Categories, Definitions and Mathematics:
Student Reasoning about Objects in Analysis

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Contents

Figures and Tables

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Declaration

Abstract

This thesis has two integrated components, one theoretical and one investigative.

The theoretical component considers human reason about categories of objects. First, it proposes that the standards of argumentation in everyday life are variable, with emphasis on direct generalisation, whereas standards in mathematics are more fixed and require abstraction of properties. Second, it accounts for the difficulty of the transition to university mathematics by considering the impact of choosing formal definitions upon the nature of categories and argumentation. Through this it unifies established theories and observations regarding student behaviours at this level. Finally, it addresses the question of why Analysis seems particularly difficult, by considering the relative accessibility of its visual representations and its formal definitions.

The investigative component is centred on a qualitative study, the main element of which is a series of interviews with students attending two different first courses in Real Analysis. One of these courses is a standard lecture course, the other involves a classroom-based, problem-solving approach. Grounded theory data analysis methods are used to interpret the data, identifying behaviours exhibited when students reason about specific objects and whole categories. These behaviours are linked to types of understanding as distinguished in the mathematics education literature. The student's visual or nonvisual reasoning style and their *sense of authority*, whether "internal" or "external" are identified as causal factors in the types of understanding a student develops. The course attended appears as an intervening factor. A substantive theory is developed to explain the contributions of these factors. This leads to improvement of the theory developed in the theoretical component.

Finally, the study is reviewed and the implications of its findings for the teaching and learning of mathematics at this level are considered.

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Chapter 1
Overview

1.1 Introduction

This study is about the transition from school to university mathematics, which is known to be difficult even for those who have had considerable success in school. Specifically, it is about the learning of Analysis, which is recognised as one of the most difficult subjects encountered at the beginning of a mathematics degree. The study came about due to the instigation of an innovative pedagogical approach. Beginning in 1997, the University of Warwick's Mathematics Institute chose to teach Analysis not by the standard lecture method but by placing students in a classroom environment in which they worked through a structured sequence of problems and thereby proved most of the results of the course themselves. At a time when there is concern about the preparedness of undergraduates enrolling for university mathematics, this qualitative change represents an interesting alternative to cutting syllabi or other quantitative changes.

From the perspective of research into the learning of mathematics, the situation is particularly interesting since a parallel lecture course in the same institution covered the same material at the same time. This study assesses the effects of these two different pedagogical strategies on the students' understanding of the material. Through a detailed, qualitative approach it builds on previously identified misconceptions and difficulties students have with the material of Analysis, investigates cognitive difficulties involved in the transition to university mathematics, and ascertains factors which lead to some students, in some circumstances, overcoming those difficulties more easily than others.

A personal aim throughout is as follows. I have seen an enormous amount of frustration while engaged in this research. Students are frustrated by their lecturers' apparent unwillingness or inability to talk to them on their level. Lecturers are frustrated that despite their best efforts at clear and fluent exposition, the students don't understand very much and the educators tell them they're getting it wrong. I hope that the work in this thesis may improve these groups' understanding of each other and thereby contribute to a reduction in frustration all round.

1.2 Structure: one study, two parts

For the purposes of reporting, this study is divided into two parallel reports. The first, appearing in part 2, posits a theoretical framework describing ways in which human beings reason about categories of objects. It then applies this to clarify the particular thinking required in advanced mathematics, thus characterising reasons for the difficulty of making the transition. The second, appearing in part 3, investigates factors influencing students' progress in coping with this transition. Part 1 introduces both reports by describing the research setting and method, and part 4 reviews and evaluates both and indicates future directions for the research. All four parts are outlined more fully below; the subsequent diagram (Figure 1) indicates how they are interrelated.

1.2.1 Part 1

Part 1 consists of this introduction, which goes on to describe the use of existing literature throughout this thesis, and three more chapters. Chapter 2 describes the setting in which the innovative course took place and gives detail of the logistics of both this and the lecture course. In particular it assesses the responsibilities accorded to the students on each as learners of mathematics. Chapter 3 reviews literature pertaining to the content of both courses, discussing proofs and proving, definitions, and student reasoning about particular mathematical concepts including convergence and limits. It also reviews more general literature on students' beliefs about their role as learners of mathematics. Chapter 4

describes the aims of the study and the investigation methods used, in addition to giving more detail on the theoretical development through the pilot and main studies.

1.2.2 Part 2

Part 2 develops the theoretical framework and its applications to advanced mathematics. Chapter 5 reviews cognitive psychology literature on human reasoning about categories of objects. It introduces the idea of a prototype as a theoretical construct and develops a description of ways in which individuals use prototypes in reasoning according to the demands of the situation. In chapter 6 this is then compared with the required form of reasoning in advanced mathematics. Assuming that students continue to reason as would be appropriate in everyday life, this leads to a unified explanation for a number of known student behaviours such as those previously discussed in chapter 3.

1.2.3 Part 3

Part 3 develops the investigation, which studies how and why different types of student make the progress they do toward the new type of reasoning required in advanced mathematics. Chapter 7 reviews literature on typologies of understanding, relating these to the theoretical framework and choosing a particular typology on this basis in order to classify behaviours seen in interview protocols. Chapter 8 introduces the factors found to be correlated with the type of understanding a student develops, via a reprise of related literature and illustrations of these emergent distinctions. Chapter 9 examines these correlations in each combination of these factors and the students' course arising in the study, suggesting theoretical explanations for the type of understanding developed in each case. Chapter 10 then gives a different perspective, isolating each factor in turn to examine its influence and highlight the consistency in these theoretical explanations. As a consequence of this data analysis, it becomes apparent that the theoretical framework is inadequate in explaining the behaviour of a subset of the students, and chapter 11 clarifies the form of this inadequacy and remedies it by returning to the mathematics education literature on conceptual development.

1.2.4 Part 4

Part 4 returns to looking at the study as a whole. Chapter 12 reviews the study, considering the issues of validity and generalisability and highlighting new questions that arise from the research. Chapter 13 considers the implications of the findings for the teaching and learning of Analysis, and of university mathematics in general.

1.2.5 Diagram

The diagram on the following page illustrates the relationships between the chapters.

PART 1: THE STUDY | PART 2: THEORY

PART 4:
CONCLUSIONS

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│ 8. Causal       │───────▶│ 9. Combinations  │
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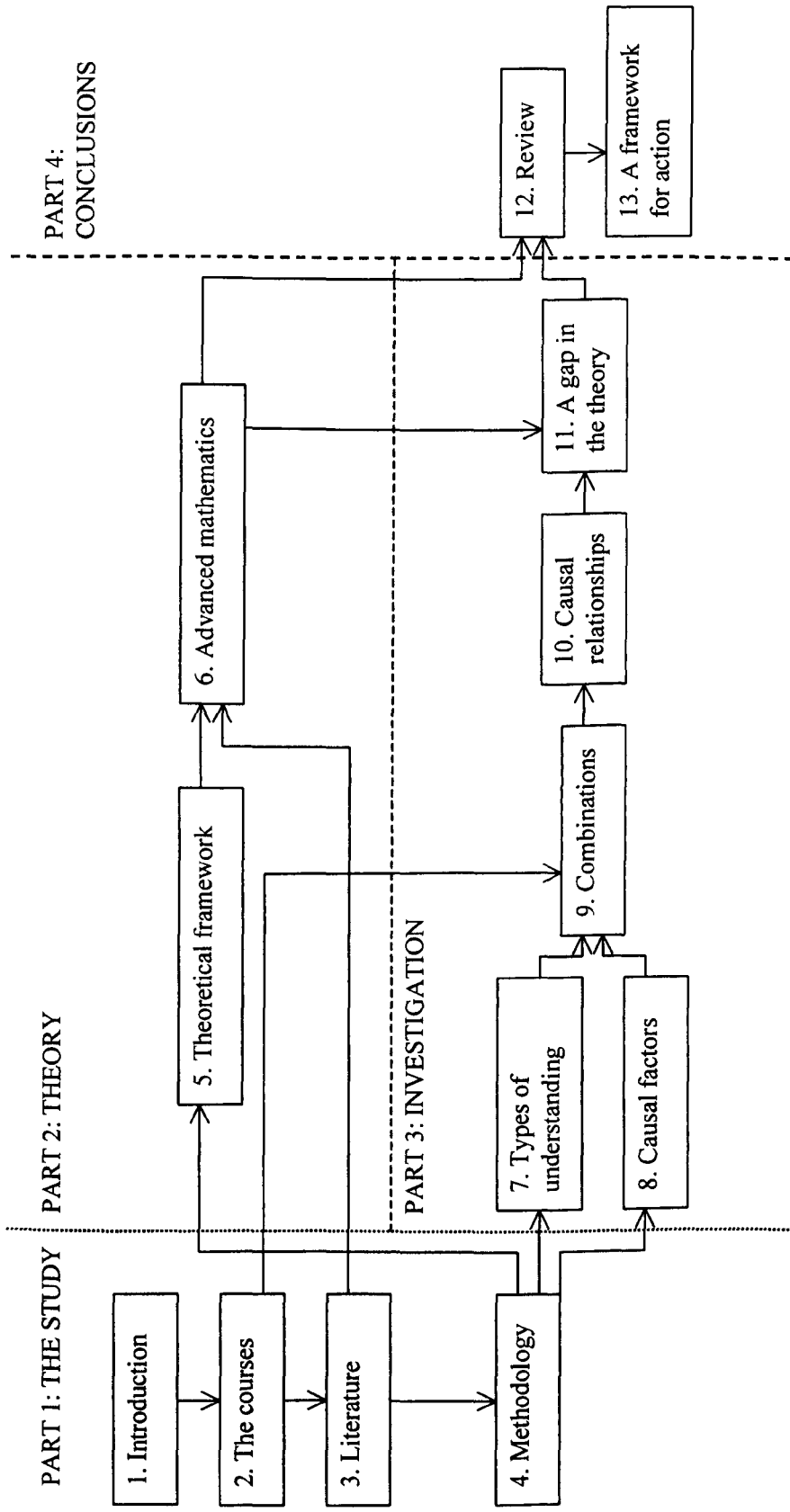
Figure 1: Diagram of the thesis

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1.3 Theory and literature

Owing to the wealth of literature pertaining to the learning of mathematics at university level in general and the material of Analysis in particular, it was decided that this study could best contribute not by investigating one particular domain but by seeking to determine how different factors interact in these students' developing understanding of Analysis. In addition, the theoretical framework is applied in an attempt to unify much of this work. The result is that significant bodies of literature from a number of areas are used, and to avoid confusion, these are introduced when they are most pertinent. The placement of different bodies of work and their use is described and summarised below.

1.3.1 Part 1

Chapter 2, on the logistics of the courses, briefly covers literature on pedagogical approaches and nonstandard teaching styles, to clarify the particular characteristics of the courses studied. Chapter 3, the most "standard" review, covers literature directly relevant to the content of the courses and likely student interpretations of this. That is to say, it covers proofs and proving (including a section on visual justifications), definitions, convergence and limits, and student beliefs about their role as learners of mathematics. It also contains a first review of literature on understanding in mathematics, which informs the methodology in chapter 4. Chapter 4 then discusses opinion on the merits and detractions of qualitative methods and interviewing.

1.3.2 Part 2

Chapter 5, developing the theoretical framework, uses cognitive psychology literature on categorisation and reasoning. In chapter 6 this is applied to advanced mathematics by drawing on the literature on the place of definitions and proofs within mathematical theories. Chapter 6 also returns to the work covered in chapter 3 to explain how it can be related to this framework.

1.3.3 Part 3

Chapter 7, on the understanding manifested by the students, returns to the literature on understanding, giving a detailed account of suggested typologies and relating these to the theoretical framework. Chapter 8, on the causal factors in students' acquisition of understanding, reviews literature on classification of visual and nonvisual reasoning approaches and reprises the work on student beliefs about mathematics and their own role as learners. Chapter 11 covers in more detail the literature on process-object theories and pseudo-empirical abstraction, using this to further fill out the theoretical framework.

1.3.4 Part 4

Chapter 13 does not introduce substantial new literature, but makes reference to other theoretical constructs that might be incorporated into future extensions of this research.

1.3.5 Summary

- Chapter 2: pedagogical approaches
- Chapter 3: proof, definitions, limits, beliefs, understanding
- Chapter 4: qualitative methods
- Chapter 5: categorisation, reasoning
- Chapter 6: advanced mathematics
- Chapter 7: typologies of understanding
- Chapter 8: visualisation, reprise of beliefs
- Chapter 11: process-object theories

Chapter 2
The courses

2.1 Introduction

This chapter describes the research setting. The first section focuses on the students' experience as first year mathematics undergraduates. The second describes the circumstances surrounding the inception of the new course, both nationally, in light of concern about student competence, and locally, examining the particular circumstances which prompted the change and made it viable. The third section gives logistical detail of the new course and the parallel lecture course, examining the demands made of the students in each. The fourth compares this pedagogical strategy with other non-lecture-based approaches.

2.2 Mathematics at Warwick

2.2.1 Being an undergraduate

The Mathematics Institute at Warwick is large by UK standards, currently employing 40 full-time faculty, providing for approximately 100 graduate students and admitting around 200 undergraduates per year on to its single honours mathematics degree. These students are selected solely according to their A-level grades; there is no formal interviewing process. In 1998, when the main study for this thesis was carried out, the requirement was a grade A in mathematics, A and B in two other subjects and a grade 2 in a Special or STEP paper in mathematics. The institute also provides teaching for approximately 150 students enrolled on degrees titled Physics with Mathematics, Mathematics and Statistics and MORSE (Mathematics, Operational Research, Statistics and Economics). The entry grades

for these students are generally slightly lower but still demanding at A in mathematics, and B in two other subjects.

Students taking these degrees join a total of approximately 8500 full time undergraduates at Warwick. As first years, virtually all will live in campus accommodation, sharing a kitchen with a "corridor" of around fifteen peers who are likely to form their social group for at least the first term. Campus life has advantages and disadvantages from the perspective of teachers wishing their students to study hard. All essential services are easily accessible, cutting down on the time needed for everyday chores, and travel time to lectures is minimal. In addition most live near other students on mathematics-related degrees so can collaborate on homework assignments if they so choose. However "collaborate" can mean different things according to the student's goals, so this can be a disadvantage to individual thinking. A student's work might also be disrupted by the large number of active societies and students' union events on campus.

2.2.2 Degree structure

Students on all of the aforementioned degrees attend a number of common core modules in their first year, although due to the large numbers involved these are usually taught in separate lectures/classes for the mathematicians and combined degree students. All attend modules in Analysis (the courses under investigation here both cover the material of "Analysis I"; "Analysis II" follows in the second term). They also take modules in Linear Algebra, "Foundations" (a basic course on sets, functions and groups), differential equations and geometry. All students are required to take a certain percentage of courses in their home departments; this includes further core modules for those studying for combined degrees. Information is provided on which modules form a good basis for work in future years, but there is considerable flexibility in the system and some take more unusual options in business, computing or languages. Students usually attend lectures for more modules than required, deciding which examinations to take later in the year.

Assessment for first year mathematics modules typically comprises some credit for concurrent weekly assignments (around 15% of the total) with the remainder resting on an examination in the Summer term. Analysis I and Foundations are the exception; they are examined together in one 3-hour examination at the beginning of the Spring term.

2.2.3 Teaching

Contact hours for most modules are limited to three 1-hour lectures per week. Typically of large pure mathematics courses in the UK, it is not usual for there to be much student participation in these lectures. In addition to this time, the students have an academic tutor in their home department who they see for up to an hour per week. The form these tutorials take is at the discretion of the tutor, who may help the students with their coursework, introduce them to other mathematics or attend to administrative matters. In a system which is generally agreed to be one of the teaching strengths of the department, each student is also assigned to a group of four which meets with a graduate student "supervisor" for one or two hours per week (one in the case of joint degree students, two for single honours mathematicians). These supervisions cover the work for core mathematics courses and some of the options, and the supervisor marks the students' assignments for the core courses. It is not an infallible system and occasionally students do not feel that supervisions provide good quality teaching, but it means that students have a specified time when they can receive help with their mathematics in a small group and with a young teacher.

2.3 Background to the innovation

2.3.1 National: university mathematics in the UK

For a number of years, there has been concern in the UK about the mathematical competence of students entering university mathematics. In the mid-90s an LMS report gave the following summary of the ways in which those in higher education found their students wanting:

1. A serious lack of technical facility – the ability to undertake numerical and algebraic calculation with fluency and accuracy;

2. A marked decline in analytical powers when faced with simple problems requiring more than one step;

3. A changed perception of what mathematics is – in particular the essential place within it of precision and proof.

(London Mathematical Society, 1995)

These weaknesses are attributed largely to a shift in school curricula during the 1990s (LMS, *ibid.*). In particular, A-level syllabi changed during this time. Typically this means that techniques of differentiation and integration are retained in the core but have reduced importance, complex numbers and vectors are no longer in the core and modules on problem solving and data handling have been introduced or expanded (SMP, 1990; SEAC, 1993). Hoyles notes that within the new curricula, proof is often limited to investigation-based sections. She suggests that this detaches it from the material of mathematics as a whole and means that it may be learned as part of a social ritual in which particular forms of presentation are required (Hoyles, 1997). Simpson acknowledges the view of some educators that the newer syllabi may help more students to devleop skills of critical analysis and explanation which will be useful to them in their different futures (Simpson, 1995). He suggests that problems now arise for students entering university mathematics because they face a "broken route" to proof. Whereas an earlier emphasis on sums and techniques led through proof techniques and on to the predicative and propositional calculus at university, he suggests that students may now be more prepared for "proof through reasoning", where formalising appears as a final step after exploring, discovering patterns, explaining and justifying. He suggests that the emphasis on beginning with formalism at university does not form a natural continuation of this approach (Simpson, *ibid.*), as it would have done of more traditional curricula.

Whatever the benefits and detractions of either approach, universities have felt that there is a serious problem. In response, some have moved formal Analysis to later in the degree, or even abandoned it altogether. Warwick's response has been to retain it as a first year, first term course, but to spread the material out so that it is covered over three terms in two academic years rather than two terms in one year. As part of a general strategy across all subjects in the first year it has also expanded the requirement of coursework for credit. This has been a successful change, dramatically cutting the number of students required to take resit examinations. However, Analysis is still perceived to be a particularly difficult topic which students often do not understand well, and specific changes were made to this course as are outlined below.

2.3.2 Local: history of the new course

With the national situation in the background, the changes to Warwick's Analysis course came about as a result of a personal initiative by Professor David Epstein, a member of the department for over thirty years. Professor Epstein had begun to be disheartened with the lecture method, both because of the recognised passivity of students in lectures and because repeated attempts to improve his explanations did not lead to improved student performance in examinations. He first began using novel teaching practices in a second year course in Metric Spaces in 1989, after being inspired by work by the Geometry Center in Minneapolis: Bill Thurston, John Conway, Peter Doyle and Jane Gilman were running a course for talented high school students, who were introduced to abstract mathematical concepts through group work on practical tasks (Epstein, 1998, personal communication). Various strategies tried by Professor Epstein met with limited success, but he was keen to instigate an experiment using a completely different pedagogical style for Analysis for various reasons.

First, Analysis is recognised to be a difficult subject. Second, the climate was right: an increased emphasis on improving teaching was taking hold in the department. Third, and perhaps most crucially, an appropriate text was available. The course would be based

closely on R.P. Burn's "Numbers and Functions – Steps into Analysis" (Burn, 1992), a text previously used successfully by another department member, Mr. David Fowler, in his tutorials with first year students. The book consists mainly of questions, with compressed solution "hints" and a very short summary of the main ideas at the end of each chapter. The questions develop rationales for the main definitions, lead students to construct proofs of the theorems and require them to use those theorems in subsequent arguments. The following question, leading to the formulation of the ratio test for sequences, is typical of the style:

(a) Use a calculator or computer to evaluate terms of the sequence $\left(n^2/2^n\right)$. Calculate the values for $n = 1, 2, 5, 10, 20, 50$. Would you conjecture that the sequence is null?

(b) Verify that the ratio $a_{n+1}/a_n = \frac{1}{2}(1 + 1/n)^2$.

(c) Find an integer N such that, if $n \geq N$, then the ratio $a_{n+1}/a_n < \frac{3}{4}$.

(d) By considering that $a_{N+1} < \frac{3}{4}a_N$ and $a_{N+2} < \frac{3}{4}a_{N+1}$, etc. prove that $a_{N+n} < \left(\frac{3}{4}\right)^n a_N$.

(e) Why is $\left(\left(\frac{3}{4}\right)^n a_N\right)$ a null sequence?

(f) Use the sandwich theorem, qn 28, to show that $\left(n^2/2^n\right)$ is a null sequence.

(g) What numbers might have been used in place of $\frac{3}{4}$ in part (c) (perhaps with a different N) which would still have led to a proof that the sequence was null?

(Burn, 1992, p.49)

The new course was trialled with 26 students in the Autumn term of 1996 (this study and the researcher's involvement commenced one year later). All of the students joining the mathematics degree in that year were invited to take the course, and the eventual participants were randomly selected from the 52 who volunteered. Help was enlisted from Warwick's Mathematics Education Research Centre: Adrian Simpson and Professor David Tall advised on the format of the classes, and attended as helpers and interested observers. The 26 students attended two 2-hour sessions per week during the 10-week term, working through the material in the relevant chapters of *Numbers and Functions*. The remainder of their peers attended lectures as usual. All were required to submit solutions to the same assignments each week, and all were examined in June of 1997 in a 3-hour examination

covering the material of Analysis I and its second term counterpart Analysis II, which was taught in lectures as usual.

Data from the assignments and examinations were treated with caution and not assumed to be conclusive about the merits of the course. The volunteer effect was taken into consideration – those volunteering to attend may be at least more confident and motivated – and some of those on the experimental course were known to have attended the standard lectures. In the final examination those from the new course had a mean score approximately 10% higher than that for the whole year, but only 4% different from that of those who had originally volunteered for the course. However the benefits were deemed sufficient for a proposal to be put forward to run the course for the entire cohort of single honours students in the following year.

2.4 The two courses

2.4.1 Material

The material covered was the same for both courses, and is summarised in the table below.

Table 1: Material covered in Analysis I

Inequalities (approx. 1 week)	Sequences (approx. 3 weeks)	Completeness (approx. 1 week)	Series (approx. 4 weeks)
Axioms for inequalities and proofs of further results from these, Triangle inequality, Bernoulli's inequality	Monotonicity, Boundedness, Tending to infinity, Convergence, Algebra of limits	Decimal expansions, Completeness, Cauchy sequences, Bolzano-Weierstrass theorem	Series convergence, Tests for convergence of series with positive terms (ratio test, comparison test etc.), Absolute convergence, Conditional convergence Representations of e

Summarising the philosophy, the syllabus for Analysis as a whole states:

At the beginning of the nineteenth century the familiar tools of calculus, differentiation and integration began to run into problems. Mathematicians were unsure of how to apply these tools to sums of infinitely many functions. The origins of Analysis lie in their attempt to formalise the ideas of calculus purely in the language of arithmetic and to resolve these problems. You will study the ideas of the mathematicians Cauchy, Dirichlet, Weierstrass, Bolzano, D'Alembert, Riemann and others, concerning sequences and series in term one, continuity and differentiability in term two and integration in term one of your second year. By the end of the year you will be able to answer many interesting questions: what do we mean by 'infinity'? How can you accurately compute the value of π or e or $\sqrt{2}$? How can you add up infinitely many numbers, or infinitely many functions? Can all functions be approximated by polynomials? There will be considerable emphasis throughout the module on the need to argue with much greater precision and care than you had to at school. With the support of your fellow students, lecturers and other helpers, you will be encouraged to move on from the situation where the teacher shows you how to solve each kind of problem, to the point where you can develop your own methods for solving problems. You will also be expected to question the concepts underlying your solutions, and understand why a particular method is meaningful and another not so. In other words, your mathematical focus should shift from problem solving methods to concepts and clarity of thought.

2.4.2 Logistics

Data collection for the pilot study of this thesis took place in the Autumn of 1997. This was the first year in which all of the single honours mathematics students attended the new classes, and the credit structure for the course was rather complicated. The students both kept portfolios of their answers to questions they were instructed to attempt from the book, and handed in separate assignments for which the questions were often very similar. The year of the main study this was streamlined, and the table below summarises the logistics of

both courses in the Autumn of 1998 (numbers of students are those who took the end-of-year examination in June):

Table 2: Logistics of the two courses

Course	Number of students entered	Teaching time	Assignment of credit (for Analysis I and II combined)
New	194 Mathematics 10 Maths/Statistics[1]	2 2-hour classes per week 1 1-hour lecture per week	15% weekly workbooks/ 2nd term assignments 25% January examination 60% June examination
Lecture	58 MORSE	3 1-hour lectures per week	15% weekly assignments 25% January examination 60% June examination

Those on the lecture course all attended the standard 30 lectures and submitted assignments for marking each week as usual. At times the students received handouts covering some of the material, which the lecturer would go through in class while the students made extra notes, but in the majority of lecture time the students copied the notes the lecturer wrote on the board. In keeping with standard practice in the department, attendance at lectures was not recorded.

Those on the new course attended classes with between 30 and 35 students, and their lecture served to round up the material covered in the classes and look ahead to future work, rather than to present new material. In the classes they worked in groups of between 3 and 5 to answer the questions in their booklets, under the supervision of a teacher (a member of staff or a graduate student with teaching experience) and one or two "peer tutors". Peer tutors were second year students who had taken the course the previous year and performed well in

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[1] A total of 33 students studying mathematics and physics also took one or other Analysis course (all were given the choice to opt for the new course). Information is unavailable on which students chose which, so relevant information for this section of the cohort will be given separately throughout.

examinations. Prior to the course, both the teachers and peer tutors received an intensive day's training on teaching in this type of environment. They primarily spent time offering assistance to groups and to individual students, "teaching" the whole class only when they felt that the majority were experiencing the same difficulties.

The classes varied in their exact format but for most students the majority of the time was spent working through their booklets with their classmates. Observations of a number of classes and conversations with class teachers indicated that students' organisation of their work varied: some worked together on all problems, others worked individually and used each other as a resource when they became stuck. A small number of students chose to work alone; they were encouraged to collaborate with their classmates but were not pushed into this if it was not their preference. In addition to work on booklet questions, a small amount of class time each week (perhaps half an hour in one of the classes) was used for other activities, including student presentations at the board. These were usually demonstrations of answers to booklet questions, which would be critiqued by the remainder of the class and commented upon by the teacher. An informal register was taken in the classes, and this information indicates that the vast majority of students reliably attended.

2.4.3 Students' responsibilities

The underlying *didactic contract* (Brousseau, 1997) in each course was therefore quite different. In the lecture course, the student's role was to attend carefully to the lecturer's presentation and take good notes. An ideal student would write down both the notes the lecturer makes and their additional helpful comments, read over their notes after the lecture and seek help with any parts they did not understand. The degree to which the students adhere to this model was not monitored: there was no requirement to attend the lectures and, other than the requirement to submit assignments, there is no system to enforce extra work.

The assignments for these students do not involve proving any of the major results of the course; this is done by the lecturer. They do involve using these results, sometimes in a relatively straightforward way, as in these questions about series convergence:

Use the Alternating Series Test to show that the following series are convergent:

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\left(\sqrt{n+1}-\sqrt{n}\right)$.

(b) $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{\left(n^{2}+1\right)\pi}{n}\right)$.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}n^{2}}{n^{3}+1}$.

(Lecture course, week 8 assignment, question B5)

In other cases, questions require more creativity in the use of known results to derive others, as in this case:

Let $s_{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ and let $t_{n}=1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{(-1)^{n+1}}{n}$.

Prove that $t_{2n}=s_{2n}-s_{n}$.

Deduce from [*theorem in notes*] that $t_{2n}\to\log 2$ as $n\to\infty$.

Deduce further that $t_{2n+1}\to\log 2$ as $n\to\infty$.

Hence prove that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\log 2$. (Hint: use [*question from previous sheet*]).

Each type of question is likely to be more difficult than those which students were exposed to in school. In this course the student is provided with statements of theorems, but not with "worked examples" to follow, even in cases like the first. They themselves are responsible for working out how to apply the new results in answering the questions.

In the new course, class time is spent not listening to a lecturer and taking notes, but working through booklets and answering questions. The role of the teacher is therefore quite different. They may occasionally take it upon themselves to go over their version of an answer, but primarily their role is shifted from "sage on the stage" to "guide on the side" (*cf.* Davis & Maher, 1997). Hence the students are responsible for proving most of both the major and minor results of the course. In particular, in addition to "applications" questions like those received by the lecture class, these students also have many more questions of the type seen above from Burn's book. This further example, from the week 8 workbook, requires students to use the definitions of convergence and absolute convergence in order to establish a general result, rather than to use that result to draw conclusions about specific series.

A series $\sum a_n$ for which $\sum |a_n|$ is convergent is said to be *absolutely convergent.* The argument used in qn 62 can be generalised to establish that a series *which is absolutely convergent is necessarily convergent.* The clue, as we have seen, is to construct the series formed by the positive terms and the series formed by the negative terms.

If we define $u_n = \frac{1}{2}\left(|a_n| + a_n\right)$, what values may u_n take?

If we define $v_n = \frac{1}{2}\left(|a_n| - a_n\right)$, what values may v_n take?

Now use the convergence of $\sum |a_n|$ to prove that $\sum u_n$ and $\sum v_n$ are convergent and hence that $\sum (u_n - v_n)$ is convergent. What is $u_n - v_n$ equal to?

To answer these questions the student must become more involved in proving results from definitions and building up a network of results.

In neither course are the students explicitly required to either memorize the material as they go along or to attempt to synthesize it. There are no formal closed-book tests during the course and both groups have either their notes or their previous workbooks available when they are working. In the case of the new course, results from the previous week are printed

at the beginning of the next workbook so the students have access to these despite having submitted their previous workbook for marking. Further, in both cases, requirements apply to the student body *as a whole*. In either course an individual may work essentially alone, or may submit answers that were produced either cooperatively with others or completely by others. Also, once work is submitted, both groups may rely on their supervisors to check their answers and provide final arbitration, although for those on the new course official marking covers only a subset of their answers and if they require help with reviewing others they must specifically request it.

2.4.4 Related research

In practical terms, the new course is quite different from a standard lecture course, although the degree to which this impacts upon any individual student depends upon that person's decisions in pursuing their own work. This section compares the new course with other reported pedagogical approaches.

First, note that the focus of the course remains the material of Analysis. It does not, for example, involve open-ended investigations like the course described by Hirst or recommended by Bell (Hirst, 1981, Bell, 1976). While the students are required to formulate answers to questions rather than absorb presented material, they have no say in the form or detail of these questions and they are not required to pursue any of the work in directions of their own choosing. Neither are they required to submit evidence of reflection on the process of developing this type of mathematics, unlike those students enrolled in Hirst's investigative course or Chazan's geometry course in which inductive and deductive proofs were specifically contrasted (Hirst, *ibid.*, Chazan, 1993).

Similarly, the course is not about problem solving per se like those discussed by Schoenfeld, Mason, Burton and Stacey, or Mohammad Yusof and Tall, in which Pólya's heuristics are adapted and taught (Schoenfeld, 1992 & 1998, Mason, Burton & Stacey, 1982, Mason, 1988, Mohd Yusof & Tall, 1996, Pólya, 1945). While it is hoped that the students on the

new course will develop strategies which enable them to work more efficiently and solve different types of problem, such heuristics are not specifically taught and the questions in the course are short and closed in relation to the type used in such courses.

The new course does have elements in common with "bridging courses", such as those described by Moore and recommended by Chazan (Moore, 1994, Chazan, 1993). It provides students with increased direct experience of working with the concepts of the course, in an environment in which discussion is encouraged. They thus have opportunities to contribute to the construction of what Yackel and Cobb call *sociomathematial norms* (Yackel & Cobb, 1996, Yackel, Rasmussen & King, 2000); they can try out their arguments and justifications and attempt to refine these if they do not meet the standards imposed by their peers and the teacher. Hence it may be hoped that they will more quickly become enculturated into the mathematical community, learning its values by experience more closely approximating apprenticeship than could be found in a lecture course (*cf.* Schoenfeld, 1992, Bishop, 1991). This hope is implicit in this extract from the "course philosophy" as described in a handout given to students at the beginning of the new course:

"In first term Analysis we will teach you very little and help you to learn a lot. Throughout the course there will be an emphasis on doing mathematics as opposed to just hearing about it or copying down someone else's notes. You will learn most of the material through thinking about and solving problems and sharing your insights with others."

However, as stated above, the focus of discussion in the classes remains on the material of Analysis rather than on proof itself.

The questions-based pedagogical approach is designed with something of the espoused philosophies of Movshovitz-Hadar and Leron in mind: Leron argues that proofs given in lectures often appear as very clever answers to questions which have never been asked

(Leron, 1985b). In this course the questions come first, which may contribute to Movshovitz-Hadar's goal that on learning something new, it is desirable if students not only *are* wiser but also *feel* wiser. That is to say, questions are raised in their minds and answers sought (Movshovitz-Hadar, 1988). It would be misleading to say that the questions in this course all arouse great curiosity or even necessarily a desire for proof, but they are raised and the students do seek answers for themselves.

The students' role is discussed further in the next chapter, which continues to relate this study to existing work by reviewing literature pertinent to the material of Analysis and to students' own beliefs regarding this role.

2.5 Main points of chapter 2

- Mathematics and mathematics-related students at Warwick usually attend courses involving three one-hour lectures per week, submit weekly assignments and receive additional help from a tutor and supervisor.

- The new course began against a background of concern about the standards of students entering university mathematics, and was made possible by support in the department and the availability of an appropriate textbook.

- Students on the new course attended two two-hour classes per week, in which they worked in small groups on a structured sequence of problems.

- Students on the new course had more responsibility than those on the lecture course for proving, as well as using, the results of Analysis.

- While the new course has elements in common with investigative and problem-solving-based courses reported in the literature, its focus is the material of Analysis rather than metacognitive issues or proof *per se*.

Chapter 3
Literature

3.1 Introduction

As described in the introduction, this chapter contains the first substantial review of relevant literature. The ideas covered here provide a basis for the interview questions used in the investigative study, as described in the next chapter on method. Some of these ideas, such as the notion of *concept image* and student conceptions of proof, will be revisited in chapter 6. There they will be related to the theoretical framework developed through part 2. Others, such as those referring to the student's interpretation of their role as a learner, will later inform the substantive theory developed in part 3. In this chapter, synthesis of the ideas presented is limited to those which are obviously linked by subject matter.

3.2 Content

This section covers literature related to the material of Analysis. It covers proofs, proving and definitions, then focuses more specifically on convergence and limits since these concepts will have a central role in the rest of the thesis.

3.2.1 Proof

The centrality of proof in the university curriculum, and the abruptness of the transition to this type of mathematics, has led to much mathematics education research in this area. This research can be divided roughly into three areas, each addressing one of Bell's three stated functions of proof: *verification, illumination* and *systematisation* (Bell, 1976). This section looks at illumination, that is to say, whether a proof leads to increased understanding of a

result for a given audience. Illumination corresponds to *explaining* in Hersh's assertion that proving is *convincing* and *explaining* (Hersh, 1993). The next section discusses verification, that is, what brings conviction for an individual. Systematisation, referring to the role proof plays in building mathematical theories, is discussed to some degree in the section on definitions and is expanded upon further in chapter 6.

Various authors discuss the idea that a proof should be not only correct but also illuminating. Some point out that the debates over the status of computer-assisted and probabilistic proofs attest to the importance of this aspect for mathematicians: there is dissatisfaction if a proof must be believed correct without it providing any further insight or global view of the problem or theorem (Hanna, 1989, Hersh, 1993). Then, naturally, there is particular concern that proofs should be illuminating *to students*, from both researchers and students themselves (Chazan, 1993, Hoyles, 1997). Various types of proof have been dissected, investigated for their particular inherent difficulty and reformulated in attempts to make them more accessible.

Leron, for example, notes the cognitive strain associated with working in a "false world" in a contradiction proof. He advocates alleviating this by performing the appropriate positive construction first, then considering the negative assumption, at which point the impossibility of this statement should be obvious (Leron, 1985a). Tall's suggestion that the use of generic examples can be helpful is based on a similar constructive principle (Tall, 1995). Leron further advocates the standardisation of good practices in exposition. He suggests that the presentation proceed in a top-down fashion, first identifying the pivotal required construction and then moving to more detailed levels in making this construction. This, he suggests, places the emphasis on the structure of the proof, which is often difficult for students to identify from a standard presentation (Leron, 1985b).

Tall goes further with the idea of adapting the type of proof to the level of the student, arguing that formal proof is one of only many possible types. He describes *enactive* proofs

involving some physical action, *visual* proofs involving imagined actions or re-focusing attention, *manipulative* proofs using algebra and *generic* proofs using a specific statement as typical of a class of statements. He argues that any of these might be appropriate at different times for allowing insight into and understanding of a result, so that although they may not conform to the standards of the mathematical community, they can be appropriate for learners at different stages of development (Tall, 1995).

3.2.2 Proving

This section discusses Bell's *verification*, which corresponds to Hersh's *convincing* (Bell, 1976, Hersh, 1993). What brings conviction or a sense that a result has been verified varies with the individual and the situation. It is not fixed even in mathematics; as Thurston points out, mathematicians do not produce proofs written in formal predicate calculus (Thurston, 1995). In reality, a sense of the authority of the author, the importance of the result and its sitting well with other known results can influence the acceptance of a proof (Hanna, 1989, Hanna & Jahnke, 1993).

Here we are interested in what brings conviction to a student and how this leads them to behave in mathematical situations encountered at beginning undergraduate level, where the standards are relatively fixed. Harel and Sowder have produced a typology of *proof schemes* that a student may hold (Harel & Sowder, 1998). Since this typology is comparatively exhaustive, this section follows its structure, relating the various schemes to other research and exploring links between what students offer by way of proof and what they judge to be correct when a proof is provided. It also touches on student beliefs, such as the perceived role of authorities; such ideas are discussed in more detail in section 3.3.

The first broad classification Harel and Sowder make is that of *external conviction* proof schemes, in which conviction for the student comes from some source other than the content of the proof itself. In the first such scheme, conviction is derived from the ritual of argument presentation. Hoyles refers to this phenomenon in her speculation that curricula in

which proof appears only in a particular type of investigative work can lead to students mimicking the standard format more or less meaninglessly when proof is required (Hoyles, 1997). Similarly Segal has noted that both students and mathematicians are liable to accept incorrect proofs which are written in an appropriate style, (Segal, 2000), and Leron gives this excellent example of "writing nonsense in a mathematical style": asked to prove that the sets N and E (the natural and even numbers) are equivalent, a student wrote:

"Define a function $f : N \to E$ by $f(n) \in E$ for all $n \in N$. Then f is one-to-one since $f(m) = f(n)$ implies $m=n$. Also, f is *onto*, for given any $e \in E$ let $n \in N$ satisfy $f(n) = e$; then n is a source for e."

(Leron, 1985b)

In the second external conviction scheme, doubts are removed by the word of an authority. It is recognised that students in all disciplines must liberate themselves from the view that "right answers" exist and should be provided by authorities (Perry, 1970 & 1988). This is liable to occur at a delay in mathematics since a curriculum which deals in "truth rather than reasons for truth" (Harel & Sowder, 1998) allows students to maintain beliefs that mathematics is about learning methods to quickly answer a given range of questions (Skemp, 1976, Schoenfeld, 1992). For students with such beliefs, the external authority has an essential place as the disseminator of these methods.

In the third external conviction scheme, conviction is derived from symbolic manipulation according to standard algorithms. Facile symbol manipulation provides vast potential for compression in mathematics and therefore for alleviation of cognitive load (Harel & Kaput, 1992). However, if the meanings of the symbols used are not sufficiently well understood this can lead to errors. Harel and Sowder's cite an example of a student who, in working with the matrix equation $x_1 A_1 + \cdots + x_n A_n = 0$, writes $x_1 = -\dfrac{x_2 A_2 + x_3 A_3 + \cdots + x_n A_n}{A_1}$

(Harel & Sowder, 1998).

Harel and Sowder's next set of proof schemes are characterised as *empirical*. In the first, conviction is attained inductively; the student considers that evidence from a number of examples constitutes proof. Fischbein uses the similar term *empirical inductive conviction* to describe conviction resting on this type of evidence (Fischbein, 1982). Arguing by reference to examples is a widely noted phenomenon throughout the literature on student-produced proof; Balacheff suggests that the student often behaves as practical man rather than mathematician, seeking only sufficient evidence for a conjecture to be largely true or useful (Balacheff, 1986). Similarly Galbraith notes that younger children will judge a conjecture to be "partly right" rather than rejecting it in the face of a counterexample (Galbraith, 1981). Galbraith, Bell and Chazan all discuss inductive justifications produced by students, with adequacy ranging from minimal consideration of limited cases to more sophisticated strategies such as checking a result for every known subclass of the objects under consideration (Galbraith, 1981, Bell, 1976, Chazan, 1993).

The relationship between empirical evidence and deductive proof also gives rise to interesting phenomena. Several authors observe that in addition to offering inductive evidence in place of proof, students are known to continue checking for confirmatory evidence after a deductive proof has been presented (Chazan, 1993, Fischbein, 1982). This may even be the case when the student herself is responsible for the original production of the proof: Duffin and Simpson report on a case in which a girl produces an algebraic argument but apparently continues to think of the variables used as representing the particular numbers she started with, later checking for other cases (Duffin & Simpson, 1993). Development towards more sophisticated awareness of the relative status of inductive and deductive arguments is evident in a study by Segal. She finds that through their beginning years at university, students increasingly rated inductive arguments as convincing but *not* as valid, showing some appreciation of the agreed standards of mathematical argumentation (Segal, 2000).

Empirical proof schemes also encompass those characterised as *perceptual*, in which conclusions are based on observation of static examples. Conviction derived in this way would be termed *intuitive intrinsic* by Fischbein, resting as it does on affirmatory intuition that a statement is self-evident (Fischbein, 1982). Various authors have made the case for increased value to be accorded to visual argumentation in mathematics, as it already plays a large role in the investigation of new problems (Dreyfus, 1991, Neubrand, 1989). Dreyfus however notes that there are problems inherent in this type of reasoning, since standards for this type of argument have not been so well developed within the community as they have for algebraic proofs (Dreyfus, 1991). Tall is more specific, pointing out that even an algebraic argument derived from a diagram may only be valid for the range of examples for which that diagram is prototypical (Tall, 1995). Presmeg's extensive work on visualisation in mathematics confirms this observation; she investigates the fact that while visual imagery has mnemonic advantages and can be used to great effect, students making use of it are liable to focus on irrelevant detail or show inflexible thinking related to a particular concrete image (Presmeg, 1986a&b). These issues will be discussed in more detail in chapters 8, 9 and 10, which describe the emergence in this study of an inclination towards visual or nonvisual reasoning as a causal factor in the development of a student's understanding of Analysis.

Harel and Sowder term their most sophisticated proof schemes *analytic*. Proofs generated within such schemes may rely on images, but in this case these are dynamic rather than static, resembling Tall's visual proofs involving imagined actions or re-focusing of attention (Tall, 1995). Presmeg too observes that students who make good use of imagery may use it in this way (Presmeg, 1986b). Her work suggests that those who use imagery appropriately will frequently translate back and forth between visual and nonvisual representations, thus resembling Krutetskii's *harmonic* thinkers who are able to take advantage of both types of representation (Presmeg, 1986b, Krutetskii, 1976). In chapter 9 it will be demonstrated that the most successful "visual" students in this study also reason in this way.

Finally, proof schemes may make use of axiomatic systems. In such systems it is necessary to suspend judgement of the truth of a conjecture until this has been proved within the theory, and this is notoriously difficult. It requires strong discipline of mind to prove a result from statements with arbitrarily assigned preferred status, particularly in cases in which the result is considered more self-evident than those statements (Harel & Tall, 1991, Fischbein, 1982). Harel and Sowder make a further distinction between those who still think of the axioms as describing objects which are real in some way, and those who can genuinely think of them as forming part of a formal system without the necessity for a meaning beyond this (Harel & Sowder, *ibid.*; also *cf.* Hanna, 1989). Again, examples will be seen in chapter 9.

3.2.3 Structure, language and logic

Where some students can cope with abstract systems in these sophisticated ways, for others the structure of mathematics remains opaque. In addition to the problems of working within axiomatic systems, students may fail to appreciate logical direction in theorem statements. This phenomenon is prevalent for existence or nonexistence theorems, and can result in students invoking such theorems inappropriately (Hazzan & Leron, 1996). A substantial contributory factor to such breakdowns in logical reasoning is the use of language in mathematics, and particularly the difference between this and its use in everyday life. Logical connectives are used much more strictly in mathematics: Zepp, Monin and Lei attribute misinterpretations of "or" statements to the fact that these may be inclusive or exclusive in everyday speech. Similarly they attribute underappreciation of the inherent direction of "if...then" statements to the everyday use of these to indicate if and only if (Zepp, Monin & Lei, 1987). Dubinsky, Elterman & Gong discuss a course developed to improve student facility in handling universal and existential quantifiers, with particular emphasis on building up to complex statements such as those associated with the definitions of Analysis (Dubinsky, Elterman & Gong, 1988). They emphasise the encapsulation of subclauses that must be achieved for such statements to be understood.

Without appreciation of these linguistic conventions, students can struggle to handle mathematics at university level. Selden and Selden discuss the essential place of inferring "surpressed quantifiers" in interpreting theorem statements and thus in being able to ascertain whether a theorem is proved by a given argument. For example in the statement "differentiable functions are continuous", a universal quantifier is understood by convention (Selden & Selden, 1995). In this study the most important type of such standardised language will be the use of "show that this sequence is convergent" to mean precisely "show that this sequence satisfies the definition of convergence". This thesis argues, following Alcock & Simpson, that this simple suppression is central to the difficulties in making the transition to university mathematics (Alcock & Simpson, 1999). The argument is presented in part 2.

3.2.4 Definitions

In order to use definitions in proofs, students need to learn them, and often they do not (Moore, 1994, Bills & Tall, 1998). Moore observes that students often find it difficult to understand verbal and symbolic definitions before they have had sufficient experience with examples, and that their understanding of images, definitions and associated procedures often remains much less integrated than that of their teachers (Moore, 1994). Dahlberg and Housman discuss this disparity between a student's and a lecturer's experience in terms of the number of examples a more experienced mathematician has access to. They investigate student reactions to new definitions, observing that while some generate examples to assist them in reasoning about the defined concept, others work with the symbolic form of the definition in isolation (Dahlberg & Housman, 1997). Students who do this necessarily miss out on the advantages of having multiple representations for a concept: Winicki-Landman and Leikin's study on equivalent definitions shows that students find it more natural in some circumstances to reason about equivalence of sets rather than the definitional properties themselves (Winicki-Landman & Leikin, 2000).

In cases where the use of a definition is restricted to particular symbolic tasks, the status of definitions in mathematical theories can become obscured. Alcock and Simpson observe that while definitions have a dual role (an object is a member of a mathematical set *if and only if* it satisfies the property associated with that set) they are likely to be used in only one direction when first introduced (Alcock & Simpson, 1998). For example, students often initially gain practice in using the definition of convergence for a sequence by showing that example sequences are convergent, which uses the only the "if" direction (Davis & Vinner, 1986). In this situation students may assimilate the symbolic forms of the definition as associated with this procedure, failing to appreciate that they are also used as a starting point in proving results about *all* convergent sequences (Alcock & Simpson, *ibid.*). Any pedagogical strategy falls victim to similar problems since concepts cannot be assimilated as pre-formed wholes – one has to start somewhere (Davis & Vinner, *ibid.*). However *didactical obstacles* such as this often arise when knowledge that is useful in some circumstances proves inadequate in others (Cornu, 1992, Sierpinska, 1987).

When experience with examples is available, especially if it is extensive and occurs before the introduction of the definition, almost exactly the opposite didactical obstacle arises. In such cases students may reason using a *concept image* built up from this experience, where the term indicates the "cognitive structure" in an individual's mind which is associated with a given concept (Tall & Vinner, 1981). This phenomenon is observed by Vinner in the case of students who make misassessments of whether given examples are functions. Students often do not use the definition in making these assessments even though they may be able to state it. Instead they apparently compare new examples with some informal idea of the form a function should take. Common beliefs are that a function should be represented by a single formula or that its graph should be continuous; it is easy to see how these could be derived from a student's initial experiences of the function concept (Vinner, 1983). Analogous phenomena with respect to the concepts of rational and irrational numbers are reported by Pinto and Tall (Pinto & Tall, 1996).

Vinner suggests that when a definition is introduced for an already familiar concept, the concept image may change in response to this, or may remain as it is with the definition being stored separately and perhaps evoked only when it is specifically requested (Vinner, 1992). One result is that a student may have a concept image which is inconsistent with the formal definition, since both need to be evoked simultaneously for such conflicts to be recognised (Tall & Vinner, 1981). Pinto and Tall discuss further variations on the possible relationships between an individual's concept image, the formal concept definition and the way these are used. They note that a student may have:

- *informal imagery* not deduced from a definition, which may be further subdivided into imagery *consistent* or *inconsistent* with the formal theory,

- *distorted imagery* produced from a distorted personal definition or by faulty reasoning from a correct definition,

- *pseudo-formal imagery*, which may seem consistent with formal theory but is not ultimately deduced from the definition by formal reasoning,

- *formal imagery* deduced formally from the definition.

(Pinto & Tall, 1996).

Examples of students using these types of imagery will be seen in chapter 9.

Another result is that where a teacher would wish a student to address a given task by consulting the definition, either in isolation or with additional input from their concept image, the student may evoke some part of their concept image and use this alone (Vinner, 1983 & 1992). Vinner attributes this to students' previous experience with definitions being limited to the type used to introduce concepts. In these cases, once a concept image is established the definition remains inactive or may be forgotten. Indeed, many everyday concepts do not have definitions in the technical sense (Vinner, 1992, Skemp, 1979a). Vinner argues that the layman views lexical definitions as statements of fact about the

world, and that students may interpret mathematical definitions in this way rather than as generative of formal concepts (Vinner, 1976).

Fundamental to the discrepancy is that when definitions are interpreted as lexical, the objects remain primary. In mathematics, formal definitions have primacy and formal concepts are constructed by deduction from these (Tall, 1995, Gray et. al., 1999). Since students only see the construction part of this process (Tall, 1995), the arbitrary nature of the mathematical definition is obscured (Vinner, 1977) as is the decision about precisely which objects are to be included as instances of the concept (cf. Lakatos, 1976, Balacheff, 1986). This aspect of the mathematical enterprise is not usually explicitly discussed in mathematics curricula (Vinner, 1976) and there have been suggestions that students should be set tasks encouraging them to recognise it. Wood recommends that students should examine definitions from outside mathematics, for example legal terms, in conjunction with mathematical definitions, in all cases considering their audience, efficacy in promoting understanding of a concept and utility in a range of circumstances (Wood, 1999). Winicki-Landman and Leikin describe a course in which teachers were encouraged to consider equivalent definitions and their relative appropriateness in introducing geometrical concepts. They report some success in terms of improved awareness of these issues (Winicki-Landman & Leikin, 2000).

3.2.5 Convergence and limits

This section addresses manifestations of the above phenomena in the case of limits and convergence. Like *function*, these concepts are vulnerable to the didactic transmission of cognitive obstacles (Cornu, 1992). In particular, early experience with the limit concept may lead students to focus on the manipulative aspect of working with algebraically expressed sequences, coming to believe that "finding the limit" is central to the theory of limits, and/or that a sequence must be expressed by a single "formula" (Davis & Vinner, 1986, Sierpinska, 1987).

The student's conception of infinity may also form an epistemological obstacle to the acquisition of the limit concept. Tirosh observes that students often hold notions of infinity which are incompatible with the formal Cantorian notions of cardinal and ordinal infinities, as a consequence of our minds' natural adaptation to finite objects. Both she and Tall note that misconceptions such as the idea that 0.999... is an infinitesimal distance from 1 are only misconceptions relative to the existing formal theory; in nonstandard analysis such ideas are entirely legitimate (Tirosh, 1992, Tall, 1980). Sierpinska considers in detail the interaction between students' conception of infinity, their beliefs about mathematics and their understanding of the limit concept. She describes beliefs such as the "potentialist" model in which the limit is approached in time, so it is not reached because of the impossibility of running through infinity in a finite time (Sierpinska, 1987).

Such obstacles are compounded by yet another contributory factor: the use of the terms "converges", "tends to", "approaches" and "limit" in everyday life (Monaghan, 1991). The everyday meanings of these terms mean that the concept image is built up not only by experience with the mathematical concepts, but also under the influence of *spontaneous conceptions* derived from these meanings (Cornu, 1992, Davis & Vinner, 1986). Hence "limit" is often associated with notions of speed limit (a theoretically impassible point), "approaches" and "tends to" with dynamic conceptions involving movement towards an object which may or may not be reached, and "converges" with continuous objects which come nearer and in most cases touch (Monaghan, 1991). In addition a variety of dynamic and static conceptions may be attributed to more formal teaching. For example Robert reports dynamic models such as "a convergent sequence is an increasing sequence bounded above (or decreasing bounded below)" and "the values approach a number more and more closely", and static models such as "the elements of the sequence end up by being found in a neighbourhood of *l*" (Robert, 1982, cited in Cornu, 1992).

Cottrill et. al. report an attempt to address the difficulties associated with learning the limit concept in the context of continuous or discontinous functions. They utilise a research

paradigm based on establishing a *genetic decomposition* of the constructions a learner needs to make to understand the concept, and providing learning environments designed to foster these constructions (Cottrill et. al., 1996, Asiala et. al., 1997). The underlying theoretical framework is *APOS theory*, which posits that mathematics is learned through a sequence in which *A*ctions performed on specific objects are interiorised to form *P*rocesses which can be imagined operating in general; processes are then encapsulated to form mathematical *O*bjects and all of these may be coordinated into a *S*chema for a given concept (Dubinsky, 1992). Process-object theories in general will be discussed in more detail in chapter 11, which discusses their power in explaining the difficulty of certain mathematical concepts and uses them to explain the behaviour of certain students in this study.

In the study by Cottrill et. al., students received instruction using computer environments. Among various tasks, they were required to write short programs to evaluate a function at a set of points successively closer to a given point, and to investigate $\varepsilon - \delta$ "windows" around a point (Cottrill et. al., *ibid.*). Cottrill et. al. report limited success in improving student understanding of the limit concept, and discuss revisions of their genetic decomposition based on student responses to the instructional programme. For example, they introduce a new initial step to help students for whom the process of finding the limit at a point was equated with the single action of evaluating the function at that point. In limits of sequences, this corresponds to the idea that the limit of the sequence (a_n) is found by evaluating a sort of "a_∞" term (Davis & Vinner, 1986). Overall, such an approach is not likely to lead to complete success when the concept concerned is a formally defined one. As discussed, definitions have primacy in formal theory and formal objects are constructed from these (Harel & Tall, 1991). As such, any experience with specific examples, while it may extend a student's concept image and perhaps bring it into line with the formal definition, does not address the extra step required to change the way in which the student reasons about the concept (Vinner, 1992, Gray et. al., 1999, Alcock & Simpson, 1999). Characterisation of this extra step, and examination of student behaviour in tasks that require it to have been made, are central in the remainder of this thesis.

For the time being, note that in order to achieve this step the student must realise that their existing thinking is inadequate according to the mathematical community. This is nontrivial. First, in order to become aware of a conflict between their own understanding and the formal concept, the learner must develop a *concept definition image* (Tall & Vinner, 1981). To do this they must overcome difficulties in comprehending the complex quantified statement which forms the definition of convergence (Dubinsky, Elterman & Gong, 1988). In order to then recognise the conflict, this concept definition image and their contradictory images must be evoked simultaneously. Hence a significant compression or "chunking" (Baddeley, 1997, Ashcraft, 1994) must have taken place for these to be held in their mind and then compared[2]. Even if this is achieved and the concept image is reconstructed appropriately, earlier conceptions will not be abandoned and may still be retrieved and used inappropriately (Davis & Vinner, 1986, Davis, 1980). However, conflict between existing knowledge and new information may not be recognised, or if recognised may not prompt the reorganisation of the existing knowledge (Sierpinska, 1987 & 1990, Duffin & Simpson, 1993). Occurrences of this sort and corresponding explanations are discussed in the next section.

3.3 The role of the learner

The way in which a student responds to conflict between their existing knowledge and new information is best viewed in the context of their wider beliefs about their role as a learner of mathematics. This subject was broached in section 3.2.2 on proving, which touched on the relationship between the learner and the authorities. Similarly, sections 2.4.3 and 2.4.4 discussed the students' responsibilities on the two courses and compared this with other research on pedagogical strategies. This section reviews literature regarding this relationship and other beliefs; chapter 8 refines this overview by describing the ways in which different combinations are evidenced in this study.

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[2] Although the term "chunking" is more commonly associated with information-processing models of human thought, the notion is compatible with the more constructivist approach taken in this study.

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3.3.1 Approaches to learning mathematics

Extensive research by Perry reveals dynamics of the relationship with authority for college students of all disciplines (Perry, 1970). Perry describes a nine-step developmental scheme through which students come to a mature set of "philosophical assumptions". The nine positions are divided into four main stages. *Dualists* believe that there are truths in the world and it is the teacher's job to present these truths for them to learn. *Multiplists* recognise that sometimes there are no right answers, and believe that in such cases all opinions are equally valid. *Relativists* realise that it is necessary to be critical of their own thinking. The final stage Perry calls *Commitment*, in which the student becomes aware that they will have to make choices in an uncertain world, and, eventually, that this will always be the case (Perry, 1988).

Resistance to taking responsibility for making relativistic decisions is well documented in literature on mathematics learning. The mathematics curriculum, with its emphasis on "truth rather than the reasons for truth" (Harel & Sowder, 1998), seems to make it viable for a student to maintain a dualistic perspective into later educational years. Students want to know "the answer": they resist attempts at explanation, wishing to be told "what to do" (Skemp, 1976, Dreyfus, 1991), and may consider proof *per se* redundant, relying on the word of a teacher to provide conviction (Harel & Sowder, 1998). This resistance, sometimes culminating in angry altercations between students and their teachers, is recognised sympathetically by Perry and explained in terms of the trauma engendered by accepting changes to one's world-view: such changes almost inevitably bring greater insecurity along with greater freedom (Perry, 1970 & 1988). Copes also recognises the difficulty of accepting such changes, and places emphasis on the difficulty of recognising the need for them. He likens each stage in Perry's scheme to a lens through which the world is perceived, meaning that contradictory information may not be recognised or understood, and that only when a certain amount of information peripheral to the current view has accumulated is this likely to precipitate a change (Copes, 1982). Like Perry, he emphasises

the fact that any classroom is likely to include students at a number of these stages, and he suggests ways in which tasks in mathematics can be set at different levels in order to accommodate this and encourage development.

A dualistic perspective in mathematics is compatible with what Skemp calls *instrumental learning*. This is the behaviour associated with rote learning and which leads to *instrumental understanding*; the student attempts to learn only *how* to follow mathematical procedures without understanding *why* they work (Skemp, 1976). This type of "understanding" is recognised as such by Skemp since many students and their teachers use the term with this meaning. In addition, the successful use of such routines means that they cannot have been stored in isolation (Byers, 1980). However the inflexible nature of knowledge acquired in this way restricts its use to situations for which a specific procedure has been taught (note the importance of an authority here) (Skemp, *ibid.*). Where transfer does occur, a student's answers may contain "bugs" deriving from "spurious generalisation" of known procedures (Nickerson, 1985). This can also occur due to retrogressive corruptions in memory, for instance when students begin to add fractions by adding the numerators and the denominators only after learning the analogous procedure for multiplication (Byers & Erlwanger, 1985, Davis, 1980, Nickerson, *ibid.*).

This dualistic or instrumental view of mathematics is supported by instrumental explanations from teachers and in textbooks (Skemp, 1976, Schoenfeld, 1992). Consequential beliefs amongst students include the convictions that any mathematical question should be solvable using a specific procedure, and that producing such a solution should only take a few minutes if one knows this procedure (Schoenfeld, *ibid.*, Mohd Yusof & Tall, 1996, Mason, Burton & Stacey, 1982). Correspondingly students report anxiety when faced with unpredictable questions (Mohd Yusof & Tall, *ibid.*, Skemp, 1971), and engage in counterproductive behaviour in such situations: they often pursue unproductive attempts to follow up an initial idea with little or no consideration of where this will lead, and without monitoring their progress towards a solution (Schoenfeld, 1992). In line with

the view that "mathematics is what is left over after problem solving activity" (Davis, 1992) courses have been designed to address this anxiety and improve metacognitive awareness of the problem-solving process by emphasising the investigative aspect of the mathematical enterprise (Mason, Burton & Stacey, 1982, Hirst, 1981, Schoenfeld, 1998). Success is often evident in a maturing of student attitudes in response to such experience (Hirst, *ibid.*), but can be mitigated on return to a standard curriculum in further courses (Mohd Yusof & Tall, 1996, Meel, 1998). In addition, Copes' lens metaphor explains reports that attempts to teach problem solving heuristics can easily be subverted by students who learn *these* instrumentally (Copes, 1982, Mohd Yusof & Tall, *ibid.*, Gray et. al., 1999). In such cases, rather than resulting in more flexible thinking, problem-solving experiences simply leave the student with more routines to remember (Gray et. al., *ibid.*).

Instrumental learning is viable because the learner often gains immediate success in the use of a new procedure, enabling them to reach the "right answers" and thus to gain the external approval which is the measure of success to a dualist (Skemp, 1976 & 1979b, Perry, 1970). However they face a difficult task in the longer term, since this type of learning necessarily involves much of what Harel and Tall term *disjunctive generalisation*: as with the problem-solving heuristics above, when a new type of situation is encountered, a new procedure is learned separately rather than as a generalisation or special case of a previously recognised relationship (Harel & Tall, 1991). This means that instrumental learners are eventually more liable to suffer *cognitive strain* than Skemp's other type, *relational learners* (Gray et. al., 1999, Skemp, 1976). Relational learners are those whose goal is to acquire understanding both of how to perform mathematical operations and of why these methods work (Skemp, 1976, 1979b). This means that routines are not learned in isolation but related to more general mathematical relationships (Byers & Herscovics, 1977). This type of understanding is initially harder to acquire as it often involves *expansive* or *reconstructive* generalisation; existing knowledge must be reconceptualised as applying more widely, or modified in response to new information (Harel & Tall, 1991). However it is ultimately easier to remember as any piece of knowledge forms part of a connected whole (Skemp,

1976). This also means that those who learn relationally are more likely to be able to make use of *reconstruction* in another sense: they are able to forget the details of much of their knowledge since these can be reconstructed from the more general relationships (*cf* Duffin & Simpson, 2000). Such effects of different learning strategies on student understanding will be examined in part 3.

3.3.2 Response to cognitive conflict

The notion of the reconstructions sometimes required in relational learning brings us back to the significance of conflict between new information and existing knowledge. Duffin and Simpson introduce the notion of three types of experience a learner may encounter: a *natural* experience, which makes sense in terms of their existing mental structures, a *conflicting* experience, which may contradict existing knowledge or be explained in different ways by different structures, and an *alien* experience, which appears to have no relationship to these structures (Duffin & Simpson, 1993). Simpson later suggests that the terms natural and alien might be used to caricature types of learner, with "natural" learners always attempting to make sense of their experiences by connecting them to existing structures, and "alien" learners being more willing to build up isolated structures dealing with just those experiences, linking these structures later in response to conflict (Simpson, 1995a). The distinction has similarities with that between relational and instrumental learners, but this should not be carried too far: a student who does not understand the internal steps of procedures but who links them through the observance of similarities and differences might be described as instrumental and natural, and one who learns areas of mathematics separately but with meaningful internal connections as relational and alien.

The essential notion is the role of conflict in the learning of mathematics. Duffin and Simpson consider and reject the idea that conflict should be avoided: in keeping with other authors they view it as historically and epistemologically inevitable (Duffin & Simpson, 1993, Davis & Vinner, 1986, Tirosh, 1992). Instead their focus is on the way a learner recognises and responds to conflicting experiences. As alluded to in the previous section, a

conflict only exists relative to a person who recognises it. Hence where a teacher sees a conflict a student may see none, or vice versa (Balacheff, 1986). Persuading a learner to recognise a conflict may thus be highly nontrivial: people are naturally unwilling to consider their existing knowledge erroneous or inadequate and can exhibit considerable ingeniousness in avoiding this conclusion (Tall & Vinner, 1981, Davis, 1980). This is not unreasonable in view of the fact that this knowledge may previously have served them very well; it is well recognised that knowledge with this status can act as a cognitive obstacle at a later stage (Cornu, 1992, Davis, 1992, Davis & Vinner, 1986).

Alternatively, a learner may recognise a mathematical conflict but remain unperturbed by this, as in the case reported by Duffin and Simpson in which a boy calculates

$$
\begin{array}{r}
526 \\
-249 \\
\hline
323
\end{array}
$$

and writes "but the real answer is 277" (Duffin & Simpson, 1993). In such cases the conflict remains unresolved due to the student's interpretation of the *didactical contract*: they see their role in the classroom as being to follow the given instructions and do not necessarily recognise that anything is wrong if these give unexpected answers (Herbst & Kilpatrick, 1999). In effect some of the procedures learned are therefore assimilated in an alien way, with new structures being built up for dealing separately with such situations.

So conflicts, and the opportunities for improved understanding that they provide, may go unnoticed by learners. It is largely inevitable that this will happen on occasion, as may be seen by considering Davis' "jigsaw metaphor". He suggests that a new piece of information will be useful to a learner if it immediately fits into a known gap or if it can be placed in the correct general area, but if it has neither of these properties it is liable to be ignored or at least quickly lost as it is not integrated into the existing cognitive structure (Davis, 1992). It is also inevitable that reconstructions in response to conflict will often be difficult, at the level of small pieces of knowledge as well as at the level of Perry's philosophical

assumptions. Duffin and Simpson liken such reconstructions to the learning of a new tennis serve: while they may produce improved performance once mastered, there may well be a period of adjustment during which competence is reduced and errors are made (Duffin & Simpson, 1993). Such periods demand sensitivity on the part of teachers, who should be aware of when to encourage further reassessment and when to offer support but essentially allow the student to resolve these difficulties for themselves (Duffin & Simpson, 1993, Davis, 1992, Maher & Martino, 1996). Hence the role of the teacher remains central, but is quite different from that anticipated by the instrumental learner with dualistic beliefs.

3.4 Understanding

The previous section touched upon the ways in which a student's approach to learning may influence the understanding they are able to acquire. The intention of this study was to investigate the understanding of Analysis developed by the students attending the two different courses, and discussion of the term recurs throughout the thesis. The next chapter, on method, examines the use of qualitative methods in assessing understanding, and discusses in the abstract the refinement of the term towards its eventual use in this study. This eventual use is described in chapter 7, where it is integrated with literature on typologies of understanding. The present section introduces the topic with an overview of literature on understanding, considering different authors' uses of the term.

3.4.1 Usage of the term "understanding"

"Understanding" is a slippery concept. Byers says that "...the word 'understanding' is often used rather loosely; any attempt to endow it with the degree of precision expected of a technical term encounters a number of difficulties" (Byers, 1980). This is reflected in the different senses in which it is used.

Some authors use "understanding" as a verb, in the sense of *coming to understand*, as when Skemp says that to understand something means to assimilate it into an appropriate schema (Skemp, 1971), or Davis states that understanding occurs when a new idea can be fitted into

a larger framework of previously assembled ideas (Davis, 1992). This is compatible with Tall's emphasis on the dynamic aspects of understanding (Tall, 1978), Pirie and Kieren's analysis of stages a student goes through in coming to understand a new piece of mathematics (Pirie & Kieren, 1994), and Sierpinska's conclusion that understanding should be thought of as an *act* (Sierpinska, 1990 & 1994). This conception is also implicit in Davis' argument that we can learn more from asking how a student's thinking develops over a period of years than by taking "snapshots" of the state of their understanding at any given moment (Davis, *ibid.*).

Other authors have used "understanding" as a noun, describing a state or structure existing in the mind. Nickerson, for example, suggests that "one way of thinking about understanding evokes the notion of expertise. One understands a concept (principle, process, or whatever) to the degree that what is in one's head regarding that concept corresponds to what is in the head of an expert in the relevant field" (Nickerson, 1985). This conception is implicit in characterisations of understanding based on mental links (Skemp, 1979a). In this case understanding is conceptualised as being a matter of degree, with concepts which are better linked with other cognitive structures being better understood (Byers, 1980). In particular, having access to multiple representations of the same concept and the ability to translate flexibly between them is seen as an indicator of good understanding (Nickerson, 1985).

The nodes and links analogy is central to Skemp's distinction between *instrumental* and *relational* understanding. As discussed in chapter 3, the distinction is that between knowing how, and knowing both how and why. Skemp further characterises the different networks of knowledge involved in these types, suggesting an analogy with a street map: an individual with instrumental understanding has access to a finite number of plans for getting from certain points to certain others, whereas one who understands relationally can flexibly generate a potentially much larger number of plans for dealing with a wider range of problems (Skemp, 1976, 1979a). In response to Skemp's classification a number of authors

suggested extensions or modifications (Byers & Herscovics, 1977, Buxton, 1978, Backhouse, 1978), and Skemp himself incorporated these to produce a more detailed typology integrating an analysis of the learner's goals (Skemp, 1979a,b). All of these are discussed in chapter 7, which relates these types to the theoretical framework developed in part 2 and establishes their meaning within this study.

3.4.2 Observing understanding

Both verb and noun uses of the word "understanding" appear in Duffin and Simpson's discussion, where they are distinguished as *building* and *having* understanding (Duffin & Simpson, 2000). Duffin and Simpson also introduce the idea of *enacting* understanding in discussing how we might observe understanding in a student[3]. The building of new understanding is often recognised by the "aha!" moment, and having understanding is also associated with affective responses. Duffin and Simpson list:

> When I understand

- I feel comfortable

- I feel confident

- I feel able to forget the detail, feeling sure that I can reconstruct it whenever I need it

- I feel that the thing belongs to me

- I can explain it to others

(Duffin & Simpson, *ibid.*)

However as Nickerson points out, "sometimes we are embarassed to discover that something we thought we understood well, we really did not understand at all" (Nickerson, 1985). In this study we will see students who have a sense of understanding, although the way in which they understand is quite removed from the expert view. Since approximation

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[3] They do not consider these as necessarily occuring separately in time, stating that: "in solving a problem a learner may use some recalled facts, enact some of their understanding, get stuck, find and resolve conflicts by building new connections, enact the understanding inherent in those new connections, bring in more recalled ideas and so on."

to an expert view is the goal of the teaching on both courses, affective aspects of understanding belong to a different sphere and will be discussed in chapter 8 as part of a characterisation of student beliefs.

Hence the study primarily relies upon inferring students' understanding as they enact it. As described above, the mathematics education literature provides a good initial knowledge of the types of behaviour students exhibit at this level. This means that questions can be written which allow students who possess an approximately expert understanding to demonstrate this, while highlighting misconceptions on the part of those whose understanding is weaker. The questions used in this study are detailed in the next chapter.

3.5 Main points of chapter 3

- There is concern that proofs should be illuminating, and strategies have been suggested for making different types of proof more accessible to students.
- For a student, conviction of the truth of a result may be derived from sources other than deductive proof.
- Students may make errors because they are unaware of linguistic conventions in mathematics.
- Students often do not know important definitions, and may learn to work with them only in restricted ways.
- Students often reason using their concept images (perhaps derived experience with examples) rather than formal definitions, even when they do know these definitions.
- Student conceptions of the limit concept may be influenced by their understanding of infinity and by spontaneous conceptions derived from everyday use of the term.
- The logical complexity of the definition makes the formal limit concept difficult to acquire.
- Students often believe that mathematics is about learning a fixed set of procedures. This approach can bring short term success but in the long term is likely to lead to cognitive strain.

- Individuals may ignore conflicts between their existing knowledge and new experience.
- The term "understanding" is used with various emphases (here the emphasis will be upon comparing a student's understanding with that of an expert).
- Understanding is difficult to observe, but may be indicated by emotional responses and when it is enacted.

Chapter 4
Methodology

4.1 Overview

The study followed a standard pilot study/main study format, with data collection for the pilot study taking place in the Autumn of 1997. The principal data came from semi-structured interviews with twelve students that were conducted regularly for the duration of their respective courses. Interviews were recorded on audiotape and fully transcribed, and analysed using a top-down approach to identify recurrent themes (Schoenfeld, 1985). Data collection for the main study took place one year later, with improved consistency in interview questions and a more rigorous grounded theory-based bottom-up data analysis procedure (Glaser & Strauss, 1967, Strauss & Corbin, 1990). In addition, lectures and classes were observed, groupwork in classes was tape-recorded and new types of task were added to the interviews. The interview data remained the focus of the inquiry throughout, due to the study's focus on individual learning, and to the eventual efficacy of grounded theory analysis of this data in producing theory that not only fits the data well, but is integrated with and strengthens the theoretical framework.

4.2 Investigating understanding: qualitative methods

The aim of this study is to build on the previously identified misconceptions and difficulties students have with the material of Analysis; to establish relationships among factors leading to a more or less expert understanding, with particular attention to the influence of the two different pedagogical strategies. Quantitative methods and/or gathering data on the entire populations were considered inappropriate to this aim for various reasons.

First, the research setting does not provide a genuinely experimental situation. The students on the two courses do not, by and large, enter with the same A-level grades. They are not taking the same, or even the same number of, other mathematics modules. Different faculty teach the different courses. Controlling for the effects of these influences would be next to impossible (Wolcott, 1990, Patton, 1990), especially considering the level of interaction the students have with the mathematics and each other outside the classroom.

Second, students on the new course spend more time in class than those on the lecture course. Hence they might be expected to attain better examination results due to increased opportunity to memorise proofs and other material (it is recognised that students pass examinations in this way, cf. Byers, 1980, Nickerson, 1985). If we also see a qualitative difference in students' mathematical thinking, this would be evidence that the new course can lead to their better development as mathematicians. To see such a qualitative change, that is to have access to the cognitive processes involved in demonstrating more than instrumental understanding of the material, methods allowing more detailed access to thinking were required than those offered by short or written responses (Ginsburg, 1981, Swanson et. al., 1981).

Third, the study was to be exploratory as regards the relationships between factors such as the students' experiences of learning mathematics and their understanding of Analysis concepts. Inferring such relationships from any data is always somewhat tenuous and dependent upon the researcher's theories about the situation (Wolcott, 1990). It is more valid/reliable where these may be confirmed through revisiting issues and evaluating consistency of responses for a smaller sample of students (Schofield, 1990).

Hence qualitative methods using a relatively small sample of students were considered appropriate, and reflective of a qualitative tradition that is now well established in education research (Schofield, 1990). Since the focus of the study was individual cognitive

development, interviewing was chosen as a central strategy. Naturalistic observation has the advantages of minimal interference in the learning process (Ginsburg, 1981, Cohen & Manion, 1994), but is difficult to pursue because of the private nature of the participants' thought (Ginsburg, *ibid.*). In this case it was necessary to impose a certain amount of standardisation across the questions asked to students from each course, and choosing a semi-structured interview environment (*cf.* May, 1997, Cohen & Manion, 1994) allows for this. It is also "intended to facilitate rich verbalisation which may shed light on underlying process" (Ginsburg, 1981), taking advantage of the fact that "the interviewer...can seek both *clarification* and *elaboration* on the answers given. This enables the interviewer to have more latitude to *probe* beyond the answers and thus enter into a dialogue with the interviewee" (May, 1997, p.111).

There are natural questions about the validity and reliability of interview data, resting as it does on self-reporting of thought processes. While it may be that "over a wide range of conditions and situations people are reasonably good at telling what they believe and think" (Swanson et. al., 1981), various constraints must be considered. Among these are epistemological questions regarding what may legitimately be inferred about cognitive processes from verbal data (Philips, 1987, Schoenfeld, 1985), and the essential unrepeatability of such investigations (Wolcott, 1990). These issues, and the way they are dealt with in the study, are described in the remainder of this chapter. Further consideration of the validity, reliability and generalisability of the theory generated from this data appears in section 4.5.1 and in chapter 12.

4.3 The pilot study

The pilot study took place in the Autumn of the 1997/98 academic year, the first year in which every first-year mathematics student attended the new style course. Its aim was to situate the researcher in the investigation context and to provide a framework for designing the main study. Its results did not contribute directly to either the theoretical framework described in part 2 or the substantive theory developed in part 3. Hence this description of

this stage of the research is not exhaustive; material that is included serves primarily as an explanation for the decisions made regarding the form of the main study. Hence detail is provided on the nature of the interviews, which remained fairly constant, but is restricted regarding the particular questions used and the data analysis, as in this regard the pilot study served as a springboard for moving to different procedures in the main study.

4.3.1 Participants

There is a tension in this type of study between depth and breadth: enough participants must be chosen to expose a range of behaviour, but enough data must be gathered on each participant to facilitate assessment of their developing understanding. An analogous choice must be made regarding the size of the interview groups, where "a balance must...be struck between the group being too small for interactive study or too large thus preventing all group members from participating in the discussion" (May, 1997, 113 on focus group interviews). In this case it was decided to select two groups of three students from each of the courses. This would restrict the amount of information on any one student in some ways (Wolcott, 1994) but was likely to increase it in others: the students should not feel so intimidated by the interview situation and conversation would be possible amongst the students with minimal participation from the interviewer. This was to be particularly important in task-based sections, as the requirement to work together naturally draws out problem-solving decisions, whereas interviewer requests to a single individual to elaborate on these is more likely to interfere with these decisions (Schoenfeld, 1985).

Participants in the pilot study were volunteers drawn from the lecture class and from two of the classes on the new course. Some volunteered in pairs, others as individuals. While using volunteers may skew a sample towards those who are naturally more confident (Cohen & Manion, 1994), this was deemed appropriate at this stage since participants were paid only in free biscuits and since:

- Available information on previous mathematics attainment was not strongly discriminatory: all of these students have very high previous mathematics grades.

- Information on other potential factors such as beliefs about mathematics was not available. While it may have been possible to survey for this, analysis of such data and the subsequent time taken finding appropriate students who were prepared to take part would delay commencement of the interviews. It was considered important to start as soon as possible, and since this was a first term course for these students it was impractical to seek such information prior to the beginning of the course.

- The students were being asked for a lot of their time. While there are undoubtedly gains for them in taking part in this sort of research (see section 4.5.1 for a discussion of ways in which the participation in interviews is likely to affect their learning), this would not be immediately apparent and students coerced into participation would not be likely to contribute a genuine picture of their views and mathematical competencies (May, 1997).

4.3.2 Timing

The students attended interviews, each lasting approximately one hour, in weeks 2,3,5,7 and 9 of a 10-week term. This pattern was decided upon because:

- The first and last weeks of term are best avoided since in the first the students have administrative tasks to complete, and in the last they are tired, likely to be busy socially and unlikely to be concentrating as normal on the mathematics.

- Asking them to attend more than once a fortnight would be unfair to them as it takes up time that might be better spent elsewhere.

- Having a gap of two weeks between interviews cuts down on "training effects"; it was important that the students should not be tempted to "prepare" for interviews in order to improve their performance (Schoenfeld, 1985).

Unfortunately, timetabling constraints meant that it was not possible to see all of the students in a short time, so interviews took place throughout the week. This meant that some students had covered marginally more work than others when they arrived for a given

interview. The questions were written so as to be accessible using material which had been covered in the previous week, and discrepancies in exposure to the new material were only seen to have an effect was when a student had studied something relevant on the same day. Where this is deemed to be a factor it is indicated in the data analysis sections.

4.3.3 Recording

The interviews were recorded using audiotape. Videotaping was considered because of the additional access it provides to nonverbal communication (Maher & Martino, 1996, Davis, 1992). However the presence even of a tape recorder may be intimidating and cause unnatural behaviour (Schoenfeld, 1985), so while video can be essential in the case of young children who may not have sufficient command over language to be able to articulate their thoughts (Ginsburg, 1981), it was decided that with these students audiotaping would be used during the pilot study in order to avoid this extra pressure.

Attempts were made to make notes during interviews (cf. Cohen & Manion, 1994), but this proved impersonal and distracting: it impeded the interviewer's participation in the conversation and made the interviewees self-conscious. Hence it was abandoned in favour making notes on atmosphere and any particularly striking incidents immediately after the interview. Exceptions occurred when the interviewer noted points of interest during the interviewees' collaborative work on task-based sections, and wished to question the students about this later rather than interrupt and potentially disrupt their work (Ginsburg, 1981, Schoenfeld, 1985).

4.3.4 Introductory interview

All of the interviews were semi-structured: questions were designed beforehand to maintain a degree of comparability between interview groups, but were flexible in the sense that each generated further questions contingent on the students' responses, and in the sense that the order could be varied if students themselves brought up topics which the interviewer had planned to discuss later (cf. Ginsburg, 1981, May, 1997).

The first interview did not require the students to attempt any mathematics. Instead it first addressed the issue of "cognition", explaining to the students what was required of them in the social encounter of the interview (May, 1997). In particular students were told that:

- They would be anonymised, and nothing that happened in the interviews would be reported to the mathematics department.

- They should not be concerned about impressing me, or each other, since I wanted to see how they were genuinely getting on.

- They should not prepare in any way for the interviews.

- If they happened to know any other students who were volunteers for the study, they should avoid discussing the details of the questions with them.

The remainder of the interview focused on the students' mathematical background and initial experience of university and university mathematics. This provided valuable information on the students' views on mathematics and on themselves as mathematicians, and more importantly allowed the students to answer open questions with descriptions and opinions. Thus it struck a balance addressing the tension between establishing a trusting rapport and the practical constraints of the inquiry (May, 1997).

Establishing this rapport is essential in research centred on interviewing, since the "experimental contract" set up will influence the subjects' responses (Balacheff, 1986). The subject should feel that their answers are valued, and their interest in the conversation should be maintained (May, 1997, Ginsburg, 1981). In fact this study encountered minimal problems in this area. The interviewer had the advantage of being of a relatively similar age to the participants and, having been an undergraduate in the same institution, was able to genuinely empathise with their concerns (Ginsburg, 1981, Patton, 1990). In addition, this is an exciting time in the lives of these people and far from being inhibited in their answers, most seemed to genuinely enjoy talking about their experiences, both good and bad, for the benefit of an interested researcher.

Having said this, it is important to carefully consider the interviewer's persona in interviews and the likely effects of this upon the participants (Henwood & Pidgeon, 1993). I aimed to enter the participant's world "not as a person who knows everything, but as a person who has come to learn; not as a person who wants to be like them, but as a person who wants to know what it is like to be them" (Bogdan & Biklen, 1992, p.79). The participants knew me only as a researcher; they were aware that I taught for the mathematics department but none had this form of contact with me, and while I laughed at their jokes and sympathised with their difficulties, I was careful to avoid value judgments related to their responses. In addition I ensured that enough time was allowed for the students to give full answers, with a view to gathering detailed information and establishing the degree to which initial responses were deeply held (Ginsburg, 1981). Elaboration and clarification were encouraged by asking further contingent questions or simply providing opportunities for elaboration with judicious use of interjections such as "hm?", "right..." and "okay...".

4.3.5 Subsequent interviews

Following this initial interview, subsequent sessions typically began with an introductory section in which students were asked how they felt about their progress, and to speak about recent material. Following this was a task-based section in which they worked on a question related to the material but written so as to require more creativity than typically needed in assignment questions. A final section reviewed their work on this task and led to more general questions about the definitions involved and the students' experience of proofs and proving. In each case the questions were written approximately one week before the interview took place, and based on both the material studied and issues arising in previous interviews.

The degree of interviewer intervention varied with the activity. The overall strategy followed Wolcott's advice to "talk little, listen a lot" (Wolcott, 1990). Discursive sections thus proceeded as in the first interview, guided by pre-decided but relatively open-ended

questions, and introducing these flexibly so that these sections resembled natural conversations (Ginsburg, 1981).

Conducting interviews with three students present meant that this strategy could be continued into the task-based sections. The students were presented with the question printed on an otherwise blank sheet of paper, provided with pens and instructed to attempt the problem by "thinking out loud" and negotiating with each other rather than the interviewer. Following Schoenfeld, they were *not* prompted to explain the reasons for all of their decisions (Schoenfeld, 1985); for approximately 10-15 minutes intervention was limited to requests for clarification of mumbled comments and reminders to express their thinking out loud for the benefit of the tape. Direct questions *to* the interviewer were treated with discretion. Those about the nature of the task were answered briefly in recognition of the need for comprehension of this and the fact that objective equivalence in questioning does not guarantee subjective equivalence (May, 1990, Ginsburg, 1981), those requesting suggestions or judgements of strategies were deflected to the group. In this way the students were given the opportunity to structure the task in the way they saw fit, with minimal chance that the interviewer would bias their responses (Ginsburg, *ibid.*).

The exact process associated with moving to the review sections was more variable. Some students came close to complete solutions to the problem and so the interview proceeded to this stage immediately. Others did not, and before proceeding the interviewer attempted to give the participants more opportunities to demonstrate their capacities by rephrasing questions and asking others which would lead towards an acceptable answer (*cf.* Ginsburg, 1981). In all cases a strategy of questioning rather than "telling" was adhered to at this stage; requests for "the answer" were deflected with other questions or with suggestions that the relevant material would be in the students' notes.

In review sections the students were prompted to expand on the reasons for their decisions during the problem solving, usually with particular emphasis on the use of definitions and

their approach to justification and/or proof. Such reflections do not disturb the problem-solving process as much as requests for justification when decisions occur (Schoenfeld, 1985, Swanson et. al., 1981), but are open to question as the subject may knowingly or unknowingly fabricate reasons if they cannot remember or did not have access to the originals (Swanson et. al., *ibid.*). This problem is less likely in this study than in those of young children, since the subjects are old enough to understand that the purpose of the investigation is to find out what they think rather than for them to attempt to "please" the interviewer by giving a certain type of answer. In fact, although rationales given after the fact may not be accurate reports of mental processing, they certainly provide interesting information about the knowledge and beliefs of the subjects (Swanson et. al., 1981). This is particularly evident in this study, when the students are only beginning to develop an understanding of what constitutes a "good" answer in this context (*cf.* Yackel & Cobb, 1996, Yackel, Rasmussen & King, 2000) and so reveal much about their thinking in this way.

4.3.6 A note on anxiety

Task-based sections merit careful consideration as they are likely to induce anxiety in students, especially when questions are designed to reveal depth or lack of understanding (Schoenfeld, 1985). An advantage of interviewing over naturalistic observation is the increased opportunity to establish *competence* rather than idiosyncratic performance, but this advantage is easily lost if the subject does not feel secure in the interview situation or if they fail to take the task seriously (Ginsburg, 1981). It is particularly important to minimise this anxiety where time pressure is a problem and the interviewer wishes to move on to subsequent questions without interference from any distress. In fact despite the level of challenge this proved to be relatively unproblematic in this study, for the following reasons:

1. Evidence from throughout the study indicates that the students tended to view the interviews as separate from and not indicative of their progress in Analysis. In some cases this was related to beliefs regarding the storage of the required knowledge in their notes; most were happy to accept suggestions that they would find the relevant material there if they wished to pursue it. This is further reinforced

by the fact that they knew the interviewer only as an interviewer and not as someone who had tried to teach them the material.

2. While the interviewer knows what the student should be able to do and what their peers have achieved, the student is considerably less well-informed in this respect. Hence it was not uncommon for the students who performed the least competently to be among the least concerned about their facility with the material.

3. In combination with the previous factors, it appears that once a suitably light-hearted atmosphere has been established in the interview setting, there is little anxiety which is not immediately alleviated by a sympathetic but non-judgemental comment along the lines of "Oh dear that was a bit hard wasn't it? Would you like a jammie dodger?"

4.3.7 Analysis

To facilitate analysis all interviews were transcribed, and analysis then proceeded in a top-down fashion. Following Schoenfeld's description of research into problem solving (Schoenfeld, 1985), each transcript was parsed into around 8-10 episodes of relatively consistent student behaviour. These episodes were then summarised for each student, and drawing on these summaries and research literature an initial network of hypothesised relationships between aspects of student behaviour was developed. In brief, this comprised:

A top level in three sections: student **beliefs** about what it means to do mathematics, their sense of **self** regarding their competence in this and their relationship with **authority**. The second level covered the degree to which the student engaged in **reflection** on and **organisation** of the material. The extent to which they would do this would depend on the top level since they may or may not feel that this was part of their role as a learner. Also on this level was the student's **proof scheme**, which may depend on their perceived relationship with the authorities, as described in chapter 3. On the third level were the **heuristics** employed by the student in solving the problems set and the degree to which they engaged in metacognitive **monitoring** of the progress of the solution attempt. Once again

these would depend on top-level beliefs regarding what they expected doing mathematics to involve. This level also contained the student's capacity with **logic**, their awareness of the **primacy** of definitions in advanced mathematics and the way in which they used **concept images** and **concept definitions**. These latter were related to their proof scheme and also to their reflection on the new material they were encountering. At the bottom level were the most concrete outcomes of all of these factors. **Problem-solving success**, depending on their available heuristics and success in monitoring their progress, ability to **follow proofs**, involving their logical competence and proof scheme, **acquisition** of the material, depending upon their understanding of the primacy of definitions and on the degree to which they attempted organisation, and their understanding of **limits**, which would depend upon their inclination to use concept images or definitions.

At this stage other researchers (the author's doctoral supervisor and several graduate students engaged in similar study in mathematics education) were invited to rate sections of transcript for student success in a limited number of these areas. While there was deemed to be satisfactory inter-rater agreement upon coding of these short sections, this served to highlight the fact that the categories chosen were not sufficiently refined to tightly "fit" with this data, and that therefore they were inadequate to the task of assessing whether the hypothesized relationships genuinely existed. Hence it was decided that for the main study, a more comprehensive and rigorous data analysis procedure was required. The grounded theory-based procedures followed in order to address this in the main study are described in sections 4.5.2 to 4.5.5.

4.4 Main study data collection

The main study data collection took place in the Autumn term of 1998. This section describes developments in the data gathering made in view of the pilot study, the next describes the data analysis procedures and the resulting substantive theory.

4.4.1 Methodological refinements

While the data analysis procedures were considered inadequate for revealing relationships among the factors influencing a students' learning, the basic format of the data collection was considered to be a success. Data collection for the main study therefore proceeded along the same lines, but with a number of refinements.

First, it was decided that students should be interviewed in pairs rather than groups of three. Having three students contributed to security in the interview setting but did not optimise the information gained about any one student, particularly on the task-based sections which are arguably the most important.

Decisions on the selection of participants were more difficult at this stage. From the pilot study it was evident that there was considerable heterogeneity within what could be assumed is a fairly homogeneous population. It would have been ideal to select subjects according to matching across the courses, but the time-related problems with achieving this were the same as in the pilot study and as described, pilot study data analysis had not progressed to an extent that suitable factors could be identified. Hence the main study would remain exploratory regarding these factors, and it was felt that it would be misleading to make claims on this front. Instead, it was decided that the participants would still be volunteers, but their number should be increased. Four pairs from the lecture course and five from the new course were randomly chosen from a volunteer population, in the hope that the expansion both in numbers and in the time available to hear each student in the interviews would provide a range of adequately detailed information. In fact, numerical data on students' position in comparison to the rest of the cohort at the beginning and end of the courses indicate that a satisfactorily representative sample was chosen. This data may be found in appendix A1.

The selected students were given pseudonyms, from the beginning of the alphabet for those on the new course and the end for those on the lecture course. Short profiles of the pairs of students and their reactions to the interview situation are given in appendix A2.

Extra care was taken regarding *accessibility*, the term used by May to describe whether the interviewee has access to information interviewer seeks (May, 1997, Swanson et. al., 1981). It was evident in the pilot study that on occasion interviewees interpreted questions differently from the intention of the interviewer. This is not a problem *per se* in this type of research; such instances can often be revealing of the way in which a student interprets their experience of the mathematics (Swanson et. al., *ibid.*). However in this case students were explicitly told that sometimes they might be asked questions which did not make sense to them, and that if this was the case they should say so. I was also more alert to the potential need to employ "flexible verbal presentation and interrogation" in order to ensure that responses to subjectively equivalent questions were obtained from all students (Ginsburg, 1981).

The pattern of interview timing was retained, and despite the increased number of groups was successfully adhered to in all except one case: Unwin and Vic did not attend the final week 9 interview. The recording methods also remained the same. Audiotaping had proved essential in allowing detailed review of student responses while being relatively unobtrusive. It was deemed to be additionally satisfactory since reviewing the tapes revealed that I quite naturally commented on instances when nonverbal communication was used in a significant way. For instance I described, or asked students to describe, their gestures, and commented on striking facial expressions. Once again, this was facilitated by the good-natured atmosphere maintained throughout the interviews. May notes such phenomena in his comment that "if people feel valued, their participation is likely to be enhanced" (May, 1997, p.117). Like him I also experienced the bizarre twist of circumstances in which the interviewees thought I was doing them a favour. Jenny and Kate regularly thanked *me* as the left the interviews. Even free biscuits surely wouldn't have

made up for the fact that they were required to attempt difficult mathematics, so it seems reasonable to assume that they actually found the experience as a whole quite pleasant.

4.4.2 Questions

At around the time of completion of the pilot study, related literature searches had expanded into cognitive psychology work on human categorisation, and the theoretical framework to be explained in part 2 was being developed on this basis. This confirmed the centrality of formal definitions in explaining the difficulty of the transition to university mathematics, and correspondingly reduced the importance of general problem solving behaviours for this research. As a result, where the questions in the pilot study were related to recent material and so included all the major topics of the course, it was decided the task-based section in the main study would focus on one central topic in order to facilitate observation of progress in student handling of definitions. Limits and convergence was chosen as this topic, since these concepts are central to the course and appear throughout in the discussion of both sequences and series. Questions on these topics appeared in all the interviews barring the first. The second interview, in week 3, was also slightly different from the remainder in that the task-based section was a puzzle-type problem rather than being Analysis-based, since students had not long encountered these central concepts by this time. Nonetheless, limits and convergence were covered in the discussions of recent material.

The main study interviews were designed with a view to consistency and comparability: each covered the major themes emerging from the analysis of pilot study data. The week 7 interview was typical of the structure, and the questions are reproduced in the table below along with the themes they were intended to address and approximate timings for the sections. Similar tables for the remaining interviews can be found in appendix B.

Table 3: Week 7 interview questions

Introductory section (5 minutes)	How are you feeling about Analysis these days?	*Self*
	What have you been doing about since I last saw you?	*Acquisition*
	Is there anything you've done so far that stands out as something you understand well?	*Acquisition, organisation, reflection*
	What makes that make sense?	*Organisation, reflection*
	Does that help you understand new things?	*Organisation, reflection*
Main task-based section (20 minutes)	CONVERGENT IMPLIES BOUNDED QUESTION	*Acquisition*
	Can you prove it?	*Heuristics, monitoring, success, primacy, concept image, limits*
	Can you explain the whole thing back to me now?	*Following, proof scheme*
Review section (5 minutes)	Why did you decide to use the definition at that point?	*Primacy*
	Can you explain in words what it says?	*Limits, concept image*
	What do you think of these definitions in general?	*Limits, logic, concept image, primacy*
Reflective section (5 minutes)	How are you getting on with proofs now?	*Following, proof scheme*
	Can you judge when you've got something right / what do you go by?	*Authority, self*
	How does all of this compare to school maths now?	*Reflection, beliefs*
	Has the way you think about maths changed?	*Beliefs*
Further task	LOGIC CARD GAME[4]	*Logic*

The main task-based questions formed the central data in assessment of student progress in understanding the material. They are reproduced and discussed below; each will henceforth be referred to by the underlined names.

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[4] This task does not contribute significantly to the thesis. For more detail see appendix B.

Week 5 (1):   sequence question

Let $a_n = \dfrac{1 + \cos n}{nx}$, where x is a real number which remains fixed as n

varies.

When is the sequence (a_n) convergent?

(You should provide as complete an answer as possible.)

This question has a deliberate ambiguity in the use of the word "when"; it is necessary for the students to give some thought to what the question is actually asking. They are also unlikely to have met sequences with a variable in addition to the n; in effect this is asking them to look at infinitely many sequences. However all students should have had practice at using results from the course to find out about the eventual behaviour of such sequences. To be fully correct they need to note that the expression is not defined for $x=0$.

Week 5 (2):   proof correction

Check this proof and make corrections to it where appropriate:

Theorem: $(\sqrt{n}) \to \infty$ as $n \to \infty$.

Proof: We know that $a < b \Rightarrow a^m < b^m$.

So $a < b \Rightarrow \sqrt{a} < \sqrt{b}$.

$n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all n.

So $(\sqrt{n}) \to \infty$ as $n \to \infty$ as required.

This question generated some very interesting responses, as will be demonstrated in the data analysis chapters. It is confusing to many students because the result is true and is immediately recognised as such by most. In such a situation, they are unaccustomed to being presented with complete yet faulty "proofs". This "proof" requires qualifications (the first line is only true with restrictions on the values of a, b and m). More importantly however, it is invalid since the jump from the third to the last line relies on the assumption that a strictly increasing sequence necessarily tends to infinity. The question thus tests for

sensitivity to logical antecedence and an understanding of tending to infinity that is compatible with the formal definition.

Week 7:   convergent implies bounded question

Consider a sequence (a_n). Which of the following is true?

a) (a_n) is bounded $\Rightarrow$ (a_n) is convergent,

b) (a_n) is convergent $\Rightarrow$ (a_n) is bounded,

c) (a_n) is convergent $\Leftrightarrow$ (a_n) is bounded,

d) none of the above.

Justify your answer.

This question is set out in a way that is different from the work the students are generally exposed to or required to do in class, in that they are required to decide which answer is correct before embarking on a justification. It is revealing of the way that they approach mathematical reasoning, and in particular of misconceptions associated with convergence of sequences and infinity.

Week 9:   series question

When does $\sum \dfrac{(-x)^n}{n}$ converge?

Justify your answer as fully as possible.

This question returns to some of the elements from week 5, in that it is necessary to interpret "when" to mean "for what values of x". The students should recognise that for certain values of x this is a series they have studied, and that convergence or otherwise for the remaining values may be determined by making comparisons with these.

4.5 Main study data analysis

This main study data analysis was undertaken using a "grounded theory" approach. A grounded theory is defined in Strauss and Corbin as:

"one that is inductively derived from the study of the phenomenon it represents. That is, discovered, developed, and provisionally verified through systematic data collection and analysis of data pertaining to that phenomenon. Therefore, data collection, analysis and theory stand in reciprocal relationship with each other. One does not begin with a theory, then prove it. Rather, one begins with an area of study and what is relevant is allowed to emerge."

(Strauss & Corbin, 1990)

The essential element is that the researcher is disciplined in seeking to "render the attachment between theory and data as close as possible" (May, 1997, p.29): the paramount consideration is that of keeping close to the data to generate a theory that "fits" (Henwood & Pidgeon, 1993). This is achieved by analysing the data in a scientifically systematic way (Strauss & Corbin, 1990), so that "the procedures impose order on the management of data, no matter how unruly the data themselves" (Wolcott, 1994, p.27). In this way Strauss and Corbin argue that a qualitative study may expect to achieve the attributes of a "scientific method", in the sense that it may be significant, produce theory that is compatible with observations, be generalisable and reproducible, have precision and rigour and be verifiable. They also state that a theoretical formulation based on concepts emerging from the data is a good basis for explaining the reality of a situation (as nearly as this can be assessed) and for providing "a framework for action" (Strauss & Corbin, ibid.). In all, a grounded theory approach is appropriate to the aims of the study, and the degree to which these ideals are achieved is reviewed in chapter 12.

A common misconception regarding the development of a grounded theory is that the researcher must enter the field with no prior knowledge of that research setting, in order that

they might carry out research uncorrupted by existing views and theories. If this were the case, this study would certainly open to this criticism as these procedures for data analysis were only used at this relatively late stage in the study. However it is not. A grounded theory approach does not preclude the extensive use of existing knowledge and literature to inform the development of theory. Both literature and previous experience with the situation are needed to form an initial questioning strategy, but use of pre-existing concepts from the literature in description and analysis is suspended until these have been seen to emerge from the study of the data (Strauss & Corbin, 1990).

Constraints on the use of this method were however imposed by this research situation. Ideally, developing a theory that is grounded in data should involve alternating data collection and analysis as the theory develops. This was not possible in this situation, since the aim was to follow the students' learning over the course of one term. This means that the methods are restricted both by the time during which data collection can take place, and the students' cognitive development. It is not possible to revisit the same students at a later stage to clarify some point; by then their thinking will have moved on. In this way aspects of developing a grounded theory that require total flexibility in returning to the field are beyond the scope of this study.

4.5.1 Validity and reliability in grounded theory

In qualitative research in general and grounded theory in particular, *validity* (the extent to which findings can be considered true) and *reliability* (the extent to which the research endeavour and findings can be reproduced (Merrick, 1999)), are generally held to be less appropriate measures than for quantitative investigations. As Henwood and Pigeon state, "...criteria for judging the quality of [qualitative] research cannot be reduced to tactics for eradicating observer bias" (Henwood & Pidgeon, 1993). Rather, the presence and influence of the researcher is usually acknowledged and discussed. Further, Schofield argues that "the goal is *not* to produce a standardized set of results that any other researcher in the same situation or studying the same issue would have produced. Rather it is to produce a coherent

and illuminating description of and a perspective on a situation that is based on and consistent with detailed study of that situation" (Schofield, 1990). Wolcott echoes this point in describing his own approach to qualitative research, saying,

"What I seek is something else [*rather than validity*], a quality that points more to identifying critical elements and wringing plausible interpretations from them, something one can pursue without becoming obsessed with finding the right or ultimate answer, the correct version, the Truth".

(Wolcott, 1990)

He characterises this approach as trying to "understand, rather than to convince".

These values do not render qualitative research less valuable than quantitative. As Goldman points out, "over and over...qualitative methods such as observations and interviews offered facts and insights that would not have emerged from quantified data from tests, questionnaires, and other sources" (Goldman, 1999). Neither does it render all considerations associated with validity obsolete. Questions remain regarding obtaining accurate data, assessing the validity of data already collected, drawing valid meaning from data and validating theoretical ideas (Wolcott, 1990). The latter, concerning the validity of the theory developed, are considered in detail in chapter 12. This chapter as a whole may be said to address the issues of validity in the data, but the reliance on interview data means that this merits some specific remarks.

Sections 4.3.4-4.3.6 discussed the problem that this type of data is always mediated by the student's own interpretation of the rules and conventions of the context (Swanson et. al., 1981, Schoenfeld, 1985), and reviewed measures taken to avoid inducing participants to consciously misrepresent their thinking. The success of these measures is corroborated by student views on the process. From a researcher's point of view, the interviews were bound to impact student learning to some extent. In particular, the participants were asked to

reflect upon their learning experience, and their responses to these questions indicate that some would not otherwise have done so. Also, for certain of the "weaker" students, there were occasions when the interviewer's questions prompted conceptual thought that was clearly new to the student. While this was kept to a minimum, at times it was necessary in order to draw out what understanding the student had[5]. Finally, all of the students probably benefited from discussing their ideas out loud. A subconscious awareness of this may have been the trigger prompting some students to thank the interviewer upon leaving.

However, the students were asked in week 9 to give their view on the impact of the interviews on their learning. It was explained that the ideal would be for the interviews to have no influence at all, but that as this was unrealistic it was important that they should be as honest as possible in their answers. For most of the participants this appeared to be an unexpected question, and most drew a blank when they first tried to respond. Following this, there was agreement that the interviews had not influenced them much, although a number commented that they thought they should have done: that the experience of failing to answer questions should have prompted them to work harder but had actually made little difference. For those students who did notice a difference, this was primarily affective and related to some relief in talking about their experience of the work[6].

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[5] For an illustration of such an instance, see appendix E.

[6] The following are examples of student responses of these types. The letters represent the students present, the first number the week of the interview, and the second number the text unit number at which the quote begins. Interviewer interjections have been removed.

    I: Erm, yes, how, do you feel coming to these interviews has affected you? Maths-wise?
    D: Erm...
    C: Not,
    D: Not, a lot,
    I: Not a lot.
    D: I don't think. Erm...a bit of kind of, self-analysis. Erm, but...I don't know.
                                                                        CD9, 1362

    S: I think it should have done...Because I think, oh no, we're going to have to do one of these
        questions again. Should have worked for it...!
                                                                        ST9, 1392

    J: No, it's nice to get things...off your chest at the end of the week...It hasn't affected I don't
        think my whole mathematical view.
                                                                        JK9, 1419

Hence it seems reasonable to assume that taking part in the research benefited the students to a small degree, but did not significantly influence their learning habits or view of mathematics. This is consistent with the observation in section 4.3.6 that the interviews were often viewed as quite detached from the "official" learning of Analysis in lectures or classes. So the data may be thought of as valid within the restrictions of such a study. The way in which they were interpreted is now described.

4.5.2 Open coding

Grounded theory generation begins with open coding of data. At this stage the researcher is involved with the data on an extremely detailed level. In keeping with Patton's comment that "the role of qualitative researchers is to assume an empathic stance in the interaction with their participants and a neutral stance in the analysis of the data" (Patton, 1990), paragraphs, sentences and even words are considered for possible alternative meanings (Strauss & Corbin, 1990, Glaser & Strauss, 1967). In particular, care is taken to avoid assuming that the respondents are necessarily interpreting the questions as they were intended (Wolcott, 1990), and to attend to the forms of words used by the respondents without assuming that their classifications and ideas are necessarily indicative of the reality of the situation (Becker, 1990, Swanson et. al., 1981). Hence, although the research sets out with goals of identifying cognitive processes to establish participants' competence regarding the mathematics, at this stage of analysis the researcher is disciplined to work with a view to discovering intellectual phenomena (Ginsburg, 1981).

This type of initial coding addresses some of the potentially problematic issues associated with the use of interview data. Some information is directly available to subjects: while everything reported is subject to their own interpretations and categorisations (Swanson et. al., 1981, Becker, 1990), it is reasonable to assume that they can give accurate factual information such as the typical practical form of their previous mathematics lessons. Hence it may be appropriate to assign in vivo codes (classifications named using the subjects' own

words or expressions) to this data (Glaser & Strauss, 1967). Other information is not directly accessible to the subjects and necessarily not to the researcher: a students' justificatory practices depend on the concepts and structures they employ in the domain and these must be inferred indirectly (Swanson et. al., 1981). Detailed coding facilitates this process, minimising the temptation to employ previously conceived theories in classifying and explaining behaviour.

At this stage the researcher's goal is conceptualisation; concepts are generated via the constant comparative method, continually making comparisons between observed phenomena and asking questions about their possible meaning in relation to each other (Glaser & Strauss, 1967). Naturally this leads to the identification of a potentially unwieldy number of concepts and many overlapping categories, but essential to this type of theory generation is the importance of remaining "with the data" at this stage. Premature naming and organisation of larger categories is avoided in order to prevent a situation in which new data are perceived only through this categorisation, risking the loss of important or interesting information.

In this study two complete sets of interviews were analysed in this way, those of new-course students Adam and Ben and lecture-course students Wendy and Xavier. These pairs were selected because during transcription it became apparent that they encompassed a wide and interesting range of phenomena. For each interview, concepts were noted and organised under appropriate headings; these headings were not compared until this process was completed. At this stage they were synthesised to form consistent categories, and the coding was approaching saturation, that is new data no longer revealed substantial new concepts (Strauss & Corbin, 1990). Naturally, this could have been an artefact of reviewing data for only a minority of the participants involved, so one additional interview from each of two other pairs of students was reviewed in the same way.

When analysis of these further interviews resulted in only minor additions to the category system rather than major shifts of emphasis, internal triangulation was performed (*cf.* Swanson et. al., 1981). Two other coders reviewed each section according to a coding manual written for this purpose, after attending seminars in which the coding system was explained and the categories illustrated. Some discrepancies arose from the difficulty of adequately explaining the nuances associated with the categories in the principle researcher's mind. All such discrepancies were resolved to the satisfaction of all concerned, and recorded information on differences of opinion and coding can be found in appendix C2. An abridged version of the coding manual is reproduced in appendix C1; below is a section outlining descriptions of the main categories (note that these differ from those originally used to structure interview questions, as is expected in grounded theory analyses).

Three of the categories cover issues which arise in the data but are not central to my study. These are:

- Background: what factors not directly related to mathematics at university might influence the student's behaviour?

- Affective: how does the student feel about themselves in relation to mathematics?

- Problem solving: does the student have good general mathematical problem-solving skills?

One category is small but is of central importance:

- Transition: What has the student noticed about the transition to university mathematics, and how do they feel about this?

Finally there are three major categories:

- Uptake: to what extent is the student picking up both the material of Analysis and the more general ideas of advanced mathematics?

- Formalism: is the student engaging in rigorous mathematical thinking?

- Role: what does the student think is their role in studying mathematics at university?

4.5.3 Axial coding 1: coding all the data

The next stage in the grounded theory data analysis is called *axial coding*. At this stage, a central core category is identified and others are systematically related to this (Strauss & Corbin, 1990). In this study this was a long process happening in three overall stages.

First, the entire body of interview data was coded using the computer system "NUD*IST" according to the existing categorisation (QSR NUD*IST 4 User Guide, 1997). Once again this led to minor amendments and additions to the categories, but since saturation had been reached, no major changes were required. Throughout this task coding task memos were kept summarising each interview. These were then used with reference to the original data to write short summaries of the information obtained about each student in each main category. These summaries were further edited to form table entries for the categories *transition, uptake, formalism* and *role* for the four interviews involving task-based sections. This enabled an overview of a student's behaviour to be seen on one page. For examples of the documents produced at each of these stages see appendix D.

At this stage, as is predicted by proponents of grounded theory methods, the researcher had developed a strong sense of patterns within the data, identifying groups of students by collections of behaviours which regularly occurred in conjunction with each other. However, no one category had yet emerged as the core; this was identified in the second phase of axial coding.

4.5.4 Axial coding 2: emergence of theoretical distinctions

Identification of the core category came about as a result of the grounded theory approach to negative case analysis: apparently aberrant cases are used to refine theory by ensuring that it is conceptually dense, rather than to reject hypotheses outright (Henwood & Pidgeon, 1993). Two such cases were identified through regular discussion with the author's doctoral supervisor. (Regular review and discussion of emerging concepts took place throughout data analysis, with a view to uncovering any bias, misjudgement or unsupported conclusions

(Goldman, 1999).) Closer examination of these cases led to a sharpening of theoretical distinctions and a change in structure of the system of categories, as described below.

The first case was that of Wendy, who in some instances appeared among the most competent of those attending the lecture course. She tackled certain of the interview task-based sections with flair, showing good use of strategies learned in the course and good monitoring of progress towards solutions (*cf.* Schoenfeld, 1985, 1987). However she showed a relatively weak perceptual proof scheme (Harel & Sowder, 1998), arguing on the basis of an inadequate visual concept image when working with convergent sequences (*cf.* Presmeg, 1986b, Tall & Vinner, 1981), and resisting encouragement to use definitions or think more generally about other types of example.

The apparent inconsistency in Wendy's performance led to a closer examination of the cognitive requirements of the task-based section questions. Through this it was recognised that the theoretical framework suggested an obvious distinction: where the proof correction and convergent implies bounded questions required reasoning about a whole category of objects, the sequence and series questions required consideration of specific objects in relation to a given category (the subtleties of these distinctions are discussed in chapter 7). Wendy performed well on the latter type but poorly on the former, so her levels of competence no longer seemed inconsistent.

The second apparently anomalous case was that of Cary, who also employed visual representations of the concepts but who showed a much better standard of mathematical argumentation, attempting to provide general arguments to support his conjectures (*cf.* Harel & Sowder, 1998, Krutetskii, 1976). He also showed good awareness of the importance of definitions in proofs at this level (*cf.* Vinner, 1992). However a cursory examination of his performance in official tests indicated that his attainment was not reflective of this apparent ability. On closer inspection, it became apparent that Cary's actual use of definitions in mathematics was lagging behind his abstract knowledge of their role: he continued to

depend upon his concept images to provide him with a sense of understanding, and did not forge significant links between the two (cf. Vinner, 1992, Pinto & Tall, 1996). Hence, while Cary was very capable in *mathematical thinking*, he was less successful in learning the *product of mathematical thought*; he made inadequate progress in acquiring the structure of Analysis as agreed upon by the mathematical community.

These distinctions together prompted a return to the literature on typologies of understanding, in particular the aspect called *formal understanding* by Byers and Herscovics and later incorporated as *logical understanding* by Skemp. Both terms are introduced to make a distinction between understanding of the content of mathematics and having the ability to represent this understanding correctly (Byers & Herscovics, 1977, Skemp, 1979b). With one extension, these types could now be related in an obvious way to the category-based theoretical framework. This led to the confirmation of *understanding type* as the core category in the study; the typology used is explained in chapter 7.

Further, Cary's case raised the question of why such a capable student should fail to make the link between his concept images and the concept definitions. Joint consideration of his case with that of Fred, who employed similar reasoning strategies with similar problems, led to the identification of the student's relationship with authority as the deciding factor. Both of these students demonstrated a very strong internal sense that they understood the material and were able to judge the correctness or otherwise of conjectures and arguments (cf. Duffin & Simpson, 2000, Nickerson, 1985). However both also had a comparatively immature relationship with the external authorities: Cary was keen to show off his capacity for swiftness in reaching conclusions, Fred was trusting that his teachers would provide him with appropriate conventional methods. In both cases this resulted in an inadequate examination of the detail of the algebraic material. Detail on these two cases appears in section 9.2.6.

4.5.5 Selective coding: emergence of the substantive theory

Relationships between the core category and other categories were confirmed through *selective coding* (Strauss & Corbin, 1990). Establishment of the distinctions described above prompted a return to the original data to draw out instances bearing on types of understanding (including the handling of specific objects, standards of logical argumentation, and use or otherwise of definitions), and the student's relationship with the authorities. Further consideration of this data in conjunction with the previously noted groups of behaviour patterns led to the following reorganisations in the category system:

- Behaviours in the category *role* were characterised as indicating either an *internal* or an *external sense of authority*. This new distinction incorporated information from the student's *background* as a precursor to their place regarding this distinction, as well as their *affective* responses. It was also correlated with the extent to which the student reflected upon the enterprise of doing mathematics at this level, thus incorporating responses previously appearing under *transition*. Sense of authority then formed the first causal factor in a student's acquisition of different types of understanding.

- Behaviours in the *uptake* and *formalism* categories were split into those pertaining to the different types of understanding now distinguished and those indicating whether the student tended to reason *visually* or *nonvisually*. Where previously this latter distinction had been conflated with behaviours regarding proving and the use of definitions, it now appeared as an overall preference which formed the second causal factor in the types of understanding acquired.

At this stage the groups of students previously identified by their common behaviours were distinguishable by their places in relation to these factors. Further splitting of groups in consideration of the course attended confirmed this as an *intervening* rather than a *causal* factor (Strauss & Corbin, 1990). Neither course altered the students' existing predispositions, but they did intervene to encourage more or less development in some types of understanding.

At this stage a grounded theory had been produced, and the results of the processes described above are presented in part 3. The next stage in a grounded theory study would be *theoretical sampling*, in which new data are gathered on the basis of the concepts that emerged as relevant to the evolving theory. For example, students from a number of categories would be selected early and followed in more detail, or wider investigation of the spread within the cohort would be investigated. This stage is unfortunately beyond the scope of this study, but the form such a continuation would take is discussed in chapter 12.

4.6 Main points of chapter 4

- The study used qualitative methods in order to study individual cognitive development.

- The primary data was gathered using a series of audiotape-recorded semi-structured interviews with students from both the new course and the lecture course.

- Care was taken to ensure that participants did not feel intimidated by the interview situation and were therefore able to demonstrate their full mathematical competence.

- The pilot study provided a basis for design of the main study, but the top-down analysis used did not make a substantial theoretical contribution to the thesis.

- For the main study students were interviewed in pairs, and questions were designed to cover a consistent range of issues throughout the term. In particular, convergence and limits was chosen as the central topic.

- Main study interview data was analysed using a bottom-up, grounded theory approach. This involved open coding, axial coding and selective coding.

- Analysis of apparently aberrant cases led to the identification of *type of understanding* as the core category, with *sense of authority* and *visual or nonvisual reasoning* appearing as causal factors.

Part 2
Theory

Introduction to part 2

This part forms the backbone of the thesis, characterising human reasoning about categories of objects and applying this characterisation to explain student behaviours in university mathematics.

Chapter 5 develops the theory in general, examining the way in which human beings reason about categories in everyday contexts. It makes use of the notion that individuals use *prototypes* in this reasoning. This notion is examined in detail, since the idea of a prototype has been used in various ways and subjected to various criticisms. Here it is used as a theoretical construct that is useful in developing a framework for discussing the difference between everyday and mathematical reasoning. At the end of chapter 5, the cognitive requirements of reasoning in these two contexts are compared, to more precisely characterise what is particular to mathematical argumentation.

Chapter 6 continues this argument into university mathematics, exploring the further complications generated by the introduction of formal definitions at this level. It uses the theory developed in chapter 5 as a basis for a unified explanation of known student behaviours at this level. The theory developed through both these chapters is then applied in part 3 to provide a classification of behaviours seen in this study.

Chapter 5
Theoretical framework

5.1 Overview

This chapter draws on literature from cognitive psychology and mathematics education to develop a framework for describing the way that human beings reason about categories of objects. It begins by considering the distinction between school and university mathematics, characterising this as a shift from calculations on specific objects to proofs about whole categories of objects. Categorisation is thus established as a sensible starting point for a theoretical exploration of the difficulty of the transition to this type of mathematics. Reviewing research on categorisation leads to a discussion of "prototype effects" and attempts to account for these effects using different models of human reasoning about categories. This leads to a discussion of the type of cognitive structure used to perform this reasoning. Various terminology which has been used to describe such structures is considered, and the word "prototype", with caveats as to its meaning, is settled upon as the most appropriate for this thesis.

This framework is then established as consistent with ideas about learning by exposure to examples, and with theories accounting for human deductive capacity. Finally two ways in which prototypes may be used are distinguished, and related to generalisation and abstraction respectively. This permits clarification of the distinction between everyday argumentation and the standards expected in mathematics. The next chapter examines

further complications arising from the fact that mathematics is a corporate enterprise incorporating formal definitions, and applies the theory so developed to account for phenomena discussed in the earlier literature chapter (in particular, in sections 3.2.2 and 3.2.4 on proving and definitions respectively).

5.2 Categories and prototypes

5.2.1 Categories, prototypes and the thesis

The theoretical framework developed in this part of the thesis posits theoretical answers to two questions: that of why the transition to university mathematics should be difficult, even for those students who have proved outstanding in school mathematics and who subsequently go on to be talented mathematicians, and that of why Analysis should be particularly challenging. The first question may be addressed by asking what exactly university mathematics requires that school mathematics did not. An obvious initial answer to this is "proof", and as explored in chapter 3, much research has focused on this area. However, while many behaviours are recognised and are attributed to factors such as student's educational experiences and their relationship with the authorities, this does not provide a coherent explanation of what students are attempting to do, or why it seems so difficult for them to adopt the mathematical community's standards of proof.

At a deeper level, the question can be refined by asking what exactly *proofs* at university involve that school mathematics did not. As described in chapter 4, analysis of data in this study led to the position that the difference may be characterised as that between reasoning about specific objects and reasoning about whole categories of objects. Before proceeding to an outline of the argument, it is appropriate to clarify the use of some terms:

- The term *specific object* is used in the sense of Sfard when she describes the way in which mathematicians treat abstract notions *structurally*, as though they are "static structures, existing somewhere in space and time" (Sfard, 1991). Its meaning resists clarification, since different individuals are capable of treating different mathematical

constructs as objects. For instance, the function $f(x) = x^2$ may be thought of as an object by one student, but be associated only with actions of evaluation for another (Dubinksy, 1992). This study will simplify by associating the meaning of object with the logical rather than psychological structure of Analysis. Hence, although an individual may or may not be able to treat them as such, specific objects at this level include specific sequences (such as $(1/n)$ or $(\cos(n))$), specific series (such as $\sum n$ or $\sum \frac{1}{2^n}$), and specific real numbers (such as $0.999\ldots$).

- A *category* of objects is a collection of specific objects, usually associated with a (mathematical) term. So the category of convergent sequences contains all possible convergent sequences. The relevance of making this definition may be seen by considering that a student's idea of which objects belong to a certain category may or may not correspond to the formal idea in agreed mathematical theory.

- An object might have certain *properties*, indeed all the specific objects in a category might have properties in common. For example, a sequence might have the property of monontonicity, and every object in the category of convergent sequences has the property of being bounded.

- In general usage, the term *concept* can be associated with objects, properties or categories: one might think of "sequence" or "boundedness" or "chairs" as a concept. The distinction between objects and properties is an important one in the following discussion, and the concept "chair" is not the same as any one specific chair or as the category of all chairs. Hence, for clarity, the term "concept" will largely be avoided in favour of object, property or category.

The difference between school mathematics and proof in university mathematics may then be seen in these terms by observing that school mathematics primarily involves calculations performed upon specific mathematical objects. For example, students in school are required to integrate a specific function on a specific domain, or solve a specific differential equation (SEAC, 1993). Even the few proofs encountered at this level have this property: students are asked to prove by induction that *this* formula gives the sum of the first n terms of *this*

series, or prove that *this* trigonometric identity is equivalent to *that* one. Proof at university goes beyond this. Calculations are still required: students are asked to find the limit of a given sequence or to find the rational that is represented by a given infinite decimal. However there is now the additional requirement of working with entire categories of objects. This might involve showing that a specific object is an element of a category, for example that a given sequence is convergent, or showing that a whole category is contained within another category, for example that all convergent sequences are bounded. It might be argued that at least the first of these is essentially equivalent to a calculation on a specific object. Logically this may be the case, but it will be argued that psychologically it is not.

In the preceding and following discussions, the term "category" is used in place of the more usual "set" because the argument rests on the idea that mathematical sets are fundamentally different from everyday human categories, in structure and consequently in the way that mathematicians reason about them. The claim is that this difference accounts for the difficulty of the transition to university mathematics. Familiar inappropriate student behaviours may then be explained by assuming that students are unaware of this and that they therefore continue working as though with everyday categories.

The hypothesis, in brief, is that when reasoning about a category of objects an individual uses a mental "prototype"; an abstract representation of what is common to members of this category[7]. They inspect such a prototype to generate or evaluate conjectures about the category, and perform comparisons with them to decide whether or not a particular object is a member. There are essentially two ways to perform these functions, the first involving direct generalisation from the prototype to the entire category, the second involving the abstraction of some property from the prototype and the making of a deduction based on this property. The choice usually depends upon the demands of the situation, but in mathematics the standards are more fixed and demand that property-based justifications should always be provided. The introduction of definitions then means the categories associated with defined

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[7] This notion will be clarified in subsequent sections, after this first outline of the argument.

terms are precisely delimited, have no preferred members and are therefore fundamentally unlike everyday categories. In particular the properties used to reason about them must be the definitional properties and no others. Consideration of the peculiarly visual character of Analysis in conjunction with the usual assimilation and use of categories then permits an answer to the second original question of the particular difficulty of this subject.

As discussed in the introduction to this part, the use of the term "prototype" is contentious in view of its history in the literature on categorisation. The following sections highlight the problems associated with its use, with the aim of demonstrating that if sufficiently clarified it becomes a useful theoretical construct for addressing this problem.

5.2.2 Cultural categories

Traditionally, human categories were viewed as sets in the mathematical sense, with membership determined solely by some necessary and sufficient set of properties that the members have in common (Lakoff, 1987b, Smith, 1988). However in the 1970s the work of Rosch and her associates revealed what became knowns as *prototype effects*: within a given category some members were reliably judged to be "better" examples of the category than others. For example, robins are considered better examples of "bird" than ducks, and desk chairs are considered better examples of "chair" than rocking chairs. These ratings were consistent across various experiments: more "prototypical" members are directly rated as better examples, are identified more quickly as members in reaction time tests, and are listed first when subjects are asked to give examples from the category. In addition new information about a category member is more likely to be generalised from a representative member to a non-representative member than vice-versa, and less representative examples are often considered to have greater similarity to representative ones than the reverse (Lakoff, 1987b, Rosch, 1978). This fundamental asymmetry is not compatible with the classical view. Neither is the more obvious observation that some categories do not have well-defined boundaries, for example there is no absolute criterion for deciding who would belong in the category "tall man" (Lakoff, 1987a).

Rosch and her colleagues were investigating categories within cultures, so this leaves open the question of how any individual thinks about categories, which is the one we are interested in. Rosch was at pains to say that prototypes did not constitute any particular theory of the use of categories in cognitive processing (Rosch, 1978). However individuals must somehow cope in a society where categories are of this nature, and various models were suggested to account for this.

5.2.3 Accounting for prototype effects

The simplest such models suggested that knowledge is stored hierarchically; in a taxonomy, attributes are associated with a concept at the highest level at which they apply and subjects rely on *deductive retrieval* to conclude that they also hold for concepts at lower levels (Smith, 1988). Other models used "weighted feature comparisons" with a prototype, so that some attributes were more important than others in deciding category membership (Medin & Wattenmaker, 1987). Both types of model are implementable in computer simulations that can then mimic human decisions. However there are problems with this type of model for describing actual human cognitive processing.

First, they require decisions about which features to attend to. As Medin & Wattenmaker point out, any two objects we encounter have infinitely many things in common: plums and lawnmowers both weigh less than 1000kg, are found in our solar system, have a smell etc. (Medin & Wattenmaker, 1987). Some such decisions may be determined by our perceptual apparatus (Rosch, 1975, Lakoff, 1987a), but this does not address the fact that this type of categorisation is taxonomic, and human beings also categorise objects in different ways. For example, by their functions, by their place in a familiar temporal routine (Fivush, 1987), according to personal theories about their relationships, or even spontaneously in response to some goal. The latter is beautifully illustrated by Medin & Wattenmaker's category:

children, jewellery, portable television sets, photograph albums, manuscripts, and oil paintings[8] (Medin & Wattenmaker, 1987).

So feature comparison models constitute a way to account for prototype effects, but the existence of prototype effects does not *imply* that we operate using feature comparisons. Lakoff criticises this implicit inference in the work of Osherson and Smith and Armstrong, Gleitman and Gleitman (Osherson & Smith, 1981, Armstrong, Gleitman & Gleitman, 1983). He also argues that to equate prototype effects with the structure of categories overlooks much that is interesting about this structure (Lakoff, 1987a&b). However this does not contradict the idea that prototype effects exist, nor that prototypes can be a useful theoretical construct in accounting for certain types of behaviour. The next section describes related notions and thereby develops the way in which the term is used in the remainder of the thesis.

5.2.4 Schemas, frames, image schemata and idealised cognitive models

The problems with feature comparisons and other such models led to the suggestion that there must be some sort of larger cognitive units, probably in hierarchical relationships, for thinking to be efficient. This is consistent with the necessity of "chunking" information in order to compensate for the limited capacity of working memory (Baddeley, 1997, Ashcraft, 1994). Minsky voices such a view in the context of his work on computer vision:

"It seems to me that the ingredients of most theories both in artificial intelligence and in psychology have been on the whole too minute, local, and unstructured to account – either practically or phenomenologically – for the effectiveness of common sense thought. The "chunks" of reasoning, language, memory and "perception" ought to be larger and more structured, and their factual and

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[8] "things to take out of one's home during a fire"

procedural contents must be more intimately connected in order to explain the apparent power and speed of mental activities."

<div align="right">(Minsky, 1975)</div>

Each of the terms in the section title has been used to describe some version of these larger structures, and while the emphasis in each case varies with the author's background, they have essential elements in common. "Prototype", as used in this study, also shares these elements; it is chosen not because it is considered the most appropriate for describing reasoning in general, but because it is most evocative of the author's meaning in this restricted context. The remainder of this section clarifies its meaning by outlining usage of the alternatives. Throughout, it may be helpful to bear in mind that the most commonly discussed mathematical category in this thesis will be that of convergent sequences of real numbers, and that a student's prototype for this category is often a "generic" monotonic convergent sequence.

Baddeley chooses "schema" as the general version for these large cognitive structures, and describes schemas as

"packets of information that comprise a fixed core and some variable aspect, which can embed within one another and thus represent knowledge at all levels of abstraction, and which represent knowledge rather than "definitions or abstract rules".

<div align="right">(Baddeley, 1997)</div>

Skemp uses the terms *concept* and *schema* interchangeably, depending on the level of *interiority* he wishes to indicate. He describes a *concept* as a mental entity representing what is common to a number of objects or events. A *schema* is a mental model made up of a number of interconnected concepts within which can be represented a present state, a goal state, and other states. From this an individual can construct paths between any two states,

<div align="right">96</div>

provided that both are within this structure (Skemp, 1979a). The schema itself might preferentially be described as a concept when it is to be thought of as a single mental entity. This usage corresponds to some degree with that of Dubinsky, for whom a schema is "a more or less coherent collection of objects and processes" which is used to "understand, deal with, organise or make sense out of a perceived problem situation" (Dubinsky, 1992).

The term schema therefore proves useful in describing a cognitive structure through which an individual can navigate in order to reason about a concept (Skemp, *ibid.*). For the purposes of this thesis however, this is a larger structure than we wish to consider. Skemp's use of *concept* is more appropriate, but for reasons already explained this too could evoke too wide a range of connotations.

Minsky coined the term "frame" for describing the larger structures he thought should exist, and the notion has much in common with schemas as used above. He says

"We can think of a frame as a network of nodes and relations. The "top levels" of a frame are fixed, and represent things that are always true about the supposed situation. The lower levels have many *terminals* – "slots" that must be filled by specific instances or data. Each terminal can specify conditions its assignments must meet. (The assignments themselves are usually smaller "subframes.")"

(Minsky, 1975)

Hence, like schemas, frames can be nested hierarchically. However in the case of frames "default values" play an essential role: an evoked frame specifies the information it needs in order to be able to operate, and in the absence of any particular piece of information, assumes these values to hold. This accounts for an individual's capacity to import large amounts of information that is not explicitly stated when interpreting discourse (Davis, 1984). The notion of a prototype as used here corresponds closely to the idea of a frame-with-default-values, and this would be a viable alternative term. The former is chosen

because the focus here is almost always exactly upon with what an individual assumes in the absence of specific information, so for syntactic simplicity it is more appropriate to have a single name to describe this.

Minsky's notion of a frame has been used to explain well-known learner errors in school mathematics. Davis accounts for the phenomenon of "spurious generalisation" by suggesting that well-established frames are liable to be evoked inappropriately in similar but non-identical situations. Hence on encountering the multiplication 4x4 the learner may erroneously retrieve a better established "addition frame" and answer "8" (Davis, 1980 & 1984). The question of what cues retrieval is fascinating in itself. Davis explores it and makes suggestions such as the idea that frames related to the one currently in use are more accessible to working memory than others (Davis, *ibid*). For the sake of manageability this thesis takes advantage of the restricted field of study, assuming that a student's "convergent sequence" prototype is retrieved when encountering the words "convergent sequence". This sounds flippant in the abstract but is sufficient when the goal is to make sense of behaviour involving a very limited number of categories of objects.

Lakoff's version of the larger cognitive structures is *idealised cognitive models*, various kinds of which he claims account for prototype effects in different ways (Lakoff, 1987a&b). For example he suggests that some categories, such as *bachelor* (following Fillmore, 1982), do have necessary and sufficient conditions for membership. Prototype effects then arise in these cases from an imperfect match between the background conditions against which the idealised cognitive model is defined and other knowledge. Hence although John Paul II technically satisfies the conditions of being a bachelor, he is not properly described as such and is certainly not prototypical since he is not subject to assumed societal norms about marriage. Alternatively, some prototype effects are accounted for by describing them in terms of *cluster models*. In the case of "mother", for instance, there are several models such as the *birth model* and the *nurturance model*, and prototypes are generated by maximal overlap between these (Lakoff, 1987a,b). Lakoff's arguments highlight the range of ways in

which categories in a culture are structured, but again his formulation would cloud the issue here by covering a more complex range of phenomena than we need.

Presmeg's discussion of large cognitive structures is derived specifically from the context of mathematics. She describes the claim of many well-respected mathematicians that they think using a generalised image of an object which is abstracted from their experience with particular examples, and she terms this type of structure an *image schema* (Presmeg, 1992). Like frame-with-default-values, this closely matches the idea that *prototype* is intended to convey here, and image schema would be a sensible alternative but for the fact that the word "image" is loaded, associated as it often is with visual images (Presmeg, 1986a&b). Visual images will be an important aspect in explaining the particular challenge of Analysis, but individual views on what may be classified as a visual image vary widely (Presmeg, 1986b). So the term *image schema* may again be misleading, especially because the fact that not everyone thinks visually is important in the investigative part of this thesis.

In all, the terminology to be used is chosen to most naturally describe the phenomena under discussion, while highlighting the difference between the logical aspect of mathematics, that is its global structure as agreed upon by the mathematical community, and the psychological aspect, that is how individuals actually learn and think about it (*cf.* Skemp, 1976). Hence the discussion will be centred upon mathematical categories, student's individual ideas about what objects belong to the category denoted by a certain term, their apparent view of what is prototypical of that category and their use of their prototypes in reasoning about it. "Prototype" is used to mean a cognitive structure associated with a category, capturing what an individual assumes to be generally true for objects in the category. To clarify:

- It is not assumed that some actual prototype exists in the wider culture; prototypes are particular to the individual.

- A prototype need not be a single specific example. This might be the case, but more often it will be akin to an object-based image schema or a frame-with-default-values, so that the level of generality for a prototype is not fixed.

- For any given category, an individual may have several qualitatively different prototypes that correspond to more or less mutually exclusive subcategories. For example to answer questions about "furniture" they may consider their prototypes for "chair", "table" etc. (*cf.* Rosch's discussion of basic-level categories, 1975).

5.3 Origins and use of prototypes

5.3.1 Learning by exposure to examples

With the above in mind, the idea of a prototype is consistent with the idea that we learn about categories of objects by exposure to examples (Skemp, 1979a). This is usually an outstandingly successful process, but has inherent drawbacks in the field of mathematics. Vinner recognises the importance of the examples encountered in learning mathematical concepts, but cautions that this may account for cases in which students' *concept images* do not correspond to the formal definitions (Vinner, 1983 & 1992). In our terms it makes sense to suppose that the *evoked concept image* (Tall & Vinner, 1981) used for identifying examples and non-examples of functions is a prototypical function (which may be in the form of a visual image), or perhaps prototypes for a number of subcategories such as linear and quadratic functions (*cf.* Eisenberg & Dreyfus, 1994). In a discussion of similar ideas, Harel and Tall use the term *prototype* in a slightly different way, describing the idea that the student is initially presented with one or more *prototypes* for the abstract concept (here these would be called *specific objects*). They make the case for controlling such examples to make them maximally *generic*, facilitating the *generic abstraction* then necessary to see these examples as embodying the abstract concept (Harel & Tall, 1991).

The ideal is that students should construct prototypes that accurately or usefully represent the mathematical category agreed by the mathematical community. However, even careful choice of the specific objects to which a student is exposed cannot yield full control of this

process. As Harel and Tall point out, "these [specific objects] may function in a seriously erroneous way if the student abstracts the wrong properties". *Abstraction*, for them, is said to occur when "the subject focuses attention on specific properties of a given object and then considers these in isolation from the original" (Harel & Tall, 1991). This is not quite appropriate for describing the construction of a prototype as the term is used in this study: here such properties would be thought of as incorporated into a prototype rather than considered in isolation (in frame terms, they would be instantiated as default values (Minsky, 1975)). However, the problem remains that we have little control of which aspects of an object a student will attend to. The potential of this as a source for serious difficulties can be seen in work with very young children, in whom a focus on concrete referents rather than number symbols can make the difference between failure and success in elementary arithmetic (Pitta & Gray, 1997, Gray & Tall, 1994). Part 3 discusses the way in which the aspects of a presentation that a student attends to can affect their understanding of Analysis (see especially section 10.2).

5.3.2 Deduction

The notion of using a prototype to reason about categories of objects is also consistent with certain, though not all, theories of human deductive capacity. Overall, such theories can be classified as one of two types: *rule theories* and *mental model theory*. Rule theories posit the existence of some internal representation of logical rules regarding conjunctions, disjunctions, conditionals etc., which the individual uses in making and assessing the validity of deductions (Rips, 1994). Common errors, and differences in the difficulty of deductions (*modus tollens* deductions are less likely to be made correctly than those relying on *modus ponens*, for example), are then accounted for at the first approximation by the length of the chain of deductions required. More sophisticated models refine this approximation in a number of ways. They might posit differences in accessibility of different rules (rather like weighted feature models for categorisation), use both forward and backward chains of reasoning (Rips, 1994), be specialised to avoid spawning irrelevant inferences (Rips, *ibid.*), or additionally include "independently motivated pragmatic

principles that influence interpretation of surface-structure cues and can invite some extra-logical inferences" (O'Brien, Braine & Yang, 1994). Computer implementations of such rule theories can mimic human performance, following sequences of deductions matching those made by subjects asked to write down inferences they use in evaluating a conclusion (Bonatti, 1994, O'Brien, Braine & Yang, *ibid.*).

Mental model theory, an alternative to rule theories, takes a semantic rather than rule-based approach. It accounts for deductive competence with a simple model of reasoning:

1. The reasoner creates a mental model of the situation described by the premises.

2. They formulate a conclusion based on describing an aspect of this model which is not explicitly stated in the premises.

3. They then attempt to construct alternative models to refute this putative conclusion, and if no such models can be found, judge it to be true.

(Abbreviated from Johnson-Laird & Byrne, 1991)

Relative difficulty of deductions is then explained by considering the relative number of models required. For example, a modus ponens deduction (deducing B from A and $A \Rightarrow B$) requires only one initial model. This is usually denoted

$$[A] \quad B,$$

where the square brackets indicate that all the possibilities for A are exhausted. Modus tollens, (deducing *not A* from *not B* and $A \Rightarrow B$), is more difficult because it requires fleshing out the further models

$$[A] \quad B;$$
$$not\,A \quad B;$$
$$not\,A \quad not\,B.$$

Refinements to this theory include the proposition that the choice of which models to flesh out in a given situation is determined by beginning with a maximally informative premise and continuing in an order which maintains co-reference if possible (Johnson-Laird, Byrne & Schaeken, 1994). Known errors such as confusing "if...then" statements with "and" statements are then accounted for by the fact that these have the same or similar initial models. Fixation with initial models is consistent with the fact that people can be influenced in their conclusions by re-stating premises so as to encourage more fleshing out of alternatives (Legrenzi & Girotto, 1996). Similarly, fallacious inferences are accounted for by the subject's inability to keep track of all possible models. Such are generally consistent with the original premises, and erroneous by virtue of being possible rather than necessary given these premises (Johnson-Laird, Byrne & Shaeken, 1994).

There are problems with both mental model and rule theories, and many of the refinements outlined above arose in response to criticisms of each by advocates of the other (Bonatti, 1994, Johnson-Laird, Byrne & Shaeken, ibid.). In particular, each is open to the criticism that observation of behaviours does not permit direct conclusions about mental operations (Phillips, 1987). For the purposes of this thesis, it is sufficient to observe that mental model theory provides an account of human deductive capacity that is consistent with the idea that we reason about categories of objects using prototypes. The prototype corresponds to the initial model of the premises, putative conclusions about the category are drawn by making novel observations about this prototype, and these conclusions are checked against less prototypical members of the category. In addition, the mental models approach does not preclude the learning of the rules of logical inference. Johnson-Laird and Byrne (1991) suggest that formal logic as a discipline arose from a metacognitive awareness that we sometimes make errors in our deductions.

5.3.3 Generalisation and abstraction

Questions arising from the previous section include that of the extent to which checks of putative conclusions are instigated, and what indicators prompt an individual to an awareness that their conclusion is in doubt. This is addressed in this section through a discussion of *metonymy*, a mode of reasoning in which

"a part (a subcategory or member or submodel) stands for the whole category - in reasoning, recognition, and so on."

(Lakoff, 1987a)

Lakoff cites social stereotypes, typical examples, submodels and salient examples as structures that are used metonymically (Lakoff, 1987a). In keeping with the description of levels of generality in prototypes, some of these are specific cases and some are not.

Presmeg discusses the occurrence of metonymy in school mathematics, citing cases of over-generalisation which are similar to those discussed in the sections on proving and concept image in chapter 3 (Presmeg, 1992 & 1997). The key to the relevance of metonymy (and in fact the whole of this chapter) for university mathematics is the recognition of two different types of metonymy, termed *synecdoche* and *metonymy proper* by Presmeg (Presmeg, 1997). Synecdoche involves generalisation: it occurs when an individual deems a given conjecture true for their prototype(s), and by generalisation assumes it to be true for the entire category. Similarly when the individual decides whether a novel object is a member of the category by comparing it with these prototypes. In this case, no further effort is directed towards justification, and this approach corresponds to the *general cognitive strategy* as described by Alcock & Simpson (1999). It is represented diagrammatically in figure 2.
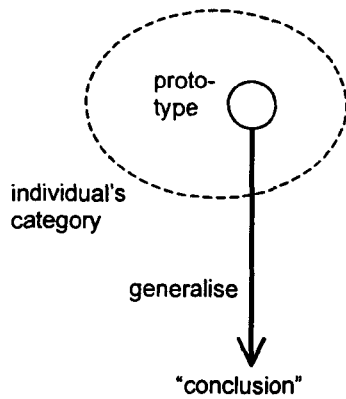
Figure 2: Generalisation from a prototype

Metonymy proper refers to using some property of the object to stand for the object itself in communication. In a mathematical context, this may be seen as the *abstraction* of some property of the prototype, and using this in further argumentation. This differs from Harel and Tall's description of abstraction in which properties of *objects* are *isolated* (Harel & Tall, 1991). Here the property abstracted from the individual's prototype is not necessarily isolated from the prototype for them, although it may effectively be so for a listener. Figure 3 contrasts reasoning using metonymy proper with synecdoche.
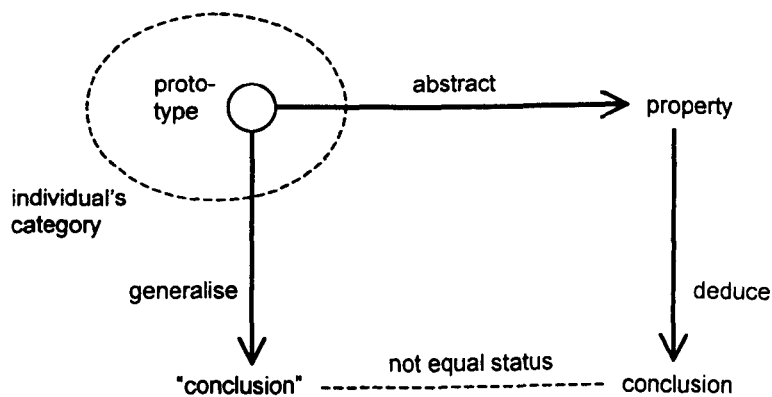
Figure 3: Contrasting generalisation with abstraction and deduction

Metonymy proper is incorporated into the basic structure of mathematics and synecdoche is favoured in general reasoning. There are very good reasons for this, resting upon the goals of any piece of reasoning. In mathematics, as in any technical field, accurate communication is necessary. Participants in the field must be sure that they intend to indicate the same objects when using a category word. Hence, while they may think using their individual prototypes, they communicate by abstracting verbally formulated properties and arguing in terms of these.

In everyday situations this type of precision is not usually necessary, and may be a hindrance to cognitive efficiency. In their own reasoning an individual may generalise directly from prototypes without needing to explicitly consider what properties of these make a conclusion sensible. Usually conclusions reached in this way will be correct, as is apparent from the discussion of concept images in chapter 3. In cases where communication is required, in general the reasoner relies on the other party's prototype being similar to their own, or on citing specific cases and inviting agreement with the generalisation. Again, most of the time this communication will succeed, so the approach is quick and efficient. This efficiency is usually valued over guaranteed accuracy in everyday reasoning (Balacheff, 1986).

This does not mean that in mathematics one always abstracts and in everyday situations one always generalises. It is a matter of degree and of the much more well-defined status of different types of arguments in mathematics. In an everyday context, if there is doubt of the validity of a conclusion, an individual may strengthen their argument by highlighting properties of their prototype that make it clearer. The extent to which this is required is

variable, and since a generalisation often suffices, we usually converse in this mode: conversations would take forever and sound like academic texts otherwise. In mathematics it is also acceptable in "conversation" to make generalisations, and as in everyday situations these rely on sufficient similarity of participants' prototypes. However, in written mathematics and other more formal presentations one cannot guarantee that an audience will understand as expected, so the conventional form is to argue in terms of abstracted properties, avoiding unexplicated references to prototypes. This does not mean that illustrations using visual or algebraic prototypes are not valuable, but they must not form a critical part of the argument[9].

Hence there is a fundamental difference in the flexibility of discourse standards between mathematical and everyday situations. The next chapter considers the way in which the corporate nature of the mathematical enterprise further complicates this situation, and identifies known student behaviours as manifestations of their lack of awareness of these issues.

5.4 Main points of chapter 5

- School mathematics involves calculations on specific objects, university mathematics involves proofs about whole categories of objects.
- Cultural categories exhibit prototype effects.
- Prototype effects can be accounted for using weighted-feature models, but these are unsatisfactory to explain human reasoning.
- It has been proposed that larger mental structures are used: these are variously referred to as schemas, frames, image schemata and idealised cognitive models.
- Here the term "prototype" is used to indicate such a structure: this corresponds to a frame-with-default-values, and the term is used with care given its history.

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[9] Section 3.2.2 cited caveats to this statement: mathematical arguments are rarely expressed fully formally. However, in a first course in Analysis, the standards are relatively fixed.

- The idea that such prototypes are used in reasoning is consistent with learning by exposure to examples, and with the mental models theory of human deductive capacity.

- Two different ways of using prototypes are considered here: direct generalisation from the prototype to the category (corresponding to synecdoche) and abstraction of a property from the prototype followed by deduction from this property (corresponding to metonymy proper).

- Generalisation is more common in everyday argumentation, since it is faster; abstraction is required in mathematical argumentation because of the extra precision it affords.

Chapter 6
Advanced mathematics

6.1 Overview

This chapter applies the framework developed in chapter 5 to the learning of mathematics at university level. In such mathematics the requirements extend beyond abstracting properties: here arguments must derive from an agreed framework of definitional properties. Categories delimited by definitions differ from everyday cultural categories both structurally (each object either is or is not a member and there is no logical recognition of prototypes), and in the way we reason about them (arguments must be traceable to the definitions). In this chapter, mathematical definitions are constrasted with dictionary definitions to provide an explanation for the breakdown in communication which occurs between lecturers and their students.

The second half of the chapter then uses the theoretical framework to give a plausible explanation for the student behaviours reviewed in chapter 3. It is suggested that in identification and reasoning tasks involving categories of objects, an individual's prototype appears as the first-evoked part of the concept image. The reasoner then may or may not perform a counterexample check or formulate their answer in terms of properties. Where such a formulation is inadequate or uses inappropriate properties, this produces the behaviours commonly associated with proofs and proving.

Finally, this chapter suggests a theoretical answer to the question of why Analysis should be particularly difficult. This answer will be further developed in chapter 11, after the theoretical framework has been linked to the investigative part of the study in chapters 7 to 10.

6.2 Definitions

This section examines the role of definitions within mathematical theory, and the impact that making such definitions has upon the logical status of mathematical categories. It relates this to the suggested psychological function of prototypes, and considers the logical and psychological functions of definitions appearing in dictionaries. Through contrasting these with the function of mathematical definitions, it offers a characterisation of student interpretations of advanced mathematics.

6.2.1 Defined categories and the object-property switch

In verbal communication, a good mathematical argument may proceed from any properties agreed upon by the participants. The standard to which these must be made explicit sometimes eludes students, with the result that their arguments are not universally valid for the category in question. In fact this is almost a necessary consequence, since *the act of choosing a property makes a category classical*; once a property is selected, the category of objects to which the argument applies is precisely determined. This is the mechanism by which it is *possible* for arguments in mathematics to have universal validity. This function of property specification usually remains below the level of mathematicians' consciousness, because the property is chosen to explicitly pin down some observation about a prototype. This property is then assumed to hold for the entire category, but an individual's prototype itself is not altered by this act.

Since prototypes are particular to individuals, the properties used within a community of practitioners may fluctuate. However certain choices will eventually attain the status of agreed definitions in order to facilitate communication on a large scale (*cf.* Lakatos, 1976).

Defined categories are classical in the same way as those determined by any property. A mathematical definition constitutes necessary and sufficient conditions for category membership, thus precisely determining the extension of the category and distinguishing no members as more or less prototypical. Advanced mathematics is built on this basis, taking "the notion of *property* as fundamental" (Gray et. al. 1999). Once a definition is agreed, both the original objects and any individual's prototype, constructed from experience with these objects, now become secondary in importance to the property itself. The formal category is constructed entirely by deductions from this property, and conjectures regarding the category do not gain the status of assertions until they are proved on this basis (Harel & Tall, 1991).

The mathematical community benefits from the systematisation inherent in this approach (Bell 1976). Precision in communication means that individuals physically far removed from each other can all contribute to the systematic growth of the subject. There are also gains for the individual, since as Harel and Tall observe in discussing the process of defining:

1. Any arguments valid for the abstracted properties apply to all other instances where the abstracted properties hold, so (provided that there are other instances) the arguments are more general.
2. Once the abstraction is made, by concentrating on the abstracted properties and ignoring all others, the abstraction should involve less cognitive strain.

(Harel & Tall, 1991)

Of course, this does not mean that individual mathematicians actually think in terms of definitions. Tall suggests that approaches vary, so that some proceed in an analystic fashion, but but many still use prototypes as a guide for formulating new ideas (Tall, 1992, after Poincaré, 1913). The essential point is that in all cases the mathematician is aware that arguments should be universally valid for the formal category. Hence their prototypes are

likely to have been constructed or reconstructed in accordance with the definition (*cf.* Harel & Tall, 1991), and when formulating conjectures based on a prototype they will take care to perform adequate counterexample checks (*cf.* Davis, 1984). There is no inconsistency here: individuals can learn to formulate arguments within the logical structure of mathematics while still reasoning using the same psychological strategies as in other contexts. Hence, even categories that are genuinely classical within a logical theory can exhibit prototype effects arising from psychological sources. Both Lakoff, and Armstrong, Gleitman and Gleitman note this effect in the case of "odd number" although they account for it in different ways (Lakoff, 1987a, Armstrong, Gleitman & Gleitman, 1983).

So, the introduction of formal definitions does more than subtley alter the meanings of some words (Skemp, 1979): it *changes the nature of the category under consideration*. This permits an explanation for the difficulty of the transition to university mathematics. Where in everyday contexts either generalisation or abstraction from prototypes can produce acceptable arguments, in advanced mathematics individuals must behave as though they have no prototypes and are working only with the formal categories. This is a fundamental difference, and moreover the requirement can remain largely opaque to students. This is partly due to the fact that the privelege of deciding the form of the defining properties is restricted to a small number of individuals. Students themselves do not ordinarily have this privelege and must work with the definitions as dictated by their teachers (Harel & Tall, 1991). The consequences of this for their learning of the subject are explored below.

6.2.2 Dictionary definitions

In advanced mathematics, in order to show that an object is a member of a category, one checks that it satisfies the definition. Similarly, in order to show that a conjecture is true of all objects in a category, one makes deductions from the definition. Hence, while formulating the detail of such deductions may be difficult, the "top level" (Leron, 1985b) or "proof framework" (Selden and Selden, 1995) is trivial. The existence of formal definitions might therefore be expected to render the student's task relatively straightforward in such

cases. However, as noted in chapter 3, the shift from "show that x is an X" to "show that x satisfies the definition of X" is not one that students readily make. The status of definitions in mathematics seems to elude them.

This can be explained by considering the role of definitions in everyday discourse. As previously discussed, everyday categories do not usually have necessary and sufficient conditions for membership, despite the appearance of "definitions" in dictionaries (Vinner, 1992, Lakoff, 1987a). This means that in general, while it may be possible to speak about any given category of objects in terms of properties, there is no universal agreement as to what these properties should be. Dictionary definitions of object-terms are *descriptions* of objects that are assumed to exist. Hence, while such definitions might be widely referred to, their logical status is the same as that of the properties an individual spontaneously abstracts from their prototype in order to reason in everyday discourse. The object-property switch, in which properties take primacy over objects, is limited to mathematics and other subject areas that have technical definitions (Harel & Tall, 1991).

Dictionary definitions do serve some of the same purposes as mathematical definitions. They are used to improve communication, usually being referred to for one of two purposes:

1. To introduce a concept to someone who has not met its word before (as described in Vinner, 1992).
2. To clarify meaning when it becomes apparent to two parties that they are using the same word in different ways (*cf.* Freyd, 1983, Keil, 1987).

However they are *not* used to reason in the same way as mathematical definitions. A good student in a non-technical context does not simply learn a definition, they use further experience with examples to deepen their understanding of the concept. Indeed, once an individual has constructed a prototype, the definition is often abandoned (Vinner, 1992).

This point can be clarified by returning to Lakoff's claim that prototype effects can arise in classical categories. In such cases he claims that rather than a culture admitting better and worse examples of a category (he uses the example "bachelor"), necessary and sufficient conditions for membership exist, but examples which satisfy these this may not satisfy other background assumptions (Lakoff, 1987a, after Fillmore, 1982). There is a significant philosophical question here since the understanding of any concept cannot happen in isolation; it depends on our other knowledge. Therefore there is no absolute reason why describing a category as non-classical is "correct". One might equally well accept Lakoff's view that some categories are classical and that other "unevenness" in the world accounts for the unequal status of their members. The important thing for our purposes is that that the effects are the same. For example, the Oxford Paperback Dictionary's definition of "swan" is "a large usually white water-bird with a long slender neck". It may be that this definition is "correct" or it may be that this is not a definition in the classical sense. Either way, it is not possible to say with absolute certainty that an object is a swan by checking that it satisfies such a dictionary definition, and neither is it reasonable to assume that deductions made from it will apply to all swans[10].

6.2.3 A communication breakdown

The result of the above is straightforward: students must learn (in their written work at least) to derive arguments from definitions rather than by either generalising from their prototypes or by abstracting properties of their own choice from these (Alcock & Simpson, 1999). Their lecturers have succeeded in making this switch, and this is incorporated into the way they view mathematical situations. However it is something to which they may never have given any conscious attention. Hence students and lecturers may find themselves working at cross-purposes. The diagrams below summarise this phenomenon.

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[10] Since first year undergraduates do not often encounter undecidable propositions, the possiblity of this situation occuring in mathematics is ignored here.
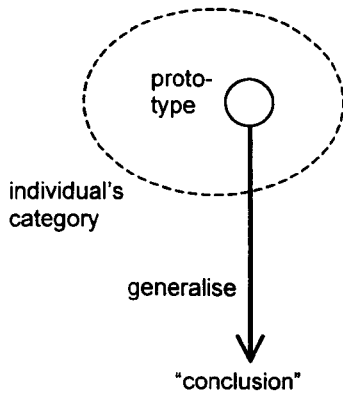
Figure 4: Generalisation

1. Under everyday circumstances, an individual usually reasons by inspecting their prototype(s) and assuming a generalisation to the whole category.
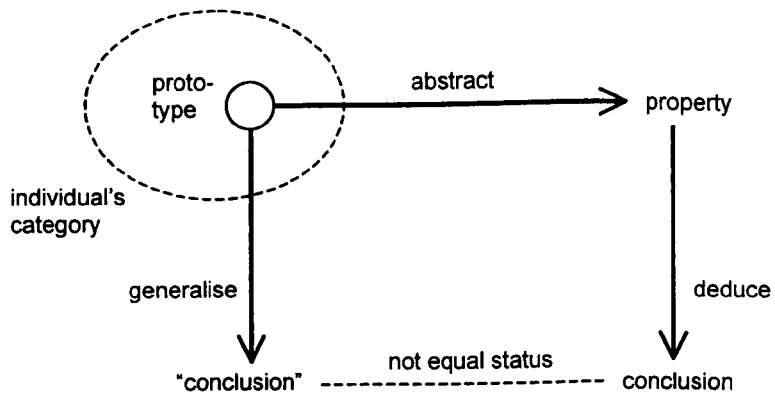
Figure 5: Abstraction

2. In mathematical argumentation, a property of the objects in question must be abstracted and a deduction made from this.
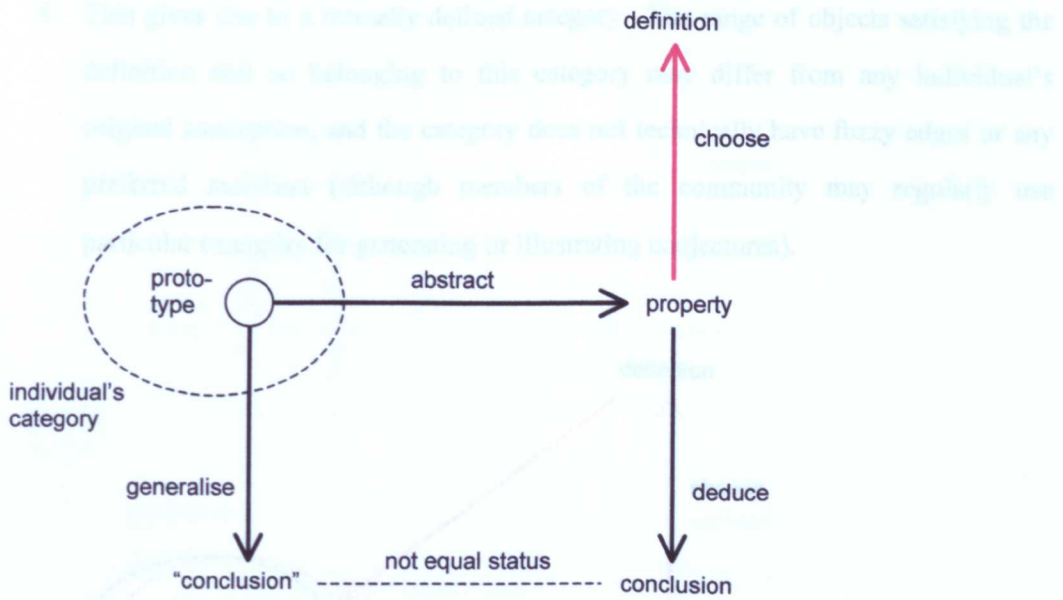
Figure 6: Choosing one property as the formal definition

3. Since mathematics is a corporate enterprise, a particular defining property is eventually agreed upon in order to facilitate communication (this new arrow therefore goes beyond individual reasoning).
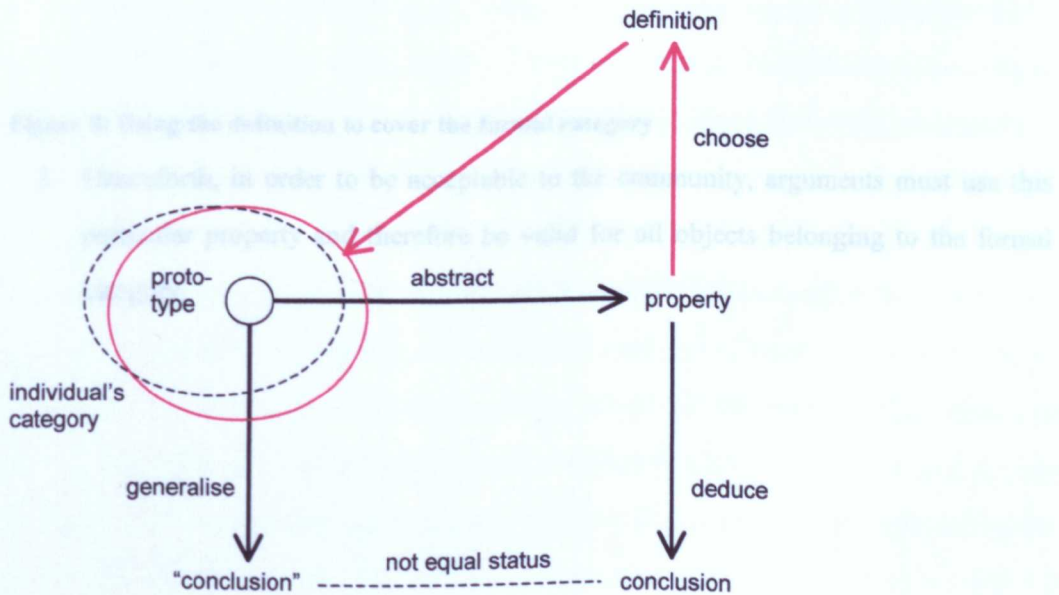
Figure 7: Constructing the formal category from the definition

4. This gives rise to a formally defined category. The range of objects satisfying the definition and so belonging to this category may differ from any individual's original conception, and the category does not technically have fuzzy edges or any preferred members (although members of the community may regularly use particular examples for generating or illustrating conjectures).
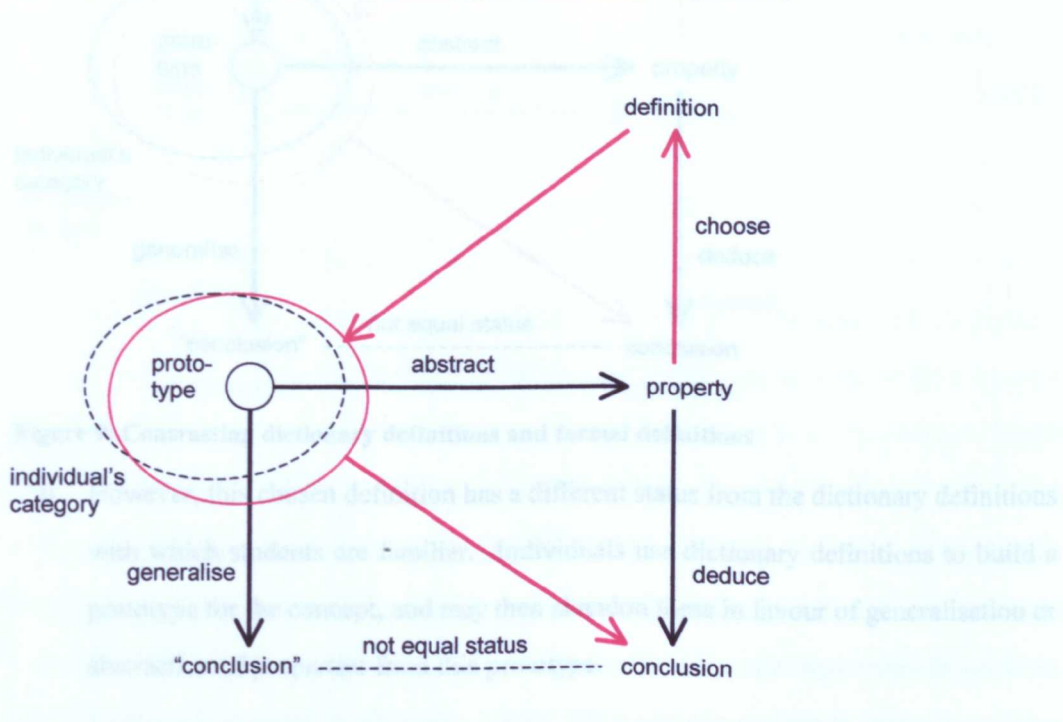
Figure 8: Using the definition to cover the formal category

5. Henceforth, in order to be acceptable to the community, arguments must use this particular property and therefore be valid for all objects belonging to the formal category.

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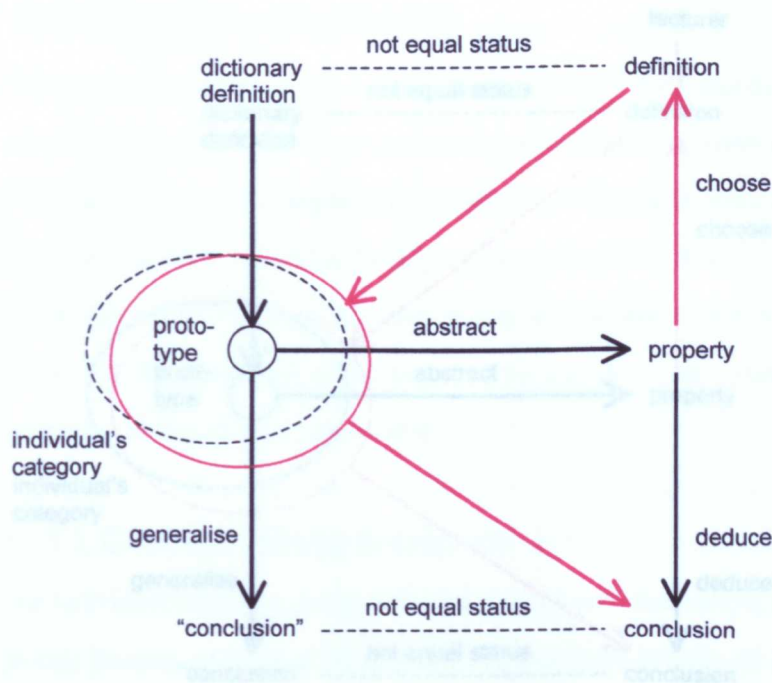
Figure 9: Contrasting dictionary definitions and formal definitions

6. However, this chosen definition has a different status from the dictionary definitions with which students are familiar. Individuals use dictionary definitions to build a prototype for the concept, and may then abandon these in favour of generalisation or abstraction of properties from this prototype.

Figure 10: Lecturer's input

7. So a lecturer gives the student a definition, expecting them to construct the formal concept from this definition and to work with it in future (to follow the red arrows). He or she may also give examples, with the intention that these should serve as illustrative (Alcock & Simpson, 1999). However the student is trying to build a prototype for use in the usual way, and may abandon the definition once a prototype is established, or even construct this prototype on the basis of the examples and ignore the definition altogether.

Crucially, it may not be apparent to either party that this communication breakdown has occurred. Particularly in cases where the technical meaning of a term is similar to (in all probability derived from) the everyday meaning, there will be much overlap between a students' idea of what is in the category and the formal version. Hence there may be very few cases where conflict seems to arise, and both student and teacher may feel that they are communicating successfully.

6.3 Recontextualising

The previous section concluded the development of the theoretical framework, explaining the difficulty of the transition to university mathematics in terms of the change in the nature of categories that accompanies the introduction of formal definitions. This section returns to the second question raised at the beginning of chapter 5, that of why Analysis should prove particularly difficult. First, however, it sites the framework within the literature reviewed in chapter 3, by viewing the phenomena discussed there as consequences of the everyday reasoning strategies described above.

6.3.1 Concept image and limits

An individual's concept image has been defined in various ways, in some cases focusing on visual imagery (Vinner, 1983), in other cases more broadly to include all of a student's experience with the concept (Vinner, 1992, Tall & Vinner, 1981). In the latter description the concept image is akin to Skemp's or Asiala et. al.'s notion of a schema (Skemp, 1979a, Asiala et. al., 1996). In the former it corresponds more closely to the notion of a prototype as used here. In either case, an individual's prototype is likely to be the part of the concept image *evoked* (Tall & Vinner, 1981) in the context of tasks involving identifying an object as a member of a category, or evaluating a conjecture for a whole category. In these cases it functions in the same way as the initial mental model in Johnson-Laird and Byrne's description of the operation of human deduction (Johnson-Laird & Byrne, 1991). The fact that students reason using their *concept images* rather than the concept definitions therefore arises as a consequence of the general cognitive strategy of examining a prototype and generalising to the whole category (Alcock & Simpson, 1999).

This makes it clear that far from being ignorant or willfully perverse, the student who reasons using this part of their concept image is employing strategies that on a daily basis prove extremely successful and efficient. Vinner's observation, that reasoning in this way often brings correct answers, then further clarifies why it should be so difficult to adjust to

the *modus operandi* of advanced mathematics (Vinner, 1992). Not only does adjusting to a theory based on defined categories require a fundamental shift in perspective, but in any given situation the gains from this are not visible: the potential for systematisation (Bell, 1976) is unlikely to be seen as a sensible goal by the student who is struggling to understand a single definition but who can already answer the majority of the questions they are asked about the defined category.

6.3.2 Proofs and proving

The goal of verification is usually more visible than that of systematisation, but the rigorous way in which this is implemented in advanced mathematics through the use of properties is not. Again student behaviours may be explained within the theoretical framework outlined above. In terms of categories, a proof should be convincing to an individual if it applies to each object that they think belongs to the category under discussion. There are exceptions, since this is not universally the standard students apply: in the case of *external conviction proof schemes*, conviction depends on the word of an outside authority or conventions of algebraic manipulation (Harel & Sowder, 1998). In these cases students are not relying upon their own reasoning about categories of objects, so the framework does not apply here[11].

Empirical proof schemes, resting on inductive arguments or perceptual observations (Harel & Sowder, *ibid.*), may however be explained within this framework. In these cases the student reasons by direct generalisation from their prototype to the category. This does not necessarily indicate that they believe their argument to be universally valid (Bell, 1976); it is possible that the student does not conceive of universal validity as a desirable goal and is only aiming to show that the conjecture is true for some central or useful range of cases (Galbraith, 1981). They may not even consider exhaustiveness attainable, either in the case in hand or in general, as indeed it is usually not in nonclassical categories.

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[11] These phenomena are explored further in part 3, which examines the relationship between prototype-based reasoning and authority-guided proofs for such students. See in particular section 9.2.1.

Alternatively the student may believe in the possibility of exhaustiveness and in its desirability as a goal, and honestly believe that this is what he or she has achieved; that the example(s) used is (are) generic, and the argument will generalise (*cf.* Tall, 1995). This may be an invalid assumption if the student's idea of the category does not correspond sufficiently well with the formal one, and it may be recognised as such by teachers who are aware of the existence of different types of example. However the student may be right and may have shown considerable insight, but the argument may remain unsatisfactory because it relies on direct generalisation rather than abstraction of properties (Moore, 1994).

From the point of view of everyday reasoning, in either case the student's behaviour is sensible and not at all random. The goal might be to establish that the conjecture works in "most" cases (with belief or not in the possibility of exhaustiveness), or it might be to provide this exhaustiveness, but be based on an inadequate understanding of the formal category or of the standards of argument required. In either case the steps taken by the student are moving them towards their goal. The two goals are indicative of very different approaches to mathematics however, with the person who believes they are producing an exhaustive check being much closer to an expert mathematical approach.

The framework also encompasses a phenomenon that seems pathological to expert mathematicians: that of making further checks despite having access to a deductive proof (Chazan, 1993, Fischbein, 1982). In such cases it may be that the student simply does not understand the details or the structure of the proof itself (Bell, 1976, Leron, 1985a&b). For a student who does not understand the proof but is aware that the issue is that they should be convinced of the validity of the conjecture, checking that it holds for other members of their category may be the only available way of reaching this sense of conviction. Alternatively, the student may understand the proof but believe it applies only to a specific figure drawn to illustrate the argument (Chazan, 1993). In this case, the problem is that the student fails to recognise the first of Harel and Tall's advantages of the property-based approach, that the

argument applies in all cases in which the property holds (Harel & Tall, 1991). In effect they fail to generalise in a way that *is* mathematically acceptable by failing to view the properties as central.

Even proofs based on more advanced *transformational proof schemes* (Harel & Sowder, 1998) may fail to meet the standards expected at university level, by virtue of depending upon properties spontaneously abstracted from prototypes rather than upon the agreed definitions. Cary's case, as described briefly in chapter 4, is an excellent illustration of this phenomenon and will be expanded upon in section 9.2.6.

6.3.3 Why is Analysis hard?

So the theoretical framework unifies theoretical explanations of a number of known phenomena in university level mathematics education. It also provides a natural answer to the question of why Analysis, of the subjects studied in the first year of a mathematics degree, should prove particularly challenging.

This answer rests on the observation that the definitions of Analysis are logically complex (Dubinsky, Elterman & Gong, 1988), whereas *visual* representations of objects and results are compelling and readily available. Hence prototypes can be constructed directly from perceptual experience (Tall, 1995). Such visual prototypes are conducive to arguing by direct generalisation: it is easy to convince oneself that an increasing sequence which is bounded above must converge, without recourse to abstracting properties or formulating algebraic arguments. Anyone who has tried to persuade students to generate proofs in Analysis will recognise a result of this, as they will have taken part in exchanges akin to the following:

> Teacher: And *why* is that true?
> Student: It just *is*!

Considerable exasperation results when a student's (visual) prototypes provide a strong feeling of intuitive intrinsic conviction (Fischbein, 1982), but they are asked to justify their assertions precisely. The requirement to use the complex definitions then means that they are often in a position from which they must prove something they consider obvious using algebraic formulations which make them feel insecure (Gray et. al., 1999).

The relative intensification of this problem in Analysis is highlighted by comparing it with, for example, elementary Group Theory (Alcock & Simpson, 2001b). Definitions in Group Theory may be long, but they are logically simpler than those in Analysis, and other types of representation are less readily available. Hence in Group Theory it is likely that more students will produce work which competently makes use of the definitions. This should not be taken as an indicator that they necessarily understand the structure of the subject or the role that definitions play within it, only that these are easier to handle and that they have no obvious other option. Indeed it would be interesting to investigate the consequences of the difference for student complaints. Since the visual representations used in Analysis occupy "an intermediate position between realistic pictures and verbal/symbolic representations" (Gibson, 1998), these appear more concrete and we may find that complaints are distributed so that Analysis attracts more "It's obvious but I don't know how to prove it" and Group Theory more "It's just too abstract".

So, paradoxically, Analysis may be difficult not only because the material is complex *per se*, but because it is initially *less* "abstract" than other beginning university subjects; because the availability of visual representations means that *more* students initially have access to a way of meaningfully understanding the concepts. In addition, the understanding gained in this way means that they feel less need for the complex algebraically-expressed properties from which the formal categories are constructed. This latter point can be clarified by considering cases in which properties are used or sought by a learner. In everyday argumentation, properties are expressed when results are not obvious or more justification is required to convince a second party. As argued above, the student is unlikely to often feel a

need for this in Analysis. Alternatively, a learner may seek properties when learning about new categories if these categories are very abstract or if distinctions between them are perceived to be fine, so that it is difficult to construct prototypes to perform identification tasks reliably. In such cases, if an authority can provide properties, this makes the task easier: checking to see whether these are satisfied is quicker than constructing and using prototypes and serves as a shortcut until this can be achieved and the properties can be abandoned[12]. Once again, in Analysis the categories do not seem too abstract to be accessible. Also, in cases such as convergence of sequences, using the provided defining properties does not, by any stretch of the imagination, constitute taking a shortcut.

So in Analysis, the learner is likely to be attracted by the simple visual representations and repulsed by the complex properties used. This compounds the difficulty of adjusting to the fundamental change in the nature of categories as one makes the transition to university mathematics, thereby plausibly accounting for the particular difficulty of this topic.

6.4 Main points of chapter 6

- The act of abstracting a property renders a category classical and so allows arguments to be universally valid.

- Within the mathematical community, certain abstracted properties gain the status of definitions.

- The choice of definitions permits a gain in systematisation and reduced cognitive strain for the individual, but it changes the nature of the categories involved.

- Individuals may still think using prototypes, but arguments presented more widely must handle categories using the appropriate definitions.

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[12] This use of properties became clear to me after a conversation I had with my brother, Keith, while watching MTV Base. I asked him to explain the difference between rap and hip hop, and when he struggled to answer, I began asking questions: were the lyrics and music more integrated in one? Was it to do with the beat? When he rejected each of these possiblities, I realised that it was going to take a long time to learn the difference by listening to examples, and that what I was looking for was a property to use as a shortcut. Even then, I was *not* expecting an answer that would let me be right in identification tasks all the time, since I do not expect such categories to have clearly defined boundaries.

- Dictionary definitions have some properties in common with mathematical definitions, but are really descriptions of objects: the objects remain primary.

- This discrepancy results in a communication breakdown between students and their lecturers: students may be trying to build prototypes to use in reasoning, rather than acquiring formal categories by construction from the definitions.

- In reasoning about membership of categories, the use of prototypes manifests itself as reasoning using concept images.

- In producing and assessing proofs, reasoning using prototypes manifests itself in "proving by example" (where the student assumes a generalisation to the rest of the category) and checking of particular cases in the presence of a deductive proof (where it is not recognised that the properties abstracted apply more widely).

- Analysis is hard because its definitions are logically complex and visual representations are simple and widely available: the latter are ideal for forming prototypes and for reasoning by generalisation, so the student is drawn towards these and is less likely to adopt the formal formulations.

Part 3

Investigation

Introduction to Part 3

This part details the qualitative investigation into the understanding acquired by the students on the two different courses. It begins by describing the classifications of student behaviours that emerged from the grounded theory analysis of interview data. Subsequent chapters describe and illustrate the relationships between those behaviours, in substantive detail and in general. Finally, the theoretical framework is strengthened by considering a group of students whose behaviour it does not yet encompass. More detail is given below.

Chapter 7 forms a bridge between the theoretical and investigative parts of the thesis. As described in section 4.5.4, the data analysis prompted a return to the literature on typologies of understanding; chapter 7 begins by reviewing this literature, relating it to the theoretical framework developed in part 2. It then describes the typology chosen to classify behaviours in the investigative study, and illustrates this with extracts from interview protocols. Chapter 8 comprises one section for each of the emergent causal factors as briefly described in section 4.5.5. First, the student's visual or nonvisual approach to reasoning. Second, their interpretation of their own role as a learner of mathematics, characterised by whether they have an internal or an external sense of authority. Each of these sections is similar in structure to chapter 7, comprising a reprise of the relevant literature and illustrations of the distinction made here.

Both chapters aim to make the distinctions clear, so the differences are presented starkly at this stage. This renders the work vulnerable to the criticism that such broad distinctions overlook the complexity of real situations, as in criticisms that the relational/instrumental dichotomy ignores the dynamic aspects of the process of understanding (Tall, 1978), or that the simplistic division of students into visualisers and nonvisualisers does not explain how learning does and should take place (Zazkis, Dubinsky & Dautermann, 1996). However the dichotomies used in this study are well grounded in student behaviours and, as is detailed through chapters 7 and 8, are theoretically viable as representing a genuine split in the student population. In addition a clear and relatively simple conceptual basis is necessary to render complex social situations comprehensible.

Chapters 7 and 8 make no attempt to discuss interactions or relationships amongst the factors described. This is done in chapters 9 and 10, where descriptions of different students as exemplifying cases incorporate individual variation within the distinctions used. Chapter 9 describes groups of students, classified by their position regarding the causal factors and the intervening factor of the course they attended. By considering each combination arising in turn, it essentially describes a function:

{visual, nonvisual} x {internal, external} x {lecture, new} → types of understanding

reasoning style   *sense of authority*   *course*   *learning outcomes*

As described in chapter 4, the data is not sufficiently detailed to completely observe the process by which the "input" factors lead to the "outcome". However, drawing on existing literature, theoretical suggestions are made regarding the operation of this process.

Chapter 10 takes a different slice through the data, considering the influence of each causal and intervening factor in turn. In this way it strengthens the explanations of process by establishing their consistency throughout the groups described. At this stage it is possible to

predict types of understanding which would be acquired under the combinations of factors not arising in this study, and such predictions are given.

Chapter 11, the final chapter in this part, returns once more to the theoretical framework. It uses the framework directly to characterise the types of reasoning employed by students displaying different combinations of causal factors. In this way it identifies a group whose behaviour cannot yet be explained in these terms, and strengthens the theoretical framework by using literature on process-object theories and pseudo-empirical abstraction to fill this gap.

Chapter 7
Types of understanding

7.1 Typologies of understanding

Skemp's paper on relational and instrumental understanding (Skemp, 1976) sparked a debate regarding both the utility of distinguishing types of understanding (Tall, 1978), and the particular types which may be usefully distinguished in this way. The former was discussed in section 3.4.1, which distinguished noun and verb uses of the word "understanding". Owing to the tension between breadth and depth in the study, as discussed in section 4.3.1, process ("understanding" as a verb (Sierpinska, 1990 & 1994)) is not realistically accessible. Hence the latter discussion becomes more pertinent at this stage, in that we wish to compare student understanding at various points in the term with that of an expert (Nickerson, 1985). This chapter reviews the debate on typologies of understanding, relating those suggested by various authors to the theoretical framework established in part 2. Through this it explains the classification used in this study. Each of the authors discussed here is also concerned with the implications of their classifications for learning and teaching. In this thesis, these issues belong to the spheres of beliefs about mathematics and what can be learned regarding the appropriateness of different pedagogical strategies. Hence they are discussed in the next chapter and in part 4. Here the focus is on the cognitive aspects of the posited types of understanding.

7.1.1 Suggested types of understanding

All of the authors involved in the debate recognise the value in distinguishing between instrumental and relational understanding. Their contributions add nuances to the meaning of relational understanding, and more substantially introduce a further type. The former are discussed first.

On the basis of discussion of Skemp's model with teachers, Byers and Herscovics (1977) suggest extending it with two further types of understanding. The first of these they term *intuitive understanding* and define as "the ability to solve a problem without prior analysis of the problem". They cite Bruner's definition:

> "Intuition implies the act of grasping the meaning or significance or structure of a problem without explicit reliance on the analytic apparatus of one's craft...It precedes proof; indeed, it is what the techniques of analysis and proof are designed to check."
>
> (Bruner, 1965)

In this sense, *intuitive understanding* would seem to incorporate both Fischbein's *anticipatory* and *affirmatory* intuitions. The first of these indicates the feeling of having solved a problem and the second occurs when a statement is seen as self-evident, but in both cases the intuition occurs in the absence of, or prior to, the availability of explicit justification (Fischbein, 1982). In the context of this thesis, concerned as it is with reasoning about categories of objects rather than more complex problems, affirmatory intuitions are therefore most relevant. Intuitive understanding in this context might be said to correspond to direct generalisation from a prototoype, since it is on this basis that an individual considers a statement self-evident for a category. Using diagrams from chapter 6, this might be represented as in figure 11 below.
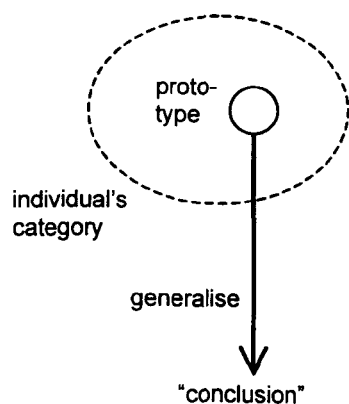
Figure 11: Generalisation (self-evidence or an affirmatory intuition)

A concept similar to Byers and Herscovics' intuitive understanding is Buxton's *observational* understanding, at which stage a pattern is perceived but not yet explained (Buxton, 1978). Buxton contrasts this with a further type termed *insightful*, which he characterises as fully relational and at which stage it is known *why* the pattern noticed at the previous level exists. In terms of reasoning about categories of objects, this seems to more closely depend upon explicit use of properties, and is therefore closely related to the branch of the diagram involving abstraction and deduction, as in figure 12. In a later paper, Skemp also incorporates this distinction by distinguishing *intuitive* and *reflective* levels of understanding as well as the underlying types (Skemp, 1979b).
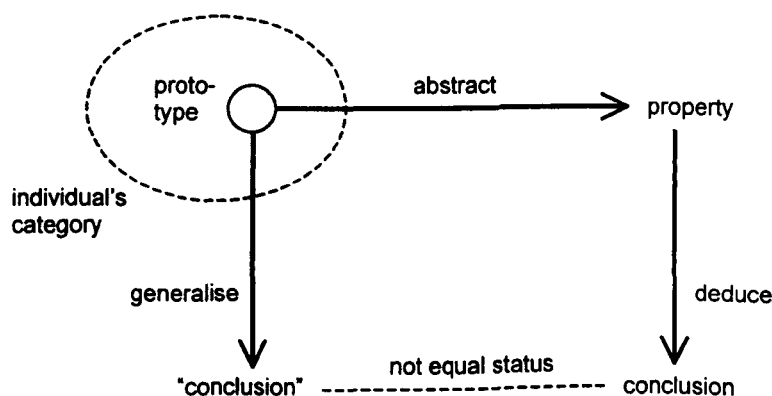
Figure 12: Abstraction and deduction (insightful or reflective understanding)

A type of understanding which is more clearly related to abstraction and deduction is Skemp's *logical* understanding. This is derived from Byers and Herscovics' second proposed extension, which they term *formal* understanding. This, they argue, should be distinguished since relational and instrumental understanding are concerned with the content of mathematics, and may exist in the absence of an ability to express this understanding correctly. They illustrate this using the hypothetical example of the student who, when solving $x+3=7$, writes:

$x+3=7$

$=7-3$

$=4.$

Such a student would be said to have relational but not formal understanding (Byers & Herscovics, 1977). Backhouse uses the term *formal* in a slightly different way, in relation to the ability to recognise a concept when represented in different written forms. There is no real conflict here: Backhouse simply places the emphasis on the translation into written form rather than the way in which logical deductions are expressed (Backhouse, 1978). In the context of reasoning about categories of objects, it makes sense to suggest that Backhouse would emphasise the abstraction or recognition of a property of some prototype, and Byers and Herscovics the writing of deductions made from this property.

Skemp accepts this as a valid different type of understanding on the basis of the student's goals: he relates the difference between relational and what he calls logical understanding to that between being convinced oneself and being able to convince other people (Skemp, 1979b). This is consistent with Buxton's (again slightly different) use of the term *formal*: Buxton uses it in relation to the proofs of formal mathematics, and says that this type of understanding is attained when the learner sees the need for, and the nature of, this type of proof (Buxton, 1978). This type might then be said to incorporate both the ability to argue as in the right hand branch of figure 12, and the awareness that this is necessary in mathematics. It is not clear whether either Buxton or Skemp intend this to include an

awareness of the formal definitions agreed by the mathematical community. Buxton's example does not explicitly use these and so could involve only spontaneously-abstracted properties, and overall the discussion is based in school mathematics where formal definitions are rarely used.

7.1.2 The chosen typology

The typology used henceforth incorporates Skemp's definitions of *instrumental, relational* and *logical* understanding (Skemp, 1979b). To maintain a degree of simplicity and coherence through the data analysis, further nuances such as Skemp's *levels* are not used explicitly, although they will be referred to cases where they are particularly relevant. However the diagram central to the theoretical framework indicates that one further *type* of understanding can usefully be distinguished in discussions of university mathematics. Hence in this study:

- *Instrumental understanding* is evidenced by the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works[13].

- *Relational understanding* is evidenced by the ability to deduce specific rules or procedures from more general mathematical relationships.

- *Logical understanding* is evidenced by the ability to demonstrate that what has been stated follows *of logical necessity*, by a chain of inferences, from (i) the given premises, together with (ii) suitably chosen items from what is accepted as established mathematical knowledge.

(Skemp, 1979b)

- *Formal understanding* is evidenced by the ability *and inclination* to reason about categories of objects using the definitions agreed by the mathematical community.

---

[13] The phrase "is evidenced by" is due to Backhouse, who points out that "we are unable to observe our pupils'...schemas directly" (Backhouse, 1978). Following this observation, it might also be noted that "ability" is similarly unobservable, but that we can observe the actual application of rules.

It might seem inappropriate to use the term *formal* with yet different connotations, but this seems to be the most appropriate adjective given the common usage of the term "formal definitions". To clarify, it is not to be confused with the use of the term *formal* by Byers and Herscovics or Backhouse, since these authors use it to label aspects associated with Skemp's logical understanding (Byers & Herscovics, 1977, Backhouse, 1978, Skemp, 1979b). Here the term indicates the ability and inclination to construct formal categories based on the definitions and to reason using these definitions:
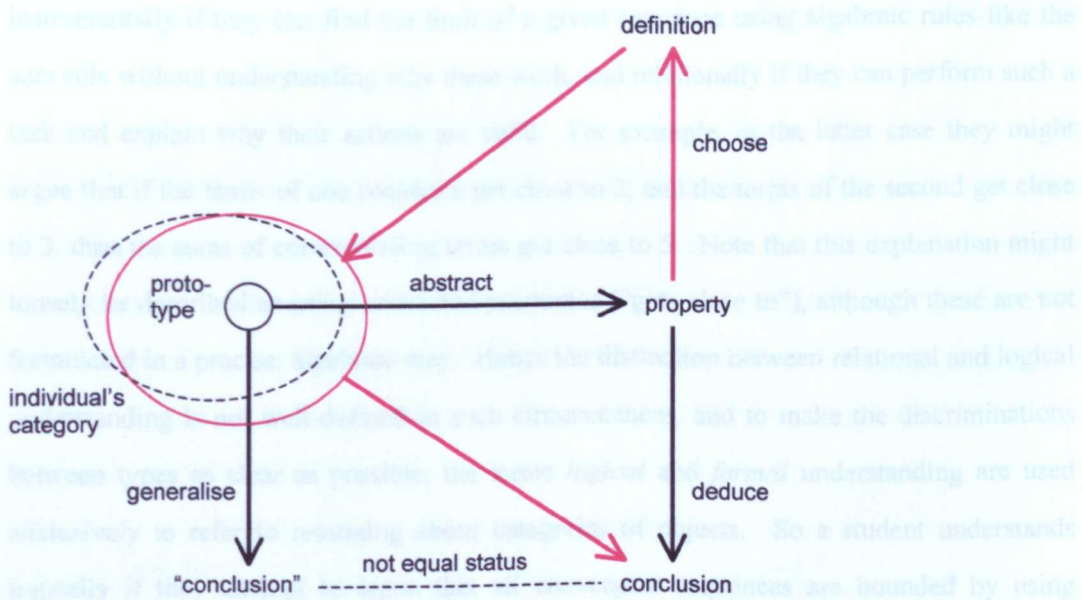
Figure 13: Formal understanding

This usage is also distinct from Piaget's in his description of the *formal operations* stage in cognitive development (Philips, 1969). Piaget describes a stage at which an individual can reason logically from given premises without confusion due to their internal meaning. In the terminology used here, this ability belongs to logical understanding. Formal understanding, by contrast, refers to adherence to the meanings ascribed to terms by agreed definitions. To see the distinction, consider that from the information that "All boys like spinach" and "John is a boy" one can conclude that "John likes spinach" (logical

understanding) without the need for a community to agree on definitions of "boy" and "spinach" (formal understanding)[14].

In the remainder of the thesis, the terms are used as follows. The data analysis remains as faithful as possible to the original referents of the terms *instrumental* and *relational* understanding. These were introduced to describe understanding in school mathematics, and they are used here to describe students' competence in handling calculations involving specific mathematical objects. So, for instance, a student is said to understand instrumentally if they can find the limit of a given sequence using algebraic rules like the sum rule without understanding why these work, and relationally if they can perform such a task and explain why their actions are valid. For example, in the latter case they might argue that if the terms of one sequence get close to 2, and the terms of the second get close to 3, then the sums of corresponding terms get close to 5. Note that this explanation might loosely be described as using abstracted properties ("gets close to"), although these are not formulated in a precise, algebraic way. Hence the distinction between relational and logical understanding is not well-defined in such circumstances, and to make the discriminations between types as clear as possible, the terms *logical* and *formal* understanding are used exclusively to refer to reasoning about categories of objects. So a student understands logically if they attempt to argue that all convergent sequences are bounded by using properties associated with convergent sequences, and formally if these properties are the agreed definitions.

Finally, it should be emphasised that the four types of understanding are <u>not</u> assumed to be hierarchically related. Hence for any particular topic a student might have, say, good instrumental understanding, adequate relational understanding, poor logical understanding

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[14] In fact such syllogistic reasoning is not the most interesting aspect of logical understanding for this study. Short examples of the more central aspects are difficult to give, as students who have logical but not formal understanding often do not succeed in completing their arguments. Extended illustrations will be seen in section 9.2.6.

and good formal understanding. In practice of course, certain combinations are unlikely, and combinations which do occur are discussed through the remainder of this part.

7.2 Manifestations of the types in this study

Where the previous section established links between the theoretical framework and posited types of understanding, this section explains the use of the chosen typology in the investigative part of the study. Each type of understanding is illustrated with interview extracts. To reiterate the point made in the introduction to this part, at this stage no attempt is made to explain how and why these behaviours arise. Such analysis appears in chapters 9 and 10.

7.2.1 Relational and instrumental understanding

Since the terms *instrumental* and *relational* understanding are used to describe students' competence in handling calculations involving specific mathematical objects, the main interview questions giving information about these types are the week 5 <u>sequence question</u> and the week 9 <u>series question</u>.

Week 5

Let $a_n = \dfrac{1 + \cos n}{nx}$, where x is a real

number which remains fixed as n varies.

When is the sequence (a_n) convergent?

(You should provide as complete an

answer as possible)

Week 9

When does $\sum \dfrac{(-x)^n}{n}$ converge?

Justify your answer as fully as possible.

It is appropriate to consider these tasks as involving specific objects: although these questions require establishing membership or otherwise of the categories of convergent sequences and series respectively, in each case this may be achieved for any specific value of x by using rules learned in the course.

Relational understanding is discussed first since, for reasons which will become clear, instrumental understanding is most clearly explained in this context as the absence of relational understanding.

Presence of relational understanding

Students with relational understanding are able to make good progress with the above questions, for the following reasons.

Having relational understanding means knowing how and why rules work or results hold; in effect, assimilating these as meaningful in terms of more familiar or more general principles (Skemp, 1976, Byers & Herscovics, 1977). In the *logical* structure of Analysis, perhaps the most important such principle is that of comparisons between objects. "Facts" about certain simple objects are established: the sequence $(1/n)$ has limit zero for example. Squeeze rules and comparison tests then rely on making explicit comparisons with these simple objects. Further implicit comparisons are made when using algebraic rules for finding limits of sequences: one might achieve this by recognising that the algebraic expression defining a sequence is made up of simpler components whose limits are already known. *Psychologically*, organisation and use of knowledge according to this principle is often evidenced in the study. For instance, Tom demonstrates relational understanding when he addresses the sequence question by using known results and coordinating these with "size" comparisons.

T: Well, because it's cos n, the greatest cos n can be is 1, so the greatest that this can

be is... on the top is 2, so...whereas the bottom, is always going to get larger and

larger

ST5, 227[15]

T: Then the bottom's just going to...increase and increase and increase and even if x is

negative it's going to be increasing by so much that the top is going to become

irrelevant, and it's just going to tend to zero.

ST5, 465

For students who organise their understanding of the material relative to such principles, the common examples and rules fit together as a coherent network. This permits good recall of the material (Skemp, 1976), and facilitates reconstruction of any elements of the network which are temporarily forgotten (Duffin & Simpson, 2000). In particular, knowing why rules work depends upon knowing what properties of the object concerned make these valid. Hence Jenny's behaviour evidences relational understanding when she is able to confidently state conditions needed for tests for convergence of series, as well as specifying values for which no conclusion can be drawn.

J: It [*the alternating series test*] says it converges when that's null, decreasing and

non-negative. So we can show when that's null, non, non-negative and decreasing.

And then we'll know it's convergent...

JK9, 403

J: No I'm looking at the values for which you can't do the ratio test. Because the ratio

test tells you what happens [*when the limit of the ratios is*] between, nought and 1,

---

[15] Letters refer to the pair of student participants, the first number to the week number of the interview. Further numbers refer to text unit numbers in the NUD*IST database: in each case the text unit number of the first student contribution is given.

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and when it's greater than one…it doesn't tell you what happens when it's equal to 1, does it.

<div align="right">JK9, 565</div>

Recall of such detail means that the student with relational understanding can reliably judge the applicability of a given theorem to a new case (Nickerson, 1985). Since exemplifying objects are important in such a network of results, such students also become familiar with ways in which these may be denoted. As a result they are in a good postition to be able to translate between different representations of an object (Nickerson, *ibid.*), and thus to improve their grasp on a new problem or open up new avenues for progress on encountering difficulties. For example when confronted with the week 9 question, Wendy immediately begins to expand the given series. This allows her to more easily identify both links with known examples and potentially useful results.

I: Can you read out what you've written when you've written it?

W: Erm, just, if x is bigger than or equal to 1 then the series will be erm…minus x, plus x squared over 2, minus x cubed over 3, plus x to the 4 over 4, and then continuing on.

<div align="right">WX9, 163</div>

W: If x is 1, it'll be, minus…we've done that, that converges, doesn't it. Because we did the 1 plus a half plus a third plus a quarter plus a fifth doesn't converge yet…when you have the alternating…what was that alternating series test?

<div align="right">WX9, 179</div>

Overall, relational understanding enables robust and accurate recall and use of material dealing with specific objects.

Absence of relational understanding: attempts to gain only instrumental understanding

Instrumental understanding alone is rarely demonstrated by the students in this study, for the following reasons. To acquire instrumental understanding a learner needs exposure to repeated near-identical episodes from which methods may be memorized (Skemp, 1976, Baddeley, 1997, Ashcraft, 1994). Even if this sort of practice is available and well structured, a learner may fail to "remember" successfully since they may impose erroneous meanings of their own when initially learning the material (Byers & Erlwanger, 1985, Davis, 1992), or corrupt it retroactively by confusing it with newer, similar procedures (Byers & Erlwanger, 1985). In Analysis, such repetitive practice is *not* readily available. Lecturers rarely present "methods" or set numerous exercises to be completed by following these. Hence at this stage, the conditions for acquiring only the "how" without the "why" do not exist. The result is that while those who acquire relational understanding are able to sensibly use the rules and procedures presented in lectures, those who attempt to understand instrumentally gain very little understanding of any kind. They therefore show considerable weakness in tackling the questions on specific objects.

Such students do not link new work to familiar knowledge or more general principles, so new knowledge they acquire is often fragmentary and not well interconnected (Harel & Tall, 1991). Hence they may become aware of some specific objects and rules, but they find it difficult to recall these in detail and they do not possess the connections that would allow their reconstruction. Thus they may resort to attempting questions using only partially remembered approaches, as in this case in which Zoë does not have a good grasp on the requirements of the week 9 question and cannot recall the detail of relevant work.

> Z: In other words, what is the...what is the value of...of x or n – what is the value of n when...is that what it means?...It's not one of these erm, thingies, is it?
>
> Y: Whaties?
>
> Z: Like one of those theorems that we've been doing. You know we did...you did...it looks approximately like...

Y: Oh, yes. Oh, what on earth did we do? I don't know.

Z: You know the, u_n and v_n [16],

Y: Yes. Oh, it might be...

Z: I don't know. Shall we try?

<div align="right">YZ9, 323</div>

This excerpt also highlights the importance of notation as a cue for evoking instrumentally understood procedures. Where a student with relational understanding is able to identify the objects in a question and relate these to familiar objects and results, students with instrumental understanding do not have this type of knowledge and so often attend to surface features. Nickerson notes an analogous phenomenon in students working on physics problems: beginners tend to focus on a problem's surface structure, such as the objects explicitly referred and the physical configuration described in the problem statement, rather than major principles which could be used in its solution (Nickerson, 1985).

Attending to surface features is an unreliable way of predicting which results will be useful in a new situation, since similar notation may represent various objects and be used in arguments with various purposes (Norman and Prichard, 1994). This means that students who try to understand instrumentally sometimes attempt to use inappropriate results. For instance, although Tom previously demonstrated relational understanding when tackling the sequence question, by week 9 this has collapsed and the notation in the series question appears to cue a confused memory for him of finding the partial sums of the series $\sum \frac{1}{n}$, which form a sequence $\log n + \lambda_n$ and hence show that the series is unbounded. He tries to use this, working for a long time without realising that neither the method nor the result is useful for this new question. He also tries to compare the limit of a series with a variable quantity, showing that he does not check that this recollection is meaningful, and loses track of the question, which does not ask for the limit of the series.

---

[16] The limit comparison test for convergence of series was presented using this notation.

T: Oh right, I know...I remember that the sum of 1 over n...tends to $\log n$ plus lambda n. Or...it's the sum of brackets minus 1 close brackets, to the power of n. Divided by n... tends to...$\log n$ plus lambda, n. And I think it's the first one.

ST9, 386

T: Okay this is just the formulas, not what they tend to. Formula 1 is the sum of 1 over n, and formula 2 is the sum of brackets minus 1 close brackets n, over n. To the power of n or whatever. So, I think, formula 2 is very similar to the question, except for x has been given a number, 1, and we've just got to use formula 1...to create formula 2, to see what it tends to.

ST9, 406

Students who focus only on the surface features of the material are also slow in narrowing their focus to central objects and properties (Bills & Tall, 1998, Gray et. al., 1999). In this case Steve fails to recognise the standard use of sequence notation in week 5.

S: Yes, it says it varies though doesn't it?

T: It says what varies?

S: n varies.

T: Of course n varies.

S: Yes but it doesn't mean it just goes up or down all the time.

T: Yes it does. Because it's a to the n.

   Short pause.

T: I think.

S: Well n could go 1,2,1,2...

ST5, 229

Steve might be applauded for acquiring something of a mathematician's sense of pedantry, avoiding unwarranted assumptions by interpreting the idea that n varies in the widest

possible sense. However it is explicitly stated in the question that (a_n) is a sequence, and it seems likely that Steve is confusing the roles of the symbols n and a_n. This phenomenon is clearer in Zoë's case when she describes the expected form of the solution and explicitly allows the possibility of n tending to zero. In addition she describes episodes, attempting to recall *what* her teacher did rather than *why* he did it, indicating attempts to acquire only instrumental understanding.

> Z: ...whenever we do this in class, we have erm...it ends up with...like a statement saying, as n tends to infinity,
>
> I: Mm,
>
> Z: Usually to infinity or it might say sometimes to nought...erm...a to the n tends to whatever.
>
> YZ5, 581

Attempts to understand in this way can render the new material virtually meaningless to the student. Returning to Zoë's case as at the beginning of this section, after some unsuccessful attempts to recall and apply the result associated with "the u_n and v_n" she is unable to answer the question of what it *means* for a series to converge. To further contrast with the behaviour of those with relational understanding, she also does not independently consider translating into another representation, and lacks fluency in accomplishing this when the strategy is suggested.

In summary, instrumental understanding is difficult to acquire in university mathematics. Those whose intention is to acquire instrumental understanding, rather than to gain insight into why results hold, do not display accurate or useful recall of the material.

7.2.2 Logical understanding

The main interview questions involving whole categories of objects, and therefore addressing logical understanding, are the week 5 <u>proof correction</u> and week 7 <u>convergent implies bounded</u> questions.

Week 5

Check this proof and make corrections to it where appropriate:

Theorem: $(\sqrt{n}) \to \infty$ as $n \to \infty$.

Proof: We know that

$$a < b \Rightarrow a^m < b^m.$$

So $a < b \Rightarrow \sqrt{a} < \sqrt{b}.$

$n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for

all n.

So $(\sqrt{n}) \to \infty$ as $n \to \infty$ as

required.

Week 7

Consider a sequence (a_n). Which of the following is true?

a) (a_n) is bounded $\Rightarrow (a_n)$ is convergent,

b) (a_n) is convergent $\Rightarrow (a_n)$ is bounded,

c) (a_n) is convergent $\Leftrightarrow (a_n)$ is bounded,

d) none of the above.

Justify your answer.

The <u>proof correction</u> involves an argument that claims to show that a specific object is an element of a category (that of sequences which tend to infinity). The <u>convergent implies bounded</u> question involves handling categories of objects at the most abstract level. To answer it fully one must prove that everything in the category of convergent sequences also has the property of being bounded. Assessing proofs and generating one's own are different tasks, but logical understanding is demonstrated in related ways in each.

Presence of logical understanding

Logical understanding is evidenced by a good grasp on the standards of argumentation at this level. Hence, when trying to understand a proof, students demonstrate logical understanding when they concern themselves not only with its individual statements but also

with its overall structure (Selden & Selden, 1995). To achieve this it is necessary to correctly interpret the role of any notation used. Jenny and Kate therefore demonstrate logical understanding when they discuss this explicitly as they work on the proof correction. They note that the possible values of a,b and m need clarifying, but quickly go on to observe that only one value of m is relevant for the remainder of the proof.

J: If m equals nought, then you get...1 is less than 1.

Short pause.

J: Don't you?

K: If $a...b$...yes, that doesn't include nought.

J: No I was talking about,

K: Oh m...

Pause.

K: Erm...

J: Well that's irrelevant anyway, so...

Short pause.

K: Yes, because you only want it to the half.

<div align="right">JK5, 931</div>

When checking the validity of individual steps, logical understanding is indicated by an awareness that each must be universally valid for the specified category of objects. Here, for instance, Adam shows that he is aware of the status of counterexamples in mathematics.

A: It's like that upper bound thing is if it goes 0,1,0,2,0,3, that sort of thing,

B: Yes.

I: Right,

A: Then it's not tending to infinity, but there's not an upper bound. So proving that...there's no upper bound doesn't prove that it tends to infinity.

<div align="right">AB5, 693</div>

This concern with universal validity also translates to producing valid arguments. Hence when a student with logical understanding draws an initial conclusion based a prototype, they will then perform a counterexample check.
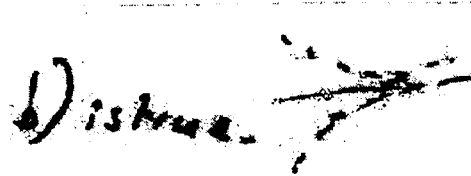
Figure 14: Cary's drawing of convergent sequences

C: I've drawn...er...convergent sequences, such that...I don't know, we have

er...curves... er...approaching a limit but never quite reaching it, from above and

below, and oscillating either side. I think that's pretty much what I've done. I was

trying to think if there's a sequence...which converges yet is unbounded both sides.

But there isn't one. Because that would be...because then it wouldn't converge.

Erm...so I'll say b) is true.

CD7, 236

Following this they will make efforts to formulate arguments in terms of properties, ensuring at all stages that the properties they choose hold for everything they want to include in the category.

C: If it converges...that has to be...well I don't suppose you can say bounded. It

doesn't have to be monotonic. Erm...

CD7, 308

C: Yes, I'm trying to think if there's like...if you can say the first term is like the highest or lowest bound but it's not. Because then you could just make a sequence which happens to go...to do a loop up, or something like that.

<div align="right">CD7, 314</div>

Essentially, those with logical understanding are aware that an argument should not rely on hidden assumptions, that reasoning should be made sufficiently explicit. The result is that they are likely to check their work to ensure that they have really said what they mean and that their use of notation is properly explained. For example, having completed a proof for the convergent implies bounded question Adam and Ben demonstrate logical understanding when they set about improving it in a variety of ways.

B: I suppose it could be nicer. It could have a bit more "n's in the natural numbers", and

A: *Laughs*...Ah yeah, for...

B: We haven't actually really stated what epsilon is...

A: Yeah you should have...probably in the definition of convergence, "for all epsilon greater than zero",

B: There exists, n

A: There exists a natural number n,

B: Such that when n is...

A: Such that...yes that's cool. And here...you've got, a minus epsilon less than a_n, less than a plus epsilon. You should say "for n greater than big N".

Pause (writing).

A: Just to get it explicit.

<div align="right">AB7, 407</div>

So those with logical understanding may not always find it easy to formulate arguments in the required way, but they are aware of the standards at this level and try to conform to them.

Absence of logical understanding

Absence of logical understanding is shown when a student fails to adhere to the required standards of mathematical argumentation. This involves more than simply not demonstrating the required reasoning strategies: it is evident in "errors" which would not be made by an individual who did possess logical understanding.

In understanding a proof, students who lack logical understanding may pay insufficient attention to its overall structure and make only a surface assessment of the way any symbols are used. This can lead to invalid assumptions about the structure of the argument, as is the case when several students demonstrate a lack of logical understanding by assuming that the n and $n+1$ in the proof correction indicate that this is an induction proof. Further, Wendy appears to think that the argument only holds for one value of n.

> W: ...They need to show that this for more values though. They need to show by erm, induction or something that it's increasing.
>
> WX5, 571

Lack of focus on the overall structure of a proof can result in students becoming bogged down in details that are not relevant to its validity. For instance in the proof correction Steve and Tom show a lack of logical understanding when they spend a great deal of time discussing irrelevant values of a,b and m (only part of this discussion is included here), and their final judgement does not include the one value of m which is needed for the remainder of the proof.

> S: It works for m, for like, everything positive, doesn't it?

Short pause.

S: Or...does it work for a power less than 1, as well?

Long pause (writing).

I: What have you written Tom?

T: Therefore *a* and *b*, have not been correctly defined.

I: Mm-hm.

T: And we'll do the same for *m*.

Pause.

T: So it should be...for all natural numbers or, for all positive integers. Shall I write that?

S: Well, it...well, yes. This works, it definitely works for...*a*, *b* and *m* all positive...integers. Doesn't it?

ST5, 744

Schoenfeld observes a similar phenomenon in expert and novice problem solvers. Where experts regularly check their progress towards a solution and may switch strategies if their chosen one seems unproductive, novices tend to pursue one strategy without monitoring their progress and therefore are liable to spend much time engaged in unproductive work (Schoenfeld, 1992).

In student-generated arguments, a lack of logical understanding may manifest itself as a lack of awareness of any need for a proof. This highlights the difference between relational and logical understanding: a student may not understand that at this level the argument is usually as important as the answer.

I: Are you happy with that answer?

X: Yeah.

W: I can't think of anything else, so...

I: Could you prove that that's true?

W: Well...in theory, yes...!

<div align="right">WX5, 341</div>

When students do expand their arguments, lack of logical understanding is apparent in "justifications" that are not recognisable as such by the standards of the mathematical community. For example, in explaining why all convergent sequences must be bounded, Wendy explicitly asks whether a diagram is enough, and "proof" seems quite detached from what brings conviction for her.

W: Well if it converges, you get closer and closer...
   Pause (drawing).
W: Is that enough to like, justify it...a little diagram, what have you?
I: Well, I'd like you to prove it, if you can.
W: Oh dear! *(laughs)*

<div align="right">WX7, 209</div>

Her reply to this prompt depends upon describing her prototypical picture rather than abstracting properties and arguing in terms of these, so that she demonstrates what Harel & Sowder would identify as a "perceptual" proof scheme (Harel & Sowder, 1998).

W: [*Draws a monotonic increasing convergent seequence*] It's convergent...yes so if
   it's convergent it's always...or...say it could be the other way round it could
   be...going down this way [*draws a monotonic decreasing convergent sequence*]. It
   converges, so it's always above that limit.

<div align="right">WX7, 231</div>

Hence students who lack logical understanding tend to produce arguments that do not satisfy the requirement of universal validity, since their prototypes may not be sufficiently general.

<div align="right">151</div>

In more subtle cases, lack of logical understanding is evident in insensitivity to the possibility of counterexamples. For example, in the excerpt below, Tom is trying to provide a justification for his assertion that if a sequence is convergent, it must be bounded. He makes a good effort to give reasons, arguing that if a sequence does not tend to infinity then it has an upper bound. Unfortunately he fails to realise that this is not actually true (Adam's earlier example is a counterexample). The rest of his answer uses names for the bounds and so *sounds* more mathematical than a direct generalisation, but has no more deductive basis than observation from a prototype. He does not provide any rationale for the existence of b based on properties of convergent sequences, simply stating that such a value exists [he uses "can be" not because he is uncertain of the existence of such a number, but because he cannot identify a particular one].

T: ...If a_n tends to big A, okay.

I: Mm.

T: Then erm... a_n does not tend to infinity, therefore there is a bound, a, lower than infinity, which a_n is...well, not is, can be bounded by. Erm...and...the second part, if a_n tends to big A, there exists a b, which is a member of the real numbers, such that b is less than big A?

I: Mm.

T: Therefore a can be bounded by b.

ST7, 759

Lack of sensitivity to counterexamples can also hamper proof checking, as in this case when Zoë shows a lack of logical understanding through her belief that the argument in the proof correction holds. She seems to feel that the proof is basically correct, and attributes a feeling of its inadequacy to there being something missing, rather than rejecting it because the step from the third to the fourth line is not universally valid.

Z: …like if you said n was a thousand okay, root of a thousand, is going to be less than root of a thousand plus 1. Therefore as n is…n is getting bigger than the square root of n…is getting bigger? Yes.

Y: Oh, I see what you mean!

Z: But yes it doesn't…it doesn't look like – it should have a few more lines, probably…!

<div align="right">YZ5, 997</div>

This phenomenon can be quite robust, so that even when counterexamples to deductions are known, a student lacking in logical understanding may still not be prepared to reject an argument. For example, Wendy and Xavier assert that a strictly increasing sequence must tend to infinity and are challenged with the counterexample $(-1/n)$. Both appear to understand this, although Wendy seems unhappy. It could be that this is because she does not see the relevance of this for this problem, since in this case the sequence is increasing *and* tends to infinity[17]. But whatever her reasons, Wendy's idea of proof does not correspond well with that of the mathematical community, as she is still not prepared to reject the proof as it stands.

I: Mm. So did this person in fact prove, that root n is a sequence that tends to infinity?
    Pause.

W: Hm…in a roundabout kind of way!

<div align="right">WX5, 811</div>

Overall, those with a lack of logical understanding may feel that they have a good understanding of the objects and results of the course, but they fail to adhere to the standards of argumentation required at this level.

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[17] Recall Johnson-Laird, Byrne and Shaeken's observation that in complex situations, "if..then" statements may be mistakenly treated as "and" statements (Johnson-Laird, Byrne & Shaeken, 1994).

7.2.3 Formal understanding

In the previous section we saw that students *with* logical understanding may reason using their own (often visual) prototypes, but nonetheless show a mature awareness of the standards of mathematical argumentation. Students with formal understanding are those who work not only with properties, but more particularly with formal definitions as agreed upon by the mathematical community. Like logical understanding, formal understanding refers to behaviour involving whole categories of objects and so is exhibited on the same questions as logical. It can also be seen in students' answers to more direct questions about the meanings of defined terms.

Presence of formal understanding

Students with formal understanding are aware of the central role of definitions in a mathematical theory, at least to the extent that they know how they should use them. For example, in these excerpts, Ben is describing why we have definitions, and Greg realises that the definition of tending to infinity should be introduced in order to complete the proof correction.

B: I suppose so...it's specified explicitly what it means. You know when you're using proofs if it's...if it's not specified exactly then you can't really use it as such I don't think.

<div align="right">AB3, 281</div>

G: ...aoh, because they're not...they're not using the definition at all of what...tending to infinity is that's, fair enough, you can't use that there...correct it...can we? Or...?

I: Well, if you think that that might be a better way to do it, then perhaps you could, show me now what you think would be a better way to do it.

G: Well. Just the, good old way of saying that erm...if something tends to infinity, then...for any C you can make it bigger than that C...

<div align="right">154</div>

When asked to explain the meaning of a given term in a mathematical context, a student with formal understanding will give its definition, or at least an answer that is consistent with it. A striking example occurs in Ben's case. The definition for the term "sequence" has only been given once and is not regularly used, but his partial answer, referring to the natural numbers, sounds remarkably formal rather than related to spontaneous conceptions about "strings of numbers" or "lists". Although he cannot give a precise answer in this case, clearly this is what he is trying to do.

B: Er…I know it's just to do with natural…the natural numbers as such. That's what…it was explained in our…. I don't actually know I mean I haven't had a good read of the beginning part of the thing yet…

AB3, 227

Since central terms have specific meanings constructed from the definitions, formal understanding is evidenced when a student takes care only to use these terms in appropriate situations. In this instance Adam corrects himself during his explanation of what the definition of convergence for a sequence means.

A: It's getting closer and closer…there's…if you pick a limit…of some distance away from – *limit was the wrong word to use*. If you pick a value sort of some distance away from…the value which it converges to, there's a point after which it's always…

AB7, 543 [*my italics*]

Similarly formal understanding is indicated by the use of definitions to reason about categories of objects. For example, in the convergent implies bounded question Adam and

Ben agree upon the correct answer, and Ben's first approach in trying to prove their assertion is to introduce the definition.

> B: Right, our definition of convergence is that...well, there exists an N such that when n is greater than big N, there...modulus of a is less than epsilon. So that leads to epsilon being a bound...plus or minus epsilon being a bound, about a. a_n.
>
> AB7, 251

Ben's recall is not quite accurate, but nonetheless this is his first tactic, and he and Adam then go on to refine this and produce a good attempt at a proof. Working with definitions in this way automatically guarantees that the category is covered, so students with formal understanding bypass some of the errors made by those who generalize directly from prototypes, and some of the difficulties encountered by those who attempt to abstract their own properties. Their work is consistent with formal mathematics.

It is worth noting that formal understanding can in principle operate without logical or even relational understanding. An individual might be able to operate from the definitions, thereby producing arguments which cover the defined category, without considering either the relationship between this and their own idea of what should be in the category or the reason that these particular properties are chosen. This approach would allow them to bypass many of the behaviours which their teachers consider to be mistakes, but has its own weaknesses, as will be demonstrated in section 9.2.9.

Absence of formal understanding

An absence of formal understanding is evidenced when students relate to the concepts as they would to everyday ones, failing to recognise the place of definitions in a mathematical theory. In particular, although a student who lacks formal understanding may introduce notation associated with the definitions, such attempts are likely to be incomplete or referred to as part of a procedure. If a student is accustomed to following procedures in mathematics

it is not surprising that the definition should be interpreted in this way, since their first encounters with the definition of convergence, for instance will almost certainly all involve showing that the definition is satisfied by particular sequences (Alcock & Simpson, 1998). This seems to be happening in Wendy's case:

W: That's right isn't it, to prove it's convergent?

X: What, subtract the limit?

W: Yes.

X: Mm...which means that a_n ...yes. So if they get, smaller...

I: Can you just read out what you've written?

W: Erm, modulus of a to the n minus the limit, which is zero, has got to be less than, epsilon.

I: Mm.

W: And, it's taking zero out...the modulus of a to the n has got to be less than epsilon, for all...n is bigger than, big N.

<div align="right">WX5, 389</div>

Lack of formal understanding is indicated when definitions are not recognised as having a special role. In Tom's case this seems to occur because superficial resemblance means they are not distinguished from other material.

T: ...the theorem and definition I think are quite similar.

I: Yes. Yes. In what way? Can you...?

T: Erm...well, this...this definition I mean...it's very similar to a...a theorem, to one of the theorems he gives us. Like, so...I can't really tell the difference between a theorem and a definition, really.

<div align="right">ST7, 1487</div>

An absence of formal understanding means that a student may have trouble answering questions about the meaning of important terms, perhaps having a sense that they understand the terms without needing to verbalise this.

I: Can you explain what it means?

V: What, the limit?

I: Well, yes.

V: It's, just... limit!

<div align="right">UV23[18], 943</div>

When explanations are given, lack of formal understanding is evidenced by imprecision. Here Wendy is *describing* what tending to infinity means for her; she seems to feel the need to keep adding more explanation (*cf.* Pinto & Tall, 1996).

W: Oh well as erm, as n gets bigger,

I: Mm.

W: The, sequence gets...to a really really big number, and it just gets bigger and bigger and...

I: Yes, yes. Is there, have you seen an actual definition of that – sorry?

W: There's no like, end to it, it just keeps on going.

<div align="right">WX3, 141</div>

Where a student *with* formal understanding recognises that defined terms have reserved meanings, a student without does not. As a result, such students may use terms carelessly, so that their statements are not consistent with the definitions and may be difficult to interpret. In this excerpt Steve and Tom are working on the <u>convergent implies bounded question</u> in week 7. They have drawn a picture and are now negotiating labels for it. Steve

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[18] Both Unwin and Vic and Steve and Tom missed the week 2 interview, so were asked these questions in week 3. UV23 denotes this composite interview.

<div align="right">158</div>

begins by using "limit" to mean "bound". Tom also appears to use it in this way, and apparently does not have "limit" associated with "what it converges to" at all.

$$b \leqslant a_n \leqslant a$$
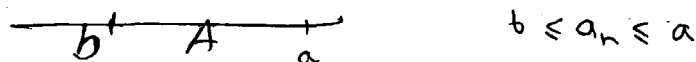
Figure 15: Tom's diagram of a limit and bounds

S: Call this lower limit b...

T: No no no, no that's nothing to do with the limit. That is what it tends to, that's what it converges to, that's what I'm saying.

S: Yes...

T: So that's nothing to do with the lower limit...but yes, call the lower limit, b if you want.

ST7, 585

Similarly, lack of formal understanding is evidenced when a student does not modify their understanding of what is in a category in response to the formal definition. If their interpretation of the term is largely based on spontaneous conceptions (*cf.* Pinto and Tall, 1996, Cornu, 1992, Vinner, 1992), they may maintain beliefs which are not consistent with the definitions.

I: Mm. What exactly does it mean for a series to converge anyway?

W: Erm...the sum...after you get beyond a certain point,

I: Mm,

W: The sum doesn't increase any more. The sum of the series...?

WX9, 577

Hence students without formal understanding do not adapt to the formal framework used by the mathematical community, so that their arguments and their understandings of the central terms may not be valid within this framework.

7.2.4 Validity of the distinctions

From a methodological perspective, it is important to ask whether the distinctions described above really exist within the population (Schoenfeld, 1985, Cohen & Manion, 1994). Here, the basic types of understanding are well-grounded in an established theoretical basis: there is agreement that there are differences between those who can use a rule and those who understand why it works, and between both of these types and those who express their understanding correctly (Skemp, 1976&1979b, Byers & Herscovics, 1977). There is also wide recognition that students frequently do not base their arguments on mathematical definitions (Vinner, 1992, Davis & Vinner, 1986). However, using these distinctions to genuinely characterise *individuals* requires more than this: any of the four types of understanding may be demonstrated on any of the material, so in principle a student may exhibit a relational understanding of one area but only instrumental understanding of another.

However, selective coding (Strauss & Corbin, 1990) on the basis of this typology revealed that, within reason, individuals reliably exhibit behaviours associated with some types of understanding but not others. This is less surprising than it might at first appear, as some types have some self-perpetuating qualities: a student who has only instrumental understanding does not have a well integrated network of principles to which they can relate new knowledge, which makes it more difficult to learn new results relationally. Similarly, a lack of formal understanding in the initial stages of the course means that an individual does not begin by focusing on key properties; this makes them more likely to suffer from cognitive overload and so may impact upon all other types of understanding (Bills & Tall, 1998). In fact, the analysis revealed that combinations of types displayed were: instrumental understanding only, relational understanding but not logical or formal, relational and logical

but not formal, all four, and formal but with weaknesses in relational and logical. Chapters 9 and 10 demonstrate that these outcomes can consistently be attributed to combinations of the causal and intervening factors outlined briefly in section 4.5.5. The next chapter describes these in detail.

7.3 Main points of chapter 7

- Types of understanding suggested by various authors may be related to the theoretical framework developed in the previous part.

- Four types of understanding are distinguished: instrumental, relational, logical and formal.

- Instrumental and relational understanding refer to reasoning about specific objects, logical and formal to reasoning about whole categories.

- Instrumental understanding alone is difficult to acquire at this level since the conditions for rote memorisation are not in place: those who attempt to gain only instrumental understanding exhibit very little understanding of any type.

- Relational understanding is evidenced by good recall and fluent use of methods for reasoning about specific objects.

- Logical understanding is evidenced by focus on the structure of proofs, awareness of the status of counterexamples and attempts to formulate arguments in terms of properties.

- Formal understanding is evidenced by the use of definitions in constructing arguments, careful use of defined terms, good correspondence between the student's category and its formal counterpart, and awareness of the status of definitions in mathematical theory.

Chapter 8
Causal factors

8.1 Introduction

This chapter describes in detail the causal factors that were outlined in the section 4.5.5 and that determine the type of understanding a student acquires. The first section deals with *visual* and *nonvisual* reasoning. The second with the student's beliefs about their own role as a learner of mathematics, classified according to whether they have an *internal* or *external* sense of authority. In each case relevant literature is briefly reviewed, following which illustrations are given of the manifestations of these factors in the study. The chapter concludes with a section that integrates substantive and theoretical perspectives in order to discuss the validity of the distinctions drawn.

8.2 Visual and nonvisual reasoning

This section describes students' use or otherwise of visual imagery in answering mathematical questions. Since this emerged as a causal factor during data analysis, literature pertaining to the topic was not covered in chapter 3 but is discussed in detail here.

8.2.1 Literature

Chapter 3 touched on visual reasoning in the context of proof generation. It noted the asymmetry in the status of visual and algebraic arguments in mathematics, and the potential problems associated with proofs based on visual images (Harel & Sowder, 1998, Dreyfus,

1991, Presmeg, 1986a&b). This section discusses in more detail the advantages and disadvantages of visual imagery, after considering arguments regarding the validity of dividing a population into visualisers and nonvisualisers.

From the perspective of cognitive psychology, the debate continues as to whether knowledge is represented in the mind in terms of propositions, or more holistic, image-like structures (Baddeley, 1997). This thesis does not make any claims at this level, concentrating instead on how student thinking may best be modelled in order to account for a variety of behaviours. At this level, which is more open to investigation, texts by and about mathematicians have made distinctions between those individuals who rely on visual or spatial imagery in their mathematics and those who prefer to proceed through algebraically expressed logical deductions (Poincaré, 1913 (cited in Tall, 1992), Clements, 1981). In light of this, various attempts have been made to identify individuals with inclinations to reason visually or nonvisually in mathematics (Krutetskii, 1976, Presmeg, 1986a&b, Eisenberg & Dreyfus, 1986). A common conclusion, exemplified by Krutetskii, is that this is only possible for those who have extreme preferences. Krutetskii also identifies talented individuals who flexibly utilise both visual and nonvisual representations, calling these *harmonic* thinkers (Krutetskii, 1976).

Some have argued that such classifications are not useful *per se* and suggested that the focus should instead be upon the visualisation *process* (Bishop, 1998, Zazkis, Dubinsky & Dautermann, 1996). Indeed, while recognising the utility of simple dichotomies, there are obvious problems with the choice of items used to determine whether an individual is inclined to visualise or not: the claim that there exist problems with equally accessible visual and nonvisual solutions (Eisenberg & Dreyfus, 1986) seems tenuous given the variation in learners' previous experience. However, studies with less stringent criteria than Krutetskii's have nonetheless yielded valuable insights into students' thinking, particularly in interaction with the apparent values of school mathematics.

Presmeg, for example, gives a wide definition of visual imagery including the following types:

- Concrete, pictorial imagery (akin to that used in Harel & Sowder's perceptual proof schemes)

- Pattern imagery (spatial representations of relationships)

- Memory images of formulae

- Kinaesthetic imagery (involving muscular activity)

- Dynamic imagery (akin to that used in Harel & Sowder's transformational proof schemes)

(Presmeg, 1986b, Harel & Sowder, 1998)

Presmeg follows Bartlett in noting both the limitations and the "peculiar excellences" of such imagery (Bartlett, 1932). She finds that students can use such imagery effectively for its mnemonic advantages, in dynamic arguments (Harel and Sowder also rate such arguments highly) and in "concretising the referent", that is, embodying an abstract idea in a concrete image (Presmeg, *ibid.*, Harel & Sowder, *ibid.*). She also notes and illustrates difficulties encountered by visualisers, specifically:

- The one-case concreteness of an image or diagram may tie thought to irrelevant details, or may even introduce false data.

- An image of a standard figure may induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram.

- An uncontrollable image may persist, thereby preventing the opening up of more fruitful avenues of thought.

(Presmeg, 1986a&b)

Examples of all of these problems in the context of this study will be seen in the following chapters.

In addition to delineating these potential pitfalls, Presmeg examines the way that visualisers may become alienated from mathematics when school presentations do not provide explanations that they can understand (Presmeg, 1986a). Similarly Dreyfus argues that the classroom emphasis on algebraic procedures cuts students off from a valuable reasoning approach (Dreyfus, 1991).

Unlike studies by Presmeg and by Eisenberg and Dreyfus, this study did not set out to distinguish between visual and nonvisual reasoning preferences (Presmeg, 1986a&b, Eisenberg & Dreyfus, 1986). Rather, the distinction came to be recognised as causal during the selective coding stage of the data analysis. This happened rather late: visual and nonvisual reasoning were often conflated with types of understanding in the original characterisation of phenomena, since there are some obvious links between the two. In particular visual reasoning often involves direct generalisation from prototypes; as discussed in chapter 6, visual prototypes are particularly conducive to this type of reasoning. However this should not be assumed to hold in all cases: students can and do use visual prototypes as a basis for abstracting properties. Neither does it mean that their arguments will necessarily be wrong or deficient: prototypes can be reconstructed so they are brought into line with the meanings of the definitions, and may then provide a good basis for reasoning within the formal framework (Gray et. al., 1999). Conversely, in general prototypes do not have to be visual. This classification arose only in this context of Analysis where visual representations are particularly available.

Similarly, nonvisual reasoning can often involve reasoning from algebraically expressed properties, including definitions. Those who take this approach are "formal learners" in the sense of Gray et. al.; they concentrate on the definition, using it and repeating it until this becomes a natural approach (Pinto, 1998, Gray et. al., 1999). In effect they construct categories from the definitions. However some nonvisual students introduce algebraic notation without understanding this as representing properties used to define categories of objects. In such cases it would be misleading to say that they are working with properties in

the sense described in chapter 3: in reality they may have no clear idea of what is achieved by certain algebraic manipulations[19].

It should be noted that the interviews sought to examine how the students think about abstract objects in Analysis rather than how they write mathematics when required to submit assignments. Hence the emphasis throughout was on oral explanations rather than written answers, and the word "justify" was often used in place of "prove" in order to elicit responses based on what the student found convincing (following Harel & Sowder, 1998, and Chazan, 1993). This means that the information gained does not necessarily reflect students' approaches to written work. The implications of this are considered in sections 12.2.2 and 12.2.3.

8.2.2 Visual reasoning

This section and the next describe the characteristics of visual and nonvisual reasoners. The first, and most obvious, characteristic of visualisers is that they introduce visual representations in their work. They often draw these on paper and are also prone to gesturing while trying to convey their thoughts. Unsurprisingly given the content of the questions, the most commonly occurring diagram is a standard picture of a (monotonic) convergent sequence.

The fact that visual solutions appear in this context could be an artefact of the research and its focus on convergent sequences. It would still be a valid observation; convergence is the central concept in the study. However there is evidence that visual inclinations extend beyond this, as those who visualise in these situations also do so in others. For example, Emma and Fred both produce visual solutions to the sequence question in week 5; their pictures and explanations are given below. Emma changes her mind about her graph (on the

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[19] Currently such an approach is not represented in the theoretical framework. Chapters 9 and 10 examine the precursors to and results of this approach, and in chapter 11 this information is used to prompt an expansion of the theoretical framework, additionally clarifying what is peculiar to Analysis.

left) and is eventually right for ill-explained reasons. Fred maintains, wrongly, that his graph is appropriate. The approach of each relies heavily on their visual representations.
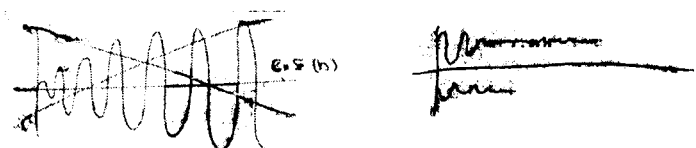
Figure 16: Emma's and Fred's pictures for the sequence question

E: That's going to get bigger. The...nx part. And...the cos n's just going to go between minus 1 and 1, so between there. So that, I would have thought, is...just increasing and you get a number between 1 and minus 1, over a big number. I think it'll be zero, still.

EF5, 377

F: So, I think it's convergent to...to 1, if x is positive, and minus 1 if x is negative. Because I think it's just a sine...well, a cos curve, just...getting smaller and smaller and smaller and smaller and smaller. Converging to 1...or converging to minus 1.

EF5, 407

Visual reasoners also explicitly express preference for visual representations, confirming Gibson's finding that these coincide with the way their "minds work" and so are easier or clearer for them than verbal or symbolic representations (Gibson, 1998). Here Emma discusses the definition of convergence for a sequence.

I: Does it [the definition of convergence]...correspond with your idea of what convergence is?

Pause.

E: Yes but it's...put in a different way, that's all.

I: Mm.

E: Because I suppose it's easier to think of it graphically.

I: Yes.

E: Not all in symbols.

<div align="right">EF7, 597</div>

Sometimes the expression of this preference is less direct, as when Adam describes the way he relates the definition to some other representation that is more meaningful for him. Evidence from his other responses suggests that this alternative representation is visual.

A: It's not usually enough to stick the definition down, you have to stick it down and then remind yourself of what it means.

I: Yes?

A: Because the definition and how you understand it are never like, exactly the same.

I: Are they not? Can you explain to me what the difference is Adam?

A: I really wish I hadn't said that now! Er, yes it's like...you understand that it gets closer and closer,

I: Yes,

A: But you can't just...the definition you can't put it gets closer and closer, you have to have the "for all epsilon greater than zero, n greater than N implies...modulus of... a_n minus a", and all that sort of stuff. It's got to be put into the mathematical terms.

<div align="right">AB7, 511</div>

8.2.3 Nonvisual reasoning

It is striking that even among students on the lecture course, who all attend the same presentation of the material, there is real difference between the aspects that individuals subsequently adopt in their thinking. In interviews, nonvisualisers stand out particularly in opposition to visual partners. For example, when Wendy produces her visual "justification"

of the fact that all convergent sequences are bounded, Xavier does not become involved in this approach and instead tries to introduce algebraic symbols relevant to the definition.

X: Isn't it something to do with that... a_n minus l, don't you remember?

Pause (writing).

X: That would be the limit, or the bound.

W: Yes...

X: And so for n bigger than N...

WX7, 305

Others, as seen in Ben's case in section 7.2.3, confidently and accurately introduce definitions as a first approach to a question.

Some students explicitly express strong nonvisual preferences, thus showing that Gibson's observation about the "natural" character of visual representations is not universal (Gibson, 1998). Hugh, for example, began to like algebra when he did well in this topic in school. He actively dislikes using visual representations and seems to feel that these are forced upon him.

I: Mm. Do you...do you like using graphs,

H: No I don't!

I: You don't?

H: No.

I: You don't.

H: But [teacher] would like to use graphs. So he uses the graph every single time we have a class.

GH5, 837

Greg also prefers to work with the algebraic expressions of the definitional properties, finding these quite natural. He can see that others work in different ways but is not disturbed by this.

G: ...I wouldn't...generally my first instinct would not be to take a graph. Everyone else's first instinct seems to be to take a graph, so,

I: Oh no! Do you think that means you're wrong or they're wrong Greg?

G: I'm wrong probably but...!

I: Do you think there is a...question of being wrong in that case?

G: No it's just the way I see it and the way other people see it.

GH5, 823

Greg and Hugh's answer to the <u>sequence question</u> contains only algebraic reasoning so can be seen to contrast with that of Emma and Fred.

Figure 17: Greg's and Hugh's work on the <u>sequence question</u>

8.2.4 Validity of the distinction

With a small number of notable exceptions[20], selective coding revealed that students in the study are reliably either visual or nonvisual in their reasoning. This is nontrivial given the well-documented difficulty of making this distinction in general. However on closer examination it makes sense that this should occur within this restricted context. What makes a student *initially* focus on visual or nonvisual representations may be a matter of some fundamental predisposition. This is a fascinating question but naturally is far beyond the scope of this thesis. However, even if the initial focus is entirely random, it does make sense that a student who begins by focusing on either visual or nonvisual representations and experiences some success in this way (either in terms of marks or by gaining a sense of understanding) should then continue to focus on the same type of representation. In addition, the complexity of the algebraic representations can contribute to the same effect: if a student does not attend to these early on, it may become increasingly difficult to understand later material in this way.

8.3 Sense of authority

This section describes the students' conceptions of their role as learners of mathematics, characterising these as indicating either an internal or external sense of authority. In fact there are three issues here: the individual's experiences of school mathematics which lead these beliefs, the beliefs themselves and the work habits and expectations the individual exhibits as they try to learn mathematics at university. This chapter will refer briefly to school experience where this proves illuminating regarding the sources of student beliefs. However the focus is on the sense of authority currently held, since this is the causal factor in determining the way the student works and the type of understanding they therefore acquire.

school experience → internal or external sense of authority → work habits and expectations

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[20] See section 9.2.8 on Adam's and Jenny's reasoning.

Again relevant literature is reprised first, following which illustrations are given of the behaviours seen in this study.

8.3.1 Literature

The literature relevant to this distinction was reviewed in section 3.3 on the student's beliefs about their own role as a learner of mathematics. That section covered Perry's developmental scheme (Perry, 1970&1988, Copes, 1982), Skemp's *instrumental* and *relational* learning (Skemp, 1976) and Duffin and Simpson's *natural* and *alien* learners (Duffin & Simpson, 1993), as well as the role of conflict in learning experiences and common student beliefs regarding the relationship between procedures and problem solving (Schoenfeld, 1992, Mohd Yusof & Tall, 1996). These beliefs and approaches to learning were related to the growth of understanding (Harel & Tall, 1991, Skemp, *ibid.*), and in a number of cases to the essential role of the teacher as authority in learning.

The distinction between an *internal* and an *external* sense of authority as described in this section is related to all of these ideas, without exactly corresponding to any of them. This is a natural consequence of the grounded theory approach to data analysis; distinctions and terminology are chosen to best describe the particular concepts emerging from the data (Glaser & Strauss, 1967).

8.3.2 External sense of authority

Like dualists and instrumental learners (Perry, 1988, Copes, 1982, Skemp, 1976), the student with an external sense of authority believes that their role as a learner is to absorb information as it is dispensed from an outside source. This makes for a passive approach to learning in which the student expects that the authorities will provide everything they need, so they are willing and in some cases keen to "accept" results which do not make sense to them rather than try to interpret these meaningfully. This is reminiscent of known cases in which students wish to be told "the answer" or "the method" without further explanation; they may have an authoritarian proof scheme in the sense of Harel and Sowder (Dreyfus,

1991, Harel & Sowder, 1998). For example, in this excerpt Wendy is explaining her lack of any desire to see proofs for results she will use.

W: I just believed what the teacher said anyway! She told me a formula and I just
   believed her!

I: Yes,

W: I just said, oh yes I'll use that.

<div align="right">WX7, 965</div>

Such students may have particular expectations for the form the material will take. Tom, for example, is annoyed that he is not being provided with methods to follow (*cf.* Schoenfeld, 1987, Mohd Yusof & Tall, 1996, Meel, 1998).

T: I've got this textbook which is rubbish.

I: What is it?

T: He recommended it – the Mary Hart one?

I: Oh Hart, yes.

T: The Guide to Analysis, or whatever. Yes and it's just the same as his notes really.

I: Yes.

T: Just, all formulas, nothing – it doesn't go through it step by step really it just goes
   straight into it.

<div align="right">ST5, 97</div>

Because the mathematics is externally provided, success for these students means making accurate reproductions of the required work or accurately predicting what the authorities will think worthy of merit. It is therefore measured by getting good marks, and in some cases students with an external sense of authority avoid answering questions about how they feel they are getting on, deferring instead to some anticipated external measure. Once again this is reflective of a dualistic mindset (Perry, ibid., Copes, ibid.).

I: Yes, what have you learnt then?

H: I don't know. We'll find out when we do the exam. If we pass the exam that means we've learned a lot.

<div align="right">GH9, 105</div>

Students with an external sense of authority often display an instrumental approach to learning: they are likely to see doing lots of exercises as a good way to imprint the required knowledge on their minds[21]. In the long term they tend to rely heavily on their notes, where guaranteed accurate versions of this information are stored. For example, in this excerpt Emma and Fred have decided on the correct answer in the <u>convergent implies bounded</u> question but Emma is almost defiant in her disinclination to engage with the problem in the absence of her workbooks.

F: Okay so we've just got to justify b). Entirely.

E: See workbook...

I: Sorry?

E: See workbook!

Pause.

E: Erm, by the definition of convergence, which is...something along those lines...erm...all convergent series are bounded. Something like that.

Long pause.

I: Can you prove it?

Long pause.

E: If we use our workbook yes, probably.

<div align="right">EF7, 457</div>

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[21] In this study, this approach to learning is seen as distinct from the acquisition of instrumental understanding (see in particular section 9.2.1).

Learning mathematics in this way is a dry rather than creative experience, and these students do not usually see it as intrinsically interesting. In particular the extrinsic nature of their motivation for learning means that they are unlikely to feel "involved" in the subject (Turner et. al., 1998). Rather they tend to view mathematics as "for" some practical application, often thinking of their degree as a stepping-stone to a future career.

S: I think we'll...we'll need it later on.... When I did A-level, stuff like differentiation...I was thinking why would anyone want to do this, what's the point? But then, starting economics now, you can see that you use it for maximum points and...

ST23, 1358

Z: ...I want to go out and I want to get a, a...highly paid job!

YZ2, 631

Z: And, I mean, some of my dad's friends have said you know the business school here is really like, well-known, so you know if we're doing something like...based in that department...because that's what, you know, our course does as well, so...it's just so that you can use the name, really...

YZ2, 683

The order in which these characteristics are described here is a convenience and it should not be assumed that the "absorption" approach to learning is central to what it means to have an external sense of authority. On the contrary, different characteristics stand out as generative for different students. For example, a student may want to get a good job, see a mathematics degree as a route to this, focus therefore on examinations and dutifully try to memorize everything they are given in order to perform well in these. Alternatively they may have gained a sense of security in their ability to "get right answers" through practice with procedures, which leads to a choice of mathematics as a degree in order to maintain

this success. In any case, the characteristics together form a consistent profile; they reinforce rather than contradict each other.

8.3.3 Internal sense of authority

Students with internal sense of authority find mathematics meaningful in and of itself, and expect that it will continue to fit together as an internally consistent, interconnected network of results. In this sense they resemble both Skemp's relational learners and Simpson's natural learners[22] (Skemp, 1976, Simpson, 1995, Duffin & Simpson, 1993). Building their understanding of such networks many not be easy but they find it satisfying (Skemp, ibid.). In the third excerpt Jenny's use of the word "nicely" indicates the emotional value she attributed to mathematics "fitting together".

J: ...maths I had to puzzle over it more, it was the one I had the least confidence in.

JK2, 193

J: [but]...it was the most rewarding.

I: Really?

J: Yes. Because it's challenging. You felt good about yourself afterwards when you worked it out!

JK2, 205

J: It's starting to fit together nicely now.

JK5, 273

Such students tend to find mathematics intrinsically interesting, and may enjoy going beyond a standard syllabus (Turner et. al., 1998).

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[22] Once again, sense of authority is about the student's approach to learning and appears as a causal factor. Relational understanding is a learning outcome which is correlated with, but not assumed to necessarily follow from, an internal sense of authority.

C: We had sort of very deep arguments during our further maths classes about whether, certain things existed...and stuff and er...mainly negative nought was the one thing.

I: Yes.

C: And, I don't know that was just...and there were some interesting books in the library on infinity and things...

CD3, 1301

This sort of self-generated construction of new knowledge makes it particularly memorable (Skemp, 1976, Byers & Erlwanger, 1985) and students with an internal sense of authority are likely to recognise this and see it as a good way to learn, expecting to able to remember the material as they go along.

G: ...I normally learn better from listening as opposed to reading,

I: Okay.

G: As a general rule. And even better from doing, basically.

I: Right.

G: So, just...finding something out is always the best way to learn, I think.

I: Right.

G: Because if you find out for yourself,

I: Okay,

G: You'll know it...you'll be able to find it again even if you don't remember it.

GH2, 671

Naturally they want to do well in examinations, but for students with an internal sense of authority the immediate goal is to understand the material (Skemp, 1979a&b). Their sense of understanding may not indicate that they actually *do* understand according to an expert viewpoint (Nickerson, 1985), but the important thing is that they rely on an internal emotional sense to measure this rather than extrinsic feedback (Turner et. al., *ibid.*). This

means that even when getting good marks, they may still express reservations about their progress.

> B: ...there's convergent and...null sequences...we did that about two weeks ago so I'm just finally getting it now,
>
> I: Right.
>
> B: So it's just taking a lot of time to fully understand it.
>
> <div align="right">AB5, 93</div>

Once again none of these factors is necessarily primary, but they form a coherent profile.

8.3.4 Validity of distinction

That this distinction should genuinely split a population is less surprising than the corresponding claims for visual and nonvisual preferences or types of understanding. As explained, the beliefs associated with each are mutually reinforcing. This means that a student who, for example, believed that mathematics was "for" something but found themselves becoming interested in it for its own sake would be likely to experience cognitive dissonance, and resolve this by moving to a more coherent profile (Sierpinska (1987) discusses a similar phenomenon regarding beliefs about infinity and limits). Naturally this does not mean that only the extremes are in evidence: those with an external sense of authority may experience some enjoyment of the subject when ideas do become linked together for them, and even a student with a strong internal sense of authority is not above occasionally handing in work they don't understand in order to "get the marks". However the vast majority of the participants in the study reliably show one set of characteristics or the other but not a mixture from both[23]. There are exceptions, but even these students do not occupy a place in the centre of a continuum. Rather, they have an internal sense of authority regarding the material, which is to say that they believe it is

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[23] This will be illustrated throughout chapter 9, and may be verified by inspected comments coded under ROLE and BACKGROUND in the NUD*IST database.

meaningful and interesting and they try to understand it, but an external sense of authority regarding the human authorities, in that approval from these authorities is still a driving force. The results of this combination are some of the most interesting and illuminating in the study, and can be found in section 9.2.6.

8.4 Main points of chapter 8

- Factors identified as causal in determining the type of understanding a student acquires are their use of visual or nonvisual reasoning and their internal or external sense of authority.

- Various studies have set out to identify visual and nonvisual preferences in reasoning; in this case this classification emerged from the data.

- Visual imagery has mnemonic advantages and may be used to embody abstract ideas, but may also tie thought to irrelevant details or lead to inflexible thinking.

- In this study, visual reasoning is indicated by the introduction of diagrams and expressed preferences for thinking "graphically".

- Similarly nonvisual reasoning is indicated by the introduction of algebraic formulations as an alternative to diagrammatic representations, and by expressed preferences for not using "graphs".

- Literature relevant to sense of authority was discussed in section 3.3.

- An external sense of authority incorporates some or all of: the belief that mathematics is dispensed from an outside source, expectations that this will be in the form of "methods", reliance on notes, extrinsic measures of understanding/progress, mathematics seen as a stepping stone to future plans.

- An internal sense of authority incorporates some or all of: interest in mathematics for its own sake, expectation that it will be meaningful, expectation of remembering new material as it is encountered, emotional/internal measures of understanding.

Chapter 9
Combinations

9.1 Introduction

This chapter begins the process of analysing the relationships between the causal factors described in the previous chapter, the course the student attends (which will be characterised as an intervening factor) and the student's acquisition of different types of understanding. Naturally, much simplification is necessary in order to manage the amount of data, but once again it should be stressed that the classifications described in the previous two chapters were not preimposed, but emerged from the data.

9.1.1 Dependencies

In the introduction to this part, it was stated that this chapter would essentially describe a function

{visual, nonvisual} x {internal, external} x {lecture, new} → types of understanding.

reasoning style   *sense of authority*   *course*   *learning outcomes*

The structure of the chapter follows this characterisation, taking combinations of inputs in turn and describing the observed learning outcomes. As described in section 4.5.5, the visual or nonvisual reasoning approach and the sense of authority emerged as causal factors; these essentially determine how well a student is likely to progress. The course the student

attends (where the differences between these were described in section 2.4) then acts as an intervening factor: students with similar profiles in reasoning and learning approach acquire slightly different types of understanding according to the influence of their course.
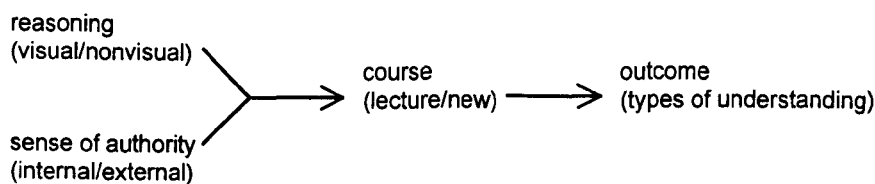
Figure 18: Relationships between causal and intervening factors and learning outcomes

The different combinations of factors are described below in an order approximately consistent acquisition of more types of understanding. This allows consistency to be discerned in the influence of each factor. The next chapter examines each factor in isolation in order to clarify this point.

9.2 Combinations

In principle, $2^3 = 8$ combinations of factors are possible. In fact, seven of these appear among the participants in the research, along with two slightly anomalous combinations. The first of these was described at the end of the previous chapter, and consists of those students whose sense of authority depends upon whether they are interacting with the mathematics or the human authorities. The second is a subset of the external, nonvisual, lecture course group. It is made up of those students who cannot be said to learn any of the material meaningfully, and whose subclassification will be termed "episodic". This group are mathematically the weakest on the study, and are examined first.

9.2.1 External, nonvisual, lecture course ("episodic")

"Episodic" learners are so named because of the type of memory they appear to attempt to use in their learning of mathematics (Baddeley, 1997, Ashcraft, 1994). They believe that

mathematics is about learning procedures using rote memorisation, which means that they rely on their teachers to provide such procedures. Since visual representations rarely form a necessary part of written work in school mathematics (Dreyfus, 1991) we would expect these students to reason nonvisually, and this explains the absence of a parallel group with visual reasoning preferences. These students try to acquire only instrumental understanding, and the result is that they acquire very little understanding of any type. Illustrations are given below.

Often the school experience of those in this group has been very regimented and they praise their teachers for giving them clear methods to follow and lots of practice.

Z: I just like the way we were taught at school…where we…we have our notes, and then we practise so many times that it's just sort of drummed into us. You just don't forget it that way…

YZ2, 705

The phrase "drummed into" is indicative of the fact that there is no meaningful processing on Zoë's part: this is memorisation by brute force. The essential place of the external authority can be seen in that these students do not expect their role to involve any creative thinking. Indeed, if information is not explicitly provided it may not occur to them to seek it out.

Z: Like, some of it we just can't do. But I mean…that's because we've not been taught it so I don't see how – it's impossible to do something that we haven't been taught.

YZ2, 799

Most significantly this group do not do anything recognisable as reasoning about objects or properties. This is apparent in their work strategy, which involves mimicking procedures or,

where these are not available, attempting to match the surface features of the question with one or more passages in their notes and fit these together.

> Z: What we usually do is when we have our notes, like sprawled out everywhere (*laughs*)...copy bits from here and here and like, put it all together. And hope it turns out right.
>
> YZ5, 521

Essentially these students work by matching symbols in a meaningless way: they do not try to judge whether or not what they have produced is sensible. Therefore they are dependent on extrinsic measures of their learning. Judging by her assignment marks, Zoë is able to assume that her progress is satisfactory.

> Z: Yes but the marks we're getting are okay, so...about 18 out of 25.
>
> I: Yes.
>
> Z: Which isn't brilliant, I mean I know other people who are getting better, but it's not disastrous.
>
> YZ5, 133

The students who fit this profile are all on the lecture course. This is probably not a coincidence: in keeping with their external sense of authority they see mathematics as a stepping-stone to a career, and have deliberately chosen a degree which involves elements of more "applied" subjects. Unfortunately the lecturer's role as dispenser of information fits very well with such students' beliefs about their teachers' responsibilities, allowing them to maintain their learning approach unchallenged (*cf.* Perry, 1970, Simpson, 1995). Certainly they do not become engaged in the process of proving: their assignments usually involve applying these results rather than proving new ones, and since they have the proofs in their notes, they anticipate that they can learn these later in order to pass examinations.

Z:	I think that once I actually…like over the Christmas holidays *(laughs)*, when I'm actually going to sit down and,

Y:	Learn it.

Z:	Learn it, then…hopefully when I come back in January I should – it should make sense.

YZ7, 1705

The result is that episodic learners display virtually no understanding of any type.	In interviews they approach every task in the same way, by attempting to recall episodes which are triggered by the symbols and words in the question.	This is evident in the regularity with which they use the phrases "we did" and "he did".	In this excerpt for example, Zoë is discussing the special case of $x=1$ in the week 9 series question.

Z:	I don't know. Yes I remember him doing something like that. I think it was 1 minus a half, plus a third…

Y:	Mm.

Z:	And then he went through that whole proof about how you can't…erm, you…

YZ9, 711

They assume that for every question they have already been told the answer or calculation procedure, and that what they should do is try to recall this.

Z:	Okay so that's going to get…that gets…I wonder if there's a proper way to do this? Which we don't, we never did?

YZ5, 275

They do not consider questions in terms of the underlying objects, and appear not to have thought about the meaning behind any of the work. Certainly they are not able to recall and use results in a way that demonstrates relational understanding. In week 7 Zoë mentions the

"last term" of a sequence and when asked about this it becomes apparent that she has misunderstood the sequence notation $a_1, a_2, \ldots, a_n, \ldots$.

I: Mm, what would the last term be anyway?

Z: Er, n.

Short pause.

Z: Wouldn't it? No it would – no it wouldn't necessarily be n,

Y: Yes,

Z: It would be the nth term...wouldn't it?

I: That would be the last term of the sequence?

Pause.

Z: Isn't it?

Y: I don't know! *(laughs)*

Z: Well I know, I suppose n could be any value, from one to whatever...inf – can you say infinity? Don't know...!

YZ7, 647

In the absence of their notes these students often cannot remember enough to even make a start. In the series question Zoë expects the symbols to have a routine attached, which means that she begins working on the task with little or no idea of what the question requires.

Z: In other words, what is the...what is the value of...of x or n – what is the value of n when...is that what it means?...It's not one of these erm...thingies, is it?

Y: Whaties?

Z: Like one of those theorems that we've been doing. You know we did...you did, it looks approximately like...

Y: Oh, yes. Oh, what on earth did we do? I don't know.

Z: You know the, u_n and v_n,

Y: Yes. Oh, it might be...

Z: I don't know. Shall we try?

<div align="right">YZ9, 323</div>

An absence of formal understanding follows as a matter of course, since these students do not work as though handling categories of objects. It is also evident in inaccurate descriptions of the concept terms and in a lack of importance attached to definitions. In this excerpt Zoë confuses *bound* with *limit*; she has not developed a sense that these terms have precise meanings, much less learned what these are. Note that this is week 7 and she will have met definitions for bounded and limits at least four weeks previously.

Z: I'm not sure about what this bounded thing is, but if it's what we think it is then...

I: Can you describe exactly what you think it is?

Z: I *think,* that it's...well from what it sounds like,

I: Yes, go on.

Z: It's that, if it's bounded above for example, then it's tending to *a*...as in the values are increasing to a certain limit.

<div align="right">YZ7, 593</div>

Throughout the course these students make little progress in understanding the aims or the details of the formal approach. In Zoë's case this is perhaps clearest in her handling of infinite objects.

Z: ...if you want to find out whether it's converging you're going to have to – I mean usually you can tell whether it's convergent just by putting in values that get reasonably high as in you don't have to...like you can do it on your calculator.

I: Right.

Z: Like do, I don't know, even if you do like a few and then do like, a hundred...n equal to a hundred, n equal to a thousand and so on,

<div align="right">186</div>

I: Mm,

Z: You can see what they're getting closer to...you don't actually have to do, infinity, as such.

<div align="right">YZ5, 735</div>

Overall, nonvisual students with an external sense of authority who try to learn by memorising episodes gain very little understanding of any type.

9.2.2 External, nonvisual, lecture course ("semantic")

This group differs from the previous one in that, although they are still nonvisual and dependent on an external authority to provide the material, they do attend to the meanings presented rather than concentrating solely on the surface features of the work (hence the label "semantic", *cf.* Baddeley, 1997, Ashcraft, 1994). The study yielded less information than would be desirable for this group since the examples are Xavier and Steve. Both of these students' partners were the more extrovert participants and were visualisers, and therefore tended to dominate in interviews (visual reasoning produces quick conclusions, as will be seen throughout this chapter). However we can see the difference between these students and the episodic learners. In particular, they are successful in acquiring relational, though not logical or formal, understanding.

Like the previous group, these students are expecting that mathematics will be predictable and based on answering particular kinds of question. For instance Xavier sees mathematics as that which is taught in classes, and questions resembling logic puzzles belong to a different realm for him.

X: They were sort of more like logic puzzles,

W: Yes.

X: As opposed to maths questions.

I: Yes. So what do you define as a maths question...?

<div align="right">187</div>

W: 2 plus 2!

I: *(laughs)* Yes.

X: Erm, I don't know, sort of…probably some of the questions you could have done, without doing A-levels maths.

<div align="right">WX2, 238</div>

They value the security inherent in this type of work and they wish to continue to do well.

X: So I thought, if you wanted to go into maths you really should do the further maths mechanics,

I: Yes.

X: Because I did A-level stats instead I thought, that's less maths related, so…

I: I think so.

X: So when you get to university because there's…vector questions, and forces questions, which I'd find…with sort of a maths/stats background,

I: Yes,

X: I'd find really difficult.

<div align="right">WX2, 390</div>

Their nonvisual reasoning strategies take time to elaborate, which means that they are often left behind by their visual partners in the initial approach to questions; conclusions based on visual prototypes can be reached quickly and nonvisualisers may not follow such arguments. Instead they introduce nonvisual representations by interrupting their partners or taking over when these people cannot produce justifications. They often then display relational understanding. For example, in working on the series question Xavier refers to the familiar series $\sum \frac{1}{n}$ in order to sensibly choose cut-off points for case analysis.

W: Shall we do that for x is 1 or would you do it just if x is bigger than nought?
Short pause.

X: Erm...yes you want to do it for like *x* is minus 1. Because then that will be 1 over *n*.

<div align="right">WX9, 117</div>

Their focus on nonvisual representations means that this group become aware of the definitions in the course, and introduce parts of these in answering questions. They thus show characteristics associated with formal understanding. However it often seems that this derives from the frequent appearance of these representations rather than from knowledge of the role of definitions in the technical context.

X: Isn't it something to do with that... a_n minus *l*, don't you remember?

<div align="right">WX7, 305</div>

In addition the lecture course provides only restricted practice in the use definitions. Hence these students' recall of them is fragmentary and they are not fluent in their use. In this instance Xavier's spoken use of the notation is awkward: he says "*e*" when he means "epsilon" and confuses his "*l*" for a digit "1".

X: Er, the term in the sequence, a_n, minus the limit...the modulus of that is less than

    epsilon where epsilon...is...any real number...and *n* is beyond any big *N* in the

    sequence,

I: Mm,

X: So then all the terms beyond large *N*. And then if you get rid of the modulus sign,

    you can write that as minus epsilon is less than *a* to the *n* minus one, is less than

    epsilon, and that can be rearranged so that *a* to the *n* is less than, *e* plus one, but it's

    also greater than one minus epsilon,

<div align="right">WX7, 343</div>

More importantly, those on the lecture course are not regularly required to use definitions in formulating arguments about whole categories of objects, and much of their experience with

<div align="right">189</div>

the definition of convergence will have been in showing that specific sequences are convergent. This leads to a weakness in logical understanding of this concept. After the above excerpt, Xavier concludes:

> X: So therefore a to the n is bounded above and below, and therefore this is the definition for the convergent sequence. So it converges.
>
> WX7, 351

It seems that once he has introduced the definition, Xavier becomes caught in the routine he has associated with it and loses track of the fact that this question requires something different. This lack of flexibility persists when the interviewer tries to prompt him to think about what is required for the sequence to be bounded.

> I: What about the terms before n is greater than big N?
>
> *Pause.*
>
> X: Erm, well they'll be…erm, the modulus will be bigger than epsilon. So you've got to go far enough along, so that you're less than epsilon.
>
> WX7, 383

This excerpt also shows that he has not fully understood the definition: he assumes that all of the terms before N are further from the limit than epsilon, which is not necessarily true.

Overall, while this group may develop good relational understanding (as the term is used in this study to describe work with specific objects), they do not develop secure logical or formal understanding of the material.

9.2.3 External, visual, lecture course

The students in this category are somewhat like those in the previous two in that they too have an external sense of authority. However their approach to reasoning is reliably based

on visual prototypes. This, combined with the lecture course's lack of requirement to produce written proofs, means that they too display relational but not logical or formal understanding.

In Wendy's case, the central factor to her external sense of authority is the idea that mathematics has practical applications.

> W: ...I thought, I didn't really want to do a straight maths degree, I wanted to do
>   something that would be useful for like...in business and stuff like that?
>
> WX2, 65

Again these students rely on a human authority to provide the material by way of notes and exercises, and are satisfied with their progress so long as they receive explanations which would allow them to handle similar exercises in future.

> W: Erm, you're sort of...not sure whether you're doing it right or not, and then you
>   think, oh what the heck hand it in like this, and then it comes back and you realise it
>   is not actually that bad, and...!
> I: Oh that's good.
> W: And they go through it in the sessions, and then you realise oh yes, well, I could do
>   another question like it now, and...
>
> WX3, 117

They attend to meaning and demonstrate relational understanding when required to work with specific objects.

> W: Erm, if...if you take x is between nought and 1, say x equal to a half, erm...and put
>   it into the series, you'd get minus a half, plus a half squared over 2, minus a half
>   cubed over 3, and so on...

X: So that's going to be smaller.

W: Erm, the terms are decreasing in size, so... x_n is bigger than x_{n+1} [*she misreads notation here*]

X: And tending to zero.

Pause (writing).

W: Converges?

In this instance Wendy is using $\sum \dfrac{\left(-\frac{1}{2}\right)^n}{n}$ as a generic example or prototype in reasoning about a category of objects. She believes her reasoning will generalise and in this instance she is correct. However she makes no move to check this or to express her argument algebraically, and here we see the weakness with this group. While they gain an informal understanding of objects, they do not attend sufficiently closely to the formal algebraic representations. For example, when describing written work they sound somewhat like episodic learners, describing the detail of the algebra and what they "had to do" without relating this to the eventual results.

W: Erm...we've done like, all the different versions? Say...like a to the n then, n to the a, and what they tend to as n goes to infinity.

W: You had, you had to erm...find an increasing and a decreasing sequence that erm, converged to the same limit,

X: Subsequence, wasn't it?

W: Yes, subsequence and prove that the...sequence would converge to the limit. And it was like, we had to find one that – a decreasing sequence that converged to the same limit as an increasing sequence. And all this other stuff going round that you had to use and...oh no!

The lack of a secure grasp of the nonvisual representations affects these students' ability to interpret notation used in proofs. We saw in the previous chapter that in the proof correction problem Wendy fails to understand that n represents a general natural number.

W: Mm…. They need to show that this…for more values though. They need to show by erm…induction or something that it's increasing.

<div align="right">WX5, 571</div>

When working with whole categories of objects, these students display a lack of logical understanding in other ways. Attending to the visual representations gives them access to visual prototypes, which means that unlike the previous groups they do work with categories of objects rather than symbolic routines. However these students do not develop an awareness of the expected standards of argumentation: they generalise directly from their prototypes and do not develop their arguments using abstracted properties.

W: Well if it converges, you get closer and closer…

Pause (drawing).

W: Is that enough to like, justify it…a little diagram, what have you?

I: Well, I'd like you to prove it, if you can.

W: Oh dear! *(laughs)* Oh right, well, if a to the n is increasing…*(writing)*…then, it's bounded…

W: It's convergent [*draws a monotonic increasing convergent sequence*]…yes so if it's convergent it's always…or…say it could be the other way round it could be, going down this way [*draws a monotonic decreasing convergent sequence*]. It converges, so it's always above that limit.

<div align="right">WX7, 209</div>

This way of reasoning is not challenged by the lecture course. A student with an external sense of authority will not naturally question anything presented in a lecture, so they do not closely examine provided proofs. Nor do they often formulate or receive feedback on their own justifications.

Similarly these students do not develop formal understanding: their visual images given them the impression that they understand the ideas, so there is no incentive for them to seek out the complicated formal representations. The result is that their explanations of central concepts are often imprecise and very unlike definitions. Here Wendy is explaining what it means for a sequence to tend to infinity.

> W: Oh well as erm...as n gets bigger,
>
> I: Mm.
>
> W: The sequence gets to a really really big number, and it just gets bigger and bigger and...
>
> I: Yes, yes. Is there, have you seen an actual definition of that – sorry?
>
> W: There's no like, end to it, it just keeps on going.
>
> <div align="right">WX3, 141</div>

They also tend to maintain beliefs that are contrary to the definitions, and their external sense of authority means that they may avoid reconstructing their understanding even when directly confronted with a conflicting example.

> I: Mm. What exactly does it mean for a series to converge anyway?
>
> W: Erm...the sum, after you get beyond a certain point,
>
> I: Mm,
>
> W: The sum doesn't increase any more. The sum of the series...?
>
> <div align="right">WX9, 577</div>

I: Right, what about the minus 1 to the n upon n one? The – the one where,

W: The alternating series?

I: Yes. Does that ever stop moving?

Short pause.

W: Well it, converges, so...! Erm, it must do.

I: I'm just intrigued by...at what point does it stop moving?

Pause.

W: No idea!

<div align="right">WX9, 647</div>

The result of this ill-developed link between visual and nonvisual representations is that these students end up feeling that the formal approach is quite alien to them. Tom, another learner with this profile, maintains a feeling that he understands the material but says that he "can't do proofs".

T: Don't like proofs.

I: Don't like proofs.

T: Still don't understand how to do proofs. Proof by induction still goes way over my head.

I: Really?

T: Anything to do with proofs.

I: Yes...

T: I can use the theorems, I just can't prove anything.

<div align="right">ST9, 878</div>

Overall, an external-visual combination is conducive to direct generalisation from prototypes, which remain based on spontaneous conceptions and are not reconstructed to take account of the definitions. Once again the lecture course does not challenge these

learning and reasoning approaches, so these students do not develop logical or formal understanding.

9.2.4 External, visual, new course

In this group we see the intervening effect of the new course. These students have an external sense of authority and a visual approach to reasoning like those in the previous group, but are more regularly required to formulate, and respond to criticism of, proofs of their own. The result is that their basic predispositions do not change, but they acquire at least some of the habits of logical and formal understanding.

In the interviews the learning and reasoning approaches of these students are manifested in familiar ways. For example in week 5 Emma gives a visual solution to the sequence question, and her external sense of authority means that she is not fazed by the fact that she and Fred disagree.

> E: I think differently to Fred.
>
> I: What do you think Emma?
>
> E: I think it'll conv – it will oscillate around 1 over nx.
>
> I: Right. And Fred thinks it will go to 1 if x is positive and minus 1 if x is negative.
>
> E: Yes.

EF5, 427

There is no attempt to keep working until she can judge whose answer is correct, because she leaves it up to the interviewer to do that.

> E: What's the answer?

EF5, 493

However the external sense of authority is manifested differently with regard to their course. In the absence of a single figurehead in the form of a lecturer, these students do not try to absorb the work and wait for feedback from supervisors. Instead they display a reliance on their peers; they try to do the work, but before submitting it will ask people they perceive to be clever to check that they are right.

> E: Well, I'm not very good at proving I don't think. But, it's a bit weird, because I don't really know when I've got there or not.
>
> I: Right.
>
> E: But erm...I did some for the stuff we'd had this week, and they seem all right, according to the clever guy on my corridor, so...!
>
> EF3, 191

This could be a coincidence, but it seems likely that it is a consequence of the increased emphasis the course places on the work the students themselves produce, in combination with the encouragement to work together in class.

Another difference stems from the fact that the students on this course are regularly required to produce proofs about categories of objects. This means that it is not viable for them to maintain a belief that mathematics is only about learning to do calcuations involving specific objects. They *do* retain the idea that their job is to learn the material as presented, but this is manifested as a belief in collecting and memorising "correct" proofs instead. The result is that like others with an external sense of authority, these students tend not to examine the material very closely and may rely on it being in notes to be learned later. Emma, for instance, seems unwilling to attempt a justification without the aid of her booklets, and postpones assessment of her progress until the examination.

> I: Can you prove it?
>
> *Long pause.*

E: If we use our workbook yes, probably.

<div align="right">EF7, 473</div>

I: ...quite a lot of work in the workbooks. Do you feel that's worthwhile? Do you

feel you learn a lot from that?

E: Tell you at the end of the exams.

<div align="right">EF9, 945</div>

However the increased requirement to work with whole categories of objects does encourage the students to focus on relationships between these. Their descriptions of the work are clearer and more conceptual than those of students on the lecture course, and focused on results rather than algebraic detail or procedures.

I: What have you been doing in Analysis?

E: Erm, just about to finish off sequences, and stuff.

I: Right.

E: Infinite series, and stuff like that.

I: Can you be more specific...?

F: Er, decimal sequences[24],

E: Mm, the ones with the sums, using geometric progressions.

I: Okay. Yes.

E: And I liked the...what is it, the ninth thing...I think that's quite good. Where

you're saying a ninth, nine ninths – no, is it nine ninths? Or point nine nine

reoccurring is really one, and stuff like that.

<div align="right">EF7, 105</div>

In fact they develop a comprehensive awareness of all the concepts studied. However because they do not seriously examine the arguments, they may know that categories are

---

[24] Fred is Emma's partner in the interviews. His own development is discussed in section 9.2.6.

related but struggle to remember exactly how, as in this instance in which Kate is aiming to argue that a convergent sequence must be bounded.

> K: Well the other one with convergent and bounded is...erm...completeness, thing. Which can be taken either way can't it, if it's increasing,
>
> J: Erm...is the completeness axiom...to the what, if and only if?
>
> K: And bounded it's convergent, if it's decreasing and bounded...no. It's just, if it's bounded and increasing it converges, if it's decreasing and bounded it converges.
>
> JK7, 519

A visual reasoning approach is displayed in the usual ways: when handling categories of objects these students begin by working with visual prototypes, and they feel that these are more natural to them.

> E: ...I suppose it's easier to think of it graphically.
>
> I: Yes.
>
> E: Not all in symbols.
>
> EF7, 105

However attending the new course means that these students receive regular feedback on their argumentation from both their teachers and their peers. This trains them to perform counterexample checks and attempt to justify their answers in terms of properties. They may not be very good at it this; generating appropriate examples and abstracting properties is difficult and may not come naturally (*cf.* Dahlberg & Housman, 1997). However they do at least attempt to instigate these measures, so showing habits associated with logical understanding. This can be seen in Kate's progress between weeks 3 and 7. In the first she confuses *limit* with *bound* in explaining their meanings. In the second she is still not fully correct but she is able use the terms in a much more mature and natural way in the context of reasoning about convergent sequences.

K: Bounds, it's got upper bounds and lower bounds...like, with what it tends to.

I: Mm. What does that mean then?

K: Like, it gets erm...a number can get higher and higher and higher, and it's getting to reach a certain number,

I: Mm.

K: And it doesn't go past that number.

<div style="text-align: right">JK3, 91</div>

K: Yes...oh, but it might...no it would be bounded wouldn't it, by its first term...

J: We don't know if it's increasing or,

K: And its last term.

J: Depends if it's increasing or decreasing doesn't it?

K: Well it would be bounded, either below – if it was decreasing it would be bounded above by...

<div style="text-align: right">JK7, 369</div>

Their development in formal understanding proceeds in a similar vein. Like the previous group of visualisers, these students do not readily use the formal work as a means of reaching answers. This discussion takes place once Emma and Fred have reached agreement on the sequence question in week 5.

I: Yes there are some rules that you might have said, that might have been useful to prove this...can you think of any of the rules that you've learned...?

F: Oh yes there's er,

E: Sandwich theorem, stuff like that.

F: No, er...er...the third one in the – the other one.

E: Sum?

F: Mm, oh, yes, yes, might...no. Kind of.

<div style="text-align: right">200</div>

I: Yes. So those didn't, you didn't use those this time but do they come to mind generally when you see this sort of thing?

E: Erm, for ques – er…if it says prove it, yes.

<div align="right">EF5, 589</div>

Similarly once they have definitions to hand they may not be very fluent in their use, or even understand them very well. For instance in answering the <u>convergent implies bounded</u> question, Kate replaces her original correct answer with an incorrect one because she misunderstands the definition of convergence.

K: Okay perhaps…I don't want to…perhaps it's false then. Because epsilon could be as big as you want it to be.

Pause.

K: Okay I've changed my mind! I think…but if epsilon's like as large as you want it to be, that could be like infinity…

J: But it could be as small as you want it as well, can't it?

K: I know but it could be, infinity. Which means that it wouldn't really have a bound.

Pause.

K: Okay perhaps that one's false and d) is true.

<div align="right">JK7, 635</div>

This is an example of what Pinto and Tall call *distorted imagery*; Kate reaches her conclusion by faulty reasoning from a correct definition (Pinto & Tall, 1996).

Overall, their experience in the new course means that these students are likely to think using definitions an appropriate way to approach a question, to know them and to have at least some facility in their use. It would be inaccurate to say that they develop good logical or formal understanding, but they do make progress in this direction.

9.2.5 External, nonvisual, new course

In this study there was only one student in this group. This is natural since the categorisations emerged *from* the data, and indicates why a grounded theory approach would ideally proceed to theoretical sampling. Additionally, Hugh is the only person in the study who appears to be less than frank in interviews. He comes across as rather cynical about the joys of learning mathematics, although this may be exaggerated in order that he does not have to give detailed explanations of the material in interviews.

However, Hugh's behaviour as a nonvisual reasoner with an external sense of authority, who attends the new course, is consistent with the pattern developed in the previous groups. He develops some of the habits of formal understanding, since the course exposes him to the need to produce proofs and his nonvisual approach means that he attends to the definitions. However his external sense of authority means that he does not tend to examine the work closely, so that his relational and logical understanding can be weak[25].

Hugh is studying mathematics at Warwick with a view to getting a good job – his aim is not to learn new things so much as to perform well in tests, and he defers to examination results as a measure of understanding.

H: Well, because I failed Oxford, so...

I: *Laughs.*

H: So Warwick is next!

I: Yes, yes. Was – would that...you would have preferred to go there would you?

H: Well because, only because if you go to Oxbridge, when you apply for a job,

I: Yes.

H: They're looking for the name. That – so if say you went to Oxbridge, you'll get a job easily.

---

[25] Nonetheless, to avoid being said to generalise from one case, this section speaks about Hugh rather than "students in this group".

I: Yes, what have you learnt then?

H: I don't know. We'll find out when we do the exam. If we pass the exam that means we've learned a lot.

He believes that mathematics involves correct or "best" answers and that learning is best accomplished through repeated exercises.

I: All these proofs that you're being shown, or the ones that you're producing, who decides whether they're correct, do you think?

H: Your supervisor.

G: Your super...?

I: How does he know?

G: He's god!

I: How does he know? Is it a he?

H: He has a model answer. So he just looks at it. If he doesn't know how to do it, he says, have you got a proof yet? And we all say yes, and he just says read the solution please.

H: Practice makes perfect so do lots of exercises every day.

Hugh is consciously nonvisual.

I: Mm. Do you...do you like using graphs,

H: No I don't!

Unfortunately information on Hugh's understanding is not as full as would be ideal, this time because Hugh's partner Greg is extremely enthusiastic in answering the interview questions whereas Hugh is more reticent. However we can see that the requirements of the course, combined with a nonvisual approach, mean that by week 5 Hugh has a more confident and complete (though still imperfect) knowledge of the definitions than students in the groups examined so far.

H: A sequence (a_n) tends to infinity for every C greater than nought, there exists an

N...for all natural num – belonging to all natural numbers, such that... a_n equal to

little n greater than big N, greater than C, when, n, little n is greater than big N.

GH5, 767

However as with the visual-external group on his course, Hugh's goal is to do well in tests by reproducing the material. This means that although he might try understand the way definitions are used, he is not looking to understand and memorise all the details as he goes along. Indeed he is positively relieved when questions can be answered without needing to expend energy in understanding proofs.

H: ...last week was [*undeciperable word: probably "series"*] theory, it's much better,

if you just write something you don't understand then...

I: Then you what sorry?

H: Then you can use the, theorem.

I: Right, okay. Even if you don't understand? Can you give me an example? Of the sort of thing?

H: I don't know...like say if you use the ratio test.

I: Right.

H: Maybe you don't really understand the proof, but you can use that,

I: Oh I see, you can still use the test to get an answer to the question. I see what you mean.

H: You can get the marks even if you don't understand.

<div align="right">GH9, 29</div>

The result is that Hugh's relational and logical understanding are weak, so that while he is aware of what has been covered, he is not confident to say whether or not given results can be used in new situations.

H: Diverges.

G: Well it's...I'm not sure.

H: It's the ratio, ratio test, thing.

G: Can we use the ratio test?

H: No...

<div align="right">GH9, 539</div>

In effect this is the opposite of a familiar problem. Where visualisers cope with the large cognitive load of the course by focusing on visual prototypes at the expense of algebraic representations, students like Hugh focus on algebraic representations at the expense of relational understanding.

9.2.6 Internal/external combination, visual, new course

For a time this was the most perplexing combination to arise in the study. These students are those described at the end of chapter 8, who have an internal sense of authority regarding the mathematics, and an external sense regarding the human authorities. The visual-new course combination means that they develop good relational and logical understanding of the material. However the sense of confidence that they gain from this, in combination with their external sense of authority, means that once again they do not examine the formal representations in detail, and so do not acquire formal understanding.

Students in this group arrive with a clear intrinsic interest in mathematics, and may also have a mature awareness of the need for proof.

> C: We had sort of very deep arguments during our further maths classes about whether...certain things existed, and stuff and er...mainly negative nought was the one thing.
>
> I: Yes.
>
> C: And...I don't know that was just - and there were some interesting books in the library on infinity and things, and it just got you thinking that you can't really assume stuff, and just the way you're looking at it.
>
> I: Yes. So.
>
> C: Again, quite happy to assume that everything's false, until proven otherwise that's why I have the er...I'm able to understand the need to prove things.
>
> CD3, 1301

They may also show *potential* for good formal understanding, coping well when terms are used in a technical sense by relating them to particular objects. Here Cary is discussing why a constant sequence is said to be increasing.

> C: Well, it's just...just following from the – it's not strictly increasing or strictly decreasing or...it's just, one or the other.
>
> I: Mm. So why do we have the increasing and decreasing definitions do you think then? If we get weird things like that?
>
> C: Because there was one sequence where it was stepping up, it went up in steps. And I suppose it just covers all eventualities.
>
> I: Yes.
>
> C: Which it does actually because otherwise if it was either just greater than or...just...then a sequence such as, 1,1,1,1,1 wouldn't have a definition.

These students exhibit an internal sense of authority as regards the material, actively trying to make sense of it and using conflicts to improve and strengthen their understanding. However they have a less mature relationship with human authorities. They are concerned about getting the best possible marks, and wish to be recognised for their ability to produce quick and intuitive answers. This latter comes from a genuine pleasure in their own intellectual capacities, but it means that they tend to jump to conclusions and do not always make appropriate checks, as illustrated in Cary's assumptions about his answer to the commuter problem[26] (Dean is subsequently able to demonstrate that these are not valid).

D: But it – I'm just wondering if that's...the only possibility.

C: Think...

D: Or not.

C: It's an answer.

D: Yes. But do we know for sure that that's the only answer?

C: Yes. Calculus will prove it.

I: Can you?

C: That's it. I could write that into a paragraph and that would be a proof.

CD3, 769

In effect, while their sense of understanding is internal, external approval is still a driving force for them.

Their visual approach gives these students a very strong sense that they can reason about the objects in the course. In week 9, Fred expresses his view that the results are simple although the proofs may not be.

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[26] See the week 3 questions in appendix B.

F: ...I don't know that no-one would understand it, because there's a lot of
     it's...simple stuff

E: I don't know it,

F: The thing is you're trying to prove it.

I: Mm.

F: So I think you could...well, as long as they had a modicum of sanity you could
     look, this is what I'm doing saying this is bigger than that...they'd be, no idea how
     you do that, but I can understand what it is you're showing.

                                                                            EF9, 127

As would be expected they reason using visual prototypes. However their internal sense of
authority in the material means that they respond to conflicts, such as those generated by
counterexamples, by reconstructing these prototypes. Hence they incorporate different
types of object into their knowledge of the category. This enables them to check for likely
counterexamples to their own reasoning, and they try to formulate arguments in terms of
properties, thus showing logical understanding.

C: If it converges...that has to be...well I don't suppose you can say bounded. It
     doesn't have to be monotonic. Erm...

                                                                            CD7, 308

C: Yes, I'm trying to think if there's like...if you can say the first term is like
     the...highest or lowest bound but it's not. Because then you could just make a
     sequence which happens to go, to do a loop up, or something like that. In the way
     that, it did...erm...

                                                                            CD7, 314

Since they reconstruct their prototypes to take account of different types of example, these
students also show some formal understanding: their categories correspond well with their

formally defined counterparts. They may also develop a good understanding of the role of definitions in the abstract. This is particularly striking in Cary's case, where it extends beyond the bounds of mathematics.

I: Do you feel that you now see maths in a different way?

C: Not maths, but arguments.

I: Right...can you explain how?

C: We had this...I walked into the kitchen. I thought, I'll have an early night, I was going to make a cup of tea,

I: Mm,

C: And there was two people around the table, arguing about whether or not law came from morals?

I: Right.

C: And erm...so I was listening to them, and I thought, they're getting this all wrong. So I started joining in, and...and I found myself, *defining* stuff, and I was like, I cannot argue with you unless I have it defined, exactly what I'm supposed to be arguing about,

<div align="right">CD7, 793</div>

However the combination of their visual reasoning and their relationship with the human authorities means that this does not have sufficient impact on what they actually *do*. As in the previous sections, students with a visual-external profile often do not attend closely to the formal representations. They may be impatient with these when they are used to prove results the student considers "obvious".

I: Mm. Can you give me an example of the pointless stuff?

E: Erm...

F: Having to prove that...multiplying a frac – multiplying an inequality by a negative number reverses the sort of sign and that. And all the inequality rules, that you just know them...because you can just look at them and say, it's about right.

<div align="right">EF3, 843</div>

Hence these students lack patience with the formal material, and do not learn the definitions very well. Fred, for example has real problems remembering the definition of convergence even at the end of the course.

F: I'm sure it's something more along the lines of a_n minus a, tending to zero or being zero and...

E: How about a_n minus l, l being the limit?

F: Well, a...that doesn't matter. Maybe it's...trying to think...erm, it's the limit – no.

I: If I say the words, "for all epsilon greater than zero", does that help?

Short pause.

F: It confuses me.

I: Does it?

F: Erm...hang on a sec, write down...for all epsilon greater than zero...

Pause.

F: Er...oh, erm...it's, for all epsilon greater than zero, a_n minus a_{n+1} is less than...epsilon? Something like that...yes? For all...no. a_n minus a big N, is less than epsilon...for... when...blbl...as long as,

E: When it's,

F: a_n is bigger than big N.

<div align="right">EF7, 503</div>

He clearly has a surface recall of what it should be like, but confuses it with the definition of a Cauchy sequence and does not appear to be trying to do any meaningful reconstruction. In Fred's case this is the result of his deferral to authorities regarding the formal work; in

keeping with an external sense of authority he believes that they will provide the "best" way to approach certain questions. He therefore sees the formal approach to proving results about convergence as a "convention", which indeed it is, but not in the absolute and procedural sense that he seems to have in mind (*cf.* Harel & Sowder (1998) on external conviction proof schemes).

F: I think that they...you have to use the conventional proof. Because I think, that's the point of all these proofs, is that they're the like...they're the way you do it and there is no other way.

I: Right.

F: So if you're doing any question where you've got to prove that something tends to infinity then you do it that way, full stop.

EF5, 965

Cary is not so deferent in his reaction to the human authorities, but like Fred he does not give attention to meaning in recalling the definition. Having introduced part of it he can produce a more complete version under prompting, but still does not state this clearly.

$$|a_n - a| < \varepsilon \text{ when } n > n_0$$

Figure 19: Cary's partial definition

I: Is that the full definition?

C: Er...no. Not totally. That's...we want a bit about here...(*writes*) that...when n is greater than n_0, there exists an $n, - n_0$, such that modulus of a_n take a, is less than epsilon, when n is greater than n_0.

CD7, 406

Indeed, he subsequently returns to trying to formulate arguments based on his prototypes. When these students *do* make the connection between the definitions and their prototypes we can see the strength of using the two representations together, as the argument follows immediately.
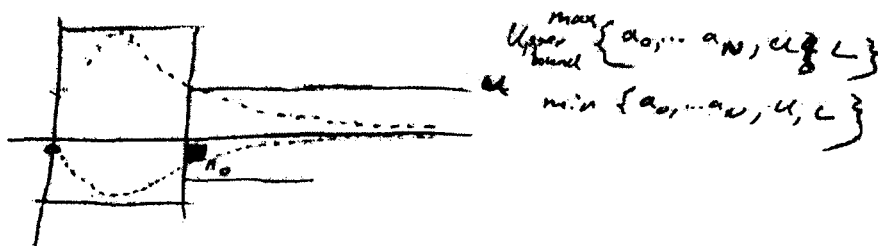
Figure 20: Cary's diagram to show that a convergent sequence must be bounded

C: ...Yes, your n_0...that could just be called your n_0 instead, so going back to your definition up there, there exists this point here, such that after that point, i.e. when n is greater than n_0, the sequence...that statement there won't be less than any epsilon which you just happen to pick.

D: Mm...

C: And so it's...and so the upper bound – so because there's finitely many terms before n_0, then er...your upper bound will either be plus or minus epsilon, or it'll be the maximum of those finite terms beforehand.

CD7, 520

However they may never successfully make this link, in which case, as we have seen with other visual groups, the formal work can remain quite detached from their own ideas about reasoning and justification. In week 7 for example, Fred and Emma decide on the correct answer for the convergent implies bounded question and are both shocked at the amount of time I think is necessary to justify a result. Fred believes it to be basically obvious, and sees the provision of a proof as quite separate from this sense of conviction.

F: Unless we try and think of exactly what the definition of a convergent is, and try and...work out if there's any other way. How long have we got, by the way?

I: About another, ten minutes or so?

Both: Ten minutes??

I: If you want. Well, have you justified your answer do you feel?

F: Well it's not really scientifically proven. Because I think...I think I'm right, but it's not... it's not...if we've got to prove it then that's a different kettle of fish altogether.

<div align="right">EF7, 359</div>

Given the progress made by essentially weaker students than these on the new course, one would expect this group to acquire a better formal understanding than they do. It appears that their relationship with the human authorities combines with the sense of understanding they derive from their visual prototypes to mean that they do not feel a need to integrate the details of the formal work into their thinking.

9.2.7 Internal, visual, lecture course

This is the second category with only one exemplifying case in the study, and unfortunately this is Vic whose week 9 interview did not take place. However, once again the results for Vic fit with the general pattern which is now emerging.

Vic's learning and reasoning approaches are similar to those of the students in the previous group, except that he has a consistent internal sense of authority. This means that his own understanding is important to him, without the inhibiting effect of the need to demonstrate speed. The result is that his arguments are considered and he adapts them reasonably in response to conflict. Unfortunately however, the lecture course does not provide him with a regular requirement to generate arguments and face such conflicts, and the result is once

again that his relational and logical understanding are good, but he does not develop good formal understanding.

Vic's internal sense of authority is evident in his response to questioning in the interviews. Whereas students with an external sense of authority do not use conflict to modify their understanding, Vic responds to challenges by re-thinking his statements and revising his knowledge to arrive at something more cohesive.

I: Yes? What does that mean, for something to tend to something?

U: It's that,

V: It gets closer and closer to it.

U: Yes.

I: Mm.

V: Never reaches it.

I: Sorry, what?

V: Never actually reaches it.

I: Never...never reaches it. Would you agree?

V: No, actually no...

U: No you must reach it, if it tends to it. Eventually.

V: It depends...if it's a convergent sequence and it's strictly increasing or decreasing then it never does.

I: Right.

V: But if it's just increasing or decreasing it can reach it.

I: Right.
     Pause.

V: Or if it's a constant sequence, but then that's increasing.

UV5, 137

As with the previous group, his visual preference gives a secure sense of access to the material. Similarly his internal sense of authority means that he gives attention to the validity of his arguments, trying to formulate reasons for his assertions and generate alternative approaches when he sees his errors. In the following excerpt he has used the definition of convergence to establish that a convergent sequence is bounded beyond a certain point, and the interviewer challenges him to extend his argument to the whole sequence.

I: ...that doesn't give you that the whole sequence is bounded does it?

V: If you took N equals one...

I: Can you do that though?

V: Yes. But you don't know what...just make epsilon big enough, so that you can.

I: How would you do that?

V: Try and...well, a_1 was...find what a is,

I: Mm,

V: Take them away, and make epsilon bigger than that.

I: Ri-ight...what if a_1 is not the biggest term of the sequence?

V: Oh that's a good point! Er,

I: (*laughs*)

V: Then I'd...find the biggest value of a_1, and the smallest,

<div align="right">UV7, 733</div>

So as with the previous group, Vic has the capacity for good logical and formal understanding; he can operate sensibly and flexibly. However, it is apparent that he has not given much consideration to how to use the definition in arguments until he is asked to do so in interviews. This highlights the fact that the lecture course does not best support his developing understanding. Indeed by week 7 he has not picked up the importance of the formal work. At this stage he still uses his visual approach (considering the result that a

convergent sequence must be bounded to be "intuitively obvious"), and feeling the need to check the meaning of defined concepts.

Figure 21: Vic's sketch of a convergent sequence

I: Well, what makes you think that one's true?

V: Just…it's intuitively obvious, basically.

I: Can you explain why it's intuitively obvious?

V: Well if you've got – can't use graphs…

I: Eh? I'm neutral mate.

V: If something converges, yes?

I: Mm,

V: Then it tends to a limit a, as n tends to infinity. Is that, like, a reasonable assumption?

U: Yes.

V: So, if it tends to a, then you're going to have stuff bigger than a, but if you can bound it…(*drawing*)…it depends what you mean by bounded as well. Has an upper bound anyway…

UV7, 383

The result is that Vic, as a visual reasoner with an internal sense of authority who attends the lecture course, shows potential with all types of understanding, but does not fulfil this regarding formal understanding.

9.2.8 Internal, visual, new course

Those who reason visually are not doomed to failure in mathematics, despite the potential pitfalls of this approach. Both Cary and Vic use their prototypes flexibly and show

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sensitivity to counterexamples in generating mathematically valid arguments, although they do not make consistent use of the formal framework. This section shows that a visual approach can also be effective in conjunction with the formal framework when the two are related in a meaningful way. The students in this group gain the relational and logical understanding demonstrated by others with a visual profile, and the new course in conjunction with a consistent internal sense of authority provides the opportunity and inclination to learn to work with the formal definitions.

Adam, an example of this group, shows his internal sense of authority by appreciating elegance in mathematics.

A: There was a lovely result where you – where you just had to say, by the result of something this, by the result of something else this, hence by the axiom of completeness this…I liked that one.

AB7, 159

His goal is to understand the material, and he does as much work as is necessary to achieve this, prioritising it over other activities.

A: I couldn't go home for the weekend and still get the work done.

I: Yes.

A: I mean I…I literally – I went to bed reasonably early on Friday and I got up before nine it must have been and worked all Saturday, to have time on Sunday…

AB7, 77

A: So one of them we didn't have to hand in, it was just Burn, and I gave up on it.

I: Yes.

A: Which I don't like to do, I normally do the Burn questions just to understand stuff. But I just couldn't get this one.

This leads to a good relational understanding as we have seen before, since the focus is on fitting objects and results together. In working on the week 5 <u>series question</u> Adam speaks informally and uses generic examples but takes care to explain his reasoning in terms of known results.

A: [*talking about x*] If it's beyond...if it's – if it's minus 1,

B: Yes,

A: Then it cancels with that minus, so you've just got 1 to the n over n, so you've just got 1 over n.

B: Yes.

A: Which is the harmonic series. Which we know diverges.

B: Yes.

A: So if this...the minus x bit is bigger, so if x is smaller than minus 1, then it's still...it's going to diverge again. So it's – that would be from minus 1, to nought.

<div align="right">AB9, 383</div>

These property-based justifications stand in contrast to the arguments of those with a visual-external profile, who exhibit relational understanding but who tend to express ideas as though they were obvious rather than offering reasons to back up their assertions. The use of properties is also fundamental to a good logical understanding, and students in the current group are aware of the standards of argumentation expected at this level. Here Adam explains his proof that all convergent sequences are bounded. His speech is informal, but he explicitly refers to the proof's overall structure.

A: Yes erm...the bit that we've done at the bottom and then stuck an arrow that should go first...is erm...you have to show that if a sequence is eventually bounded, then it is bounded for all n so...if you say that it's bounded between L and U, lower and

upper bounds, when n is greater than big N, then you can say that the whole thing is bounded by the maximum of all the terms up to big N, and that upper bound is an upper bound for the whole thing. And in the same way you've got erm...the minimum of all the terms up to L, and L, will be the lower bound for the whole lot. And then you use that later on. Erm...you've got your definition of convergence...that if a sequence a_n tends to a, then for all epsilon greater than zero there exists a natural number n, such that the modulus of a_n minus a is less than epsilon when n is greater than big N, which means...that you can get arbitrarily close. To your limit. So you can say that for n greater than big N, it's trapped between epsilon on either side of a, so you've got it bounded, eventually, and then you use the other result, to show that it is bounded.

<div align="right">AB7, 485</div>

The students in this group are undeniably visual in their approach. They draw diagrams and gesticulate in order to express their arguments. However their internal sense of authority means that they do not ignore the algebraic material. Instead they deliberately integrate this with their visual images.

A: It's not usually enough to stick the definition down, you have to stick it down and then remind yourself of what it means.

I: Yes?

A: Because the definition and how you understand it are never like, exactly the same.

I: Are they not? Can you explain to me what the difference is Adam?

A: I really wish I hadn't said that now! Er...yes it's like, you understand that it gets closer and closer,

I: Yes,

A: But you can't just...the definition you can't put it gets closer and closer, you have to have the "for all epsilon greater than zero, n greater than N implies modulus of a_n, minus a", and all that sort of stuff. It's got to be put into the mathematical terms.

This means that they may not start out with a good idea of the place of definitions in the theory, but they do develop a solid formal understanding as the course progresses. For example, at the beginning of term Adam does not understand that everything must be expressed verbally at this level, as seen here when he is talking about sequences.

A: It's one of those things you like, understand without being able to explain.

B: Yes.

I: Yes.

A: It's like, you don't understand it, explicitly like, in words, you just understand the concept.

AB3, 229

Indeed, throughout the term the objects remain primary for him, with the definitions being justified in terms of visual imagery.

A: Yes. That's cool. Do we need to prove the definition of convergence, is...could draw a pretty picture, just so we could...or not!

AB7, 401

However the link between the two representations means that such students can check that their written work is correct by making sure that it corresponds to their prototype-based understanding. In the following instance Adam pauses before reading out his definition of convergence, and explicitly recites the reason for his decision *after* it has been confirmed that he is correct, as though strengthening this particular piece of knowledge. This is very different from those with an external sense of authority who are not concerned about the *why* as long as someone has told them they are right.

A: I think that's right. Hang on, let's check.

Pause.

A: If,

B: Yes,

A: A sequence a_n tends to infinity then for all C greater than zero, there exists a natural number N such that – a natural number big N, such that little n greater than big N implies that a_n is greater than C.

I: And you scribbled out the brackets around the a_n.

A: Is that right?

I: Yes that's right.

A: Good. Yes that's...with brackets it's the sequence and without brackets it's just the term, yes...

<div align="right">AB5, 747</div>

Adam is a particularly strong student and those in this group do not necessarily develop this link so quickly. For instance, although Jenny knows and understands the definitions, she tries to generate an argument by abstracting properties from a prototype and checking for counterexamples in the <u>convergent implies bounded</u> question.

J: That's only if it's increasing though isn't it?

<div align="right">JK7, 427</div>

J: Is there such a sequence that we don't know...

<div align="right">JK7, 439</div>

J: Erm, how about something like...

K: Because it's got to go up down up down up down up down and get closer together.

J: What does...what does, this look like? (*writes*)

K: Erm...

More writing.

I: Can you explain what you've written?

J: Er…minus 1 to the power of n, times 1 over n.

<div align="right">JK7, 451</div>

Indeed there are times when she is unsure about the status of definitions in argumentation. For example in addressing the <u>proof correction</u> problem, she is unsure as to whether a proof may be correct without incorporating the definition.

J: …It doesn't meet the definition either does it – does it have to always meet the
     definition?

<div align="right">JK5, 989</div>

Jenny's case confirms a phenomenon seen in Cary's attempts to generate arguments: students who take this approach can generate a lot of work for themselves, since abstracting prototypes and generating potential counterexamples is not easy. However students in this group do try understand the detail of the formal work and are likely to eventually realise that introducing definitions early is more efficient.

J: And I am…I can immediately go to the definitions and what we know,

I: Right.

J: And writing all of that down to see how we can get to where we want to be, so I'm
     getting better.

I: Yes. Yes. Do you find that's a useful tactic in general then that, to help you do
     things?

J: Mm.

I: Is that something you weren't doing before?

J: Not as much, I was sort of looking at it waiting for…divine inspiration or
     something!

By the time they get to this stage they are in an excellent position to take advantage of the strength that visual representations can have if they are well linked to the formal algebraic ones. For instance, since she has understood the definition meaningfully, Jenny is not fazed by Kate's incorrect understanding of it in the <u>convergent implies bounded</u> question.

> K: Yes but you could fix N as the beginning...could you?
>
> J: No. Could you...? No.
>
> I: Why not Jenny?
>
> *Short pause.*
>
> J: Because we never have...!
>
> I: Ah.
>
> J: No because, erm...that's not what the definition says.

<div align="right">JK7, 791</div>

Thus a conscious and internally driven drive to fit the new material to existing understanding means that these students end up with a very secure relational, logical and formal understanding.

9.2.9 Internal, nonvisual, new course

A nonvisual approach combined with an internal sense of authority is a very good combination for success as measured by tests. Students in this group make an effort to understand the work meaningfully, and this effort is directed towards the definitions and other algebraic representations. The new course then adds to this effect by requiring regular use of definitions in producing proofs about whole categories of objects, and these students develop good formal understanding. However this does not necessarily guarantee solid logical understanding, as demonstrated below.

Like Carlson's successful beginning graduate students, students in this group are not interested by repetitive exercises, focusing instead on understanding general relationships (Carlson, 1998).

G: ...the questions got, more and more repetitive,

I: Right, okay.

G: So it's just a question of writing lots, which wasn't my idea of fun.

I: No. So you...what sort – what sort of questions do you prefer then?

G: Erm...well no I think in general the questions are pretty well done,

I: Yes...

G: But, erm, the...there have been a few times when it's been just – where there's actually only been one line which has been important in your proof, but you still have to write the whole proof out.

GH9, 887

They therefore examine the material carefully and try to make sense of the algebraic representations. As a result they tend to pick up the formal definitions early in the course, giving them a good chance of learning to express arguments in an appropriate way. In this excerpt, Greg's language is not precise, but this is only week 3 and already he is concerned with remembering "exactly" what it means for a sequence to tend to infinity. His description sounds very different from the "it gets bigger and bigger" descriptions given by other students.

I: Have you done anything about sequences tending to infinity?

G: Yes, sort of.

I: Sort of. Sort of as in,

G: Yes!

I: Yes. *(laughs)*

G: Yes but I can't remember exactly what they are now.

I: Right, yes. Can you give...can you remember the idea?

G: That, when something's...when it's going to infinity you can prove that, for any value, you can always find one bigger than it. Whichever value it is.

I: Right, yes.

G: That's the definition anyway.

GH3, 239

The result is that these students not only learn the formal representations, but also gain a solid grasp of their meaning and use. The new course supports this by requiring lots of work with the definitions in their full range of roles, and these students quickly come to understand their place in a formal theory.

I: Why do you think we have those? Things like that?

G: What that definition, or...?

I: That definition, yes.

G: Well because otherwise you don't know exactly what you're doing. Having that definition means...that you basically just relate it to the definition, and see whether that's the case.

I: Yes.

G: As opposed to sort of wandering around in the dark, almost.

GH3, 284

Previous sections demonstrated that other students are aware of the role of definitions in the abstract; this group have this awareness, but their natural inclination to algebraicise means that they also regularly act upon it. The result is quick and competent starts on the task-based interview questions. This is Greg's strikingly effortless attack on the <u>convergent implies bounded question</u>.

G: Right. It's easiest to use the definition. Just say definition of...of convergence is that eventually...okay!

I: *Laughs*

G: The definition of convergence is eventually you'll find epsilon such that...for all n bigger than big N, epsilon – no. a_n minus a is smaller than epsilon. So you've got it bounded between... a_n is... a plus epsilon – a minus epsilon even. And a plus epsilon. And for... n bigger than big N. And for all n smaller than big N, you know that a_n has a minimum, and maximum because it's finite. Er...

<div align="right">GH7, 233</div>

This combination is therefore more than enough to guarantee success in the course. What it does not guarantee is satisfaction for the student. Recall that in principle a formal understanding could be demonstrated without the student relating this to any objects. In practice this doesn't happen to this extent: these students do develop a good relational understanding and the ability to express it algebraically. However they may fail to acquire a good logical understanding, for almost exactly the opposite reason to the weaker visual students. While Wendy generalises from prototypes without abstracting properties, these people have the definitional properties available but they may not have a sense of the objects which these properties describe. The result is that they may be perfectly able to operate with the definitions and therefore bypass the problems visualisers have in providing universal validity while lacking the internal sense of understanding which comes from having access to a corresponding visual representation.

Ben, for instance, is good at Analysis by anyone's standards, but when he works with the algebraic representations of the concepts he does not think about what they mean. In fact, he actually says that convergence did not have a meaning for him before he began this course.

I: Do you find there's a bit of a difference Ben, between how you think about convergence and what the definition says?

B: Not really no.

I: Not really. Did you to begin with?

B: Erm...to begin with I didn't understand convergence at all. I thought okay, and just accepted it and then...now it's just become a sort of natural way of thinking about it.

<div align="right">AB7, 549</div>

This makes him unusual since the majority import their everyday meanings for terms and may or may not reconstruct these on encounter with the definitions. Ben, it appears, gives himself a clean slate for learning mathematical concepts, which means that he constructs formal objects rather than reconstructing spontaneous-conception-based prototypes (Gray et. al., 1999). Spontaneous conceptions do have advantages as well as disadvantages however, as they give another representation to assist in reasoning. Ben lacks this, so that while he is technically capable he sometimes has an uncomfortable sense of not really understanding.

B: ...I understand how to sort of...use it to my advantage, what I'm told.

I: Right.

B: But...I don't actually understand the concept fully as such. I...I suppose it just takes some time.

I: Can you give me an example?

B: Erm, sort of like...when it's converge – null sequences converging,

I: Mm,

B: Erm...before I just couldn't have the faintest idea of how when a number is less than epsilon,

I: Okay,

B: That shows that.... Now I know, okay, if you want to show a null sequence converges then you have to do that,

I: Yes,

B: But, I'm still not quite on top of that,

<div align="right">AB5, 29</div>

Thus although Ben introduces definitions, his arguments are occasionally improved when his partner Adam notices and corrects small inaccuracies. Having said this, it is not the case that Ben has no sense of logic. Far from it, he is more than capable of making sure that arguments are properly structured and understanding them as coherent wholes. It is simply that he is lacking some of the advantages of having two different ways of understanding.

Hence nonvisual, internal, new-course students may show slight weakness in logical understanding, but they make rapid progress in acquiring formal understanding.

9.3 Summary

For this chapter, the main points are the outcomes for different combinations of factors. These are summarised in the table below.

Table 4: Summary of chapter 9

Authority	Reasoning	Course	Instrumental	Relational	Logical	Formal
External "episodic"	Nonvisual	Lecture	N	N	N	N
External "semantic"	Nonvisual	Lecture	Y	Y	N	N
External	Visual	Lecture	Y	Y	N	N
External	Visual	New	Y	Y	Some	some
External	Nonvisual	New	Y	Y	N	Some
Int/ext	Visual	New	Y	Y	Y	N
Internal	Visual	Lecture	Y	Y	Y	N
Internal	Visual	New	Y	Y	Y	Y
Internal	Nonvisual	New	Y	Y	Some	Y

Combinations not arising in this study are treated theoretically in section 10.5. Chapter 10 also draws together the phenomena observed in this chapter to formulate consistent descriptions of the influence of each factor in isolation.

Chapter 10
Causal relationships

10.1 Introduction

The previous chapter described a function

{visual, nonvisual} x {internal, external} x {lecture, new} → types of understanding

reasoning style   *sense of authority*   *course*   *learning outcomes*

by describing triples of inputs. This chapter goes beyond straightforward data analysis and into drawing out substantive theory. Data is referred to only sparingly, and the chapter takes each factor in turn, essentially describing three functions

{visual, nonvisual}   →   types of understanding

{internal, external}   →   types of understanding

{lecture, new}   →   types of understanding.

In each case, each possibility for the factor is discussed in turn, considering its influence on the types of understanding a student attains. Hence there is nothing in this chapter which was not covered in chapter 9, but separating out these factors allows easier assessment of which influence the acquisition of types of understanding of Analysis. This allows predictions to be made regarding the likely progress of students whose combinations did not

appear in this study, and facilitates the discussion of what might be done to assist students in making the most of their own potential. This latter discussion is taken up in chapter 13.

10.2 Visual and nonvisual reasoning

Evidence in the previous chapter supports Krutetskii's conclusion that there is nothing inherently good or bad about the use of visual imagery in mathematics; that a student may succeed with either tendency (Krutetskii, 1976). However in written mathematics there is an asymmetry in the value accorded to the two ways of reasoning (Dreyfus, 1991, Davis, 1993). Relative to this background, individuals with different preferences exhibit different strengths and weaknesses.

10.2.1 Visual reasoning

Visual imagery has "peculiar strengths" (Bartlett, 1932). It can enable the user to grasp a situation holistically and to feel that they understand it (Presmeg, 1986b). Weaknesses arise because this sense of understanding appears whether or not the student's understanding is compatible with that of an expert, and whether or not they can formulate it algebraically.

For instance, confidence in the answers of students who reason visually indicates that visual images provide them with a strong sense that they understand the concepts of Analysis. However these prototypes may be used inadequately. First, the sense that certain results are "obvious" from a picture means that these students do not experience any drive to give more than this by way of demonstration (Gray et. al., 1999). Recall Wendy's remark:

W: Is that enough to like, justify it...a little diagram, what have you?

WX7, 213

Hence their arguments often amount to generalisation from prototypes, rather than the more appropriate abstraction of properties and deduction from these. Second, these prototypes may not be sufficiently general: they may be based entirely on spontaneous conceptions and

particular examples, and not reconstructed in response to the relevant definitions (Gray et. al., *ibid.*, Vinner, 1992). This means that even when verbal arguments are generated, these may not be sufficiently general. In effect, the prototype is not sufficiently representative of all the elements in the formal category (Tall, 1995, Presmeg, 1986b).

In addition, the holistic nature of such images can be a handicap in constructing a linear argument. A picture does not have any inherent direction. The student may perceive that all the relevant properties hold, but fail to distinguish some as necessitating the presence of others. This is the root of Tom's inadequate argument as seen in section 7.2.2.

T: ...if a_n tends to big A, there exists a b, which is a member of the real numbers, such that b is less than big A?

ST7, 763

Although Tom attempts to formulate arguments by labelling parts of a diagram, he does not show any sense of logical primacy. He simply *describes* the entire situation, and this is not sufficient to justify his claims on a logical basis.

So the sense of understanding derived from visual representations, combined with the complexity of their algebraic counterparts and the problem of avoiding cognitive strain, can mean that the visual student becomes increasingly alienated from the formal work. Mathematical proof can therefore become detached from the student's own sense of what brings conviction (Chazan, 1993, Fischbein, 1982). This is particularly likely to occur in cases where the student has an external sense of authority, and so is inclined to accept material from authorities without interpreting it relative to their existing understanding. Tom and Fred reach this stage for these reasons.

T: I can use the theorems, I just can't prove anything.

ST9, 890

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F: ...if we've got to prove it then that's a different kettle of fish altogether.

<div align="right">EF7, 367</div>

Visual students with an internal sense of authority (at least regarding the mathematics) fare better, but can find themselves with a huge amount of work to do. They are likely to reach conclusions quickly, but find it laborious to generate supporting arguments since they try to do this by abstracting properties afresh each time. Recall Cary's search for a property that would allow him to prove that a convergent sequence must be bounded:

C: Yes, I'm trying to think if there's like...if you can say the first term is like

the...highest or lowest bound but it's not. Because then you could just make a

sequence which happens to go, to do a loop up, or something like that. In the way

that, it did...erm...

<div align="right">CD7, 314</div>

The cases of Jenny and Cary show that these students may be good at logical reasoning, and such students would probably enjoy and perform well in a course whose focus was investigative work or problem solving. However neither course in this study is of this type: the (measured) goal in both cases is the acquisition of certain material in a relatively short time, and these visual thinkers do not make optimal use of their cognitive resources in such a situation.

Jenny's eventual progress and Adam's strong performance throughout the term demonstrate that the use of visual representations *can* be a strong basic strategy. However, the student must be able to translate these into the accepted formal algebraic representations. If a visual student consciously attends to the formal work, realising that this is central and modifying their prototypes to take account of this, they can benefit from the flexibility provided by having more than one way of thinking about a problem (Nickerson, 1985).

<div align="right">233</div>

10.2.2 Nonvisual reasoning

The nonvisual approach confers some immediate advantages upon students who take it, because of the asymmetry in what is acceptable to the mathematical community. Students who attend to and remember this aspect of the work are likely to produce something recognisably formal, whereas a weak student with visual preferences may not learn any of the formal material at all. This approach does not lead to quick conclusions in the manner of visual reasoning, but does provide an opportunity to bypass the misconceptions and inadequate arguments that arise easily in visual work. In addition, stronger nonvisual students are more likely to attend to definitions, and therefore have a way to start on problems. More subtly, with the exception of those with an "episodic" profile, none of the nonvisualisers in this study expressed a feeling of complete inadequacy regarding proofs. For nonvisual students, perhaps proofs appear at least as sequences of algebraic manipulations. These would therefore be related to their usual ways of reasoning, even though no overall structure may be discernible (Simpson, 1995b).

For the weakest students in the study, the "episodic" learners, symbolic reasoning is "superficial and mathematically vacuous" (Harel & Sowder, 1998). A student who attends *only* to the surface features of the presentation, perhaps in anticipation that this will cue routines to follow, does not stand in good stead to learn university mathematics. A lack of focus on the objects described means that they will not be able to use notation effectively or identify its role in any particular argument. Therefore they may not gain good relational *or even instrumental* understanding. Attempts to learn in this way may well account for the "writing nonsense in a mathematical style", described by Leron (1985b). In this study, we saw that Zoë uses symbols as the basis for a matching process that makes no reference to their meanings in terms of mathematical objects.

> Z: You know the, u_n and v_n,
>
> YZ9, 331

A nonvisual student who does attend to the objects behind the notation may attain good relational understanding, while remaining weak in logical understanding. Those who are expecting procedures to follow will find some of these. For instance, we saw in section 9.2.2 that Xavier is able to introduce the definition of convergence appropriately, but attempts to work with it in a rather fixed way that is inadequate for more general questions.

Facile symbol manipulation can be used to great effect by those who do have a meaningful understanding of the underlying mathematical objects and processes. This reflects the potential of symbols as powerful alleviators of cognitive load, and as a pivot between a process and the corresponding encapsulated object (Harel & Kaput, 1992, Gray & Tall, 1994). In particular, students who are able to see the role of definitions in advanced mathematics are likely to produce good work. Those who quickly assimilate the "meta-procedure" of writing down definitions and working from these are often able to produce essentially correct proofs, as seen in Ben's case in section 9.2.9. They may however be prone to small errors that go unchecked because they do not have an alternative representation against which to check their assertions. For some this problem does not cause distress; those with an external sense of authority may prefer to write without thinking about meaning. For others, however, it produces a sense of discomfort.

B: ...I understand how to sort of...use it to my advantage, what I'm told.

I: Right.

B: But...I don't actually understand the concept fully as such. I...I suppose it just takes some time.

AB5, 29

Of course this need not be the case. In this study, Greg prefers nonvisual methods but is always complete and accurate in his recall of definitions and construction of meaningful arguments. Interestingly, Greg was the only student in the study to have a non-British

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educational background, having come instead through the French Baccalaureate system in Niger. Perhaps he therefore experiences a more traditional, unbroken route to proof "through logic" (Simpson, 1995) than his peers.

10.3 Sense of authority

Where visual and nonvisual approaches are both confirmed as viable, the study supports a value judgement regarding sense of authority: those with an internal profile are more likely to acquire logical and formal understanding. This section examines this phenomenon, as well as briefly noting the possible sources of these beliefs. First, some general remarks.

Overall, sense of authority emerged as more important than the amount of work that a student puts into their learning. Undoubtedly the time spent studying does have an influence: if Tom had worked harder, maybe he would have acquired as much relational understanding as Wendy, and if Jenny had not put in the time necessary to generate arguments from spontaneously-abstracted properties, maybe she would not have come to see the importance of definitions. Indeed, those with an internal sense of authority may be more intrinsically motivated, so may be prepared to work harder (Turner et. al., 1998). However, beyond such quantitative differences, an internal or external sense of authority has a more profound effect upon the way in which the student interprets their experience of university mathematics.

This can be attributed to the restrictions that a student's sense of authority places on their interpretation of their mathematical experience. In effect, it acts as a lens, in the sense in which Copes applies this metaphor to Perry's development scheme (Copes, 1982, Perry, 1988). The student looks out through this lens in expectation of encountering particular kinds of information and experience, and any incoming experience is interpreted according to what the student expects to find. As with the visual/nonvisual dichotomy, the two sets of students distinguished in this way thus behave quite differently in response to university mathematics, focusing on different aspects of the presentation. In this case, this happens on

two levels: their reaction to the material itself, and their perception of, and response to, changes in their own role as learners.

10.3.1 External sense of authority

Common elements can be noted in the previous experience of those with an external sense of authority, which explain some of their behaviour when they reach university. This section notes these common elements, before exploring the consequences and relating these to other aspects of "external" beliefs and behaviours.

Those with an external sense of authority often come from classrooms in which practice and repetition is emphasised (*cf.* Byers & Erlwanger, 1985). Recall Zoë's comment:

Z: I just like the way we were taught at school...where we...we have our notes, and then we practise so many times that it's just sort of drummed into us.

YZ2, 705

This type of classroom incorporates the assumption that the teacher will take responsibility for what the student has to learn, and provide methods through which success is guaranteed. This type of experience provides the student with security, and many speak highly of teachers who were "organised" and "always went over any exercises we were stuck with". Some students go so far as to seek this out:

X: ...I actually switched maths groups.

WX2, 286

X: ...in the long run it was much better to have a teacher who was much more...you know he wanted to give you homework and stuff.

WX2, 306

This does not mean that such students do not take responsibility for their own learning. Xavier's story indicates that he takes his learning seriously and will do what he thinks best to promote it. However, for the student with an external sense of authority, taking this responsibility may not extend beyond dutifully learning material as it is presented. In this study, such students do not usually expect to have to do much processing of the information they are given, relying on the authorities to present them with what they need to pass the course. This is not an unreasonable assumption, especially if such learning has led to success in the past. However at university this is not enough. A student cannot expect to fully understand a subject like Analysis without giving serious thought to the concepts and their relationships. Those who anticipate that they will simply be able to learn the contents of their notes over the Christmas vacation are usually vastly underestimating the amount of work that this will involve.

Z: ...hopefully when I come back in January I should – it should make sense[27].

YZ7, 1709

The issue then is not responsibility for learning but for *understanding*: for judging whether one understands or not, and whether or not one's answer or proof is correct. For the student with an external sense of authority, this is not usually the goal. As seen in the previous chapters, often such students will tolerate uncertainty on the basis that the answers are "in their notes" to be learned later, and will happily accept corrections at a delay rather than try to be fully satisfied with their first attempts. Similarly, they consider judging correctness to be the job of the authorities, and attempt to learn "what they want you to write" rather than to develop a real sense of the required standards of argumentation. The result, seen in interviews, is that they are occasionally unwilling to engage with questions, and often unwilling to commit to their answers.

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[27] Zoë's mark in the January examination was only 28%. See section 12.2.2 for further discussion of examination marks.

E: If we use our workbook yes, probably.

<div align="right">EF7, 477</div>

W: Are you happy with that?

X: Mm.

W: I don't think I could tell...anything better.

<div align="right">WX7, 333</div>

An external sense of authority can be particularly restrictive in cases where the student has very specific expectations of what the mathematics will involve. This occurs in the case of Zoë, who assumes that for any given question she has been given a procedure, and tries to recall episodes rather than examining the meaning of a new question. Similarly Tom expects to be provided with detailed "methods" to follow. He latches on to these when they do appear, and is thus able to maintain this belief throughout the term. For instance, in a later interview Tom claims that he likes group theory:

T: Don't mind group theory. I'm getting to grips with it because we had a...what's it called...a supervision,

I: Yes,

T: And we er...we got given a sheet by [teacher],

I: Yes,

T: And we all had a group. And you had to prove that what you got given was a group.

I: Yes.

T: Because we had one each, we all went over them, and it meant that we got to grips with all the things that show it's a group...

<div align="right">ST9, 168</div>

It seems that repetitive practice in showing that given structures satisfy the group axioms has given him the impression that he understands group theory, although this part of it may be the extent of what he has mastered.

In effect, because there are parts of the mathematics that can be learned in this way, such students are able to ignore the conflict engendered by the fact that much of the work is not composed of such methods. Hence they can avoid changing or re-evaluating their beliefs about their own role in the learning process. So just as students may ignore conflicts within their understanding of the *material* (Duffin & Simpson, 1993, Sierpinska, 1987), they may do the same regarding the meaning of "learning mathematics" (Perry, 1988, Copes, 1982).

The result of such expectations that "methods" will be provided is that the student does not expect to have to think creatively in answering questions. This is apparent in Zoë's reliance on recall, and also in Tom's inclination to stop when he encounters a situation for which he has no specific instructions, rather than look for a way around the problem (Alcock & Simpson, 2001a). As Byers remarks in discussing a similar situation, "Apparently facts and skills cannot prevail over gaps in a student's general logico-mathematical structures" (Byers, 1980).

More subtly, this belief in methods, combined with a sense that mathematics is primarily useful for its practical applications, can cause difficulty for students in constructing proofs for whole categories of objects. For example, reviewing the interviewer's attempts to guide Wendy to a proof that all convergent sequences are bounded, it is striking that she seems to be looking not for a general proof, but for an algorithm which could be followed for any particular sequence[28]. While this might be logically equivalent, it is psychologically quite different. For Wendy, the difficulty of showing that a given convergent sequence must be bounded varies according to the number of terms before "big N": three is easy, three million is not.

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[28] See WX7, 669 to 723

Those with an external sense of authority tend to view proofs as chunks of knowledge to be memorised.

Y: But that erm...[lecturer] was saying that, it's not like you have to *learn* the proofs, it's you have to learn how to do the proofs so you can do them yourselves.

I: Ah. Does that sound like a possibility then?

Y: Not really at the moment!

Z: That sounds even worse! I think I'd rather just learn the proofs.

YZ7, 1731

Even Kate, who does value the experience of working things out for herself, reverts to type by the end of the course and expresses concern that for some proofs she only has her version available.

K: The fact that they don't acknowledge whether you've got anything right and stuff like that, so you could be revising from a set of things that's completely wrong.

JK9, 1113

In Kate's case this is particularly sad, as by this stage she could probably assess her early proof attempts with some degree of competence. The fact that she does not even consider doing this reflects the assumption of those with an external sense of authority that it is ultimately not their job to pass judgement on any mathematics, but rather to learn the "correct" version.

This view cuts such students off from the valuable uses of conflict at both the material and philosophical levels. Mathematical counterexamples do not necessarily faze them, as they may be expecting conjectures or methods to be useful rather than universally valid. Hence their knowledge of which objects belong to which mathematical categories is not readily

reconstructed to resemble the formal notions. Similarly, they may fail to re-evaluate their position as a learner, even under continued experience that conflicts with their expectations regarding this role. Hence they do not become involved in the process of proving or in other creative thinking.

The result is that, even for those who work hard, an external sense of authority seems at best to lead to competent performance on questions involving specific objects. Such students, particularly those on the new course, may develop some habits associated with logical and formal understanding (checking for counterexamples to conjectures, for example), but their view of what it means to learn mathematics remains essentially unchanged.

10.3.2 Internal sense of authority

Those with an internal sense of authority stand in better stead to make progress in university mathematics. These are the students who take responsibility not only for accepting material, but also for processing it in such a way that it becomes meaningful to them. What has led them to this position is open to speculation, but quite a common experience in the background of these students is that of a somewhat informal, if not outright disorganised A-level class.

> B: ...he [the teacher] was very laid back. I mean to start off with, sort of first two or
> three years we'd like get formal notes or whatever, but after that it was
> just...discussions for the last four years we had him. No, no real note taking, it's
> sort of like, all right you don't understand this so we'll go through it, and it just sort
> of clicks with him.
>
> AB2, 183

It makes sense that to achieve success in such an environment, the student is obliged to take more responsibility for organising their own work and making sure they are satisfied with

their own understanding. Ironically, this suggests that if Xavier had stuck with his original A-level class, he may have arrived at university as a more mature mathematician[29].

Regardless of what has brought them to this position, the student with an internal sense of authority is expecting the material to make sense, and so is relatively intolerant of conflict. This means that if an experience does not make sense in terms of their existing mental structures, they will expend effort in reconstructing these to accommodate it (Simpson, 1995). Carlson recognises this phenomenon in higher level students, who often value understanding over memorisation and are prepared to put in long hours of work on difficult problems to achieve this (Carlson, 1998). The result is in accordance with Baddeley's observation that "the requirement to organise material leads to good retention, even when the subject has not been told to learn it" (Baddeley, 1997). Students with an internal sense of authority are much more likely to be able to remember and use relevant information in interviews.

Students who take responsibility for their own understanding in this way develop good relational understanding almost as a matter of course. They are also more open to noticing changes in what is required of them, and hence to taking responsibility for judging the correctness of mathematical arguments and developing the logical understanding necessary to achieve this.

More interesting is their progress regarding formal understanding. Visual students in particular are vulnerable to a problem here, in that attendance to visual aspects of a presentation can satisfy the internal urge to understand, and can mean that the student pays insufficient attention to the formal algebraic formulations. The result is that the student may develop a good correspondence between the formal categories and their own knowledge of what objects belong to which, without a mature formal understanding of how to handle these

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[29] Of course he may equally well have failed his A-level: how best to encourage a student to develop some intellectual autonomy, without leaving them unsupported, is a complex question.

categories using the definitions. Cary's case, in particular, is a good illustration of this problem (as was seen in section 9.2.6).

Students with an internal sense of authority who do attend to the nonvisual representations often perform well in interviews. (This is not restricted to nonvisual students – both Jenny and Adam are naturally visualisers but also devote time to understanding the algebraic formulations they encounter.) Those who reach this stage may still experience difficulty in meeting the expected standards in written mathematics at this level; Jenny, for instance, remarks that she is not sure how much detail she needs to include in her arguments.

J: I still find I'm not quite sure what counts as being rigorous enough.

I: Right.

J: So I have an idea, and it makes complete sense to me, but I'm not sure if I can put that down as a justifiable reason, or whether it needs more.

JK7, 305

However, unlike those with an external sense of authority, they try to reach a consensus between their own sense of conviction and the demands of their teachers. The very fact that they are aware of this as a problem is a good start. As Nickerson remarks, "awareness of ignorance at one level can be evidence of understanding at another level" (Nickerson, 1985).

Overall, those with an internal sense of authority quickly realise that both the structure of the mathematics, and their own role as a learner, has changed, and work to resolve this conflict. This means that hard work for these students brings good progress towards at least one of logical and formal understanding.

10.4 The courses

As with sense of authority, this study supports a value judgement regarding the relative efficacy of the two courses. In this case however, the nature of its influence is different:

while the new course gives rise to some relative improvement in logical and formal understanding, it does not override the students' initial dispositions. Hence the course attended is characterised as an intervening factor. This is in keeping with other observations on the influence of learning environments. Byrne, for example, suggests that educational situations are well represented by the metaphor of cultivation: we cannot force particular plants to flourish in a given place, but we can provide a suitable environment to them the best chance of doing so (Byrne, 1993).

10.4.1 The lecture course

What follows is not intended as a criticism of the teacher of this course. He was an experienced lecturer, held in high regard for his teaching by peers and students alike. Neither is it intended to form a criticism of the lecture method in general. There are interesting questions regarding the relative responsibility of students and their teachers in university courses, and these are taken up in chapter 13. This section simply reports on the effects of teaching Analysis in this format to these students.

The influence of the lecture course may be summarised by saying that it allows the maintenance of unproductive beliefs and approaches, and does not provide enough practice with the right kind of tasks for the development of logical or formal understanding.

Unproductive beliefs include those associated with an external sense of authority, such as the idea that mathematics is about learning procedures, and that it is the teacher's job to present the work and judge correctness. A lecture course is a natural experience for those with such beliefs: the lecturer is unchallenged and "facts" are supposedly absorbed by the class (Duffin & Simpson, 1993). Of course this is not the ideal: the lecturer might reasonably expect that the students would work through their notes and think carefully about the justifications of the results presented. However, numerous student remarks in this study indicate that this hardly ever occurs. Most students know that this is what they should be doing, but in reality only read their notes when required to attempt homework.

X: Yes, the homeworks are useful, because they force you to go back over the - over your notes.

W: Mm.

I: Right, yes.

X: Otherwise they would just sit there.

<div align="right">WX9, 789</div>

In some cases, even this involves only cursory reading of the material and attempts to mimic answers based on surface features of the work. In effect, while everyone concerned knows what the students *should* be doing, such work relies entirely upon the student's motivation. With the exception of assignment marks (which can be artificially inflated by judicious copying from the work of peers), there is little in the system that requires students to engage in this deeper thinking. So the lecture course does not actively encourage the development of intellectual autonomy; it allows students to maintain an external sense of authority.

The lecture course also proves weak in encouraging the development of logical and formal understanding, for the following reasons. As Davis remarks, "when the instructional program does not promote the development of appropriate ways of understanding, students...invent their own ways of understanding" (Davis, 1992). In this case, the most common bases upon which students build such understanding are the everyday meanings of the mathematical terms, and the visual images used to represent the concepts. Such understandings are often in conflict with an adequate formal understanding of the material, constituting misconceptions such as the assumption that all convergent sequences are monotonic (Vinner, 1992, Davis & Vinner, 1986). Such misconceptions are much in evidence among students on the lecture course, where the lack of two-way communication means that they go unchallenged.

Similarly, the course does not provide much practice in proving results about whole categories of objects, or (necessarily) much feedback on student attempts in this task. This sometimes results in a glaring gap between students' relational understanding, as evidenced by their ability to handle questions involving specific objects, and their logical understanding. In sections 9.2.2 and 9.2.3 we saw that both Wendy and Xavier show a good grasp of standard objects and strategies, performing confidently and competently in the sequence question and series question. However both show serious inadequacies in constructing general arguments. For visual students such as Wendy the result is perhaps more damaging, since their visual tendency means that they feel the proofs of the course to be detached from what brings them conviction.

Most of the students in the study who attended the lecture course also had an external sense of authority[30]. This means that the only real evidence for the effect of this course upon a student with an internal sense of authority comes from the interviews with Vic. This is unfortunate, but this evidence is consistent with what might be expected. Vic shows flexibility in his thinking, and a good basis of logical understanding. He can answer questions regarding the categories of objects in the course, and can adapt arguments sensibly when they are shown to be insufficiently general. However, his responses suggest that the lecture course has not provided him with the encouragement he needs to ask such questions himself. This means that his formal understanding is not very far advanced: he is not quick to access potential counterexamples, and he makes errors showing that by week 7 he does not have a solid grasp of the definition of convergence. Comparing this performance with similar-profiled students on the new course (Jenny and Adam), it seems reasonable to suggest that while Vic is lazy and would not have been happy about the volume of work on the new course, he would probably would have developed better logical and formal understanding in that environment.

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[30] This could be a coincidence. It could be that those with an external sense of authority choose a more "applied" degree course since this is consistent with their belief that mathematics is "for" something. It could be that the new course has made a difference in this respect by the time the students are interviewed, but this is unlikely given their remarks in week 2 regarding their experience of mathematics in school, and the persistence of "external" beliefs as described in section 9.2.4.

10.4.2 The new course

The influence of the new course may be summarised by saying that it provides an environment more likely to encourage the development of logical and formal understanding, across all the students involved. To reiterate, this is not to say that this environment *causes* such development, although in some cases it will be argued that it does achieve this to a limited extent. Primarily its achievement is to provide students with experiences that lead them to make the most of their natural capacities, by practising the processes of argumentation and proving.

For many students, the new course more or less enforces a renegotiation of the didactic contract (Brousseau, 1997, Sierpinska, 1994). They are expected to take responsibility for the production of the majority of the material, so that they should at least attempt to generate answers and proofs for themselves. More subtly, the inclusion of such work as part of supervised time sends a message that the authorities value it. This stands out in comparison with a lecture course, in which a student spends all of his or her supervised time watching someone else do mathematics.

The result, for some students, is the desired effect. Those with an internal sense of authority respond well to the challenge. They try to make sense of both the material and this new learning experience, using the support available to construct and reconstruct new understanding. However this is not universally the case. Others, with an external sense of authority, subvert this practice, maintaining their previous role by transferring the authority from a single teacher to a wider community, and relying heavily on their "clever" peers. In some cases (Emma's, for example), this is due to their strong belief in the existence of "correct" answers; in others (as in Kate's case) simply to the student's lack of confidence to judge their own work, although they might recognise the value of trying to produce their own mathematics.

So the new course is not enough to guarantee that a student will develop independence as a learner, and the implications of this conclusion for teaching and learning at this level are discussed in chapter 13. However, even for these students it does have some success in shifting the focus from procedures to concepts, definitions, and proof. This is not to say that it radically changes the way they handle categories of objects, but they do acquire some habits of logical and formal understanding. In particular the majority develop an awareness of the need to check for counterexamples, and show more fluency in discourse on the concepts of Analysis than their lecture course-attending peers do. While this does not necessarily extend to an ability to produce good proofs using the definitions of the course, it does constitute a step in the right direction for these students. Correspondingly, their counterparts with an internal sense of authority exhibit more advanced development in these types of understanding. The arguments put forward by such students demonstrate that they have a good awareness of which objects are members of the formal categories, and at least one of good logical and good formal understanding.

The success of the new course in these respects may be attributed to the fact that it involves a lot of communication *on the part of the students*. They are required to do much more writing of mathematics than their peers on the lecture course, with a particular emphasis on writing proofs. Moreover, during the classes they engage in verbal communication with both their peers and their teachers. The extent to which this occurs varies according to the habits of the student and their group; as discussed in chapter 2, some chose to work more collaboratively than others do. However for four hours of the week they are encouraged to engage in this practice.

The result is that these students receive lots of feedback on their own arguments. This can be expected to contribute to both logical and formal understanding, as follows. While the students do not have any say in what constitutes an acceptable proof in mathematics at this level, they do have the opportunity to test their understanding of what this is against challenges from each other and their teachers (*cf.* Yackel & Cobb, 1996, Yackel, Rasmussen

& King, 2000). This means that if they are prone to making unsupported generalisations, or to omitting to consider certain types of example in their arguments, they will probably be corrected quite regularly. This can generate some uncertainty: at various times students in the study remarked that they were unsure about a result because their first thoughts were often wrong. But once again, this indicates that they are in the process of reconstructing their understanding of acceptable argumentation at this level, i.e. of developing logical understanding.

The development of formal understanding may be attributed to similar processes. Freyd's work indicates that when a community works together using certain terms, the meanings of these terms will gradually come to resemble formal definitions in order to facilitate communication (Freyd, 1983). In the logical structure of Analysis this has already occurred, and the students are not at liberty to change these meanings. However, it is reasonable to suggest that since they are obliged to communicate with each other and their teachers, they are likely to notice that others use terms in different ways, and to become engaged in debating these meanings. The guidance and encouragement to use the existing definitions does not appear to be enough to radically alter their first approach to assessing a given conjecture, but it does seem sufficient to make them aware of the definitions and willing to use them when prompted. Certainly it improves their awareness of the types of object included in central mathematical categories.

So, the new course does not alter the students' fundamental dispositions regarding visual or nonvisual reasoning or sense of authority, but it does provide an environment in which they are more likely to develop logical and formal understanding of the material.

10.5 Predictions

Having examined the influence of different factors in developing student understanding, this chapter concludes by considering predictions that may be made regarding the likely

development of students with profiles which did not arise in this study. Those combinations that did arise are:

- The external-nonvisual profile on both courses, including "episodic" and "semantic" subgroups on the lecture course [*Zoë (episodic), Xavier (semantic), Hugh (new)*].
- The external-visual profile on both courses [*Wendy (lecture), Emma and Kate (new)*].
- The internal/external combination in conjunction with visual reasoning on the new course [*Cary and Fred*].
- The internal-visual profile on both courses [*Vic (lecture), Adam and Jenny (new)*].
- The internal-nonvisual profile on the new course [*Ben and Greg*].

The remaining profiles are considered below.

10.5.1 External, nonvisual, new course ("episodic")

There is no absolute reason why students with an "episodic" profile should not appear on the new course, although it may be argued that such students would be less likely to meet the higher entry requirements. While these students all attained an "A" grade in A-level mathematics, this type of learning may not easily bring success with STEP papers, which require more co-ordination of strategies and are thus perhaps not so easily tackled by dedicated rote learning of procedures. If such a situation did arise, it is likely that the student would find the new classes disorienting, as teacher-provided work is minimal. They may well resort to copying from their peers in an attempt to collect proofs to be learned later, in much the same way as Emma. In this way they may gain in terms of recall of definitions and results, but it is unlikely that they would consistently display relational, logical or formal understanding.

10.5.2 Internal/external combination

The internal/external combination occurred in this study only with visual reasoners and only on the new course. That both students in this group are visual reasoners is unlikely to be a

coincidence. The essence of the combination is that there is a very strong internal conviction in one's own understanding, and a desire to impress authorities by reaching quick, "intuitive" conclusions. This combination fits extremely well with a visual approach, as visual prototypes are ideal for providing conviction and for quick generalisation. Nonvisual, algebraic approaches facilitate neither of these goals, and it seems unlikely that a nonvisual student who had such a strong and immediate sense of understanding would also be unable to communicate this using suitable properties.

It is entirely possible however that an internal/external-visual student could be present in the lecture course. In this situation it may be predicted that their achievements would fall a little short of those of corresponding individuals with the purely internal and visual profile. They would enter with similar talents, but would not be encouraged by the course to focus on definitions or to develop their own arguments involving whole categories of objects. Thus they may gain a sense of understanding based on exposure to visual representations, but they would be unlikely to develop good formal understanding. Their desire to demonstrate speed may exacerbate this problem, leading them to avoid the more time-consuming tasks of attending to, and formulating, algebraic arguments involving definitions.

10.5.3 Internal, nonvisual, lecture course

Students with an internal-nonvisual profile may be present on the lecture course, although it might be predicted that they would not be there in great numbers: the MORSE course is popular with those who have a sense of mathematics as useful for a future careers, which is more consonant with an external sense of authority. If they do exist, such a person may be predicted to gain more from the lecture course than any of those examined in this study. As observed, internal-visual is a particularly strong profile given the bias towards nonvisual presentations in written mathematics at all levels. Such a student would be likely to devote considerable attention to the provided proofs, and thus may gain good formal understanding. However, as in Ben's case this may not give the learner the satisfaction associated with access to visual prototypes.

10.6 Main points of chapter 10

- Visual reasoning offers a sense of understanding the material. If this is well linked to the formal representations the student can benefit from working in two representations, if not their arguments may be insufficiently general and proof may become quite detached from what brings them conviction.

- Nonvisual reasoning leads the student more directly to the formal algebraic representations. If they come to understand the role and usage of definitions this can put them in a strong position. If they attend only to surface features they may not gain any meaningful understanding.

- An internal sense of authority is better than an external sense of authority as a basis for learning Analysis, since the student needs to examine the material in detail and take responsibility for judging the correctness of their own work.

- The new course provides a better environment than the lecture course for the development of logical and formal understanding, since the student is encouraged to generate their own arguments and receives regular feedback on these.

- The new course is not enough to override the student's initial beliefs about their role as a learner of mathematics.

- On the basis of the theory now developed, predictions can be made regarding the combinations of factors that did not appear in this study.

Chapter 11
A gap in the theory

11.1 Introduction

This chapter completes the investigative part of the thesis by relating the grounded theory now developed to the original theoretical framework from part 2. That framework identifies two different strategies an individual might use in everyday reasoning about categories of objects: direct generalisation from a prototype, and abstraction of properties from a prototype and deduction from these. The framework also incorporates the effect of mathematical definitions upon categories, and the distinction between mathematical definitions and dictionary definitions.
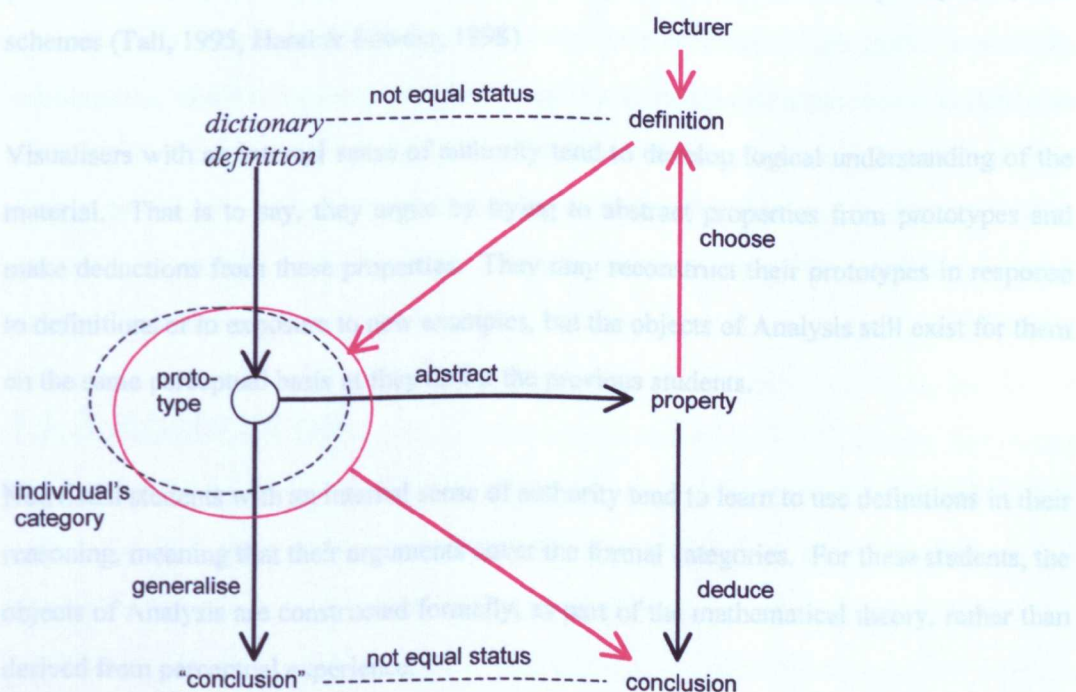
Figure 22: Diagram illustrating the theoretical framework

Having now developed a substantive theory of the influence of different factors on the types of understanding a student acquires, it is possible to identify a gap in the theoretical framework. While the framework covers the reasoning of most of the students in the study, there is one group for whom this is not the case. This chapter discusses this group and characterises their behaviour using other theories from mathematics education.

11.2 Reasoning about categories of objects

First, it is appropriate to summarise the ways in which different combinations of causal factors lead to different ways of reasoning about categories of objects.

Visualisers with an external sense of authority tend to generalise from visual prototypes. They do not adhere to expected standards of argumentation because they are convinced by what they see, and because they view the production of such justifications as either unnecessary or as the job of an external authority. Mathematical objects exist for them on a perceptual basis, as described by Tall in his discussion of the individual's construction of geometric concepts, and by Harel and Sowder in their discussion of perceptual proof schemes (Tall, 1995, Harel & Sowder, 1998).

Visualisers with an internal sense of authority tend to develop logical understanding of the material. That is to say, they argue by trying to abstract properties from prototypes and make deductions from these properties. They may reconstruct their prototypes in response to definitions or to exposure to new examples, but the objects of Analysis still exist for them on the same perceptual basis as they do for the previous students.

Nonvisual students with an internal sense of authority tend to learn to use definitions in their reasoning, meaning that their arguments cover the formal categories. For these students, the objects of Analysis are constructed formally, as part of the mathematical theory, rather than derived from perceptual experience.

So all of these behaviours are represented in the theoretical framework.

This leaves nonvisual students with an external sense of authority. There is considerable variation in the behaviour of these students. In some cases they focus only on the surface features of the work, which means that their discussions do not meaningfully involve mathematical objects. In other cases, *specific* objects appear as exemplars or as inputs and outputs for procedures. These specific objects differ from the prototypes used by the visual students, which are usually vaguer in character. Such students may use definitions, but they tend to view these as parts of fixed algebraic routines. Hence it would be inaccurate to say that these students construct formal objects, unlike those with the nonvisual-internal profile.

The theoretical framework is about reasoning about categories of objects, so this raises a question: where are the categories of objects for students with this profile?

This question can be answered with reference to the same background assumption that led to the description of the generalisations and abstractions performed by the visual groups. That is to say, these students are far from stupid – they have all attained high grades in previous examinations, and the literature on school-level mathematics education makes it clear that this is far from a trivial achievement. So what they are doing is presumably explicable as sensible behaviour. Once again it can be traced to strategies that are sensible under other circumstances, in this case the rather obvious circumstance of the students' previous mathematical success.

11.3 Abstraction

The explanation begins by considering alternative uses of the word "abstraction". This section discusses these, before considering how theory based on one particular use can account for the behaviour of the nonvisual-external students.

Thus far in this thesis, "abstraction" has been used to describe the process in which an individual focuses on (and perhaps verbally expresses) some property of its prototype for a category of objects. However the term is also used to describe the process by which such prototypes are originally created. That is, the recognition that certain objects have properties or holistic structure in common, and the creation of a mental structure to represent the associated concept (Skemp, 1979, Presmeg, 1992).

When this latter process is based on perception of physical objects, it is termed *empirical abstraction* (Dubinsky, 1992, Tall, et.al., 2000, *both after* Piaget, 1985). Empirical abstraction may thus be said to occur when a student learns basic concepts in geometry (Tall, 1995), or when students in this study observe the general shape of graphs of convergent sequences and incorporate this into their prototype. Alternatively, such abstraction may be based not on perception of objects, but on noting what is common to a series of *actions*. This is termed *pseudo-empirical abstraction* (Dubinsky, *ibid.*, Tall, *ibid.*) For example, a child learns the concept of number by repeated practice with counting. In this case it is not the objects counted, but the action of counting, which leads to this concept.

Various authors discuss this type of abstraction in the context of what may collectively be termed *process-object theories*. These theories identify stages in the process of pseudo-empirical abstraction, and hinge upon the dual status of mathematical concepts as both processes or operations, and objects within a logical structure. Sfard, for example, describes the way in which familiarity with a process leads to it being *interiorised*, so that it may be thought of without being actually performed, then *condensed* so that it is no longer necessary for the individual to think about its details. Following this, the condensed process may be *reified*, that is thought of as an object in their own right (Sfard, 1991). Dubinsky and his colleagues describe the same process in their *APOS* theory. They describe the way in which a learner performs *actions* on certain objects, *interiorises* these actions to give *processes* (which may, at this stage, be coordinated or reversed). These processes are then *encapsulated* to form new *objects*, and these actions, processes and objects may together

form schemas, that is, larger aggregates of knowledge (Dubinsky, 1992, Asiala et. al., 1996). Both theories emphasise parallels between individual learning and the historical development of mathematics.

There is much to be gained for the learner in performing these reifications or encapsulations: cognitive strain is eased considerably by the capacity to think of a concept as a single object rather than a lengthy process (Sfard, 1991). Also Gray and Tall note that *procepts* (their term for combined process-object entities), and in particular their associated symbols, lie at the heart of the competent mathematician's capacity for flexible thinking (Gray & Tall, 1994). However, encapsulation is notoriously difficult. As Sfard points out, it is qualitatively different from the earlier increase in familiarity associated with interiorisation and condensation. It requires an ontological shift, and the driving forces in making this shift form a "vicious circle": in order to perform higher level actions and processes, lower level ones must already be encapsulated to form objects. However, this encapsulation is only necessary because of the need to perform higher level processes (Sfard, *ibid.*). The resulting difficulty can be used to account for many well-known points of difficulty in the school curriculum, such as the introductions of negative numbers and fractions (Gray & Tall, *ibid.*).

To return to the argument of this chapter, note that the emphasis on algebraic procedures in school mathematics means that this largely involves pseudo-empirical rather than empirical abstraction. Those who go on to mathematics at university have proved extremely talented in this respect, succeeding in encapsulating the process of division to form fractions, in coping with the "lack of closure" associated with algebraic sums such as 2+3x, and even in handing functions as objects in order to perform the operations of calculus (Tall et. al., 2000). Of course they are not aware of this talent in these terms, and judging by research with students at a similar level, it may be expected that not all of them will yet have managed the last of these encapsulations fully (Eisenberg, 1992, Cuoco, 1994). But they have succeeded in learning mathematical concepts via pseudo-empirical abstraction, and the

behaviour of the external-nonvisual students may be explained by positing that they are attempting to carry on learning in this way.

This attempt results in another approach to learning the material of Analysis, and explains the place of categories in these students' thinking. These students attempt to learn algebraic actions, continued experience of which will eventually lead them through the process of pseudo-empirical abstraction. In order to achieve this they focus on the algebraic representations, and the effect is that at this stage there *are* no categories of objects for these students. The objects derived from pseudo-empirical abstraction appear *after* a lot of experience with routines, in contrast with those derived from empirical abstraction, which may be constructed almost immediately from perceptual experience.

Once again, the question of why some students take one approach and some take another is fascinating. It may, as suggested earlier, simply be a more or less random phenomenon established early in the course and perpetuated by success or a sense of understanding. It may be that preferences are established much earlier. Perhaps some students are "naturally" visualisers, and others are "naturally" more inclined to use algebraic and verbal thinking. Whatever the genesis of this phenomenon, its consequences allow further elaboration on the reasons for the recognised difficulty of Analysis.

11.4 Why is Analysis hard? (2)

Analysis, unlike the vast majority of school mathematics, contains material amenable to empirical abstraction: one may look at diagrams, perceive commonalities and build visual prototypes based on this experience. As seen in chapters 7 through 10, some students attend to this aspect of the presentation and gain an understanding in this way. In fact, chapter 6 suggested that Analysis should prove particularly hard because a larger than usual number of students will latch on to the visual representations, owing to the difficulty of the algebraic ones. Certainly it seems that for some students, this subject provides an opportunity to return to their preferred visual strategies after the predominance of algebraic representations

in school (Dreyfus, 1991, Presmeg, 1986a). Students who learn in this way can construct objects through perceptual experience, but this does not lead to logical or formal understanding. This is not surprising: these students have very limited experience in the use of visual representations in mathematics (Dreyfus, *ibid.*), so are not sensitive to the potential pitfalls of visual reasoning, or trained in translating visual arguments into verbal or algebraic formulations.

We can now see that the alternative approach of learning Analysis through pseudo-empirical abstraction will also be difficult, depending as it does upon experience with actions. As remarked by various authors, concepts at this level are typically introduced using definitions, assuming "ready-made objects" rather than building up to these through actions and processes (Sfard, 1991, Tall et. al., 2000). Attempts have been made by the proponents of APOS theory to address this situation, providing exercises (including computer programming tasks) with the aim of taking the learner through the actions necessary to construct notions associated with quantification and the limit concept (Dubinsky, 1992, Cottrill et. al., 1996).

Such efforts highlight the difficulties students face at this level. *Genetic decompositions*, aiming to characterise the constructions a learner might make in coming to understand a given concept, can involve numerous nontrivial steps (Dubinsky, *ibid.*, Cottrill et. al, *ibid.*). A larger problem, however, is that such approaches still rely upon experience with specific objects. If the objects are carefully chosen, this may lead to improvements in a student's knowledge regarding which objects belong to a given mathematical category. However, they do not address the issue regarded as fundamental in this thesis, that of learning to handle categories of objects exclusively via their definitions.

So, large numbers of students will find that their sensible approaches to learning Analysis are ineffective. What about the students who do succeed in acquiring formal understanding? This study indicates that the key combination is an internal sense of authority, which causes

the student to use conflict in a constructive way, and a course which provides practice in reasoning about whole categories, encouragement to use definitions and feedback on the student's own arguments. With such a combination, visualisers are provided with challenges to strengthen and better express their arguments, and all students gains more experience in using algebraic definitions in their full range of roles.

11.5 Main points of chapter 11

- Visualisers with an external sense of authority tend to generalise from prototypes, visualisers with an internal sense of authority tend to try to abstract properties from prototypes, nonvisualisers with an internal sense of authority tend to reason from definitions.

- The theoretical framework did not explain how nonvisual students with an external sense of authority handle categories of objects.

- The behaviour of these students may be explained in terms of process-object theories, in which objects arise from pseudo-empirical rather than empirical abstraction.

- In Analysis, visual representations are available as a perceptual basis for empirical abstraction. Students' inexperience with visual reasoning can mean that they succumb to its pitfalls.

- Material for pseudo-empirical abstraction is less readily available since objects are introduced "ready-formed" by way of definitions. Even further experience with different specific objects may not lead to formal understanding as this demands a change in the way categories are handled.

Part 4
Conclusions

Introduction to part 4

This study set out to investigate the learning of Analysis in two different pedagogical settings, with a view to identifying more closely what it is that makes the transition to university mathematics so difficult, and why Analysis should seem particularly hard. The findings have now been presented. Part 2 developed a theoretical framework, and part 3 reported on the investigative part of the study. Links between the two were explained throughout.

The aim of this final part is to further draw together these two strands as well as exploring potential directions for extending the research. It consists of two chapters, the first of which reviews the study, considering the methodological concerns of validity and generalisability from a grounded theory perspective. The second concludes the thesis by examining the practical implications of the findings, for both the design of pedagogical strategies and for use in standard lecturing.

Chapter 12
Review

12.1 Introduction

The aim of this chapter is to review the study, returning to the methodological issues of validity, reliability and generalisability. Some of these issues were discussed in chapter 4, which explained the measures taken to maximise the validity of the data gathered. At this stage, it is appropriate to assess the theory generated against Strauss and Corbin's list of attributes a grounded theory might expect to achieve. First, a short summary of this theory is given.

Part 2 developed the theoretical framework:

- First, it considered the ways in which people reason about categories of objects. It suggested that in everyday life the standards of argumentation are variable, with emphasis on direct generalisation, whereas standards in mathematics are more fixed and require abstraction of properties.

- Second, it accounted for the difficulty of the transition to university mathematics by considering the impact of choosing definitions upon the nature of categories and argumentation. Through this it unified established theories and observations regarding student behaviours at this level.

- Finally, it addressed the question of why Analysis is so difficult by considering the relative accessibility of its visual representations, which are ideal for building

prototypes and direct generalisation, and the formal definitions, which are logically complex.

Part 3 reported the investigative component of the research:

- This component used grounded theory data analysis methods to identify behaviours exhibited when students reason about specific objects and whole categories. These behaviours were linked to types of understanding.
- Part 3 identified causal and intervening factors in the development of these types of understanding, illustrated indicators of these factors and examined why they form coherent behaviour patterns.
- It developed a theory of why each factor contributes as it does, by considering different combinations in detail then isolating the factors in turn.
- Through the latter it identified a gap in theoretical framework and used other research and theory to explain these previously unaccounted-for behaviours.

According to Strauss and Corbin, if the study has been adequately carried out, the theory so developed should be significant, be compatible with observations, be generalisable and reproducible, have precision and rigour and be verifiable. It should also be a good basis for explaining the reality of the situation, and for providing "a framework for action" (Strauss & Corbin, 1990). With one exception, these issues are all covered in this chapter (some are gathered together under one heading). In addition, consideration of generalisability leads naturally to a final section on the possibilities for extending the research. The use of the theory as a framework for action is discussed in the next chapter, which offers final words on the implications of the findings for the teaching and learning of mathematics at this level.

12.2 Significance, precision and rigour

The attainment of precision and rigour in a grounded theory should be a matter of course provided that adequate time is devoted to open coding. If the procedures are followed

correctly, the detailed comparison of many incidents and phenomena renders the distinctions between different concepts as precise as possible. Also, continuing with such analysis until saturation is reached means that the concepts are rigorously attached to the data rather than arbitrarily assigned (Glaser & Strauss, 1967, Strauss & Corbin, 1990). Hence, while this study did not have the scope to proceed to theoretical sampling, it represents a precise and rigorous characterisation of the data gathered. In this case rigour was further enhanced by the detailed consideration of apparently anomalous cases. This led to greater richness in the explanations of relationships between causal and intervening factors and learning outcomes, and hence improved the theory's flexibility in accounting for more subtle variations within the data.

Whether the theory is significant is not for the author to judge, as such significance for any theory only becomes apparent over time, when it is or is not adopted as useful by others in the field. However it is hoped that it does at least represent a significant attempt to unify existing theories and observations of student behaviours, and that its findings are useful beyond the immediate context. The latter issue is discussed further in section 12.4 and the next chapter.

12.3 Compatibility with observations

Despite efforts to make interview data as valid as possible, caution is necessary in interpreting this data, as would be the case from any one method (Schoenfeld, 1985, Cohen & Manion, 1994). Hence it is appropriate to consider whether the theory is compatible with other observations. Other data gathered in the study were not analysed in the same detail as the interview transcripts, but it is illuminating to consider some of this information as it provides basic support for the theory while raising further questions.

12.3.1 Examination scores

The most obvious tactic for triangulating the claims of the theory is to determine whether the study's observations regarding student progress are reflected in the marks awarded to

them by the university system. The first potential measure for this is the students' marks for the assignments they submitted during the term. However, a brief perusal of these marks reveals this to be inappropriate. That they do not necessarily reflect the students' performance in interviews should not be entirely surprising: students' comments on their work habits make it clear that what they submit cannot reliably be taken as representative of their individual work. However the assignment marks also differ widely from the examination marks, which *are* a reflection of the students' performance in an individual setting. The latter are considered below.

The January examination was taken by all of the students in the first week of the new term, after the four-week Christmas vacation. They were allowed 90 minutes to complete 3 out of 4 questions (if they attempted all four, the best three marks were taken), and in the same sitting they also attempted a 90-minute Foundations paper. Participants' marks for this January examination are given in appendix A1, where it is noted that they represent a reasonable spread in relation to the marks attained by the entire cohort. Here these marks are compared with each other and with what might be expected given the theory.

Comparing across the participants attending the new course, the results show an unexpectedly high (qualitative) correlation with the theory. With the exception of Dean[31], top scores go to the nonvisualisers (Greg on 75%, Hugh on 74, Ben on 68). Following these are the visualisers with an internal sense of authority (Jenny on 65, Adam on 52). The visualisers with at least a partially external sense of authority score the lowest (Cary on 39, Emma on 37, Kate on 32, Fred on 28).

This pattern is consistent with the idea that concentration on the nonvisual representations used in the course is beneficial, and that those who focus on visual representations and rely on others to judge their work are in the weakest position. It does provide one or two

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[31] Dean's case was not discussed in detail as Cary's effusiveness meant that comparatively little information was gained on his thinking. His lower mark (49) could be due to his increasing involvement in university social issues during the term.

surprises: first, Adam scores lower than would be expected given his strength in linking visual and nonvisual representations, and second, sense of authority appears not to be a great influence in the marks of the nonvisual students. Interestingly, the marks for the later June examination (covering both first year Analysis modules) seem to indicate that the expected balance is reached later[32]. In that case, Adam scores 77 and Hugh 35. This suggests that Adam's learning has provided a strong basis for understanding later material, and that Hugh's "practice makes perfect" approach to memorising mathematics does not stand up well when numerous examinations are held close together.

For those on the lecture course, the scores are for the most part lower, confirming the observation that they do not acquire such a good knowledge of the content as their peers on the new course. The interesting result in this case is that the marks are considerably less correlated with the causal factors. The population is split into two quite cleanly:

- those who pass comfortably[33]: Wendy on 71, Unwin on 59 and Xavier 53.
- those who fail rather badly: Zoë on 28, Tom on 27, Steve on 25, Yvonne on 23 and Vic on 20.

The majority of the actual scores are not surprising. Of these students, Wendy and Xavier showed the most competence with the material during interviews. However Unwin's score is surprisingly high (Unwin's work was not shown in detail in part 3 but he would best have been classified as an "episodic" learner), as is Wendy's when compared with similar students on the new course. Vic's score is surprisingly low, as he is the only student in this group who could be said to have an internal sense of authority. In addition, the split in this case is not along the lines of visualisers and nonvisualisers (the visualisers are Wendy, Tom and Vic).

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[32] The June examination marks are not discussed in detail as the students were not followed in detail beyond the end of the first term so it would be misleading to offer more than speculations on their performance after this time.
[33] The usual pass mark is 40%.

This raises the question of what might account for these results. It is unlikely that the students were seriously misclassified according to the category system, since this system emerged from study of their responses. It could be that the classification is not a good predictor of examination performance, and that the results for the students on the new course are misleading. However, perhaps the most interesting possibility is that the classification is not a good predictor of success *for those on the lecture course*. This would fit with the suggestions in chapters 9 and 10 that students on the lecture course are less likely to fulfil their potential regarding logical and formal understanding, although they may acquire good relational understanding. In view of Wendy's score in particular, this then raises the further question of what precisely were the demands of the examination; if a relatively high score could be gained through the use of procedures, perhaps Wendy's focus on this aspect of the work led to her good performance. These questions are addressed further in the next section, which looks at the examination paper in more detail.

12.3.2 Examination scripts

Examination of the students' written work is important in this case since the participants' mathematical behaviour in the interview situation does not necessarily mirror that when they are doing their work for the course[34]. It could be argued that the work they do when aiming to "get marks" would show more evidence of awareness of the standards required, as well as demonstrating the students' competence when they have more time to work on a question.

Copies of assignment work were collected for various questions. Unfortunately, at the time these were chosen in an attempt to match these across the courses (recall that the weekly assignments were different for those on the lecture and new courses). This meant that the work collected largely involved questions about specific objects, which are of limited interest since learning to reason about whole categories is central to the transition to

[34] This does not have undue impact on the purpose of the study: the aim was to find out how the students think about the concepts of Analysis rather than what they are learning to produce by way of written work.

university mathematics. Because of this, and because these assignment scripts cannot be considered representative of the students' individual work, once again this section concentrates on the January examination scripts.

With a view to discerning whether the structure of the examination allowed Wendy to perform well through competence with procedures, the students' examination scripts were used to further break down the marks according to whether question parts demanded work with specific objects or more conceptual work involving whole categories. Summaries of this breakdown and the students' scores are given in appendix F. However this classification did not prove illuminating regarding the performance of different students. Students largely chose the questions dealing with material from the beginning of the course, and the marks gained across these followed a similar pattern for most students[35].

It is however illuminating to consider one question in further detail. The first part of question 2 required students to prove that a convergent sequence must be bounded, as was required in the week 7 interview. This was approximately 8 weeks before the examination and the students had had a chance to revise, so it could theoretically be that their responses are quite different by this time. However it turns out that they are remarkably consistent with the students' interview performances, and examples are included below to illustrate this.

Adam's solution is perfectly correct, and its fluency reflects his confident logical and formal understanding[36].

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[35] It is worth noting that Wendy did perform well in questions requiring deriving limits for specific sequences, and performing an integration in the context of the integral test.
[36] See sections 9.2.8, 9.2.4, 9.2.1 and 9.2.1 for detail on Adam, Kate, Xavier and Zoë respectively.

(2)a(i) If (a_n) converges to a limit, ℓ say, then for an
$\varepsilon > 0$ there exists a natural number N such
that $|a_n - \ell| < \varepsilon$ whenever $n > N$. If we let
$\varepsilon = 1$, it follows that there exists N_i such
that $|a_n - \ell| < 1$ whenever $n > N_i$

This means that $-1 < a_{N+n} - \ell < 1$, and by adding
ℓ we have $\ell - 1 < a_{N+n} < \ell + 1$
This means that a_{N+n} is bounded between
$\ell - 1$ and $\ell + 1$
If we let $V = \min \{a_1, a_2, \ldots, a_N, \ell - 1\}$
then V is a lower bound for a_n.
Similarly if we let $U = \max \{a_1, a_2, \ldots, a_N\}$
then U is an upper bound for a_n.
4   Hence a_n is bounded

Figure 23: Adam's examination solution

Kate's solution incorporates explicit references to prototypes, showing that she relies on these in her reasoning despite exposure to the formal work.

2a-(i) Def^n of convergence says: ?
$\exists U$ (upper bound of a_n) st $a_n \leq U$
$\exists L$ (lower bound of a_n) st $a_n \geq L$
$\therefore L \leq a_n \leq U$

If a_n is convergent, it must have a start value
which will be either it's upper or lower bound
All the following values will be on one side
(above or below) this starting value.
As a_n tends to a limit (converges) there must
be a value U (usually the limit, except in
oculating sequences) which a_n does not exceed.
$\therefore$ If (a_n) is convergent then it is bounded

Figure 24: Kate's examination solution

Xavier's solution is complete and correct, but somewhat sparse in terms of explanation. This perhaps reflects his relative inexperience in writing such arguments.

$2.$ Suppose $a_n \to a$ as $n \to \infty$ ___ Let ε

$a_1)$ $|a_n - a| < \varepsilon = 1$ ___ $|a_n - a + a| \le |a_n - a| + |a| < \ + |a|$

$\forall n$ ___ $|a_n| \le \max\{|a_1|, |a_2|, |a_3| \dots 1 + |a\}$ ___ $\square$

$a_{ii})$ $b_n \to 0$ as $n \to \infty$ ___ $c_i \le c$

$b_n c_n - bc$ ___ $|b_n c_n| < \varepsilon$

Figure 25: Xavier's examination solution

Finally, Zoë's solution incorporates an incorrect "definition", and some correct algebra along with some nonsense. This seems consistent with the idea that she has attempted to memorise rather than meaningfully interpret the work.

Q2. a) i). (a_n) is said to be convergent if, as $n \to \infty$, it tends to a finite limit, L.

(a_n) is said to be bounded if ~~∃M∈ℕ∃~~ s.t.

$\forall M > a$ ~~#~~ $\exists M > 0$ s.t.

$\forall M > N$ ___ $|a_n| \le M$

~~suppose~~ Given $\varepsilon > 0$ ~~∀~~ $n > N \Rightarrow |a_n - L| < \varepsilon$

$\therefore$ $L - \varepsilon < a_n < L + \varepsilon$

if $|M| \ge \varepsilon > 0$

$M \le L - \varepsilon + < a_n < L + \varepsilon \le M$

$\therefore$ $|a_n| \le M$.

Figure 26: Zoe's examination solution

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12.3.3 Explaining the reality of the situation

The theory's compatibility with other observations goes some way towards demonstrating its appropriateness for explaining the reality of the situation. It seems odd that Strauss and Corbin should claim that a grounded theory should also be verifiable, since even theories in the positivist tradition do not have this attribute. Indeed, their suggested use of anomalous cases to adapt and enhance a grounded theory rather than refute it appears incompatible with this aim (Strauss & Corbin, 1990). However, further short remarks may be made regarding the former property, which perhaps corresponds most closely to the traditional use of "validity", in the sense of whether the theory may be considered "true".

This question may be addressed on two levels in this study. First, are the distinctions made actually representative of splits in the population? Chapters 7 and 8 addressed this point, considering the internal consistency of the set of beliefs associated with an external sense of authority for example. In addition, the distinctions made here reflect and build on other notions in the existing literature, as described in chapter 3 and throughout part 2. It is thus "compatible with observations" from beyond this study, which further adds to its credibility as a reasonable explanation of student behaviours.

Second, the theory benefits from being integrated across various levels of abstraction. The abstract work on reasoning about categories is applied to classify types of understanding, which are identified in the detail of student responses, and attributed to the presence of other factors which are also identified through this detail and linked to existing theory. Hence, while the theory cannot claim to be verifiable as representing the workings of the students' brains, it does provide a coherent explanation of the reality of the situation.

12.4 Generalisability and reproducibility

Reproducibility is a difficult question, and once again it seems strange that Strauss and Corbin should incorporate it given their arguments regarding the unique nature of any grounded theory study (Strauss & Corbin, 1990). Certainly a different researcher beginning

with the same situation would produce a theory that was different in the terms it used and the detail of the distinctions, since this is the nature of this type of work. However it is hoped that the classifications of student behaviours were made clearly enough that another researcher studying the learning of Analysis would be able to classify students as having a visual or nonvisual reasoning approach, and an internal or external sense of authority. They should then be able to assess whether the students showed the predicted behaviours in reasoning about categories of mathematical objects, and hence to "reproduce" or invalidate the findings in this sense.

12.4.1 Generalisability of grounded theory

Regarding generalisability, Strauss and Corbin state that further research would be necessary to truly determine whether theory developed in one context is applicable in another (Strauss & Corbin, 1990). Schofield echoes this view, saying that "...there is broad agreement that generalisability in the sense of producing laws that apply universally is not a useful standard or goal in qualitative research" (Schofield, 1990). Indeed, as stated in chapter 4, the goals of this study were in line with Wolcott's aim of gaining an understanding of the situation rather than producing a "true" account of it (Wolcott, 1990). However, generalisability remains an interesting question; in qualifying her comments Schofield observes that many qualitative researchers do hope to shed light on issues generally and not just as experienced at one site (Schofield, 1990). Becker goes further, saying that "if...you don't generalise about schools, in some crucial respect you haven't solved the problem most educational researchers have in mind when they talk about generalising" (Becker, 1990). Certainly it is appropriate to consider the value of this research for informing practice at other universities, especially since at first sight Warwick appears to be a rather non-typical institution.

12.4.2 Atypicality of the site

The students on all the degree programmes discussed here arrive with excellent academic records. As described in section 2.2.1, the entry grades for all of these courses are very

high. In particular, in mathematics it is not uncommon to come across students with four grade A's at A-level. At first sight therefore, the research setting appears to detract from the potential for informing practice in other institutions. Schofield, for instance, recommends studying the typical to study *what is* (Schofield, 1990), and such students could not be considered typical of the population of mathematics students nationwide. This affects any arguments regarding the effectiveness of the new course, since even though it seems to encourage improvement in logical and formal understanding in these students, it could be argued that this is because they are already further ahead and so are able to take on this style of work. Mathematicians from other institutions have expressed concern that their students would not be able to cope with such a demanding course. As an educator my first reaction to this is one of horror, since it seems tantamount to an admission that we do not expect most students to be able to do more than copy and memorise proofs. However it highlights a serious question. This *is* a demanding course and although it is set up to require more than memorisation from students, it may not be the only or most effective way of improving their learning.

However this study does more than consider the nominal influence of the two different courses, and as Schofield goes on to say, "...studying a site chosen for its special characteristics does not necessarily restrict the application of the study's findings to very similar sites. The degree to which this is the case depends on the degree to which the findings appear to be linked to the special characteristics of the situation" (Schofield, 1990). Henwood and Pigeon and Becker make similar points, suggesting that a rich grounded theory will suggest its own sphere of relevance and application (Henwood & Pigeon), and that generalisations can be the same although the results might be quite different (Becker, 1990). This latter point is key to the transferability of the theory developed here: this theory does not depend upon measures of the students' previous attainment, it depends upon their tendency to argue visually or nonvisually and their sense of authority. Students anywhere will have these attributes, and their interpretations of Analysis can thus be predicted. It might be that students who score lower by previous measures are more likely to have an

external sense of authority, but it is worth noting that the new course seemed to benefit the thinking of even these students, encouraging them to develop at least the habits of logical and formal understanding.

12.5 Extending the research

The issue of how generalisability might be increased can be addressed by considering possible directions for extending this research.

First, the study could be broadened. Having established certain factors as important in determining the ways in which students interpret the material of Analysis, an obvious question is what proportion of students have which preferences. Such a study would not need to be so detailed: having identified traits shared by those with an internal or external sense of authority, for instance, a questionnaire could be written to provide an indication of the numbers holding each view. Establishing visual or nonvisual preferences would be harder, but it might be possible to teach a lesson incorporating both aspects and have students write summaries, to see which aspect they seemed to focus upon. Indeed it would be interesting methodologically to combine this with a more detailed study, to ascertain whether it is possible to identify such beliefs and preferences from more cursory examination. In any case, if such information could be established it would be useful to teachers in predicting the ways in which large sections of their class are likely to interpret the work, at the original site and beyond.

A second direction for extension is that suggested by the procedures of grounded theory, which is to proceed into theoretical sampling and so examine phenomena in more depth. Having established a substantive theory, the aim would be to enrich and strengthen this by sampling with a focus on the issues that emerged as important. Hence students would be selected early on for their place regarding the causal factors, and followed in more detail throughout the course. This would be useful in providing more detail of the process by which understanding grows, and thus in confirming or rejecting the proposed relationships

between the causal and intervening factors and the learning outcomes. Hence such a study could establish more detail on the conditions that would need to be in place for the findings to be generalised. It could also be aimed at answering particular questions, such as what prompts a visualiser to adopt the regular use of definitions. Such information might be useful in designing amendments to pedagogical strategies.

Finally, a more comprehensive longitudinal study could be undertaken. Beyond generalising to other students learning Analysis, it would be particularly useful to establish whether the students themselves generalise from a module like the new Analysis course in their longer term approach to learning. Would the habits associated with logical and formal understanding, which appear to be encouraged by the new course, persist into the students' work in other modules? This is a serious question, since proponents of theories of situated cognition suggest that transfer of knowledge from one domain to another happens rarely if at all (Lave, 1988). Indeed it has been noted by various authors in mathematics education that while a course with emphasis on conceptual understanding or problem-solving skills may impact upon a student's thinking in its own context, any benefit from this is likely to be lost when further courses revert to a more standard format (Meel, 1998, Mohd Yusof & Tall, 1996). Hence when further courses do not place similar emphasis on the students inventing proofs themselves, it may be that many revert to memorisation.

Such a study may of course reveal positive results. First, the students on the new course do seem to more rapidly become conversant with the material of Analysis. This can only stand them in good stead for future courses that build on this material. Second, although the effect is not universal, the requirements of the new course do seem to assist students in learning the central definitions and recognising their importance in mathematical theory. If *this* knowledge transfers then it should be of great benefit to them in appropriately interpreting all future mathematical material (Alcock & Simpson, 1999). It would be interesting to see whether this is indeed the case.

Ways in which teachers in various settings might help their students to develop in such ways are discussed in the next and final chapter.

12.6 Main points of chapter 12

- The procedures of grounded theory data analysis render a theory precise and rigorous.

- The theory developed here is largely compatible with observations from participants' examination marks and scripts; interesting questions are raised by discrepancies between the correspondence for the new and lecture course students.

- The theory provides an explanation of the situation that is consistent with the data, with other observations and with the wider literature.

- Although the site is atypical, the theory is generalisable.

- Generalisability could be further enhanced by extending the research in three directions: breadth, to ascertain the numbers of students for whom different causal factors apply, depth, to gain more detail on the process of developing logical and formal understanding, and length, to address the question of transfer.

Chapter 13
A framework for action

13.1 Introduction

This final chapter addresses the last of Strauss and Corbin's points regarding the attributes a grounded theory may be expected to attain, that of its usefulness in providing a framework for action. It begins by reviewing what may be inferred from the study regarding pedagogical strategies in university mathematics, and concludes by considering the way in which the research might inform ongoing teaching at this level.

13.2 Implications

This section is about what we can learn from the study, specifically, what sort of environments and tasks might be provided in order to best enable students to develop logical and formal understanding of this material. Naturally it must be borne in mind that nothing will work for everyone. In fact the thesis highlights the acuity of the student remark cited by Duffin and Simpson: when a peer failed to understand a teacher's explanation, the student said "perhaps he's thinking about it in a different way from you and that's why he can't understand what you're saying" (Duffin & Simpson, 1993). The investigative part of this study confirmed the fact that students attend to different aspects of a presentation, and interpret material in highly idiosyncratic ways. This aim of this section is to suggest ways in which this knowledge might be put to positive use, both in planning courses and activities, and through bringing it to the attention of the students themselves as a means of encouraging them to reflect on their own ways of thinking.

13.2.1 The value of innovations

First, the innovative course itself contributes much to the discussion of how to improve student learning at undergraduate level. In part 3 it was argued that, while the course did not alter the students' predispositions, it did provide an environment in which they were better able to fulfil their potential for logical and formal understanding. Even those whose external sense of authority prevented them from fully taking responsibility for judging their own mathematical argumentation did at least develop some of the habits of both. This ties in with what might be expected, given the large body of research recommending that the role of teacher should be commuted from dispenser of information to, in Duffin and Simpson's words:

- Challenging the students' current [mental] structures, with which they are comfortable and seemingly secure, so that they can develop by overcoming the challenges.
- Supporting their new ways of thinking by helping to strengthen them through success.

(Duffin & Simpson, 1993)

The latter hints at another benefit of the new course: the more natural environment it provides for teachers to support students on an individual basis, as well as for students to support each other. This aspect was not the focus of the present inquiry, but remarks on student feedback sheets indicate that these social aspects are appreciated. Students feel it eases the transition to university work to have at least some classes in a more school-like environment, where they are encouraged to work together and so get to know more of their fellow students, and where help is on hand.

Finally, such well thought-out innovations are to be commended in and of themselves. They make practical use of the vast amount of formal and informal knowledge gained by experienced teachers, whose dedication in bringing them to fruition can only benefit

students. They can also have much benefit for the development of theory. The contrast between such work and more traditional courses highlights the benefits and the limitations of both, and provides an excellent impetus for studies such as this one to be undertaken.

13.2.2 The link between visual and nonvisual representations

This section and next take up issues highlighted in chapter 10. They use the demonstrated influence of the causal factors as a basis for discussing what might be done to improve student understanding.

The principal observation regarding visual and nonvisual reasoners is that almost everyone would benefit from learning the skills of the opposite group. Those who tend to focus on the visual representations would do well to learn to express their reasoning in definition-based language. Similarly, if they learn to incorporate diagrams into their thinking, those who do not naturally use visual representations may be able to understand more holistically, make discoveries more naturally, and perform checks that their algebra produces sensible conclusions.

So can we, as Gibson suggests, explicitly teach students to use diagrams as an intermediary between their own thinking and the formal mathematics (Gibson, 1998)? While it is important to consider the limitations of prescribing tasks to address particular issues like this (students may simply learn more procedures, rather than understand with more depth), perhaps students would benefit from tasks that specifically require them to draw diagrams illustrating algebraic statements, or write such statements to describe diagrams. Students, of course, may not see the benefit of such tasks. While a student like Ben may recognise the gains since he is already aware that his understanding is not what it could be, he may well find this sort of task difficult. Similarly, while students with an external sense of authority may be willing to accept this as just another authority-dictated task, they may also have little patience with spending time on diagrams when the algebraic forms are eventually more valued (cf. Eisenberg & Dreyfus, 1986).

An alternative potentially useful tactic would be the introduction of example generation tasks. Dahlberg and Housman suggest that example generation in response to a new definition is a feature of the thinking of stronger students (Dahlberg & Housman, 1997), and the theory developed in this thesis suggests that there is sometimes a gap in the tasks we set for students in this respect. Students *are* asked to manipulate specific objects to obtain others, to show that categories are subsets of other categories, and to show that specific objects are members of certain categories. It seems that we might also regularly ask questions which begin with a category and ask the student for specific objects. Mason suggests such an approach in advocating the use of "undoing" and "freedom and constraint" questions which require the student to consider what types of object satisfy certain conditions (Mason, 2000). Such questions may be useful for developing formal understanding, in the sense of a correspondence between the student's idea of what objects belong to a given category and the formal extension of that category. They might encourage visualisers to look beyond their prototypes and consider other types of example, and benefit weaker nonvisual students by encouraging them to think in terms of mathematical objects rather than algebraic routines.

Finally, a more student-centred approach would be to capitalise on the different strategies already in use by different students. Language used in interviews seemed to indicate that students are not commonly aware of such differences: often a student would say "when *you* do your work at home, *you...*", using the word "you" in the general sense but proceeding to finish the sentence with an observation quite inapplicable to other participants. This occurs even in the classes where students work together. For instance, Jenny appeared to be capable of using definitions long before she regularly chose to. Perhaps if her attention had been directed to the strategies of other students who picked up the tactic of "write the definitions down first" earlier, she would not have become disheartened with the amount of time she was having to spend on the work.

13.2.3 Moving towards an internal sense of authority

Those who gain most from the new course are those with an internal sense of authority. These students are best placed to take advantage of the environment and try to use the available feedback to generate their own arguments and improve their own understanding. This is not to devalue the course, as those with an external sense of authority benefit too. However, the latter still do not engage with the material to the same extent. Thus this study confirms the well-recognised view that development of intellectual autonomy should be major goal of education.

The issue of sense of authority is also particularly important when considering generalisability. In this study, the majority of the participants on the lecture course had an external sense of authority. They also had slightly lower A-level grades. This alone does not suggest that there is a significant correlation between the two, as much more extensive study would be needed to make such a claim. However it raises questions about student attitudes at universities in general, as it is certainly true that many students lack the intellectual curiosity to wonder why a theorem or formula is true (Harel & Sowder, 1998).

So we would like students to develop an internal sense of authority, but we must recognise that this will not be easy to achieve. People in general do not take kindly to suggestions that they should change well-established parts of their world-view. As discussed throughout the thesis, they will go to some lengths to avoid recognising conflict between their expectations and their experiences. In particular, chapters 9 and 10 demonstrated that those with an external sense of authority quite easily subvert the new course's emphasis on relying on their own thinking. Another obstacle to encouraging a move to an internal sense of authority is the internal consistency of an external set of beliefs. Indeed, students with such beliefs are constrained by a number of vicious circles: if an individual is not looking for meaning in the work, they are not going to be upset if they do not see any, and if they are not upset by not seeing meaning, they are unlikely to make an effort to look for it. Similarly, surface-only processing of the work means that an individual will probably not

remember it very well, making them more dependent on notes. But once they rely on everything being in their notes, they lose impetus to try to understand or remember as they go along.

Authors working in the field of student beliefs often express sympathy with students who are required to make transitions (Perry, 1988, Copes, 1982). This certainly seems appropriate in this case: many students arrive under the impression that they know what they have to do in order to do well, and they follow this approach in good faith only for it not to bring them success at all. In fact, in this respect these high-achieving students may be at a disadvantage, since they have enjoyed exceptional success on the basis of their existing learning habits. It is therefore not unlikely that being "good at maths" is important to their self-image, especially in a culture where this is thought somewhat mysterious and often attributed to genetic luck. Finding that this is not the case by the new measures in force in university mathematics could turn them off the subject altogether.

However, despite all these negative points there is hope that an internal sense of authority can be encouraged. At a basic level, and reflecting their A-level performance, the "weaker" students in this study are hardly incapable. Even Zoë, with her "episodic" approach to learning, can answer questions about meaning if these are asked at the right (albeit very simple) level, and can thus improve her understanding?[37]. If she had learned to ask such questions herself, she might therefore have developed much stronger relational understanding at least.

Further, this study confirms Perry's observations about ways in which students proceed through his levels, and Copes' application of these to the mathematics classroom (as described in section 3.3.1). Most students, for instance, do recognise the value in producing their own work, even where they do not believe that they should be the final judge of its worth. This corresponds closely to a position noted by Perry, in which the dualistic student

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[37] See appendix E.

retains their belief that right answers exist, but is willing to accept that the authorities want them to find these answers "for themselves" (Perry, 1970, Copes, 1982). Similarly, students may move towards relativism through first producing critiques of their own answers rather cynically, because this is "what the authorities want", then recognising the value of this approach across their studies. Hence the simple requirement to engage in this sort of task may be effective in encouraging development for at least some students.

At a more conscious level, it is also worth noting that most of the students in the study expressed at least some reflection on the change in the mathematics by the end of the course. Zoë, for instance, commented that if what they are doing now is mathematics, then what they did at school could not really count as such. Similarly, as noted in chapter 9, Fred has realised by this stage that the ideas are very muddled in his head, and Jenny has recognised the importance and value of using definitions. This raises the question of whether, as suggested with visual and nonvisual preferences, it is possible to harness the range of approaches existing in a class in order to encourage students to think more reflectively about their own strategies and possible alternatives. Indeed such emphases, particularly regarding discussion of what counts as a mathematically acceptable proof, have been successfully incorporated at all levels in the curriculum (Yackel & Cobb, 1996, Yackel, Rasmussen & King, 2000, Maher & Martino, 1996, Hirst, 1981).

So student-led discussion of the structure and standards of university mathematics may be effective in encouraging a movement towards an internal sense of authority, particularly as instigating such discussions and taking student contributions seriously emphasises the point that the student's own thinking is of value. However more direct, teacher-led approaches can also be useful. Various authors, for instance, recommend the introduction of some form of standardised, repetitive questioning. The idea is that after repeatedly being asked questions such as "What (exactly) are you doing? Can you describe it precisely?", students learn to anticipate these questions and seek answers without prompting (Schoenfeld, 1992, Mason & Davis, 1987).

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With any of these strategies, students cannot be expected to suddenly jump from one set of beliefs to another: as Copes and Perry point out, they may go through various stages and need considerable support in this process. Also it is inevitable, as demonstrated here, that students in one class will be at quite different stages, so it is impossible to provide an environment that will be maximally beneficial to everyone. However, it seems that work on developing students' ideas about their own role in the learning process, whether explicit or implicit, is of value. I believe that this sort of discussion should be incorporated into existing courses across the undergraduate curriculum, so that students might recognise it as an important part learning mathematics. Ways in which such discussions might be incorporated into lectures, a less obvious environment for promoting them, are discussed in the conclusion.

13.3 Conclusion

This chapter has considered both implications of this research for pedagogical strategies, and extensions of the research that might further improve our understanding of the situation. This final section concludes the thesis by considering ways in which the research might inform everyday, ongoing teaching.

When I began this research, I believed that there was something inherently bad about lecturing, involving as it did the students simply watching rather than doing mathematics. The research has led me to revise this view considerably. I now believe that lectures can be the ideal way of conveying information to a large number of people, but that the effectiveness of this approach is severely restricted in the case of first year students, as is outlined below.

This argument is best seen in the context of a constructive criticism of a common mathematicians' approach to issues in education. It has been my observation that mathematicians, when discussing teaching issues, do not often talk for more than ten

minutes before getting into a debate about the order in which material should be presented. This means that they are discussing logical rather than psychological issues, and it can appear that the students do not have a place in the discussion at all. To identify reasons for this, it is appropriate to make a parallel observation to the one made in the introduction: just as the students are not stupid, neither are their teachers. These are highly intelligent people who are highly reflective about their own learning: the problem is that such people know what their own difficulties are, and naturally assume that others have difficulties of the same type.

Unfortunately this turns out to be an inaccurate assumption when applied to first year undergraduates. Good organisation may well be the central issue in whether a mathematician does or does not understand a talk given by another, but for a typical first year, there are much more basic things to be learned than the relationships between the concepts in any new topic. Critically, such a student is unlikely to understand the role of mathematical definitions in determining the extension of categories, and the consequent standards for constructing arguments about these. So, where a mathematician may have problems attaching meaning to a concept encountered by way of a formal definition, a student *may not know that that is what they are supposed to be doing*. They may already have a meaning that they have learned elsewhere, or they may simply be trying to learn to perform mechanical operations with the algebra. Either way they will not interpret the material in the way intended by their lecturer.

This is not to say that good organisation is not a valuable goal; Skemp advocates that the teacher should take time to consider the logical structure of their topic and how best to present the relationships therein (Skemp, 1979a). However once a basically sensible order of presentation is established, it is debatable whether further adjustments are more than superficial from a pedagogical perspective. As Byers comments, "Unfortunately, superior organization of knowledge in a textbook, a teacher's notebook, or in a curriculum committee does not necessarily improve the organization of knowledge in a student's head" (Byers,

1980). This is true on a rather grand scale at this stage in a student's mathematical education, where they must learn to reason about categories of objects in a new way.

So a lecture, no matter how clear and well-presented, may be unsuccessful in conveying the desired information if this issue has not been addressed. It is my hope that the material in this thesis may contribute to increased recognition of this problem, as well as providing a way of viewing the situation that is coherent enough to be useful to teachers in conceptualising likely student interpretations of the work. In particular, I hope to contribute something to teachers' understanding that their students' behaviour is usually reasonable, and to highlight the ways in which advanced mathematical contexts differ from those requiring everyday argumentation. This should be useful since it seems that if we do not know why people are behaving in a certain unproductive way, we are pretty much limited to saying "don't do that, it's no good, do this instead." This is not very satisfactory – why should anyone believe us when presumably their experience has led them to believe that this is a good way to proceed? If we do know why they do something, we are in a better position to say "I understand why you're doing that, and it's a good idea, but this is better because…". This surely lends more weight to an argument that we want a student to change their approach.

With these ideas as a basis, it is possible to make some recommendations on how this problem might be addressed. Of course, it must be recognised that it is not possible to say "do this, and the students will understand". Just as mathematics is "not a spectator sport", neither is mathematics education. It is not possible to give methods to follow which will always be successful in a certain range of circumstances. At the end of the day, a teacher needs to be sensitive, and well-informed regarding potential student misconceptions.

Given this, there is much that can be done. Awareness of common misconceptions means that these can be shared with the students. It is perfectly possible to say "People often think that…. You should be careful to avoid this because…". And, I would say, to write it down!

It simply is not realistic to assume that students take in everything that is said in lectures, and if they have this sort of information in their notes too it will be there when they come to revise. This also holds for the more fundamental information regarding the role of definitions. It may not be possible (or desirable) to get into a full-blown philosophical discussion on the nature of argumentation in a lecture, but it is possible to say "If you want to show something is an X, you have to show that it satisfies the definition of X, and if you want to show that all Xs have property Y, you start with the definition of X and show that you can get property Y from it". And, indeed, to say this over and over again, and once again to write it down.

As noted in chapter 3, this latter knowledge renders many proof frameworks (Selden & Selden, 1995), if not the proofs themselves, virtually trivial. It may therefore seem condescending to state it so simply, and naturally it is good to avoid patronising the students. However it does seem from this research that there is generally an underestimation of the level of advice needed by students at this stage.

Of course, it cannot be expected that students will readily take in and make use of this information. They do not interpret much of the mathematics in the way we would want, so it is unrealistic that we should expect to do any better with meta-mathematical issues. However, it seems important to try, since students do listen to their teachers sometimes, and we can send the message that we think something is worthy of their attention if we regularly emphasise it. Certainly teacher-student exchanges along the lines of "What is a proof?", "A proof is a convincing argument" miss the point that there are conventions the student must learn in order to produce such an argument. These conventions simply fall below the consciousness of most mathematicians, most of the time. As they should: the interesting part is the concepts and their relationships, not the standard way in which arguments must be expressed. However this makes it all the more important that students should acquire this solid base for reasoning, and that mathematicians should be aware that they are unlikely to have it when they first arrive at university.

One further caveat before concluding: from a mathematician's point of view, the apparent increase in complexity engendered by the use of certain definitions is worthwhile, because of what it yields in systematisation and reduced cognitive strain once it is mastered. It should be remembered however that we cannot expect all of our students to share our enthusiasm on this score: many choose mathematics as a degree course because they were good at it at school, which often means that they were good at recognising which of a set of procedures was appropriate to a given situation, and then applying it correctly. They may not have any interest in building complex mental structures of interlinked concepts, and it is not realistic to think that everyone will gain an appreciation of mathematics at this level.

However we can do our best to explain clearly what is required, to offer support and encouragement, and be understanding when students do not pick up the new standards straight away. It would be desirable if students who decided that university mathematics was not for them did so because they understood and rejected these goals, rather than because they were simply never able to make sense of the material.

In all, I hope that the material contained in this thesis makes a contribution to the personal goal outlined in the introduction: that of reducing frustration on the part of those involved in teaching and learning university mathematics.

Part 5
Appendices and references

Appendix A: Participants

A1. Participants' test scores

The data displayed below come from the "diagnostic test" and the January examination in Analysis. The diagnostic test is a short test administered by the mathematics department in the students' first week at the university. It consists of sections on differentiation, integration, trigonometry and inequalities.

Because of the small number of participants in the study, the data is not amenable to statistical estimates of how well the sample represents the entire population. Even a comparison of means would be questionable when sample sizes are as small as 8 and 10, and would reveal nothing about the spread of the participants' marks relative to those of the entire cohort. Hence the data is displayed "raw" in the charts below. In each case, the marks of the corresponding study participants are listed in ascending order below the chart. The students on the mathematics and physics combined degree are treated separately since, as explained in chapter 2, information as to which of these students attended which course is not available. In the first two cases the students represent a reasonable spread of the marks for their group; both of the Physics students score low in the January examination.

new course diagnostic test (maths and maths/stats students)
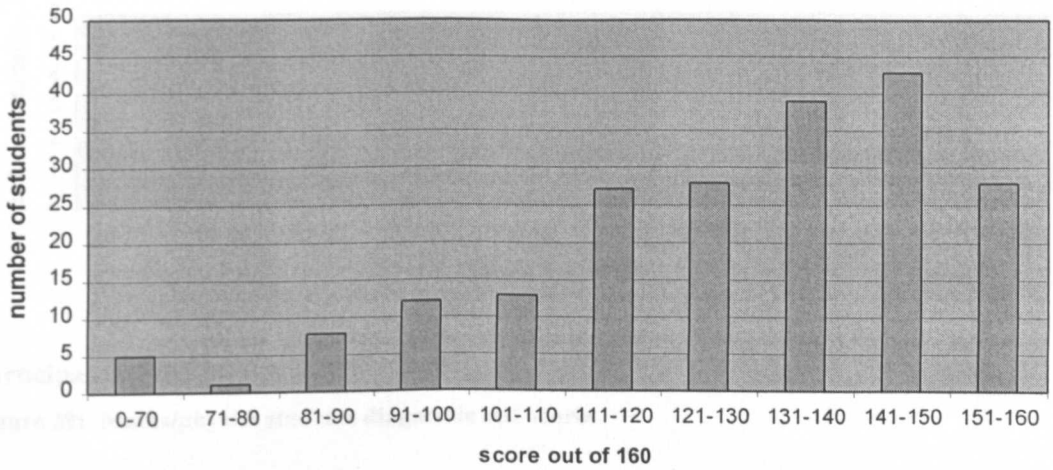
Participants: Kate 85, Cary 95, Adam 120, Dean 123, Greg 124, Ben 137, Jenny 139, Hugh 141

Figure 27: New course diagnostic test scores

lecture course diagnostic test (MORSE students)
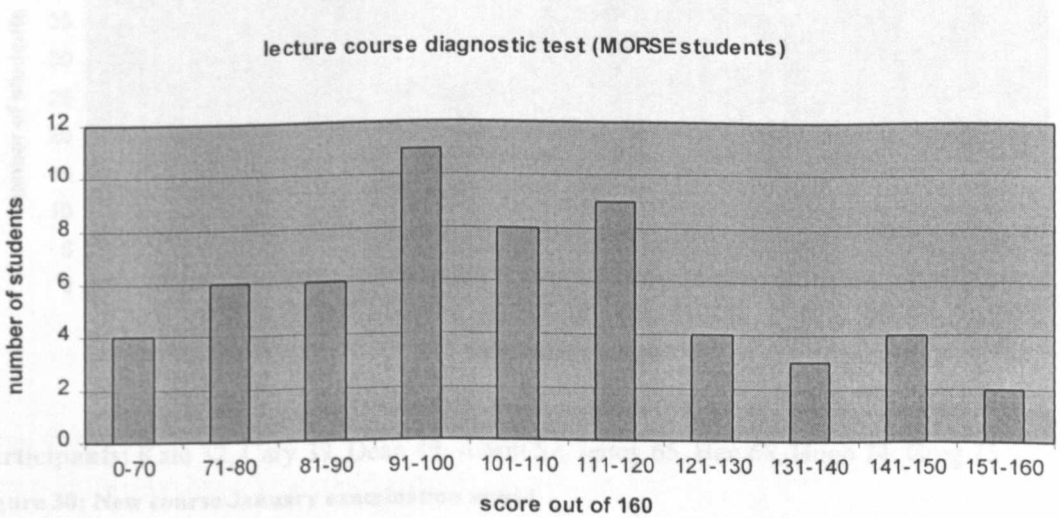
Participants: Vic 72, Tom 78, Unwin 84, Steve 100, Zoë 107, Yvonne 115, Xavier 133, Wendy 155

Figure 28: Lecture course diagnostic test scores
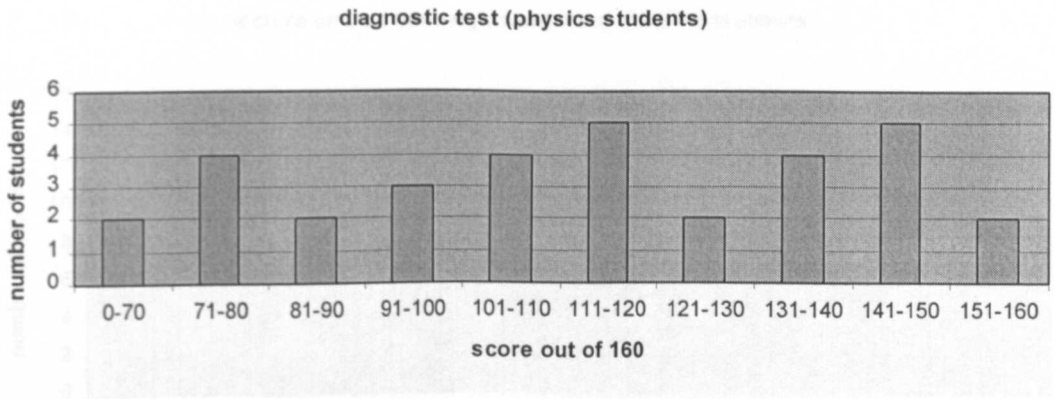
Participants: Fred 71, Emma 120

Figure 29: Maths/physics students diagnostic test scores

new course January examination (maths and maths/stats students)
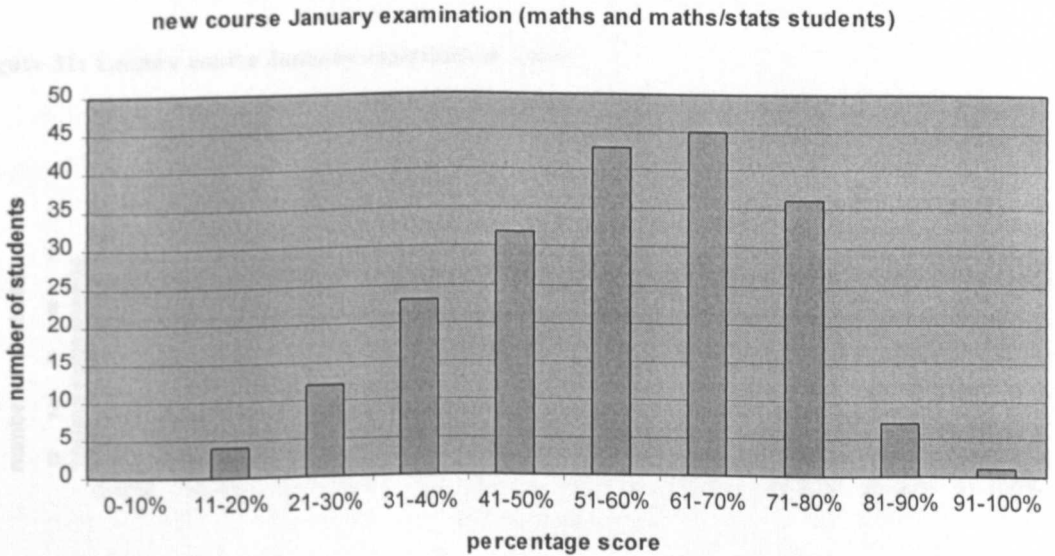
Participants: Kate 32, Cary 39, Dean 49, Adam 52, Jenny 65, Ben 68, Hugh 74, Greg 75

Figure 30: New course January examination scores

lecture course January examination (MORSE students)
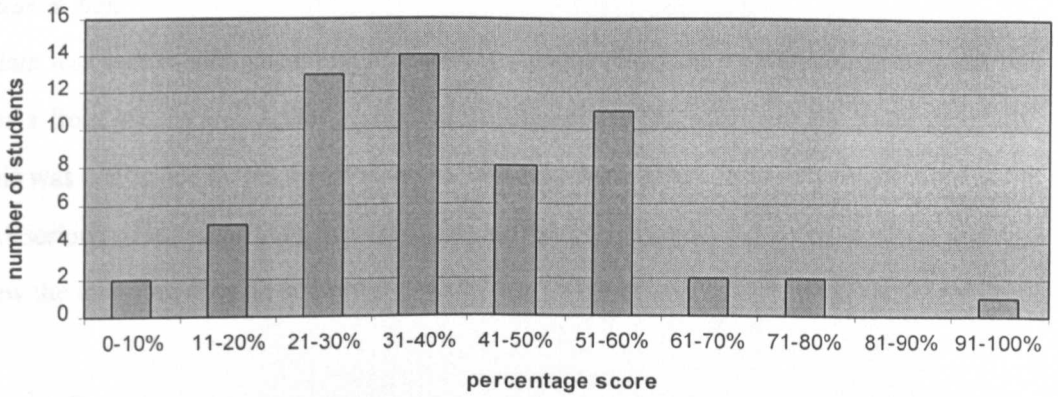
Participants: Vic 20, Yvonne 23, Steve 25, Tom 27, Zoë 28, Xavier 53, Unwin 59, Wendy 71

Figure 31: Lecture course January examination scores

January examination (physics students)
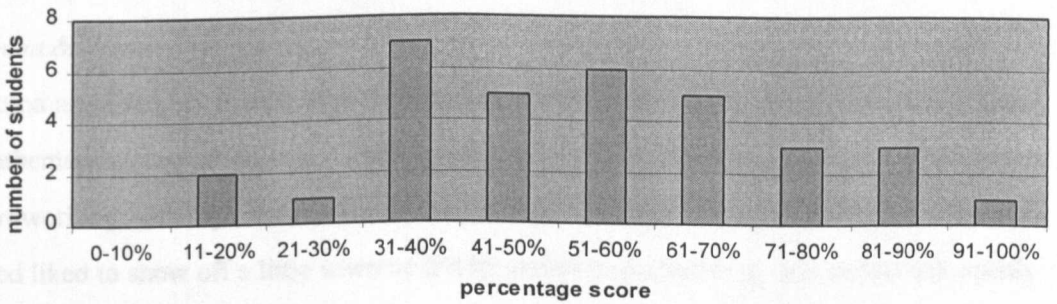
Participants: Fred 28, Emma 37

Figure 32: Maths/physics students January examination scores

A2. Profiles: reactions to the interview situation

Adam & Ben

Adam was somewhat cynical in interviews, in a rather charming way that suggested that this was a front and he was actually very enthusiastic about the work and life more generally. This was confirmed by his involvement in task-based questions. Ben was rather more polite and serious. He would join in with light-hearted conversation but seemed much more to view the interviewer as an authority.

Cary & Dean

Cary was positively effusive in interviews and would chatter a great deal both in general discussions and in task-based sections. This meant that he tended to dominate in interviews, to the extent that it was occasionally necessary to ask him to be quiet so that Dean could answer. Dean was less exuberant but always thoughtful and willing to contribute his thoughts and opinions.

Emma & Fred

Emma and Fred knew each other before they came to university, having been in the same mathematics class at college. Both considered Fred to be "cleverer" and Emma more hardworking, although their performance in interviews did not necessarily bear this out. Fred liked to show off a little when he felt he understood something, and would talk openly about his experience of the course. Emma tended to be more reserved: more specific probes were needed to gain more detailed answers.

Greg & Hugh

Greg was happy and enthusiastic in interviews. He was clearly enjoying the course and university life in general, and would chat to the interviewer as though to an equal. Hugh was much more reserved. He may have been slightly intimidated by Greg's quick uptake of the mathematics, and it was often necessary to use questions directed to him to draw him back into the discussion.

Jenny & Kate

Jenny and Kate volunteered together and obviously got on well, tending to be giggly in interviews. Both came across as determined in their work despite the fact that they thought others found it easier, and they willingly shared their experiences of the classes and their attempts to understand the mathematics. Jenny was very determined to understand the detail of the work, Kate more willing to accept that she would not understand all of it.

Steve & Tom

Steve and Tom lived on the same "corridor" on campus and volunteered together. Tom was the much more exuberant of the two; he seemed to be very much enjoying the social side of university life and did not apparently work very hard. Steve was quieter and more serious in interviews, and seemed to have a better awareness of how difficult he was finding the work.

Unwin & Vic

Unwin and Vic also lived together on campus and volunteered together. Unwin was the more hardworking of the two, although he often became nervous in interviews when asked about the material. He had missed the first week of term because of a death in the family, and had also had a year out before attending university. This meant that he was serious about working but clearly very worried about having to catch up. Vic was much lazier, expressing a lack of passionate interest in the work and in university life more widely. He became quite interested when engaged with a question, but it would take him some while become involved.

Wendy & Xavier

Wendy and Xavier attended the same college before coming to university, and volunteered together. Wendy was very chatty in general discussion, and would happily tell animated stories of her experience of university life. When discussing the work she came across as hardworking but not particularly interested in the mathematics for its own sake. Xavier was

more reserved and always considered in his responses. He would always answer questions but would not naturally expand these answers.

Yvonne & Zoë

Yvonne and Zoë had clearly become good friends since arriving at university. Both were giggly and seemed to be very much enjoying the social side of university mathematics rather than getting involved with doing more work than necessary. In interview tasks, Zoë would usually take the lead and be more serious and persistent in trying to answer. Yvonne often did seriously contribute to such attempts, although she would always be nominally engaged in the conversation.

Appendix B: Interview questions

B1. Week 2 (introductory interview)

Table 5: Week 2 interview questions

Introductory section	Have students agree to conditions	*This section based on the interviewer talking, so not addressing research themes but putting students at ease and making clear what is expected of them*
	Explain my position and what I want	
	Explain what I want	
	Explain what student gets in return (biscuits and help session at end of term)	
	Explain what will happen in interviews	
Life in maths	When did you start to like maths?	*Self*
	What was it like at school?	*Self, authority, beliefs*
	Why did you decide to carry on and do this degree?	*Self, beliefs*
Teachers	Who was your best ever maths teacher?	*Authority, beliefs*
	What did you like about the way he/she taught?	*Authority, beliefs*
	Who was the worst and why?	*Authority, beliefs*
Aims	What do you hope to get out of your time here?	*Self*
Learning (1)	What do you think is the best way for you to learn a new piece of mathematics?	*Beliefs, reflection, organisation*
Confidence	Are you feeling particularly confident or anxious about doing maths at university?	*Self*
	Has anything in particular made you feel like this?	*Self*
Learning (2)	Can you describe an incident in your learning of maths before university which stands out in your memory?	*Reflection, organisation*

B2. Week 3

Table 6: Week 3 interview questions

Introductory section (5 minutes)	How have you got on this week?	*Self*
	What have you been working on in Analysis?	*Acquisition, organisation*
	Can you make me a quick list of what you've done so far? (Have you done any work on sequences?)	*Acquisition, organisation*
	What does it mean for a sequence to tend to infinity?	*Concept image, primacy*
	Have you seen an actual definition of that?	*Concept image, primacy*
	Do you understand it?	*Primacy, logic*
	Why do we have it, do you think?	*Primacy*
Task (10 minutes)	LOGIC CARD GAME[38]	*Logic*
Task (20 minutes)	LATE COMMUTER PROBLEM[39]	*Heuristics, monitoring, success*
	Can you explain the answer back to me now?	*Following*
Reflective section on the work (5 minutes)	How are you finding proofs in general?	*Following, proof scheme*
	Do they seem necessary?	*Authority, beliefs*
	Who decides whether a proof is correct? Can you?	*Authority, self, beliefs*
Reflective section on experience (5 minutes)	Do you think you'll like university maths?	*Self, reflection*
	How does it compare to school?	*Beliefs, reflection*

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[38] Students were required to state what could be said about the colour of the reverse side of various cards, given certain restrictions. For example, given the possible colours *black, red* and *green*, they were shown a green card and asked what they could say given that "If the front is not green, the back is not red" (idea derived from Zepp, Monin & Lei, 1987).

[39] Problem reads thus: A commuter is in the habit of arriving at his suburban station each evening exactly at five o'clock. His wife always meets the train and drives him home. One day he takes an earlier train, arriving at the station at four. The weather is pleasant, so instead of telephoning home he starts walking along the route always taken by his wife. They meet somewhere on the way. He gets into the car and they drive home, arriving at their house ten minutes earlier than usual. Assuming that the wife always drives at a constant speed, and that on this occasion she left just in time to meet the five o'clock train, can you determine how long the husband walked before he was picked up?

B3. Week 5

Table 7: Week 5 interview questions

Introductory section (5 minutes)	How are you getting on in Analysis now?	*Self*
	Does it seem different from the maths you did at school? Can you describe how?	*Reflection, authority*
	Which do you prefer? Why?	*Beliefs, reflection, authority*
Task-based section (15 minutes)	SEQUENCE QUESTION[40]	*Heuristics, monitoring, success, primacy, concept image, limits*
	Why did you decide to make that your first approach?	*Heuristics, reflection*
Task-based section (10 minutes)	PROOF CORRECTION	*Proof scheme, following, logic, limits, concept image, primacy, authority, self*
Reflective section (5 minutes)	Can you explain it out loud now that it's corrected?	*Following*
	What did you decide was wrong at this point?	*Concept image, primacy, following*
	Can you remember the actual definition of "tends to infinity"?	*Concept image, primacy*
Further task (8 minutes)	CONCEPT MAPS[41]	*Acquisition, organisation*

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[40] Major questions in task-based sections are presented and discussed in section 4.4.2.
[41] The students were asked to draw concept maps for Analysis so far. They were allowed to use their notes during this time.

B4. Week 9

[The table for the week 7 interview appears in chapter 4.]

Table 8: Week 9 interview questions

Introductory section (5 minutes)	So term's nearly finished – how are you feeling about Analysis now?	*Self*
	Do you think you have learned anything? What?	*Beliefs, reflection*
Main task-based section (20 minutes)	SERIES QUESTION	*Heuristics, monitoring, success*
	Can you summarise those results now and explain how you got them?	*Following, proof scheme*
	Can you prove something you've used in detail?	*Acquisition, proof scheme*
Review section (5 minutes)	What exactly does it mean for a series to converge anyway?	*Concept image, limits*
	Are you more comfortable with this sort of definition now?	*Primacy, concept image*
	Why was/is it hard, do you think?	*Concept image, logic*
	What about proofs?	*Following*
Reflective section (10 minutes)	What have you gained from the lectures/classes and workbooks/homeworks?	*Reflection, authority, self*
	How long have you spent working on Analysis in an average week?	*[Practical]*
	What about your other courses?	*[Practical]*
	How well do you feel you understand all your courses? What's your measure for that?	*Reflection, self*
	Have these interviews affected the way you work or the way you think about maths would you say?	*Reflection*
Further task (12 minutes)	CONCEPT MAPS	*Acquisition, organisation*

Appendix C: Triangulation

C1. Coding manual

Space restrictions do not permit the reproduction of the entire manual. Hence an abridged version is included which gives the flavour of the most important sections. The full version may be found on the accompanying CD-ROM.

1.3 Coding the text

The following is a suggestion for how you should go about coding your transcript section. It is a guideline rather than a series of instructions; you are free to go about this in whatever way you see fit, but it might help you get started. A good piece of general advice is to keep "memos" at every level as you go along, about your current thoughts and anything that stands out. No-one else needs to see these so they don't have to make sense to anyone other than yourself! I expect the whole task to take two to three hours, although it can be mentally tiring and you may wish to do it in more than one stint.

1. Read through the section in full to get an idea of what it is about. Don't forget to make memos about anything which comes to mind, for example which major categories are represented.

2. Divide the text roughly into "episodes" where a certain person is talking about a specific topic or is explaining a piece of reasoning. (This is not necessary but it helps to break up the task.)

3. Write codes at whatever level you feel is appropriate for the whole episode.

4. Go back over the episode in detail, spending time thinking about each line and whether it is indicative of anything besides the coding for the whole passage. You can code any episode, line or even word at as many categories and subcategories as you feel is appropriate. Look out for peculiar uses of language, emotional content and difficulty or ease of expression as indicators of how the student might be thinking.

5. Remember to write memos to yourself – these can easily become the basis of the summaries.

6. Write a summary for each student for each major category. (Remember that you may not have information on all categories). Also you may like to add any hypotheses you have formed about links between the categories for these students, note anything which is conspicuously absent and/or indicate any points which you feel are particularly important or which particularly stood out.

2.2 The categories

UPTAKE (UPT)

Uptake is an umbrella term for the uptake both of the material of Analysis and for more general topic-independent ideas of advanced mathematics. *Note the emphasis here is on new material or at least that which was not a necessity at A-level.*

MATERIAL (MAT)

How much material from the Analysis course is the student picking up?

- RECALL (REC)

How much (recent) material from the course does the student recall?

  - How much can the student offer when asked what they are learning?
  - On what level of detail is this recalled material?
  - Is the recalled material fragmentary or does the student appear to have a coherent, global view?

- FACILITY (FAC)

Can the student work with the material?

  - Does the student know how to apply results/theorems/reasoning from the course?

- Can the student manipulate and use definitions?
- Can the student express themselves using the new terms?
- Does the student recognise situations in which new knowledge might apply?

● MEANING (MEA)

Does the student have a meaningful understanding of the material?

- Does the student have a meaningful understanding of the new terms and their definitions?
- Does the student understand the ideas of the course (informally)?
- Does the student appear to understand the links and differences between concepts and results?

● ERRORS (ERR)

Does the student seem to make any particular errors, or have any particular misunderstandings, either in recall or in use of the material?

3. Illustration of transcript coding

3.1 Introduction

The transcript section is divided into four episodes, with the sentences in italics briefly describing what happens in each. The codes for the categories appear in the third column, and in the fourth are my descriptions of what is going on which should clarify why those codes appear. Please note that this section has been chosen to illustrate a large number of categories - your section may not cover so many.

Summaries by category for both Steve and Tom follow.

3.2 Information about the transcript section

The transcript is taken from an interview in week 5 of term 1.
Steve and Tom are discussing the sequence

$$\left(\frac{1+\cos n}{nx}\right).$$

(In fact they have been discussing this between themselves for some time; this is the point where I join in and attempt to guide them to a solution.)

This sequence tends to zero as *n* tends to infinity for all values of *x* except 0, for which no term is defined.

The students should be able to establish this using the squeeze rule, multiplication by a constant and the known limit of the sequence (1/n).

Tom has stated that the sequence always tends to zero.

We begin as I have just asked Steve his opinion.

3.3 The transcript and coding

1. Steve talks about positive and negative values of x.

Table 9: Illustration of coding (manual)

S:	Well if...if x is positive then...it definitely increases to infinity...	UPT/ TID/ HAB	Steve's explanation is imprecise; he doesn't clarify what "it" is.
I:	What does?		
S:	The, sequence. No, if x is positive then the bottom,	UPT/ MAT/ FAC	Steve confuses the bottom and the whole expression, suggesting that he does not have facility with the new material.
I:	Mm,		
S:	Tends to infinity.		
I:	Yes.		
S:	So the top is just irrelevant.	UPT/ MAT/ MEA	Steve does seem to have informal meaning for the relative importance of finite and infinite quantities however.
I:	Mm.		
S:	But if x is, negative...		
T:	It increases to minus infinity.		
S:	Yes.		
	Pause.	UPT/ MAT/ FAC	
I:	Which means...?		
T:	The same.		
I:	The same. Does it Steve?		Throughout **Tom** is

S:	Mm?		confident and is largely correct.
I:	Does it mean the same, thing happens?	PRS/	
S:	No.	GSP&	
I:	Hm?	ENG	
S:	Might – well, it can either tend to minus infinity or infinity.		Steve is uncertain and loses track of the argument.
I:	What about the whole thing though?	UPT/	
	Pause.	MAT/	
S:	Erm...that just depends on, whether x is positive or negative.	REC& FAC	
			Steve still has his erroneous belief that the sign of x matters by the end of this section.

Summary for Steve

UPT/TID   Steve's speech is imprecise; he does not exhibit the behaviour of clarifying which objects he is discussing as a mathematician would.

UPT/MAT   Steve seems very uncomfortable with the material in general. He becomes confused and makes erroneous judgements in his own reasoning and does not appear to follow Tom's very easily. This seems to indicate that he is lacking either in meaningful understanding of the concepts involved or in facility in reasoning about these concepts, or both. The fact that he does not contribute a great deal unless prompted is consistent with this interpretation. He does show signs of an informal understanding of arguments similar to those in the course but only when they are explicitly pointed out to him.

PRS   Steve does not seem to have a good grasp of the problem or be particularly engaged in solving it. [In this case this is likely to be due to his inability with this material.]

Overall impression: Steve is not adept with the new material and is aware of this, which makes him disinclined to engage and really try to contribute. The feeling is that he is no longer expecting it to make any sense and has become detached because of this. (So implicit ROL/CTR and TRN).

Issues raised: once a student has become detached this presumably inhibits their progress since they will not try so hard to understand further material.

C2. Triangulation documents

The examples below illustrate the coding by other researchers and the subsequent discussion. Presented are one page of coding of the opening section of CD9 by Hazel Howat and a discussion sheet comparing the coding of Lara Alcock, Hazel Howat and Chris Bills.

D:	One thing, I would say, that I've learnt, across the whole of the courses is proofs.
I:	Yes?
D:	Because, before I mean, we were occasionally shown a proof, but it's just kind of, if you're interested in a proof that it definitely works and you, didn't...
I:	Yes.
D:	Whereas, we've done a lot more now.
I:	Yes.
D:	So, that's kind of, Foundations as much as Analysis.
I:	Yes. So when you say you've *learned* proofs, what do you mean by that?
D:	Well, learned to produce them. On your own, and, learned some standard methods,
I:	Mm,
D:	Er...
I:	And do you feel confident with that now?
D:	Erm, I wouldn't say I was totally confident, but gradually getting more and more.
I:	Mm. What do you think Cary?
C:	Yeah...yes it's okay actually. Like I said at that point where I just suddenly understood it,
I:	Right,
	And that's...
I:	Yes. Can you put your finger on what it is that's made it, become easier for you Dean? To do that sort of, proofs and stuff.
D:	Erm. I think it's just practice.
	Mm.
D:	I think. Erm...yes because just, I don't know there's usually one or two little, things in a, lots of it's just kind of just manipulation but usually one or two little twists in a proof,
I:	Ah right, yes.
D:	Erm, may, maybe it's just practice.
I:	Yes.
D:	Because, I don't know. No two proofs are the same, or not often the same,
I:	Yes,
I:	And yet something...
D:	Yes.
	Yes.
D:	Maybe it's just the way we're thinking. Different now, to when we arrived.
I:	Maybe it is. Do you feel like you're thinking differently?
D:	Erm...I don't, I don't feel, like I am. But I probably am.

FOR/ PRF Identifies proof as content running thro' all courses

TRNS/ PVS Identifies use of proof as a difference between university maths & school maths.

FOR/PRF
ROL/WIM LER Implies learning procedures for proof

AFF/ TRN/ PVS Feeling confident Seems to be improving confidence with university maths.

TRN/ PVS Cary seems to have held point at which maths being done starts to make sense.

ROL/WIM/ LER Dean sees practice as important - uni maths has become easier for him

FOR/ PRF Implies proof a step by st procedure?

ROL/WIM/ LER
FOR/ PRF Practice necessary proof

TRN/ PVS Dean feels that he is thinking differently from when he was at university.

Figure 33: Hazel Howat's coding of a transcript excerpt

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Coding triangulation

Interview section: CD9   Student: Dean

Coders present: Chris, Hazel

Areas of agreement

Category	Student's position
AFF	Gaining in confidence (with prog). Enjoys doing work of Analysis.
BKG	Humour. Interviewer an authority but confident to ask me a question.
TRN	Recognised introduction of prog as new to him as an activity. Thinks he's thinking differently, so do we, as in he seems to be coping me with the transition.
FOR	Overview of process of prog
MAT	Overview recall demonstrated
ROL	Wants control. Maths still has outside purpose though is enjoyable.

Areas of disagreement Thinks practice has helped him.

Category	Disagreements	Resolved?
FOR	Whether he does progs procedurally. In this section, stuff about method, practice, "learned" prog can be associated with procedural learning. But also could be more an overview of doing progs.	I agree you can see this 'here'. Difference in backgro of coders.
ROL	Feeling from L & C is getting control but difficulty finding specific evidence	

Behaviours coded in different categories by different coders

Categories	What coded there?	Resolved?
MAT, FOR	as with Gary	
FOR, TRN	prof issues, realising its an important thing	

Comments on using the category system

Didn't take too long, no inexplicable disagreements.
Hazel liked the questions to 'describe the categories, thought it was clear.
Problem is boundaries of categories.

Date: 6.7.00   Signed:

Figure 34: Comparing the coding of three researchers

Appendix D: Examples of documents produced during data analysis

All documents refer to the week 5 interview with Wendy and Xavier. Where only extracts are given, these are from the opening discussion. Where only one student's information is given, this is Wendy's.

D1. Open coding

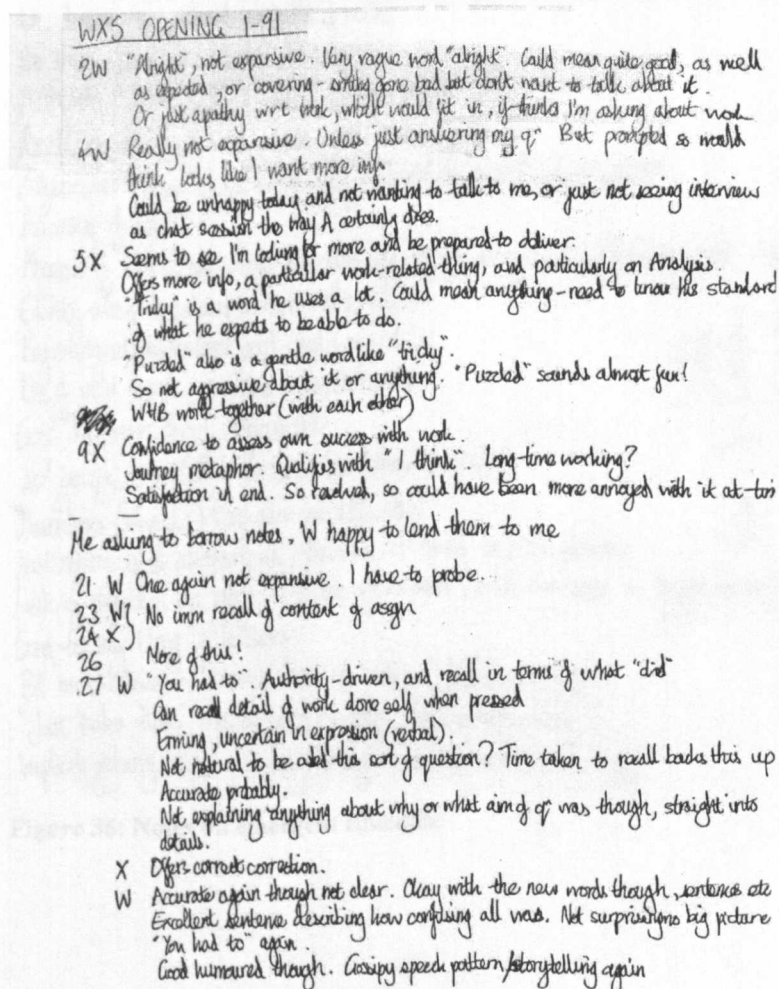Reproduction of original open coding notes on the opening section of the week 5 interview with Wendy and Xavier.

WX5 OPENING 1-91

2W "Alright, not expansive. Very vague word "alright". Could mean quite good, as well as expected, or covering - something gone bad but don't want to talk about it. Or just apathy wrt task, which would fit in, if thinks I'm asking about work

4W Really not expansive. Unless just answering my q? But prompted so would think looks like I want more info. Could be unhappy today and not wanting to talk to me, or just not seeing interview as chat session the way X certainly does.

5X Seems to see I'm looking for more and be prepared to deliver. Offers more info, a particular work-related thing, and particularly on Analysis. "Today" is a word he uses a lot. Could mean anything - need to know the standard of what he expects to be able to do. "Puzzled" also is a gentle word like "tidy". So not aggressive about it or anything. "Puzzled" sounds almost fun! W&B work together (with each other)

9X Confidence to assess own success with work. Journey metaphor. Qualifies with "I think". Long-time working? Satisfaction if end. So radical, so could have been more annoyed with it at-tn

He asking to borrow notes, W happy to lend them to me

21 W Once again not expansive. I have to probe...

23 W} No imm recall of content of assgn
24 X)

26 More of this!

27 W "You had to". Authority-driven, and recall in terms of what "did"
Can recall detail of work done self when pressed
Erring, uncertain in expression (verbal).
Not natural to be asked this sort of question? Time taken to recall backs this up
Accurate probably.
Not explaining anything about why or what aim of was though, straight into details.

X Offers correct correction.
W Accurate again though not clear. Okay with the new words though, sentences etc
Excellent sentence describing how confusing all was. Not surprising big picture
"You had to" again.
Good humoured though. Gossipy speech pattern /storytelling again

Figure 35: Open coding notes

D2. Emergent concepts

Reproduction of original notes of emergent concepts made while open coding opening section of the week 5 interview with Wendy and Xavier.

WX5 OPENING 1-91   EMERGENT STUFF

Unexpansiveness at beginning again?
Poss interpretation of q's as narrow maths-restricted
Prob of not knowing their expectations when say it was hard
Willingness to admit to having difficulties
Emotion about these difficulties.
W/B working with other people.
Conf in assessing own work
Qualification of statements; mild form of covering or just quest for accuracy?
Unexpansiveness about details of work
No imm recall of content of assgn
Authority: "you had to" when talking about work
Recall of detail (of work done self) when pressed
Unfamiliar type of question, being asked to recall what did?
Accuracy of recall
Absence of big-scale what-the-aim-was in q"; very much following small instruc
Clarity when speaking about mathematics
Beginningness of whether good expln or not
Use of new words, sentence-structure etc.
Good humour about difficulties
W gossipy/storytelling general 'you' speech pattern.
Describing (vividly) confusion on difficult q"
Instinctive good assessment, then not wanting to overestimate
Lack of reflection on difficult thing afterwards; still confusion in trying to explain
Conf to say Uing in lectures
Bit more active in lectures than in WX3? Bit more grippy?
"Just takes time" phenomenon/feeling. Implicit here though.
Implicit getting-used-to-way-things-are-around-here stuff

X has opinion-qualities all the time.

last time v
Perception of self as making progress

Figure 36: Notes on emergent concepts

D3. Categorising emergent concepts

Reproduction of original organisation of emergent concepts into categories for whole of week 5 interview with Wendy and Xavier.

Figure 37: Organising emergent concepts

D4. Memo summarising interview

Sample page of memo kept while coding whole of week 5 interview with Wendy and Xavier. The full memo and those for the other interviews can be found on the accompanying CD-ROM.

```
*WX5

*Wendy and Xavier, Friday 2-3 Week 5


+++ Document Memo:

9:47 am, Jul 31, 2000.
```

At the very beginning we see Wendy being generally quite noncommittal compared to some of them about how it's been, and Xavier going straight into talking about the maths, so that type of reaction to me.

They remember a homework being hard but not immediately what it was about, and then we get Wendy talking at the details of what you had to do level rather than the conceptual level. She was obviously quite confused by it, talking in her usual anecdotey way. It's all "you had to" as well (to 59).

It really does seem to be a big thing with Wendy that if she loses it for a minute in lectures she's completely lost. I'm not sure what this shows. Perhaps that she is still trying to keep up with everything as it goes, which is good, but perhaps also that she's not seeing the larger structure of what's being done, staying at the detailed line-to-line level.

When Wendy doesn't get into stories she's pretty quiet, just replying with one word answers (to 121). Both seem just not to be the type of people to get all enthused about things generally, their outside activities not being talked about in long

answers either. This could be general personality or it could be partly an interview reaction (to 137).

Xavier talks about the lack of examples. It's not clear but I think he is talking about worked examples because of the previous discussion about questions and what the work in school was like (to 173). He talks about it being more difficult to "get hold of" proofs, but unfortunately I don't ask him to expand on what this means.

The two of them are humorous about the pens. They're just quieter in general than a lot of them.

Wendy very quickly grasps what the "when" in the problem refers to, and Xavier illustrates this with a generic example. She also picks out that it won't work at nought. They really get sensibly into this very quickly. This might be a good illustration for something at one of my talks. Xavier seems to have a good grasp on how the work that they've done applies to this problem.

It seems that Wendy's approach is more to think about what's happening visually and try to work it out, Xavier's to use ideas from the course (to 288).

They seem pretty inclined to write things down. I think these guys are pretty well trained in some ways, and they do seem to be doing pretty well with the ideas from the course at this stage (to 331).

We get the point where they are unwilling to commit themselves, not apparently because they don't think they're right but more out of a concern about possibly being wrong and an urge to hedge against this (to 339).

At a request to prove in this context, Wendy says "something like" and introduces the usual bit of the definition. I wonder whether this is somehow overly associated with the "proof" cue at this stage for a lot of them? The approach definitely changes at this request to trying to remember things that they've done, so some of that "this is what the

authorities want when they ask for this" thing.

Wendy actually gets the for all n bigger than big N part of the definition too (to 403). They don't really use it though, going back to using strategies from the course, correctly but not really seeming to link the two, Wendy saying they "haven't really but an epsilon in it anywhere, but yes!" when asked if they're happy with it.

In the proof correction Wendy spots where the problem really is at 545, and then Xavier makes the same mistake as Kate (I think) did in assuming it's a comparison with root of n+1.

D5. Summary for Wendy

Summary of the week 5 interview for Wendy, split according to main categories prior to axial and selective coding.

BKG

Wendy has a mix of speaking styles in the interview. Sometimes she's gets into anecdotey amusing stories about, for example, how difficult a question was. Other times she just answers my questions directly with one-word answers. Either in the interview or in general she seems not to be an exuberant person, not getting into chatting about her outside interests either - things tend to be "okay". She does have humorous moments, it's not like she's miserable, just not generally enthused about things.

AFF

Wendy seems pretty happy with how things are going. She says that the work is doable if you read through it and then leave it for a couple of days, and having a sense that she understands lectures now and it's all sinking in, so some progress.

PRS

Wendy quickly gets a grasp on the first problem and also spots the problem with dividing by zero. She draws pictures and tries particular cases. Her language is a bit hedgy when it comes to committing herself on anything (see ROL and UPT).

TRN

Wendy compares that they had a lot of questions to do at school whereas now it's all proving "theorems about things". She seems not to have grasped that it's not okay to just take square roots in the problem, so not really aware of this goal of building it all up and checking everything perhaps. The proof-checking shows up other problems (see UPT and FOR).

UPT

Wendy has an interesting mix of abilities and weaknesses. At the beginning she talks
at the detail-no concepts level about a piece of work she found difficult. She has
some strategies in the first problem (see PRS) although it's really Xavier who leads
with the introduction of ideas from the course. She can remember most of the
definition of

convergence (see also FOR). She makes some quite serious errors in the proof
correction, apparently thinking of this as for only one value of n and misusing n in
her notation where she also lacks precision. She also suggests an inappropriate
alternative method relying on square roots. She has a very good case of this classic
problem of a lack of mathematician's method in that she can say something is wrong
with the proof but not what it is (see also FOR).

FOR

Wendy is aware of the definition of convergence to the point of remembering the n>N
part but not the for all epsilon. She seems to introduce it at a request to prove,
reasoning prior to this taking place in the first problem by way of visual and
example-based reasoning. She does not do well with the proof correction, being able
to say there is a jump between the third and last lines but not to do anything about
it (see UPT). Could be under misconception that since the result is right the proof
does give something towards it, feeling that there is a gap rather than something
fundamentally wrong. After I have pointed out what is wrong she still is not sure
and ends up saying that it proves it "in a roundabout way". She thus appears to have
something of an idea of what a proof about these concepts should contain without the
meaning attached or a sense of what the criteria are for something actually being
proved. Also bad correspondence in tthat she says something brief about needing less
than or equal to for convergence, suggesting she perhaps thinks the terms are
eventually all equal, and agrees with Xavier that if the sequence is strictly

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increasing then it does tend to infinity.   Proving that something is convergent appears to be assimilated as a procedure: "that's right isn't it, to prove that it's convergent?"

ROL

The maths comes from elsewhere for Wendy, who talks about the need to concentrate in lectures and what "you had to" do.   When it comes to reaching a conclusion in a problem she becomes hesitant even if she has sounded sure before, unwilling to commit herself.   So a lack of confidence in her own judgement, which also fits with the idea of

arbitration coming from elsewhere.   She's horrified at the thought of someone staying up till 4am doing Analysis.   She's had a tough week trying to get the Analysis done but is pleased "with how it is" when she's handed it in, so she's not thinking she's done nothing.

D6. Summary table for Wendy

Table summarising four main categories across all four standard interviews for Wendy. Corresponding tables for the remaining participants may be found on the accompanying CD-ROM.

	WEEK 3	WEEK 5	WEEK 7	WEEK 9
Uptake	Can say what they've been doing but she doesn't express it well. She remarks on the new notation as a reason why it's sometimes hard to follow.	Speaks at detail level about a piece of work she found difficult. Has some strategies and can remember most of definition of convergence but mis-reads and -uses notation in proof correction. Lacks ability to pin down what is wrong with proof.	Explanations often unclear. "You use" a lot for something learned, so procedural. Thinks of sequences as graphs but with domain R and tries to use tan to find a sequence with an infinite term. Implies error on if...then statements but can correct.	Material all "spinning round" in her head at the moment. Can use case analysis, generic examples, informal reasoning and particular examples from course in problem. Explanation of alt series test is details without what this shows.
Transition	Finding course much harder since beginning; mostly okay but some things go over her head. Thinks she'll like the maths "as long as it's not all proofs"; no identification with aims of formal maths; prefers stats and being told "the formula".	Compares doing lots of questions at school to "all proving theorems about things" here. Suggests an inappropriate strategy in the proof correction suggesting she hasn't grasped the idea of building up and not assuming things.	Not really made transition to university maths. Uses visual reasoning to an inappropriate extent and seems not to think of results in general. Struggles to describe what maths is about now. She'd be happy just to accept things still.	Feels made progress and got into the thinking, but describes material gained rather than this. She's no longer thinking "oh no not another proof!" Notation and applying earlier things appear as new, second suggests not much of this in school.
Formalism	Description of tending to infinity very informal though does relate it to the value of n. Can't remember the definition. Except induction proofs something you "follow through" at the moment. No internal need for proof; happy to be told thing.	Begins with visual and example-based reasoning, introduces most of definition at request to prove and seems to have this as procedure. Feels gap in proof correction, but bad correspondence with definition for tending to infinity and after correct answer on leading question still says this proves it "in a roundabout way".	Persistent visual image for convergence. Definition as criterion and proof as moving on from that. Worried reaction to "proof" but goal of general proof eludes her. Describes picture without deducing from properties. When I guide them seems to think of doing this for a particular sequence. Exasperated by me pressing for detail.	Convergent series as eventually constant: not shaken by counter-example but I make her uncertain eventually. Proving not internal, she likes to see applications. Uses generic examples a lot; does it well but doesn't check choices appropriate as a mathematician would. Get an "I would say" rather than formal checking.
Role	Mentions problem of trying to follow beginning and losing rest; has to follow things through after lectures. Some things (epsilon over 2) mysterious. Satisfied when feels could do similar question after supervision. Gives usual judge-ment criterion. Does most of work alone, asks Xavier or maths neigh-bours for hints. Maths has a point.	Maths external; needs to concentrate in lectures and talks about what "you had to "do. Unwilling to commit herself to answers even if confident in working. Had a tough week trying to get the Analysis done but pleased to have handed it in.	Worried by assignments till cover work. Doesn't try to improve own work in interview. Maths practical, thinks of working with particular sequences. Ends up doubting she has right answer when I keep pushing, asks what I "want" rather than sharing my goal of proof. Gives usual "faffing" criterion for judging whether something right.	Likes seeing theorems applied. Happy to believe me about some-thing can't remember. Informal reasoning and attempted recall on problem; wants a specific page of the notes; no attempt to reconstruct. Assessment of courses by quality of explanation, and understanding by ability to do questions and follow line to line.

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D7. Selective coding for week 5 interview with Wendy

Excerpts drawn from the week 5 interview with Wendy and Xavier during the selective coding stage, to highlight Wendy's work with specific objects and whole categories, use of formal definitions and relationship with the authorities.

I: Can you remember what it was about?

Short pause.

W: You had, you had to erm, find an increasing and a decreasing sequence that erm, converged to the same limit,

X: Subsequence, wasn't it?

W: Yes, subsequence and prove that the, sequence would converge to the limit. And it was like, we ??? find one that, a decreasing sequence that converged to the same limit as an increasing sequence. And all this other stuff going round that, you had to use and...oh no!

W: Well at, erm, school and college, we had loads and loads of questions to do,

I: Mm,

W: Erm, you don't really seem to get any questions in Analysis really it's all erm, proving,

I: Mm,

W: Erm, theorems about things, and...

W: X remains fixed...that'll be for which values of x is that convergent. Won't it?

Pause.

X: So it'll be like 1 plus cos n over 2 x.

W: Yes. No over 2 n.

W: Well if it's nought, it won't be, it won't work will it?

W: Hm...well if you draw like the graph...(draws) that's the cos,

X: Mm,

W: And you have 1 plus cos, would be that...

W: Well do you know that that actually, converges? Because if you draw...1 plus cos n, over n...so you divide that by n, and, must be something like that, won't it?

X: Yes but n's increasing, so it would get smaller and smaller as you divided it.

W: Oh, true.

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W: Surely it'll be the same for all the, reals, if you don't include nought. Will they all go to nought? They should do, shouldn't they?

I: Are you happy with that answer?

X: Yeah.

W: I can't think of anything else, so...

I: Could you prove that that's true?

W: Well, in theory, yes...!

I: How about in practice, Wendy?! (laughs) Would you have any idea where to start?

W: Erm, you'd have to show that, a to the n, minus the limit, was less than – the mod of it was less than, epsilon. Something like that.

W: That's right isn't it, to prove it's convergent?

X: What, subtract the limit?

W: Yes.

X: Mm...which means that a n, yes. So if they get, smaller...

I: Can you just read out what you've written?

W: Erm, modulus of a to the n minus the limit, which is zero, has got to be less than, epsilon.

I: Mm.

W: And, it's taking zero out, the modulus of a to the n has got to be less than epsilon, for all, n is bigger than, big n.

W: It's got to be less than or equal to, 2 over, n x, is it?

X: Mm-hm.... Say n x tends to infinity. Assuming it's constant.

W: Yes. No 1 over, n x goes to, zero.

X: Mm. The bottom goes to infinity. Well minus infinity, if x is negative...

W: 2 over n, x...infinity...(writing) goes to zero...goes to infinity...and because x is fixed...
Pause.

I: Are you happy with that?

W: Well we haven't really put an epsilon in it anywhere, but yes!

W: Oh. Well you know that, that's always less than that. So if that tends to infinity, then that tends to zero, that's got to, tend to zero.

X: Because it's,

W: By the squeeze rule.

W: Yes, we've done that.

X: Yes...

I: Done what sorry?

W: We had the homework on that.

X: Yes.

I: The top line?

W: Yes.

W: No, because that wouldn't necessarily,
because root n is less than – they need
to show that something smaller goes to
infinity. And then, by...you'd be able
to say that root n would.
Pause.
More pause.

W: Although that does show, that it's
always increasing. And if it always
carries on increasing...

X: It's going to tend to infinity...

W: Mm.... They need to show that this,
for more values though. They need to
show by erm, induction or something
that it's increasing.

W: Needs to show that...the sequence is always
increasing,

X: So n tends to infinity therefore, root n...no.
Short pause.

W: You could do it, for the sequence of n, n
plus 1, n plus 2. Check that that goes to
infinity.

W: They haven't proved that it, actually does
tend to infinity. They've just said that root
n, is less than root n plus 1.

W: You could just take the erm...a n, square it,
so you get like, n, the sequence n, n plus 1, n
plus 2,

X: Mm.

W: Show that that goes to infinity, and then say
therefore if you, root it, the er, a n will go to
infinity.

X: So, the sequence n, n plus 1, n plus 2 tends
to infinity, so...

W: Because it's an increasing sequence, and it's
not bounded above.

I: Mm. I think the previous line says, each term's bigger than the one before.

W: Yes, yes.

I: So does that show it tends to infinity then?

W: Well it could be convergent.

I: Ah.

W: But then it would be less than or equal, to... (uncertain). So yes it shows that it's strictly increasing.

I: I...don't think so. I'm sorry I'm in a mean mood this afternoon. What about he sequence *minus* 1 upon n?

W: Minus sorry?

I: 1 over n.

Short pause.

X: It'll tend to zero.

I: Mm.

Pause.

I: You see I think that's a strictly increasing sequence though.

X: Mm, true.

W: Mm, yes. (sounds unhappy!)

I: Can you remember the actual definition of a sequence tending to infinity, while we're at it?

W: Erm...I don't think I can! Erm...

I: How about the definition of a sequence having a limit? Being convergent, that sort of thing?

X: Convergent and bounded...

W: It's, if it's convergent...coming back to this again! The modulus of a n minus the limit, will be less than, erm, epsilon which is a positive number.

I: Right.

W: For all, little n is, bigger than big N. You can find a big N where, erm, the, term a to the n, minus the limit,

I: Mm,

W: Is so small that, it's just...

Appendix E: Teaching Zoë

Transcript illustrating the interviewer's "teaching", used occasionally when a student struggles with a question. Also demonstrates Zoë's capacity to understand the work when asked the right questions.

I: So, what does it mean for a sequence to be convergent anyway?

Z: It means, whatever values you put in,

Y: Goes to a limit.

I: As what happens?

Pause.

Z: As n, what...?

Y: It gets closer and closer to,

Z: A value.

Y: A value.

I: Right. In this case, what will the sequence a_n be? What will the terms be?

Pause.

Z: What do you mean?

I: Well what will a_1 be, what will a_2 be, what will a_3 be?

Z: Er, erm, well it'll be 1 plus, cos of 1 over x.

I: Yes.

Z: Or,

I: Do you want to write that down?

Z: Erm...(writing). And then a_2 will be...do we know how to work that out? What's – we can't really do any more. a_2 equals...and then it just goes on and on and on.

I: Mm.

Pause.

I: So what are we talking about when we're talking about that sequence being convergent?

Pause.

Z: As n, as n increases,

I: Mm,

Z: Erm, whatever this comes out as,

I: Mm,

Z: Erm...will, tend to a value. Will end up, getting closer and closer to a value.

I: Yes. So, are we talking about n going to nought, or n going to infinity, or n going to minus infinity, or...?

Z: We're talking about n going to infinity, aren't we?

I: Yes. We are.

Z: Right, yes.

I: But you were talking a minute ago about n going to zero, sometime...

Z: I know!

Laughter.

Z: I just didn't understand...I see, yes.

<div align="right">YZ5, 597</div>

Appendix F: Examination scores

[Appendix A1 gives the participants' scores in comparison with the cohort.]

F1. Participants' January and June examination scores

Table 10: January and June examination scores

New course			Lecture course		
	January	*June*		*January*	*June*
Adam	52	77	Steve	25	25
Ben	68	69	Tom	27	17
Cary	39	44	Unwin	59	46
Dean	49	31	Vic	20	38
Emma	37	33	Wendy	71	68
Fred	28	13	Xavier	53	57
Greg	75	74	Yvonne	23	---[42]
Hugh	74	35	Zoë	28	12
Jenny	65	92			
Kate	32	44			

F2. Detailed examination marks

The list below gives a summary of the parts of the January examination questions, marking these C or S according to whether they require work with whole Categories or Specific objects. The table gives the participants' [43] *marks for each of these parts (each whole question is marked out of 25, the result from the best 3 questions is scaled up to give the final percentage mark.)*

1a: definitions of sequence properties   C

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[42] Yvonne dropped out of the course before the June examination.

[43] Fred's paper is not available so his work is not covered in this section.

1b: stating limits of sequences (no proofs)   S

1c: finding limit for unfamiliar type of sequence   S

2a: proofs of simple sequence results   C

2b: deriving limits of familiar types of sequence   S

3a: explanation and partial proof of integral test   C

3b: use of integral test   S

4a: proofs of series convergence from definition   C

4b: series convergence using tests   S

Table 11: Breakdown of examination scores

Question	A	B	C	D	E	G	H	J	K	S	T	U	V	W	X	Y	Z
1a (6)	6	6	6	2	6	6	6	6	6	6	0	6	3	1	5	2	6
1b (13)	12	13	10	13	10	13	9	12	11	4	11	12	7	13	8	5	12
1c (6)	2	2	0	2	0	0	0	3	2	0	0	1	0	1	2	2	2
2a (13)	13	13	4.5	11	4	13	2.5	6	2	-	4	7	2	10	5	4	1
2b (12)	3	11	3	4	3.5	2	7	0	3	-	2	6	0	9	8	2	0
3a (15)	-	-	5	5	-	-	10	12	-	4	-	11	3	10	-	-	-
3b (10)	-	-	1	0	-	-	1	10	-	3	-	1	0	9	-	-	-
4a (14)	3	3	-	-	3	14	3	-	1	2	3	-	-	-	11	2	0
4b (11)	0	3	-	-	1	7	3	-	1	0	0	-	-	-	1	0	0

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