Some studies on the consequences of internal symmetry and supergauge invariance in particle physics

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Some Studies on the Consequences of Internal Symmetry and Supergauge Invariance in Particle Physics.

by

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Preface

The work presented in this thesis was carried out in the Department of Mathematics of the University of Durham between October 1972 and May 1975 under the supervision of Dr. D.B.Fairlie.

This material has not been submitted previously for any degree in this or any other university. It is claimed to be original except for chapters one and four and other places where explicitly referenced. Some of this work has been published in two papers by the author, one of which was written in collaboration with Dr. J.D.Jenkins. Relevant unpublished work by the author is also included.

The author wishes to express his gratitude to Dr. Fairlie and Dr. Jenkins for their help, guidance and encouragement during the course of this work. He would also like to thank his colleagues for numerous discussions over the last three years. He also thanks the Science Research Council for providing a research studentship.

Abstract

The object of this thesis is twofold. The first part concerns the improvement of the unitary operator of Buccella et al. This operator is an example of a Melosh transformation connecting the algebras of the constituent and current quarks. The second part of this thesis examines the structure of the multiplets and the corresponding Lagrangians arising from the enlarged supersymmetry algebra incorporating isospin.

Chapter One is a general introduction to the Melosh transformation and the work done by Buccella et al. The second chapter examines the difficulties involved with the Buccella transformation and contains a discussion on its possible improvement. In the third and final chapter on the Buccella transformation these improvements are implemented successfully, giving a unitary transformation which is correct to the second order of a perturbation expansion. Using this transformation we are able to obtain mass equations which are in good agreement with experiment in addition to the usual successful predictions for axial couplings typical of Melosh transformations in general.

Chapter Four gives a general introduction to the concept of supersymmetry, describing the successes of the original model and also its special points of interest. The fifth chapter looks in detail at the structure of the multiplets arising from the larger super-algebra incorporating isospin, suggested by Salam and Strathdee. In the sixth chapter we try to form "super-invariant" Lagrangian densities from these multiplets which are physically applicable. Finally there is a discussion of our conclusions.

Contents

CHAPTER ONE

Introduction: In this chapter we shall introduce the Melosh transformation and also that of Buccella et al and describe the connection between them. We shall then look in greater detail at the work done on the Buccella transformation leading to predictions for the mass spectrum.

Section 1.1

It is now more than ten years since Gell-Mann first introduced the concept of quarks in order to give a schematic model for the meson and baryon multiplets of SU(3) (ref 1). It was soon realised (ref 2) that these "constituent quarks" are able to give a very successful classification scheme for all known hadrons and that these hadrons could be arranged in the larger multiplets which are representations of $SU(6)_W \otimes O(3)$. It was also realised that this symmetry cannot be exact, since, for example, the predictions it would give for axial couplings are clearly not physical.

It has also been useful to use quarks to describe the experimental results of deep inelastic lepton scattering. In this case, using a "current quark" model, it is possible to form an $SU(3) \otimes SU(3)$ algebraic structure from the vector and axial vector charges. This can be generalised in the infinite momentum frame to $SU(6)_W$.

Recently H.J.Melosh (ref 3) has postulated the existence of a unitary transformation connecting the generators of $SU(6)_{W;\text{currents}}$ and $SU(6)_{W;\text{constituents}}$ and has explicitly constructed the transformation for free quarks.

At the same time attempts were being made by Buccella et al to find a transformation to connect the "hadron states" classified by the constituent quarks and the hadron states which are actually observed (ref 4-7). This second approach has been called "phenomenological" since it was originally begun in order to account for empirical observations, though it is now possible to see that the theoretical basis of this work is the same as that introduced by Melosh.

The success of both these transformations has been in the prediction of axial couplings which are in good agreement with experiment. In addition, Buccella et al have attempted to use their transformation in order to predict the masses of the mesons (ref 6) but they had only limited success.

The aim of this work is to examine again the problem of predicting meson masses using the Buccella transformation. In particular, we are able to show that it is possible to derive mass equations, involving the observed mesons, which are in good agreement with experiment.

In the remainder of this chapter we shall expand upon the ideas already introduced, showing the difference between constituent and current quarks and the connection between the work of Melosh and that of Buccella et al. Then we shall look at the transformation of Buccella et al in more detail as a preliminary to the following work.

Section 1.2

i) Constituent Quarks

The constituent quark classification scheme implies that mesons could be formed by a quark-antiquark pair and thus fit into simple multiplets represented by $SU(6)_W \otimes O(3)$. Here the quarks themselves are represented by $SU(6)_W$ and $O(3)$ represents the orbital angular momentum between the quark-antiquark pair.

Many difficulties arise if one attempts to test this scheme against experimental observation. We shall just note here that the symmetry group implies that all the members of the ground state multiplet (π, ρ, ω) have the same mass. Also the predictions for axial couplings are readily seen to disagree with experiment since they are zero between different multiplets. Nevertheless, as a classification scheme it is remarkably successful.

We shall restrict our attention throughout this work to the mesons which could be generated by non-strange constituent quarks (though the extension to the full scheme should be a technical problem not involving any new theory). Thus we are considering the hadrons classified according to the $\underline{15}$ representation of $SU(4)$ and the states obtained by exciting these with orbital angular momentum, L.

The ground state multiplet (L=0) contains π, ρ and ω with η as the corresponding singlet. The next multiplet (L=1) contains,

J^{PG}	I=1	I=0
2^{+-}	A_2	f
1^{+-}	A_1	D
0^{+-}	A_0	σ
1^{++}	B	(H)

where H is the corresponding singlet. We should note that some of these states have not been definitely observed but we include them in our classification in order to establish notation.

There is general agreement that the isospin zero member of an SU(3) octet mixes to some extent with the corresponding SU(3) singlet. We shall not include this mixing explicitly in our work but shall note this possibility.

ii) Current quarks

The algebraic structure of the current quarks was first suggested by Gell-Mann (ref 8) for the vector and axial vector charges,

$$Q^i(t) = \int d^3x \; \mathcal{J}^i_0(x) \quad , \quad Q^{i5}(t) = \int d^3x \; \mathcal{J}^{i5}_0(x) \, ,$$
$$(i = 1, \ldots\ldots 8)$$

where $\mathcal{J}^i_\mu(x)$ and $\mathcal{J}^{i5}_\mu(x)$ are octets of local current densities which can in principle be measured in weak and electromagnetic transitions. In particular, in a current quark model these densities can be written as

$$\mathcal{J}^i_\mu(x) = \bar{q}(x)\gamma_\mu \frac{\lambda_i}{2} q(x) \quad , \; \mathcal{J}^{i5}_\mu(x) = \bar{q}(x)\gamma_\mu\gamma_5 \frac{\lambda_i}{2} q(x) \, .$$

With canonical equal-time anticommutation relations for quark fields it can be shown that

$$\left[Q^i(t) \, , Q^j(t) \right] = i f_{ijk} \; Q^k(t)$$
$$\left[Q^i(t) , \; Q^{j5}(t) \right] = i f_{ijk} \; Q^{k5}(t)$$
$$\left[Q^{i5}(t), \; Q^{j5}(t) \right] = i f_{ijk} \; Q^k(t)$$

which is an SU(3)$\otimes$SU(3) algebraic structure.

It was hoped that this algebraic structure could be used for the physical vector and axial vector charges independently of the possible existence of the current quark field. This has not been tested directly but, using P.C.A.C. it leads to the successful Adler-Weisberger relation (ref 9).

If one tries to expand the SU(3)$\otimes$SU(3) group to U(12) using as generators the integrals over the local densities,

$$\bar{q}(x)\Gamma\lambda_i q(x) \qquad (i=0,1 \ldots,8),$$

in the infinite momentum frame many of these operators lead to vanishing matrix elements. We call the operators which do not vanish when taken between finite mass states "good" operators and the remainder are "bad" operators. Following Gell-Mánn (ref 10) we can show that,

S:	$q^{+}\beta\lambda_i q$	$\approx \frac{1}{P_\infty}$	"bad"	
P:	$q^{+}\beta\gamma_5\lambda_i q$	$\approx \frac{1}{P_\infty}$	"bad"	
V:	$q^{+}\lambda_i q$	≈ 1	"good"	identical
	$q^{+}\alpha_z\lambda_i q$	≈ 1	"good"	
	$q^{+}\alpha_{x,y}\lambda_i q$	$\approx \frac{1}{P_\infty}$	"bad"	
A:	$-q^{+}\gamma_5\lambda_i q$	≈ 1	"good"	identical
	$q^{+}\sigma_z\lambda_i q$	≈ 1	"good"	
	$q^{+}\sigma_{x,y}\lambda_i q$	$\approx \frac{1}{P_\infty}$	"bad"	
T:	$-iq^{+}\beta\alpha_x\lambda_i q$	≈ 1	"good"	identical
	$q^{+}\beta\sigma_y\lambda_i q$	≈ 1	"good"	
	$-iq^{+}\beta\alpha_y\lambda_i q$	≈ 1	"good"	identical
	$q^{+}\beta\sigma_x\lambda_i q$	≈ 1	"good"	
	$q^{+}\beta\alpha_z\lambda_i q$	$\approx \frac{1}{P_\infty}$	"bad"	
	$q^{+}\beta\sigma_z\lambda_i q$	$\approx \frac{1}{P_\infty}$	"bad"	

We can see that the "good" charges generate an $SU(6)_W$ algebra which we shall call $SU(6)_{W;\text{currents}}$ (to distinguish it from $SU(6)_{W;\text{constit.}}$ introduced previously). This larger algebra requires the existence of tensor currents, for which the experimental evidence is not clear.

Section 1.3

The similarity between $SU(6)_{W;\text{constituents}}$ and $SU(6)_{W;\text{currents}}$ leads one to suggest that they may be equivalent, that is, it is possible that we can equate the generators. There are many examples that can be given to show that this is not feasible in practice. As already noted, if we take $SU(6)_{W;\text{constituents}}$ as more than a classification scheme we run into the difficulty that all the masses in the ground state multiplet (π, ρ, ω) are predicted to have the same value. Also, many decays are forbidden since the axial couplings between members of different multiplets are zero (for example, $A_1 \not\to \rho\pi$, $B \not\to \omega\pi$).

Melosh suggested the possibility of relating the generators of the two $SU(6)_W$ algebras using a unitary transformation, thus allowing both descriptions to be equally valid. The difficulty is to determine the form of this unitary transformation.

The "Melosh transformation" (ref 3) was derived by assuming that the quarks are free. It is then possible to determine the form of the transformation explicitly. Using a technique similar to the Foldy-Wouthuysen transformation (ref 11) to exclude "bad" operators, Melosh shows that

$$|q\,\underline{k}\rangle = U(\underline{k})\,|\tilde{q}\,\underline{k}\rangle \qquad U(\underline{k}) = e^{i\theta(k)\,(\underline{\sigma}\wedge\underline{k})_z/2k}$$

$$\theta(k) = \arctan k/m \qquad k = |\underline{k}| \tag{1.1}$$

where q denotes a constituent quark with effective mass m, $\tilde{q}$ denotes a current quark and $\underline{k}$ is the transverse momentum of both quarks. (Note, we are only concerned with states moving with infinite momentum in the z-direction.) In order to apply this transformation in a realistic manner one has to abstract the important characteristics, the importance being judged by the success of the predictions that follow.

The unitary transformation proposed by Buccella et al (ref 5) arose independently of the Melosh transformation. It was found that the introduction of mixing between the "hadron states" classified by constituent quarks could produce some successful predictions for the axial coupling constants (ref 4). In addition, it was possible to make some mass predictions but these were very unphysical, suggesting that a greater degree of mixing was necessary. Therefore, in order to introduce a general mixing scheme without, at the same time, introducing an infinite number of arbitrary parameters, a unitary operator was suggested (ref 5).

In fact, Buccella et al did not consider the full $SU(6)_W$ algebras, since in their "phenomenological" approach it was only necessary to look at the subalgebras $SU(3) \otimes SU(3)$. For convenience they confined their attention even further to the non-strange subalgebras $SU(2) \otimes SU(2)$, though, as already noted, the extension to states with strangeness should simply be a technical problem requiring no new theory. In this work we shall restrict our attention to the $SU(2) \otimes SU(2)$ subalgebras throughout.

The unitary operator, U, is defined such that

$$Q^5_a = U\, A(\sigma_z \tau_a)\, U^\dagger \quad , \quad Q_a = A(\tau_a) \tag{1.2}$$

where $A(\sigma_z \tau_a)$ and $A(\tau_a)$ are the generators of the chiral subalgebra $SU(2) \otimes SU(2)$ of $SU(4)$ for the constituent quarks. The particular form of the unitary operator of Buccella et al is

$$U(\theta) = \exp(\theta Z) \tag{1.3}$$

$$Z = i(\underset{\sim}{W} \wedge \underset{\sim}{M})_z = M_+ W_- - M_- W_+ \tag{1.4}$$

where $\underset{\sim}{M}$ is a vector under $O(3)$ and $\underset{\sim}{W}$ is similar to the w-spin of Lipkin and Meshkov; θ is an undetermined parameter. The effect of this operator is to mix states with the same quantum numbers, except for parity and spin which can vary because of excitation by additional orbital angular momentum.

Using this operator at first order in a perturbation expansion, it was shown to be possible to duplicate and improve upon the good predictions for axial couplings obtained in ref.4 . For example, by fixing one parameter so that $q^2_{\rho\pi}(0) = 1/9$ * in agreement with experiment, Buccella et al predict (ref 5)

$$\sum_h q^2_{A_1\rho}(h) = \tfrac{1}{2} \qquad (\text{expt.} \leq 0.48)$$

$$\sum_h q^2_{B\omega}(h) = 1/3 \qquad (\text{expt.}\ \ 0.33)$$

$$\sum_h q^2_{A_2\rho}(h) = 1/6 \qquad (\text{expt.}\ \ 0.13)$$

$$q^2_{\sigma\pi} = 2/9 \qquad (\text{expt.} \gg 0.1) \tag{1.5}$$

and also the ratio $\left|\dfrac{q_{A_1\rho}(1)}{q_{A_1\rho}(0)}\right| = \dfrac{1}{2}$ (expt. 0.48 ± 0.13).

We can show, following C.A.Savoy (ref 12), that there is a close similarity between the Melosh transformation for mesons and the "phenomenological" transformation of Buccella et al. We will assume that hadrons, in the infinite momentum frame, are a simple system of constituent quarks which interact with currents like free quarks. This is probably not a realistic assumption but it would seem to be a good starting point and sufficient for our purposes. If the $q\bar{q}$ pair of constituent quarks in a meson have transverse momentum $\underline{k}$ and $\bar{\underline{k}}$ respectively and the meson has no transverse momentum then $\underline{k} = -\bar{\underline{k}}$.

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* where $\langle\beta|Q^5_\alpha(h)|\alpha\rangle = \delta_{t_\beta,a}\, q_{\beta\alpha}(h)$ for $I_\alpha=0, I_\beta=1$

$\langle\beta|Q^5_a(h)|\alpha\rangle = i\,\varepsilon_{t_\beta,a t_\alpha}\, q_{\beta\alpha}(h)$ for $I_\alpha=I_\beta=1$

with $q_{\beta\alpha}$ hermitian.

The current quark content can now be written using the Melosh transformation,

$$|q\underset{\sim}{k}, \bar{q}\underset{\sim}{\bar{k}}\rangle = U(\underset{\sim}{k})\,\bar{U}(\underset{\sim}{\bar{k}})\,|\tilde{q}\underset{\sim}{k}, \tilde{\bar{q}}\underset{\sim}{\bar{k}}\rangle$$
$$= e^{i\theta(k)\left[(\underset{\sim}{\varepsilon}-\underset{\sim}{\bar{\varepsilon}})\wedge \underset{\sim}{k}\right]_z/2k}\,|\tilde{q}\underset{\sim}{k}, \tilde{\bar{q}}\underset{\sim}{\bar{k}}\rangle$$

where $\tilde{\bar{q}}$ represents a current antiquark. We can now compare this unitary operator with that of Buccella et al (eqn 1.3,1.4). The two operators are equal if

$$2\theta\underset{\sim}{M} = \theta(k)\,\underset{\sim}{k}/k.$$

There are a number of points which arise from this comparison;-

i) $\theta\underset{\sim}{M}$ is independent of isospin

ii) $[M_i, M_j] = 0$

iii) $\theta\underset{\sim}{M}$ is not a vector under rotations

iv) a perturbative approach is justified only if $k \ll m$.

Unfortunately, we do not know how much importance to give to these implications, since the derivation for the Melosh transformation for mesons involved strong assumptions. Also, we know that there must be corrections at the second order in θ in the Buccella transformation in order for it to satisfy helicity conditions (ref 7; see section 1.5). We shall therefore not demand that these implications are satisfied but consider their importance at a later stage.

We should emphasise again the strong similarity between the form of the two transformations. This implies that the transformation of Buccella et al can now be understood in terms of the theoretical arguments introduced by Melosh, instead of having to rely on purely phenomenological arguments.

There have been a number of successful attempts to produce results similar to those described above using differing assumptions but based on the Melosh transformation (ref 13). The particular success of the Buccella transformation is that it is not only able to produce results concerning axial couplings but that it is also able to give predictions relating to the masses of the mesons (ref 6). This has not been demonstrated by other similar models.

The method used by Buccella et al is to expand the unitary operator to second order in θ and apply constraints resulting from Weinberg's equation (ref 14),

$$\left[Q_a^5 , \left[Q_b^5 , m^2 \right] \right] \propto \delta_{ab} \,. \tag{1.6}$$

To do this it is necessary to give a strict interpretation to the operators $\underset{\sim}{M}$ and $\underset{\sim}{W}$ which initially are not completely defined.

In the next section we shall summarize the necessary theory related to Weinberg's equation which will be used in the following chapters. Then, in sections 1.5 and 1.6, we shall look at the attempts, first by Buccella, Celeghini and Savoy (ref 6) and then by Celeghini, Sorace and Zappa (ref 7), to formulate definitions for $\underset{\sim}{M}$ and $\underset{\sim}{W}$ and predict mass relations for the mesons.

Section 1.4

In this section we shall summarize the theory resulting from Weinberg's equation which we shall need in the following work and which was first introduced in this form in ref. 6. We can rewrite Weinberg's equation (eqn 1.6) as

$$\left[A(\sigma_a\tau_a),\ \left[A(\sigma_b\tau_b),\ U^+m^2U\right]\right] = \frac{1}{3}\delta_{ab}\left[A(\sigma_c\tau_c),\ \left[A(\sigma_c\tau_c),\ U^+m^2U\right]\right] \tag{1.7}$$

using equation(1.2). Equation (1.7) implies (ref 14) that U^+m^2U must transform as the sum of a chiral scalar and the fourth component of a chiral four-vector under the SU(2)$\otimes$SU(2) algebra.

The SU(2)$\otimes$SU(2) content of states belonging to the $\underline{15}\oplus\underline{1}$ representation of SU(4) is,

	$\underline{I=1}$	$\underline{I=0}$
$\underline{S=1,\ S_z=0}$	$\|\bar{t}\rangle = \frac{1}{\sqrt{2}}\left[(1,0)-(0,1)\right]$	$\|\bar{s}\rangle = (0,0)$
$\underline{S=1,\ S_z=\pm1}$	$\|\vec{v},S_z=\pm1\rangle = (\frac{1}{2},\frac{1}{2})_{I=1}$	$\pm\|v_4,S_z=\pm1\rangle = \pm(\frac{1}{2},\frac{1}{2})_{I=0}$
$\underline{S=0}$	$\|t\rangle = \frac{1}{\sqrt{2}}\left[(1,0)+(0,1)\right]$	$\|s\rangle = (0,0)$.

It is easily seen that the absence of $(1,1)_{I=0}$ in U^+m^2U causes constraints to be made on the following terms in order to satisfy equation (1.7)

$$\langle(1,0)|\,(1,1)_{I=0}\,|(0,1)\rangle\ ,\quad \langle(\tfrac{1}{2},\tfrac{1}{2})|\,(1,1)_{I=0}\,|(\tfrac{1}{2},\tfrac{1}{2})\rangle.$$

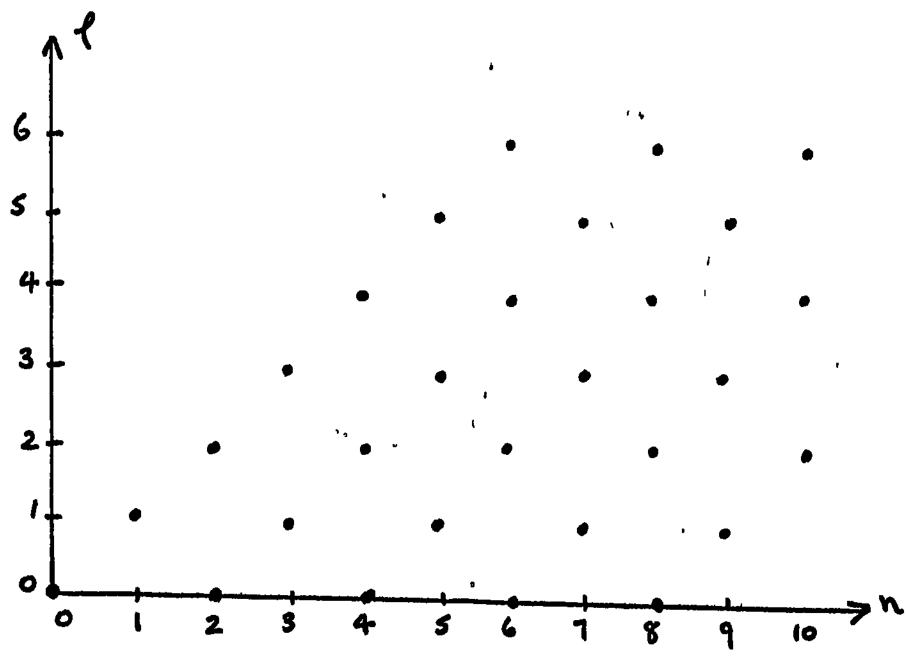
These constraints are

$$\langle t\,n'\,l'\,l_z|\,U^+m^2U\,|t\,n\,l\,l_z\rangle = \langle\bar{t}\,n'\,l'\,l_z|\,U^+m^2U\,|\bar{t}\,n\,l\,l_z\rangle \tag{1.8}$$

$$\langle \vec{v}\, s_z'\, n'\, l'\, l_z' | U^{+} m^2 U | \vec{v}\, s_z\, n\, l\, l_z \rangle$$
$$= \langle v_4\, s_z'\, n'\, l'\, l_z' | U^{+} m^2 U | v_4\, s_z\, n\, l\, l_z \rangle \tag{1.9}$$

$$\langle t\, n'\, l'\, l_z | U^{+} m^2 U | \bar{t}\, n\, l\, l_z \rangle$$
$$= \langle \bar{t}\, n'\, l'\, l_z | U^{+} m^2 U | t\, n\, l\, l_z \rangle. \tag{1.10}$$

In these equations (l, l_z) represents the orbital excitation of the states. The quantum number n counts the number of times one has to apply the operator $\underset{\sim}{M}$ to reach a state, starting from the ground state (L=0), on the (n,l) lattice. ($\underset{\sim}{M}$ will be defined such that each time it is applied $\Delta l = 1$, $\Delta n = 1$.)



The introduction of this lattice structure allows for the existence of more than one set of states at each value of l, thus we can use daughters.

The approach used throughout this work is a perturbative one and the $(\text{mass})^2$ operator is expanded as

$$m^2 = m_0^2 + \theta^2 m_2^2 + \theta^4 m_4^2 + \ldots\ldots \tag{1.11}$$

When we have defined the unitary operator, $U(\theta)$, we shall adopt the procedure of examining the nature of the above constraints at each power of the parameter, θ. We shall assume in this work that $U(\theta)$ is adiabatic, that is, it is a continuous function in θ and $U(\theta) \rightarrow 1$ as $\theta \rightarrow 0$.

Section 1.5

Before going on to complete the definition of the unitary operator, we shall first display explicitly the conditions that it must satisfy (ref 7),

$$[J_3, U(\theta)] = 0 \;, \quad [\eta, U(\theta)] = 0 \;, \quad [G, U(\theta)] = 0 \qquad (1.12)$$

$$\left[J_+, \left[J_+, U A(\sigma_a \tau_3) U^+\right]\right]_{\Delta n = 0} = 0 \qquad (1.13)$$

where J is the total angular momentum, G is G-parity and $\eta = P e^{-i\pi J_2}$ represents inversion of the y-axis.

We shall find the first three conditions easy to satisfy. But the final condition will necessitate the introduction of a correction at the second order in θ to the unitary operator (eqn 1.3). So the unitary operator now takes the form,

$$U(\theta) = 1 + \theta Z + \tfrac{1}{2}\theta^2 Z^2 + \theta^2 b + O(\theta^3). \qquad (1.14)$$

Furthermore, ref.7 shows that such corrections must be introduced at all even powers of θ.

Celeghini, Sorace and Zappa then begin to construct a mathematical formalism in which such a structure of an infinite series of operators can be handled. The advantage of such a formalism is that it could be used in the whole class of Melosh transformations and not just the Buccella transformation. We shall not follow this line but instead concentrate here on trying to improve the Buccella transformation.

<u>Section 1.6</u>

In this section we shall begin to tackle the problem of defining the terms in our unitary operator. This was first attempted in ref.6, where the following definitions were suggested;

$$\begin{aligned} &\underline{I=1}: \quad W_\pm |t\rangle = |\vec{v}, s_z = \pm 1\rangle \; ; \quad W_\pm |\bar{t}\rangle = 0 \\ &\underline{I=0}: \quad W_\pm |s\rangle = \pm\lambda |v_4, s_z = \pm 1\rangle \; ; \quad W_\pm |\bar{s}\rangle = 0 \end{aligned} \tag{1.15}$$

and for the $\underset{\sim}{M}$ operator,

$$\langle I=1; n' \, l \; l_z \pm 1 | M_\pm | I=1; n, l-1, l_z \rangle = = \frac{1}{\sqrt{2}} \sqrt{(l \pm l_z)(l \pm l_z + 1)} \; C^{n' \; l}_{n \; l-1}$$

$$\langle I=0; n' \, l \; l_z \pm 1 | M_\pm | I=0; n \; l-1 \; l_z \rangle = = \frac{1}{\sqrt{2}} \sqrt{(l \pm l_z)(l \pm l_z + 1)} \; \tilde{C}^{n' \; l}_{n \; l-1} \tag{1.16}$$

where λ, C and $\tilde{C}$ are real numbers.

The constraints of the Weinberg equation (1.8-1.10) imply that

$$m_0^2 = A_0 + \Delta n + B(-1)^{l+1}(\underset{\sim}{L}.\underset{\sim}{S} + \tfrac{1}{2}) \tag{1.17}$$

$(A_0, \Delta, B \text{ constants})$

for both isospin one and isospin zero states. We can also deduce that

$$\begin{aligned} \lambda \, \tilde{C}^{n \; l}_{n-1 \; l-1} &= C^{n \; l}_{n-1 \; l-1} = \frac{f_+(n+l)}{\sqrt{4l^2-1}} \\ \lambda \, \tilde{C}^{n-1 \; l}_{n \; l-1} &= C^{n-1 \; l}_{n \; l-1} = \frac{f_-(n-l)}{\sqrt{4l^2-1}} \, , \end{aligned} \tag{1.18}$$

where f_+ and f_- are arbitrary functions such that $f_-(-1)=0$, so that there is no tendency for states to mix with "ancestors" above the leading trajectory. It is evident from these equations that the attempt to make a difference between isospin one and isospin zero in the definitions of $\underset{\sim}{W}$ and $\underset{\sim}{M}$ has not worked. Hence, $\lambda = 1$ and $\underset{\sim}{M}$ is independent of isospin, as suggested by the comparison in section 1.3

between the Buccella and Melosh transformations.

So far we have not considered the constraints on the system imposed by considering the Weinberg conditions at second order when $n'=n$ and $\ell'=\ell$. This reqires that,

$$(-1)^{\ell+1}\beta\left[f_+^2(n+\ell+2)+f_-^2(n-\ell-1)-f_+^2(n+\ell)-f_-^2(n-\ell+1)\right]=$$

$$= 2\Delta\left[\frac{f_+^2(n+\ell+2)-f_-^2(n-\ell-1)}{2\ell+3}-\frac{f_+^2(n+\ell)-f_-^2(n-\ell+1)}{2\ell-1}\right]. \tag{1.19}$$

Clearly, one solution of this equation is,

$$f_\pm(n\pm\ell)=\sqrt{n\pm\ell+1} \tag{1.20}$$

and this solution ensures that $f_-(-1)=0$.

We shall show in the next chapter that this solution has undesirable consequences and attempt to find other solutions of equation (1.19). We should note that the work in ref.6 did not take account of the correction in the unitary operator that was introduced in the last section but similar results can be obtained if the correction is included.

CHAPTER TWO

In this chapter we shall introduce a new constraint on the functions, $f_{\pm}$, which appear in the definition of the $\underline{M}$-operator. The original solution for the functions, $f_{\pm}$, does not satisfy this constraint, hence we shall look for further solutions. Finally there is a discussion of difficulties encountered.

Section 2.1

The problem has now been clearly outlined. It is basically one of increasing the degree of accuracy of the unitary operator so that it is not only compatible with the experimental values for axial couplings amongst the mesons but also compatible with the masses of the mesons. We shall show in this chapter the difficulties encountered by the solution proposed in ref.6 and then go on to discuss improvements which will be incorporated in the following chapter.

In this section we shall introduce a new constraint upon the functions, $f_{\pm}$, which appear in the definition of $\underline{M}$ and show that this necessary constraint is not satisfied by the solution of ref.6 (eqn 1.20). In section 2.2 we shall look again at the problem of solving equation (1.19) and then in section 2.3 we shall consider the changes necessary in order to improve this scheme.

There are two conditions which must be satisfied by the functions, $f_{\pm}$. The first, as we have already seen, is that $f_{-}(-1)=0$. This ensures that there is no possibility of states mixing with "ancestors" which are above the leading trajectory. The second condition is that $f_{\pm}$ must be bounded above by a constant. Since this condition was not used in ref.6 we shall make this point more explicitly.

The aim of the mixing is to obtain an operator of the form

$$U(\theta)|{}_m Z_i^{\ell}\rangle = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{j} {}_n a_j^r \,|{}_n Z_j^r\rangle \tag{2.1}$$

where $|{}_n a_j^r| \leq 1$ and $\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{j} ({}_n a_j^r)^2 = 1$

and ${}_m Z_i^{\ell}$ is a meson state, i, excited by orbital angular momentum, ℓ, with the additional quantum number equal to m. But, since

$$\begin{aligned} U(\theta)|{}_m Z_i^{\ell}\rangle &= |{}_m Z_i^{\ell}\rangle - \theta\,(M_- W_+ - M_+ W_-)|{}_m Z_i^{\ell}\rangle + O(\theta^2) \\ &= |{}_m Z_i^{\ell}\rangle - \theta \sum_{\substack{m'=m\pm1 \\ \ell'=\ell\pm1}}\sum_{j} {}_{m'}\beta_j^{\ell'}\,|{}_{m'} Z_j^{\ell'}\rangle + O(\theta^2), \end{aligned} \tag{2.2}$$

if the ${}_{n'}\beta_j^{l'}$ coefficient becomes unbounded then the only possibility to cancel this unboundedness is between the various terms of the series involving different powers of θ. But, the condition of adiabaticity suggests that there should be a solution possible for all θ as $\theta \to 0$, $U(\theta) \to 1$. Therefore the functions, $f_\pm$, are bounded above by a constant.

This condition excludes the solution,

$$f_\pm(n \pm l) = \sqrt{n \pm l + 1}$$

which had been considered originally (ref 6). In the next section we shall re-examine equation (1.19) in order to try to find a physically acceptable solution.

Section 2.2

We return now to the problem of solving equation (1.19) in the hope of obtaining an acceptable solution. There are two possibilities, either n and ℓ are independent of each other or they have some kind of interdependence. In the latter case the only real alternative is that $n-\ell =$ constant, i.e. $f_-(x) = 0$, and this will be considered later.

We shall look first at the case in which n and ℓ are independent (except $n+\ell$ will remain either odd or even throughout the lattice). Equation (1.19) can be rewritten in the form

$$F(n,\ell) \equiv \frac{2\Delta}{2\ell-1}\left[f_+^2(n+\ell) - f_-^2(n-\ell+1)\right] + (-1)^\ell B\left[f_+^2(n+\ell)+f_-^2(n-\ell+1)\right]$$
$$= F(n,\ell+2). \tag{2.3}$$

Since n and ℓ are independent, $F(n,\ell+2)$ is independent of ℓ and so

$$F(n,\ell) = F(n,n+2) = \frac{2\Delta}{2n+3}\, f_+^2(2n+2) + (-1)^n B\, f_+^2(2n+2).$$

If $n-\ell = A$, for some $A \geqslant 0$, as $n \to \infty$ then it can be shown that

$$B\, f_-^2(A+1) = 0 \tag{2.4}$$

i.e. either $B=0$ or $f_-^2(x) \equiv 0$. (In the latter case n and ℓ are linearly interdependent and so we will assume that $B=0$ here).
If $n \to \infty$ but ℓ remains finite then

$$\frac{2\Delta}{2\ell-1} \lim_{n\to\infty}\left[f_+^2(n+\ell) - f_-^2(n-\ell+1)\right] = 0 \tag{2.5}$$

i.e. either $\Delta = 0$, and the whole equation is degenerate, or the limit is zero.

If $\Delta \neq 0$ then

$$\frac{2}{2\ell-1} \lim_{n\to\infty} n^r\left[f_+^2(n+\ell) - f_-^2(n-\ell+1)\right] = \lim_{n\to\infty} f_+^2(2n+2)\,n^{r-1}$$

which together with the condition that $f_{\pm}$ are bounded above by a constant implies that $f_+^2(x) - f_-^2(x) \equiv 0$.

Hence there are three distinct possibilities:-

i) $f_-^2 \equiv 0$ and then for the leading trajectories, $n=\ell$,

$$f_+^2(2\ell) = \frac{\lambda(4+\ell^2-1)}{[B(2\ell-1)-2\Delta][B(2\ell+1)+2\Delta]} \quad \text{or} \quad \frac{\lambda(4+\ell^2-1)}{[B(2\ell-1)+2\Delta][B(2\ell+1)-2\Delta]} \tag{2.6}$$

depending on whether ℓ is even or odd. In order to guarantee the positivity of f_+^2 , $|B| > 2\Delta/3$

ii) $B = \Delta = 0$

iii) $f_+ = f_- \equiv 0$ for $\ell > 0$, but $f_{\pm}^2(x) \not\equiv 0$ for $\ell = 0$ where the equation does not necessarily apply since the helicity is zero. It is necessary that $B=0$ if there is to be mixing at the very lowest level and also $f_+(n+1) = f_-(n)$ for agreement with the equation (1.19), but Δ remains unconstrained.

These alternatives should now be discussed with reference to the spectrum at zero order given by m_0^2 (eqn 1.17). This spectrum is completely degenerate in the second case. In case (i) the slopes of the trajectories, Δ and $\Delta \pm B$, are too widely spaced to be considered approximations to the observed situation.

The final solution is merely a statement that at low orders there is no mixing except amongst the members of the ground state multiplet. This solution is clearly unsatisfactory since, although it allows the π-ρ mass split and spaces all other multiplets in a way that is consistent with no mixing, it does not give the required axial couplings between the different multiplets.

Hence we have explored all the possibilities contained within the set of definitions suggested in ref.6 and have been unable to extract any system which meets with our requirements.

Section 2.3

There are two sources of difficulty emerging from the work of Buccella et al in ref.6 . The first is the problem of obtaining a feasible scheme which is consistent with equation (1.19). It is worthwhile noting that, if we did not have to satisfy this equation, there would be a much greater possibility of obtaining a solution which is physically applicable.

The second difficulty is that in the above discussion we have omitted to take account of the additional term, b, which was introduced in equation (1.14) in order to ensure the helicity conditions (eqn 1.13) are satisfied. The reason for not including this term is that its form is not straightforward because of the nature of the definitions for the operators $\underset{\sim}{M}$ and $\underset{\sim}{W}$. The process involved in solving equation (1.19) has been shown to be one of increasing degeneracy, cutting away the possibility of "fine structure" for the $(\text{mass})^2$ operator. It is reasonable to suppose that the further complication of an additional term would not simplify equation (1.19) but, instead, increase the rate at which the "fine structure" disappeared. In particular, it will be shown in the next chapter, in a modified system, that the presence of the b-term causes $B=0$ at an early stage and subsequently plays no part in the calculations up to second order.

The difficulty found in constructing the b-term can be removed by redefining the $\underset{\sim}{W}$-operator in a way such that the $\underset{\sim}{W}$ and spin operators together form an $SU(2)\otimes SU(2)$ algebra. We shall show in the next chapter that, by this redefinition of the $\underset{\sim}{W}$-operator and also redefining the $\underset{\sim}{M}$-operator in an analogous way, it is possible to obtain a system which is free of the difficulties encountered in this chapter.

CHAPTER THREE

In this final chapter on the transformation of Buccella et al we shall introduce the improvements previously noted and find as a consequence that we are able to produce mass equations which are in good agreement with experiment.

<u>Section 3.1</u>

In this chapter we shall construct a unitary operator which gives predictions that are compatible with both the axial couplings and the meson masses. In the next section we shall introduce our new definitions for $\underset{\sim}{W}$ and $\underset{\sim}{M}$, explaining their advantages over the previous definitions. Then, in section 3.3, we shall work through the process of checking the effects of the constraints from Weinberg's equation (eqn 1.8-1.10). Section 3.4 will be an analysis of the consequences of the new approach, giving a brief comparison with the results of ref.6 . Finally, in the last section, we shall summarize the achievements of our work on the Buccella transformation.

This section is devoted to generalising the procedure used in ref.4 to find the unmixed $(\text{mass})^2$ equations of any multiplet of mesons. In ref.4 this problem was only considered for multiplets of mesons which could be constructed from constituent quarks with orbital angular momentum, $\ell = 0,1$. We shall generalise this procedure to all values of angular momentum, ℓ.

In order to determine the chiral content of a general multiplet, we must consider the application of orbital angular momentum to the ground state multiplet, $\ell = 0$. This multiplet can be represented as follows,

$\underline{h=1}$	$\underline{h=0}$	$\underline{h=-1}$
$\rho = v_1^A$	$\pi = t'$, $\eta = s'$	$\rho = v_1^B$
$\omega = v_0^A$	$\rho = t$, $\omega = s$	$\omega = v_0^B$

where h is the helicity. Giving this system angular momentum, ℓ, results in a multiplet containing eight particle states, Z_i^ℓ , where i=1,2, ...,8 labels the states within each multiplet such that,

Spin	I = 1	I = 0
$\ell+1$	Z_1^ℓ	Z_4^ℓ
ℓ	Z_2^ℓ , Z_7^ℓ	Z_5^ℓ, Z_8^ℓ
$\ell-1$	Z_3^ℓ	Z_6^ℓ

(Note, the Z_7^ℓ results from giving angular momentum to the π and Z_2^ℓ from the ρ.)

If we consider the Clebsch-Gordan coefficients we are ~~lead~~ led to the following set of equations at helicity, m,

$$Z_1^\ell = \sqrt{\frac{(\ell+m+1)(\ell+m)}{2(\ell+1)(2\ell+1)}}\,v_1^A + \sqrt{\frac{(\ell+1)^2-m^2}{(\ell+1)(2\ell+1)}}\,t + \sqrt{\frac{(\ell-m+1)(\ell-m)}{2(\ell+1)(2\ell+1)}}\,v_1^B$$

$$Z_2^\ell = -\sqrt{\frac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}}\,v_1^A + \frac{m}{\sqrt{\ell(\ell+1)}}\,t + \sqrt{\frac{(\ell+m+1)(\ell-m)}{2\ell(\ell+1)}}\,v_1^B$$

$$Z_3^\ell = \sqrt{\frac{(\ell-m+1)(\ell-m)}{2\ell(2\ell+1)}}\,v_1^A - \sqrt{\frac{\ell^2-m^2}{\ell(2\ell+1)}}\,t + \sqrt{\frac{(\ell+m)(\ell+m+1)}{2\ell(2\ell+1)}}\,v_1^B$$

$$Z_7^\ell = t' \qquad (3.1)$$

where $v_1^A = |11\rangle|\ell\ m-1\rangle$ $\qquad v_1^B = |1-1\rangle|\ell\ m+1\rangle$

$t = |10\rangle|\ell\ m\rangle_{I=1}$ $\qquad s = |10\rangle|\ell\ m\rangle_{I=0}$

using the notation $\quad |s\,s_z\rangle|\ell\ m\rangle$.

There are also a similar set of equations for the isospin zero states obtained by changing $v_1 \to v_0$, $t \to s$, $t' \to s'$, for the states labelled 4,5,6,8 respectively.

From these equations it is possible to find the following mass relations by inverting the matrix of Clebsch-Gordan coefficients,

$$m^2_{v_1^A} = \frac{(\ell+m+1)(\ell+m)}{2(\ell+1)(2\ell+1)}\,m^2_{Z_1^\ell} + \frac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}\,m^2_{Z_2^\ell} + \frac{(\ell-m+1)(\ell-m)}{2\ell(2\ell+1)}\,m^2_{Z_3^\ell}$$

$$m^2_t = \frac{(\ell+1)^2-m^2}{(\ell+1)(2\ell+1)}\,m^2_{Z_1^\ell} + \frac{m^2}{\ell(\ell+1)}\,m^2_{Z_2^\ell} + \frac{\ell^2-m^2}{\ell(2\ell+1)}\,m^2_{Z_3^\ell}$$

and $$m^2_{t'} = m^2_{z_7^\ell} \tag{3.2}$$

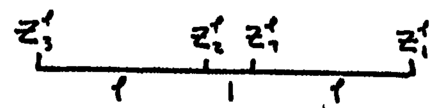
together with the equation given by changing $v_1^A \to v_1^B$, $m \to -m$. There are also four equivalent equations for the isospin zero states.

If there is no mixing then $m^2_t = m^2_{t'}$, $m^2_{v_1^A} = m^2_{v_0^A}$, $m^2_{v_1^B} = m^2_{v_0^B}$,

which implies that $m^2_{z_1^\ell} = m^2_{z_4^\ell}$, $m^2_{z_2^\ell} = m^2_{z_5^\ell}$, $m^2_{z_3^\ell} = m^2_{z_6^\ell}$, $m^2_{z_7^\ell} = m^2_{z_8^\ell}$

and
$$(2\ell+1)\, m^2_{z_2^\ell} = \ell m^2_{z_1^\ell} + (\ell+1)\, m^2_{z_3^\ell}$$
$$(2\ell+1)\, m^2_{z_7^\ell} = (\ell+1)\, m^2_{z_1^\ell} + \ell m^2_{z_3^\ell} \,. \tag{3.3}$$

From these equations it can readily be shown that the spacing of the squared-masses within the ℓ-th multiplet is



This result is equivalent to saying that, if there is no mixing,

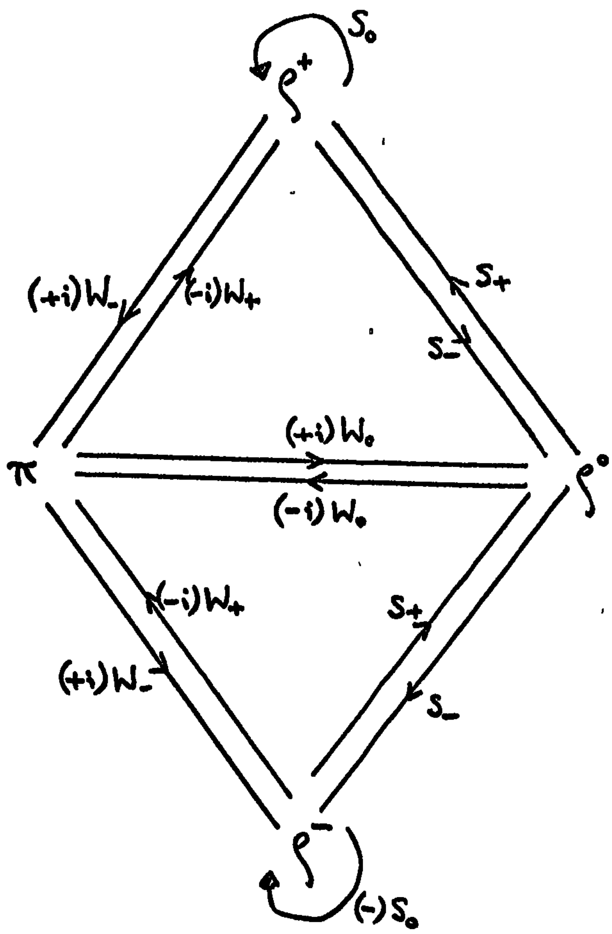
$$m_0^2 = A(\ell) + B(\ell)\, \underline{L} \cdot \underline{S} \tag{3.4}$$

but the above method of derivation differs from that used to obtain this equation previously (ref.2).

Section 3.2

In this section we shall introduce new definitions for the operators $\underset{\sim}{M}$ and $\underset{\sim}{W}$. These are essentially only modifications to the definitions previously used but we will find that they are sufficient to make significant improvements.

The new definitions for the $\underset{\sim}{W}$-operator are most concisely expressed in the following diagrammatic form;



where the constant in brackets is the coefficient associated with that particular operation, for example, $W_+|\rho^-\rangle = -i|\pi\rangle$.
The $\underset{\sim}{S}$-operators form the usual SU(2) spin algebra and the $\underset{\sim}{S}$ and $\underset{\sim}{W}$ operators combine to form SU(2)$\otimes$SU(2),

$$[S_i, S_j] = i\varepsilon_{ijk}S_k \quad , \quad [S_i, W_j] = i\varepsilon_{ijk}W_k \quad , \quad [W_i, W_j] = i\varepsilon_{ijk}S_k \ .$$

The operations on the isospin zero states are exactly the same.

In order to simplify the algebraic properties of $\underset{\sim}{M}$, we shall modify the definitions of ref.6, as follows,

$$\langle n+1, \ell+1, m' | M_{\substack{\pm\\0}} | n\, \ell\, m\rangle = \mp i \sqrt{\frac{\ell+1}{2\ell+3}}\, C^{\ell\;\;1\;\;\ell+1}_{m\;\;\;\;\;m'}\, f_+(n+\ell+2)$$

$$\langle n-1, \ell+1, m' | M_{\substack{\pm\\0}} | n\, \ell\, m\rangle = \mp i \sqrt{\frac{\ell+1}{2\ell+3}}\, C^{\ell\;\;1\;\;\ell+1}_{m\;\;\;\;\;m'}\, f_-(n-\ell-1)$$

$$\langle n+1, \ell-1, m' | M_{\substack{\pm\\0}} | n\, \ell\, m\rangle = \mp i \sqrt{\frac{\ell}{2\ell-1}}\, C^{\ell\;\;1\;\;\ell-1}_{m\;\;\;\;\;m'}\, f_-(n-\ell+1)$$

$$\langle n-1, \ell-1, m' | M_{\substack{\pm\\0}} | n\, \ell\, m\rangle = \mp i \sqrt{\frac{\ell}{2\ell-1}}\, C^{\ell\;\;1\;\;\ell-1}_{m\;\;\;\;\;m'}\, f_+(n+\ell) \qquad (3.5)$$

where $C^{\ell\;1\;\ell'}_{m\;\;\;m'}$ are Clebsch-Gordan coefficients. It can easily be shown that $M_+^\dagger = M_-$ and $M_0^\dagger = M_0$ if the functions, $f_\pm$, are real.

The presence of the Clebsch-Gordan coefficients, giving the helicity dependence, ensure that the commutation relations with the orbital angular momentum operators, $\underset{\sim}{L}$, are

$$[L_+, M_-] = M_0 \quad , \quad [L_3, M_\pm] = \pm M_\pm \, .$$

Alternatively, setting $M_\pm = (M_1 \pm i M_2)/\sqrt{2}$ in the usual way

$$[L_i, M_j] = i\,\varepsilon_{ijk} M_k \, .$$

Commuting the M-operators amongst themselves gives the following,

$$\langle n, \ell, m' | [M_0, M_\pm] | n, \ell, m\rangle = \pm \langle n, \ell, m' | L_\pm | n, \ell, m\rangle\, g(n,\ell)$$

$$\langle n, \ell, m | [M_+, M_-] | n\, \ell\, m\rangle = \langle n\, \ell\, m | L_3 | n\, \ell\, m\rangle\, g(n,\ell) \qquad (3.6)$$

and zero for all other possible matrix elements, where

$$g(n,\ell) = \frac{1}{2\ell+1}\left[f_+^2(n+\ell+2) + f_-^2(n-\ell-1) - f_-^2(n-\ell+1) - f_+^2(n+\ell) \right].$$

It is evident that if the function, $g(n,\ell)$ is a non-zero constant, then we have an $SU(2)\otimes SU(2)$ algebra very similar to that of the $\underline{W}$ and $\underline{S}$ operators just introduced. Alternatively, if $g(n,\ell)$ is identically zero, then the M's commute.

If $g(n,\ell)$ is a non-zero constant for all values of n and ℓ, then the sum of four bounded terms must be greater than a linearly increasing term, $2\ell+1$, and clearly this is not possible, If instead $g(n,\ell)$ is identically zero, then either $f_{\pm}^{2}(x) = \text{const.}(x+1)$ (cf. ref.6) but this is not bounded, or $f_{\pm}^{2}(x) = \text{constant}$. In the second case, we are lead to consider "ancestors" unless $f_{-}(x)\equiv 0$, i.e. unless there is no mixing between states on different trajectories. This is possible but we shall consider this as a special case later. Although we have not been able to obtain an algebra for the $\underline{M}$ and $\underline{L}$ operators in general, this does not prevent us from using equations(3.6), which are in a very convenient form.

The new definitions for the $\underline{W}$ operator are similar to those adopted in ref.7 (see Appendix A), where they were introduced specifically for the purpose of creating an algebraic structure which would allow easy calculation of the correction, b, in the unitary operator. The authors were unaware of the consequences in the calculations of ref.6 since they used the definitions for $\underline{M}$ that had been derived in ref.6 . In this work, we have adopted a slightly different set of definitions in order to simplify the "spin" content of the axial generator, $A(\sigma_{z}\tau_{3})$. This now takes the form $S_{0_{\Delta I=1}} + W_{0_{\Delta I=0}}$ (ref.7).

Equation (1.13), the helicity condition on the unitary operator, can now be written,

$$\left[J_{+}, \left[J_{+}, U\left(S_{0_{\Delta I=1}} + W_{0_{\Delta I=0}}\right)U^{+}\right]\right]_{\Delta\kappa=0} = 0 \quad . \qquad (3.7)$$

If we note that $\left[J_{+}, \underline{M}.\underline{W}\right]=0$ and $\left[J_{+}, \left[J_{+}, Z\right]\right]=0$, we can show that the condition is satisfied at second order in θ

if $$\not{b} = -\tfrac{1}{2}\left[M_0W_0 , \underset{\sim}{M}.\underset{\sim}{W}\right] \tag{3.8}$$

where $$\underset{\sim}{M}.\underset{\sim}{W} = M_0W_0 + M_+W_- + M_-W_+ .$$

At order θ^2 the helicity condition can be written, for $\Delta I = 1$,

$$\begin{aligned}
&\left[J_+, \left[J_+, \tfrac{1}{2}[z,[z,S_0]] + [\not{b},S_0]\right]\right] = \\
&\quad = \tfrac{1}{2}\left[J_+, \left[J_+, \left[z, M_+W_- + M_-W_+\right] + \left[z, M_0W_0\right]\right]\right] \\
&\quad = \tfrac{1}{2}\left[J_+, \left[J_+, \left[z, \underset{\sim}{M}.\underset{\sim}{W}\right]\right]\right] \\
&\quad \equiv 0
\end{aligned}$$

and similarly, for $\Delta I = 0$,

$$\begin{aligned}
&\left[J_+, \left[J_+, \tfrac{1}{2}[z,[z,W_0]] + [\not{b},W_0]\right]\right] = \\
&\quad = \tfrac{1}{2}\left[J_+, \left[J_+, \left[\underset{\sim}{M}.\underset{\sim}{W}, M_-S_+ - M_+S_-\right]\right]\right] \\
&\quad \equiv 0
\end{aligned}$$

(Note that if $[M_i, M_j] = 0$, then $\not{b} = -\tfrac{1}{2}M_0(M_-S_+ - M_+S_-)$ cf. ref.7).

Finally in this section, we shall look back to the implications of comparing the Melosh transformation with the transformation of Buccella et al that we discussed in Chapter 1. As we noted there, the implications of this comparison are not necessarily valid because of the number of assumptions made. Nevertheless, it is interesting to see to what extent they are compatible with the definitions that we have been lead to introduce here.

The first point to note is that $\underset{\sim}{M}$ is independent of isospin. Originally provision was made so that there could be a difference between $\underset{\sim}{M}$ acting on isospin one and isospin zero states, but it was found that this distinction was unnecessary (ref.6).

The commutativity of the $\underset{\sim}{M}$-operators amongst themselves is feasible with our present definitions, as we have already seen. Without imposing further conditions on the functions, $f_{\pm}$, equations (3.6) show us that the $\underset{\sim}{M}$'s do not commute but the function, $g(n,\ell)$, on the right hand side will cause the commutator to be $O(1/\ell)$. Hence, there is some agreement here even in the general case.

The problem of the nature of $\theta \underset{\sim}{M}$ under rotations can be answered by noting the existence of the additional terms to be introduced into the transformation of Buccella et al at all even orders of θ . Thus we can have $\underset{\sim}{M}$ as a vector under rotations and still not disagree with this comparison.

The final point, concerning whether or not the transverse momentum of the quark is much less than the effectime mass of the quark, is not clear. Instead, we shall judge the correctness of the perturbative approach by its degree of success in predicting experimental observables.

Section 3.3

We are now in a position where we can begin to look at the consequences of applying the constraints resulting from Weinberg's equation (eqns 1.8-1.10). As might be expected, this series of calculations is similar in outline to those undertaken in ref.6 but, as we shall note, there are important distinctions.

The details of the procedure of the calculation can be found in Appendix B. We shall display here the results of those calculations;

$$m_0^2 = A_0 + \Delta n \qquad (A_0, \Delta \text{ constants}) \tag{3.9}$$

$$m_1^2 = \alpha(I\,n\,\ell) + \beta(I\,n\,\ell)\,\underset{\approx}{L}\cdot\underset{\approx}{S} + \tfrac{1}{2}\,\delta(I\,n\,\ell)\,S^2 + \gamma(I\,n\,\ell)\,(\underset{\approx}{L}\cdot\underset{\approx}{S})^2 \tag{3.10}$$

where $\gamma(I=0, n, \ell) = \gamma(I=1, n, \ell) \equiv \gamma(n,\ell)$

$$= \frac{\Delta}{2(2\ell+1)}\left[\frac{f_+^2(n+\ell+2) - f_-^2(n-\ell-1)}{2\ell+3} + - \frac{f_+^2(n+\ell) - f_-^2(n-\ell+1)}{2\ell-1}\right] \tag{3.11}$$

and $$\delta(I=1, n, \ell) = -\frac{\Delta}{2\ell+1}\left[\frac{(\ell+1)^2}{2\ell+3}\left(f_+^2(n+\ell+2) - f_-^2(n-\ell-1)\right) + - \frac{\ell^2}{2\ell-1}\left(f_+^2(n+\ell) - f_-^2(n-\ell+1)\right)\right] \tag{3.12}$$

and $$\alpha(I=1, n\,\ell) + \delta(I=1, n\,\ell) = \alpha(I=0, n\,\ell) + \delta(I=0, n\,\ell). \tag{3.13}$$

These results represent the conditions imposed on our system by Weinberg's equation up to, and including, second order in θ. Use of these conditions at higher order is not necessarily valid because of the need to introduce further corrections to the unitary operator at higher orders to satisfy the helicity condition (eqn 3.7).

We will note explicitly that there is no equation emerging from these calculations of a form similar to equation (1.19). The reason for this is a direct consequence of the redefinitions introduced in section 3.2 . Hence, we now have a solution which is completely valid to second order in θ and which is free of further constraints. In the next section, we shall look at the implications of this solution on the predictions for meson masses.

Section 3.4

It has been our intention to introduce a mixing scheme for all the non-strange mesons but, since we only have experimental data for the lowest lying states on the leading trajectories, we must look at the predictions on these offered by our calculations. Appendix C contains a table listing the $(\text{mass})^2$ predictions, with accuracy up to second order in θ. This table also gives the predictions resulting from the additional constraint of $f_{\pm} = \sqrt{x+1}$ in the second column (as suggested in refs. 6 & 7). In the last column, we have set $f_+ \equiv$ constant, $f_- \equiv 0$. This case was suggested in section 3.2 because it would imply that the M-operators commuted.

We can deduce the following $(\text{mass})^2$ equations from all three cases; (to simplify notation we shall write m_π^2 as π, m_ρ^2 as ρ, etc.)

$$\begin{aligned} \rho &= \omega \\ A_2 + A_0 - A_1 &= f + \sigma - D \\ A_2 + A_1 &= f + D \end{aligned} \tag{3.14}$$

In addition, the second column gives the relations,

	experimentally l.h.s.	experimentally r.h.s.	
$A_1 + A_2 - 2B = 2(\rho - \pi)$,	-0.2	1.2	(3.15)
$2A_2 + A_0 - 3B = 3(\rho - \pi)$,	0	1.8	

Alternatively, the third case gives the additional equations,

$A_1 + A_2 - 2B = \frac{2}{5}(\rho - \pi)$,	-0.2	0.2	(3.16)
$2A_2 + A_0 - 3B = 9/5(\rho - \pi)$,	0	1.1	

It is readily seen that the effect of specifying the functions, $f_{\pm}(x)$, in either of the ways suggested, is to suppress the mass difference between π and ρ. But, since this difference is one of the prime factors which we wish to be compatible with our work, these particular definitions for $f_{\pm}(x)$ must be ruled out.

Equations (3.14), on the other hand, are acceptable. We know that, in addition to the mixing described above, there can also be mixing between the isospin zero state of an SU(3) octet and the corresponding singlet state. This implies that we can be satisfied with approximate agreement for isospin zero states. Eliminating the mass of the D-meson, which has not been firmly established, we have

$$\rho = \omega$$

$$2A_2 + A_0 = 2f + \sigma$$

These are both well satisfied and we can therefore consider the above scheme successful to the degree of accuracy claimed.

If we try to improve the predictions of ref.5, for the axial couplings, by increasing the order of θ, we unfortunately find that we are unable to do so because of the large increase in new parameters. Though, of course, the predictions given in equation (1.5) are still true up to the first order in θ.

Finally, we note that it is possible to obtain the same predictions if $f_-(x) \equiv 0$ but $f_+(x)$ is unspecified, that is, if we do not allow mixing between states on the leading trajectory and daughter states.

Section 3.5

In this last section of our work on the transformation of Buccella et al we shall summarize the results of the last three chapters and give some indications of the directions in which it could be continued.

The basis of this work has been the Melosh assumption that it is possible to introduce a unitary operator that connects the generators of the algebras $SU(6)_{W;currents}$ and $SU(6)_{W;constituents}$. It has been demonstrated that using this assumption it is possible to obtain good predictions for the axial couplings between mesons in a variety of models. We have shown, in the particular case of the Buccella transformation, that we can also consider the masses of the mesons and we have produced equations relating these mesons which are in good agreement with experiment.

The transformation used involves the unitary operator, $U(\theta)$, defined such that,

$$Q_i^5 = U(\theta) A(\sigma_z \lambda_i) U(\theta)^\dagger \quad , \qquad Q_i = A(\lambda_i)$$

and which also satisfies the necessary helicity equations. It has been shown that, if this operator is expanded in terms of the parameter, θ ,it is possible to use a perturbative approach to making it compatible with Weinberg's equation,

$$[Q_a^5 , [Q_b^5 , m^2]] \propto \delta_{ab} .$$

Unfortunately, in previous attempts to use this method, the operators used in the unitary transformation were defined in such a way as to lead to a poor set of predictions.

In this work, we have undertaken to review the definitions of the operators used in the construction of the unitary transformation. As a consequence, we have been able to extract the following predictions,

$$m_\rho^2 = m_\omega^2$$

$$2m_{A_2}^2 + m_{A_0}^2 = 2m_f^2 + m_\sigma^2$$

up to the second order in θ, which are consistent with experimental observations.

In view of the success of this scheme, it is worthwhile considering further improvement of the unitary operator by introducing corrections at orders θ^3 and θ^4. The calculation of these corrections is not straightforward and it is possible that these terms would not be unique. Nevertheless, they would help to constrain further the functions already introduced and perhaps lead to further predictions.

The approach used in determining the transformation first proposed by Buccella et al is "phenomenological". That is, it involves a number of assumptions which are introduced to simplify the transformation. This enables us to use it to a greater extent than would otherwise be possible. Our conclusion is that we should re-assess the importance of the Buccella transformation in view of the work presented here, since it has been shown to be capable of giving good mass predictions which are, so far, beyond the scope of other Melosh transformations.

CHAPTER FOUR

This chapter is a general introduction to supersymmetry, giving a summary of the successes of the original super-algebra and also its special points of interest. There is also a description of the algebra and formalism associated with the larger super-algebra which incorporates internal symmetry and a review of the work of Dondi and Sohnius in which they derive the reducible scalar multiplet.

Section 4.1

The idea of supersymmetry first emerged from consideration of the Neveu-Schwarz-Ramond dual model (ref 15). When this model is written as a two-dimensional field theory, there arises, in addition to the linear Klein-Gordan and Dirac equations and a generalisation of the gauge condition, an extra "supergauge" condition.

Gervais and Sakita (ref 16) interpreted this condition as a set of transformations under which the Lagrangian density for free fields is invariant. Wess and Zumino then introduced the idea of auxiliary fields in order to obtain a closed group structure for this set of transformations in two dimensions and, by a process of trial and error, generalised the system to four dimensions (ref 17). In the generalised form, the transformations are no longer a local gauge symmetry but a global "supersymmetry".

There are several reasons for finding interest in this new symmetry. The multiplet of fields, associated with each closed set of transformations, combines fields of integral and half-integral spin, i.e. bosons and fermions. Also, models can be constructed which are highly renormalisable. A third point of interest is the form of the "super-algebra", which contains the Poincaré algebra as a subalgebra. This allows us to have a symmetry containing relativistic-spin which is consistent with unitarity. In section 4.4 we shall look at some of these points in more detail.

Initially the super-algebra contained the conformal algebra and the transformations involved a parameter, ξ, which is a totally anticommuting Majorana spinor satisfying

$$(\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu - g_{\mu\nu} \gamma_\lambda \partial^\lambda)\, \xi = 0 . \tag{4.1}$$

In later work, attention has been restricted to the special case when ξ is constant. This removes the limitation to massless particles in the Lagrangian and also generally allows a more compact and manageable description of the theory. The effect of setting ξ to a constant is to have a super-algebra which contains just the Poincaré algebra, instead of the full conformal algebra. In addition to the usual generators of the Poincaré algebra, $J_{\mu\nu}$ and P_μ, the super-algebra also has a spinor super-charge, S_α. This extends the Poincaré algebra to include

$$\begin{aligned} [S_\alpha, P_\mu] &= 0 \\ [S_\alpha, J_{\mu\nu}] &= \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta \\ \{S_\alpha, S_\beta\} &= (\gamma_\mu C)_{\alpha\beta} P_\mu \end{aligned} \tag{4.2}$$

where S_α is a Majorana spinor, i.e. $S_\alpha = C_{\alpha\beta}\bar{S}^\beta$. In these equations the matrix C denotes the charge conjugation matrix;(a summary of the notation used in the following chapters can be found in Appendix D).

In section 4.3 we shall give a brief description of the details arising from this super-algebra, which will be useful for comparison in later chapters. But, the main part of our work on supersymmetry is concerned with the super-algebra incorporating internal symmetry, which was first introduced by Salam and Strathdee (ref 18). In this case, the super-charge spinor, $S_{\alpha i}$, has an additional suffix relating to isospin, thus giving the spinor a total of eight components; as well as the usual commutation rules involving $J_{\mu\nu}$, P_μ and $\underline{I}$ only, we have

$$
\begin{aligned}
[S_{\alpha i}, P_\mu] &= 0 \\
[S_{\alpha i}, J_{\mu\nu}] &= \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta S_{\beta i} \\
[S_{\alpha i}, \underline{I}] &= \tfrac{1}{2}(\underline{\tau})_i{}^j S_{\alpha j} \\
\{S_{\alpha i}, S_{\beta j}\} &= 2i\,\varepsilon_{ij}(\gamma_\mu\gamma_5 C)_{\alpha\beta} P^\mu
\end{aligned}
\tag{4.3}
$$

where $S_{\alpha i}$ satisfies the "Majorana" condition,

$$
S_{\alpha i} = i\,\varepsilon_{ij}(\gamma_5 C)_{\alpha\beta}\,\bar{S}^{\beta j}
$$

The object of the work in the following chapters is to attempt to duplicate, using this algebra, the successful results arising from the original super-algebra (eqns 4.2). The hope is to find a Lagrangian model which is physically realistic, incorporating internal symmetry, and is invariant under supersymmetry transformations, thus having the interesting properties already noted.

Section 4.2

The introduction of the concept of the superfield by Salam and Strathdee (ref 19) represented a major step forward in the understanding of the supersymmetry transformations. Previously these transformations had been found by a process of inspired guess-work. The superfield approach made it possible to derive the results that had already been determined and go on to look at larger multiplets, including those incorporating isospin.

In this section we shall briefly look at the superfield formalism using an approach analogous to that described in refs. 19,20 in the absence of isospin. The superfield is a function, $\Phi(x,\theta)$, of a space-time variable, x, and a totally anticommuting "Majorana" spinor, $\theta_{\alpha i}$, i.e.

$$\theta_{\alpha i} = i\,\varepsilon_{ij}\,(\gamma_5 C)_{\alpha\beta}\,\bar{\theta}^{\beta j}. \tag{4.4}$$

Since θ is totally anticommuting, it is possible to expand the superfield as a terminating series in θ . If we regard the functions in x, which are the coefficients in this expansion, to be the fields of particles, then each superfield represents a multiplet of such fields which are closed under supersymmetry transformations.

Although it is possible to obtain all the following results using $\theta_{\alpha i}$, we shall follow ref.20 and use instead $\theta_{a i}$ and its conjugate $\bar{\theta}^{\dot{a}}_{i}$, which are four-component complex spinors. These new four-component spinors, θ and $\bar{\theta}$, can be recombined to give the "Majorana" spinor,

$$\theta_{\alpha i} = \begin{pmatrix} \theta_{a i} \\ \bar{\theta}^{\dot{a}}_{i} \end{pmatrix}. \tag{4.5}$$

Proceeding in the same way to translate the spinor supercharge, $S_{\alpha i}$, into the dotted and undotted SL(2,C)-spinor notation, we write

$$S_{\alpha i} = \begin{pmatrix} S_{a i} \\ \bar{S}^{\dot{a}}{}_{i} \end{pmatrix} .$$

This implies the following changes to equations (4.3) for the large super-algebra

$$\begin{aligned} \{ S_{a i} , \bar{S}_{\dot{b}}{}^{j} \} &= 2\,\delta_i{}^{j}\,(\sigma_\mu)_{a\dot{b}}\,P^\mu \\ \{ S_{a i} , S_{bj} \} &= \{ \bar{S}_{\dot{a} i} , \bar{S}_{\dot{b} j} \} = 0 \\ S_{a i}{}^{\dagger} &= \bar{S}_{\dot{a}}{}^{i} \end{aligned} \qquad (4.6)$$

where $(\sigma_\mu)_{a\dot{b}} = (1, -\underline{\sigma})_{a\dot{b}}$ and $\underline{\sigma}$ are the Pauli matrices. It follows from equation (4.6) that, if η_{ai} and $\bar{\eta}_{\dot{a}i}$ are completely anticommuting parameters, then*

$$[\eta S, \bar{S}\eta] = -2\eta\sigma_\mu\bar{\eta}P^\mu \qquad (4.7)$$

where $\eta S = \eta^{ai}S_{ai}$, $\bar{S}\bar{\eta} = \bar{S}^{\dot{a}i}\bar{\eta}_{\dot{a}i}$, $\eta\sigma_\mu\bar{\eta} = \eta^{ai}(\sigma_\mu)_{a\dot{b}}\bar{\eta}^{\dot{b}}{}_{i}$.

* Throughout this work, whenever a quadratic form in spinors occurs, the spinor indices will be omitted on the understanding that the spinors on the left and right have respectively upper and lower indices, unless otherwise is evident from the nature of the quadratic form. Note that equation (4.7) provides an example of the exception.

A group element may be written in three different ways,

$$\Phi(x,\theta,\bar\theta) = \exp i(\theta S + \bar S\bar\theta - x.P) \tag{4.8a}$$

$$\Phi'(x,\theta,\bar\theta) = \exp i(\theta S - x.P)\ \exp i\bar S\bar\theta \tag{4.8b}$$

$$\Phi''(x,\theta,\bar\theta) = \exp i(\bar S\bar\theta - x.P)\ \exp i\theta S. \tag{4.8c}$$

We can show, using equation (4.7), that these forms are connected by

$$\Phi(x,\theta,\bar\theta) = \Phi'(x - i\theta\sigma\bar\theta, \theta, \bar\theta) = \Phi''(x + i\theta\sigma\bar\theta, \theta, \bar\theta). \tag{4.9}$$

Operating on the left of equations (4.8) with the group element

$$g = \exp i(\xi S + \bar S\bar\xi)$$

leads to the following infinitesimal transformation laws

$$\delta\Phi = \left(\xi\frac{\partial}{\partial\theta} + \bar\xi\frac{\partial}{\partial\bar\theta} + i(\xi\sigma_\mu\bar\theta - \theta\sigma_\mu\bar\xi)\partial^\mu\right)\Phi \tag{4.10a}$$

$$\delta\Phi' = \left(\xi\frac{\partial}{\partial\theta} + \bar\xi\frac{\partial}{\partial\bar\theta} - 2i\theta\sigma_\mu\bar\xi\partial^\mu\right)\Phi' \tag{4.10b}$$

$$\delta\Phi'' = \left(\xi\frac{\partial}{\partial\theta} + \bar\xi\frac{\partial}{\partial\bar\theta} + 2i\xi\sigma_\mu\bar\theta\partial^\mu\right)\Phi''. \tag{4.10c}$$

These expressions are now abstracted and taken as the basic infinitesimal transformation properties of the superfields.

It is clear that it is possible to pass from any one of these three transformation laws to any one of the others by "shifting" the variable, x, according to equation (4.9). Since the "shift" $i\theta\sigma_\mu\bar\theta$ is pure imaginary one can require Φ to be real, as its transformation properties show that it will remain real; but Φ' and Φ'' are essentially complex and, in fact, each transforms as the complex conjugate of the other.

When constructing Lagrangians from the superfields it is important to know the transformation properties of two or more superfields. The transformation rules show that the product of two superfields of the same type (i.e. transforming in the same way) is also a superfield of that type. In order to multiply superfields of different types one must first transform them by "shifting" to superfields of the same type.

It can be seen that there are two super-invariant derivatives on each type of superfield. These are as follows,

$$\frac{\partial}{\partial\theta} - i\sigma_\mu\bar{\theta}\partial^\mu \quad \text{and} \quad \frac{\partial}{\partial\bar{\theta}} + i\theta\sigma_\mu\partial^\mu \quad \text{on} \quad \Phi \tag{4.11a}$$

$$\frac{\partial}{\partial\theta} - 2i\sigma_\mu\bar{\theta}\partial^\mu \quad \text{and} \quad \frac{\partial}{\partial\bar{\theta}} \quad \text{on} \quad \Phi' \tag{4.11b}$$

$$\frac{\partial}{\partial\theta} \quad \text{and} \quad \frac{\partial}{\partial\bar{\theta}} + 2i\theta\sigma_\mu\partial^\mu \quad \text{on} \quad \Phi'' \tag{4.11c}$$

The formalism we have described here will be used extensively throughout the next three chapters and is basic to our understanding of supersymmetry. It would be extremely difficult to proceed with our study of the larger super-algebra without using the superfield formalism and its importance in giving structure to all work on supersymmetry theories should be stressed.

Section 4.3

In this section we shall briefly summarize the results emerging from the original super-algebra (ref 21). Our aim in the following chapters is to repeat the success of these results using the super-algebra which incorporates internal symmetry. We shall find it interesting to compare our new results with those set out below; (see also Appendix E for more details).

The general scalar superfield contains two irreducible scalar superfields and one spinor superfield. The irreducible scalar multiplet contains a scalar and a pseudoscalar field, a Majorana spinor and a scalar and a pseudoscalar auxiliary field. If the parameter, ξ (of eqn 4.1) is constant, then it is possible to construct an interacting Lagrangian density that is "super-invariant", i.e. invariant under supersymmetry transformations up to a total derivative. In this Lagrangian all the fields have the same mass and the coupling constants are all related. The equations of motion of the auxiliary fields can be found and used to eliminate these fields from the Lagrangian density. It is this model which has been studied in detail and found to be renormalizable to all orders of perturbation theory (ref 22).

The spinor multiplet contains a vector field and a Majorana spinor together with a scalar auxiliary field. It is possible to form a massless free Lagrangian from these fields which is invariant under supersymmetry transformations up to a total derivative.

It has been suggested by Wess and Zumino (ref 23) that the fields of the spinor multiplet could represent a photon and a "neutrino". They have combined the fields from the two different multiplets to form a Lagrangian which is both "superinvariant" and invariant under ordinary gauge transformations. This model is shown to be renormalizable in the one-loop approximation.

Section 4.4

We are going to give a description here of the special points of interest emerging from supersymmetry which were noted in the introduction.

We have already seen, in the earlier chapters on $SU(6)_W$ theory, that it is possible to classify all mesons and baryons in multiplets of $SU(6)_W \otimes O(3)$. The important distinction between any proposed super-multiplets and the $SU(6)_W \otimes O(3)$ multiplets is that, in the latter case, fermions and bosons never appear in the same multiplet, whereas, in super-multiplets, they will always appear together.

In 1965, O'Raifeartaigh examined the general problem of combining the inhomogeneous Lorentz algebra, L, and an internal symmetry algebra, T, into a larger symmetry algebra of finite order, E, (ref 25). He was prompted by the current interest in large symmetry algebras and the mass-splitting which is necessarily required within multiplets in order to get agreement with experiment. The basis of O'Raifeartaigh's work was Levi's radical-splitting theorem. This states that every Lie algebra, E, of finite order, is the semi-direct product of a semi-simple Lie algebra, G, and an invariant solvable algebra, S. The main consequences arising from this theorem are that any physical large symmetry group, E, is probably a direct sum of L and T and that there can be no mass-splitting without the introduction of symmetry breaking phenomena. There are other possibilities for the symmetry group, E, but these were not considered to be physically applicable.

We have already seen that there is no mass-splitting amongst the members of a super-multiplet. But, the super-algebra is not the direct sum of the Lorentz algebra and an internal symmetry algebra. Originally, it was suggested that this discrepancy was due to the fact that the super-algebra contained a combination of commutators and anticommutators and so O'Raifeartaigh's theorem did not apply. Recently Goddard has shown (ref 24) that it is possible to rewrite the superalgebra as a Lie algebra involving only commutators, hence the theorem must apply. O'Raifeartaigh's theorem does include the possibility of the algebra, E, taking the form of our super-algebra. But, this was excluded from further consideration on the grounds that it appeared unphysical and unlike any higher symmetry group that had been suggested. Hence, one of the interesting features of the super-algebra is that it explores a possibility that has not been considered previously.

A further point of interest arises from the renormalizability of the "super-invariant" Lagrangians. In particular, the interacting Lagrangian formed from the original irreducible multiplet (ref 26) has been shown to be renormalizable to all orders of perturbation theory and only one renormalization constant is needed (ref 22).

The approach used in ref.22 considers the component fields of the multiplet explicitly. This work has since been repeated using a powerful technique that introduces Feynman graphs for the superfields (refs 27,28). The advantage of this approach is that one superfield diagram corresponds to several component field graphs and the cancellation of the associated infinities is implicitly contained in this one graph.

The technique of using "supergraphs" has been used with great effect in the more complex models involving the original spinor multiplet (ref 29) and the reducible scalar multiplet arising from the super-algebra containing internal symmetry (ref 30). Unfortunately, in both these cases, the conclusion is that the models are non-renormalizable though the divergences are considerably less than might be expected from power-counting.

In chapter 6 we shall attempt to construct "super-invariant" Lagrangians using the multiplets of fields from the larger super-algebra which incorporates isospin. It will be interesting at that stage to look more closely at the work of Capper and Leibbrandt (ref 30) and the approach they adopt. But first we shall begin our analysis of the supermultiplets of the larger super-algebra by reviewing a paper by Dondi and Sohnius in which they derive the reducible scalar multiplet (ref 31).

Section 4.5

In considering the original super-algebra (eqns 4.2) a great deal of importance was given to the scalar multiplet corresponding to the superfield $\Phi'(x,\theta,\bar{\theta})$ that satisfied the super-invariant constraint

$$\frac{\partial}{\partial\bar{\theta}}\,\Phi'(x,\theta,\bar{\theta}) = 0 \; . \tag{4.12}$$

This was found to be the smallest irreducible multiplet, containing only two scalar fields, two auxiliary scalar fields and a Majorana spinor field. Therefore, when the larger super-algebra incorporating isospin (eqns 4.3) was introduced the first step was to calculate the equivalent scalar multiplet. This work was done by Dondi and Sohnius (ref 31) and we will present their results here for completeness.

The superfield, $\Phi'(x,\theta,\bar{\theta})$, which satisfies the super-invariant constraint, eqn 4.12, (where now θ_{ai} and $\bar{\theta}^{\dot{a}}{}_i$ are each complex four-component spinors) has the expansion,

$$\begin{aligned}\Phi'(x,\theta) = {} & A(x) + \theta\psi(x) + \frac{1}{2!}\,\theta\,\tau\,\theta\,.\,F(x) + \\ & + \frac{1}{2.2!}\,\theta\Sigma_{\mu\nu}\theta\,\phi^{\mu\nu}(x) + \frac{1}{3!}\,\theta^{ai}\theta_{aj}\theta^{bj}\chi_{bi}(x) + \\ & + \frac{1}{4!}\,\theta^{ai}\theta_{aj}\,\theta^{bj}\,\theta_{bi}\,G(x)\end{aligned} \tag{4.13}$$

where $\frac{1}{2}(\Sigma_{\mu\nu})_a{}^b$ is the undotted-spinor representation of the $J_{\mu\nu}$. If ξ_{ai} is the completely anticommuting infinitesimal parameter, the constituent fields have the supersymmetry transformation properties, as follows,

$$\delta A(x) = \xi \psi(x)$$

$$\delta \psi(x) = -2i\, \sigma_\mu \bar{\xi}\, \partial^\mu A(x) - \underline{\tau} \cdot \xi\, \underline{E}(x) + \tfrac{1}{2} \Sigma_{\mu\nu}\, \xi\, \phi^{\mu\nu}(x)$$

$$\delta \underline{E}(x) = i\, \bar{\xi}\, \sigma_\mu^{T}\, \underline{\tau}\, \partial^\mu \psi(x) + \tfrac{1}{4} \xi\, \underline{\tau}\, \chi(x)$$

$$\delta \phi^{D}_{\mu\nu}(x) = -i\, \bar{\xi}\, C \bar{\sigma}_\rho\, \Sigma_{\mu\nu}\, \partial^\rho \psi(x) - \tfrac{1}{4} \xi\, \Sigma_{\mu\nu}\, \chi(x)$$

$$\delta \chi(x) = 4i\, \underline{\tau}\, \sigma_\mu \bar{\xi} \cdot \partial^\mu \underline{E}(x) - 2i\, \Sigma_{\mu\nu}\, \sigma_\rho \bar{\xi}\, \partial^\rho \phi^{\mu\nu}(x) + \xi\, \varsigma(x)$$

$$\delta \varsigma(x) = 2i\, \bar{\xi}\, \bar{\sigma}_\mu\, \partial^\mu \chi(x) \tag{4.14}$$

where the superfix, T, denotes matrix transpose, C is the lowering matrix for undotted (dotted) SL(2,C)-spinor indices, $\bar{\sigma}_\mu \equiv (\bar{\sigma}_\mu)^{\dot{a}b} = (1, \underline{\sigma})^{\dot{a}b}$ and where

$$\phi^{D}_{\mu\nu}(x) = \phi_{\mu\nu}(x) + \tfrac{i}{2}\, \varepsilon_{\mu\nu\lambda\rho}\, \phi^{\lambda\rho}(x). \tag{4.15}$$

CHAPTER FIVE

In this chapter we shall analyse the irreducible multiplets emerging from the larger super-algebra incorporating internal symmetry. In addition, we shall find the independent Casimir operators of this algebra.

Section 5.1

The aim of the following two chapters is to comtinue the process suggested by Salam and Strathdee (ref 18) and already begun by Dondi and Sohnius (ref 31) of working out the consequences of introducing isospin into supersymmetry theory in a non-trivial way. In this section we shall begin by looking at the new representations of the Casimir operators of the larger super-algebra. We shall go on to consider the different types of irreducible multiplets contained within the most general scalar superfield and derive their supersymmetry transformations. Then in the next chapter we shall be able to attempt to construct "super-invariant" Lagrangian densities using these multiplets in the hope of finding a form which is physically applicable.

The independent Casimir operators of the direct product of the Poincaré and isospin algebras which are normally used are P^2, W^2 and $\underset{\sim}{I}^2$, where W_μ is the Pauli-Lubanski spin-operator,

$$W_\mu = -\frac{1}{2}\,\varepsilon_{\mu\nu\lambda\rho}\,P^\nu J^{\lambda\rho} \, . \tag{5.1}$$

When considering the super-algebra (eqns 4.3) only P^2 is super-invariant. Therefore we must find super-invariant generalisations of W^2 and $\underset{\sim}{I}^2$.

To generalise W^2, first define

$$W'_\mu = W_\mu - \frac{i}{8}\,\bar{S}\gamma_\mu\gamma_5 S \, , \tag{5.2}$$

It can be seen from the super-algebra that the transverse part of W'_μ ,written in the form

$$K'_{\mu\nu} = P_\mu W'_\nu - P_\nu W'_\mu \tag{5.3}$$

is super-invariant and its square provides a super-invariant generalisation of W^2. (Note that for convenience we are here adopting the "Majorana" notation.)

Similarly, in order to generalise $\underset{\sim}{I}^2$, we first define

$$\underset{\sim}{g}_\mu = P_\mu \underset{\sim}{I} - \tfrac{1}{8} \bar{S} \gamma_\mu \underset{\sim}{\tau} S \tag{5.4}$$

and it follows from consideration of the super-algebra that the longitudinal part,

$$\underset{\sim}{g}' = P^\mu \underset{\sim}{g}_\mu \tag{5.5}$$

is super-invariant and its square gives us a super-invariant generalisation of $\underset{\sim}{I}^2$.

Now we want to translate these operators into their corresponding representations when acting on the superfield, $\Phi(x,\theta,\bar{\theta})$. We have already seen (section 4.2) that the representation for the spinor super-charge, $S_{\alpha i}$ is

$$S_{\alpha i} \rightarrow -\frac{\partial}{\partial \bar{\theta}^{\alpha i}} - i(\gamma_\mu \theta)_{\alpha i} \partial^\mu \tag{5.6}$$

and the super-invariant derivative is defined to be

$$D_{\alpha i} \rightarrow -\frac{\partial}{\partial \bar{\theta}^{\alpha i}} + i(\gamma_\mu \theta)_{\alpha i} \partial^\mu \tag{5.7}$$

We can therefore deduce from the super-algebra that the following representations are valid,

$$\begin{aligned} P_\mu &\rightarrow i\partial_\mu \\ J_{\mu\nu} &\rightarrow J''_{\mu\nu} + \tfrac{1}{2} \bar{\theta} \sigma_{\mu\nu} \frac{\partial}{\partial \bar{\theta}} \\ \underset{\sim}{I} &\rightarrow \underset{\sim}{I}'' + \tfrac{1}{2} \bar{\theta} \underset{\sim}{\tau} \frac{\partial}{\partial \bar{\theta}} \end{aligned} \tag{5.8}$$

where the double-primed operators have no dependence on terms involving super-variables θ and $\bar{\theta}$ or their derivatives.

Inserting these representations into the terms introduced to define the above super-invariant operators we find

$$K_{\mu\nu}' \longrightarrow K_{\mu\nu}'' + \tfrac{1}{8}\bar{D}\, X_{\mu\nu}\, D$$

and

$$\underset{\sim}{\mathfrak{g}}' \longrightarrow \underset{\sim}{\mathfrak{g}}'' - \tfrac{i}{8}\bar{D}\, \partial\!\!\!/\, \underset{\sim}{\tau}\, D \tag{5.9}$$

where $$X_{\mu\nu} = (\partial_\mu \gamma_\nu - \partial_\nu \gamma_\mu)\gamma_5 .$$

In this form the super-invariance of these terms is manifest.

Section 5.2

In the original supersymmetry theory it was shown to be feasible to explicitly find the supersymmetry transformations of the most general scalar superfield (ref 19). This contained only 16 independent components if the superfield was real. In the supersymmetry theory which includes isospin intrinsically the most general real scalar superfield contains 256 independent components. In practice, this means that we can only give detailed consideration to the smaller superfields which are contained within the general superfield.

The most straightforward way to show the reduction of the general superfield, $\Phi(x,\theta,\bar{\theta})$, is to consider it to be complex and to adopt a tabular form, indicating the fields which are the coefficients of the various powers of θ and $\bar{\theta}$ (using the dotted and undotted spinor notation).

		θ	$\theta\theta$	$\theta\theta\theta$	$\theta\theta\theta\theta$
	A	ψ	$E, F_{[\mu\nu]}$	χ	ς
$\bar{\theta}$	ψ'	$\underset{\sim}{A}_\mu, V_\mu$	$\underset{\sim}{\chi}, \chi_{[\mu\nu]}$	$\underset{\sim}{B}_\mu, W_\mu$	λ
$\bar{\theta}\bar{\theta}$	$\underset{\sim}{F}', F'_{[\mu\nu]}$	$\underset{\sim}{\chi}', \chi'_{[\mu\nu]}$	$\underset{\approx}{E}, E_{[\mu\nu]}$ $\underset{\sim}{F}'_{[\mu\nu]}, F_{[\mu\nu][\varsigma\eta]}$	$\underset{\sim}{\lambda}', \lambda'_{[\mu\nu]}$	$\underset{\sim}{\varsigma}, \varsigma_{[\mu\nu]}$
$\bar{\theta}\bar{\theta}\bar{\theta}$	χ'	$\underset{\sim}{B}'_\mu, W'_\mu$	$\underset{\sim}{\lambda}, \lambda_{[\mu\nu]}$	$\underset{\sim}{C}_\mu, V'_\mu$	ξ
$\bar{\theta}\bar{\theta}\bar{\theta}\bar{\theta}$	ς'	λ'	$\underset{\sim}{\varsigma}', \varsigma'_{[\mu\nu]}$	ξ'	B .

In the above table greek letters represent complex spinor fields and latin letters complex boson fields. Isovector labels are denoted by underlining the field, for example, $\underline{F}$ is an isovector, $\underline{\underline{F}}$ is an isotensor and $\underline{\chi}$ is a spinor with an additional isospin label. Fields which have greek suffices are tensors and square brackets denotes antisymmetry with respect to the intershange of two of these indices. For example, B_μ is a vector, $F_{[\mu\nu]}$ is an antisymmetric second rank tensor. $F_{[\mu\nu][\rho\eta]}$ is a fourth rank tensor which is antisymmetric under the interchange of either μ and ν or ρ and η. $\chi_{[\mu\nu]}$ is a spinor with an additional antisymmetric tensor label.

Now consider the effect of imposing the constraint

$$\frac{\partial}{\partial\bar{\theta}}\,\Phi(x,\theta,\bar{\theta}) = 0$$

This leaves us with a complex 16-component multiplet, $\Phi^{(1)}(x,\theta)$, in the first row of the table and nothing elsewhere. This is exactly what was described in section 4.5 (assuming that $\Phi^{(1)}(x,\theta)$ transforms according to equation (4.8b), otherwise $\partial/\partial\bar{\theta}$ would not be a super-invariant derivative).

Similarly, the first column can be seen to be a multiplet of the same form, except the fields are coefficients of $\bar{\theta}$ in this case. We could regard the column-multiplet as satisfying

$$\frac{\partial}{\partial\theta}\,\Phi(x,\theta,\bar{\theta}) = 0$$

where $\Phi(x,\theta,\bar{\theta})$ must transform according to equation (4.8c).

Adopting the first of these two alternatives, we can expand the general unconstrained superfield in terms of superfields of the basic form $\Phi^{(1)}(x,\theta)$,

$$\Phi(x,\theta,\bar\theta) = \Phi^{(1)}(x,\theta) + \bar\theta^{\dot a i}\,\bar\Phi^{(2)}_{\dot a i}(x,\theta) + \frac{1}{2!}\bar\theta\underline{\tau}\bar\theta.\,\underline{\Phi}^{(3)}(x,\theta) +$$
$$+ \frac{1}{2.2!}\bar\theta\Sigma^{\mu\nu}\bar\theta\;\Phi^{(4)}_{[\mu\nu]}(x,\theta) + \frac{1}{3!}\bar\theta^{\dot a i}\bar\theta_{\dot a j}\bar\theta^{\dot b j}\bar\Phi^{(5)}_{\dot b i}(x,\theta) +$$
$$+ \frac{1}{4!}\bar\theta^{\dot a i}\bar\theta_{\dot a j}\,\bar\theta^{\dot b j}\bar\theta_{\dot b i}\,\Phi^{(6)}(x,\theta). \qquad (5.10)$$

It has been shown that $\Phi^{(1)}(x,\theta)$ is a superfield; that is, it can give rise to a series of supersymmetry transformations which close upon the constituents of the multiplet all contained within $\Phi^{(1)}(x,\theta)$. If suffices are attached to the basic scalar superfield we know that these new functions will also be superfields. Therefore the coefficients of $\bar\theta$ in the above expansion could each be regarded as an independent superfield. In other words, each row (or, alternatively, each column) could represent an independent superfield. (It should be noted that if we were to derive the supersymmetry transformations of all the fields in the unconstrained scalar superfield explicitly, we would find that the independent multiplets do not emerge until the fields have been redefined, subtracting out the terms which are dependent on fields in other multiplets.)

Hence we have found that the general complex scalar superfield can be reduced into two smaller scalar superfields, an isovector superfield, an antisymmetric tensor superfield and two spinor superfields.

Section 5.3

In this section we shall look in detail at the supersymmetry transformations of the different kinds of multiplets. These transformations are obtained using equation (4.10) acting on the superfield expanded in powers of θ and $\bar{\theta}$. We will omit the formal derivation here and just quote the resulting transformations.

For the smaller scalar superfield $\Phi^{(1)}(x,\theta,\bar{\theta})$ satisfying

$$\frac{\partial}{\partial\bar{\theta}} \Phi^{(1)}(x,\theta,\bar{\theta}) = 0$$

we can write

$$\begin{aligned}\Phi^{(1)}(x,\theta) = \; & A(x) + \theta\psi(x) + \frac{1}{2!}\,\theta\,\Sigma\,\theta.\underline{E}(x) + \\ & + \frac{1}{2.2!}\,\theta\,\Sigma_{\mu\nu}\,\theta\,F^{[\mu\nu]}(x) + \frac{1}{3!}\,\theta^{ai}\theta_{aj}\theta^{bj}\chi_{bi}(x) + \\ & + \frac{1}{4!}\,\theta^{ai}\theta_{aj}\,\theta^{bj}\theta_{bi}\,\varphi(x).\end{aligned} \qquad (5.11)$$

As noted, this is just the multiplet examined by Dondi and Sohnius (ref 31); we will find it more convenient to express this in the "Majorana" notation, rather than the dotted and undotted spinor notation used previously in ref.31 . In the "Majorana" notation it becomes clear that this scalar multiplet is reducible.

The supersymmetry transformations for the reducible scalar multiplet can be written as follows,

$$\delta A_1 = i\bar{\xi}\gamma_5\psi$$

$$\delta A_2 = -i\bar{\xi}\psi$$

$$\begin{aligned}\delta\psi = \; & \gamma_5 \partial\!\!\!/ A_1\xi + \partial\!\!\!/ A_2\xi + i\Sigma.\underline{E}_2\xi + i\gamma_5\Sigma.\underline{E}_1\xi \\ & - \frac{i}{2}\sigma_{\mu\nu}F_2^{[\mu\nu]}\xi - \frac{i}{2}\gamma_5\sigma_{\mu\nu}F_1^{[\mu\nu]}\xi\end{aligned}$$

$$\delta \underset{\sim}{F}_1 = \tfrac{1}{2}\bar{\xi}\,\partial\!\!\!/\,\gamma_5\,\underset{\sim}{\tau}\,\psi + \tfrac{1}{8}\bar{\xi}\,\underset{\sim}{\tau}\,\chi$$

$$\delta \underset{\sim}{F}_2 = -\tfrac{1}{2}\bar{\xi}\,\partial\!\!\!/\,\underset{\sim}{\tau}\,\psi + \tfrac{1}{8}\bar{\xi}\,\underset{\sim}{\tau}\,\gamma_5\chi$$

$$\delta F_1^{[\mu\nu]} = \tfrac{i}{2}\bar{\xi}(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\gamma_5\psi + \tfrac{i}{8}\bar{\xi}(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\frac{\partial\!\!\!/}{\partial^2}\chi$$

$$\delta F_2^{[\mu\nu]} = -\tfrac{i}{2}\bar{\xi}(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\psi + \tfrac{i}{8}\bar{\xi}(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\frac{\partial\!\!\!/}{\partial^2}\gamma_5\chi$$

$$\delta\chi = \tfrac{1}{2}G_1\xi - \tfrac{1}{2}\gamma_5 G_2\xi - 4i\,\partial\!\!\!/\,\underset{\sim}{\tau}\cdot\underset{\sim}{F}_1\,\xi + 4i\gamma_5\,\partial\!\!\!/\,\underset{\sim}{\tau}\cdot\underset{\sim}{F}_2\,\xi$$
$$- 2i\gamma_5\,\partial\!\!\!/\,\sigma_{\mu\nu}F_2^{[\mu\nu]}\xi + 2i\,\partial\!\!\!/\,\sigma_{\mu\nu}F_1^{[\mu\nu]}\xi$$

$$\delta G_1 = -2i\,\bar{\xi}\,\partial\!\!\!/\,\chi$$

$$\delta G_2 = -2i\,\bar{\xi}\,\partial\!\!\!/\,\gamma_5\chi \tag{5.12}$$

where the boson fields have been written in terms of their real and imaginary parts, for example, $A = \frac{1}{2}(A_1 + iA_2)$ and $\underset{\sim}{F} = \frac{1}{2}(\underset{\sim}{F}_1 + i\underset{\sim}{F}_2)$.

As we noted in section 4.2, the product of two superfields of the same type, $\Phi^{(1)}(x,\theta)$ and $\Phi^{(2)}(x,\theta)$, is also a superfield of that type, $\Phi^{(3)}(x,\theta)$. The procedure for obtaining the combination rules, displaying the relationship between the component fields of the three superfields, is basically one of comparison of the coefficients of θ . We will find it extremely useful to use the explicit combination rules for two reducible scalar superfields in the next chapter when we consider the problem of obtaining a "super-invariant" Lagrangian density. These combination rules are given in Appendix F.

If we consider the spinor field

$$\tilde{\chi} = \chi - 4\gamma_5\,\partial\!\!\!/\,\psi\,, \tag{5.13}$$

it becomes evident that there are two multiplets separating out. If we try setting $\tilde{\chi}$ identically to zero together with E_1, $F_1^{[\mu\nu]}$, we obtain a new irreducible scalar superfield.

This process of reduction can be expressed formally by demanding that the reducible scalar superfield, $\Phi^{(1)}(x,\theta)$, satisfies the super-invariant relation

$$\frac{1}{2.4!}\frac{\partial}{\partial\bar{\theta}^{\dot{a}i}}\frac{\partial}{\partial\bar{\theta}_{\dot{a}j}}\frac{\partial}{\partial\bar{\theta}^{\dot{b}j}}\frac{\partial}{\partial\bar{\theta}_{\dot{b}i}}\tilde{\Phi}^{(1)}(x+2i\theta\sigma\bar{\theta},\bar{\theta}) = \partial^2\Phi^{(1)}(x,\theta) \tag{5.14}$$

where $\tilde{\Phi}^{(1)}(x+2i\theta\sigma\bar{\theta},\bar{\theta})$ is obtained from $\Phi^{(1)}(x,\theta)$ by first taking the hermitian conjugate and then "shifting" the space-time variable. Using equation (5.11) and the identity,

$$\frac{1}{4}\frac{\partial}{\partial\bar{\theta}^{\dot{a}i}}\frac{\partial}{\partial\bar{\theta}_{\dot{a}j}}\frac{\partial}{\partial\bar{\theta}^{\dot{b}j}}\frac{\partial}{\partial\bar{\theta}_{\dot{b}i}}\left(\bar{\theta}_{\dot{a}k}\bar{\theta}_{\dot{j}\ell}\bar{\theta}_{\dot{a}m}\bar{\theta}_{\dot{f}n}\right) = \varepsilon_{k\ell}\varepsilon_{mn}\left(C_{\dot{i}\dot{i}}C_{\dot{j}\dot{f}} + C_{\dot{i}\dot{f}}C_{\dot{j}\dot{i}}\right) +$$

$$+\varepsilon_{km}\varepsilon_{\ell n}\left(C_{\dot{j}\dot{i}}C_{\dot{i}\dot{f}} + C_{\dot{j}\dot{i}}C_{\dot{i}\dot{f}}\right) + \varepsilon_{kn}\varepsilon_{\ell m}\left(C_{\dot{i}\dot{j}}C_{\dot{f}\dot{i}} + C_{\dot{i}\dot{i}}C_{\dot{f}\dot{j}}\right),$$

the following explicit relations are found,

$$\begin{aligned}
G^{+}(x) &= 8\partial^2 A(x)\\
\sigma_\mu\partial^\mu\bar{\chi}(x) &= 4i\partial^2\psi(x)\\
-\partial^2 E^{+}(x) &= \partial^2 E(x)\\
-\sigma_\lambda\Sigma^{\tau}_{\mu\nu}\bar{\sigma}_\rho\,\partial^\lambda\partial^\rho F^{[\mu\nu]+}(x) &= \Sigma_{\mu\nu}\partial^2 F^{[\mu\nu]}(x)
\end{aligned} \tag{5.15}$$

where $\quad (\Sigma_{\mu\nu})_{\dot{a}}^{\ \dot{b}} = -(\Sigma_{\mu\nu})_a^{\ b*}$.

It is immediately evident that this is the same reduction as that implied by setting $\tilde{\chi}$ to zero.

Consistently with equations (5.15) and the transformation laws (eqns 5.12), the expansion of a superfield of type $\Phi^{(1)}(x,\theta)$, subjected to equation (5.14), may be written as,

$$\hat{\Phi}(x,\theta) = \tfrac{1}{2}A_1(x) + \tfrac{i}{2}A_2(x) + \theta\psi(x) + \tfrac{1}{2}\theta\tau\theta.\underset{\sim}{F}_2(x) + $$
$$+ \tfrac{i}{4}\theta\Sigma_{\mu\nu}\theta F_2^{[\mu\nu]}(x) - \tfrac{2i}{3}\theta^{ai}\theta_{aj}\theta^{bj}(\sigma_\mu)_{bi}\partial^\mu\bar{\psi}^c_i(x) +$$
$$+ \tfrac{1}{6}\theta^{ai}\theta_{aj}\theta^{bj}\theta_{bi}\partial^2\big(A_1(x) - iA_2(x)\big) \qquad (5.16)$$

with the boson fields all hermitian and $F_2^{[\mu\nu]}(x)$ satisfying

$$\varepsilon_{\mu\nu\lambda\rho}\partial^\nu F_2^{[\lambda\rho]}(x) = 0 \ . \qquad (5.17)$$

The infinitesimal transformation laws satisfied by the constituent fields of $\hat{\Phi}$ may be deduced from equations (5.12),

$$\delta A_1(x) = \bar{\xi}\, i\gamma_5\psi(x)$$
$$\delta A_2(x) = -i\bar{\xi}\psi(x)$$
$$\delta\psi(x) = \gamma_5\gamma_\mu\xi\partial^\mu A_1(x) + \gamma_\mu\xi\partial^\mu A_2(x) + i\underset{\sim}{\tau}\xi.\underset{\sim}{F}_2(x)$$
$$- \tfrac{i}{2}\sigma_{\mu\nu}\xi F_2^{[\mu\nu]}(x)$$
$$\delta\underset{\sim}{F}_2(x) = -\bar{\xi}\gamma_\mu\underset{\sim}{\tau}\partial^\mu\psi(x)$$
$$\delta F_2^{[\mu\nu]}(x) = -i\bar{\xi}(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\psi(x) \ . \qquad (5.18)$$

Note that these equations provide an immediate check that the constraint (eqn 5.17) is super-invariant.

The procedure involved in obtaining the supersymmetric transformations is a very long process involving much algebraic manipulation. In practice, it is found to be a very convenient check to look at the double-transform equation (4.7). In Appendix G we shall show how the transformation laws (eqns 5.18) satisfy equation (4.7).

Before moving on to look at another set of transformation laws we should consider the nature of the product of two superfields of the form of equation (5.16). If we demand that the superfields, $\Phi^{(1)}(x,\theta)$ and $\Phi^{(2)}(x,\theta)$, of Appendix F, both satisfy equation (5.14), then the product is still a reducible scalar superfield of the same form as equation (5.11). This is an important point of difference between the irreducible scalar multiplet (eqns 5.18) and the scalar multiplet of the original super-algebra. In the latter case the product of two irreducible scalar superfields of the same type is also an irreducible scalar superfield of that type. This particular property was exploited by Wess and Zumino (ref 23) to obtain an extension of quantum electrodynamics which is super-invariant using the original superfields. It has not been found possible to duplicate the work of ref.23 using the irreducible scalar multiplet of the supersymmetry which incorporates internal symmetry.

An alternative way of reducing the superfield of equation (5.11) is to demand that it satisfies the super-invariant relation which is obtained from equation (5.14) by multiplying both sides by a Green's function for the wave-equation and integrating over space-time. The superfield, so reduced, has the expansion,

$$\begin{aligned}\hat{\Phi}_1(x,\theta) = \; & \frac{1}{32}\frac{1}{\partial^2}\left(\varphi_1(x) - i\varphi_2(x)\right) - \frac{i}{4}\frac{1}{\partial^2}\,\theta\gamma_\mu\partial^\mu\bar{\chi}(x) + \frac{i}{2}\,\theta\tau\theta.\underline{F}_1(x) + \\ & + \frac{i}{4}\,\theta\Sigma_{\mu\nu}\theta\,F_2^{\mu\nu}(x) + \frac{1}{3!}\,\theta^{ai}\theta_{aj}\theta^{bj}\chi_{bi}(x) + \\ & + \frac{1}{2.4!}\,\theta^{ai}\theta_{aj}\theta^{bj}\theta_{bi}\left(\varphi_1(x) + i\varphi_2(x)\right)\end{aligned} \tag{5.19}$$

with $F_2^{[\mu\nu]}(x)$ satisfying the constraint

$$\varepsilon_{\mu\nu\lambda\rho}\,\partial^\nu F_2^{[\lambda\rho]}(x) = 0$$

and all boson fields hermitian.

The infinitesimal transformation laws again follow from equation (5.12), they are

$$\delta q_1(x) = -2i\bar{\xi}\not{\partial}\chi(x)$$

$$\delta q_2(x) = 2i\bar{\xi}\gamma_5\not{\partial}\chi(x)$$

$$\delta\chi(x) = \tfrac{1}{2}\xi q_1(x) - \tfrac{1}{2}\gamma_5\xi q_2(x) + 4i\gamma_5\gamma_\mu\tau\xi\partial^\mu \underset{\sim}{F}_2(x)$$
$$- 2i\gamma_5\gamma_\lambda\sigma_{\mu\nu}\xi\partial^\lambda F_2^{[\mu\nu]}(x)$$

$$\delta\underset{\sim}{F}_2(x) = \tfrac{1}{4}\bar{\xi}\tau\gamma_5\chi(x)$$

$$\delta F_2^{[\mu\nu]}(x) = -\frac{i}{4}\bar{\xi}(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\frac{1}{\partial^2}\gamma_5\not{\partial}\chi(x) \quad (5.20)$$

By introducing minus signs into equation (5.14), we can produce similar multiplets involving the other boson fields, $\underset{\sim}{F}_1$ and $F_1^{[\mu\nu]}$; the only difference in the supersymmetry transformations is the appearance of extra γ_5 -matrices. The exact structure of these transformations can easily be obtained by considering the reducible multiplet equations (5.12).

The antisymmetric tensor superfield and the isovector superfield have similar supersymmetry transformations, and can also be reduced by a similar procedure to that used above, by merely appending the appropriate indices onto the fields each time they appear.

The remaining superfield which we have to consider is the spinor superfield. This is by far the largest irreducible superfield in the supersymmetry system we are considering, containing 128 independent complex components. Its supersymmetry transformations can be written as follows,

$$\delta\psi = \gamma^\rho\left[\underline{\tau}.\underline{A}_\rho + V_\rho + \gamma_5\underline{\tau}.\underline{A}'_\rho + \gamma_5 V'_\rho\right]\xi$$

$$\delta\underline{A}_\rho = \tfrac{1}{4}\bar{\xi}\not\partial\gamma_\rho\underline{\tau}\psi + \bar{\xi}\tau_i\underline{\tau}\gamma_\rho\chi_i + \bar{\xi}\sigma_{\xi\eta}\gamma_\rho\underline{\tau}\chi^{[\xi\eta]}$$

$$\delta\underline{A}'_\rho = \tfrac{1}{4}\bar{\xi}\not\partial\gamma_\rho\underline{\tau}\gamma_5\psi + \bar{\xi}\tau_i\underline{\tau}\gamma_\rho\gamma_5\chi_i + \bar{\xi}\sigma_{\xi\eta}\gamma_\rho\underline{\tau}\gamma_5\chi^{[\xi\eta]}$$

$$\delta V_\rho = \tfrac{1}{4}\bar{\xi}\not\partial\gamma_\rho\psi + \bar{\xi}\underline{\tau}\gamma_\rho.\underline{\chi} + \bar{\xi}\sigma_{\xi\eta}\gamma_\rho\chi^{[\xi\eta]}$$

$$\delta V'_\rho = \tfrac{1}{4}\bar{\xi}\not\partial\gamma_\rho\gamma_5\psi + \bar{\xi}\underline{\tau}\gamma_\rho\gamma_5.\underline{\chi} + \bar{\xi}\sigma_{\xi\eta}\gamma_\rho\gamma_5\chi^{[\xi\eta]}$$

$$\delta\chi_i = \tfrac{1}{8}\gamma^\rho\partial_\sigma\left[\underline{\tau}.\underline{A}_\rho + V_\rho + \gamma_5\underline{\tau}.\underline{A}'_\rho + \gamma_5 V'_\rho\right]\gamma^\sigma\tau_i\xi$$
$$+\tfrac{1}{8}\gamma^\rho\left[\underline{\tau}.\underline{B}_\rho + W_\rho + \gamma_5\underline{\tau}.\underline{B}'_\rho + \gamma_5 W'_\rho\right]\tau_i\xi$$

$$\delta\chi_{[\kappa\lambda]} = \tfrac{1}{32}\gamma^\rho\partial_\sigma\left[\underline{\tau}.\underline{A}_\rho + \gamma_5\underline{\tau}.\underline{A}'_\rho + V_\rho + \gamma_5 V'_\rho\right]\sigma_{\kappa\lambda}\gamma^\sigma\xi$$
$$-\tfrac{1}{32}\gamma^\rho\left[\underline{\tau}.\underline{B}_\rho + \gamma_5\underline{\tau}.\underline{B}'_\rho + W_\rho + \gamma_5 W'_\rho\right]\sigma_{\kappa\lambda}\xi$$

$$\delta\underline{B}_\mu = \tfrac{1}{4}\bar{\xi}\gamma_\mu\underline{\tau}\lambda + \bar{\xi}\not\partial\tau_i\underline{\tau}\gamma_\mu\chi_i - \bar{\xi}\not\partial\sigma_{\xi\eta}\gamma_\mu\underline{\tau}\chi^{[\xi\eta]}$$

$$\delta\underline{B}'_\mu = \tfrac{1}{4}\bar{\xi}\gamma_\mu\underline{\tau}\gamma_5\lambda + \bar{\xi}\not\partial\tau_i\underline{\tau}\gamma_\mu\gamma_5\chi_i - \bar{\xi}\not\partial\sigma_{\xi\eta}\gamma_\mu\underline{\tau}\gamma_5\chi^{[\xi\eta]}$$

$$\delta W_\mu = \tfrac{1}{4}\bar{\xi}\gamma_\mu\lambda + \bar{\xi}\not\partial\underline{\tau}\gamma_\mu.\underline{\chi} - \bar{\xi}\not\partial\sigma_{\xi\eta}\gamma_\mu\chi^{[\xi\eta]}$$

$$\delta W'_\mu = \tfrac{1}{4}\bar{\xi}\gamma_\mu\gamma_5\lambda + \bar{\xi}\not\partial\underline{\tau}\gamma_\mu\gamma_5.\underline{\chi} - \bar{\xi}\not\partial\sigma_{\xi\eta}\gamma_\mu\gamma_5\chi^{[\xi\eta]}$$

$$\delta\lambda = \gamma^\mu\partial_\sigma\left[\underline{\tau}.\underline{B}_\mu + \gamma_5\underline{\tau}.\underline{B}'_\mu + W_\mu + \gamma_5 W'_\mu\right]\gamma^\sigma\xi. \tag{5.21}$$

It is worthwhile noting that there is a difference between the spinor multiplet of the original supersymmetry theory and the multiplet described above. In the original theory the equivalent spinor multiplet was reducible and could be regarded as the combination of two irreducible spinor superfields, each containing a Majorana spinor, a vector field and an auxiliary scalar field. It was possible to construct from one of these irreducible multiplets a massless free Lagrangian density which is "super-invariant". In the present case we have derived a very large spinor multiplet which is irreducible and, as we shall indicate in the next chapter, it is not possible to form a physical Lagrangian density that is "super-invariant". An additional problem is the spinor field with the additional isovector label which is found in the spinor multiplet. Whereas all the other fields could be regarded as corresponding in some sense to physical fields, the appearance of this spinor field is difficult to understand in physical terms.

Finally in this section we shall consider the most general <u>real</u> scalar superfield. It might be suggested that, since the 256 independent components of the real scalar superfield are themselves real, the problems are not so difficult and, in fact, the first superfield calculations were undertaken on a general real scalar superfield (ref 19). Unfortunately, because of the complexity of the calculations, we are not able to make direct derivations in this case. Therefore we found it easier to use the complex superfield written in the tabular form, which gave us a clear indication of the steps to take. Nevertheless, we should note the form of the general real scalar superfield and show that it can be reduced into the multiplets we have described above.

We shall not write the superfield out formally but instead note the coefficients for each power of the totally anticommuting "Majorana" spinor θ, adopting the same notation for the fields used in section 5.2,

$$
\begin{array}{ll}
 & A \\
\theta & \psi \\
\theta^2 & E, E' \quad F_{[\mu\nu]} \quad F'_{[\mu\nu]} \quad \underline{A}_\mu \quad \underline{V}_\mu \\
\theta^3 & \underline{\chi} \quad \chi_{[\mu\nu]} \quad \chi \\
\theta^4 & \rho \quad \rho' \quad \underline{B}_\mu \quad \underline{B}'_\mu \quad W_\mu \quad W'_\mu \quad \underline{E}_{[\mu\nu]} \quad \underline{E}'_{[\mu\nu]} \quad \underline{\underline{E}} \quad F_{[\mu\nu][\rho\sigma]} \\
\theta^5 & \underline{\lambda} \quad \lambda_{[\mu\nu]} \quad \lambda \\
\theta^6 & \underline{\rho} \quad \underline{\rho}' \quad \rho_{[\mu\nu]} \quad \rho'_{[\mu\nu]} \quad \underline{C}_\mu \quad V'_\mu \\
\theta^7 & \xi \\
\theta^8 & B.
\end{array}
$$

It is not possible to give a clear indication of how these fields form into irreducible multiplets without explicit calculation. But, noting the reduction of the complex scalar superfield, it can be seen that the real scalar superfield is composed of two irreducible scalar superfields, a spinor superfield , an irreducible isovector superfield and an irreducible antisymmetric tensor superfield.

CHAPTER SIX

In this final chapter on supersymmetry we shall look at the possibilities of forming Lagrangian densities from the multiplets derived in the previous chapter. We shall compare the results with the work of Capper and Leibbrandt and also with the achievements arising from the original super-algebra.

Section 6.1

In this chapter we are going to look at the possibilities for forming a physical Lagrangian density which is "super-invariant" from the multiplets which we derived in the previous chapter. The formal procedure for obtaining Lagrangian densities for super-symmetry multiplets was described in the early papers (ref 19,26), but with the introduction of internal symmetries this process becomes very long. We will introduce here a shorthand way of displaying the contents of the Lagrangian density. Though the exact coefficients still have to be determined, this technique does help to determine all possible "super-invariant" Lagrangian densities and thus eliminate quickly the unphysical ones.

The procedure in all cases is to find the coefficient of the highest power of the θ's and $\bar{\theta}$'s of a product of superfields of the same type. This coefficient can be extracted, up to a total derivative, by applying the appropriate number of super-invariant derivatives and this forms the Lagrangian density. There are two possibilities, that is, either the product of superfields contains only θ's (or $\bar{\theta}$'s) or it contains both θ's and $\bar{\theta}$'s. In the first case only four super-invariant derivatives are required but in the second case eight derivatives are necessary. Clearly, if both types Lagrangian are going to be considered together, extra derivatives or mass operators must be inserted for dimensional reasons.

For illustration, we shall consider here products of two superfields of the form $\Phi(x,\theta)$, where

$$\Phi(x,\theta) = A(x) + \theta\psi(x) + \frac{1}{2!}\theta\underset{\sim}{\Gamma}\theta.\underset{\sim}{F}(x) + \frac{1}{2.2!}\theta\Sigma_{\mu\nu}\theta F^{[\mu\nu]}(x) +$$

$$+\frac{1}{3!}\theta^{ai}\theta_{aj}\theta^{bj}\chi_{bi}(x) + \frac{1}{4!}\theta^{ai}\theta_{aj}\theta^{bj}\theta_{bi}\, G(x).$$

In the first case, in order to obtain the contents of the Lagrangian density,

$$\mathcal{L} = \mathcal{D}^4 \left[\Phi(x,\theta)\, \Phi(x,\theta) \right],$$

simply multiply the multiplet "top to tail", i.e.

$$\left\{ \begin{matrix} A & \psi & \underline{E}, F_{[\mu\nu]} & \chi & \varsigma \\ \varsigma & \chi & \underline{E}, F_{[\mu\nu]} & \psi & A \end{matrix} \right\}.$$

The Lagrangian density, $\mathcal{L}$, in this case contains

$$A\varsigma,\ \psi\chi,\ \underline{E}^2,\ F_{[\mu\nu]}^2 .$$

In the second case it is necessary to return to the tabular form. Insert the component fields of $\Phi(x,\theta)$ in the first row and those of $\Phi^*(x,\theta)$ in the first column. Then fill the last row and column with the appropriate derivatives of the component fields (the number of derivatives is determined by dimensional arguments). The object of this method is to find the contents of

$$\mathcal{L} = \mathcal{D}^8 \left[\Phi(x,\theta)\, \Phi^*(x + 2i\theta\sigma\bar{\theta}, \bar{\theta}) \right]$$

and it is straightforward to do this by extracting the coefficient of $\theta^4\bar{\theta}^4$ from the following table. It can be seen that the Lagrangian density contains,

$$A\partial^4 A^*,\quad A^*\partial^4 A,\quad \underline{E}\partial^2\underline{E}^*,\quad \underline{E}^*\partial^2\underline{E},\quad \varsigma\varsigma^*$$

$$\bar{\chi}\not{\partial}\chi,\quad \bar{\psi}\not{\partial}\partial^2\psi,\quad F_{[\mu\nu]}\partial^2 F^{[\mu\nu]*},\quad F^*_{[\mu\nu]}\partial^2 F^{[\mu\nu]}.$$

		θ	θ^2	θ^3	θ^4
	$A^{(*)}$	ψ	$\underline{E}, F_{[\mu\nu]}$	χ	ρ
$\bar{\theta}$	ψ				$\sigma_\mu \partial^\mu \chi$
$\bar{\theta}^2$	$\underline{E}^*, F_{[\mu\nu]}$				$\partial^2 \underline{E}, \partial^2 F_{[\mu\nu]}$
$\bar{\theta}^3$	χ				$\sigma_\mu \partial^\mu \partial^\nu \psi$
$\bar{\theta}^4$	ρ^*	$\sigma_\mu \partial^\mu \chi$	$\partial^2 \underline{E}^*, \partial^2 F^*_{[\mu\nu]}$	$\sigma_\mu \partial^\mu \partial^\nu \psi$	$\partial^4 A^{(*)}$

These techniques are not precise but they do allow us to examine the alternatives which are available and to discount those that are clearly not going to lead to a model that is physically applicable.

Section 6.2

In this section we shall look in detail at the possible Lagrangian densities that can be formed from the multiplets derived in the previous chapter. The first thing to note is that the contents of the Lagrangian densities which were given in the last section are not, in their present form, physically applicable. It is possible to introduce a system which involves cancellations (ref 30) and we shall consider this later, but here we shall consider Lagrangian densities which appear without needing such cancellations.

First we will look at the multiplet arising from the superfield of equation (5.16). A "super-invariant" free Lagrangian density for this multiplet may be written as follows

$$\mathcal{L}_0 = -\frac{1}{4}\frac{1}{4!}\, D^{ai} D_{aj} D^{bj} D_{bk} \left(\hat{\Phi}(x,\theta)\, \hat{\Phi}(x,\theta) \right) + \text{h.c.} \tag{6.1}$$

where

$$D_{ai} = \frac{\partial}{\partial \theta^{ai}} - 2i (\sigma_\mu \bar{\theta})_{ai}\, \partial^\mu \tag{6.2}$$

is a super-invariant derivative acting on superfields which transform as equation (4.10b). Using the combination rules of Appendix F, the free Lagrangian density can be written explicitly as

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu A_1 \partial^\mu A_1 + \frac{1}{2}\partial_\mu A_2 \partial^\mu A_2 + \frac{i}{2}\bar{\psi}\not{\partial}\psi + \frac{1}{2}E_2^2$$
$$- \frac{1}{4} F_{2[\mu\nu]} F_2^{[\mu\nu]} + \text{divergence terms} \, . \tag{6.3}$$

This gives a massless free theory, and it would be natural to try to introduce mass terms in a super-invariant way. However, this cannot be done, as it is easily seen from the supersymmetry transformations (eqns 5.18) that the construction of a non-trivial quadratic form in the fields, which does not involve derivatives, is impossible.

Next we shall consider couplings. The simplest self interaction of the superfield $\hat{\Phi}(x,\theta)$ is given by the following "super-invariant" interaction Lagrangian density,

$$\mathcal{L}_I(x) = -\frac{1}{3}\frac{1}{4!} D^{ai} D_{aj} D^{bj} D_{bi} \left(\hat{\Phi}(x,\theta)\right)^3 + h.c. \tag{6.4}$$

In terms of the constituent fields this is given explicitly by

$$\mathcal{L}_I = i\bar{\psi}\not{\partial}\psi A_1 + F_2^2 A_1 - \frac{1}{2} F_{2[\mu\nu]} F_2^{[\mu\nu]} A_1 + A_1\partial_\mu A_1 \partial^\mu A_1 +$$
$$+ A_1 \partial_\mu A_2 \partial^\mu A_2 - i\bar{\psi}\gamma_5 \not{\partial}\psi A_2 + \frac{1}{2}\bar{\psi}\gamma\gamma_5\psi . F_2 +$$
$$+ \frac{1}{4}\bar{\psi}\gamma_5 \sigma_{\mu\nu}\psi F_2^{[\mu\nu]} + \frac{1}{4}\varepsilon_{\mu\nu\kappa\lambda} F_2^{[\mu\nu]} F_2^{[\kappa\lambda]} A_2 +$$
$$+ \text{ divergence terms } . \tag{6.5}$$

It is interesting to note that if the field-equations for F_2 and $F_2^{[\mu\nu]}$, which follow from equation (6.3) together with equation (6.5), are used to eliminate these auxiliary fields from the total Lagrangian, the resulting interaction is non-polynomial in the fields $A_1(x)$ and $A_2(x)$.

It should also be noted that it is possible to give an alternative interpretation to the field $F_2^{[\mu\nu]}$ and regard it instead as

$$F_{2[\mu\nu]} = \partial_\mu v_\nu - \partial_\nu v_\mu , \qquad \partial_\mu v^\mu = 0 . \tag{6.6}$$

This would imply that the new vector field, v_ν , has the supersymmetry transformation

$$\delta v_\nu = i\bar{\xi}\gamma_\nu\psi - i\frac{\partial_\mu\partial_\nu}{\partial^2}\bar{\xi}\gamma^\mu\psi ,$$

moreover, this definition is consistent with equation (5.17).

The consequence of considering the vector field, v_μ , instead of the antisymmetric tensor field, $F_2^{[\mu\nu]}$ is that the latter appears as an auxiliary field but the former can be regarded as a physical field. We are not able to give any insight into this apparent ambiguity.

Now we shall consider the multiplet represented by the irreducible superfield of equation (5.19). As we have seen, this is essentially the same superfield as that used above, only with the fields redefined. We shall see that the free Lagrangian densities are of an entirely different character. A "super-invariant" free Lagrangian density for this superfield is given by

$$\mathcal{L}_o(x) = -\frac{1}{64}\,\frac{1}{4!}\,D^{ai}D_{aj}\,D^{bj}D_{bi}\left(\hat{\Phi}_1(x,\theta)\,\partial^2\,\hat{\Phi}_1(x,\theta)\right) + \text{h.c.}$$

$$\text{i.e.}\quad \mathcal{L}_o(x) = \frac{1}{8}\left(G_1^2 + G_2^2\right) + \frac{1}{2}\,\bar{\chi}\,\not{\partial}\chi + 8\left(\partial_\mu \underset{\sim}{F}_2\right)^2 - 4\left(\partial_\sigma F_2^{[\mu\nu]}\right)^2 + \text{divergence terms.} \qquad (6.7)$$

Note that in this multiplet the roles assumed by the boson fields in the reduced multiplet are interchanged. Whereas previously $\underset{\sim}{F}_2(x)$ and $F_2^{[\mu\nu]}(x)$ were auxiliary fields and the others "physical", here $\underset{\sim}{F}_2(x)$ and $F_2^{[\mu\nu]}(x)$ are the "physical" fields and the others are the auxiliary fields.

The simplest self interaction is given by

$$\mathcal{L}_I(x) = -\frac{1}{48}\,\frac{1}{4!}\,D^{ai}D_{aj}\,D^{bj}D_{bi}\left(\hat{\Phi}_1(x,\theta)\left[\partial^2\hat{\Phi}_1(x,\theta)\right]^2\right) + \text{h.c.} \qquad (6.8)$$

It can be shown, using the combination rules, that this interaction contains terms which are clearly undesirable,for example, $(\partial_\mu G_1)^2\frac{1}{\partial^2}G_1$. Whereas in the free Lagrangian density the symmetry takes care of such terms and ensures they do not appear in the final version, in this case such terms are unavoidable. Thus we are lead to discount this formulation as unphysical.

We can form a Lagrangian density from the reducible scalar multiplet by redefining the first two component fields such that

$$\hat{\Phi}_2(x,\theta) = \frac{1}{16}\frac{1}{\partial^2}\mathcal{D}(x) - \frac{i}{4\partial^2}\,\theta\gamma_\mu\partial^\mu\bar{\lambda}(x) + \frac{1}{2}\theta\Sigma\theta.\underline{E}(x) +$$
$$+\frac{1}{4}\,\theta\Sigma_{\mu\nu}\theta\,F^{[\mu\nu]}(x) + \frac{1}{3!}\,\theta^{ai}\theta_{aj}\,\theta^{bj}\chi_{bi}(x) +$$
$$+\frac{1}{4!}\,\theta^{ai}\theta_{aj}\,\theta^{bj}\,\theta_{bi}\,\mathcal{G}(x). \qquad (6.9)$$

Using the tabular technique we can see that this Lagrangian density would contain

$$\bar{\lambda}\not{\partial}\lambda\,,\quad \bar{\chi}\not{\partial}\chi\,,\quad \underline{E}_1\partial^2\underline{E}_1\,,\quad \underline{E}_2\partial^2\underline{E}_2\,,\quad \mathcal{G}_1^2\,,\ \mathcal{G}_2^2\,,\ \mathcal{D}_1^2,\ \mathcal{D}^2,$$
$$\partial^2F_{2[\mu\nu]}\,F_2^{[\mu\nu]}\,,\quad \partial^2F_{1[\mu\nu]}\,F_1^{[\mu\nu]}\,.$$

Note that if $\lambda \equiv \chi\,,\ \mathcal{D} \equiv \mathcal{G}^*\,,\ \underline{E}_1 \equiv 0\,,\ F_1^{[\mu\nu]} \equiv 0\,,$ then $\hat{\Phi}_2(x,\theta) = \hat{\Phi}_1(x,\theta)$. Therefore, using equation (5.14)

$$\frac{1}{2.4!}\,\mathcal{D}^4\left(\frac{\partial}{\partial\bar{\theta}}\right)^4\left[\hat{\Phi}_1(x,\theta)\,\hat{\Phi}_1(x+2i\theta\sigma\bar{\theta},\bar{\theta})\right] = \mathcal{D}^4\left[\hat{\Phi}_1(x,\theta)\,\partial^2\hat{\Phi}_1(x,\theta)\right]$$

and thus we can see the connection between this Lagrangian and the previous one. Unfortunately, as in the previous case, the difficulty arises in forming an interaction Lagrangian and we must discount this formulation also.

All the above calculations can be made applicable for the multiplets of the isovector and antisymmetric tensor superfields by just adding the appropriate indices.

Finally, we shall briefly consider the spinor superfield. The only method of forming a possible Lagrangian without cancellations is to construct an appropriate second spinor multiplet so that when these two are multiplied together they produce a Lagrangian density which could be physical. Unfortunately, this Lagrangian density would contain the term $\bar{\chi}\not{\phi}.\chi$ and, since we cannot give this any physical interpretation, we are lead into discounting this Lagrangian and also this multiplet.

The conclusion of this section is that, of all the Lagrangians we have considered, only the ones formed from the irreducible scalar multiplet could be regarded as physical (eqns 6.3, 6.5). All the other multiplets have to be excluded because they do not give both a free and an interacting Lagrangian density that could be regarded as physically applicable.

The successful multiplet forms a massless interacting Lagrangian using a procedure similar to that introduced for the scalar multiplet of the original supersymmetry theory (ref 19), though in that case the fields could be massive. In the next section we shall describe the work undertaken by Capper and Leibbrandt (ref 30) in which they try to form a massive interacting Lagrangian from the reducible scalar multiplet.

Section 6.3

In this section we shall give a brief summary of the attempts of Capper and Leibbrandt to form a massive "super-invariant" Lagrangian density (ref 30). Unlike the approach used in section 6.2, the procedure used in ref.30 does not restrict attention to Lagrangians which could be produced without cancellations. On the contrary, the approach used is to introduce additional "super-invariant" terms into the Lagrangian density specifically to cancel the undesirable terms from the first attempt at construction.

The massive free Lagrangian density proposed takes the form

$$\mathcal{L}_0 = \frac{1}{40}(\bar{D}\Sigma D)^4 \Phi_+ \Phi_- + (\bar{D}\Sigma D)^2\left[\Phi_+(\partial^2+2m^2)\Phi_+\right] + (\bar{D}\Sigma D)^2\left[\Phi_-(\partial^2+2m^2)\Phi_-\right]. \qquad (6.11)$$

where Φ_+ and Φ_- are reducible scalar superfields such that $\Phi_+^* = \Phi_-$.

The form of the Lagrangian density is such that the fourth
? order derivatives from the first term (which are ghosts) are cancelled by the second term. The auxiliary fields in this free Lagrangian density do not have such simple equations of motion as in the previous cases we have considered, since many mixed products occur, but when they are removed the form of the Lagrangian density is physical. (In fact, the spinor fields appear in an undiagonalised manner but no ghosts appear if these are diagonalised).

The most promising interaction Lagrangian density is

$$\mathcal{L}_I = (\bar{D}\Sigma D)^2\left(\Phi_+^3 + \Phi_-^3\right) \qquad (6.12)$$

and when the auxiliary fields are removed from the sum of the free and interaction Lagrangian densities the resulting expression is very complicated, involving quartic interactions and cubic terms with quadratic derivatives.

Capper and Leibbrandt show, using supergraph techniques, that, though the divergences are considerably less than expected, this does not lead to a renormalizable theory. Also, the choice of another interaction is unlikely to improve this conclusion. The important distinction between this case and the original model is that here the mass plays an essential role and cannot be set to zero. If, in fact, we consider what happens when the mass tends to zero we are lead to the irreducible scalar multiplet which we have already considered.

Section 6.4

We are now in a position to give an assessment of the results emerging from consideration of the larger super-algebra incorporating isospin. In making this assessment we shall draw a comparison between the results derived in this work and those found in the original supersymmetry theory (see section 4.3 and Appendix E).

Our object in this work has been to duplicate the successes of the original supersymmetry theory using the larger super-algebra which incorporates isospin intrinsically.

We have found that there are two basic types of irreducible multiplets emerging from the larger super-algebra, these form the scalar and spinor superfields. If we impose the super-invariant constraint

$$\frac{\partial}{\partial\bar{\theta}} \Phi(x, \theta, \bar{\theta}) = 0$$

on a general scalar superfield, $\Phi(x, \theta, \bar{\theta})$, that transforms as equation (4.10b), the resulting super-multiplet is reducible (unlike the original supersymmetry theory). In order to obtain the irreducible scalar superfield we must also impose the constraint (eqn 5.14)

$$\frac{1}{2.4!} \frac{\partial}{\partial\bar{\theta}^{ai}} \frac{\partial}{\partial\bar{\theta}_{aj}} \frac{\partial}{\partial\bar{\theta}^{bj}} \frac{\partial}{\partial\bar{\theta}_{bi}} \tilde{\Phi}(x + 2i\theta\sigma\bar{\theta}, \bar{\theta}) = \partial^2 \Phi(x, \theta)$$

where $\tilde{\Phi}(x+2i\theta\sigma\bar{\theta}, \bar{\theta})$ is obtained by first taking the hermitian conjugate and then shifting the space-time variable.

We noted in section 5.3 that, because of this further constraint, it is not possible to combine two irreducible scalar multiplets to give a third irreducible scalar multiplet. Instead this product would give the reducible scalar multiplet. This is in contrast to the original theory were it was possible to make such a combination and this property was used by Wess and Zumino (ref 23) in order to extend quantum electrodynamics in a super-invariant way. Hence, we are not able to repeat the work of ref.23 using the irreducible scalar multiplet of the larger super-algebra.

Nevertheless, we are able (eqns 6.3, 6.5) to form a successful massless interacting Lagrangian with the irreducible scalar multiplet in a manner analogous to that used in the original theory (cf. Appendix E). In fact, this was the only suitable interacting Lagrangian which we were able to extract from the superfields without requiring cancellations.

In section 6.3 we described an attempt by Capper and Leibbrandt to construct a massive interacting theory using the reducible scalar multiplet. In this model the mass plays an essential role and cannot be set to zero. This differs from the original model since there the massive free Lagrangian was constructed in two independent parts and if the mass is set to zero the Lagrangian is still valid (see Appendix E). This important difference results in the conclusion that, unlike the original model, the Lagrangian model of ref.30 is non-renormalizable. (It is interesting to note that the conclusions of ref.30 are similar to those of Adjei and Akyeampong (ref 29) when they used the same approach with the reducible spinor multiplet of the original theory.)

It remains for us to consider the Lagrangian densities of equations (6.3 & 6.5) as a possible model. In fact, these equations are similar to the original interacting Lagrangian model (see Appendix E) and it is interesting to compare them in some detail. Both models are based on irreducible scalar multiplets with a scalar and a pseudoscalar field and a spinor field. In the original case the spinor is Majorana, i.e.

$$\psi_\alpha = C_{\alpha\beta}\bar{\psi}^\beta$$

whereas, in the case in which we are incorporating isospin, the spinor satisfies

$$\psi_{\alpha i} = i\varepsilon_{ij}(\gamma_5 C)_{\alpha\beta}\bar{\psi}^{\beta j}.$$

In the isospin case the auxiliary fields are an antisymmetric tensor and an isovector field. This differs from the original model in which the two auxiliary fields were a scalar and a pseudoscalar field. It is readily seen that, once these auxiliary fields have been removed from the massless free Lagrangian densities, the two models are quite similar.

It is interesting to note that, when the auxiliary fields are removed from the interacting Lagrangian densities, the result in the original case is still a polynomial, whereas in the isospin case the interaction Lagrangian density must now be non-polynomial in the fields $A_1(x)$ and $A_2(x)$.

Therefore the new Lagrangian shares many characteristic features of the original one, including the nature of the "physical" fields involved. Unfortunately, the scalar bosons which are present in both models are not found to play an important role in physics. Whilst we can superimpose higher spin and isospin labels on the basic multiplet and also re-interpret the auxiliary antisymmetric tensor field as a "physical" vector field, we then have the difficulty of interpreting the spinor fields with these additional labels.

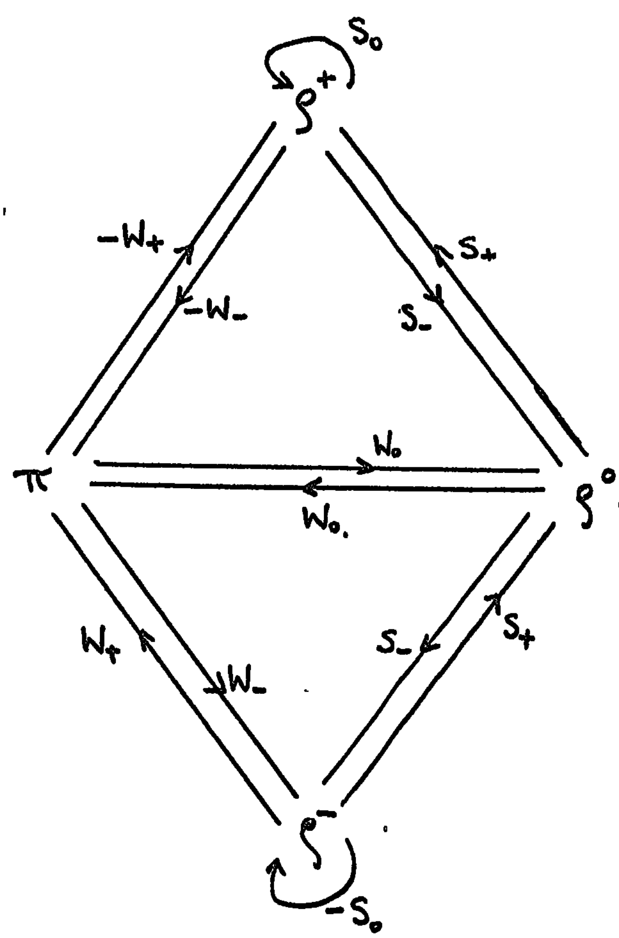
Hence as our conclusion we are lead to suggest that the massless interacting Lagrangian densities of equations (6.3 & 6.5) form the most promising model to emerge from the large super-algebra incorporating isospin. Unfortunately, this model is not physically applicable, since the isospin content which we hoped to incorporate into the model does not appear on the boson fields which could be interpreted as physical. Whilst we could superimpose the isospin labels this would lead to the problem of understanding the spinor fields with these additional labels. Therefore we have not been able to give supersymmetry any additional importance in describing nature.

Nevertheless, we have successfully introduced isospin into the original supersymmetry theory and thus we do have a theory which incorporates all the special points of interest noted in section 4.4 . The theory contains bosons and fermions together in a manner which is in general less divergent than might be expected from power counting. Also, the form of the super-algebra is not the direct sum of an internal symmetry algebra and the Lorentz algebra and thus supersymmetry is intrinsically different to previous theories. For these reasons it is useful to attempt to extend the scope of the work presented here to a general SU(N) internal symmetry (ref 18) and work is progressing in this direction (ref 32).

Appendix A

The set of definitions for $\underset{\sim}{M}$ and $\underset{\sim}{W}$ given in section 3.2 are not unique in having all the required properties. In this appendix we shall display the definitions for $\underset{\sim}{W}$ suggested in ref.7 and the corresponding version of the $\underset{\sim}{M}$-operator. Together these are able to generate the same solution as the set of definitions proposed in section 3.2 .

Using the diagrammatic form, the $\underset{\sim}{W}$-operators could be defined as follows,



where the $\underset{\sim}{W}$ and $\underset{\sim}{S}$ operators form an $SU(2)\otimes SU(2)$ algebra.

The $\underset{\sim}{M}$-operators which are analogous to those presented in equations (3.5) are,

$$\langle n+1, \ell+1, m | M_{\substack{+\\-\\0}} | n \ell m \rangle = \substack{-\\+\\+}\sqrt{\frac{\ell+1}{2\ell+3}}\; C_{m}^{\ell}\,{}^{1}\,{}_{m'}^{\ell+1}\; f_{+}(n+\ell+2)$$

$$\langle n-1, \ell+1, m | M_{\substack{+\\-\\0}} | n \ell m \rangle = \substack{-\\+\\+}\sqrt{\frac{\ell+1}{2\ell+3}}\; C_{m}^{\ell}\,{}^{1}\,{}_{m'}^{\ell+1}\; f_{-}(n-\ell-1)$$

$$\langle n+1, \ell-1, m | M_{\substack{+\\-\\0}} | n \ell m \rangle = \substack{+\\-\\-}\sqrt{\frac{\ell}{2\ell-1}}\; C_{m}^{\ell}\,{}^{1}\,{}_{m'}^{\ell-1}\; f_{-}(n-\ell+1)$$

$$\langle n-1, \ell-1, m | M_{\substack{+\\-\\0}} | n \ell m \rangle = \substack{+\\-\\-}\sqrt{\frac{\ell}{2\ell-1}}\; C_{m}^{\ell}\,{}^{1}\,{}_{m'}^{\ell-1}\; f_{+}(n+\ell).$$

These definitions differ significantly from those given in ref.7 . Firstly, we have made no attempt here to determine the functions, $f_{\pm}$, Secondly, the definitions are such that $M_{+}^{\dagger} = M_{-}$, $M_{0}^{\dagger} = M_{0}$ and so

$$Z = M_{+}W_{-} - M_{-}W_{+} = -Z^{\dagger}.$$

There is only one point of disagreement between the sets of definitions in the series of calculations outlined in Appendix B. But, since this effects the nature of the function, $B(n,\ell)$ (eqn 3.4), there is no difference finally as in both cases $B(n,\ell)$ is necessarily zero.

Appendix B

In this appendix we shall give the details of the series of calculations performed in order to ensure that our unitary operator is compatible with the constraints resulting from Weinberg's equation (eqn 1.8-1.10).

Using the general form for the zero-th order $(\text{mass})^2$-squared equation,

$$m_0^2 = A(n,\ell) + B(n,\ell)\,\underset{\sim}{L}\cdot\underset{\sim}{S}$$

As already noted, the approach used is a perturbative one and we shall look at all three constraint equations (1.8-1.10) at each order of θ.

At order θ Eqn 1.10 with $n' = n \pm 1$, $\ell' = \ell \pm 1$

$$B(n,\ell) = (-1)^{n+1} B = (-1)^{\ell+1} B \qquad (B \text{ constant}) \qquad \text{(B.1)}$$

(Note that using the definitions of Appendix A we would obtain $B(n,\ell)$ = constant.)

The other equations offer no constraints at this order.

At order θ^2 Eqn 1.8 off diagonal i.e. $\langle t n' \ell' m | t n \ell m \rangle = 0$

$$A(n,\ell) + A(n\pm 2, \ell+2) - 2A(n\pm 1, \ell+1) + B(-1)^{n+1} = 0$$

and $$A(n,\ell) + A(n,\ell\pm 2) - A(n+1,\ell\pm 1) - A(n-1,\ell\pm 1) + B(-1)^{n+1} = 0$$

$$\Rightarrow A(n,\ell) = A_0 + \Delta n + \frac{B(-1)^{n+1}}{4} \qquad \text{(B.2)}$$

$(A_0, \Delta$ constants$)$.

<u>At order θ^2</u>   <u>Eqn 1.9</u>   off-diagonal, $s_a = -s_a'$

$$\Rightarrow \qquad B = 0 \tag{B.3}$$

(Note, this is true in both systems of definitions and they are now effectively equivalent.)

The other off-diagonal equations up to second order do not give any further constraints.

We shall now consider the equations at second order when $n' = n$, $\ell' = \ell$. The most general form of m_t^2 is, (ref 6),

$$m_t^2 = \alpha(I n \ell) + \beta(I n \ell)\,\underline{L}.\underline{S} + \gamma(I n \ell)\,(\underline{L}.\underline{S})^2 + \tfrac{1}{2}\,\delta(I n \ell)\,S^2. \tag{B.4}$$

<u>Eqn 1.9</u> $s_a = s_a'$

$$\alpha(I{=}1, n\ell) + \delta(I{=}1, n\ell) - \alpha(I{=}0, n\ell) - \delta(I{=}0, n\ell) =$$

$$= \left[\gamma(I{=}0, n\ell) - \gamma(I{=}1, n\ell)\right]\langle J n \ell m|(\underline{L}.\underline{S})^2|J n \ell m\rangle$$

$$\Rightarrow \quad \gamma(I{=}0, n\ell) = \gamma(I{=}1, n\ell)$$

and $\quad \alpha(I{=}1, n\ell) + \delta(I{=}1, n\ell) = \alpha(I{=}0, n\ell) + \delta(I{=}0, n\ell)$

since true for all helicities.   (B.5)

<u>Eqn 1.9</u> $s_a = -s_a'$

$$\gamma(n,\ell) = \frac{\Delta}{2(2\ell+1)}\left[\frac{f_+^2(n+\ell+2) - f_-^2(n-\ell-1)}{2\ell+3} + \right.$$

$$\left. - \frac{f_+^2(n+\ell) - f_-^2(n-\ell+1)}{2\ell-1}\right]. \tag{B.6}$$

Eqn 1.8

$$\delta(I=1, n\,\ell) + \gamma(n,\ell)\left[\ell(\ell+1) - m^2\right] =$$

$$= -\frac{\Delta}{2}\left[\frac{(\ell+2)(\ell+1)+m^2}{(2\ell+1)(2\ell+3)}\right]\left(f_+^2(n+\ell+2) - f_-^2(n-\ell-1)\right) +$$

$$+\frac{\Delta}{2}\left[\frac{\ell(\ell-1)+m^2}{(2\ell-1)(2\ell+1)}\right]\left(f_+^2(n+\ell) - f_-^2(n-\ell-1)\right). \qquad \text{(B.7)}$$

Inserting $\gamma(n,\ell)$ we can see that there is <u>no</u> helicity dependence in the equation (cf. ref.6).

Hence,

$$\delta(I=1, n\,\ell) = -\frac{\Delta}{2\ell+1}\left[\frac{(\ell+1)^2}{2\ell+3}\left(f_+^2(n+\ell+2) - f_-^2(n-\ell-1)\right)\right.$$

$$\left. -\frac{\ell^2}{2\ell-1}\left(f_+^2(n+\ell) - f_-^2(n-\ell+1)\right)\right]. \qquad \text{(B.8)}$$

Equation (1.10) does not impose any further constraints. Hence, we have now considered all the relevant conditions resulting from Weinberg's equation up to, and including, second order in θ.

Appendix C

In this appendix we derive the consequences of the calculations of Appendix B on the meson states in the L=0,1 multiplets. The chiral content of these multiplets is,

	h = 2	h = 1	h = 0
L = 0		$\rho = \upsilon$	$\rho = \bar{t}$
		$\omega = \upsilon_4$	$\omega = s$
			$\pi = t$
L = 1	$A_2 = \upsilon'$	$A_2 = \frac{\upsilon' + \bar{t}}{\sqrt{2}}$	$A_2 = \frac{1}{\sqrt{3}}\left(\frac{\upsilon' + \upsilon^{-1}}{\sqrt{2}} + \sqrt{2}\,\bar{t}\right)$
	$f = \upsilon_4'$	$f = \frac{\upsilon_4' + s}{\sqrt{2}}$	$f = \frac{1}{\sqrt{3}}\left(\frac{\upsilon_4' - \upsilon_4^{-1}}{\sqrt{2}} + \sqrt{2}\,s\right)$
		$A_1 = \frac{\upsilon' - \bar{t}}{\sqrt{2}}$	$A_1 = \frac{1}{\sqrt{2}}(\upsilon' - \upsilon^{-1})$
		$D = \frac{\upsilon_4' - s}{\sqrt{2}}$	$D = \frac{1}{\sqrt{2}}(\upsilon_4' + \upsilon_4^{-1})$
		$B = t$	$B = t$
			$A_0 = \frac{1}{\sqrt{3}}(\upsilon' + \upsilon^{-1} - \bar{t})$
			$\sigma = \frac{1}{\sqrt{3}}(\upsilon_4' - \upsilon_4^{-1} - s)$

Hence,
$$m^2_{A_2} = \langle \upsilon^{\pm1};\, 1\ \pm1 |\, m^2 \,| \upsilon^{\pm1};\, 1\ \pm1 \rangle$$
$$= A_0 + \Delta + \theta^2(\alpha_1' + \beta_1' + \gamma_1' + \delta_1')$$

and
$$m^2_B = \langle t;\, 1\ l_z |\, m^2 \,| t;\, 1\ l_z \rangle$$
$$= A_0 + \Delta + \theta^2 \alpha_1'$$

(where the prime refers to isospin one, otherwise isospin zero, and the subscript is the value of n = ℓ). Similarly we can find the masses for all the states.

Physical particle	Calculated $(\text{mass})^2$, up to order θ^2.	Predicted $(\text{mass})^2$	$(\text{mass})^2$ if $f_\pm = \sqrt{x+1}$	$(\text{mass})^2$ if $f_- \equiv 0$, $f_+ \equiv \frac{\Delta'}{\Delta}$, const.
π	$A_0 + \theta^2 \alpha_0'$	—	π	π
η	$A_0 + \theta^2 \alpha_0$	—	—	—
ρ	$A_0 + \theta^2(\alpha_0' + \delta_0')$		$\pi - \theta^2 \Delta$	$\pi - \theta^2 \Delta'/3$
ω	$A_0 + \theta^2(\alpha_0 + \delta_0)$	$\omega = \rho$	$\pi - \theta^2 \Delta$	$\pi - \theta^2 \Delta'/3$
A_2	$A_0 + \Delta + \theta^2(\alpha_1' + \beta_1' + \gamma_1' + \delta_1')$	$K + \theta^2(\beta_1' + \gamma_1)$	$K + \theta^2 \beta_1'$	$K + \theta^2(\beta_1' - 2\Delta'/15)$
f	$A_0 + \Delta + \theta^2(\alpha_1 + \beta_1 + \gamma_1 + \delta_1)$	$K + \theta^2(\beta_1 + \gamma_1)$	$K + \theta^2 \beta_1$	$K + \theta^2(\beta_1 - 2\Delta'/15)$
A_1	$A_0 + \Delta + \theta^2(\alpha_1' - \beta_1' + \gamma_1' + \delta_1')$	$K + \theta^2(-\beta_1' + \gamma_1)$	$K - \theta^2 \beta_1'$	$K - \theta^2(\beta_1' + 2\Delta'/15)$
D	$A_0 + \Delta + \theta^2(\alpha_1 - \beta_1 + \gamma_1 + \delta_1)$	$K + \theta^2(-\beta_1 + \gamma_1)$	$K - \theta^2 \beta_1$	$K - \theta^2(\beta_1 + 2\Delta'/15)$
A_0	$A_0 + \Delta + \theta^2(\alpha_1' - 2\beta_1' + 4\gamma_1' + \delta_1')$	$K + \theta^2(-2\beta_1' + 4\gamma_1)$	$K - 2\theta^2 \beta_1'$	$K - \theta^2(2\beta_1' + 8\Delta'/15)$
σ	$A_0 + \Delta + \theta^2(\alpha_1 - 2\beta_1 + 4\gamma_1 + \delta_1)$	$K + \theta^2(-2\beta_1 + 4\gamma_1)$	$K - 2\theta^2 \beta_1$	$K - \theta^2(2\beta_1 + 8\Delta'/15)$
B	$A_0 + \Delta + \theta^2 \alpha_1'$	—	$K + \theta^2 \Delta$	$K - \theta^2 \Delta'/15$
H	$A_0 + \Delta + \theta^2 \alpha_1$	—	—	—
	where the prime refers to isospin one (otherwise isospin zero) and the subscript is the value of $n = \ell$.	where $K = A_0 + \Delta + \theta^2(\alpha_1' + \delta_1')$ $= A_0 + \Delta + \theta^2(\alpha_1 + \delta_1)$ in all columns	If $f_\pm = \sqrt{x+1}$, then $\delta' = -\Delta$, $\gamma \equiv 0$, $\pi = A_0 + \theta^2 \alpha_0'$	If $f_- \equiv 0$, $f_+ = \Delta'/\Delta$, then $\delta_0' = -\Delta'/3$, $\delta_1' = \Delta'/15$, $\gamma \equiv -2\delta$, $\pi = A_0 + \theta^2 \alpha_0'$

Appendix D

Summary of notation used in Chapters 4,5 and 6 .

$$\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$$

Pauli matrices:

$$\sigma_i \sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\,\sigma_k$$

$$\sigma_k \sigma_i \sigma_k = -\sigma_i$$

Dirac γ-matrices:

$$\{\gamma^A\} = \{1, \gamma^\mu, \gamma^5, \sigma^{\mu\nu}, i\gamma^\mu\gamma^5\}$$

where $\gamma^5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\,\varepsilon_{\mu\nu\rho\sigma}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$

$$\varepsilon_{0123} = 1$$

and $\sigma^{\mu\nu} \equiv \frac{i}{2}\,[\gamma^\mu, \gamma^\nu]$

For any γ^A define γ_A so that $\gamma_A\gamma^A = +1$ (no summation).

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Useful relations:

$$\sigma^{\mu\nu}\gamma^5 = -\tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,\sigma_{\rho\sigma}\,.$$

$$\sigma^{\mu\nu}\gamma^\rho = -i\left(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu\right) + \varepsilon^{\mu\nu\rho\lambda}\,i\,\gamma_\lambda\gamma^5\,.$$

$$[\sigma^{\mu\nu}, \gamma^\rho] = 2i\left(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu\right)$$

$$\sigma^{\mu\nu}\sigma^{\kappa\lambda} = i\big(\eta^{\nu\kappa}\sigma^{\mu\lambda} + \eta^{\mu\lambda}\sigma^{\nu\kappa} - \eta^{\nu\lambda}\sigma^{\mu\kappa} - \eta^{\mu\rho}\gamma^\nu + \eta^{\mu\kappa}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\kappa} + \varepsilon^{\mu\nu\kappa\lambda}\gamma^5\big)\,.$$

$$\sigma^{\mu\nu}\sigma_{\mu\nu} = 12$$

$$\sigma_{\kappa\lambda}\gamma_{\varsigma}\gamma_{\eta}\sigma^{\kappa\lambda} = 16\eta_{\varsigma\eta} - 4\gamma_{\varsigma}\gamma_{\eta}$$

$$\gamma_{\kappa}\gamma_{\lambda}\gamma_{\varsigma}\gamma^{\kappa} = 4\eta_{\lambda\varsigma}$$

$$\gamma^{\mu}\sigma^{\kappa\lambda}\gamma_{\mu} = 0$$

$$\sigma_{\mu\nu}\gamma_{\varsigma}\sigma^{\mu\nu} = 0 .$$

Properties of the charge conjugation matrix, C,

$$C^{-1}\gamma^{\mu}C = -\gamma^{\mu T}$$

$$C^{+} = C^{-1}$$

$$C^{T} = -C$$

hence

$$C^{-1}\gamma^{5}C = \gamma^{5T}$$

$$C^{-1}\sigma^{\mu\nu}C = -\sigma^{\mu\nu T}.$$

It is useful to note that the matrices $\gamma_{\mu}C$ and $\sigma_{\mu\nu}C$ are symmetric, whilst C, $\gamma_{5}C$ and $i\gamma_{\mu}\gamma_{5}C$ are antisymmetric.

Hence

$$\bar{\psi}_1\gamma^{A}\tau_{B}\psi_2 = +\bar{\psi}_2\gamma^{A}\tau_{B}\psi_1$$

for $\gamma^{A}\tau_{B} \in \{\tau_{K}, \tau_{K}\gamma^{5}, \tau_{K}\gamma^{\mu}, \sigma^{\mu\nu}, i\gamma^{\mu}\gamma^{5}\}$

and

$$\bar{\psi}_1\gamma^{A}\tau_{B}\psi_2 = -\bar{\psi}_2\gamma^{A}\tau_{B}\psi_1$$

for $\gamma^{A}\tau_{B} \in \{1, \gamma^{5}, \gamma^{\mu}, \tau_{K}\sigma^{\mu\nu}, i\tau_{K}\gamma^{\mu}\gamma^{5}\}$.

Also there exists the re-arrangement formula

$$\psi_1 \bar{\psi}_2 = -\frac{1}{8} \sum_{A,B} (\bar{\psi}_2 \gamma_A \tau_B \psi_1) \gamma^A \tau_B$$

where

$$\psi_{\alpha i} = i\, \varepsilon_{ij} (\gamma_5 C)_{\alpha\beta} \bar{\psi}^{\beta j}$$

for both ψ_1 and ψ_2.

Note: no claim for originality is claimed for this appendix but it is included for completeness.

Appendix E

In this appendix we shall give some details of the work done on the original superfields. No claim for originality is made for this work. It is included as a basis for comparison for the work presented in chapters 4,5 and 6 . In order to aid this comparison we shall display this summary in a way similar to that adopted in the main text.

The general complex scalar superfield can be written in the following tabular form,

		θ	θ^2
	A	ψ	F
$\bar{\theta}$	ψ'	V_μ	λ'
$\bar{\theta}^2$	F'	λ	D

and can be expanded as

$$\Phi(x,\theta,\bar{\theta}) = \Phi^{(A)}(x,\theta) + \bar{\theta}^{\dot{a}i}\,\bar{\Phi}^{(B)}_{\dot{a}i}(x,\theta) + \tfrac{1}{2}\bar{\theta}^{\dot{a}i}\bar{\theta}_{\dot{a}i}\,\bar{\Phi}^{(C)}(x,\theta) \qquad (E.1)$$

(cf. eqn 5.10). If we impose the super-invariant derivative,

$$\frac{\partial}{\partial\bar{\theta}}\,\Phi\,(x,\theta,\bar{\theta}) = 0,$$

then we separate out $\Phi^{(A)}(x,\theta)$ and this is the irreducible scalar multiplet. The supersymmetry transformations of this multiplet can be written, in the Majorana notation, where ψ is a Majorana spinor,

$$\begin{aligned}
\delta A_1(x) &= i\bar{\xi}\psi(x) \\
\delta A_2(x) &= i\bar{\xi}\gamma_5\psi(x) \\
\delta\psi(x) &= \not{\partial}(A_1(x)+\gamma_5 A_2(x))\xi + (F_1(x)+\gamma_5 F_2(x))\xi \\
\delta F_1(x) &= i\bar{\xi}\not{\partial}\psi(x) \\
\delta F_2(x) &= i\bar{\xi}\gamma_5\not{\partial}\psi(x)
\end{aligned} \tag{E.2}$$

where A_1 and A_2 are the real and complex parts of A, similarly for F.

From this set of fields we can form the following "super-invariant" Lagrangian densities (ref 26),

$$\mathcal{L}_0 = -\tfrac{1}{2}(\partial_\mu A_1)^2 - \tfrac{1}{2}(\partial_\mu A_2)^2 - \tfrac{i}{2}\bar{\psi}\not{\partial}\psi + F_1^2 + F_2^2 \tag{E.3}$$

$$\mathcal{L}_m = m\left(F_1 A_1 + F_2 A_2 - \tfrac{i}{2}\bar{\psi}\psi\right) \tag{E.4}$$

$$\mathcal{L}_g = g\left[F_1(A_1^2 - A_2^2) + 2F_2 A_1 A_2 - i\bar{\psi}(A_1 - \gamma_5 A_2)\psi\right]. \tag{E.5}$$

The sum of these three terms is an interacting Lagrangian density and if we use the equations of motion for the auxiliary fields, F_1, F_2

$$\begin{aligned}
F_1 + mA_1 + g(A_1^2 - A_2^2) &= 0 \\
F_2 + mA_2 + 2gA_1A_2 &= 0
\end{aligned} \tag{E.6}$$

then we can eliminate these fields and obtain the following "super-invariant" Lagrangian density

$$\begin{aligned}
\mathcal{L} = &-\tfrac{1}{2}(\partial_\mu A_1)^2 - \tfrac{1}{2}(\partial_\mu A_2)^2 - \tfrac{i}{2}\bar{\psi}\not{\partial}\psi \\
&- \tfrac{1}{2}m^2(A_1^2 + A_2^2) - \tfrac{i}{2}m\bar{\psi}\psi \\
&- gmA_1(A_1^2 + A_2^2) - \tfrac{g^2}{2}(A_1^2 + A_2^2)^2 - ig\bar{\psi}(A_1 - \gamma_5 A_2)\psi
\end{aligned}$$

Note that all the masses are equal and the couplings are all related as a consequence of demanding that the Lagrangian density is "super-invariant". It is this model which has been shown to be renormalizable to all orders of perturbation theory (ref 22).

The spinor multiplet can be reduced to a multiplet containing only an antisymmetric tensor field, a Majorana spinor field and an auxiliary scalar field.

$$\begin{aligned} \delta v_{[\mu\nu]} &= i\partial_\mu(\bar{\xi}\gamma_\nu\lambda) - i\partial_\nu(\bar{\xi}\gamma_\mu\lambda) \\ \delta\lambda &= -\tfrac{1}{2} v_{[\mu\nu]}\gamma^\mu\gamma^\nu\xi + D\gamma_5\xi \\ \delta D &= i\bar{\xi}\gamma_5 \not{\partial}\lambda \end{aligned} \tag{E.7}$$

where the tensor field can also be interpreted as a vector field

$$v_{[\mu\nu]} = \partial_\mu v_\nu - \partial_\nu v_\mu \;, \qquad \partial_\mu v^\mu = 0 .$$

From this multiplet we can form the massless free Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4} v_{[\mu\nu]} v^{[\mu\nu]} - \tfrac{i}{2}\bar{\lambda}\not{\partial}\lambda + \tfrac{1}{2}D^2 . \tag{E.8}$$

Appendix F

We shall display here the combination rules for two reducible scalar superfields. As noted in section 4.2, the combination of two superfields of the same type is a superfield of that type. We shall consider here the product

$$\Phi^{(3)}(x,\theta) = \Phi^{(1)}(x,\theta)\,\Phi^{(2)}(x,\theta)$$

where each of the superfields is a reducible scalar superfield which is a function of x and θ but not $\bar{\theta}$ (eqn 5.11). Expressing both sides of equation (5.11) as an expansion in θ, we can equate the coefficients and determine the following combination rules, transformed here into the "Majorana" notation for convenience,

$$A_1^{(3)} = A_1^{(1)} A_1^{(2)} - A_2^{(1)} A_2^{(2)}$$

$$A_2^{(3)} = A_1^{(1)} A_2^{(2)} + A_2^{(1)} A_1^{(2)}$$

$$\psi^{(3)} = (A_1^{(1)} - \gamma_5 A_2^{(1)})\psi^{(2)} + (A_1^{(2)} - \gamma_5 A_2^{(2)})\psi^{(1)}$$

$$E_1^{(3)} = A_1^{(1)} E_1^{(2)} + A_1^{(2)} E_1^{(1)} - A_2^{(1)} E_2^{(2)} - A_2^{(2)} E_2^{(1)} + \tfrac{1}{2}\bar{\psi}^{(1)} \tau \psi^{(2)}$$

$$E_2^{(3)} = A_1^{(1)} E_2^{(2)} + A_1^{(2)} E_2^{(1)} + A_2^{(1)} E_1^{(2)} + A_2^{(2)} E_1^{(1)} + \tfrac{1}{2}\bar{\psi}^{(1)} \gamma_5 \tau \psi^{(2)}$$

$$F_{1[\mu\nu]}^{(3)} = A_1^{(1)} F_{1[\mu\nu]}^{(2)} + A_1^{(2)} F_{1[\mu\nu]}^{(1)} - A_2^{(1)} F_{2[\mu\nu]}^{(2)} - A_2^{(2)} F_{2[\mu\nu]}^{(1)} + -\tfrac{1}{2}\bar{\psi}^{(1)} \sigma_{\mu\nu} \psi^{(2)}$$

$$F_{2[\mu\nu]}^{(3)} = A_1^{(1)} F_{2[\mu\nu]}^{(2)} + A_1^{(2)} F_{2[\mu\nu]}^{(1)} + A_2^{(1)} F_{1[\mu\nu]}^{(2)} + A_2^{(2)} F_{1[\mu\nu]}^{(1)} + -\tfrac{1}{2}\bar{\psi}^{(1)} \sigma_{\mu\nu} \gamma_5 \psi^{(2)}$$

$$\chi^{(3)} = (A_1^{(1)} - \gamma_5 A_2^{(1)})\chi^{(2)} + (A_1^{(2)} - \gamma_5 A_2^{(2)})\chi^{(1)} +$$
$$+ 4i(\gamma_5 \underset{\sim}{F}_1^{(1)} + \underset{\sim}{F}_2^{(1)}).\underset{\sim}{\Sigma}\psi^{(2)} + 4i(\gamma_5 \underset{\sim}{F}_1^{(2)} + \underset{\sim}{F}_2^{(2)}).\underset{\sim}{\Sigma}\psi^{(1)} +$$
$$+ 2i(\gamma_5 F^{(1)}_{1\,[\mu\nu]} + F^{(1)}_{2\,[\mu\nu]})\sigma^{\mu\nu}\psi^{(2)} +$$
$$+ 2i(\gamma_5 F^{(2)}_{1\,[\mu\nu]} + F^{(2)}_{2\,[\mu\nu]})\sigma^{\mu\nu}\psi^{(1)}.$$

$$q_1^{(3)} = A_1^{(1)} q_1^{(2)} + A_1^{(2)} q_1^{(1)} - A_2^{(1)} q_2^{(2)} - A_2^{(2)} q_2^{(1)} +$$
$$+ 16\,\underset{\sim}{F}_1^{(1)}.\underset{\sim}{F}_1^{(2)} - 16\,\underset{\sim}{F}_2^{(1)}.\underset{\sim}{F}_2^{(2)} +$$
$$+ 4\,\varepsilon_{\mu\nu\kappa\lambda} F_1^{(1)[\mu\nu]} F_2^{(2)[\kappa\lambda]} +$$
$$+ 4\,\varepsilon_{\mu\nu\kappa\lambda} F_2^{(1)[\mu\nu]} F_1^{(2)[\kappa\lambda]}$$
$$- 8 F^{(1)}_{1\,[\mu\nu]} F_1^{(2)[\mu\nu]} + 8 F^{(1)}_{2\,[\mu\nu]} F_2^{(2)[\mu\nu]}$$
$$+ 2i\,\bar{\psi}^{(1)}\gamma_5\chi^{(2)} + 2i\,\bar{\psi}^{(2)}\gamma_5\chi^{(1)}.$$

$$q_2^{(3)} = A_1^{(1)} q_2^{(2)} + A_1^{(2)} q_2^{(1)} + A_2^{(1)} q_1^{(2)} + A_2^{(2)} q_1^{(1)} +$$
$$+ 16\,\underset{\sim}{F}_1^{(1)}.\underset{\sim}{F}_2^{(2)} + 16\,\underset{\sim}{F}_1^{(2)}.\underset{\sim}{F}_2^{(1)} +$$
$$- 4\,\varepsilon_{\mu\nu\kappa\lambda} F_1^{(1)[\mu\nu]} F_1^{(2)[\kappa\lambda]} + 4\,\varepsilon_{\mu\nu\kappa\lambda} F_2^{(1)[\mu\nu]} F_2^{(2)[\kappa\lambda]}$$
$$- 8 F^{(1)}_{1\,[\mu\nu]} F_2^{(2)[\mu\nu]} - 8 F^{(1)}_{2\,[\mu\nu]} F_1^{(2)[\mu\nu]}$$
$$- 2i\,\bar{\psi}^{(1)}\chi^{(2)} - 2i\,\bar{\psi}^{(2)}\chi^{(1)}.$$

In this very general form the combination rules look very unwieldy but, in fact, all the information we shall need to obtain the Lagrangian densities for the various irreducible scalar multiplets can easily be extracted.

Finally, we shall note that the product of two irreducible superfields of the form of equation (5.16) is not itself an irreducible superfield but rather a reducible superfield of the general form, as equation (5.11). This can readily be seen by noting that if $\underset{\sim}{F}_1^{(1)}$, $\underset{\sim}{F}_1^{(2)}$ are identically zero then

$$\underset{\sim}{F}_1^{(3)} = - A_2^{(1)} \underset{\sim}{F}_2^{(2)} - A_2^{(2)} \underset{\sim}{F}_2^{(1)} + \tfrac{1}{2} \bar{\psi}^{(1)} \underset{\sim}{\Sigma} \psi^{(2)}$$

$$\underset{\sim}{F}_2^{(3)} = A_1^{(1)} \underset{\sim}{F}_2^{(2)} + A_1^{(2)} \underset{\sim}{F}_2^{(1)} + \tfrac{1}{2} \bar{\psi}^{(1)} \gamma_5 \underset{\sim}{\Sigma} \psi^{(2)}$$

and clearly neither of these is zero hence the superfield cannot be irreducible and must be of the form of equation (5.11).

Appendix G

In this appendix we shall give two examples to show the way in which equation 4.7 is satisfied by the supersymmetry transformations we have derived in section 5.3 (eqns 5.18). Formally these transformations must satisfy equation (4.7) by the nature of their construction but in practice it has been found very useful to consider the following type of calculation as a relatively simple check on the long algebraic derivation.

We shall rewrite equation (4.7) in the form

$$[\delta_1 , \delta_2] = -2i\, \bar{\xi}_2 \not{\partial} \xi_1$$

where now ξ_1 and ξ_2 are the completely anticommuting parameters. Consider first the commutator acting on a boson field,

e.g.

$$\begin{aligned}
[\delta_1 , \delta_2] A_1 &= i\, \bar{\xi}_2 \gamma_5\, \delta_1 \psi - i\, \bar{\xi}_1 \gamma_5\, \delta_2 \psi \\
&= i\, \bar{\xi}_2 \gamma_5 \left[\gamma_5 \not{\partial} A_1 + \not{\partial} A_2 + i \underline{\Sigma} \cdot \underline{E}_2 - \tfrac{1}{2} \sigma_{\mu\nu} F_2^{[\mu\nu]} \right] \xi_1 \\
&\quad - i\, \bar{\xi}_1 \gamma_5 \left[\gamma_5 \not{\partial} A_1 + \not{\partial} A_2 + i \underline{\Sigma} \cdot \underline{E}_2 - \tfrac{1}{2} \sigma_{\mu\nu} F_2^{[\mu\nu]} \right] \xi_2 \\
&= -2i\, \bar{\xi}_2 \not{\partial} \xi_1 A_1
\end{aligned}$$

using the relations shown in Appendix D.

All the boson fields behave in a similar manner under the double-transform but the fermion fields are more complicated and involve the use of the rearrangement formula (see Appendix D).

$$\delta_1 \delta_2 \psi = \gamma_5 \not{\phi} (\bar{\xi}_1 i \gamma_5 \psi) \xi_2 + \not{\phi}(-i \bar{\xi}_1 \psi) \xi_2 + \\ + i \underline{\tau} . (-\bar{\xi}_1 \not{\phi} \underline{\tau} \psi) \xi_2 - \tfrac{i}{2} \sigma^{\mu\nu} \left(- \bar{\xi}_1 (\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu) \psi \right) \xi_2$$

then using the rearrangement formula,

$$[\delta_1 , \delta_2] = - \frac{2i}{8} \sum (\bar{\xi}_1 \gamma^A \tau_B \xi_2) \begin{bmatrix} \gamma_5 \not{\phi} \gamma_A \tau_B \gamma_5 - \not{\phi} \gamma_A \tau_B \\ - \underline{\tau} \gamma_A \tau_B \not{\phi} \underline{\tau} \\ + \frac{1}{2} \sigma^{\mu\nu} \gamma_A \tau_B (\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu) \end{bmatrix} \psi$$

where $\gamma_A \tau_B = \{ 1, \gamma_5 , \gamma_\eta , \gamma_5 \gamma_\eta \tau_i , \sigma_{\eta\xi} \tau_i \}$,

the other values having cancelled on the introduction of the commutator. By considering the coefficient of each $\bar{\xi}_1 \gamma^A \tau_B \xi_2$ it can readily be shown that

$$[\delta_1 , \delta_2] \psi = - 2i \bar{\xi}_2 \gamma_\mu \partial^\mu \xi_1 \psi .$$

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