Conformal Metrics and True "Gradient Flows" for Curves

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Abstract

We wish to endow the manifold M of smooth curves in $\mathbb{R}^n$ with a Riemannian metric that allows us to treat continuous morphs (homotopies) between two curves c_0 and c_1 as trajectories with computable lengths which are independent of the parameterization or representation of the two curves (and the curves making up the morph between them). We may then define the distance between the two curves using the trajectory of minimal length (geodesic) between them, assuming such a minimizing trajectory exists. At first we attempt to utilize the metric structure implied rather unanimously by the past twenty years or so of shape optimization literature in computer vision. This metric arises as the unique metric which validates the common references to a wide variety of contour evolution models in the literature as "gradient flows" to various formulated energy functionals. Surprisingly, this implied metric yields a pathological and useless notion of distance between curves. In this paper, we show how this metric can be minimally modified using conformal factors the depend upon a curve's total arclength. A nice property of these new conformal metrics is that all active contour models that have been called "gradient flows" in the past will constitute true gradient flows with respect to these new metrics under specfic time reparameterizations.

1 Introduction

Ever since the introduction of snakes by Kass, Witkin, and Terzopoulos [7], active contours [1] have played a prominent role in a variety of image processing and computer vision tasks, most notably segmentation. Early reasearch on active contours saw the tansition from parameterization dependent models to geometric models independent of the parameterization of the evolving curve. Next, there were many efforts to incorporated region based image information to make the active contour depend upon global information about the image rather than just the traditional locally computed edge descriptors. In recent years, the latest trend in active contour research seems to be that of incorporating global shape priors into the active contour paradigm. This has brought up non-trivial questions such as how to define an "average shape" or how to characterize "variations in shape". All of these questions ultimately lead to a more basic and fundamental question of how to measure the distance between two given curves.

In this paper we study geometries on the manifold M of curves. This manifold contains curves c, which we parameterize as $c : S^1 \to \mathbb{R}^n$ (S^1 is the circle). Given a curve c, we define the tangent space T_cM of M at c including in it deformations $h : S^1 \to \mathbb{R}^n$, so that an infinitesimal deformation of the curve c in direction h will yield the curve $c(\theta) + \varepsilon h(\theta)$.

We would like to define a *Riemannian metric* on the manifold M of curves: this means that, given two deformations $h_1, h_2 \in T_cM$, we want to define a scalar product $\langle h_1, h_2 \rangle_c$, possibly dependent on c. The Riemannian metric would then entail a *distance* $d(c_0, c_1)$ between the curves in M, defined as the infimum of the length $\text{Len}(\gamma)$ of all smooth paths $\gamma : [0, 1] \to M$ connecting c_0 to c_1. We call *minimal geodesic* a path providing the minimum of $\text{Len}(\gamma)$ in the class of γ with fixed endpoints.

Surprisingly, almost two decades of literature on variational approaches to active contours and active surfaces (whether they be for image segmentation, stereo reconstruction [4], or other computer vision tasks) suggests a consistent metric on the space of curves. However, the rather unanimous suggestion of this common underlying metric structure is made implicitly, perhaps even unknowingly in many cases. Many authors have defined Energy Functionals on curves (or surfaces) and utilized the Calculus of Variations to derive curve evolutions to minimize the Energy Functionals; often referring to these evolutions as Gradient Flows. For example, the well known Geometric Heat Flow [6], popular for its smoothing effect on contours, is often referred as the *gradient flow for length*, or the more general Geodesic Active Contour model [2, 10] is described as the *gradient flow* for a conformally weighted length based upon image data.

The reference to these flows as *gradient flows* implies a certain Riemannian metric on the space of curves; but this fact has been largely overlooked. We call this metric H^0

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henceforth. If one wishes to have a consistent view of the geometry of the space of curves in both Shape Optimization and Shape Analysis, then one should use the H^0 metric when computing distances, averages, and morphs between shapes. To our knowledge, however, this consistently suggested metric has not been used for these purposes in the shape analysis literature.

In this paper we first introduce the metric H^0 and immediatly remark that, even more surprisingly, it does not yield a well defined metric structure, since the associated distance is identically zero[1], [8, 11] making it pointless to even talk about geodesics between curves (optimal morphs from one curve to another). We propose instead in a class of *conformal metrics* H^ϕ that fix the above problems while minimally altering the earlier flows: in fact the new gradient flows will amount to simple time reparameterizations of the earlier flows. As such, contour evolutions models that have thusfar been referred to as gradient flows will constitute true (time-reparameterized) gradient flows with respect to these conformal metrics. In addition the conformal metrics that we propose have some nice numerical and computational properties: distances measured between curves are defined using only first order derivatives (and therefore the resulting optimality conditions involve only second order derivatives); as a consequence, flows designed to converge towards these optimality conditions are second order, thereby allowing the use of Level Set methods [9]: we indeed show such an implementation and a numerical results for an experimental example.

As a final preface, we wish to point out that a vast literature is starting to arise on shape metrics for the purpose of shape analysis (see for example [3, 5, 12]) that have little or no relation to the conformal metrics we are presenting here. It is not our intent to make comparisons between these metrics in terms of their utility and performance for *shape analysis* tasks. They are all quite different from each other and such a comparison would not only be difficult, but would detract from the primary point we wish to make in this paper. Namely, while the only metric used thusfar in *shape optimization* is unsuitable for shape analysis, certain conformal modifications of this metric can solve this problem *while still being consistent* with prior shape optimization techniques. This consistency is lost in any other class of metrics.

2 The Unspoken Curve Metric H^0

2.1 The manifold of curves

We will denote by M the manifold of smooth curves c in $\mathbb{R}^n$. The tangent space T_cM to M at the curve c will consist of all possible infinitesimal deformations of the curve c along the direction of its unit normal N at each point. We may represent such elements of T_cM as vector fields along the curve c which are also normal to the curve c. The restriction to the normal direction is necessary if we wish to treat curves geometrically (i.e. without regard to their parameterization). It is well known in curve evolution theory that deformations of a curve along its tangent direction at each point do nothing more than reparameterize the curve.

2.2 Trajectories on the manifold of curves

Let $C : [0,1] \times [0,1] \to \mathbb{R}^n$, denote a homotopy (continuous morph) $C(u,v)$ between two boundary curves c_0 and c_1 where u parameterizes each individual curve in the homotopy and v parameterizes the homotopy itself.[2] Such a "curve of curves" represents a trajectory on the manifold M between the points c_0 and c_1. In what follows, we will let $T(u,v) = C_u/\|C_u\|$ denote the unit tangent vector at a point $C(u,v)$ along a particular curve in the homotopy, and we will let $L(v)$ denote the total arclength of a particular curve in the homotopy.

2.3 Specifying a Riemannian metric

To obtain a Riemannian metric on M, we must define a scalar product $\langle h_1, h_2 \rangle$ for all $h_1, h_2 \in T_cM$ at each point c in M. We initially propose the scalar product

$$\langle h_1, h_2 \rangle_{H^0} = \int_c h_1(s) \cdot h_2(s)\, ds \qquad (1)$$

where s denotes the arclength parameter of the curve c. From now on, when we speak of the H^0 metric, we will be implying this last definition.

Once we have specified an scalar product, we are able to calculate the length of trajectories by integrating the speed of the trajectory (the norm of its derivative according to the specified scalar product). For a trajectory $C(u,v)$ of curves and for the H^0 metric, we obtain the following expression for its length (noting that $ds = \|C_u\| du$).

$$\text{Len}(C) \doteq \int_0^1 \sqrt{\int_0^1 \left\|C_v - (C_v \cdot T)\, T\right\|^2 \|C_u\|\, du}\; dv$$

Related to the length is also the energy E of the homotopy.

$$E(C) \doteq \int_0^1 \int_0^1 \left\|C_v - (C_v \cdot T)\, T\right\|^2 \|C_u\|\, du\, dv \qquad (2)$$

Unfortunately, it has been noted in [8, 11] that the metric H^0 does not define a distance between curves, since

$$\inf E(C) = 0.$$

[1] This striking fact was first described in [8]

[2] i.e. $C(u,0) = c_0(u)$ and $C(u,1) = c_1(u)$

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In other words, for any $\varepsilon > 0$ it is possible to find a homotopy (morph) C between any two simple curves c_0 and c_1 such that $E(C) < \varepsilon$.

2.4 The implicit omnipresence of H^0

There is a good reason to focus our attention on the properties of this metric (1) for curves. Namely, this is precisely the metric that is implicitly assumed in formulating *gradient flows* of contour based energy functionals in the vast literature on shape optimization. Consider for example the well known *geometric heat flow* ($c_t = c_{ss}$ or $c_t = \kappa N$) in which a curve evolves along its inward normal N with speed equal to its signed curvature κ. This flow is widely considered to be the *gradient descent* of the Euclidean arclength functional [6]. Its smoothing properties have led to its widespread use within the fields of computer vision and image processing. The *only* sense, however, in which this is *a true gradient flow* is with respect to the H^0 metric as we see in the following calculation (where $L(t)$ denotes the time varying arclength of an evolving curve $c(u,t)$ parameterized by the time-independent parameter $u \in [0,1]$).

$$
\begin{aligned}
L(t) &= \int_c ds = \int_0^1 |c_u|\, du \\
L'(t) &= \int_0^1 \frac{c_{ut} \cdot c_u}{|c_u|}\, du = \int_0^1 c_{tu} \cdot c_s\, du \\
&= -\int_0^1 c_t \cdot c_{su}\, du = -\int_c c_t \cdot c_{ss}\, ds \\
&= -\Big\langle c_t, c_{ss} \Big\rangle_{H^0}
\end{aligned}
$$

If we were to change the metric then the inner-product shown above would no longer correspond to the inner-product associated to the metric. As a consequence, the above well-known and well-loved curvature flow could no longer be considered the *gradient flow* for arclength with respect to the changed metric!

Similar calculations will show that the *geodesic active contour* [2, 10] flow $[c_t = \phi\kappa N - (\nabla\phi \cdot N)N]$ is the gradient descent of the geometric energy $\int_c \phi\, ds$ *only for* the H^0 metric. However, this model does not stand out alone in this respect. To our knowledge, all other variational active contour flows that have been derived as gradient flows of various other constructed energies functionals suffer the same problem. They are gradient flows *only with respect to the H^0 metric*!

2.5 Pathologies of H^0

As we have already pointed out, the infimum of the energy of all possible homotopies between two curves c_0 and c_1 is zero, and therefore, unfortunately, no minimizing homotopy (geodesic) exists. On the other hand the pathologies of the H^0 metric are revealed in an interesting and instructive manner if we attempt to derive in blissful ignorance a flow that drives any initial homotopy C between two curves c_0 and c_1 to a homotopy of minimal energy, and therefore minimal length, according to the H^0 metric (in other words, a trajectory shorting flow in the space of curves).

2.5.1 Geometric parameters s and v_*

We have denoted by $u \in [0,1]$ the parameter which traces out each curve in a parameterized homotopy $C(u,v)$ and we have denoted by $v \in [0,1]$ the parameter which moves us from curve to curve along the homotopy. Note that both of these parameters are arbitrary and not related to the geometry of the curves comprising the homotopy. We now wish to construct more geometric parameters for the homotopy which will yield a more meaningful and intuitive expression for the minimizing flow we are about to derive. The most natural substitute for the curve parameter u is the arclength parmeter s. We must also address the parameter v, however. While v as a parameter ranging from 0 to 1 seems to have little to do with the arbitrary choice of the curve parameter u, the differential operator $\frac{\partial}{\partial v}$ depends heavily upon this prior choice. The desired effect of differentiating along the homotopy is mixed with the undesired effect of differentiating along the contour if flowing along corresponding values of u between curves in the homotopy requires some motion along the tangent direction. To see the dependence of $\frac{\partial}{\partial v}$ on u, note that $C(u,v)$ and $\hat{C}(u,v)$ where

$$\hat{C}(u,v) = C\left(u^{(1+v)}, v\right)$$

constitute the same homotopy geometrically, yet $\frac{\partial C}{\partial v} \neq \frac{\partial \hat{C}}{\partial v}$.

We will therefore introduce the more geometric parameter v_* whose corresponding differential operator $\frac{\partial}{\partial v_*}$ yields the most efficient transport from one curve to another curve along the homotopy regardless of "correspondence" between values of the curve parameters. It is clear that such a transport must always move in the normal direction to the underlying curve since tangential motion along any curve does not contribute to movement along the homotopy. More preceisely, we define the parameteres s and v_* in terms of u and v as follows.

$$\frac{\partial}{\partial s} = \frac{1}{\|C_u\|}\frac{\partial}{\partial u} \qquad \text{and} \qquad \frac{\partial}{\partial v_*} = \frac{\partial}{\partial v} - \Big(C_v \cdot C_s\Big)\frac{\partial}{\partial s}$$

2.5.2 The unstable H^0 minimizing flow

Suppose we now consider a time varying family of homotopies $C(u,v,t) : [0,1] \times [0,1] \times (0,\infty) \to \mathbb{R}^n$ and compute the derivative of the H^0 energy along this family. Note

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that the H^0 energy, in terms of the new parameters s and v_* may be simply expressed as follows

$$E(t) = \int_0^1 \int_0^L \left\|C_{v_*}\right\|^2 ds\, dv \tag{3}$$

After some tedious calculations, one may show that the time-derivative of E may be written as

$$E'(t) = -2 \int_0^1 \int_0^L C_t \cdot \left(C_{v_* v_*} - (C_{v_* v_*} \cdot C_s)C_s - (C_{v_*} \cdot C_{ss})C_{v_*} + \frac{1}{2}\|C_{v_*}\|^2 C_{ss}\right) ds\, dv$$

In the planar case, C_{v_*} and C_{ss} are linearly dependent (as both are orthogonal to C_s) which means that

$$(C_{v_*} \cdot C_{ss})C_{v_*} = (C_{v_*} \cdot C_{v_*})C_{ss} = \|C_{v_*}\|^2 C_{ss}$$

and therefore

$$E'(t) = -2\int_0^1\int_0^L C_t \cdot \left(\left(C_{v_* v_*} - (C_{v_* v_*} \cdot C_s)C_s\right) - \frac{1}{2}\|C_{v_*}\|^2 C_{ss}\right) ds\, dv$$

by which we derive the minimization flow

$$C_t = C_{v_* v_*} - (C_{v_* v_*} \cdot C_s)C_s - \frac{1}{2}\|C_{v_*}\|^2 C_{ss}$$

which is geometrically equivalent to the following more simple flow (by adding a tangential component):

$$C_t = C_{v_* v_*} - \frac{1}{2}\|C_{v_*}\|^2 C_{ss} \tag{4}$$

Note that the flow (4) consists of two orthogonal diffusion terms. The first term $C_{v_* v_*}$ is stable as it represents a forward diffusion along the homotopy, while the second term $-\|C_{v_*}\|^2 C_{ss}$ is an unstable backward diffusion term along each curve.

3 Conformal Versions of H^0

Given the pathologies of H^0 we have no choice but to propose a new metric if we wish to construct a useful Riemannian geometry on the space of curves. However, we may seek a new metric whose gradient structure is as similar as possible to that of the H^0 metric. In particular, for any functional $E : M \to \mathbb{R}$ we may ask that the gradient flow of E with respect to our new metric be related to the gradient flow of E with respect to H^0 by only a time reparameterization. In other words, if $c(t)$ represents a gradient flow trajectory according to H^0 and if $\hat{c}(t)$ represents the gradient flow trajectory according to our proposed new metric, then we want

$$\hat{c}(t) = c(f(t))$$

for some positive time reparameterization $f : \mathbb{R} \to \mathbb{R}$, $\dot{f} > 0$. The resulting gradient flows will then be related as follows.

$$\hat{c}_t = \dot{f}(t)\, c_t \tag{5}$$

The only class of new metrics that will satisfy (5) are *conformal* modifications of the original H^0 metric, which we will denote by H^0_ϕ. Such metrics are completely defined by combining the original H^0 metric with a positive *conformal factor* $\phi : M \to \mathbb{R}$ where $\phi(c) > 0$ may depend upon the curve c. The relationship between the inner products is given as follows.

$$\Big\langle h_1, h_2 \Big\rangle_{H^0_\phi} = \phi(c) \Big\langle h_1, h_2 \Big\rangle_{H^0} \tag{6}$$

Note that for any energy functional E of curves $c(t)$ we have the following equivalent expressions, where the first and last expressions are by definition of the gradient and the middle expression comes from the definition (6) of a conformal metric.

$$\frac{d}{dt}E(c(t)) = \left\langle \frac{\partial c}{\partial t}, \underbrace{\nabla^\phi E(c)}_{\text{Conformal Gradient}} \right\rangle_{H^0_\phi} = \tag{7}$$

$$= \phi \left\langle \frac{\partial c}{\partial t}, \underbrace{\nabla^\phi E(c)}_{\text{Conformal Gradient}} \right\rangle_{H^0} = \left\langle \frac{\partial c}{\partial t}, \underbrace{\nabla E(c)}_{\text{Original Gradient}} \right\rangle_{H^0}$$

We see from (7) that the *conformal gradient differs only in magnitude* from the original H^0 gradient

$$\nabla^\phi E = \frac{1}{\phi}\nabla E$$

and therefore the *conformal gradient flow differs only in speed* from the H^0 gradient flow.

$$\frac{\partial c}{\partial t} = -\nabla^\phi E(c) = -\frac{1}{\phi(c)}\nabla E(c)$$

As such and as we desired, the *solution differs only by a time reparameterization* f given by

$$\dot{f} = \frac{1}{\phi(c)}$$

The obvious question now is how to choose the conformal factor. A first requirement is that the distance $d(c_1, c_2)$ induced by the new conformal metric should be positive for curves c_0, c_1 that are different:

Proposition 1 [3] *Suppose there exists an $a > 0$ such that*

$$\phi(c) \geq a\, L(c), \qquad \Big(L = length(c)\Big) \tag{8}$$

[3] We thank Prof. Mumford for suggesting this result.

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for all curves c. Consider a homotopy C connecting two curves $c_0 = C(\cdot, 0), c_1 = C(\cdot, 1)$, and its H^0_ϕ–energy

$$\int_0^1 \phi(C(\cdot, v)) \int_0^1 |C_{v_*}|^2 |C_u| \, du \, dv$$

Up to reparameterization, $|C_u| = L$ so we can rewrite the energy (using the relation (4.10) in [11]) as

$$\int_0^1 \frac{\phi}{L} \int_0^1 |C_v \times C_u|^2 \, du \, dv \geq a \int_0^1 \int_0^1 |C_v \times C_u|^2 \, du \, dv$$
$$\geq a \left(\int_0^1 \int_0^1 |C_v \times C_u| \, du \, dv \right)^2.$$

The last term is the square of the area swept by the homotopy. So if $c_0 \neq c_1$, and there do not exist a homotopy connecting c_0, c_1 with zero area, then $d(c_1, c_2) > 0$.

We will therefore look for a conformal factor that satisfy (8). Moreover we may hope to counteract the unstable element (backward curvature flow) in the H^0 minimizing flow, by choosing a conformal factor dependent upon arclength L which will thereby add a forward curvature flow term.

3.1 The conformal minimizing flow

We now define the conformal H^0_ϕ energy of a homotopy $C(u, v)$ between curves c_0 and c_1 (when the conformal factor ϕ is a function of the arclength L of each curve) as

$$E_\phi = \int_0^1 \phi(L) \int_0^L \left\| C_{v_*} \right\|^2 ds \, dv \tag{9}$$

After another set of tedious calculations, we may write the derivative of E_ϕ for a time-varying family of homotopies $C(u, v, t)$ as

$$E'_\phi(t) = -\int_0^1 \int_0^L C_t \cdot \Big(2\phi' L_{v_*} C_{v_*} + 2\phi C_{v_* v_*} - 2\phi (C_{v_* v_*} \cdot C_s) C_s$$
$$-2\phi (C_{v_*} \cdot C_{ss}) C_{v_*} + (\phi m + \phi' M) C_{ss} \Big) \, ds \, dv$$

where

$$m = \|C_{v_*}\|^2 \qquad \text{and} \qquad M = \int_0^L m \, ds = \int_0^L \|C_{v_*}\|^2 \, ds.$$

As before, we now consider the planar case in which C_{v_*} and C_{ss} are linearly dependent and therefore $(C_{v_*} \cdot C_{ss}) C_{v_*} = m C_{ss}$, yielding

$$E'(t) = -2 \int_0^1 \int_0^L C_t \cdot \Big(\phi \big(C_{v_* v_*} - (C_{v_* v_*} \cdot C_s) C_s \big)$$
$$+ \phi' L_{v_*} C_{v_*} + \frac{1}{2} (\phi' M - \phi m) C_{ss} \Big) \, ds \, dv$$

from which we obtain the following minimizing flow (after adding a tangential term).

$$C_t = \phi C_{v_* v_*} + \phi' L_{v_*} C_{v_*} + \frac{1}{2} (\phi' M - \phi m) C_{ss} \tag{10}$$

3.2 A stabilizing conformal factor

To stabilize the flow (10), we look for a ϕ such that

$$\phi' M - \phi m \geq 0 \qquad \text{for all } (s, v_*) \tag{11}$$

or (assuming $M \neq 0$)

$$\frac{\phi'}{\phi} = (\log \phi)' \geq \frac{m}{M} \qquad \text{for all } (s, v_*) \tag{12}$$

One way to satisfy this is to choose

$$(\log \phi)' = \max_{s, v_*} \frac{m}{M} \doteq \lambda \tag{13}$$

giving us

$$\phi = e^{\lambda L} \tag{14}$$

yielding the following flow of homotopies

$$C_t = e^{\lambda L} \Big(2 C_{v_* v_*} + 2 \lambda L_{v_*} C_{v_*} + (\lambda M - m) C_{ss} \Big) \tag{15}$$

Note that the choice $\phi = e^{\lambda L}$ satisfies (8), and then induces a non-degenerate distance of curves.

3.3 Level set implementation

We note that the minimizing flow (10) consists of two stable diffusion terms and a transport term. As such, we have the option to utilize level set methods in the implementation of (10). We represent the evolving homotopy $C(u, v, t)$ as an evolving surface $S(u, v, t) = (C(u, v), v, t)$. We then perform a *Level Set Embedding* [9] of this surface into a 4D scalar function ψ such that

$$\psi\big(C(u, v, t), v, t\big) = 0.$$

The goal is now to determine an evolution for ψ which yields the evoluton (10) for the level sets of each of its 2D cross-sections. Differentiating

$$\frac{d}{dt} \Big(\psi \big(x(u, v, t), y(u, v, t), v, t \big) = 0 \Big) \longrightarrow \psi_t + \nabla \psi \cdot C_t = 0$$

where $\nabla \psi = (\psi_x, \psi_y)$ denotes the 2D spatial gradient of each 2D cross-section of ψ, and substituting (10), noting that $N = \nabla \psi / \|\nabla \psi\|$, yields the corresponding *Level Set Evolution*.

$$\psi_t = \psi_{vv} - \frac{2\psi_v}{\|\nabla \psi\|^2} (\nabla \psi_v \cdot \nabla \psi) + \frac{\psi_v^2}{\|\nabla \psi\|^4} \big(\nabla^2 \psi \nabla \psi \big) \cdot \nabla \psi$$
$$-\frac{1}{2} \left(\frac{\psi_v^2}{\|\nabla \psi\|^2} - \lambda \int_0^L \frac{\psi_v^2}{\|\nabla \psi\|^2} \, ds \right) \nabla \cdot \left(\frac{\nabla \psi}{\|\nabla \psi\|} \right) \|\nabla \psi\|$$
$$+ \lambda L_v \psi_v \tag{16}$$

Note that for simplicity we have dropped the factor $e^{\lambda L}$ from (10) since we are guaranteed that this factor is always positive. As a result, we do not change the steady-state of the flow by omitting this factor.

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4 Experimental Results

In this section we show experimental results of using the time evolution PDE (16) to compute the geodesic homotopy between two rather different closed curves c_0 and c_1. The two boundary curves c_0 and c_1 are displayed above the caption in figure 1, while the geodesic homotopy computed between these two curves is displayed below (in a left-to-right then top-to-bottom visualization). In figure 2 we show the homotopy surface S represented by the zero level-set of the function $\psi(x, y, v, t)$ after running the evolution (16) to steady state. Note that the curves visualized in figure 1 we obtained by taking cross-sections of this homotopy surface at evenly spaced values of v. The initial condition used by the evolution equation (16) was a simple linear interpolation of the signed distance transforms of the two boundary curves c_0 and c_1. We chose λ to satisfy (13) at time $t = 0$ and found that this stabilized the flow until convergence.

References

[1] A. Blake and M. Isard, *Active Contours*, Springer-Verlag, London 1998.

[2] V. Caselles, R. Kimmel, G. Sapiro, "Geodesic snakes," *Int. J. Computer Vision*, 1998.

[3] G. Charpiat, O. Faugeras, and R. Keriven, "Approximations of shape metrics and application to shape warping and empirical shape statistics", *INRIA report 4820*, 2003.

[4] O. Faugeras and R. Keriven, Variational principles, surface evolution pdes, level set methods and the stereo problem. *INRIA Report*, 3021:1–37, 1996.

[5] E. Klassen, A. Srivastava, W. Mio, and S. Joshi, "Analysis of Planar Shapes Using Geodesic Paths on Shape Spaces," *IEEE Trans Pattern Anal. Machine Intell.*, vol. 26, pp. 372–383, 2004.

[6] M. Grayson, "The heat equation shrinks embedded plane curves to round points," *Journal of Differential Geometry*, vol. 26, pp. 285–314, 1987.

[7] M. Kass, A. Witkin, and D. Terzopoulos, "Snakes: Active contour models," *International Journal of Computer Vision*, vol. 1, pp. 321–331, 1988.

[8] P. Michor and D. Mumford, "Riemannian geometries on spaces of plane curves',' ESI Preprint 1425, arXiv:math.DG/0312384, Dec. 2003.

[9] S. Osher and J. Sethian, "Fronts propagation with curvature dependent speed: Algorithms based on Hamilton-Jacobi formulations," *Journal of Computational Physics*, vol. 79, pp. 12–49, 1988.

[10] A. Yezzi, S. Kichenassamy, A. Kumar, P. Olver, and A. Tannenbaum, "A Geometric Snake Model for Segmentation of Medical Imagery", *IEEE Trans. Medical Imaging*, vol. 16, no. 2, pp. 199–209, 1997.

[11] A. Yezzi and A. Mennucci, "Metrics in the space of curves," *arXiv:math.DG/0412454*, May 2005.

[12] L. Younes, "Computable elastic distances between shapes," *SIAM Journal of Applied Mathematics*, vol. 58, pp. 565–586, 1998.

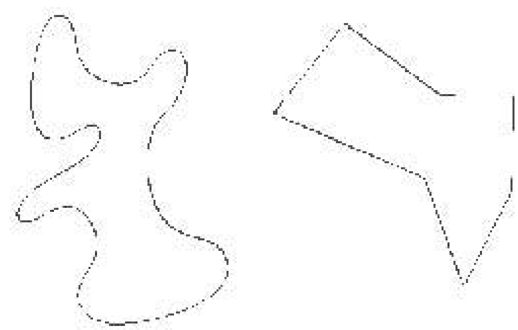

Figure 1. Visualization of the geodesic homotopy.

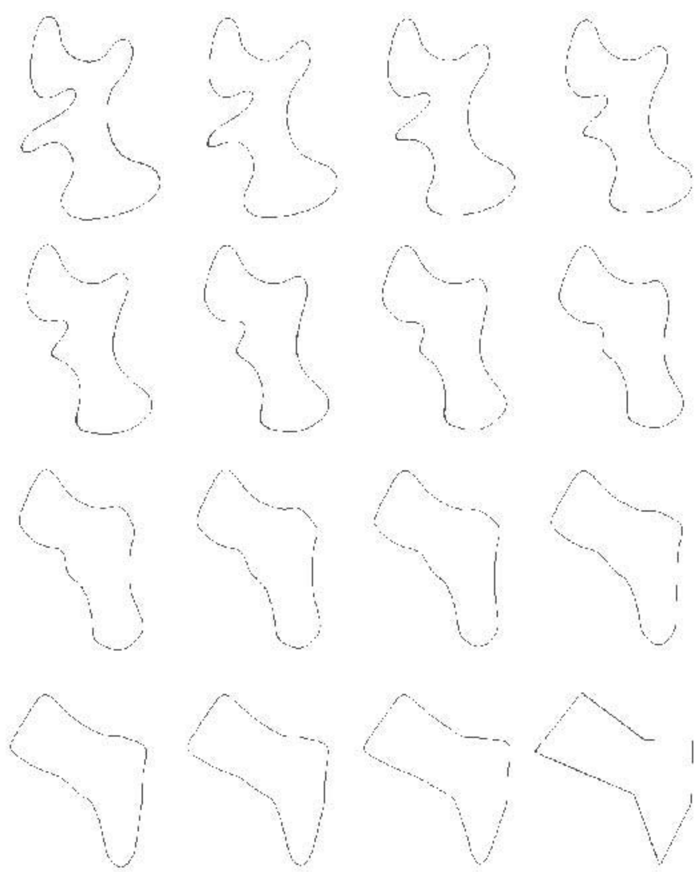

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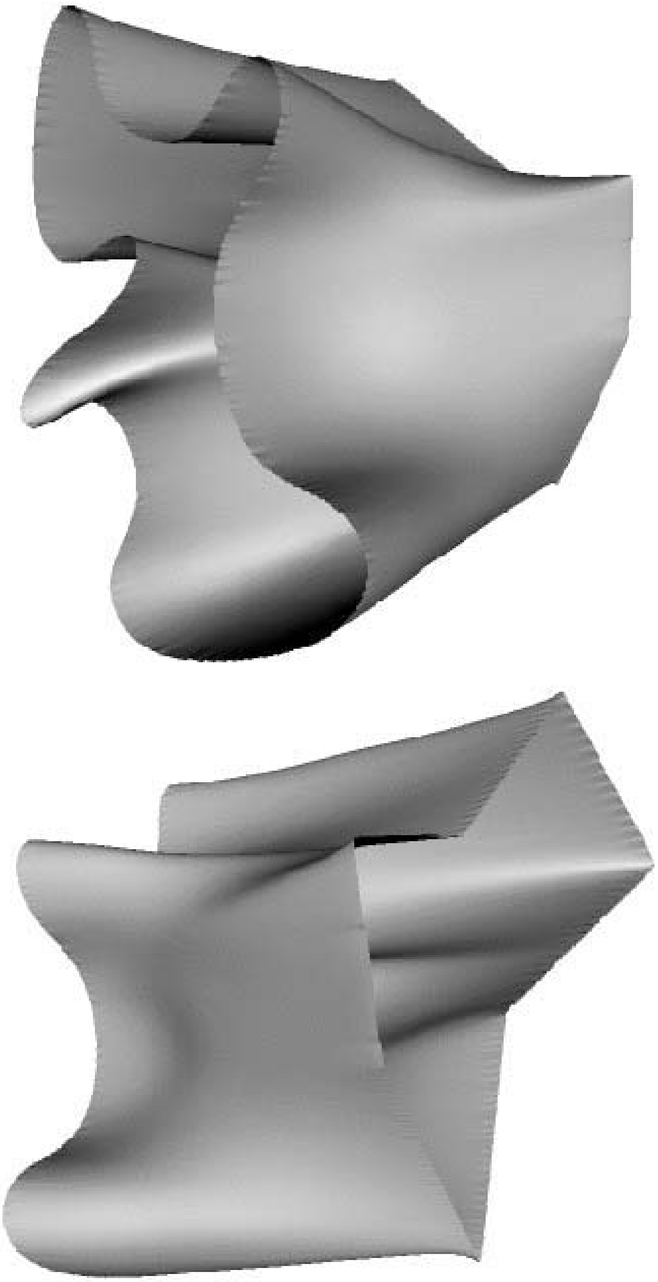
Figure 2. Computed homotopy surface (two views).



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