^2$ and then (5) gives the correct expansion of G_F. From (5) and (4) we can define the regularised axial current of (1) by [60]

$$J^5_\alpha(x) = -\frac{1}{\Gamma(s)}\int_0^\infty dt\; t^{s-1}\; tr\left[\gamma^5\gamma_\alpha\,\gamma\cdot D\,\mathcal{G}_F(x,y,t)\right]_{x=y}\Big|_{s=1}. \tag{8}$$

Using $\frac{\partial}{\partial x_\alpha} f(x,x) = \left[\frac{\partial}{\partial x_\alpha} f(x,y) + \frac{\partial}{\partial y_\alpha} f(x,y)\right]_{x=y}$ and the trace property

$$\partial_\alpha J^5_\alpha(x) = -\frac{1}{\Gamma(s)}\int_0^\infty dt\; t^{s-1}\; tr\left[\gamma^5\gamma_\alpha D_\alpha\,\gamma\cdot D\,\mathcal{G} - \gamma^5\gamma_\alpha\,\gamma\cdot D\,\mathcal{G}\,\overleftarrow{D}_\alpha\right]_{x=y}\Big|_{s=1}. \tag{9}$$

By representing $\mathcal{G}_F$ in terms of a complete set of eigenfinctions of $\gamma\cdot D$ ($\gamma\cdot D\psi_n = i\lambda_n\psi_n$, λ_n real) it is easy to see that

$$\gamma\overrightarrow{D}\,\mathcal{G}_F = \mathcal{G}_F\,\overleftarrow{D}\cdot\gamma. \tag{10}$$

Then using the Dirac trace (9) becomes

$$\partial_\alpha J^5_\alpha(x) = -\frac{2}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\, tr\left[\gamma^5(\gamma\cdot D)^2 \mathcal{G}_F(x,y,t)\right]_{x=y}\Big|_{s=1}$$
$$= \frac{2(s-1)}{\Gamma(s)}\int_0^\infty dt\, t^{s-2}\, tr\left[\gamma^5 \mathcal{G}_F(x,x,t)\right]\Big|_{s=1} \qquad (11)$$

where we have integrated by parts in the last step using (6) and the fact that $(\gamma\cdot D)^2\Pi=0$. Again we need to find the residue of the pole of the integral at $s=1$ which arises from the small t part of the integration. The residue comes from the constant term in the small t expansion of $\mathcal{G}_F(x,x,t)$. The constant term is

$$\frac{1}{16\pi^2}\, a_{2F}(x,x) - \Pi(x,x) \qquad (12)$$

where a_{2F} is defined in the same way as a_2 of section 17 and Π arises because $\Pi(x,y)$ must be subtracted from the whole expansion corresponding to (17.1) to ensure (7) is satisfied. $a_{2F}(x,x)$ can be calculated in the same way as $a_2(x,x)$ in (17.4) giving

$$a_{2F}(x,x) = \frac{1}{12}F_{\alpha\beta}F_{\alpha\beta} + \frac{1}{2}\left(\sigma_{\alpha\beta}F_{\alpha\beta}\right)^2 + \frac{1}{6}D^2\left(\sigma_{\alpha\beta}F_{\alpha\beta}\right) \qquad (13)$$

with σ as given in Appendix A. Taking the required trace in (11) of (13) and using the properties in Appendix A

$$tr\left(\gamma^5 a_{2F}(x,x)\right) = tr\left(F_{\alpha\beta}\tilde{F}_{\alpha\beta}\right). \qquad (14)$$

Inserting (12) into (11) yields the equation for the axial anomaly [60,61]

$$\partial_\alpha J^5_\alpha(x) = \frac{1}{8\pi^2}\, tr\left(F_{\alpha\beta}\tilde{F}_{\alpha\beta}\right) - 2\sum_i \varepsilon_i\, tr\left(\psi_i(x)\psi_i^+(x)\right) \qquad (15)$$

where the zero modes ψ_i are chosen to have definite chirality $\varepsilon_i=\pm 1$ given by

$$\gamma_5\psi_i = \varepsilon_i\psi_i\,. \qquad (16)$$

(15) may be integrated to give the index theorem relating the topological charge k to the difference in the number of positive and negative chirality solutions (16) of $\gamma\cdot D\psi=0$.

In the case of self-dual solutions we can give explicit expressions for the various terms of (15) [3]. All the solutions of $\gamma\cdot D\psi=0$ have negative charality as Π is given by (9.20). Thus using the expression for the non-zero block of Π and the result (6.19)

$$tr\,\gamma_5\,\Pi(x,x) = -\frac{1}{\pi^2}\, tr_2\left(f\, b^+ P b f\, b^+ b\right)$$
$$= \frac{1}{8\pi^2}\,\partial^2\, tr_2\left(f\, b^+ b\right)\,. \qquad (17)$$

We may use the result (9.18) of Brown et. al. [36] expressing the Dirac Green function S_F in terms of the scalar Green function G for the self-dual case.

$$S_F(x,y) = \gamma \vec{D}\, G(x,y)\, \frac{1+\gamma_5}{2} + G(x,y)\, \gamma \overleftarrow{D}\, \frac{1-\gamma_5}{2}. \tag{18}$$

So the expression corresponding to (8) is

$$J^5_\alpha(x) = -\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\, \mathrm{tr}\left(\gamma^5\gamma_\alpha\, \gamma\cdot\vec{D}\, \mathcal{G}\, \frac{1+\gamma_5}{2} + \gamma_\alpha\gamma_5\, \mathcal{G}\, \overleftarrow{D}\cdot\gamma\, \frac{1-\gamma_5}{2}\right)\Big|_{s=1} \tag{19}$$

where $\mathcal{G} = e^{tD^2}$ is given by (17.1). (The negative sign of (19) is due to a negative sign in the regularised Green function corresponding to (5) since G is defined as the inverse of the negative definite operator D^2). The trace over the Dirac matrices gives

$$\begin{aligned} J^5_\alpha(x) &= -\frac{2}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\, \mathrm{tr}\left(\vec{D}_\alpha \mathcal{G} - \mathcal{G}\overleftarrow{D}_\alpha\right)_{x=y}\Big|_{s=1} \\ &= +\frac{2(s-1)}{\Gamma(s)}\int_0^\infty dt\, t^{s-2}\int d^4z\left[\vec{D}_\alpha G(x,z)\,\mathcal{G}(z,y,t) - \mathcal{G}(x,z,t)\,G(z,y)\overleftarrow{D}_\alpha\right]_{x=y}\Big|_{s=1} \end{aligned} \tag{20}$$

where again we have integrated by parts. By using the same techniques as in section 17 we can show that the singular part $-\Phi(x,y)/4\pi^2(x-y)^2$ of $G(x,y)$ does not contribute leaving

$$\begin{aligned} J^5_\alpha(x) &= -\frac{2}{4\pi^2}\,\mathrm{tr}\left[\vec{D}_\alpha R(x,y) - R(x,y)\overleftarrow{D}_\alpha\right]_{x=y} \\ &= \frac{2}{4\pi^2}\,\mathrm{tr}\left[v^+ b\,\partial_\alpha f\, b^+ v\right] \end{aligned} \tag{21}$$

by the expression for R (18.8). This can be written as a derivative using the properties of the self-dual matrices giving

$$J^5_\alpha(x) = \frac{1}{4\pi^2}\,\partial_\alpha\, \mathrm{tr}\left(P_b f b^+\right). \tag{22}$$

Inserting (22) and (17) into the expression (15) for the self-dual case, it becomes

$$\partial_\alpha J^5_\alpha = \frac{1}{4\pi^2}\,\partial^2\, \mathrm{tr}\left(P_b f b^+\right) = \frac{1}{8\pi^2}\,\mathrm{tr}\left(F_{\alpha\beta}F_{\alpha\beta}\right) - \frac{2}{8\pi^2}\,\partial^2\,\mathrm{tr}\left(f b^+ b\right) \tag{23}$$

which on rearrangement gives us the required result (6.13), (6.14)

$$\mathrm{tr}\left(F_{\alpha\beta}F_{\alpha\beta}\right) = 2\partial^2\,\mathrm{tr}\left(f b^+ P_b + f b^+ b\right). \tag{24}$$

CHAPTER 5 - CONCLUSION

Section 22 - Recent Progress on the Determinant

To proceed further in non-perturbative calculations for QCD by the philosophy of chapter 1, one would like to be able to integrate the determinant given by (18.18) completely. This might enable one to study the statistical mechanics of the instantons in the same way as has been done for the CP^1 model in [62,63]. Recently some further progress towards this end has been achieved by Osborn [23] and Berg and Lüscher [24].

When the variation of the determinant in the form of (18.13) is examined, one can see that there is a conformally invariant term [23,24]

$$\delta\theta = \frac{1}{12\pi^2}\int d^4x\ \varepsilon_{\alpha\beta\gamma\delta}\ \mathrm{tr}\left(\delta g\, k_\alpha k_\beta k_\gamma k_\delta\right) \tag{1}$$

with

$$\begin{aligned} k_\alpha &= f\,\partial_\alpha f^{-1} = f(\Delta^+ b e_\alpha + e_\alpha^+ b^+ \Delta) = f\,\mathrm{tr}(\Delta^+ b e_\alpha) \\ \delta g &= f\,\delta f^{-1} = f(\Delta^+\delta\Delta + \delta\Delta^+\Delta) = f\,\mathrm{tr}(\Delta^+\delta\Delta) \end{aligned} \tag{2}$$

which at first does not appear expressible as a variation. In addition, after some very lengthy algebra the remainder of (18.13) can be integrated so that [23,24]

$$\delta\zeta'(0) = \frac{1}{12\pi^2}\int d^4x\ \delta\left[-\frac{1}{4}\mathrm{tr}\left(k_\alpha k_\alpha k_\beta k_\beta\right) + 5\,\mathrm{tr}\left(f\nu f\nu\right)\right] + \delta\theta \tag{3}$$

$(b^+b = \nu 1_1)$. In the first term the variation can be removed from the integral (which is convergent). The behaviour of the integrand without the variation is

$$-4\,\mathrm{tr}(1_k)\cdot\frac{x_\alpha}{x^2}\cdot\frac{x_\alpha}{x^2}\cdot\frac{x_\beta}{x^2}\cdot\frac{x_\beta}{x^2} + 5\,\mathrm{tr}(1_k)\,\frac{1}{x^2}\cdot\frac{1}{x^2} = \frac{k}{x^4} \tag{4}$$

at infinity, as

$$k_\alpha \sim 1_k\,\frac{2x_\alpha}{x^2}\ ,\qquad f\nu \sim 1_k\cdot\frac{1}{x^2}\qquad \text{as } |x|\to\infty \tag{5}$$

so the removal of the variation gives a divergent answer. But since the divergence arising from (4) is independant of the instanton parameters (for fixed k) it can be subtracted without changing (3). Thus the first term in (3) gives a contribution of [23,24]

$$\alpha(k) + \frac{1}{12\pi^2}\int d^4x\left[-\frac{1}{4}\mathrm{tr}\left(k_\alpha k_\alpha k_\beta k_\beta\right) + 5\,\mathrm{tr}\left(f\nu f\nu\right) - \frac{k}{(1+x^2)^2}\right] \tag{6}$$

to the determinant where $\alpha(k)$ is some undetermined parameter independant constant.

Berg and Lüscher [24] have succeeded in writing $\delta\theta$ as a variation of a five dimensional integral

$$\theta = 5\int_0^1 dt \int d^4x \; \varepsilon_{\alpha\beta\gamma\delta} \; tr\left[K^{-1}\partial_t K . K^{-1}\partial_\alpha K . K^{-1}\partial_\beta K . K^{-1}\partial_\gamma K . K^{-1}\partial_\delta K\right] \tag{7}$$

where

$$K(x,t) = (1-t)(1+x^2)1_k + t f^{-1} \tag{8}$$

is invertible for all x and t.

In the special case of the JNR solutions (section 10) one can show that $\delta\theta$ vanishes and the term corresponding to (6) can be simplified to give [23,24]

$$-\ln\det(-D^2) = \zeta'(0) = \tilde{\alpha}(k) - \frac{1}{12.16\pi^2}\int d^4x \ln\det(f\nu).\partial^2\partial^2 \ln\det(f\nu)$$
$$-\frac{1}{12}\sum_{i\neq j}\ln(x_i - x_j)^2 - \frac{1}{6}\sum_0^k \ln\lambda_i^2 + \frac{k+1}{6}\ln\sum_0^k \lambda_i^2 \tag{9}$$

where $\tilde{\alpha}(k)$ can be related to $\alpha(k)$. This is essentially the same as that first derived by Brown and Creamer [15] and is also given (with some errors) in [20].

Berg and Lüscher also compute the constant $\alpha(k)$. To do this one observes that for fixed k an SU(n) potential for $n\leq 2k$ can be embedded in a 2k x 2k matrix and considered as an SU(2k) potential, and for $n>2k$ an SU(n) self-dual potential can always be reduced to an SU(2k) solution (see section 10). Also from the representation of the determinant of (16.4) one can see that in the ratio

$$\frac{\det(-D_k^2)}{\det(-D_0^2)} \tag{10}$$

(A_κ in D_k has topological charge k and in D_0 has charge 0) the denominator for SU(n) is $[\det(-\partial^2)]^n$. This is because $A_\kappa^{k=0}$ is a pure gauge and

$$\zeta(s) = \frac{1}{\Gamma(s)}\int dt\, t^{s-1}\, \mathrm{Tr}\, e^{tD^2} \tag{11}$$

is gauge invariant so the D^2 in this can be transformed to $\partial^2.1_n$. Then the internal index trace in Tr gives the factor n in the zeta function and hence in the logarithm of the determinant. Similarly when an SU(n) potential is reduced or extended to SU(2k) as above

$$\ln\det -D_k^2\big|_{SU(n)} = \ln\det -D_k^2\big|_{SU(2k)} + (n-2k)\ln\det(-\partial^2). \tag{12}$$

Hence the ratio of interest (10) is

$$\frac{\det(-D_k^2)}{\det(-D_0^2)}\bigg|_{SU(n)} = \frac{\det(-D_k^2)\big|_{SU(2k)}}{\left[\det(-\partial^2)\right]^{2k}} = \frac{\det(-D_k^2)}{\det(-D_0^2)}\bigg|_{SU(2k)}. \tag{13}$$

Then the construction of Drinfeld and Manin [31] in section 10 gives all the SU(2k) solutions and one can argue from this that the parameter manifold is connected. So $\alpha(k)$ is the same for all k-instanton solutions irrespective of the group SU(n) so only needs to be calculated for one particular example. A suitable choice is the SU(2k) instanton solution [24]

$$\begin{array}{cc} & \leftarrow k \text{ times} \rightarrow \\ A_\alpha^{2k} = & \begin{pmatrix} A_\alpha^2 & 0 & \cdots & 0 \\ 0 & A_\alpha^2 & & \\ \vdots & & \ddots & \\ 0 & & & A_\alpha^2 \end{pmatrix} \end{array} \tag{14}$$

where A_α^2 are SU(2) 1-instanton solutions. One can see for this that it is self-dual with topological charge k and the logarithm of its determinant is k times $\ln\det(-D_{k=1}^2)$. Using the result (6) (θ vanishes for this example) we see that $\alpha(k) = k\alpha(1)$. Since the 1-instanton determinant has been calculated completely for SU(2) [5-12] one can compute $\alpha(1)$ and hence $\alpha(k)$ [24].

Appendix A - Properties of Quaternions and Dirac Matrices.

The conventions used here follow closely those of [1,2,3] and many of the following formulae occur there.

Call the generators of the quaternions e_α $(\alpha = 0,1,2,3)$ which can be represented by 2×2 matrices

$$e_0 = 1_2 \qquad e_a = -i\sigma_a \quad (a = 1,2,3) \tag{1}$$

where σ_a are the Pauli matices. Then an arbitrary quaternion is given by

$$x = x_\alpha e_\alpha \tag{2}$$

where x_α are the four real components of the quaternion x.

The quaternion conjugate $^+$ corresponds to the hermitian conjugate on 2×2 matrices also denoted by $+$. The 2×2 transpose will be denoted by ^{t} and the complex conjugate by * so $^+ = {}^{t*}$. Various properties of the quaternions follow from those of the Pauli matrices.

$$\begin{aligned} &e_0^+ = e_0 \quad , \quad e_a^+ = -e_a \\ &\varepsilon e_\alpha \varepsilon^{-1} = e_\alpha^* \quad , \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \varepsilon^t = \varepsilon^{-1} . \end{aligned} \tag{3}$$

We can make a connection between the quaternions e_α and the generators $\bar{\eta}, \eta$ of self-dual and anti-self-dual antisymmetric tensors introduced by 't Hooft [5,6]:

$$\begin{aligned} e_\alpha e_\beta^+ &= \delta_{\alpha\beta} 1_2 + i\bar{\eta}_{\alpha\beta} \\ e_\alpha^+ e_\beta &= \delta_{\alpha\beta} 1_2 + i\eta_{\alpha\beta} \end{aligned} \tag{4}$$

where

$$\eta_{\alpha\beta} = \sigma_a \eta^a_{\alpha\beta} \quad , \quad \bar{\eta}_{\alpha\beta} = \sigma_a \bar{\eta}^a_{\alpha\beta} \tag{5}$$

and $\eta^a_{\alpha\beta}$ is defined by

$$\eta^a_{\alpha\beta} = -\eta^a_{\beta\alpha} \quad , \quad \eta^a_{bc} = \varepsilon_{abc} \quad ; \quad \eta^a_{0b} = -\delta_{ab} . \tag{6}$$

$\bar{\eta}$ is defined similarly except $\bar{\eta}^a_{0b} = +\delta_{ab}$. $\bar{\eta}$ and η are respectively self-dual and anti-self-dual:

$$\begin{aligned} \bar{\eta}_{\alpha\beta} &= \tfrac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} \bar{\eta}_{\gamma\delta} \\ \eta_{\alpha\beta} &= -\tfrac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} \eta_{\gamma\delta} . \end{aligned} \tag{7}$$

Some useful properties are

$$\begin{aligned} &\text{(i)} && \eta^a_{\alpha\beta} \eta^a_{\alpha\beta} = 4\delta_{ab} \\ &\text{(ii)} && \eta^a_{\alpha\beta} \eta^a_{\alpha\gamma} = 3\delta_{\beta\gamma} \\ &\text{(iii)} && \eta^a_{\alpha\beta} \eta^b_{\alpha\gamma} = \delta_{\beta\gamma}\delta_{ab} + \varepsilon_{abc}\eta^c_{\beta\gamma} \\ &\text{(iv)} && \eta^a_{\alpha\beta} \eta^a_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} - \varepsilon_{\alpha\beta\gamma\delta} . \end{aligned} \tag{8}$$

The same results hold with $\bar{\eta}$ except in (8iv) the ε symbol occurs with positive sign. A consequence of (8iv) is that $\frac{1}{4}\bar{\eta}^a_{\alpha\beta}\bar{\eta}^a_{\gamma\delta}$ and $\frac{1}{4}\eta^a_{\alpha\beta}\eta^a_{\gamma\delta}$ are respectively the projections onto self- and antiself-dual antisymmetric tensors.

From (4) we can immediately deduce

$$\begin{aligned} e_\alpha e_\beta^+ + e_\beta e_\alpha^+ &= 2\delta_{\alpha\beta}1_2 = e_\alpha^+ e_\beta + e_\beta^+ e_\alpha \\ e_\alpha e_\beta^+ - e_\beta e_\alpha^+ &= 2i\bar{\eta}_{\alpha\beta} \\ e_\alpha^+ e_\beta - e_\beta^+ e_\alpha &= 2i\eta_{\alpha\beta} \\ tr(e_\alpha e_\beta^+) &= 2\delta_{\alpha\beta} = tr(e_\alpha^+ e_\beta) . \end{aligned} \tag{9}$$

tr is the 2×2 or quaternionic trace and where it may be confused with other higher dimensional traces it will be denoted by tr_2 . The four matrices e_α form a complete set of 2×2 complex matrices. Following from this

$$\begin{aligned} (e_\alpha)_{AB}(e_\alpha^+)_{CD} &= 2\delta_{AD}\delta_{BC} \\ (e_\alpha)_{AB}(e_\alpha)_{CD} &= 2\varepsilon_{AC}\varepsilon_{BD} \end{aligned} \tag{10}$$

which may be written equivalently as

$$\begin{aligned} e_\alpha a e_\alpha^+ &= 2tr_2 a \\ e_\alpha a e_\alpha &= 2\varepsilon a^t \varepsilon^{-1} \end{aligned} \tag{11}$$

for any 2×2 matrix a. As a consequence

$$\begin{aligned} e_\alpha e_\alpha^+ &= 4\cdot 1_2 \\ e_\alpha e_\beta^+ e_\alpha &= -2e_\beta \\ e_\alpha e_\beta^+ e_\gamma e_\alpha^+ &= 4\delta_{\beta\gamma}\cdot 1_2 \end{aligned} \tag{12}$$

and other such relations hold.

The Euclidean Dirac matrices can be represented by

$$\gamma_\alpha = \begin{pmatrix} 0 & e_\alpha \\ e_\alpha^+ & 0 \end{pmatrix} \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \tag{13}$$

which satisfy

$$\begin{aligned} &\gamma_\alpha^+ = \gamma_\alpha \quad , \quad \gamma_5^2 = 1_4 \\ &\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}.1_4 \\ &\{\gamma_\alpha, \gamma_5\} = 0 \end{aligned} \tag{14}$$

where $\{\ ,\ \}$ is the anticommutator, and

$$[\gamma_\alpha, \gamma_\beta] = 4\sigma_{\alpha\beta} \quad , \quad \sigma_{\alpha\beta} = \frac{i}{2}\begin{pmatrix} \bar{\eta}_{\alpha\beta} & 0 \\ 0 & \eta_{\alpha\beta} \end{pmatrix} . \tag{15}$$

Appendix B - Large |x| Behaviour of the Self-dual Functions

One can solve the defining equations (5.2) and (5.3) for $v(x)$ in a power series in $1/|x|$. This enables us to deduce the large $|x|$ behaviour of the self-dual expressions occurring in the text.

We write

$$v(x) = v^0(x) + v^1(x) + v^2(x) + \cdots \tag{1}$$

where $v^r(x)$ is $\sim |x|^{-r}$ as $|x| \to \infty$. Also we choose $\Delta(x)$ to be in the canonical form (7.11) and write

$$a = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} , \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

where the upper block is $n \times 2k$ and the lower block is $2k \times 2k$. Similarly v^r is written in block form

$$v^r = \begin{pmatrix} v_0^r \\ v_1^r \end{pmatrix} \tag{3}$$

where the upper and lower blocks are $n \times n$ and $2k \times n$ respectively.

Solving

$$\begin{aligned} (a^+ + x^+ b^+) v &= 0 \\ v^+ v &= 1 \end{aligned} \tag{4}$$

iteratively gives

$$v^0 = \begin{pmatrix} u(x) \\ 0 \end{pmatrix} , \quad v^1 = \begin{pmatrix} 0 \\ -\frac{1}{x^2} x a_0^+ u(x) \end{pmatrix} , \quad v^2 = \begin{pmatrix} -\frac{1}{2x^2} a_0 a_0^+ u(x) \\ +\frac{1}{x^4} x a_1^+ x a_0^+ u(x) \end{pmatrix} , \quad \cdots \tag{5}$$

where $u(x)$ is an arbitrary $n \times n$ unitary matrix so it is ~ 1 as $|x| \to \infty$. Substituting (5) into the expression (5.1) for the gauge potential gives

$$A_\alpha(x) = u^+ \partial_\alpha u + \frac{1}{2x^4} u^+ a_0 (x^+ e_\alpha - e_\alpha^+ x) a_0^+ u + O\left(\frac{1}{|x|^4}\right). \tag{6}$$

Then A_α is a pure gauge which is $O(\frac{1}{|x|})$ plus a term of $O\left(\frac{1}{|x|^3}\right)$. Because in general throughout the text we assume $A_\alpha(x)$ is regular everywhere all the topological charge must come from the behaviour at infinity (see (3.14), (3.13)). Due to the result (6) only the pure gauge part of A_α will contribute in (3.14) thus to give the charge k required by (6.18) the unitary matrix $u(x)$ must have winding number k at infinity (ie. it must cover an SU(2) subgroup of SU(n) k times as x runs over a large sphere S^3). For example such a matrix is

$$u(x) = \begin{pmatrix} \left(\frac{x}{|x|}\right)^k & 0 \\ 0 & 1_{n-2} \end{pmatrix} \tag{7}$$

where x is the 2×2 quaternion form of x_α and $x/|x|$ covers SU(2) once.

The combination b^+v occurs in many expressions. By (2) and (5) this is

$$b^+v = -\frac{1}{x^2}x\,a_0^+u(x) + \frac{1}{x^4}x\,a_1^+x\,a_0^+u + O\left(\frac{1}{|x|^3}\right) \tag{8}$$

so it is $O\left(\frac{1}{|x|}\right)$. The full behaviour of any term can then be seen by expanding $f(x)$ (5.13)

$$\begin{aligned} f(x) = {} & \frac{1}{x^2}(b^+b)^{-1} - \frac{1}{x^4}(b^+b)^{-1}(a^+bx + x^+b^+a)(b^+b)^{-1} \\ & + \frac{1}{x^6}(b^+b)^{-1}(a^+bx + x^+b^+a)(b^+b)(a^+bx + x^+b^+a)(b^+b)^{-1} - \frac{1}{x^4}(b^+b)^{-1}(a^+a)(b^+b)^{-1} + O\left(\frac{1}{|x|^5}\right) \end{aligned} \tag{9}$$

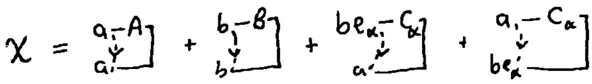
giving

$$\begin{aligned} \rho_b &= -\left(1 - b(b^+b)^{-1}b^+\right)\frac{ax^+}{x^2} + O\left(\frac{1}{|x|^2}\right) \\ b^+\rho_b &= +\frac{xa^+}{x^2}\left(1 - b(b^+b)^{-1}b^+\right)\frac{ax^+}{x^2} + O\left(\frac{1}{|x|^3}\right). \end{aligned} \tag{10}$$

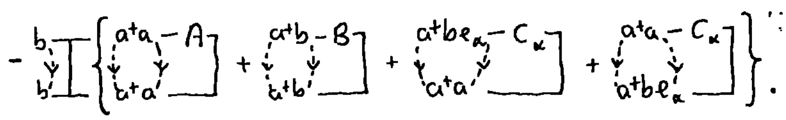
(Note $b^+v = -\frac{xa^+v}{x^2}$ by (4)).

Appendix C - Algebraic Calculations for Section 14

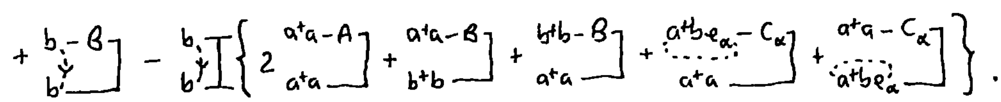
As required by the construction of the tensor product self-dual matrices in section 14, the matrix $\mathcal{M}$ defined by (14.6) must have exactly $4k_1k_2$ unit eigenvectors. They are of the form


(1)

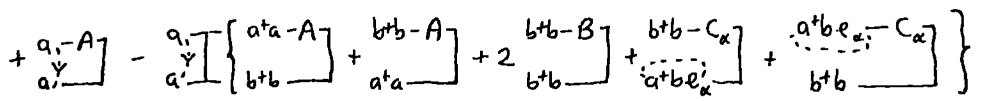
where A, B, C_α are $k_1 \times k_2$ complex matrices. Taking the last term of $\mathcal{M}$ in (14.6) and multiplying by χ gives


(2)

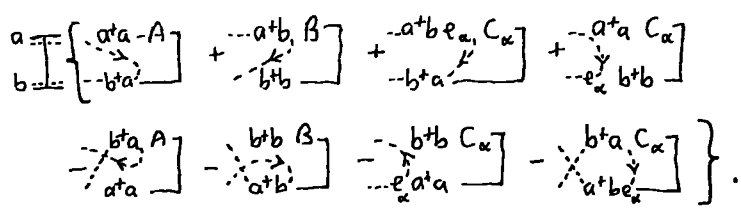
Using (13.14) in the second term of this it becomes


(3)

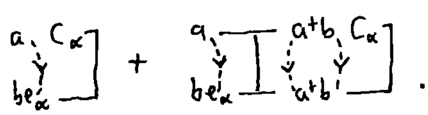
Similarly the second last term of $\mathcal{M}$ in (14.6) times χ gives


(4)

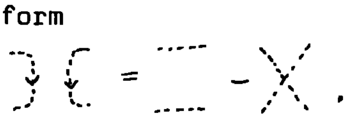
The first and third terms of $\mathcal{M}$ give


(5)

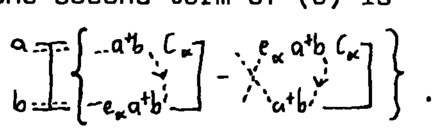
In (5) the fourth and seventh terms combine by (13.14) into


(6)

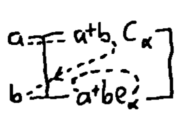
To proceed we use the relation $\varepsilon_{AB}\varepsilon_{CD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}$ which has the diagrammatic form


(7)

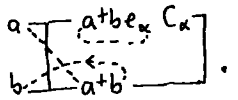
Then the second term of (6) is


(8)

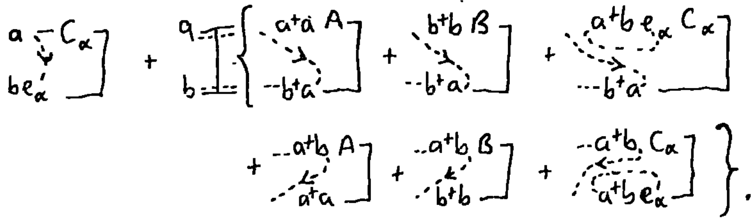
The first term of this combines with the third term of (5) due to (7) and (13.17) to give


(9)

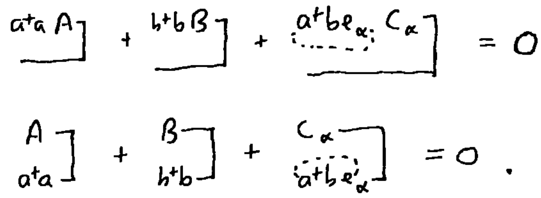
and the second term of (8) combines with the eighth term of (5) to give


(10)

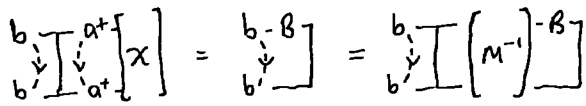
Hence (5) becomes


(11)

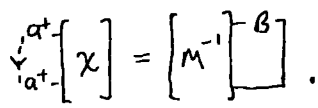
Similarly the second and fourth terms of $\mathcal{M}$ times χ yield (11) with the top row now interchanged with the bottom. We see from (3), (4) and (11) that $\mathcal{M}\chi=\chi$ provided


(12)

These constitute $2k_1k_2$ equations for the $6k_1k_2$ unknowns in A, B, C_α giving $4k_1k_2$ solutions. That they are linearly independent can be seen by proving that $\chi=0$ implies $A, B, C_\alpha=0$. Using (12) in (3) the latter becomes


(13)

so calculating in the same way gives


(14)

Hence $\chi=0$ implies $B=0$ as M is invertible. Similarly using (4) and (11) with the condition (12) we can see that $\chi=0$ implies $A=0$ and $C_\alpha=0$. Thus we have proven $\mathcal{M}\chi=\chi$ has $4k_1k_2$ independent solutions giving

$$(1-\mathcal{M})\chi=0 \tag{15}$$

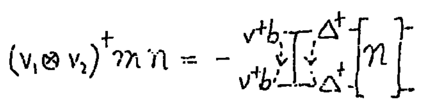
with χ as in (1).

The remaining results required in section 14 are proven in a similar fashion to the above result so they are only sketched here. Next we check that $\hat{\Delta}$ in (14.16) satisfies (14.18) provided (14.19) holds. We first note that if we replace a by Δ in $\mathcal{M}$ which leaves it unchanged (see the comment after (14.6)) and a by Δ in χ then the proof $\mathcal{M}\mathcal{X}=\mathcal{X}$ follows through exactly as above provided a is replaced by Δ in the conditions (12). This appears to give us new eigenvectors $\mathcal{X}$ for each choice of x in Δ, but the replacement of a by Δ in (1) just gives us $\mathcal{X}'$ with

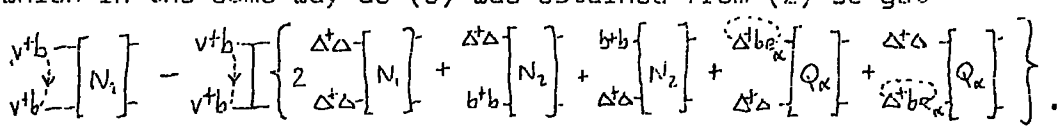
$$\begin{aligned} A' &= A \\ B' &= B + 2x_\alpha C_\alpha + x^2 A \\ C'_\alpha &= C_\alpha + x_\alpha A\ . \end{aligned} \tag{16}$$

Also the conditions (12) with a replaced by Δ are the same as (12) with A, B, C_α replaced by A', B', C'_α of (16).

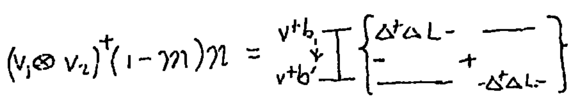
Since $\mathcal{N}$ of (14.17) is of the same form as $\mathcal{X}$ we can replace a by Δ and check that (14.18) is true when the conditions (14.19) have the same replacement. This simplifies the caculation of $\tilde{v}^+\tilde{\Delta}$. Now

 (17)

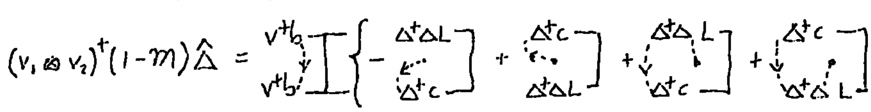
which in the same way as (3) was obtained from (2) we get

 (18)

Thus

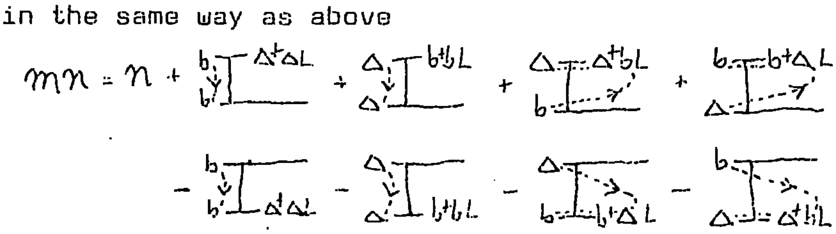
 (19)

where L_1 and L_2 are given by (14.19) with a replaced by Δ. Then using this with (14.16) for $\hat{\Delta}$ we see

 (20)

which vanishes as $\Delta^+\Delta$ is proportional to the unit 2×2 matrix so confirming (14.18).

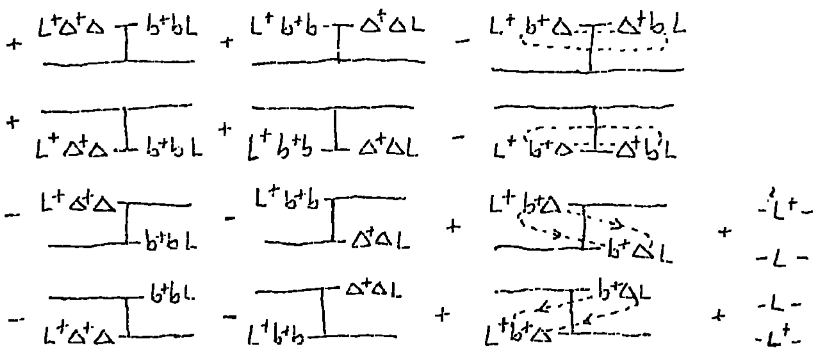
We now wish to calculate $(1-\mathcal{M})\mathcal{N}$. By using the corresponding relations in the calculation of $\mathcal{M}\mathcal{X}$ with a replaced by Δ we see that in the same way as above

 (21)

so $(1-\mathcal{M})\mathcal{N}$ and hence $(1-\mathcal{M})\hat{\Delta}$ depends on N_1, N_2, Q_α only through L_1, L_2

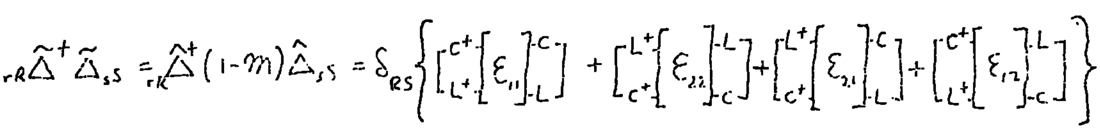
as asserted after (14.17).

For the calculation of $\tilde{\Delta}^+\tilde{\Delta}$ (14.20) we need $n^+(1-m)n$. This is obtained from (21) again using the results for $\chi^+ m = (m\chi)^+$. The answer is



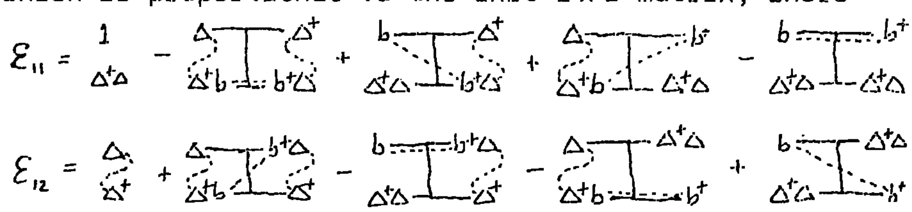
(22)

and then by this and (21) and $\hat{\Delta}$ given by (14.16) we obtain after a lengthy calculation, using (7) many times, that



(23)

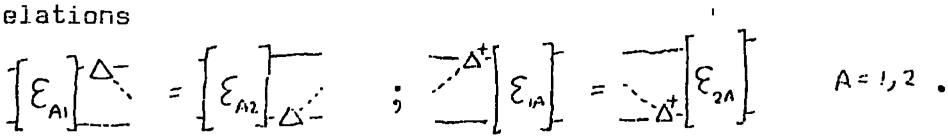
which is proportional to the unit 2×2 matrix, where



(24)

and $\mathcal{E}_{22}$, $\mathcal{E}_{21}$ are the same as $\mathcal{E}_{11}$, $\mathcal{E}_{12}$ respectively but with the top row interchanged with bottom. From (24), $\tilde{\Delta}^+\tilde{\Delta}$ appears to be quartic in x, but it is easy to check that in each of the $\mathcal{E}_{AB}$ the quartic and cubic parts both cancel leaving only those up to quadratic. Similarly (21) and (22) are in fact independant of x as implied in their construction, so are the same with Δ replaced by a.

We can easily show that the $\mathcal{E}$ matrices satisfy the following relations



(25)

These are of the form of the equations (14.14) for c_1, c_2 and since (c_1^r, c_2^r), $1 \le r \le \tilde{k}$ form a complete set of solutions we must be able to write

$$\mathcal{E}_{AB} = (c_A^r) X_{rs} (c_B^s)^+ \qquad 1 \le A, B \le 2 \tag{26}$$

for some matrix X. To determine X consider for example

$$\left[\mathcal{E}_{11}\right]\begin{matrix}-\rho c\\ -f\end{matrix}\Big] + \left[\mathcal{E}_{11}\right]\begin{matrix}-f\\ -c\end{matrix}\Big] . \tag{27}$$

Using (25) and (14.14) we immediately see that (27) simplifies to $\overset{-c}{\rfloor}$ thus from (26)

$$c_1^r X_{rs}\left\{\begin{bmatrix} c_s^\dagger \rho c_t \\ f \end{bmatrix} + \begin{bmatrix} f \\ c_s^\dagger c_t \end{bmatrix}\right\} = c_1^t . \tag{28}$$

The three other relations corresponding to (27) and (28) have the same expression in the curly brackets of (28) which is just Ω_{st} of (14.23). Hence $X_{rs} = (\Omega^{-1})_{rs}$ and using this in $\mathcal{E}_{AB}$ of (26) and then substituting in the expression (23) for $\tilde{\Delta}^\dagger\tilde{\Delta}$ we see that the latter becomes the same as (14.21).

The tensor product Dirac zero mode (14.24) is calculated by using $(v_1 \otimes v_2)^\dagger (1-m)n$ from (19). Then since $\hat{b}$ is the coefficient of the linear term in x of $\hat{\Delta}$ (14.16) we obtain

$$(v_1 \otimes v_2)^\dagger (1-m)\hat{b}\,\mathcal{E}_A = \begin{matrix} v^\dagger b\, L \\ v^\dagger c \end{matrix}\Big]_A + \begin{matrix} v^\dagger c \\ v^\dagger b\, L \end{matrix}\Big]_A$$

$$+ \begin{matrix} v^\dagger b \\ v^\dagger b \end{matrix}\Big]\left\{ - \begin{matrix} \Delta^\dagger \Delta L \\ b^\dagger c \end{matrix}\Big]_A + \begin{matrix} b^\dagger c \\ \Delta^\dagger \Delta L \end{matrix}\Big]_A + \begin{matrix} \Delta^\dagger b L \\ \Delta^\dagger c \end{matrix}\Big]_A + \begin{matrix} \Delta^\dagger c \\ \Delta^\dagger b L \end{matrix}\Big]_A \right\} . \tag{29}$$

To show the equality of (14.24) with (14.26) we multiply the latter by $\tilde{\Delta}^\dagger\tilde{\Delta} = Z^\dagger \Omega^{-1} Z$. Then the resulting

$$\left\{ \begin{matrix} v^\dagger c \\ v^\dagger b f \end{matrix}\Big] + \begin{matrix} v^\dagger b f \\ v^\dagger c \end{matrix}\Big] \right\} \Omega^{-1} Z \tag{30}$$

can be calculated by the completeness relations (26) giving

$$\begin{matrix} v^\dagger \\ v^\dagger b f \end{matrix}\left\{ \left[\mathcal{E}_{11}\right]\begin{matrix}-c\\ -L\end{matrix}\Big] + \left[\mathcal{E}_{12}\right]\begin{matrix}-L\\ -c\end{matrix}\Big] \right\} + \begin{matrix} v^\dagger b f \\ v^\dagger \end{matrix}\left\{ \left[\mathcal{E}_{21}\right]\begin{matrix}-c\\ -L\end{matrix}\Big] + \left[\mathcal{E}_{22}\right]\begin{matrix}-L\\ -c\end{matrix}\Big] \right\} . \tag{31}$$

By the expressions (24) for $\mathcal{E}_{AB}$ and with the help of (7) this is identical to (29) so completing the proof.

The completeness relation (9.20) for the Dirac zero modes can be calculated for the tensor product case by using the tensor product Green function given by (13.1) and (13.10). The calculation proceeds in an exactly analogous fashion to the fundamental case so need not be repeated here (cf. [39]). The result can be written for the complete orthonormal set of zero modes as

$$\pi^2 \sum_r \tilde{\psi}^r_A(x)\, \tilde{\psi}^r_B(y) = \begin{array}{r} v^\dagger(x) \\ \cdot A \\ v^\dagger(x) b f(x) \end{array} \left[\; \bar{\mathcal{E}}_{11} \; \right] \begin{array}{l} v^\dagger(y) \\ B \\ f(y) b^\dagger v(y) \end{array} + \begin{array}{r} v^\dagger(x) b f(x) \\ A \\ v^\dagger(x) \end{array} \left[\; \bar{\mathcal{E}}_{22} \; \right] \begin{array}{l} f(y) b^\dagger v(y) \\ B \\ v(y) \end{array}$$

$$+ \begin{array}{r} v^\dagger(x) \\ \cdot A \\ v^\dagger(x) b f(x) \end{array} \left[\; \bar{\mathcal{E}}_{12} \; \right] \begin{array}{l} f(y) b^\dagger v(y) \\ B \\ v(y) \end{array} + \begin{array}{r} v^\dagger(x) b f(x) \\ A \\ v^\dagger(x) \end{array} \left[\; \bar{\mathcal{E}}_{21} \; \right] \begin{array}{l} v(y) \\ B \\ f(y) b^\dagger v(y) \end{array} \tag{32}$$

where $\bar{\mathcal{E}}_{AB}$ are the same as $\mathcal{E}_{AB}$ given by (24) but with Δ replaced by b and b replaced by a. One can show that the same completeness relations (26) hold for $\bar{\mathcal{E}}_{AB}$ except that Ω (which is the inverse of χ) is replaced by $\bar{\Omega}$ in which all the Δ's are replaced by b. Hence (32) simplifies to

$$\sum_r \tilde{\psi}^r_A(x)\, \tilde{\psi}^r_B(y) = \frac{1}{\pi^2} \left(\begin{bmatrix} v^\dagger c_r \\ A \\ v^\dagger b f \end{bmatrix} + \begin{bmatrix} v^\dagger b f \\ A \\ v^\dagger c_r \end{bmatrix} \right)_x \bar{\Omega}^{-1}_{rs} \left(\begin{bmatrix} c_s^\dagger v \\ B \\ f b^\dagger v \end{bmatrix} + \begin{bmatrix} f b^\dagger v \\ B \\ c_s^\dagger v \end{bmatrix} \right)_y \tag{33}$$

which is consistent with (14.29) and (14.30).

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