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Inhomogeneous parabolic equations on unbounded metric measure spaces

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We study the inhomogeneous semilinear parabolic equation

$$u_t = \Delta u + u^p + f(x),$$

with source term f independent of time and subject to $f(x) \geqslant 0$ and with $u(0, x) = \varphi(x) \geqslant 0$, for the very general setting of a metric measure space. By establishing Harnack-type inequalities in time t and some powerful estimates, we give sufficient conditions for non-existence, local existence and global existence of weak solutions, depending on the value of p relative to a critical exponent.

1. Introduction

In recent years, the study of partial differential equations on self-similar fractals has attracted increasing interest (see, for example, [7–9, 13, 14]). We investigate a class of nonlinear diffusions with source terms on general metric measure spaces. Diffusion is of fundamental importance in many areas of physics, chemistry and biology. Applications of diffusion include sintering, i.e. making solid materials from powder (powder metallurgy, production of ceramics); catalyst design in the chemical industry; diffusion of steel (e.g. with carbon or nitrogen) to modify its properties; doping during production of semiconductors.

Let (M, d, μ) be a *metric measure* space, that is, (M, d) is a locally compact separable metric space and μ is a Radon measure on M with full support. We consider the following nonlinear diffusion equation with a source term f on (M, d, μ):

$$u_t = \Delta u + u^p + f(x), \quad t > 0 \text{ and } x \in M, \tag{1.1}$$

with initial value

$$u(0, x) = \varphi(x), \tag{1.2}$$

where $p > 1$ and $f, \varphi \colon M \to \mathbb{R}$ are non-negative measurable functions. With an appropriate interpretation of weak solutions of (1.1) on (M, d, μ), we shall investigate the non-existence (or blow-up) of solutions, the local and global existence of weak solutions to (1.1) and (1.2), and the regularity of these solutions. Although we were partly motivated by a series of earlier papers [1,9–11,15–17], there are new ideas in this paper which allow conditions for the existence and non-existence of nonlinear parabolic equations to be developed for the very general setting of metric measure spaces.

Recall the definition of the heat kernel which will be central to our approach. For $(\mu \times \mu)$-almost all $(x, y) \in M \times M$ and for all $t, s > 0$, a function $k(\cdot,\cdot,\cdot)\colon \mathbb{R}_+ \times M \times M \to \mathbb{R}$ is called a *heat kernel* if the following conditions are satisfied.

(k_1) *Markov property:* $k(t,x,y) > 0$, and $\int_M k(t,x,y)\,\mathrm{d}\mu(y) \leqslant 1$.

(k_2) *Symmetry:* $k(t,x,y) = k(t,y,x)$.

(k_3) *Semigroup property:* $k(s+t,x,z) = \int_M k(s,x,y)k(t,y,z)\,\mathrm{d}\mu(y)$.

(k_4) *Normalization:* for all $f \in L^2(M,\mu)$,

$$\lim_{t\to 0^+} \int_M k(t,x,y)f(y)\,\mathrm{d}\mu(y) = f(x) \quad \text{in the } L^2(M,\mu)\text{-norm}.$$

We assume that the heat kernel $k(t,x,y)$ considered in this paper is jointly continuous in x, y, and hence the above formulae in (k_1)–(k_4) hold for *every* $(x,y) \in M \times M$.

Two typical examples of heat kernels in $\mathbb{R}^n$ are the Gauss–Weierstrass and the Cauchy–Poisson kernels:

$$k(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right),$$
$$k(t,x,y) = \frac{C_n}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-(n+1)/2}, \quad C_n = \frac{\Gamma(\frac{1}{2}(n+1))}{\pi^{(n+1)/2}}.$$

Jointly continuous sub-Gaussian heat kernels exist on many basic fractals, for example, on the Sierpínski gasket [4] and on Sierpínski carpets [2,3]. For other fractals, see [12,13]. For non-sub-Gaussian heat kernels, see [5,6].

A heat kernel k is called *conservative* if it satisfies the following.

(k_5) *Conservative property:*

$$\int_M k(t,x,y)\,\mathrm{d}\mu(y) = 1$$

for all $t > 0$ and all $x \in M$.

We will also assume that the heat kernel satisfies the following estimates.

(k_6) *Two-sided bounds:* there exist constants $\alpha, \beta > 0$ such that, for all $t > 0$ and all $x, y \in M$,

$$\frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right) \leqslant k(t,x,y) \leqslant \frac{1}{t^{\alpha/\beta}}\Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right), \tag{1.3}$$

where Φ_1 and Φ_2 are strictly positive and non-increasing functions on $[0,\infty)$.

The parameter α in (1.3) is the *fractal dimension*, and β is the *walk dimension* of M (see, for example, [11]).

Two-sided estimates (1.3) hold on various fractals where

$$\Phi_i(s) = C_i \exp(-c_i s^{\beta/(\beta-1)}) \quad \text{for all } s \geqslant 0$$

for constants $C_i, c_i > 0$, $i = 1, 2$, and $\beta > 2$ is the walk dimension.

To prove the regularity of solutions, we need to assume that the heat kernel k is Hölder continuous in the space variables.

(k_7) *Hölder continuity:* there exist constants $L > 0$, $\nu \geqslant 1$ and $0 < \sigma \leqslant 1$ such that

$$|k(t, x_1, y) - k(t, x_2, y)| \leqslant L t^{-\nu} d(x_1, x_2)^{\sigma}$$

for all $t > 0$ and all $x_1, x_2, y \in M$.

Given a heat kernel k, the operator Δ in (1.1) is interpreted as the *infinitesimal generator* of the *heat semigroup* $\{K_t\}_{t\geqslant 0}$ in $L^2 := L^2(M, \mu)$. Thus, we let

$$K_t g(x) = \int_M k(t, x, y) g(y) \, \mathrm{d}\mu(y), \quad t > 0, \ g \in L^2, \tag{1.4}$$

and define Δ by

$$\Delta g = \lim_{t \downarrow 0} \frac{K_t g - g}{t} \quad \text{in } L^2\text{-norm.} \tag{1.5}$$

Observe that $\{K_t\}_{t>0}$ is a strongly continuous and contractive semigroup in L^2, that is, for all $s, t \geqslant 0$ and all $g \in L^2$,

$$\left.\begin{aligned} K_{s+t} &= K_s K_t, \\ \lim_{t\to 0^+} \|K_t g - g\|_2 &= 0, \\ \|K_t \phi\|_q &\leqslant \|\phi\|_q \quad \text{for all } 1 \leqslant q \leqslant \infty. \end{aligned}\right\} \tag{1.6}$$

The domain of Δ is dense in L^2. Note that the operator Δ defined in this way is not necessarily local, unlike in the classical case.

A function $u(t, x)$ is termed a *weak solution* to (1.1), (1.2) if it satisfies the following integral equation:

$$u(t, x) = K_t \varphi(x) + \int_0^t K_\tau f(x) \, \mathrm{d}\tau + \int_0^t K_{t-\tau} u^p(\tau, x) \, \mathrm{d}\tau, \tag{1.7}$$

where K_t is the heat semigroup defined in (1.4).

We identify critical exponents for the problem which depend only on α and β:

$$p_0 = \begin{cases} \alpha/(\alpha - \beta) & \text{if } \alpha > \beta, \\ +\infty & \text{if } \alpha \leqslant \beta. \end{cases}$$

In § 2 we show the non-existence of weak global solutions to (1.1), (1.2) for $1 < p \leqslant p_0$. In § 3 we obtain various sufficient conditions for the local and global existence of solutions for a range of parameters p, particularly global solutions for $p > p_0$ for sufficiently small source terms f and initial values φ. Finally, in § 4 we investigate the Hölder continuity of weak solutions.

Our results extend to a very general setting the familiar situation where $M = \mathbb{R}^n$ and μ is n-dimensional Lebesgue measure, and where the heat kernel k is the Gauss–Weierstrass function, so that Δ is the usual Laplacian. In this case, $\alpha = n$ and $\beta = 2$, with critical exponent $p_0 = \infty$ if $n \leqslant 2$ and $p_0 = n/(n-2)$ if $n > 2$ (see [1,10,15,16]). See also [9] for the case where M is a fractal and μ is an α-dimensional Hausdorff measure, and where k is the Gauss-type heat kernel on M.

Notation. The letters C, C_i, $i = 1, 2, \dots$, denote positive constants whose values are unimportant and may differ at different occurrences.

2. Non-existence of solutions

In this section we give sufficient conditions for the non-existence of essentially bounded solutions. Writing d_s for the *spectral dimension* $d_s = 2\alpha/\beta$, the exponents $p = 1+\beta/\alpha = (d_s+2)/d_s$ (where $\alpha, \beta > 0$) and $p_0 = \alpha/(\alpha-\beta) = d_s/(d_s-2)$, where $\alpha > \beta > 0$, which occur in the heat kernel bounds (1.3), play a crucial role in our analysis (see theorem 2.2). First, we establish lemma 2.1, where condition (k_6) is our only assumption on the heat kernel k (we do not need the conservative property of k at this stage).

The following properties of the functions Φ_1 and Φ_2 in condition (k_6) may or may not hold. There exist positive constants a_i, b_i and c_i such that, for all $s, t \geqslant 0$,

$$\Phi_1(s) \geqslant a_1 \Phi_2(a_2 s), \tag{2.1}$$

$$\Phi_2(s+t) \geqslant b_1 \Phi_2(b_2 s) \Phi_2(b_3 t), \tag{2.2}$$

$$\Phi_1^p(s) \geqslant c_1 \Phi_2(c_2 s). \tag{2.3}$$

Note that if (2.1) holds, then $0 < a_1 \leqslant 1$ by letting $s = 0$ and using the fact that $\Phi_2(0) \geqslant \Phi_1(0)$. Without loss of generality, we may assume that $a_2 > 1$ in (2.1), since if (2.1) holds for some $a_2 \leqslant 1$, it also holds for any constant $a_2 > 1$ by the monotonicity of Φ_2.

The *Gauss-type* functions

$$\Phi_1(s) = C_1 \exp(-C_2 s^\gamma), \quad \Phi_2(s) = C_3 \exp(-C_4 s^\gamma), \quad s \geqslant 0, \tag{2.4}$$

for constants $\gamma > 0$ and $C_i > 0$, $1 \leqslant i \leqslant 4$ satisfy properties (2.1) and (2.3). The *Cauchy-type* functions

$$\Phi_1(s) = C_1(1+s)^{-\gamma}, \quad \Phi_2(s) = C_2(1+s)^{-\gamma}, \quad s \geqslant 0 \tag{2.5}$$

for constants $\gamma > 0$ and $C_i > 0$, $i = 1, 2$, satisfy properties (2.1) and (2.2), but not (2.3) if $p > 1$.

Condition (k_6) and inequality (2.1) lead to the following key lemma.

LEMMA 2.1. *Assume that the heat kernel k satisfies condition* (k_6) *and* (2.1)*. Then, for all non-negative measurable functions g on M, and for all $t > 0$, $x \in M$,*

$$K_t g(x) \geqslant A_1 K_{Bt} g(x), \tag{2.6}$$

$$\int_0^t K_\tau g(x)\, \mathrm{d}\tau \geqslant A_2 t K_{B^2 t} g(x), \tag{2.7}$$

where $A_1 = a_1 a_2^{-\alpha} < 1$, $A_2 = a_1 a_2^{-2\alpha}(1 - a_2^{-\beta}) < 1$ *and* $B = a_2^{-\beta} < 1$. *Consequently, for all non-negative measurable functions* φ,

$$K_t\varphi(x) + \int_0^t K_\tau g(x)\,\mathrm{d}\tau \geqslant A[K_{B_1 t}\varphi(x) + tK_{B_1 t}g(x)], \tag{2.8}$$

where $A = \min\{A_1^2, A_2\} < 1$ *and* $B_1 = B^2 = a_2^{-2\beta}$.

Proof. It follows from condition (k_6) and (2.1) that

$$\begin{aligned} K_t g(x) &= \int_M k(t,x,y)g(y)\,\mathrm{d}\mu(y) \\ &\geqslant \int_M \frac{1}{t^{\alpha/\beta}} \Phi_1\biggl(\frac{d(x,y)}{t^{1/\beta}}\biggr) g(y)\,\mathrm{d}\mu(y) \\ &\geqslant a_1 \int_M \frac{1}{t^{\alpha/\beta}} \Phi_2\biggl(a_2 \frac{d(x,y)}{t^{1/\beta}}\biggr) g(y)\,\mathrm{d}\mu(y). \end{aligned} \tag{2.9}$$

which, using (k_6) again, yields

$$\begin{aligned} K_t g(x) &\geqslant a_1 a_2^{-\alpha} \int_M k(a_2^{-\beta}t, x, y) g(y)\,\mathrm{d}\mu(y) \\ &= a_1 a_2^{-\alpha} K_{a_2^{-\beta}t} g(x) \\ &= A_1 K_{Bt} g(x), \end{aligned}$$

proving (2.6).

To show (2.7), we see from (2.9) that, for all $\tau \in [a_2^{-\beta}t, t]$, using the monotonicity of Φ_2 and condition (k_6),

$$\begin{aligned} K_\tau g(x) &\geqslant a_1 \int_M \frac{1}{\tau^{\alpha/\beta}} \Phi_2\biggl(a_2 \frac{d(x,y)}{\tau^{1/\beta}}\biggr) g(y)\,\mathrm{d}\mu(y) \\ &\geqslant a_1 \int_M \frac{1}{t^{\alpha/\beta}} \Phi_2\biggl(a_2 \frac{d(x,y)}{(a_2^{-\beta}t)^{1/\beta}}\biggr) g(y)\,\mathrm{d}\mu(y) \\ &\geqslant a_1 a_2^{-2\alpha} \int_M k(a_2^{-2\beta}t, x, y) g(y)\,\mathrm{d}\mu(y) \\ &= a_1 a_2^{-2\alpha} K_{B^2 t} g(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^t K_\tau g(x)\,\mathrm{d}\tau &\geqslant \int_{a_2^{-\beta}t}^t K_\tau g(x)\,\mathrm{d}\tau \\ &\geqslant \int_{a_2^{-\beta}t}^t a_1 a_2^{-2\alpha} K_{B^2 t} g(x)\,\mathrm{d}\tau \\ &= a_1 a_2^{-2\alpha}(1 - a_2^{-\beta}) t K_{B^2 t} g(x), \end{aligned}$$

proving (2.7).

Finally, replacing t by Bt, we see from (2.6) that

$$K_{Bt}\varphi(x) \geqslant A_1 K_{B^2t}\varphi(x) = A_1 K_{B_1t}\varphi(x),$$

and thus,

$$K_t\varphi(x) \geqslant A_1 K_{Bt}\varphi(x) \geqslant AK_{B_1t}\varphi(x). \tag{2.10}$$

Adding (2.7) and (2.10), we obtain (2.8). □

Lemma 2.1 gives the following estimate (2.11) that plays an important role in proving the non-existence of global bounded solutions.

THEOREM 2.2. *Assume that the heat kernel k satisfies conditions* (k_6) *and* (2.1). *Let $u(t,x)$ be a non-negative essentially bounded solution of* (1.7) *in* $(0,T)\times M$. *Then, for all* $(t,x)\in(0,T)\times M$,

$$t^{1/(p-1)}K_{B_1t}\varphi(x) + t^{p/(p-1)}K_{B_1t}f(x) \leqslant C_1, \tag{2.11}$$

where $B_1 = a_2^{-2\beta}$ as before, and C_1 depends only on p, a_1 and a_2 (and, in particular, is independent of T, φ and f).

Proof. Observe that, by condition (k_1) and using a weighted Hölder inequality, for all $t>0$, $x\in M$, and for all non-negative functions g,

$$\begin{aligned} K_t(g^p)(x) &= \int_M k(t,x,y)g^p(y)\,\mathrm{d}\mu(y) \\ &\geqslant \left[\int_M k(t,x,y)g(y)\,\mathrm{d}\mu(y)\right]^p \\ &= [K_tg(x)]^p. \end{aligned}$$

It follows from (1.7) and (2.10) that

$$\begin{aligned} u(t,x) &\geqslant \int_0^t K_{t-\tau}u^p(\tau,x)\,\mathrm{d}\tau \\ &\geqslant A\int_0^t K_{B_1(t-\tau)}u^p(\tau,x)\,\mathrm{d}\tau \\ &\geqslant A\int_0^t [K_{B_1(t-\tau)}u(\tau,x)]^p\,\mathrm{d}\tau. \end{aligned} \tag{2.12}$$

From (1.7) and (2.8), we see that

$$\begin{aligned} u(t,x) &\geqslant K_t\varphi(x) + \int_0^t K_\tau f(x)\,\mathrm{d}\tau \\ &\geqslant A(K_{B_1t}\varphi(x) + tK_{B_1t}f(x)). \end{aligned} \tag{2.13}$$

Starting from (2.13), we shall apply (2.12) repeatedly to deduce the desired inequality (2.11). Indeed, we obtain from (2.12) and (2.13) that, using the semigroup property (1.6) of $\{K_t\}_{t\geqslant 0}$ and the elementary inequality $(a+b)^p \geqslant a^p + b^p$ for all $p\geqslant 1$

and $a, b \geqslant 0$,

$$\begin{aligned} u(t,x) &\geqslant A\int_0^t [K_{B_1(t-\tau)}u(\tau,x)]^p \,\mathrm{d}\tau \\ &\geqslant A\int_0^t [K_{B_1(t-\tau)}\{A(K_{B_1\tau}\varphi + \tau K_{B_1\tau}f)\}(x)]^p \,\mathrm{d}\tau \\ &= A^{p+1}\int_0^t [K_{B_1t}\varphi(x) + \tau K_{B_1t}f(x)]^p \,\mathrm{d}\tau \\ &\geqslant A^{p+1}\biggl\{t(K_{B_1t}\varphi(x))^p + \int_0^t \tau^p (K_{B_1t}f(x))^p \,\mathrm{d}\tau\biggr\} \\ &= A^{p+1}\biggl\{t(K_{B_1t}\varphi(x))^p + \frac{1}{1+p}t^{1+p}(K_{B_1t}f(x))^p\biggr\}. \end{aligned}$$

Repeating the above procedure, we obtain that, for all $n \geqslant 1$,

$$\begin{aligned} u(t,x) \geqslant A^{1+p+\cdots+p^n}\biggl\{&\frac{t^{1+p+\cdots+p^{n-1}}[K_{Bt}\varphi(x)]^{p^n}}{(1+p)^{p^{n-2}}(1+p+p^2)^{p^{n-3}}\cdots(1+p+\cdots+p^{n-1})} \\ &+ \frac{t^{1+p+\cdots+p^n}[K_{Bt}f(x)]^{p^n}}{(1+p)^{p^{n-1}}(1+p+p^2)^{p^{n-2}}\cdots(1+p+\cdots+p^n)}\biggr\}. \end{aligned}$$

It follows that

$$A^{(p^{n+1}-1)/(p-1)p^n} t^{(p^n-1)/(p-1)p^n} K_{Bt}\varphi(x) \leqslant u(t,x)^{p^{-n}} \prod_{i=2}^n (1+p+\cdots+p^{i-1})^{p^{-i}}, \tag{2.14}$$

$$A^{(p^{n+1}-1)/(p-1)p^n} t^{(p^{n+1}-1)/(p-1)p^n} K_{Bt}f(x) \leqslant u(t,x)^{p^{-n}} \prod_{i=1}^n (1+p+\cdots+p^{i})^{p^{-i}}. \tag{2.15}$$

Since

$$\begin{aligned} \log\prod_{i=2}^n (1+p+\cdots+p^{i-1})^{p^{-i}} &\leqslant \sum_{i=2}^\infty \frac{1}{p^i}\log(ip^i) < +\infty, \\ \log\prod_{i=1}^n (1+p+\cdots+p^{i})^{p^{-i}} &\leqslant \sum_{i=1}^\infty \frac{1}{p^i}\log((i+1)p^i) < +\infty, \end{aligned}$$

and that $u(t,x)$ is essentially bounded on $(0,T)\times M$, we pass to the limit as $n \to \infty$ in (2.14) and (2.15), and conclude that

$$t^{1/(p-1)}K_{Bt}\varphi(x) \leqslant \tfrac{1}{2}C_1, \tag{2.16}$$

$$t^{p/(p-1)}K_{Bt}f(x) \leqslant \tfrac{1}{2}C_1 \tag{2.17}$$

for some $C_1 > 0$. Adding (2.16) and (2.17), we obtain (2.11). □

We are now in a position to obtain the main results of this section.

THEOREM 2.3. *Assume that the heat kernel k satisfies conditions* (k_6) *and* (2.1). *Then the problem* (1.1), (1.2) *does not have any essentially bounded global solution in each of the following cases:*

(i) *if $p < 1 + \beta/\alpha$ and if either $\varphi(x) \gneqq 0$ or $f(x) \gneqq 0$;*

(ii) *if $\alpha \leqslant \beta$ and if $f(x) \gneqq 0$;*

(iii) *if $\alpha > \beta$ and $p < \alpha/(\alpha - \beta)(> 1 + \beta/\alpha)$ and if $f(x) \gneqq 0$.*

Proof. We prove the results by contradiction. Assume that $u(t, x)$ is a non-negative essentially bounded global solution. Replacing $B_1 t$ by t, we see from (2.11) that, for all $x \in M$ and $t > 0$,

$$t^{1/(p-1)} K_t \varphi(x) + t^{p/(p-1)} K_t f(x) \leqslant C_1, \tag{2.18}$$

where $0 < C_1 < \infty$ is independent of φ and f.

Proof of case (i). If $\varphi(x) \gneqq 0$, we see from (k_6), using Fatou's lemma, that

$$\liminf_{t\to\infty} t^{\alpha/\beta} K_t \varphi(x) \geqslant \liminf_{t\to\infty} \int_M \Phi_1\left(\frac{d(x, y)}{t^{1/\beta}}\right) \varphi(y) \, \mathrm{d}\mu(y) \geqslant C_2,$$

where $C_2 = 1$ if $\|\varphi\|_1 = \infty$, and $C_2 = \Phi_1(0)\|\varphi\|_1$ if $\|\varphi\|_1 < \infty$. However, as

$$\frac{1}{p-1} > \frac{\alpha}{\beta},$$

this is impossible by using (2.18). Hence, (1.1), (1.2) do not have any global essentially bounded solution.

If $f(x) \gneqq 0$, observe that $u(t + t_0, x)$ is a weak solution of (1.7) with initial datum $\varphi(x) = u(t_0, x)$. We may find $t_0 > 0$ such that $u(t_0, x) \gneqq 0$. Repeating the above argument, we again see that (1.1), (1.2) do not have any global essentially bounded solution.

Proof of case (ii). Observe that, by (1.7) and (2.7),

$$u(t, x) \geqslant \int_0^t K_\tau f(x) \, \mathrm{d}\tau \geqslant A_2 t K_{B_1 t} f(x). \tag{2.19}$$

We distinguish two cases: $\alpha < \beta$ and $\alpha = \beta$.

CASE 1 ($\alpha < \beta$). It follows from (2.19) and (k_6) that

$$\begin{aligned} \liminf_{t\to\infty} t^{(\alpha/\beta)-1} u(t, x) &\geqslant A_2 \liminf_{t\to\infty} t^{\alpha/\beta} K_{B_1 t} f(x) \\ &\geqslant A_2 \liminf_{t\to\infty} t^{\alpha/\beta} \int_M \frac{1}{(B_1 t)^{\alpha/\beta}} \Phi_1\left(\frac{d(x, y)}{(B_1 t)^{1/\beta}}\right) f(y) \, \mathrm{d}\mu(y) \\ &\geqslant C_3, \end{aligned} \tag{2.20}$$

where $C_3 = 1$ if $\|f\|_1 = \infty$ and $C_3 = A_2 B_1^{-\alpha/\beta} \Phi_1(0) > 0$ if $\|f\|_1 < \infty$. However, since u is globally essentially bounded and $\alpha/\beta < 1$, we see that

$$\liminf_{t\to\infty} t^{(\alpha/\beta)-1} u(t, x) = 0,$$

which is a contradiction.

Case 2 ($\alpha = \beta$). For $t > 1$, it follows from (k_6), (2.1) and the monotonicity of Φ_2 that

$$\begin{aligned} u(t,x) &\geqslant \int_0^t K_\tau f(x)\,\mathrm{d}\tau \\ &\geqslant \int_0^t \mathrm{d}\tau \int_M \tau^{-1}\Phi_1\biggl(\frac{d(x,y)}{\tau^{1/\beta}}\biggr) f(y)\,\mathrm{d}\mu(y) \\ &\geqslant a_1 \int_1^t \mathrm{d}\tau \int_M \tau^{-1}\Phi_2\biggl(a_2\frac{d(x,y)}{\tau^{1/\beta}}\biggr) f(y)\,\mathrm{d}\mu(y) \\ &\geqslant a_1 \int_1^t \tau^{-1}\,\mathrm{d}\tau \int_M \Phi_2(a_2 d(x,y)) f(y)\,\mathrm{d}\mu(y). \end{aligned} \tag{2.21}$$

Since $f(x) \gneqq 0$, we can find a point $x \in M$ such that

$$\int_M \Phi_2(a_2 d(x,y)) f(y)\,\mathrm{d}\mu(y) > 0.$$

Passing to the limit as $t \to \infty$ in (2.21), this contradicts that u is globally essentially bounded.

Proof of case (iii). It follows from (2.18) and (k_6) that

$$\begin{aligned} \liminf_{t\to\infty} C_1 t^{\alpha/\beta - p/(p-1)} &\geqslant \liminf_{t\to\infty} t^{\alpha/\beta} K_t f(x) \\ &\geqslant \liminf_{t\to\infty} \int_M \Phi_1\biggl(\frac{d(x,y)}{t^{1/\beta}}\biggr) f(y)\,\mathrm{d}\mu(y) \geqslant C_4, \end{aligned}$$

where $C_4 = 1$ if $\|f\|_1 = \infty$ and $C_4 = \Phi_1(0)\|f\|_1$ if $\|f\|_1 < \infty$. However, this is impossible since $\alpha/\beta - p/(p-1) < 0$. The proof is complete. □

In theorem 2.3, we do not know in general whether there exists any essentially bounded global solution for the two critical cases $p = 1 + \beta/\alpha$, $\alpha, \beta > 0$, and $p = \alpha/(\alpha - \beta)$, $\alpha > \beta > 0$.

However, theorem 2.3(i) may be improved to include the critical exponent $p = 1 + \beta/\alpha$ under further assumptions (2.2) and (2.3) on the heat kernel k. We first need the following property.

Proposition 2.4. *If Φ_2 satisfies (2.2), then, for all $t > 0$ and all $x, y \in M$,*

$$\frac{\Phi_2(d(x,y)t^{-1/\beta})}{\Phi_2(b_2 d(x,0)t^{-1/\beta})} \geqslant b_1 \Phi_2(b_3 d(y,0)t^{-1/\beta}), \tag{2.22}$$

where the constants b_i, $i = 1, 2, 3$, are as in (2.2).

Proof. Since Φ_2 is strictly positive and decreasing on $[0, \infty)$ and $d(x,y) \leqslant d(x,0) + d(y,0)$, we have

$$\Phi_2(d(x,y)t^{-1/\beta}) \geqslant \Phi_2(d(x,0)t^{-1/\beta} + d(y,0)t^{-1/\beta}). \tag{2.23}$$

It follows from (2.2) that

$$\Phi_2(d(x,0)t^{-1/\beta} + d(y,0)t^{-1/\beta}) \geqslant b_1\Phi_2(b_2 d(x,0)t^{-1/\beta})\Phi_2(b_3 d(y,0)t^{-1/\beta}),$$

which combines with (2.23) to give (2.22). □

THEOREM 2.5. *Assume that the heat kernel k satisfies conditions* (k_5), (k_6) *and* (2.1)–(2.3). *Then* (1.1), (1.2) *does not have any essentially bounded global solutions if $p \leqslant 1 + \beta/\alpha$ and if either $\varphi(x) \gneqq 0$ or $f(x) \gneqq 0$.*

Proof. In view of part (i) of theorem 2.3, it is sufficient to consider the critical exponent $p = 1 + \beta/\alpha$. We only consider the case $\varphi(x) \gneqq 0$ (the case $f(x) \gneqq 0$ may be treated in a similar way). Then (2.18) becomes

$$t^{\alpha/\beta} K_t \varphi(x) + t^{1+\alpha/\beta} K_t f(x) \leqslant C_1.$$

From condition (k_6),

$$\int_M \varphi(y)\,\mathrm{d}\mu(y) \leqslant C_2, \tag{2.24}$$

where $C_2 = C_1/\Phi_1(0)$. For any $t_0 > 0$, the function $v(t, x) \equiv u(t + t_0, x)$ is a weak solution to (1.7) with initial datum $\varphi(x) = u(t_0, x)$. Repeating the procedure of (2.24), we have that, for all $t_0 > 0$,

$$\int_M u(t_0, y)\,\mathrm{d}\mu(y) \leqslant C_2. \tag{2.25}$$

We claim that there exist positive constants γ, ρ possibly depending on t_0 and φ such that, for all $x \in M$,

$$u(t_0, x) \geqslant \rho k(\gamma, x, 0). \tag{2.26}$$

To see this, observe that

$$\Phi_2(d(x, 0)\gamma^{-1/\beta}) \geqslant k(\gamma, x, 0)\gamma^{\alpha/\beta},$$

and thus, using (2.22) and setting $\gamma = (a_1 b_2)^{-\beta} t_0$,

$$\begin{aligned}\Phi_2(a_2 d(x, y) t_0^{-1/\beta}) &\geqslant b_1 \Phi_2(a_1 b_3 d(y, 0) t_0^{-1/\beta}) \Phi_2(a_1 b_2 d(x, 0) t_0^{-1/\beta}) \\ &\geqslant b_1 \Phi_2(a_1 b_3 d(y, 0) t_0^{-1/\beta}) k(\gamma, x, 0)\gamma^{\alpha/\beta}.\end{aligned}$$

Using (1.7) and (2.1),

$$\begin{aligned}u(t_0, x) &\geqslant \int_M k(t_0, x, y)\varphi(y)\,\mathrm{d}\mu(y) \\ &\geqslant t_0^{-\alpha/\beta} \int_M \Phi_1(d(x, y) t_0^{-1/\beta})\varphi(y)\,\mathrm{d}\mu(y) \\ &\geqslant a_1 t_0^{-\alpha/\beta} \int_M \Phi_2(a_2 d(x, y) t_0^{-1/\beta})\varphi(y)\,\mathrm{d}\mu(y) \\ &\geqslant a_1 b_1 \left(\frac{\gamma}{t_0}\right)^{\alpha/\beta} k(\gamma, x, 0) \int_M \Phi_2(a_1 b_3 d(y, 0) t_0^{-1/\beta})\varphi(y)\,\mathrm{d}\mu(y),\end{aligned}$$

hence, inequality (2.26) holds by setting

$$\rho := a_1 b_1 \left(\frac{\gamma}{t_0}\right)^{\alpha/\beta} \int_M \Phi_2(a_1 b_3 d(y, 0) t_0^{-1/\beta})\varphi(y)\,\mathrm{d}\mu(y),$$

proving our claim.

Consider $v(t,x) \equiv u(t+t_0,x)$ such that $u(t_0,x) \gneqq 0$. Applying (2.26), we obtain

$$\begin{aligned} v(t,x) &\geqslant \int_M k(t,x,y)u(t_0,y)\,\mathrm{d}\mu(y) \geqslant \rho \int_M k(t,x,y)k(\gamma,y,0)\,\mathrm{d}\mu(y) \\ &= \rho k(t+\gamma,x,0), \end{aligned}$$

which yields, using (1.7), (k_5) and Fubini's theorem, that

$$\begin{aligned} \int_M v(t,x)\,\mathrm{d}\mu(x) &\geqslant \int_M \mathrm{d}\mu(x) \int_0^t \mathrm{d}\tau \int_M k(t-\tau,x,y)v^p(\tau,y)\,\mathrm{d}\mu(y) \\ &= \int_0^t \mathrm{d}\tau \int_M v^p(\tau,y)\,\mathrm{d}\mu(y) \\ &\geqslant \rho^p \int_0^t \mathrm{d}\tau \int_M k^p(\tau+\gamma,y,0)\,\mathrm{d}\mu(y). \end{aligned} \tag{2.27}$$

As $p = 1 + \beta/\alpha$, we see from (2.3) and (k_6) that

$$\begin{aligned} k^p(\tau+\gamma,y,0) &\geqslant (\tau+\gamma)^{-(1+\alpha/\beta)}\Phi_1^p(d(y,0)(\tau+\gamma)^{-1/\beta}) \\ &\geqslant c_1(\tau+\gamma)^{-(1+\alpha/\beta)}\Phi_2(c_2 d(y,0)(\tau+\gamma)^{-1/\beta}) \\ &= c_1 c_2^{-\alpha}(\tau+\gamma)^{-1}[c_2^{-\beta}(\tau+\gamma)]^{-\alpha/\beta}\Phi_2(c_2 d(y,0)(\tau+\gamma)^{-1/\beta}) \\ &\geqslant c_1 c_2^{-\alpha}(\tau+\gamma)^{-1}k(c_2^{-\beta}(\tau+\gamma),y,0), \end{aligned}$$

which combines with (2.27) to give

$$\int_M v(t,x)\,\mathrm{d}\mu(x) \geqslant c_1 c_2^{-\alpha}\rho^p \int_0^t (\tau+\gamma)^{-1}\,\mathrm{d}\tau. \tag{2.28}$$

Passing to the limit as $t \to \infty$, we conclude that

$$\int_M v(t,x)\,\mathrm{d}\mu(x) \to \infty,$$

which contradicts (2.25). □

3. Existence of solutions

In this section we give sufficient conditions for local existence and global existence of weak solutions.

THEOREM 3.1 (local existence). *Suppose that the heat kernel k satisfies* (k_6). *Let $b(t)$ be a continuously differentiable function on $[0,T_0)$ satisfying*

$$b'(t) = b^p(t)\left[\int_0^t \frac{\|K_\tau f\|_\infty}{b(\tau)}\,\mathrm{d}\tau + \|K_t\varphi\|_\infty\right]^{p-1} \tag{3.1}$$

with initial value $b(0) = 1$. If

$$\int_0^{T_0}\left[\int_0^s \frac{\|K_\tau f\|_\infty}{b(\tau)}\,\mathrm{d}\tau + \|K_s\varphi\|_\infty\right]^{p-1}\mathrm{d}s \leqslant \frac{1}{p-1}, \tag{3.2}$$

then (1.1), (1.2) *has a non-negative local solution $u \in L^\infty((0,T),M)$ for all $0 < T < T_0$, provided that $\|\varphi\|_\infty < \infty$.*

REMARK 3.2. By Peano's theorem, there exists some $T_0 > 0$ and some continuous differentiable function $b(t)$ such that (3.1) holds in $[0, T_0)$. Clearly, such a $b(t)$ is non-decreasing in $[0, T_0)$. On the other hand, condition (3.2) may be verified for some specific cases. For example, if (k_5) holds and if $f = 0$, $\varphi = C > 0$, then

$$b(t) = [1 - (p-1)C^{p-1}t]^{-1/(p-1)}$$

satisfies (3.1) in $[0, T_0)$, where $T_0 = (p-1)^{-1}C^{-(p-1)}$, and (3.2) also holds. As another example, let $f = 1, \varphi = 0$ and assume that (k_5) holds. Then, for $p = 2$, we see that $b(t) = 1/(\cos t)$ satisfies (3.1) for $t \in [0, \frac{1}{2}\pi)$, and that (3.2) holds.

Proof. Define

$$a(t) = b(t) \int_0^t \frac{\|K_\tau f\|_\infty}{b(\tau)} \, \mathrm{d}\tau.$$

Note that $a(0) = 0$ and $a(t) \geqslant 0$ for $t \in [0, T_0)$. Incorporating this into (3.1), we get

$$b'(t) = b(t)[a(t) + b(t)\|K_t\varphi\|_\infty]^{p-1}.$$

Moreover,

$$\begin{aligned} a'(t) &= \|K_t f\|_\infty + \frac{b'(t)a(t)}{b(t)} \\ &= \|K_t f\|_\infty + a(t)[a(t) + b(t)\|K_t\varphi\|_\infty]^{p-1}. \end{aligned}$$

Together with the initial conditions, these differential equations are equivalent to

$$a(t) = \int_0^t \|K_\tau f\|_\infty \, \mathrm{d}\tau + \int_0^t a(\tau)(a(\tau) + b(\tau)\|K_\tau\varphi\|_\infty)^{p-1} \, \mathrm{d}\tau, \tag{3.3}$$

$$b(t) = 1 + \int_0^t b(\tau)(a(\tau) + b(\tau)\|K_\tau\varphi\|_\infty)^{p-1} \, \mathrm{d}\tau. \tag{3.4}$$

Let $\mathcal{H}$ be the family of continuous functions u satisfying

$$K_t\varphi(x) \leqslant u(t,x) \leqslant a(t) + b(t)K_t\varphi(x) \quad \text{for all } (t,x) \in [0, T_0) \times M. \tag{3.5}$$

Define

$$\mathcal{F}u(t,x) = K_t\varphi(x) + \int_0^t K_\tau f(x) \, \mathrm{d}\tau + \int_0^t K_{t-\tau}u^p(\tau, x) \, \mathrm{d}\tau. \tag{3.6}$$

We claim that if $u \in \mathcal{H}$, then $\mathcal{F}u \in \mathcal{H}$, that is,

$$K_t\varphi(x) \leqslant \mathcal{F}u(t,x) \leqslant a(t) + b(t)K_t\varphi(x), \quad 0 \leqslant t < T_0, \ x \in M. \tag{3.7}$$

Using (k_1), observe that

$$\begin{aligned} &\int_0^t K_{t-\tau}[a(\tau) + b(\tau)K_\tau\varphi]^p(x) \, \mathrm{d}\tau \\ &\qquad = \int_0^t \mathrm{d}\tau \int_M k(t-\tau, x, y)[a(\tau) + b(\tau)K_\tau\varphi(y)]^p \, \mathrm{d}\mu(y) \\ &\qquad \leqslant \int_0^t [a(\tau) + b(\tau)\|K_\tau\varphi\|_\infty]^{p-1}[a(\tau) + b(\tau)K_t\varphi(x)] \, \mathrm{d}\tau. \end{aligned}$$

It follows from (3.6) and (3.5) that

$$\begin{aligned}\mathcal{F}u(t,x) &\leqslant K_t\varphi(x) + \int_0^t K_\tau f(x)\,\mathrm{d}\tau + \int_0^t K_{t-\tau}[a(\tau)+b(\tau)K_\tau\varphi]^p(x)\,\mathrm{d}\tau \\ &\leqslant \left[\int_0^t \|K_\tau f\|_\infty\,\mathrm{d}\tau + \int_0^t a(\tau)[a(\tau)+b(\tau)\|K_\tau\varphi\|_\infty]^{p-1}\,\mathrm{d}\tau\right] \\ &\quad + \left[1+\int_0^t b(\tau)[a(\tau)+b(\tau)\|K_\tau\varphi\|_\infty]^{p-1}\,\mathrm{d}\tau\right]K_t\varphi(x) \\ &= a(t)+b(t)K_t\varphi(x)\end{aligned}$$

using (3.3) and (3.4), so (3.7) holds, proving our claim.

For $n = 0, 1, 2, \dots$, define

$$\begin{aligned}u_0(t,x) &= K_t\varphi(x), \\ u_{n+1}(t,x) &= \mathcal{F}u_n(t,x).\end{aligned}$$

Using (3.6) inductively, it follows that the sequence $\{u_n(t,x)\}$ is non-decreasing in n, and, for all $n \geqslant 0$ and all $x \in M$ and $t \in [0, T_0)$, satisfies

$$K_t\varphi(x) \leqslant u_n(t,x) \leqslant a(t) + b(t)K_t\varphi(x).$$

Let $u(t,x) := \lim_{n\to\infty} u_n(t,x)$. Note that $K_t\varphi(x) \leqslant u(t,x) \leqslant a(t)+b(t)K_t\varphi(x)$. Using the monotone convergence theorem, we have

$$\lim_{n\to\infty}\int_0^t \mathrm{d}\tau \int_M k(t-\tau,x,y)u_n^p(\tau,y)\,\mathrm{d}\mu(y) = \int_0^t \mathrm{d}\tau \int_M k(t-\tau,x,y)u^p(\tau,y)\,\mathrm{d}\mu(y).$$

Since $u_n(t,x)$ satisfies

$$u_{n+1}(t,x) = K_t\varphi(x) + \int_0^t K_\tau f(x)\,\mathrm{d}\tau + \int_0^t K_{t-\tau}u_n^p(\tau,x)\,\mathrm{d}\tau, \tag{3.8}$$

we pass to the limit as $n \to \infty$ to obtain

$$u(t,x) = K_t\varphi(x) + \int_0^t K_\tau f(x)\,\mathrm{d}\tau + \int_0^t K_{t-\tau}u^p(\tau,x)\,\mathrm{d}\tau,$$

which shows that $u(t,x)$ is a non-negative local solution of (1.1), (1.2) for $t \in [0, T_0)$.

Since $a(t)$, $b(t)$ are differentiable functions on $[0, T_0)$, we see from (3.5) that, for all $t \in [0, T_0)$,

$$\|u(t,\cdot)\|_\infty \leqslant \|a(t)+b(t)K_t\varphi\|_\infty < \infty.$$

The proof is complete. □

Recall that, by theorem 2.3, (1.1), (1.2) do not have any essentially bounded global weak solution if $\alpha > \beta$ and $p < \alpha/(\alpha-\beta)$ and if $f(x) \gneqq 0$. However, we can show that (1.1), (1.2) possess an essentially bounded global solution if $p > \alpha/(\alpha-\beta)$, for small functions f and φ (cf. [17] for Euclidean spaces). To do this, we need some integral estimates which are consequences of measure bounds for small and large balls.

Recall that a measure μ on a metric measure space is *upper α-regular* if there exist some $C, \alpha > 0$ such that

$$\mu(B(x,r)) \leqslant Cr^{\alpha} \quad \text{for all } x \in M,\ r > 0, \tag{3.9}$$

and is *α-regular* if there exists a constant $C > 0$ such that, for all $x \in M$ and all $r > 0$,

$$C^{-1}r^{\alpha} \leqslant \mu(B(x,r)) \leqslant Cr^{\alpha} \quad \text{for all } x \in M,\ r > 0. \tag{3.10}$$

It was shown in [11, theorem 3.2] that if the heat kernel k satisfies (k_5) and (k_6) with $\Phi_2(s)$ satisfying

$$\int_0^{\infty} s^{\alpha-1}\Phi_2(s)\,\mathrm{d}s < \infty, \tag{3.11}$$

then the measure μ is *α-regular*. Note that, by the monotonicity of Φ_2, condition (3.11) implies that $s^{\alpha}\Phi_2(s) \leqslant C < \infty$ for all $s \in [0,\infty)$.

Proposition 3.3. *Assume that μ is upper α-regular and x_0 is a reference point in M. If $0 < \lambda_1 < \alpha$ and $\lambda_1 + \lambda_2 > \alpha$, then there exists a constant $C_0 > 0$ such that*

$$\int_M \frac{1}{d(y,x)^{\lambda_1}[1 + d(y,x_0)^{\lambda_2}]}\,\mathrm{d}\mu(y) \leqslant C_0 \quad \textit{for all } x \in M. \tag{3.12}$$

Proof. For each $x \in M$, let $\Omega_1 = \{y \in M : d(y,x) \geqslant d(y,x_0)\}$ and $\Omega_2 = M \setminus \Omega_1$. Then

$$\int_{\Omega_1} \frac{1}{d(y,x)^{\lambda_1}[1 + d(y,x_0)^{\lambda_2}]}\,\mathrm{d}\mu(y) \leqslant \int_M \frac{1}{d(y,x_0)^{\lambda_1}[1 + d(y,x_0)^{\lambda_2}]}\,\mathrm{d}\mu(y)$$

and

$$\int_{\Omega_2} \frac{1}{d(y,x)^{\lambda_1}[1 + d(y,x_0)^{\lambda_2}]}\,\mathrm{d}\mu(y) \leqslant \int_M \frac{1}{d(y,x)^{\lambda_1}[1 + d(y,x)^{\lambda_2}]}\,\mathrm{d}\mu(y).$$

Routine estimates using upper regularity (3.9) now give uniform bounds on these integrals near x_0 and x (since $\lambda_1 < \alpha$) and, for large $d(y,x_0)$ and $d(y,x)$ (since $\lambda_1 + \lambda_2 > \alpha$), to give (3.12). □

Proposition 3.4. *Assume that μ is upper α-regular and x_0 is a reference point in M. If $0 < \lambda_1 < \alpha$ and $\lambda_2 > \alpha$, then there exists a constant $C_1 > 0$ such that*

$$\int_M \frac{1}{d(y,x)^{\lambda_1}[1 + d(y,x_0)^{\lambda_2}]}\,\mathrm{d}\mu(y) \leqslant \frac{C_1}{1 + d(x,x_0)^{\lambda_1}}. \tag{3.13}$$

Proof. Fix $x \in M$. If $d(x,x_0) \leqslant 1$, then (3.13) directly follows from (3.12), since

$$\int_M \frac{1}{d(y,x)^{\lambda_1}[1 + d(y,x_0)^{\lambda_2}]}\,\mathrm{d}\mu(y) \leqslant C_0 \leqslant \frac{2C_0}{1 + d(x,x_0)^{\lambda_1}}.$$

Assume that $d(x,x_0) \geqslant 1$. If $d(y,x_0) \geqslant \frac{1}{2}d(x,x_0)$, we have that

$$\frac{1}{1 + d(y,x_0)^{\lambda_2}} \leqslant \frac{C}{[1 + d(y,x_0)^{\lambda_2-\lambda_1}][1 + d(x,x_0)^{\lambda_1}]},$$

where C is independent of x_0, y. Using proposition 3.3, it follows that

$$\begin{aligned}\int_{d(y,x_0)\geqslant d(x,x_0)/2} &\frac{1}{d(y,x)^{\lambda_1}[1+d(y,x_0)^{\lambda_2}]}\,\mathrm{d}\mu(y)\\ &\leqslant \frac{C}{1+d(x,x_0)^{\lambda_1}}\int_{d(y,x_0)\geqslant d(x,x_0)/2}\frac{1}{d(y,x)^{\lambda_1}[1+d(y,x_0)^{\lambda_2-\lambda_1}]}\,\mathrm{d}\mu(y)\\ &\leqslant \frac{C_1}{1+d(x,x_0)^{\lambda_1}}. \end{aligned} \tag{3.14}$$

If $d(y,x_0) < \frac{1}{2}d(x,x_0)$, then

$$d(y,x)^{-\lambda_1} \leqslant [d(x,x_0)-d(y,x_0)]^{-\lambda_1} \leqslant [\tfrac{1}{2}d(x,x_0)]^{-\lambda_1} \leqslant \frac{2^{\lambda_1+1}}{1+d(x,x_0)^{\lambda_1}}, \tag{3.15}$$

and hence,

$$\int_{d(y,x_0)<d(x,x_0)/2}\frac{1}{d(y,x)^{\lambda_1}[1+d(y,x_0)^{\lambda_2}]}\,\mathrm{d}\mu(y) \leqslant \frac{C_2}{1+d(x,x_0)^{\lambda_1}}, \tag{3.16}$$

where we have used that

$$\int_M \frac{1}{1+d(y,x_0)^{\lambda_2}}\,\mathrm{d}\mu(y) < \infty \quad \text{as } \lambda_2 > \alpha.$$

Adding (3.14) and (3.16), we see that (3.13) also holds if $d(x,x_0) \geqslant 1$. □

We now show the global existence of weak solutions for small φ and f.

THEOREM 3.5 (global existence). *Let $\alpha > \beta > 0$ and suppose that the heat kernel k satisfies* (k_5), (k_6) *and that Φ_2 satisfies (3.11). Let $\lambda > \alpha$ and let x_0 be a reference point in M. Then, for each $p > \alpha/(\alpha-\beta)$, there exists $\delta > 0$ such that if*

$$0 < \varphi(x), f(x) \leqslant \frac{\delta}{1+d(x,x_0)^{\lambda}},$$

then (1.1), (1.2) has an essentially bounded global solution.

Proof. Recall that conditions (k_5), (k_6) and (3.11) imply that μ is α-regular. Let the map $\mathcal{F}$ be defined as in (3.6):

$$\mathcal{F}u(t,x) = K_t\varphi(x) + \int_0^t K_\tau f(x)\,\mathrm{d}\tau + \int_0^t K_{t-\tau}u^p(\tau,x)\,\mathrm{d}\tau.$$

For $\epsilon > 0$, let S_ε be the complete subset of the Banach space $L^\infty([0,\infty)\times M)$ given by

$$S_\varepsilon = \biggl\{u \in L^\infty([0,\infty)\times M)\colon 0 \leqslant u(t,x) \leqslant \frac{\varepsilon}{1+d(x,x_0)^{\alpha-\beta}}\biggr\}.$$

We will use the contraction principle to show that, for appropriately small ϵ and δ, there exists a global solution in S_ε.

For $\lambda > \alpha$, we claim that there exists $C_2 > 0$ such that, for all $0 \leqslant g(x) \leqslant \delta/(1+d(x,x_0)^{\lambda})$, we have

$$K_t g(x) \leqslant \frac{C_2\delta}{1+d(x,x_0)^{\alpha}} \quad \text{for all } x \in M \text{ and all } t > 0. \tag{3.17}$$

To see this, let $x \in M$. If $d(x,x_0) \leqslant 1$, then (3.17) is clear, since

$$\begin{aligned} K_t g(x) &= \int_M k(t,x,y) g(y)\,\mathrm{d}\mu(y) \\ &\leqslant \int_M \frac{\delta}{1+d(y,x_0)^\lambda} k(t,x,y)\,\mathrm{d}\mu(y) \\ &\leqslant \delta \int_M k(t,x,y)\,\mathrm{d}\mu(y) \leqslant \delta \\ &\leqslant \frac{2\delta}{1+d(x,x_0)^\alpha}. \end{aligned}$$

So assume $d(x,x_0) > 1$. Using condition (k_6), we have that

$$\begin{aligned} K_t g(x) &\leqslant \int_M \frac{\delta}{1+d(y,x_0)^\lambda} k(t,x,y)\,\mathrm{d}\mu(y) \\ &\leqslant \delta \biggl\{ \int_{\Omega_1} \frac{1}{1+d(y,x_0)^\lambda} \frac{1}{t^{\alpha/\beta}} \Phi_2\biggl(\frac{d(y,x)}{t^{1/\beta}}\biggr)\,\mathrm{d}\mu(y) \\ &\qquad + \int_{\Omega_2} \frac{1}{1+d(y,x_0)^\lambda} k(t,x,y)\,\mathrm{d}\mu(y) \biggr\}, \end{aligned} \tag{3.18}$$

where $\Omega_1 = \{y \in M \colon d(y,x_0) \leqslant \frac{1}{2} d(x,x_0)\}$ and $\Omega_2 = M \setminus \Omega_1$. For $y \in \Omega_1$, noting from (3.11) that $s^\alpha \Phi_2(s)$ is bounded, we have

$$\begin{aligned} \frac{1}{t^{\alpha/\beta}} \Phi_2\biggl(\frac{d(y,x)}{t^{1/\beta}}\biggr) &= \frac{1}{d(y,x)^\alpha} \biggl(\frac{d(y,x)}{t^{1/\beta}}\biggr)^\alpha \Phi_2\biggl(\frac{d(y,x)}{t^{1/\beta}}\biggr) \\ &\leqslant \frac{C}{d(y,x)^\alpha} \leqslant \frac{2^\alpha C}{d(x,x_0)^\alpha} \\ &\leqslant \frac{2^{\alpha+1} C}{1+d(x,x_0)^\alpha}, \end{aligned}$$

and hence, using

$$\int_M \frac{\mathrm{d}\mu(y)}{1+d(y,x_0)^\lambda} < +\infty \quad \text{for } \lambda > \alpha,$$

we have

$$\begin{aligned} \int_{\Omega_1} \frac{1}{1+d(y,x_0)^\lambda} \frac{1}{t^{\alpha/\beta}} \Phi_2\biggl(\frac{d(y,x)}{t^{1/\beta}}\biggr)\,\mathrm{d}\mu(y) &\leqslant \frac{2^{\alpha+1} C}{1+d(x,x_0)^\alpha} \int_{\Omega_1} \frac{\mathrm{d}\mu(y)}{1+d(y,x_0)^\lambda} \\ &\leqslant \frac{C}{1+d(x,x_0)^\alpha}. \end{aligned} \tag{3.19}$$

For $y \in \Omega_2$,

$$\begin{aligned} \int_{\Omega_2} \frac{1}{1+d(y,x_0)^\lambda} k(t,x,y)\,\mathrm{d}\mu(y) &\leqslant \frac{2^\lambda}{1+d(x,x_0)^\lambda} \int_{\Omega_2} k(t,x,y)\,\mathrm{d}\mu(y) \\ &\leqslant \frac{C}{1+d(x,x_0)^\alpha}. \end{aligned} \tag{3.20}$$

using that $\lambda > \alpha$. Adding (3.19) and (3.20), we see that (3.17) follows from (3.18), proving our claim.

Observe that, by (k_6) and (3.11),

$$\begin{aligned}\int_0^t k(\tau,x,y)\,\mathrm{d}\tau &\leqslant \int_0^t \frac{1}{\tau^{\alpha/\beta}}\Phi_2\biggl(\frac{d(y,x)}{\tau^{1/\beta}}\biggr)\,\mathrm{d}\tau \\ &= \frac{\beta}{d(y,x)^{\alpha-\beta}}\int_{d(x,y)/t^{1/\beta}}^{\infty} s^{\alpha-\beta-1}\Phi_2(s)\,\mathrm{d}s \\ &\leqslant \frac{\beta}{d(y,x)^{\alpha-\beta}}\int_0^{\infty} s^{\alpha-\beta-1}\Phi_2(s)\,\mathrm{d}s \\ &\leqslant \frac{C}{d(y,x)^{\alpha-\beta}}, \end{aligned} \tag{3.21}$$

since

$$\int_0^{\infty} s^{\alpha-\beta-1}\Phi_2(s)\,\mathrm{d}s \leqslant \Phi_2(0)\int_0^1 s^{\alpha-\beta-1}\,\mathrm{d}s + \int_1^{\infty} s^{\alpha-1}\Phi_2(s)\,\mathrm{d}s < +\infty,$$

using the monotonicity of Φ_2 and (3.11).

Therefore, using (3.21) and (3.13) with $\lambda_1 = \alpha-\beta > 0$ and $\lambda_2 = \lambda > \alpha$,

$$\begin{aligned}\int_0^t K_\tau f(x)\,\mathrm{d}\tau &= \int_M \biggl[\int_0^t k(\tau,x,y)\,\mathrm{d}\tau\biggr] f(y)\,\mathrm{d}\mu(y) \\ &\leqslant \int_M \frac{C}{d(y,x)^{\alpha-\beta}}\frac{\delta}{(1+d(y,x_0)^{\lambda})}\,\mathrm{d}\mu(y) \\ &\leqslant \frac{C\delta}{1+d(x,x_0)^{\alpha-\beta}} \end{aligned} \tag{3.22}$$

for all $x \in M$ and $t > 0$. Similarly, for $u \in S_\varepsilon$, using (3.13) with $\lambda_1 = \alpha-\beta$, $\lambda_2 = p(\alpha-\beta) > \alpha$, we have that

$$\begin{aligned}\int_0^t K_{t-\tau}u^p(\tau,x)\,\mathrm{d}\tau &\leqslant \int_0^t\int_M k(t-\tau,x,y)\frac{\varepsilon^p}{(1+d(y,x_0)^{\alpha-\beta})^p}\,\mathrm{d}\mu(y)\,\mathrm{d}\tau \\ &\leqslant \int_M \frac{C}{d(y,x)^{\alpha-\beta}}\frac{\varepsilon^p}{(1+d(y,x_0)^{\alpha-\beta})^p}\,\mathrm{d}\mu(y) \\ &\leqslant C\varepsilon^p\int_M \frac{1}{d(y,x)^{\alpha-\beta}}\frac{1}{1+d(y,x_0)^{(\alpha-\beta)p}}\,\mathrm{d}\mu(y) \\ &\leqslant \frac{C\varepsilon^p}{1+d(x,x_0)^{\alpha-\beta}} \end{aligned} \tag{3.23}$$

for all $x \in M$ and $t > 0$. It follows from (3.17), (3.22) and (3.23) that if $u \in S_\varepsilon$, then

$$\begin{aligned}\mathcal{F}u(t,x) &\leqslant \frac{C_2\delta}{1+d(x,x_0)^{\alpha}} + \frac{C\delta + C\varepsilon^p}{1+d(x,x_0)^{\alpha-\beta}} \\ &\leqslant \frac{C_1(\delta+\varepsilon^p)}{1+d(x,x_0)^{\alpha-\beta}} \leqslant \frac{\varepsilon}{1+d(x,x_0)^{\alpha-\beta}} \end{aligned}$$

provided that $C_1(\delta+\varepsilon^p) \leqslant \varepsilon$, in which case $\mathcal{F}S_\varepsilon \subset S_\varepsilon$.

Next we show that $\mathcal{F}$ is contractive on S_ε. Indeed, for $u_1, u_2 \in S_\varepsilon$, we have

$$|\mathcal{F}u_1(t,x) - \mathcal{F}u_2(t,x)| \leqslant \int_0^t \int_M k(t-\tau,x,y)|u_1^p(\tau,y) - u_2^p(\tau,y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\tau.$$

Using the elementary inequality

$$|a^p - b^p| \leqslant p\max\{a^{p-1}, b^{p-1}\}|a-b| \quad \text{for } a,b \geqslant 0,\ p>1,$$

and the definition of S_ε, we obtain, using (3.21) and (3.12), that

$$\begin{aligned}
&|\mathcal{F}u_1(t,x) - \mathcal{F}u_2(t,x)| \\
&\quad \leqslant \|u_1 - u_2\|_\infty \int_0^t \int_M k(t-\tau,x,y) \frac{p\varepsilon^{p-1}}{[1+d(y,x_0)^{\alpha-\beta}]^{p-1}}\,\mathrm{d}\mu(y)\,\mathrm{d}\tau \\
&\quad \leqslant \|u_1 - u_2\|_\infty \int_M \frac{C}{d(y,x)^{\alpha-\beta}} \frac{p\varepsilon^{p-1}}{1+d(y,x_0)^{(\alpha-\beta)(p-1)}}\,\mathrm{d}\mu(y) \\
&\quad \leqslant C_3 p\varepsilon^{p-1}\|u_1 - u_2\|_\infty.
\end{aligned}$$

Thus, if ϵ is small enough to ensure that both $C_3p\varepsilon^{p-1} < 1$ and $C_1\varepsilon^p < \varepsilon$, and then if δ is chosen small enough so that $C_1(\delta + \varepsilon^p) \leqslant \varepsilon$, applying Banach's contraction principle to $\mathcal{F}$ on the complete set S_ε implies that (1.7), and thus (1.1), (1.2) have a global positive solution in S_ε. □

4. Regularity

In this section we discuss the regularity of weak solutions. We show that weak solutions are Hölder continuous in the spatial variable x if the source term f and initial value φ are both Hölder continuous. We adapt the method used in [9].

In order to obtain the regularity of weak solutions, we need to assume that the function Φ_2 in condition (k$_6$) satisfies the following assumption:

$$\int_0^\infty s^\alpha \Phi_2(s)\,\mathrm{d}s < \infty, \tag{4.1}$$

where α is as in condition (k$_6$). Since Φ_2 is non-increasing on $[0,\infty)$, condition (4.1) implies that $s^{1+\alpha}\Phi_2(s) = o(1)$ as $s \to \infty$.

Clearly, the Gauss-type function Φ_2 defined as in (2.4) satisfies condition (4.1) for all $\gamma > 0$, while the Cauchy-type function Φ_2 defined as in (2.5) satisfies condition (4.1) for all $\gamma > 1+\alpha$.

Note that condition (4.1) is stronger than (3.11), and hence it implies that μ is α-regular.

Proposition 4.1. *Assume that μ is upper α-regular. If Φ_2 satisfies (4.1), then, for all $\lambda \in (0,1]$,*

$$\int_M d(x,y)^\lambda \Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right)\mathrm{d}\mu(y) \leqslant C_2 t^{(\alpha+\lambda)/\beta} \quad \text{for all } x \in M,\ t>0, \tag{4.2}$$

for some constant C_2.

Proof. Let $g(r) = r^{\lambda}\Phi_2(r/t^{1/\beta})$ for $r > 0$. From (4.2), $g(r) = o(r^{-\alpha})$ so, by a standard argument using α-regularity and integration by parts [9, proposition 4.1], it follows that

$$\begin{aligned}\int_M d(x,y)^{\lambda}\Phi_2\biggl(\frac{d(x,y)}{t^{1/\beta}}\biggr)\,\mathrm{d}\mu(y) &= \int_M g(d(x,y))\,\mathrm{d}\mu(y)\\ &\leqslant C_1\int_0^{\infty} r^{\alpha}|g'(r)|\,\mathrm{d}r\\ &= C_1\int_0^{\infty} r^{\alpha}\biggl|\lambda r^{\lambda-1}\Phi_2\biggl(\frac{r}{t^{1/\beta}}\biggr)+r^{\lambda}\Phi_2'\biggl(\frac{r}{t^{1/\beta}}\biggr)t^{-1/\beta}\biggr|\,\mathrm{d}r\\ &\leqslant C_2t^{(\alpha+\lambda)/\beta}\biggl[\int_0^{\infty}\lambda s^{\alpha+\lambda-1}\Phi_2(s)\,\mathrm{d}s+\int_0^{\infty}s^{\alpha+\lambda}(-\Phi_2'(s))\,\mathrm{d}s\biggr].\end{aligned}$$

By an easy calculation, the last integral

$$\begin{aligned}\int_0^{\infty} s^{\alpha+\lambda}(-\Phi_2'(s))\,\mathrm{d}s &= -s^{\alpha+\lambda}\Phi_2(s)|_0^{\infty}+(\alpha+\lambda)\int_0^{\infty}s^{\alpha+\lambda-1}\Phi_2(s)\,\mathrm{d}s\\ &= (\alpha+\lambda)\int_0^{\infty}s^{\alpha+\lambda-1}\Phi_2(s)\,\mathrm{d}s\\ &\leqslant C_3\end{aligned}$$

using (4.1). Therefore,

$$\int_M d(x,y)^{\lambda}\Phi\biggl(\frac{d(x,y)}{t^{1/\beta}}\biggr)\,\mathrm{d}\mu(y)\leqslant C_2t^{(\alpha+\lambda)/\beta},$$

as desired. □

We now show the Hölder continuity of weak solutions of (1.7).

Theorem 4.2 (Hölder continuity). *Assume that φ, $f \in L^1(M)$ are Hölder continuous with exponents $\theta_1, \theta_2 \in (0, 1]$, respectively. Then, for all $x_1, x_2 \in M$,*

$$|\varphi(x_1)-\varphi(x_2)|\leqslant C_5d(x_1,x_2)^{\theta_1}, \tag{4.3}$$

$$|f(x_1)-f(x_2)|\leqslant C_6d(x_1,x_2)^{\theta_2}, \tag{4.4}$$

where $C_5, C_6 > 0$. Assume that the heat kernel k satisfies (k_5)–(k_7) *and that Φ_2 satisfies* (4.1) *with $\lambda = \max\{\theta_1, \theta_2\}$. Let $u(t, x)$ be a non-negative weak solution to* (1.1), (1.2) *that is bounded in $(0, T) \times M$ for some $T > 0$. Then $u(t, x)$ is Hölder continuous. For all $x_1, x_2 \in M$ and all $t \in (0, T)$,*

$$|u(t,x_1)-u(t,x_2)|\leqslant Cd(x_1,x_2)^{\theta}, \tag{4.5}$$

where $\theta = \theta_1\sigma/(\theta_1 + \nu\beta)$ and $C > 0$ may depend on T but is independent of t, x.

Proof. From (k_6), (4.3) and (4.2), there exists $C > 0$ such that, for all $t > 0$ and $x \in M$,

$$\int_M k(t,x,y)|\varphi(y)-\varphi(x)|\,\mathrm{d}\mu(y) \leqslant C_5 t^{-\alpha/\beta}\int_M d(x,y)^{\theta_1}\Phi_2\biggl(\frac{d(x,y)}{t^{1/\beta}}\biggr)\,\mathrm{d}\mu(y) \leqslant Ct^{\theta_1/\beta}. \tag{4.6}$$

By (1.7), it is sufficient to show that each of the functions u_0, u_1 and u_2 is Hölder continuous in $(0,T)\times M$, where

$$u_0(t,x) = K_t\varphi(x),$$
$$u_1(t,x) = \int_0^t K_\tau f(x)\,\mathrm{d}\tau,$$
$$u_2(t,x) = \int_0^t K_{t-\tau}u^p(\tau,x)\,\mathrm{d}\tau.$$

We first show the Hölder continuity of u_0. Indeed, for $t > 0$ and $x_1, x_2 \in M$, we see from (k_7) that

$$\begin{aligned}|u_0(t,x_1)-u_0(t,x_2)| &= \biggl|\int_M (k(t,x_1,y)-k(t,x_2,y))\varphi(y)\,\mathrm{d}\mu(y)\biggr| \\ &\leqslant Lt^{-\nu}d(x_1,x_2)^{\sigma}\|\varphi\|_1 \\ &\leqslant L\|\varphi\|_1 d(x_1,x_2)^{\sigma-\nu s_0}\end{aligned} \tag{4.7}$$

if $t \geqslant d(x_1,x_2)^{s_0}$, where $s_0 > 0$ will be specified later on. On the other hand, if $t \leqslant d(x_1,x_2)^{s_0}$, we have, using ($k_5$), (4.6) and (4.3), that

$$\begin{aligned}|u_0(t,x_1)-u_0(t,x_2)| &\leqslant \biggl|\int_M k(t,x_1,y)(\varphi(y)-\varphi(x_1))\,\mathrm{d}\mu(y) + [\varphi(x_1)-\varphi(x_2)] \\ &\qquad - \int_M k(t,x_2,y)(\varphi(y)-\varphi(x_2))\,\mathrm{d}\mu(y)\biggr| \\ &\leqslant 2Ct^{\theta_1/\beta} + C_5 d(x_1,x_2)^{\theta_1} \\ &\leqslant C[d(x_1,x_2)^{s_0\theta_1/\beta} + d(x_1,x_2)^{\theta_1}].\end{aligned}$$

Combining this with (4.7), it follows that

$$\begin{aligned}|u_0(t,x_1)-u_0(t,x_2)| &\leqslant C[d(x_1,x_2)^{\sigma-\nu s_0} + d(x_1,x_2)^{s_0\theta_1/\beta} + d(x_1,x_2)^{\theta_1}] \\ &\leqslant Cd(x_1,x_2)^{\theta_1\sigma/(\theta_1+\nu\beta)}\end{aligned} \tag{4.8}$$

for all $t > 0$ and $x_1, x_2 \in M$ with $d(x_1,x_2) \leqslant 1$, taking $s_0 = \sigma/(\nu+\theta_1/\beta)$ so that $\sigma - \nu s_0 = s_0\theta_1/\beta$, and where we have used the fact that $\theta_1 \geqslant s_0\theta_1/\beta$ for $\sigma \leqslant 1 \leqslant \nu$ and $\beta \geqslant 1$.

Next we show the Hölder continuity of u_1. As with (4.6), from (k_6), (4.4) and (4.2), we have that

$$\int_M k(\tau,x_1,y)|f(y)-f(x_1)|\,\mathrm{d}\mu(y) \leqslant C\tau^{\theta_2/\beta},$$

which yields, using (k_5) and (4.4), that

$$\begin{aligned}
&|u_1(t,x_1)-u_1(t,x_2)| \\
&\quad = \left| \int_0^t [K_\tau f(x_1) - K_\tau f(x_2)]\,\mathrm{d}\tau \right| \\
&\quad = \left| \int_0^t \mathrm{d}\tau \int_M k(\tau,x_1,y)(f(y)-f(x_1))\,\mathrm{d}\mu(y) + t[f(x_1)-f(x_2)] \right. \\
&\qquad\qquad \left. - \int_0^t \mathrm{d}\tau \int_M k(\tau,x_2,y)(f(y)-f(x_2))\,\mathrm{d}\mu(y) \right| \\
&\quad \leqslant 2C \int_0^t \tau^{\theta_2/\beta}\,\mathrm{d}\tau + C_6 t d(x_1,x_2)^{\theta_2} \\
&\quad = Ct^{\theta_2/\beta+1} + C_6 t d(x_1,x_2)^{\theta_2} \\
&\quad \leqslant C[d(x_1,x_2)^{s_1+s_1\theta_2/\beta} + d(x_1,x_2)^{s_1+\theta_2}] \qquad (4.9)
\end{aligned}$$

if $t \leqslant d(x_1,x_2)^{s_1}$, where $s_1 > 0$ will be chosen later.

On the other hand, if $t > d(x_1,x_2)^{s_1}$, and setting $t_1 = d(x_1,x_2)^{s_1}$, we obtain, using (k_7), that

$$\begin{aligned}
\left| \int_{t_1}^t [K_\tau f(x_1) - K_\tau f(x_2)]\,\mathrm{d}\tau \right| &\leqslant \int_{t_1}^t \mathrm{d}\tau \int_M |k(\tau,x_1,y)-k(\tau,x_2,y)|\,|f(y)|\,\mathrm{d}\mu(y) \\
&\leqslant \int_{t_1}^t L\tau^{-\nu} d(x_1,x_2)^\sigma \|f\|_1\,\mathrm{d}\tau \\
&\leqslant L\frac{t_1^{1-\nu}-t^{1-\nu}}{\nu-1} d(x_1,x_2)^\sigma \|f\|_1 \\
&\leqslant \frac{L}{\nu-1} d(x_1,x_2)^{s_1(1-\nu)+\sigma}\|f\|_1. \qquad (4.10)
\end{aligned}$$

It follows from (4.10) and (4.9) that

$$\begin{aligned}
&|u_1(t,x_1)-u_1(t,x_2)| \\
&\quad \leqslant \left| \int_0^{t_1} K_\tau f(x_1) - K_\tau f(x_2)\,\mathrm{d}\tau \right| + \left| \int_{t_1}^t K_\tau f(x_1) - K_\tau f(x_2)\,\mathrm{d}\tau \right| \\
&\quad \leqslant C[d(x_1,x_2)^{s_1+s_1\theta_2/\beta} + d(x_1,x_2)^{s_1+\theta_2} + d(x_1,x_2)^{s_1(1-\nu)+\sigma}] \\
&\quad \leqslant Cd(x_1,x_2)^{\sigma(\theta_2+\beta)/(\theta_2+\nu\beta)} \qquad (4.11)
\end{aligned}$$

if $d(x_1,x_2) \leqslant 1$, taking $s_1 = \sigma\beta/(\theta_2+\nu\beta)$ so that $s_1 + s_1\theta_2/\beta = s_1(1-\nu)+\sigma$, and where we have used the fact that

$$s_1 + \theta_2 \geqslant s_1 + s_1\theta_2/\beta$$

for $\sigma \leqslant 1 \leqslant \nu$ and $\beta \geqslant 1$.

Finally, we show the Hölder continuity of u_2. As $u(t,x)$ is bounded on $(0,T)\times M$, we see that

$$\int_{t-\eta}^t \mathrm{d}\tau \int_M k(t-\tau,x,y)u^p(\tau,y)\,\mathrm{d}\mu(y) \leqslant C\eta.$$

Hence, using (k_7), we obtain

$$\begin{aligned}
&|u_2(t,x_1) - u_2(t,x_2)| \\
&\quad = \bigg| \int_{t-\eta}^{t} \mathrm{d}\tau \int_M k(t-\tau, x_1, y) u^p(\tau, y)\,\mathrm{d}\mu(y) \\
&\qquad - \int_{t-\eta}^{t} \mathrm{d}\tau \int_M k(t-\tau, x_2, y) u^p(\tau, y)\,\mathrm{d}\mu(y) \\
&\qquad + \int_0^{t-\eta} \mathrm{d}\tau \int_M (k(t-\tau, x_1, y) - k(t-\tau, x_2, y)) u^p(\tau, y)\,\mathrm{d}\mu(y) \bigg| \\
&\quad \leqslant 2C\eta + L \int_0^{t-\eta} \mathrm{d}\tau \int_M |t-\tau|^{-\nu} d(x_1, x_2)^{\sigma} u^p(\tau, y)\,\mathrm{d}\mu(y) \\
&\quad \leqslant C(\eta + \eta^{1-\nu} d(x_1, x_2)^{\sigma}).
\end{aligned}$$

Taking $\eta = d(x_1, x_2)^{\sigma/\nu}$, we thus have

$$|u_2(t,x_1) - u_2(t,x_2)| \leqslant C d(x_1, x_2)^{\sigma/\nu}. \tag{4.12}$$

Combining (4.8), (4.11) and (4.12), we conclude that

$$|u(t,x_1) - u(t,x_2)| \leqslant C d(x_1, x_2)^{\theta_1 \sigma/(\theta_1 + \nu\beta)}$$

for all $t \in (0,T)$ and $x_1, x_2 \in M$ with $d(x_1, x_2) \leqslant 1$, for some $C > 0$, where we have used that

$$\frac{\theta_1 \sigma}{\theta_1 + \nu\beta} \leqslant \frac{\sigma}{\nu} \leqslant \frac{\sigma(\theta_2 + \beta)}{\theta_2 + \nu\beta}.$$

The proof is complete. □

Finally, one may show that if the heat kernel k satisfies (k_5), if $|f|_\infty < \infty$ and if $\varphi(x)$ satisfies

$$|K_{t+\delta}\varphi(x) - K_t\varphi(x)| \leqslant C\delta \quad \text{for all } t > 0,\ x \in M,$$

then the essentially bounded weak solution u of (1.7) is Lipschitz continuous in time t on $(0,T) \times M$, that is,

$$|u(t+\delta, x) - u(t,x)| \leqslant C_1\delta, \quad t \in (0,T),\ \delta > 0,\ x \in M.$$

We omit the details, which are similar to the special case considered in [9].

We note that, unlike the blow-up and the existence, the regularity of solutions is not related to the Hausdorff dimension α and the walk dimension β.

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