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Applications of Numerical Computation Methods in Microeconomic Theory

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Abstract

The solution of mathematical problems by numerical analysis is a large, intricate subject in its own right, and the substance of many Ph.D. theses in mathematics. The advancement of numerical analysis and computer technology are clearly not mutually exclusive. Moreover this combination through the growth in computer software facilities is easily within reach of a researcher with no expertise in either numerical analysis or computer programming. In particular the Numerical Algorithms Group (NAG) based in Oxford provides a library of subroutines for incorporation into source programmes across a broad spectrum of mathematics. The relevance of this development for the economist lies with the considerable scope for providing quantitative evaluations of microeconomic models outside of traditional statistical methods. To justify such a claim the thesis develops a number of applications from microeconomic theory: imperfect information in a non-sequential search framework; optimum tax with endogenous wages; a two sector general equilibrium model of union and non-union wage rate determination; Chamberlin's welfare ideal; and a quantity setting duopoly analysis of the structure conduct performance paradigm.

It is hoped that the insights gained from such diverse topics will convince the reader as to the appropriateness of applying numerical computing to microeconomic questions in general, and the usefulness of the NAG software in particular.

Acknowledgements

For their help and encouragement, as supervisors in different time periods, I should like to thank Avinash Dixit and Norman Ireland. At many stages of this work I have received helpful advice from John Cable, John Craven, Richard Disney, Tim Hopkins, Andrew Oswald, Bill Smith, Nicholas Stern, Mark Stewart, David Ulph and Kenneth Wallis. I should also like to express my appreciation to Nicola Garrish for an excellent typing job on a difficult manuscript. Finally, I should like to express my gratitude to my wife, Karin.

CONTENTS

Preface

The contents of this thesis reflect to some extent my initial research experience at the Fraser of Allander Institute, University of Strathclyde, which consisted mainly of computer-based projects, for example, Bell and Carruth (1976). While a graduate student at the University of Warwick, I provided programming assistance for Nicholas Stern's work on 'Optimum Taxation with Errors in Administration'. This was linked to the Social Science Research Council financed project on Taxation, Incentives and the Distribution of Income. By then I felt that the continued development of 'easy to use' algorithms through the Numerical Algorithms Group (NAG, 1981) library offered considerable scope to enhance theoretical work, and provide useful insights. The thesis develops a number of applications which, I hope, may support such a claim.

I have discovered that the ability to write computer programmes leads naturally to cooperation in research, which I personally enjoy, but creates the difficulty of using this work, fairly, in a thesis. Two chapters reflect this situation. Chapter 4 is based on a paper by Carruth and Oswald (1982). I have restricted the presentation to a small subset of the work which highlights the numerical computations, and can stand on its own with respect to the analytical content. Chapter 6 does not afford this luxury, so it would be right and proper to roughly indicate the areas of responsibility. It is based on a paper by Cable, Carruth and Dixit (1982). Computations and graphics were my responsibility; the style of presentation reflects Cable and to a lesser extent myself and Dixit. The original framework was due to Dixit (1979). I am indebted to my co-authors for allowing me to make use of this work. I accept sole responsibility for the way in which it has been presented in the thesis.

The diversity of subject matter has given each chapter a measure of autonomy. Therefore I have provided an introduction and conclusion in each case. Similarly the footnote numbering is exclusive to each chapter and operates in ascending order. I have also taken the liberty of treating the words 'numerical' and 'computational' as synonyms and similarly 'analysis', 'techniques' and 'methods', when used in the context of the expression 'numerical analysis'.

Finally, to illustrate the usefulness of the Numerical Algorithm Group's library of subroutines, I have used in Chapter 1 photocopies of the contents page, and the decision tree to chapter E04 from the Fortran manual at Mark 9. I should declare that my use of this material is for the personal research purposes of this thesis, and for no other reason.

ALAN CARRUTH

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Chapter 1 Computational Techniques in Economic Theory

1.1 Introduction

The numerate economist appears to see his role as the testing of and forecasting from economic models by well tried statistical techniques, often labelled econometrics. Most measurements generated can be related to standard tests of significance be they of a t -, F - or Chi-square basis. Even though strong assumptions may underlie the statistical approach, it is still, by and large, the main tool of the applied economist. It is equally evident that the application of econometrics has been considerably conditioned by the development of computer technology.[1] On the software side there has been a proliferation of easy to use packages designed for researchers with little programming experience.[2] This is exactly as it should be in a world where the division of labour has played a crucial role throughout history. However the essence of the statistical approach is always the availability of a suitable data set.

A less obvious approach to quantify qualitative predictions from economic models, which are not amenable to conventional econometric or other statistical methods, is the use of numerical analysis to find optimisation or equilibrium solutions to theoretical problems. Two reasons can be advanced for this state of affairs.

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1. Improvements in the speed of hardware have facilitated the iterative solution, for example, the estimation of complex labour supply problems with piecewise linear budget constraints, or the estimation of disequilibrium models with minimum conditions (see Atkinson, Stern and Gomulka, 1980 and Rosen and Quandt, 1978).

2. Two well known packages are Time Series Processor (TSP) and Statistical Package for the Social Sciences (SPSS).

First a greater familiarity with computer operation is required of the researcher. For example, knowledge of a high level language like Fortran may be necessary. This will always necessitate a working knowledge of the operating system at the researcher's local computer site. Those with resources can of course hire someone to carry out the computing stage of a project, and for large projects the division of labour argument would deem this a sensible course of action: but, small problems will not warrant a full-time computer specialist. Only the computations to Chapter 3 were extensive enough to gainfully employ an individual at the programming end for a considerable period of time.

Second is the difficulty that any quantification is deterministic in the sense that well behaved random errors are not part of the problem. As such the researcher will often choose key parameters upon which any quantitative assessment may be made. The applied economist is then in a very dictatorial position, and may be influenced by personal value judgments. It may then be possible to present results to fit a particular political persuasion. This can be harmful, but is not unique to the numerical analysis approach. There are many ways for economists to back their political instincts.[3] At least the use of sensitivity tests, which may not be vitally interesting, can be an important check on the robustness of the results the researcher favours. Moreover, in many computational problems one can directly appeal to the econometric work of other researchers, where there appears to be some consensus. Sometimes indirect appeals are possible, for example, with U-shaped scale curves (quadratic function say) we can easily

3. Compare the different economic forecasts computed by the Liverpool, London Business School and Cambridge models of the macro economy.

examine the effect of a unit cost penalty operating when the output rate is only 50% of minimum efficient scale.[4] Pratten (1971) has examined this question in a comprehensive empirical study of industry scale curves.

The essence of numerical analysis is an iterative, and, therefore, approximate solution to mathematical problems where suitable parameter values are available. Non-linear econometrics is based on iterative solutions; but, the two approaches part company over the availability or relevance of suitable data. This will become clear as the thesis develops the many different applications.

Finally, microeconomic problems which invite computational solutions can sometimes be usefully illustrated using computer graphics to draw contours, functions or straight lines: and even in three dimensions if necessary. This can lighten the burden of numerous tables of results, as a picture is often a preferred means of communication. So, though of pedagogic value, it does complement the computational approach. Chapters 5 and 6 attempt a demonstration of this technique.

The next section provides a brief account of how, in the last ten years, researchers have been receiving greater assistance in implementing numerical analysis methods and computer graphics. Section 1.3 briefly discusses a number of previous studies which have employed computational techniques. Finally section 1.4 sets down the scope of the thesis with respect to the microeconomic applications undertaken by the author.

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4. This is examined in detail in the appendix to Chapter 2.

1.2 The Numerical Algorithms Group (NAG, 1981) and Computer Graphic Libraries

A considerable body of microeconomic theory rests on the concepts of optimisation and equilibrium. There is also a wealth of mathematical knowledge on these subjects, which has been brought to bear at a practical level on finding solutions for the problems and experiments set up by mathematicians and natural scientists. These are the numerical analysis techniques which are a vast subject in themselves. This thesis is not concerned with the mathematical background to numerical analysis, but may be more correctly labelled as numerical computing. All we desire is to be able to use numerical methods in our research.

This makes good sense because, even though access to computer facilities is now the norm for academics, our computer user faces two main problems in any scientific computation. First, considerable experience is necessary before a computer user could transform a given algorithm into an efficient programme in terms of programme run-time and storage space on the computer. Secondly, even for an experienced computer user, considerable knowledge of numerical analysis principles and methods is required before one can guarantee to have an efficient algorithm.

Such difficulties led to the formation of the Numerical Algorithms Group (NAG from now on) project[5] which according to Ford and others (1979) has four main aims:

"1. To create a balanced, general purpose numerical algorithms library to meet the mathematical and statistical requirements of computer users, in Fortran and Algol 60.

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5. The project was initiated in 1970, but has really gained pace since around 1975.

2. To support the library with documentation giving advice on problem identification and algorithm selection, and on the use of each routine.
3. To provide a test programme library for certification of the library.
4. To implement the library as widely as user demand required" (1979, pp. 65).

The authors stress the need for collaboration between different technical communities in order to achieve and maintain these four aims.

To illustrate the comprehensive nature of the library Table 1.1 presents the contents page from the NAG Fortran Library manual at Mark 9.[6] The Chapters represent general mathematical areas, and within each area are many programme subroutines based on different algorithms or on variants of a type of mathematical problem. Notice that the present extent of the Fortran library runs to six (large!) volumes.

At the beginning of each chapter of the NAG library there is considerable guidance on how to choose the appropriate subroutine for a particular research problem. Tables 1.2a and 1.2b illustrate one aspect of this choice problem using decision trees. The chapter is 'E04' which is useful for economic optimisation models. Chapter 3 of the thesis on optimum taxation makes use of E04-routines. The Tables also demonstrate the considerable number of subroutines available to the researcher depending on the type of problem he is faced with.

Having selected an appropriate routine our researcher can then consult the appropriate section of the manual which gives full details plus an example programme on the routine's use. This will normally be

6. The members of NAG (It is a non-profit organisation) attempt to update the library approximately once a year; hence Mark 10 will be the next update and so on.

TABLE 1.1

NAG FORTRAN Library Manual

CONTENTS - FLM9

CONTENTS OF THE NAG FORTRAN LIBRARY MANUAL - MARK 9

NB: Some chapter contents documents are headed "CHAPTER CONTENTS – MARK 5". Such documents refer to chapters where lists of routine documents have not changed since Mark 5 and are equally applicable to Marks 6, 7, 8 and 9.

TABLE 1.2a

INTRODUCTION - E04

3.3. Decision Trees

3.3.1. Selection Chart for Constrained Problems

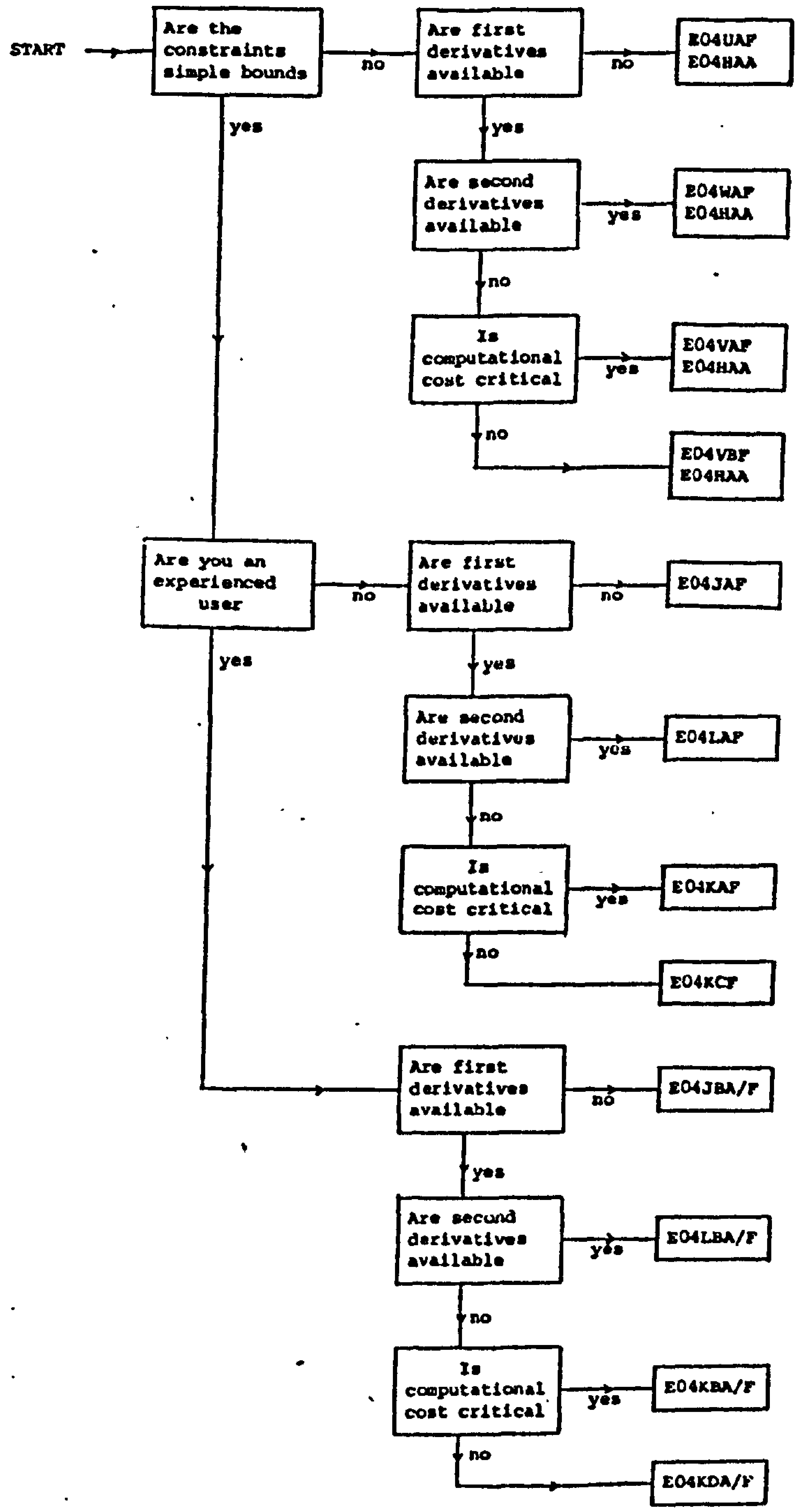
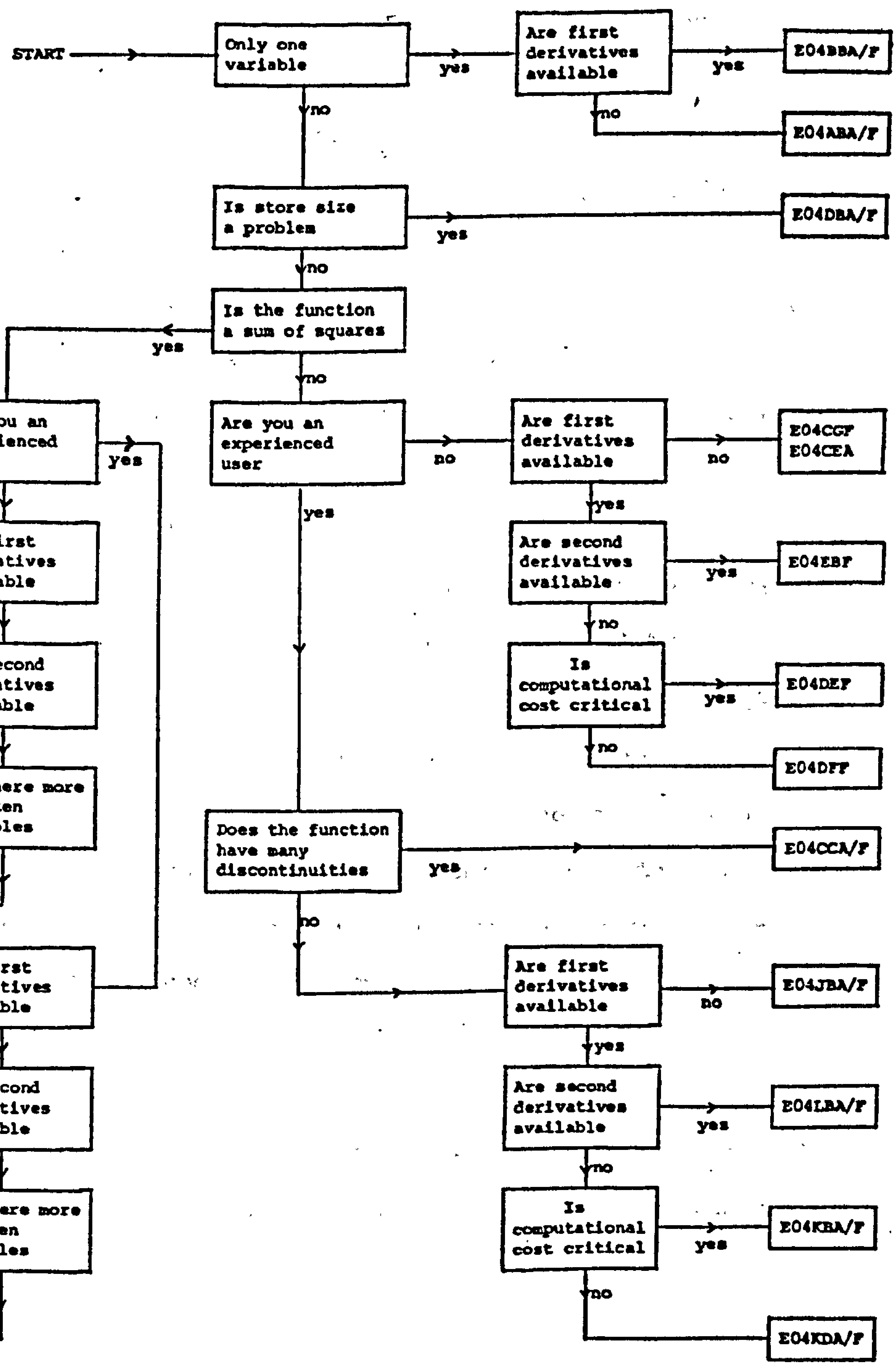


TABLE 1.2b

E04 - Minimizing or Maximizing a Function

3,3.2. Selection Chart for Unconstrained Problems



incorporated into the user's source programme with relative ease. Ideally, very sophisticated techniques can be implemented easily and efficiently by a researcher with only a limited knowledge of programming and numerical analysis.[7] Therefore through a number of microeconomic applications which implement NAG it is hoped to demonstrate that it can be put to good use by economists.

Similarly computer graphics has been developed by scientists (GHOST library (1978)) and design centres (GINO-F library (1976)). Again there is scope for economists to make use of such facilities even though it may have only teaching value. The present writer has had some experience of GINO-F (Graphical Input-Output-Fortran) originally developed at the computer aided design centre in Cambridge. The appendix to Chapter 6 lists a programme written to utilise GINO-F. This graphics library is implemented at the University of Kent. Whilst at the University of Warwick, the author worked with GHOST (Graphical Output System) developed at the U.K. Atomic Energy Authority Culham Laboratory in Oxford. GHOST has the advantage of being able to produce graphical output on any output device.[8] Moreover the link between the mathematical space of a problem and the physical space of the diagram is much simpler for GHOST. Nevertheless both systems provide a variety of facilities which the economist can put to good use.

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7. Compare the applied researcher's use of packages like TSP and SPSS.

8. The television medium is a good illustration of the usefulness of computer graphics, especially in the presentation of statistical information, eg. "The Money Programme".

1.3 Historical Perspective of Computational Applications

The present set of applications attempts to advertise the use of numerical techniques and in particular how the recent growth of the NAG library, and its general availability, can make this approach a good deal more amenable to the researcher who, more often than not, is no expert in numerical methods nor computer programming. However the application of computational techniques does not need to be based solely on NAG routines. Chapter 6 develops a duopoly problem which does not require a sophisticated solution: explicit formulae can be derived and evaluated through a Fortran source programme. Moreover early practitioners did not have the benefit of a NAG library at their local computer site.

Growth theory and dynamic problems generally were an early devotee of computational practices. For example, Mirrlees (1967), Atkinson (1969), Mirrlees and Stern (1972) and Dixit, Mirrlees and Stern (1975) attempt to complement their work with numerical analysis solutions. In particular Atkinson (1969) is concerned with the timescale of growth models, for

"In many cases we know how the major variables of the model change over time, in very few cases do we know how quickly they will change. Yet the speed of change is a prediction of the model, and by examining this we have a further test of the model's properties" (1969, pp. 137).

In one example Atkinson analyses the one sector growth model for the case where technical progress is both capital and labour augmenting. The basic prediction of this model is that one of the factor shares falls to zero over time. It has been argued that such an outcome is at variance with the reality of constant factor shares, or Bowley's Law. However as Atkinson points out we do not know how quickly the depleting factor share will be eradicated, that is how long it will take to attain the long run growth path. Therefore the model may have unattractive features and not yet necessarily be inconsistent with constant factor shares if the time path is very slow. In other words a slow decline of one factor share may not be an unreasonable approximation to reality.

Atkinson then examines the case where the capital share declines (elasticity of substitution less than unity). The differential equations derived are not tractable, therefore a computational appraisal can be most helpful, and so Atkinson seeks a solution using a numerical integration procedure. He shows that, for a specific set of parameters, the capital share takes at least 110 years to fall to half its initial value. He follows the basic result with a number of sensitivity tests on the selected parameters. This limited check does not lead him to reject the main finding that the approach to a long-run equilibrium may take rather a long time. Hence, given the length of time series data presently at our disposal, the model may not be at odds with Bowley's Law even though it can be criticised on other grounds.

Another theoretical field which becomes analytically messy is the optimum tax literature. From the seminal work of Mirrlees (1971) we can observe an attempt to incorporate the numerical analysis technique in calculating optimum tax rates for specific functional forms and selected parameters. Subsequent work has maintained the tradition, Atkinson (1973) and Stern (1976, 1982).

Among other things Stern (1976) demonstrates that the numerical estimation of tax rates could be greatly improved in the tax literature by an appeal to econometric analysis of labour supply. This stems from the rather surprising optimum tax rates derived by Mirrlees which were rather low and tended to fall at the higher end of the income distribution. Atkinson (1973) takes up the issue on the basis of the cardinality of the utility function demonstrating the limiting Rawlsian Maximin case where the utility of the worst-off individual is maximised. This increases the optimum tax rates found, but not dramatically. The Maximin criterion in the Mirrlees model yields tax rates around 50% for the median individual, see Atkinson (1973).

Stern (1976) points out that the probable influence of backward bending labour supply effects have an important bearing on the value of the elasticity of substitution between leisure and commodities, ε. The earlier work had not carried out any sensitivity tests on the value of ε and so $\varepsilon = 1$ was an arbitrary selection. Following Ashenfelter and Heckman (1973), Stern calculates ε to be approximately 0.4, certainly less than unity. Leaving aside cardinality he calculated optimum tax rates for ε in the range, (0, 1). It becomes apparent that the marginal rate of tax approaches 100% as ε tends to zero. In fact a theorem is proved that optimum taxation involves a marginal rate of tax of 100% for $\varepsilon = 0$. A most significant point for the appropriate use of numerical methods is that Stern is able to arrive at tax rates more in accord with those observed in practice by complementing the numerical approach with econometric work on labour supply, that is, by a better selection of parameter values based on known empirical results. Chapter 3 looks at some recent issues on the structure of optimum income taxation.

Numerical analysis has not been exclusively restricted to the fields of optimum growth and taxation. Dixit (1973) has employed the approach in a study of the optimum size and arrangements of a monocentric city. Nelson and Winter (1973, 1976) have used more grandiose simulation techniques to study technological change in the theory of the firm. Their work was inspired by the view that conventional models of the firm do not correspond adequately to economic reality. In a world of friction, uncertainty and feedback it is suggested that only crude economic mechanisms function reliably. Their simulation model attempts to capture these mechanisms.

Finally Fisher (1971) also explores through simulation the question of why the aggregate production function model seems to fit so well when its assumptions are considered so dubious. He demonstrates that it is the constancy of labour's share which allows the aggregate Cobb-Douglas production function to work reasonably well, particularly in explaining wages, rather than that the underlying technology is in fact Cobb-Douglas. Therefore causation is in the opposite direction, that is, not from an underlying Cobb-Douglas technology to a constancy in labour share. Hence the constancy of labour's share becomes an unexplained open question.

The above remarks have not in any way attempted to be exhaustive but demonstrate that numerical techniques have been put to some use in the past. However the recent development of the NAG library has made their application relatively more accessible to economic and other researchers. The next section will sketch a number of other applications to be developed in the thesis, which will hopefully bring out the versatility of this approach.

1.4 The Applications in Outline

It has been suggested that the terms of reference of the thesis are to demonstrate, through applications undertaken by the author across a broad range of microeconomic problems, the scope for computational techniques in the light of the NAG library development. This enables us to provide a quantitative appraisal of microeconomic models where analytic solutions may be limited or difficult to derive explicitly.

There is a tendency in the economics profession to marvel at theoretical models which have tight unambiguous results. Yet models of greater complexity and perhaps realism which do not yield elegant comparative statics are often dismissed as lacking in some way. If we can demonstrate that numerical solutions can help reduce ambiguity and provide quantitative assessments of problems which are not amenable to conventional econometric or statistical techniques, then the chapters to follow will have achieved their purpose.

Chapter 2 takes up the issue of economic models of markets with imperfect information which have increasingly involved high degrees of theoretical sophistication, yet, so far there has been no movement beyond qualitative prediction. An appeal to fairly simple numerical optimisation techniques enables us to question under what circumstances single price equilibria will exist under different assumptions about the distribution of search costs.

Chapter 3 examines some recent work by Stern (1982) and Allen (1982) and assesses the sensitivity of optimum tax rates to the production and consumption assumptions embodied in their models.

Chapter 4 considers the question if, in a partially unionised economy, union workers force up their absolute wage rate, how does this affect the wage paid in the non-union sector. Here the framework is a two sector general equilibrium model of a closed economy, and the computations attempt to lessen an analytic ambiguity.

Chapter 5 develops a model which captures the trade-off between scale economies and product variety in a world of monopolistic competition. It extends the work of Spence (1976a) to include a second best solution which requires a NAG routine, plus a U-shaped scale curve. It also shows how the entire analysis can be illustrated on diagrams using computer graphics.

Chapter 6 introduces oligopolistic interaction into the monopoly welfare loss debate which was pioneered by Harberger (1954). By postulating a specific social welfare function we can solve directly for the level of welfare (net surplus), concentration, prices and output rates. The numerical computations are almost trivial and require no appeal to the NAG library. Again the analysis can be usefully illustrated by computer graphics.

Conclusions will be drawn at the end of each chapter, especially in view of the diversity of the applications. However a short concluding chapter will bring out the more general points of the thesis.

Chapter 2 A Computational Assessment of the Quantitative Significance of Imperfect Information in a Non-Sequential Search Model

2.1 Introduction*

Retail markets are characterised, more often than not, by price dispersion, yet conventional microeconomic wisdom espouses the cause of single price equilibria. Some economists have attempted to explain how price dispersion can persist in markets where some consumers follow rational behaviour patterns. One focus has been the information structure of markets. However the seminal paper by Stigler (1961) assumed price dispersion, a priori, without questioning whether it would exist in a full equilibrium. This difficulty was remedied by Salop and Stiglitz (1977) and Braverman (1980). Other researchers, for example, Stiglitz (1974) concentrated on the efficiency properties for competitive equilibrium in the presence of imperfect information. He claimed that the results require us to modify the competitive market paradigm.

A common characteristic of all this work is its qualitative nature. We have no idea of the quantitative significance of imperfect information, such as the possibility of price dispersion. This chapter attempts to make such quantitative assessments by way of the numerical analysis approach discussed in the previous chapter. It turns out that the most we require is a numerical solution to a fairly straight-forward optimisation problem. The relevant sections have the details.

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* A shortened version of this chapter is forthcoming in Carruth (1983). The content has benefited from discussions with Avinash Dixit.

2.2 Analytical Framework

The basis of the chapter is the non-sequential search framework, the variants of which have been brought together under a fairly general specification due to Braverman (1980) and Braverman and Dixit (1981).[1] This approach embodies limited information about a homogeneous product where identical consumers know the distribution of prices charged in the market, but not the locations. Individuals enter the market only once, and can identify the lowest price store by expending a fixed sum, c, which differs across the population: or, they may select a store at random.

The consumer's decision is to

$$\max \quad \mu = x_o + U(x_i)$$
$$\text{s.t.} \quad x_o + p_i x_i = M_i$$

where i = store selected; M_i = income when buying from i; x_o = numeraire commodity bundle; p = price; x = quantity; and, $p_o = 1$. The linearity in the numeraire good removes income effects from the analysis in the sense that consumer demand is not influenced by the possibility of the search cost being met out of disposable income.

The maximand and budget constraint can be rearranged to give

$$\mu = M_i + V(p_i)$$

where $V(p) = \max_x (U(x) - px)$

$= $ 'consumer surplus'

1. Von Zur Muellen (1980) has carried out a similar exercise for a sequential search framework.

and

$$V'(p) = - x$$

$$V''(p) = - dx/dp > 0$$

so V is convex and decreasing in the price of bought commodities. If the consumer decided to search, then $M_i = M - c$ and $p_i = p_{min}$, and to select, at random $M_i = M$. Therefore an individual's appraisal of the two strategies would be based on whether

$$M + \overline{V(p_i)} \gtrless M - c + V(p_{min})$$

where $\overline{V(p_i)}$ is the expected utility from random selection, given that consumers know the price distribution. This condition can be rearranged to give

$$c \gtrless V(p_{min}) - \overline{V(p_i)}$$

which says that search is only worthwhile if the additional utility from guaranteeing purchase at the lowest price store is greater than the cost, c, of providing this guarantee. Rational behaviour means that the point of indifference can be depicted by

$$\hat{c} = V(p_{min}) - \overline{V(p_i)}$$

so $\hat{c}$ is the critical search cost which separates buyers into informed and uninformed groups.

Firms on the other hand maximise profit in a Bertrand-Nash fashion and can take account of customers information gathering responses; therefore the firm-consumer equilibrium is Stackelberg in nature. Finally unit cost curves are taken to be U-shaped. The framework has some weak features. The information structure presumes that consumers know the price distribution yet are unaware of the specific price each firm charges. Likewise any changes in the price distribution arising from firm behaviour are known without knowledge of the actual firm(s) inducing the change(s). Firms are highly sophisticated in their analysis of consumer reactions yet naive with respect to fellow competitor's pricing decisions. However the main appeal of this approach lies with its attention to industry equilibrium.

Three types of Nash equilibrium configurations can arise. First is a single price equilibrium (SPE) at the competitive price (SPCE) or at the monopolistically competitive price (SPME). Second is a two price equilibrium (TPE), where the low price is the competitive price, the high price is monopolistically competitive. Finally there is the possibility of non-existence of any Nash equilibrium.

Given specific families of demand, cost and information conditions the possibilities of the above equilibria can be examined and their relative likelihoods assessed. It is clearly of interest to enquire as to what percentage of informed individuals are required for the competitive outcome to arise through arbitrage.

It turns out that the analysis can be framed in terms of the distribution of search costs around zero. This is tied to the analytical approach which postulates a zero profit equilibrium, then attempts to reconcile whether profit maximisation operating through a firm's perceived demand curve for a contemplated price change is consistent with the initial postulated equilibrium. As such four cases can be identified following Braverman (1980) and Braverman and Dixit (1981):

(i) a group with zero search costs (positive atom at zero)

(ii) no individual has zero search costs (zero density at zero)

(iii) many consumers with arbitrarily small search costs (infinite density at zero)

(iv) a positive density at zero.

For the final case the quantitative possibilities of single price equilibria do not arise, as only two price equilibria are possible (Braverman (1980), pp. 491). Therefore the next three sections will consider the implications of (i), (ii) and (iii) above using numerical methods under particular, but sufficiently rich functional forms.

2.3 A group with zero search costs

Suppose consumers are of two types a fraction δ having zero search costs and the rest $(1 - \delta)$, positive ones.[2] Indeed the rest are assumed here to have infinite search costs, that is, not to search at all. The effect of this will be pointed out at a later stage.

If in an initial equilibrium all n stores charge the same price they will have a 1/n share of the market denoted by DD. A slight increase in price by a firm will cause it to lose all its share of the informed group, while a small cut will cause it to gain them all. Thus the perceived demand curve facing each firm, denoted by dd, will be discontinuous. If the unit cost curve is a conventional U-shape, there can be an equilibrium where each firm charges the competitive minimum average cost price if the demand curve for each firm is as shown in Figure 2.1.(a). However, for a case like Figure 2.1.(b), each firm will wish to raise its price suitably and a competitive equilibrium will not prevail.

Our aim is to find conditions on the fraction δ, in terms of demand and cost parameters, for a competitive equilibrium to exist and to find out what happens otherwise. It will be shown that, for a linear demand and quadratic cost formulation, two outcomes are exhaustive. There will either be a single price competitive equilibrium (SPCE) or a two price equilibrium (TPE). Non-existence will not arise. Price dispersion is restricted to a TPE because this limited information framework can only partition consumers into at most two groups, as was shown in the previous section. A full explanation can be found in Salop and Stiglitz (1977).

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2. Where we normalise the number in the population to unity.

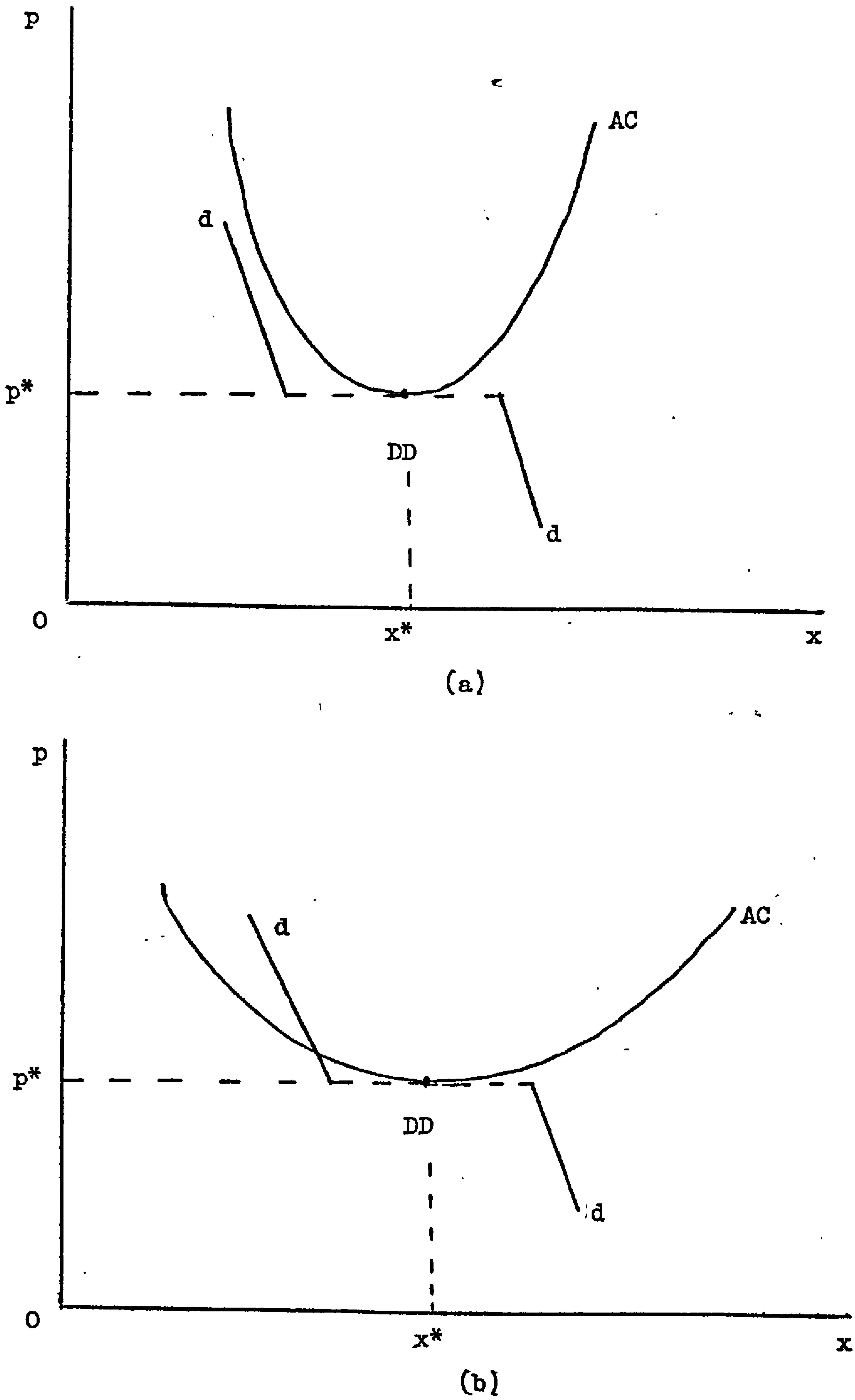

Figure 2.1 Potential outcomes for a group with zero search costs.

Let output per firm be represented by

$$x = (a - p)/2nb \qquad (2.1)$$

and total cost by

$$TC(x) = f + kx + gx^2 \qquad (2.2)$$

where n is the number of firms, p is price, x is output rate per firm and a, b, f, k, g are parameters. Figure 2.1. (a) indicates that the SPCE must coincide with minimum average cost, so we require that

$$x^* = (f/g)^{\frac{1}{2}} \text{ and } p^* = k + 2(fg)^{\frac{1}{2}} \qquad (2.3)$$

be the competitive output and price respectively. For the remainder of this chapter an asterisk will denote competitive magnitudes. Equation (2.1) can then be used to obtain n*.

The analytical procedure discussed earlier is to determine under what conditions (p*, x*, n*) can be a competitive equilibrium. Figure 2.1. makes it clear that price reductions by any firm will generate losses and such behaviour should not take place under profit maximisation; but Figure 2.1. (b) suggests the possibility of non-existence, where a deviant price-raising firm can make a profit. Here the infinite search cost assumption placed on the uninformed individuals will favour non-existence, because finite price dispersion, where there is a well defined search cost distribution, may induce more than the informed group to search. Therefore any additional reduction in the deviant's sales makes positive profit less likely which would help to maintain the SPCE. The later section on the infinite density at zero which is similar to the present case relaxes this infinite search cost assumption on the uninformed. To keep the present arguments tractable it is simpler to presume that the uninformed will not search.

It can be shown that there will be a critical percentage of consumers with zero search costs, labelled $\hat{\delta}$, which partitions the equilibrium price distributions into either a SPCE or a TPE. Non-existence will not arise. The details are left to an appendix. This critical percentage is given by

$$\hat{\delta} = 1 - 4(fg)^{\frac{1}{2}}/(a - k + 2(fg)^{\frac{1}{2}}). \qquad (2.4)$$

Any value of δ greater than $\hat{\delta}$ is sufficient to ensure that price is less than unit costs for all output rates below the competitive rate, x^*. As such deviant price-raising behaviour will not appear worthwhile, so the SPCE will hold. On the other hand when the actual value of δ is less than or equal to $\hat{\delta}$ a TPE will be supported by the market.

Equation (2.4) demonstrates that this critical percentage of informed consumers is determined entirely by the demand and cost parameters; therefore, given explicit values of the parameters, it is easy to evaluate $\hat{\delta}$. Here then is one basis for a quantitative test of the proportion of informed consumers required to maintain a competitive equilibrium.[3] Any numerical appraisal is faced with the problem of realistic parameter values. Two cases are identified dependent upon the cost penalty envisaged for firms operating at less than the competitive output rate. Details of this exercise are also left till the appendix. Suffice it to say that our treatment is in terms of a 10% or 20% cost penalty for firms which produce only one half the competitive output. This is consistent with the empirical work of Pratten (1971). Table 2.1 presents the results including the effect of varying fixed costs, f.

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3. Notice that this problem is so straightforward that powerful numerical algorithms are not required. We have an explicit solution in terms of $\hat{\delta}$ for alternative parameter values. The remaining sections require more powerful computational techniques.

Table 2.1.

Group with zero search costs results[a]

Parameter Set					
a	b	k	f	g	$\hat{\delta}$
20	1	6	1	4	0.56
			3		0.34
			5		0.22
			7		0.14
20	1	1	1	4	0.65
			3		0.46
			5		0.36
			7		0.28

a. Remember $\delta > \hat{\delta}$ implies a single price competitive equilibrium, and $\delta \leq \hat{\delta}$ means a two price equilibrium with $(0 \leq \delta \leq 1)$.

It is obvious from the table that the range of values for $\hat{\delta}$ is considerable. This is not surprising in an exercise where parameter sensitivity tests can simulate extreme effects. For example, with f = 1 in equilibrium, fixed costs comprise 20% of total costs for the first parameter set. A 10% relationship would increase $\hat{\delta}$ to approximately 78%; whereas the change to f = 7 makes fixed costs 32% of total costs. The inverse relationship between f and $\hat{\delta}$ demonstrated by Table 2.1 is in accord with earlier arguments. With rising fixed costs the average cost curve becomes steeper, therefore a potential price deviant operating with excess capacity will be more easily thwarted as smaller discontinuities in demand will generate the losses required to maintain the competitive outcome. Graphically Figure 2.1. (a) illustrates the case of a higher fixed cost to that displayed by Figure 2.1. (b). The final and perhaps unexpected conclusion from this set of results is that for a competitive equilibrium to exist through arbitrage the proportion of individuals with zero search costs may require to be fairly substantial. However, with increasing cost penalties when working at less than optimum scale (higher fixed costs), the single price outcome is more likely for a given proportion of informed individuals.

2.4 No individuals with zero search costs

This section considers the case where the density function of search costs, labelled $\mu'(\cdot)$, is such that $\mu'(0) = 0$. With the absence of demand discontinuities (due to $\mu'(0) = 0$) the analysis centers on monopolistic market structures. Braverman (1980) shows that the only possible candidate for a single price equilibrium requires a three way tangency among the market share demand (DD), perceived demand (dd) and average cost, curves.[4] Figure 2.2 illustrates a potential single price monopolistic equilibrium (SPME), which is somewhat different from the traditional Chamberlin result, and involves the following intuition.

The tangency between DD and dd is an important distinction from the well known Chamberlin result where DD cuts dd from above in equilibrium. This arises because a price changing firm in the usual Chamberlin scheme will gain or lose customers depending on the direction of price movement. On the other hand for small price changes with imperfect information captured by the present form of search cost distribution consumers will respond by gathering information. Therefore infinitesimal price changes will not gain or lose a deviant any customers, so his perceptions must reflect market share which requires the equality of DD and dd slopes at the equilibrium price-output configuration. When price is greater (less) than the monopolistic equilibrium price, dd will lie to the left (right) and below (above) DD, because a collective price increase (fall) is liable to have a smaller individual effect than a deviant price changer on his own. It is then interesting to question whether finite price reductions will break the SPME. This can occur if the inducement to search is enough to take a deviant's perceived sales inside the

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4. A rigorous demonstration of this result is given by Braverman (1980). It is probably easier to accept the result given the aims of this analysis rather than padding out the thesis with other researchers mathematical proofs. I have been unable to find mistakes in the analysis.

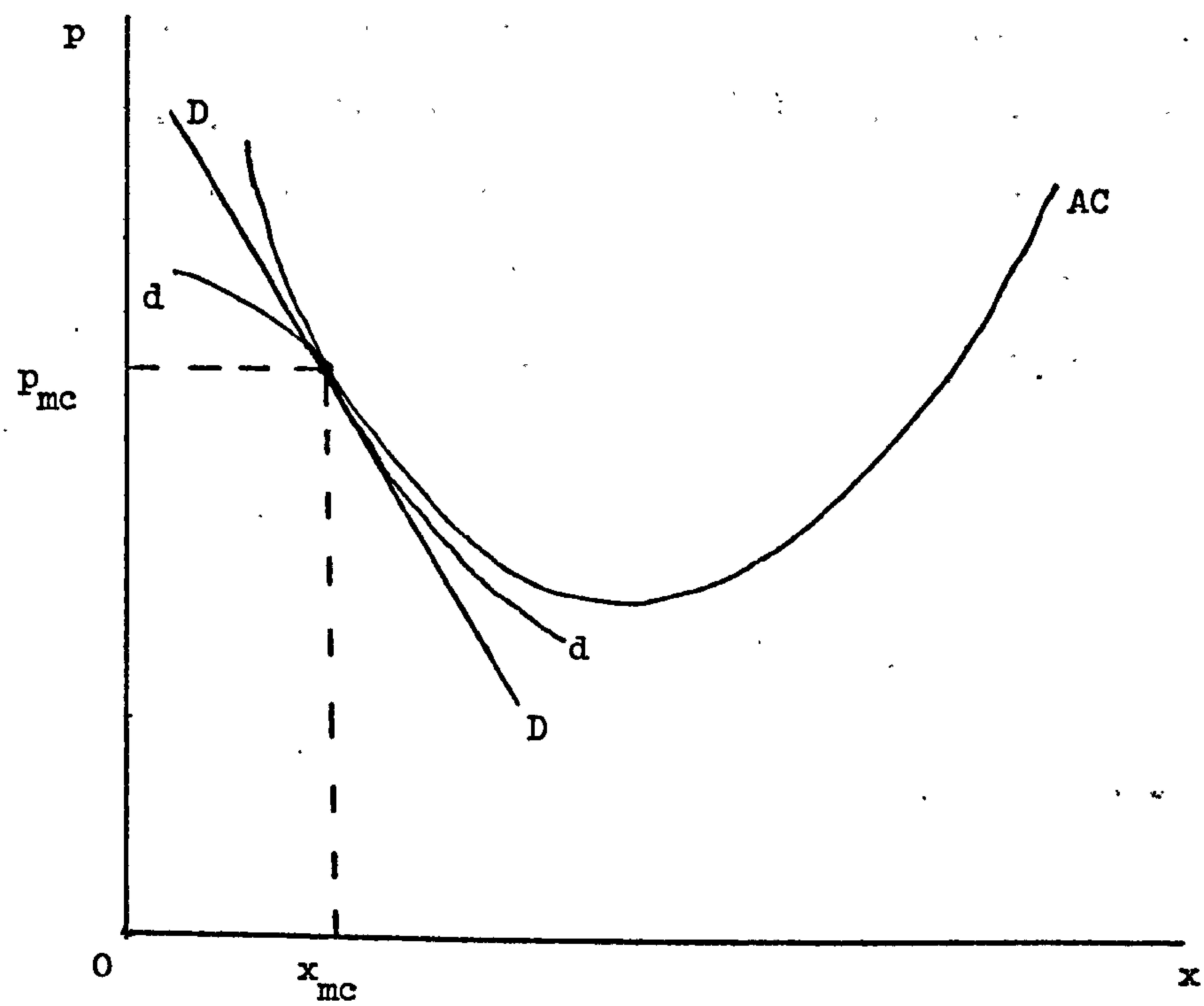

Figure 2.2. A single price monopolistic equilibrium.

average cost curve, and so violate zero profits. The rest of this section examines the quantitative possibilities of <u>finite price dispersion</u> to test under what search conditions perceived demand will remain below average cost. Notice that price cuts below the competitive price are not conceivable.

Figure 2.2. illustrates that for a SPME to exist over the price range (p^*, p_{mc})[5] the horizontal distance between average cost and perceived demand will be minimised at zero. A numerical procedure can check this explicitly for any particular case. We maintain the assumption of linear demand, quadratic cost and postulate an inverted V search cost density function. This yields the following cumulative search cost distribution

$$\mu(c) = 2c^2/\underset{\sim}{c}^2 \quad \text{for} \quad c \leqslant \tfrac{1}{2}\underset{\sim}{c} \tag{2.5}$$

$$= -1 + 4c/\underset{\sim}{c} - 2c^2/\underset{\sim}{c}^2 \quad \text{for} \quad c \geqslant \tfrac{1}{2}\underset{\sim}{c} \tag{2.6}$$

$$= 1 \quad \text{for} \quad c \geqslant \underset{\sim}{c} \tag{2.7}$$

where $\underset{\sim}{c}$ is an arbitrary value of c, the search cost fee, which determines the shape of the search cost distribution. As $\underset{\sim}{c}$ increases for a given price dispersion, the numbers induced to search falls.

At the SPME we know that price equals average cost $(f/x + k + gx)$. Solving this condition for x by selecting the smaller root due to average cost decreasing for $x < x^*$, we obtain

$$x = z(p) = ((p-k) - [(p-k)^2 - 4fg]^{\frac{1}{2}})/2g \tag{2.8}$$

---

5. p_{mc} is the monopolistically competitive price.

for all p in the range (p^*, p_{mc}). Provided price is greater than or equal to minimum average cost, the expression under the square root will be greater than or equal to zero. The next step is to consider what happens when (n_{mc} - 1) firms charge p_{mc} and one charges $p < p_{mc}$. It can be shown that with such a price dispersion consumers will search if

$$c \leqslant (n_{mc} - 1)\ (V(p) - V(p_{mc}))/n_{mc} = \hat{c} \qquad (2.9)$$

where $\hat{c}$ can be treated as a critical search cost in terms of the percentage of individuals who will search. V(p) is an individual's indirect utility function which is convex and decreasing in the price of bought commodities. Thus individuals will search if the cost of acquiring information is less than the difference in expected utility between the search and no-search strategies. For convenience consumers who are indifferent in this choice are assumed to search. Hence $\mu(\hat{c})$ individuals will search. A price cutting firm's perceived demand curve will be of the general form

$$d(p) = - V'(p)\mu(\hat{c}) - V'(p)(1 - \mu(\hat{c}))/n_{mc} \qquad (2.10)$$

where - V'(p) makes use of Roy's identity.[6] Perceived demand comprises the informed plus a $1/n_{mc}$ share of the uninformed.

Finally for either of the Table 2.1 parameter sets[7] the problem is solved in the following way. A choice of p in the range (p^*, p_{mc}) enables $\hat{c}$ to be evaluated from equation (2.9), then $\mu(\hat{c})$ is obtained from (2.5), (2.6), or (2.7) by comparing $\hat{c}$ with $\underline{c}$. Perceived demand and cost in terms of z(p) can be derived from (2.10) and (2.8)

6. $- V'(p) = + (a - p)/2b$

7. with f=1 only.

respectively. Our objective is to

$$\underset{p}{\text{Minimise}} \quad (z(p) - d(p)) \text{ for all } p \in (p^*, p_{mc}) \qquad (2.11)$$

By careful choice of $\underset{\sim}{c}$ we can ensure that this minimum will be zero at $p = p_{mc}$. It, of course, can never be negative if the SPME is to be maintained. Table 2.2. presents the results.

Equation (2.11) was solved for the different parameter sets of Table 2.2. by NAG routine EO4ABF, which searches for a minimum in a given finite interval of a single variable, continuous function. The methodology is based on quadratic interpolation and the algorithm was proposed by Gill and Murray (1973). This routine is very easy to use and requires function values only. It was also used to solve for a similar problem presented in the next section.

Table 2.2. clearly demonstrates that if $\underset{\sim}{c}$ is large enough our function minimisation procedure will select the monopolistic tangency as the minimum point, confirming the potential outcome portrayed by Figure 2.2. Intuitively the actual value of $\underset{\sim}{c}$ is of no interest. What is of interest is its implications for the proportion of the individuals in the market adopting a search strategy under finite price dispersion. The $\mu(\hat{c})$ column of the table suggests that, when a SPME exists, price dispersion equivalent to the difference between p* and p_{mc}, entices no more than 2½% of the consumer population to pay the search cost fee and become informed. The final column of the table underlines this point demonstrating that a considerable percentage price dispersion is required for even 1% of the market to find search worthwhile when the SPME is valid.

It is also apparent that the higher cost penalty case (parameter set 2) needs a lower likelihood of search to maintain the SPME. Given a steeper unit cost curve this is exactly what we should expect to happen.

In summing up this section it would seem that the single price monopolistic equilibrium is a rather fragile concept in that most consumers would require to remain oblivious to substantial price dispersion through an inability to acquire information. Yet while a steeper unit cost curve will make the single price monopolistic equilibrium less likely here, the opposite occurred in the previous section on competitive equilibrium. This is not surprising given the nature of these equilibria.

Table 2.2.

No individual with zero search costs results[a]

Parameter Set						Minimum Value of $(z(p)-d(p))$ $p\varepsilon(p^*,p_{mc})$	Estimated Position of Minimum p	$\hat{\mu}(c)$ when $p=p^*$	Extent of Price Dispersion to entice 1% of market to search
a	b	k	f	g	c			%	%
20	1	6	1	4	30	-1.826	10.00	3.6	5.2
					120	-0.0044	11.26	2.7	17.7
					131	0.0000	13.57	2.3	19.2
20	1	1	1	4	30	-7.05	5.08	100.0	4.6
					280	-0.047	5.92	3.1	33.9
					390	0.0000	10.92	1.6	44.9

a. For the first parameter set, $p^* = 10.0$, $p_{mc} = 13.57$
for the second set, $p^* = 5.0$, $p_{mc} = 10.92$
NAG routine EO4ABF provided the solutions.
Accuracy was to 14 decimal places.

2.5 Many consumers with arbitrarily small search costs

This possibility was elaborated in a note by Braverman and Dixit (1981). Where the density function of search costs is infinite at zero, the firm's perceived demand curve at some initial position has an infinite price derivative. If the starting price, p, equals the minimum average cost, and if average cost increases rapidly enough for output rates below the optimum scale, then a competitive equilibrium will result. Figure 2.3. illustrates this case.

There are of course similarities with the zero search cost group, but while

> "no consumer has literally zero search cost, ...
> there are sufficiently many with arbitrarily small
> search costs"
>
> (Braverman and Dixit (1981) pp. 658)

to yield the outcome depicted by Figure 2.3. Here we actually want to play much the same sort of game as in the previous section, but now we are interested in a deviant price-raising firm. We therefore want to check that perceived demand will always remain below and to the left of average cost for all $p > p^*$. Prices less than p^* are untenable as before. Numerically an upper bound on p is not a problem, although the output origin is clearly a constraint.

The choice of density function is again somewhat arbitrary. Pareto can meet our requirements where

$$\mu'(c) = \alpha \underline{c}^{\alpha} c^{\alpha-1} \qquad (2.12)$$

and $\underline{c}$ is the smallest value of c; for $\mu'(\underline{c}) = \alpha/\underline{c}$, which approaches

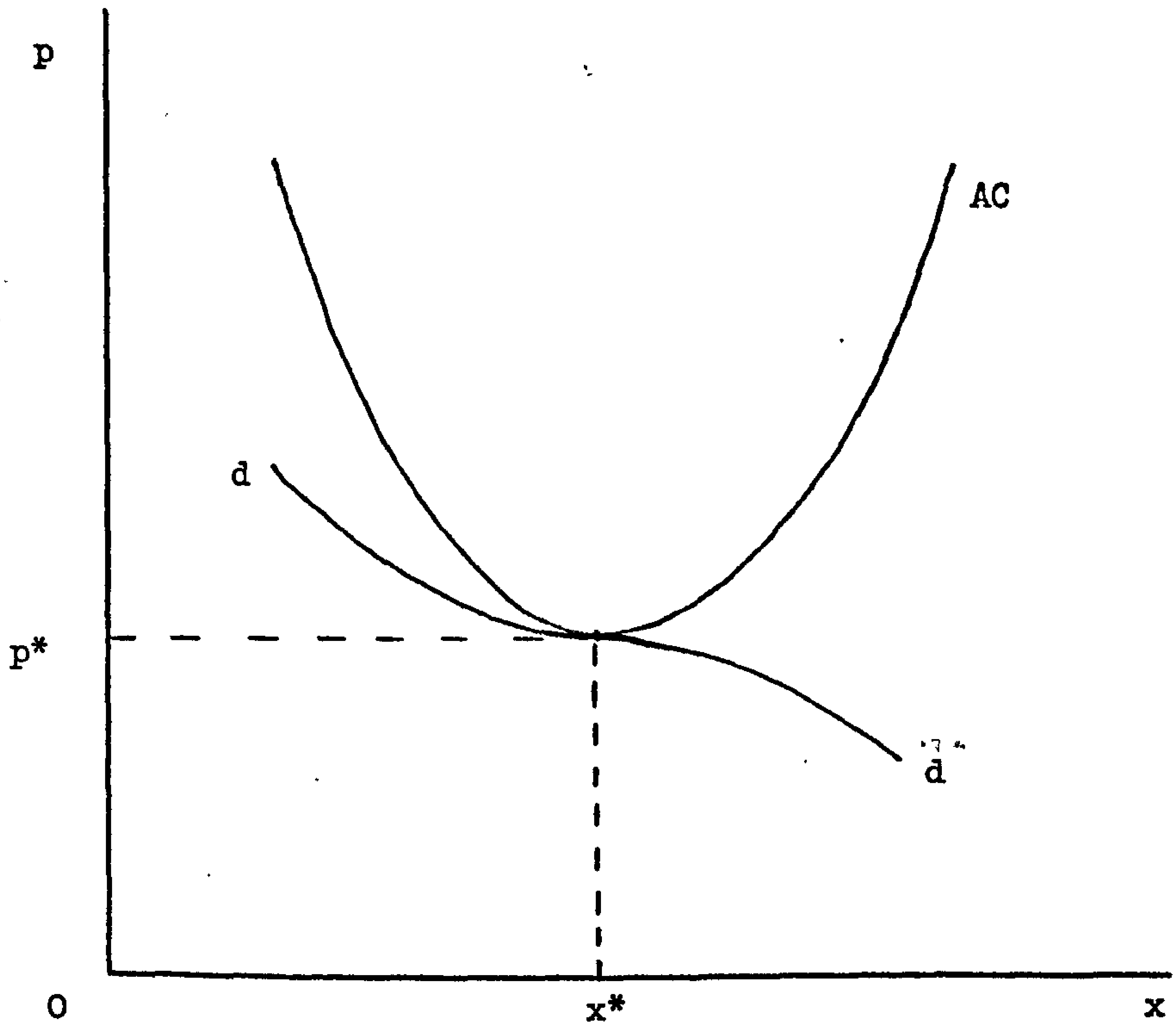

Figure 2.3. A single price competitive equilibrium with no discontinuity in perceived demand.

infinity as $\underset{\sim}{c}$ approaches zero. If we set $\underset{\sim}{c}$ to the machine accuracy[8] of the computer, then this should be a reasonably fair representation of the Braverman and Dixit scheme. The cumulative distribution function will be

$$\mu(c) = -\underset{\sim}{c}^{\alpha}/c^{\alpha} + \text{constant} \qquad (2.13)$$

A value of unity for the constant gives $\mu(\underset{\sim}{c}) = 0$ and $\mu(c \to \infty) = 1$, which is exactly what we want. Consumer willingness to search will depend upon

$$c \leqslant (V(p^*) - V(p))/n^* = \hat{c} \qquad (2.14)$$

and so perceived demand will be

$$d(p) = -V'(p)(1 - \mu(\hat{c}))/n^* \qquad (2.15)$$

A price raising deviant firm can only expect a $1/n^*$ share of the uninformed. The $z(p)$ relationship is exactly as before (equation (2.8)).

A slight problem is that the minimisation of $(z(p) - d(p))$ can only be checked for $p > p^*$, otherwise $\hat{c} = 0$, which is not defined for the cumulative distribution function. As $\underset{\sim}{c}$ is small, $\hat{c}$ can get fairly close to zero. Given the way in which the results are presented this is of little consequence. Here we can ensure that the SPCE will be maintained by our choice of α, which again means that the minimum of $(z(p) - d(p))$ can never be negative. Table 2.3. has the results.

All outcomes in the table represent situations where a SPCE

---

8. $\underset{\sim}{c} = 0.22.\ 10^{-15}$ on the ICL 2960 at Kent.

will hold. The column headed $\mu(\hat{c})$ for p 1% above p* is interesting for its close relationship to $\hat{\delta}$ of Table 2.1. As there is no longer an infinite search cost assumption, it is not surprising that the percentages are slightly less than for Table 2.1. This accords with the arguments presented at that stage. If price continues to rise up to 50% above p*, the additional search undertaken is small. The final column of the table is also rather interesting. The minimisation routine was able to find another tangency point at prices above the competitive price ($z(p) - d(p) = 0$ for $p > p^*$). From Table 2.2, for $f = 1$, we can see that this price was fairly close to the monopolistic price,[9] therefore, a TPE was a possibility. However, non-existence could not be ruled out for this case. Finally the effect of higher fixed costs was to increase the likelihood of a single price competitive equilibrium.

9. Accuracy here is actually restricted to our choice of α.

Table 2.3.[a]

Many consumers with arbitrarily small search cost results.

Parameter Set						p^*	$\mu(\hat{c})$ for p 1% above p^*	$\mu(\hat{c})$ for p 50% above p^*	$z(p)-d(p)=0$ for $p > p^*$
a	b	k	f	g	α		%	%	
20	1	6	1	4	0.0225	10.00	52.4	56.1	13.50
			3		0.01135	12.93	31.9	34.4	14.66
			5		0.00693	15.09	21.1	22.5	15.82
			7		0.00425	16.75	13.6	13.8	16.98
20	1	1	1	4	0.029	5.0	60.9	64.9	10.83
			3		0.0175	7.93	44.2	47.7	11.73
			5.		0.0125	9.94	34.5	37.4	12.56
			7		0.00915	11.58	26.8	29.1	13.39

a. Routine E04ABF found the minimum of $(z(p)-d(p))$ for $p > p^*$, in a similar fashion to the problem of Table 2.2.

2.6 Conclusion

This chapter has studied the quantitative significance of imperfect information. Some numerical results have been obtained for models brought together in a general framework by Braverman (1980) and Braverman and Dixit (1981), based on non-sequential search behaviour. No attempt was made to quantify the sequential search approach of Von Zur Muehlen (1980) which also focussed on the existence of industry equilibrium price distributions.

Initially, it was shown, for example, that up to 65% of individuals would need to be perfectly informed (zero search costs) for a competitive equilibrium to be reached through arbitrage; though substantial scale economies may help to maintain the single price competitive equilibrium. Where an atom of consumers have zero search costs, under linear demand and quadratic cost, non-existence will not arise. Similar results were found when many consumers have arbitrarily small search costs, but, with a well defined Pareto search cost distribution, non-existence could not be ruled out.

In a situation where no individuals have zero search costs, perceptions for very small price changes reflect share of the market demand (DD), so the focus is on monopolistic market structures. Here the single price monopolistic equilibrium could only exist, provided less than 2½% of the consumer population were induced to search under a price dispersion equivalent to the difference between monopolistic and competitive prices. It is then not surprising that price dispersion is the norm in markets with imperfect information.

The next chapter goes on to detail how NAG software can also be used to study the structure of optimum taxation in models with endogenous wages, and more than one type of worker.

Chapter 2: Appendix: Group with zero search costs

Suppose we start from a SPCE and take the equations of the text as given, and assume the uninformed $(1-\delta)$ will not search. Therefore a price raising firm will have sales

$$x = (a-p)(1-\delta)/2bn^* \tag{2.16}$$

where the equilibrium number of firms is the relevant magnitude. Rearranging (2.16) gives

$$p = a - 2bn^*x/(1-\delta) \tag{2.17}$$

For a SPCE we require $p < AC$ for all $x < x^*$, that is,

$$a - 2bn^*x/(1-\delta) < f/x + k + gx \qquad \text{for all } x$$

which can be expressed as

$$a < \min_x \{ f/x + k + \phi x \} \tag{2.18}$$

with $\phi = g + 2n^*b/(1-\delta)$

(2.18) states that the minimum value of the right hand side (RHS) with respect to x be greater than a if a SPCE is to survive. The minimum value of x can be obtained by differentiating the RHS of (2.18). Substitution of this value and for n^* in (18) yields the condition

$$\delta > 1 - 4(fg)^{\frac{1}{2}}/(a-k+2(fg)^{\frac{1}{2}}) \tag{2.19}$$

which is sufficient to ensure $p < AC$ for all $x < x^*$.

If (2.19) fails to hold it turns out that the outcome is a TPE, not non-existence. The easiest way to demonstrate this point is to formulate a similar condition for a TPE. Suppose there are n_1 low price firms and n_2 high price firms, then the output of a low price firm depends upon its share of the informed and uninformed customers respectively, which is

$$x^* = \delta\ (a-p^*)/2bn_1 + (1-\delta)(a-p^*)/2b(n_1+n_2) \qquad (2.20)$$

Competitive magnitudes on x and p arise because a potentially deviant low price firm perceives a demand discontinuity, the informed group imparting an effect similar to that illustrated by Figure 2.1. Hence it must operate at the optimum scale position. Substitution for x* and p* from equation (2.3) yields

$$\delta y_1 + (1-\delta)y_2 = 2b(f/g)^{\frac{1}{2}}/(a-k-2(fg)^{\frac{1}{2}}) \qquad (2.21)$$

where $y_1 = 1/n_1$ and $y_2 = 1/(n_1+n_2)$

A high price firm depends solely on the uninformed customers for its sales, so

$$x = (1-\delta)y_2(a-p)/2b \qquad (2.22)$$

As a monopolistic competitor in a market with many firms, the high price firm will obey profit maximisation (MR = MC) and normal profit (AR = AC). Equation (2.22) in inverse form is

$$p = a - 2bx/(1-\delta)y_2 \qquad (2.23)$$

which is average revenue. Marginal revenue is twice as steep as AR, so we replace 2bx with 4bx in (2.23). Marginal cost and average cost are easily obtained from (2.2). The simultaneous solution of the two profit conditions gives $x = 2f/(a-k)$, and substitution back into one of the profit conditions for x generates the result

$$y_2 = 8bf/(1-\delta)((a-k)^2 - 4fg) \qquad (2.24)$$

Now n_1, $n_2 > 0$ and $n_1 < n_1 + n_2$ are necessary and sufficient for a TPE, which is equivalent to n_1, $n_2 > 0$ and $y_1 \geq y_2$; moreover

$$y_1 \geq y_2 \Longleftrightarrow \delta y_1 + (1-\delta)y_2 \geq y_2 \qquad (2.25)$$

Equations (2.21) and (2.24) can be substituted into (2.25) to give

$$y_1 \geq y_2 \Longleftrightarrow \delta \leq 1 - 4(fg)^{\frac{1}{2}}/(a-k+2(fg)^{\frac{1}{2}}) \qquad (2.26)$$

which provides another restriction on δ, but for a TPE. Inspection of (2.19) and (2.26) demonstrates that the possibilities for this case are exhausted: non-existence will not arise. Hence we can use

$$\hat{\delta} = 1 - 4(fg)^{\frac{1}{2}}/(a-k+2(fg)^{\frac{1}{2}}) \qquad (2.27)$$

As a partition into the two equilibrium price distributions with $\delta \leq \hat{\delta}$ implying a TPE, and $\delta > \hat{\delta}$ implying a SPCE.

The final issue we wish to deal with in this appendix concerns the implications for scale or minimum efficient size (mes). This helps with our choice of parameter values. Average cost is

$$AC = f/x + k + gx$$

minimum average cost, $\min(AC) = k + 2(fg)^{\frac{1}{2}}$ at $x^* = (f/g)^{\frac{1}{2}}$

Suppose a plant operates at 50% of mes at $x = \frac{1}{2}x^*$. Therefore

$$AC = f/\tfrac{1}{2}x^* + k + \tfrac{1}{2}gx^*$$

$$= k + 5/2(fg)^{\frac{1}{2}}$$

and $$(AC - \min(AC))/\min(AC) = \tfrac{1}{2}(fg)^{\frac{1}{2}}/k+2(fg)^{\frac{1}{2}} \qquad (2.28)$$

$$= 1/(4 + 2k/(fg)^{\frac{1}{2}})$$

If a 10% cost penalty is appropriate when operating at 50% of mes, then

$$(AC - \min(AC))/\min(AC) = 1/10$$

and so $k = 3(fg)^{\frac{1}{2}}$, using (2.28). An analogous argument for a 20% cost penalty yields $k = \frac{1}{2}(fg)^{\frac{1}{2}}$.

Such restrictions help to reduce the somewhat arbitrary choice of numerical values for the parameters. The work of Pratten (1971) suggests that a cost penalty in the range 5% - 20% is consistent with his empirical cost analysis of a variety of different industries when plants are restricted to produce at 50% of mes. For the 10% cost penalty case we let $f = 1$, $g = 4$, which means $k = 6$. Similarly $k + 2(fg)^{\frac{1}{2}} = 10$ and $a > 10$ is necessary. To allow some leeway for changing f we take $a = 20$, $b = 1$, $k = 6$, $f = 1$, $g = 4$. For the 20% cost penalty we have $a = 20$, $b = 1$, $k = 1$, $f = 1$, $g = 4$.

Chapter 3 Optimum Taxation Models with Endogenous Wages

3.1 Introduction*

Mirrlees (1971) in his seminal paper on optimum income taxation formulates a model where individuals have identical utility functions but differ in their skills and pre-tax wage rates. The government chooses the income tax function to maximise the sum of utilities across the population. There is a resource constraint and in addition individuals make their own utility maximising choice of consumption and leisure given their pre-tax wages and the income tax schedule. His rigorous handling of the incentive issue did not allow the derivation of many unequivocal results. However he was able to show that marginal tax rates would be non-negative.

The Mirrlees framework assumes that the elasticity of substitution between workers of different productive abilities is constant and infinite. Consequently the ratio of the wages of any two groups of workers of different abilities is completely independent of the number of man-hours supplied by these groups (and indeed, by any other group). Researchers wishing to work with a continuum of abilities have in the main kept to this assumption for ease of exposition. However this type of approach not only ignores the impact of supply factors it also limits the role of the tax system in improving the welfare distribution to that of redistributing spending power.

The introduction of a production function with more than one type of labour means that the wages earned by the various groups of workers can also depend upon their labour supply. This allows an alternative

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* A subset of this chapter has appeared in a symposium, Carruth (1982). This version has benefited from some recent work undertaken by Heady, Ulph and Carruth (1982). For encouragement on this topic I am indebted to Nicholas Stern and David Ulph, who are in no way responsible for remaining failings.

route by which the tax system can bring about some redistribution. Feldstein (1973)** was the first to address the consequences of different, but finite, labour types and endogenous wages. In fact his discrete population included only two groups of workers, which, Heady, Ulph and Carruth (1982) have recently suggested, may be restrictive. Nevertheless Feldstein's numerical computations indicated that the effect of endogenous wage rates on optimum tax rates was of little consequence. Recent theoretical work by Allen (1982) has questioned this finding. He argues that the redistribution route through labour supplies and relative wage rates (the production effect) was submerged in Feldstein's computations due to the adoption of the Cobb-Douglas production function. Otherwise it is theoretically possible to posit outcomes which have negative marginal tax rates at the optimum. This stands in stark contrast to the Mirrlees finding. Section 3.4 will attempt to explore the circumstances of the Allen result through numerical computation.

Another application of the Feldstein framework in terms of the Cobb-Douglas production function was undertaken by Stern (1982). He compares the welfare implications of lump-sum taxation where errors in classifying individuals are committed, and income taxation, where each individual faces the same tax schedule. With no mistakes in classification and no disutility from providing information first-best welfare theory unambiguously favours lump-sum taxation. However, as horizontal inequity can occur through otherwise identical individuals receiving an incorrect lump-sum transfer, this first-best implication may no longer hold. This begs the question as to the scope for governments to commit errors in classification before optimum income taxation becomes the preferred tax structure. It would seem from Stern's results that, among other factors, much depends upon society's

** It should be noted that Feldstein worked within the linear income tax framework, whereas Mirrlees was firmly committed to non-linear income tax preference for inequality in utility levels. It is of interest to check whether his conclusions are also specific to the Cobb-Douglas formulation. Therefore sections 3.2 and 3.3 will present a computational framework to include a CES production function, which will provide an additional degree of freedom in the value of the elasticity of substitution, denoted by σ.

The computations for this chapter were dependent to a large extent on the NAG software. General details were set out in an earlier chapter. Specific information on the actual routines used to solve a problem will be given at appropriate points in the text.

3.2 The Effect of a CES Production Function on Optimum Taxation with Errors

The basis of this analysis is a comparison of

> "the welfare levels which can be achieved by two distinct tax regimes: lump-sum taxation, where one attempts to identify individuals and allocate transfers or subsidies on the basis of characteristics, and income taxation, where characteristics are not observed but incomes are measured and taxed. Where there are no errors in classifying individuals, lump-sum taxation is superior, but, where mistakes are made in the allocation of lump-sum grants or subsidies, income taxation may be more attractive." (Stern (1982), pp. 181)[1]

Both of these tax regimes have their own information requirements and administrative costs. To keep the analysis tractable we take it that administrative costs are similar for the two schemes, but that the set-up costs of each are prohibitive enough to make having both operating together undesirable. We, therefore, concentrate on the benefits of either regime.

The Lump-Sum Tax Model

In line with most adaptations of the Feldstein model the analysis is restricted to the two labour types, one skilled (subscript S), the other unskilled (subscript N). Both types are involved in the production of a single consumption good, Y. As is usual each individual maximises an identical utility function

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1. It is well known that incentives may exist for individuals not to reveal information on personal characteristics or income. We do not address this issue directly.

$$U(C,L) = \left[(1-\alpha)C^{-\mu} + \alpha(1-L)^{-\mu}\right]^{-1/\mu} \tag{3.1}$$

subject to the budget constraint

$$C_i = (1-t)\ W_i\ L_i + G_j$$

where W_i is the gross hourly wage of labour type i, t is the marginal tax rate, L_i is the amount of labour supplied, C_i is consumption and G_j is the lump-sum grant for individual type j. The indices i and j take the values S and N: if an individual is correctly classified $i = j$, if incorrectly $i \neq j$. The utility function has a standard CES interpretation of the parameters, and $\varepsilon = 1/(1+\mu)$ is the consumption-leisure elasticity of substitution. It would appear from Stern (1976) that $\varepsilon = \frac{1}{2}$ has some empirical plausibility in terms of recent econometric work on labour supply.[2]

It was suggested above that any attempt at discrimination among individuals and the likelihood of making errors in such a practice is one of the novel features of this work. As such the probability of an individual being misclassified is δ_i. With $G_S < G_N$ the asymmetry of incentive[3] to be placed in the wrong group warrants an endogenous δ. Two alternatives pursued by Stern for exogenous δ are, firstly, to have the proportion of skilled misclassified greater than unskilled ($\delta_S > \delta_N$). Secondly due to the asymmetrical incentive to be misclassified, the unskilled should always be correctly classified; so $\delta_N = 0$ and $\delta_S > 0$. In reality there may be a significant extra cost in ensuring that the unskilled are correctly screened. If society is unable or unwilling to bear such costs then it is necessary to accept $\delta_N > 0$ and

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2. This is consistent with a negative labour supply elasticity, and is based on work by Ashenfelter and Heckman (1973) and others.

3. Skilled individuals would be happy to receive G_N, but not the unskilled with G_S.

investigate its consequences. Hence computations will be presented for $\delta = \delta_S = \delta_N \geqslant 0$, $\delta_S > \delta_N$ and $\delta_N = 0$.

The production function is given by

$$Y = \gamma\{\theta(\beta L_S)^{-\rho} + (1-\theta)((2-\beta)L_N)^{-\rho}\}^{-1/\rho} \tag{3.2}$$

where γ, θ and ρ are the efficiency, distribution and substitution parameters respectively. There are β individuals of type S and $(2-\beta)$ of type N. The degree of substitutability between the two types of labour is given by $\sigma = 1/(1+\rho)$. Individual workers are paid their marginal products per hour of work supplied.

$$W_S = \theta Y^{\rho+1}/\gamma^{\rho}(\beta L_S)^{\rho+1} \tag{3.3}$$

$$W_N = (1-\theta)Y^{\rho+1}/\gamma^{\rho}((2-\beta)L_N)^{\rho+1} \tag{3.4}$$

Like all linearly homogeneous production functions, the CES in its present form will display constant returns, so factor shares will exhaust output.

The labour supply functions derived from (3.1) are

$$L_S^o = (1-a\hat{W}_S^{-\varepsilon}G_S)/(1+a\hat{W}_S^{1-\varepsilon}) \tag{3.5}$$

$$L_S^1 = (1-a\hat{W}_S^{-\varepsilon}G_N)/(1+a\hat{W}_S^{1-\varepsilon}) \tag{3.6}$$

$$L_N^o = (1-a\hat{W}_N^{-\varepsilon}G_N)/(1+a\hat{W}_N^{1-\varepsilon}) \tag{3.7}$$

$$L_N^1 = (1-a\hat{W}_N^{-\varepsilon}G_S)/(1+a\hat{W}_N^{1-\varepsilon}) \tag{3.8}$$

where $\hat{W}_i = (1-t)\, W_i$ for $i = S,N$ is the net wage and $a = [\alpha/(1-\alpha)]^{\varepsilon}$. Correctly classified persons have superscript o and those incorrectly classified a superscript 1. Consumption levels follow from (3.1). The average labour supply of type S and N groups is given respectively by the linear combinations

$$L_S = (1-\delta_S)L_S^o + \delta_S L_S^1 \tag{3.9}$$

$$L_N = (1-\delta_N)L_N^o + \delta_N L_N^1 \tag{3.10}$$

The government budget constraint is

$$\{\beta(1-\delta_S)+(2-\beta)\delta_N\}G_S+\{\beta\delta_S+(2-\beta)(1-\delta_N)\}G_N = tY-R \tag{3.11}$$

where R represents a revenue requirement outside of the transfer system.

Equations (3.2) - (3.11) represent a simultaneous system of 10 equations in 12 unknowns, the left hand sides of (3.2) to (3.10) plus t, G_N, G_S. By adopting values of t and G_N it is possible to solve for the other variables in terms of the maximand

$$\nu W_\nu = (1-\delta_S)\beta U^\nu(C_S^o, L_S^o) + \delta_S\beta U^\nu(C_S^1, L_S^1) + (1-\delta_N)(2-\beta)U^\nu(C_N^o, L_N^o) + \delta_N(2-\beta)U^\nu(C_N^1, L_N^1). \tag{3.12}$$

U is just our CES utility function: ν is Atkinson's (1970) inequality aversion parameter in terms of utility levels. Ex ante it allows society to decide the weight to attach to the lower utility levels, which in this case will be the incorrectly classified unskilled individual. With $\nu = 1$ we have the utilitarian objective whereas as $\nu \to -\infty$ we maximise the utility of the worst-off individual in society - the maximin criterion.

<u>The Non-Linear Income Tax Model</u>

This model is formulated as

$$\text{Max} \quad W(U(C_S, L_S), U(C_N, L_N)) \tag{3.13}$$

$$\text{s.t.}\quad U(C_S, L_S) - U(C_N, \frac{W_N}{W_S} L_N) \geqslant 0 \tag{3.14a}$$

$$- C_S - C_N - R + Y \geqslant 0 \tag{3.15}$$

with β=1. We require to maximise social welfare, $W(\cdot)$, subject to the resource constraint, (3.15), and constraint (3.14a) which just states that type S individuals do not want to earn C_N post tax. This reflects the skilled having a lower social marginal utility of consumption, and is related to the issue of wages monotonic increasing in ability for the Mirrlees (1971, infinite elasticity of substitution) production framework. With finite production elasticities some kinds of workers may be very abundant relative to other less able workers and so may receive a lower wage. However such an outcome raises the likelihood that skilled individuals will switch to the higher paid jobs. Stern (1982) and Allen (1982) rule out this possibility by assuming the wage is strictly increasing in skill; hence constraint (3.14a) is appropriate.

Nevertheless non-monotonicity does raise issues of absolute/ comparative advantage of workers and the potential endogeneity of job choice. This can only be examined satisfactorily for the continuum case. Heady, Ulph and Carruth (1982) have attempted to move in this direction though the analysis becomes extremely complex. Suffice it to say that in the section on Allen's (1982) two theorems we encounter the problems of computational solutions with $W_N > W_S$ and a need to use the constraint

$$U(C_N, L_N) - U(C_S, \frac{W_S}{W_N} L_S) \geqslant 0 \tag{3.14b}$$

so that type N individuals do not want to earn C_S post tax. Effectively the numerical analysis encounters regions where the monotonicity assumption is no longer viable. This is in complete sympathy with the

Heady, Ulph, and Carruth (1982) position.

Stern (1982) demonstrates that the inclusion of a production function with more than one type of labour and finite substitution elasticity violates the well-known theorems of Mirrlees (1971), positive marginal tax rates and Seade (1977), bounded income distributions have zero marginal tax rates at both endpoints. We now expect to find a marginal subsidy (negative marginal tax rate) at the top and a positive marginal tax rate at the bottom. Heady, Ulph and Carruth (1982) demonstrate that this result carries over to the continuum case. It reflects the following intuition.

"By lowering the marginal tax rate at the top the highest skilled workers are encouraged to work harder, so driving down their wage relative to that of other individuals. This narrowing of the wage distribution means that marginal tax rates can be lower elsewhere in the distribution (less redistribution required), and so the increased distortion at the top of the distribution can be traded off against the reduced distortion lower down." (1982, pp 5).

Other Outcomes

Optimum linear income tax with $G_S = G_N = G$ (and $\delta_S = \delta_N = 0$, of course) in (3.2) to (3.12) is simplified to one dimension, t; for now the government budget constraint, (3.11) provides a relation between t and G. It is also apparent from lump-sum that when $\delta_i = \frac{1}{2}$ the classification provides no information - a random allocation.[4] Such randomness suggests a solution of equal grants for all which corresponds to optimum linear income taxation. Therefore between $\delta_i = 0$ and

4. $\delta_i = b$ and $\delta_i = 1-b$ will provide the same information with the labels reversed when $b = \frac{1}{2}$.

$0 < \delta_i < \frac{1}{2}$ we should expect to move from first-best to approximately optimum linear taxation.

A final comparison for the welfare levels of lump-sum taxation is given by the point on the first-best frontier where both skilled and unskilled have equal utility levels. This is the maximin solution.

3.3 Computations for Optimum Taxation with Errors

There are now four types of optima to be calculated under the CES production function specification: lump-sum taxation with errors, non-linear income taxation, linear income taxation and first-best maximin. The lump-sum solution can be evaluated (given a set of parameter values) by searching over (t, G_N) with bounds on t from zero to 90% (0.9), and G_N, zero to 0.6. Equations (3.2) - (3.11) could only be reduced to two simultaneous non-linear equations in two unknowns, L_S, L_N. A Newton-Raphson procedure was used to provide a solution. It was this routine which displayed an element of instability from time to time; however it is well known that the basic Newton method either works very quickly, or, not at all. With this approach the unconstrained optimisation[5] NAG routine EO4JBF could be used to maximise social welfare, equation (3.12).

Another method of obtaining lump-sum solutions was to maximise social welfare subject to the four labour supply first-order conditions, (3.5) - (3.8), but with six unknowns L_S^o, L_S^1, L_N^o, L_N^1, t, G_N. Using a more sophisticated constrained optimisation routine, NAG-EO4UAF, it is also possible to place bounds on the values of the unknowns. This is useful in keeping labour supply within the (0, 1) range. This procedure was adopted for the greater complexities introduced by Allen's theorems. It is certainly very useful for economic problems which place bounds on the values of key variables.

The linear tax solution was evaluated over t running from 0 to 90%, using the NAG routine EO4ABF. When the search was widened to

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5. The EO4- NAG routines are set up for minimisation, but for maximisation all that is required is a minus sign in front of the function value. There are many different routines from which to choose, see Chapter 1, Table 1.2.

negative tax rates to examine Allen's production effect, we switched to the EO4UAF method above; but, notice we only have five unknowns when $G_S = G_N = G$. The nonlinear income tax problem was solved by EO4UAF throughout. The two constraints are equations (3.14a) and (3.15) and the two unknowns are C_S and C_N. In section 3.4 the difficulty with $W_N > W_S$ means that constraint (3.14a) has sometimes to be replaced with (3.14b). Finally with maximin the marginal tax rate equals zero and so we have a one dimensional problem in C_N which can also be solved by EO4ABF. Accuracy was to four decimal places except for computations using EO4ABF, which had 14 decimal places accuracy.

Following Stern (1982) social welfare was calibrated using the notion of the equally distributed leisurely-equivalent consumption, ${}^{o}C$, defined as

$$2U^{\nu} ({}^{o}C, 0) = \nu W_{\nu}$$

that is, "that consumption which if equally distributed and when hours of work were zero for everyone would give social welfare level W_{ν}." U is the CES utility as before. Similarly we label $\hat{\delta}$, the value of δ which gives equal welfare in both lump-sum and non-linear income tax schemes. Moreover $\delta < \hat{\delta}$ favours the lump-sum regime, and $\delta > \hat{\delta}$, non-linear income tax.

We define a <u>base run</u> parameter set as $\nu = -1$, $R = 0$, $\varepsilon = 0.5$, $\alpha = 0.5$, $\gamma = 1$, $\theta = 0.67$, $\beta = 1$ and $\sigma = 4$. To reduce a vast amount of tabulation, only results from varying the elasticity of substitution in production, σ, and the measure of attitudes to inequality, ν, will be presented in the text. It is essentially the role of σ which distinguishes this work from Stern's. Undoubtedly the use of the CES production function raises the question of appropriate values for σ.

Empirical investigation by Layard et al (1971) on the economic implications of qualified manpower indicates from their production analysis that any confidence interval for σ may be extremely large. As such we propose to work with two values of σ; one is the base run, $\sigma = 4$ above, and the other $\sigma = \frac{1}{2}$. The base run also has $\delta_S = \delta_N = \delta$, and $0 \leqslant \delta \leqslant 0.5$.

The computations for the base run are presented in Table 3.1(a) and for $\nu = 0.97$ in Table 3.1(b).[6] Tables 3.2(a) and 3.2(b) present the case $\sigma = \frac{1}{2}$. The effect of allowing $\delta_S - \delta_N = 0.1$ for the base run plus $\sigma = \frac{1}{2}$ is illustrated in Table 3.4; and, similarly for $\delta_N = 0$ in Table 3.3. The value of $\hat{\delta}$ is given at the foot of each table except for Table 3.4 where it is necessary to distinguish values of both δ_S and δ_N. Also listed is the no tax system welfare level.

An overall statement must reflect that the influence of σ does not lead to substantial qualitative differences from the results brought out by Stern's Cobb-Douglas treatment. Society's attitude towards inequality can generate significant differences in welfare levels, amply illustrated by comparing Tables 3.1(a) and 3.1(b), and 3.2(a) and 3.2(b). For Table 3.1(b), the maximand with $\nu = 0.97$, the lump-sum case with $\delta = 0.1, 0.2$ has the unskilled, incorrectly classified individual working his full one unit of time for zero consumption. In effect this is a corner solution, for it is never optimal in this framework to have individuals idle, unlike the Mirrlees formulation. However it does point to the unsatisfactory nature of the utilitarian maximand, in a world where governments can make mistakes in classification, and the degree of substitution

6. $\nu = 0.97$ is our approximation for the utilitarian maximand, $\nu = 1$. Convergence problems with the NAG routines pre-empted this approximation.

between labour types is greater than unity.

The values of $\hat{\delta}$ in Tables 3.1 and 3.2 tell a similar story. When $\nu = 0.97$ considerable misclassification is required before income taxation will be preferred: more precisely for Table 3.1(b), $\hat{\delta} = 0.379$ means that if more than 62% of individuals are correctly classified then lump-sum taxation prevails over income taxation. For Table 3.1(a) 92% of individuals need to be correctly classified to favour lump-sum taxation. Clearly when $\nu = -1$ it matters a great deal that the unskilled may face a lump-sum tax, G_S. Further evidence can be adduced from the relation between the marginal tax rate and δ under the different degrees of aversion to inequality. An equality conscious society will have tax rates rising more quickly and to higher levels with increasing δ. The influence of σ on the marginal tax rate in relation to δ is relatively minor. It appears that as σ falls the marginal tax rate rises less quickly for $\delta < 0.2$, but ultimately it attains higher levels.

The calibration of welfare in terms of consumption makes it easier to discuss the redistributive gains from taxation. Consider Table 3.1(a) (and for comparison in parentheses equivalent values for Table 3.2(a)). This brings out the importance of the elasticity of substitution. A move from no taxation to lump-sum provides a welfare gain of 0.02 (0.04) consumption units or 10.6% (19.2%). A restriction to income taxation will yield a gain of 8.1% (17.9%) for non-linear, and 3.2% (13.5%) for linear. Finally there is a 5.2% (3.3%) fall in welfare from the first best to a position where only 80% of individuals are satisfactorily screened ($\delta = 0.2$). The welfare gains from having a tax system are considerable, particularly when σ is less than unity. So the greater the extent to which individuals are trapped within their skill category, the greater the benefits to be had from a tax system.[7] Notice also that random classification ($\delta = 0.5$) corresponds roughly to the linear tax solution, as predicted. Therefore non-linear income taxation will always do better in welfare terms for the present framework.

With misclassification ($\delta \to 0.5$) gross relative and absolute wage rates of the unskilled fall in all cases. The movements are more pronounced with a falling elasticity of substitution in production. A lower degree of substitutability means that it is less easy to counteract the welfare implications of misclassification through adjustments in labour input. This point is examined in detail in the next section.

Table 3.3 demonstrates that, where the skilled alone are misclassified considerable gains in welfare are available of the order of 3% in consumption units (c.f. Table 3.3 with Tables 3.1(a) and 3.2(a) for $0 < \delta \leq 0.5$). Moreover with less strain on the redistributive function of the tax system it is not surprising that the optimum marginal tax rates fall. Table 3.4 provides a similar picture, but the orders of magnitude are rather smaller. As $\delta_N = 0$ is the extreme position, this is exactly what we should expect.

Other sensitivity tests were carried out for different values of R, β, and ε, though the tabulations have not been presented in the text. Additional needs for government revenue outside the tax system ($R > 0$) pushes marginal tax rates higher and lowers lump-sum grants for $\delta > 0$ under lump-sum tax. Labour supply and output increases but welfare declines. Both types of income taxation also require an increased output. This result is not of great significance given that

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7. This result would be useful for exponents of dual labour markets.

any benefits which might be attributed to government spending of R are ignored. An increase in ε, the elasticity of substitution between consumption and leisure reduces the marginal tax rate for $\delta > 0$ because the deadweight loss from taxation is larger. A reduction in the proportion of skilled in the population raises the marginal tax rate and lowers output. Clearly falling incomes for the unskilled increases the desirability of redistribution. Finally notice that Stern's endpoint results for income taxation continue to operate: the skilled face a marginal subsidy and the unskilled a positive marginal tax rate.[8]

The final section of the Chapter provides a computational assessment of Allen's (1982) two theorems. It turns out that the degree of substitutability between labour types along with assumptions about individual consumption-leisure choices serve to drastically alter the conventional wisdom on the structure of optimum tax rates.

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8. $(1-MTR_S) > 1$ and $(1-MTR_N) < 1$, respectively.

TABLE 3.1(a)$

The Base Run $\nu = -1$, $\sigma = 4$

Optimum Lump-Sum Taxation with Errors

δ	t	G_N	G_S	W_N	W_S	Y	^oC
0	0	0.1235	-0.1235	0.3443	0.6573	0.5896	0.2100
0.1	0.2291	0.1386	-0.0116	0.3383	0.6623	0.5542	0.2038
0.2	0.3081	0.1338	0.0321	0.3335	0.6666	0.5384	0.2001
0.3	0.3487	0.1244	0.0602	0.3302	0.6698	0.5293	0.1977
0.4	0.3689	0.1124	0.0811	0.3281	0.6719	0.5244	0.1963
0.5	0.3970	0.1022	0.1038	0.3279	0.6721	0.5187	0.1959

Optimum Non-linear Income Taxation

$(1-MTR_N)$	$(1-MTR_S)$	G_N	G_S	W_N	W_S	Y	^oC
0.7339	1.0146	0.1123	-0.0715	0.3382	0.6623	0.5706	0.2053

Optimum Linear Income Taxation

t	G	W_N	W_S	Y	^oC
0.3753	0.0981	0.3275	0.6726	0.5229	0.1959

First-Best Maxi-Min

G_N	G_S	W_N	W_S	Y	^oC
0.0938	-0.0938	0.3384	0.6622	0.5866	0.2089

$\hat{\delta} = 0.076$; No tax system welfare level: $^oC = 0.1899$

$ Notation

G_i, i=N, S = lump-sum grant intended for individuals type i

W_i, i=N, S = wage rate for individuals type i

t = marginal tax rate (MTR)

δ - proportion incorrectly classified

Y = output

^oC = equally-distributed leisurely-equivalent level of welfare

ν = inequality aversion parameter

σ = elasticity of substitution in production between different labour types

The different optima

Optimum lump-sum taxation with errors: where $\delta > 0$ some individuals receive incorrect grants. Optimum non-linear income taxation: every individual faces the same income tax schedule although they differ in their wage rates; $1\text{-}MTR_i$ is one minus the marginal tax rate and G_i is the lump-sum grant as given by the tangent to the indifference curve for individuals type i.

Optimum linear income taxation: $G = G_S = G_N$ is the common grant, so there is a one dimension optimisation with respect to t.

First-best maximin: point on the first best frontier where $V_S = V_N$.

The no tax system welfare level means that both t and G_i are equal to zero.

Other parameters

R, ε, k_o, α and β are the government revenue requirement, the elasticity of substitution between consumption and leisure, the efficiency parameter in the CES production function, the distribution parameter in the CES utility function and the proportion of individuals of each type, respectively. Excluding the final section of this chapter the

Tables present results where R = 0, $\varepsilon = 0.5$, $\gamma = 1$, $\alpha = 0.5$, $\beta = 1$. Section 3.4 distinguishes $\varepsilon_S \neq \varepsilon_N$.

TABLE 3.1(b)

$\nu = 0.97,\ \sigma = 4$

Optimum Lump-Sum Taxation with Errors*

δ	t	G_N	G_S	W_N	W_S	Y	^oC
0	0	0.5474	-0.5474	0.4930	0.6070	0.6447	0.2287
0.1	0.2183	0.4608	-0.3321	0.4249	0.6200	0.5894	0.2227
0.2	0.2025	0.4213	-0.3045	0.3818	0.6350	0.5770	0.2168
0.3	0.1906	0.3964	-0.2880	0.3561	0.6489	0.5684	0.2109
0.4	0.1819	0.3507	-0.2487	0.3368	0.6636	0.5609	0.2051
0.5	0.1860	0.0519	0.0511	0.3242	0.6761	0.5539	0.2018

Optimum Non-Linear Income Taxation

$(1-MTR_N)$	$(1-MTR_S)$	G_N	G_S	W_N	W_S	Y	^oC
0.7713	1.0012	0.1052	-0.0617	0.3369	0.6635	0.5712	0.2063

Optimum Linear Income Taxation

t	G	W_N	W_S	Y	^oC
0.1846	0.0511	0.3242	0.6761	0.5541	0.2018

First-Best Maxi-Min

G_N	G_S	W_N	W_S	Y	^oC
0.0938	-0.0938	0.3384	0.6622	0.5866	0.2089

$\hat{\delta} \approx 0.379$; No tax system welfare level: $^oC = 0.2011$

* Results for δ=0.1 and 0.2 have $C_N^1=0$, $L_N^1=1$, and for δ=0.3 and δ=0.4, equilibrium has the unskilled, misclassified individual working very hard for little return.

TABLE 3.2(a)

$\nu = -1$, $\sigma = \frac{1}{2}$

Optimum Lump-Sum Taxation with Errors

δ	t	G_N	G_S	W_N	W_S	Y	^oC
0	0	0.0803	-0.0803	0.4011	0.6041	0.5895	0.2089
0.1	0.1642	0.1196	-0.0266	0.3927	0.6114	0.5664	0.2057
0.2	0.2998	0.1417	0.0210	0.3741	0.6280	0.5425	0.2020
0.3	0.4003	0.1461	0.0625	0.3516	0.6489	0.5210	0.1985
0.4	0.4480	0.1353	0.0926	0.3326	0.6674	0.5088	0.1960
0.5	0.4596	0.1167	0.1156	0.3250	0.6751	0.5055	0.1951

Optimum Non-Linear Income Taxation

$(1-MTR_N)$	$(1-MTR_S)$	G_N	G_S	W_N	W_S	Y	^oC
0.6567	1.0771	0.1103	-0.0674	0.3983	0.6065	0.5700	0.2057

Optimum Linear Income Taxation

t	G	W_N	W_S	Y	^oC
0.4658	0.1174	0.3262	0.6738	0.5041	0.1951

First-Best Maxi-Min

G_N	G_S	W_N	W_S	Y	^oC
0.0709	-0.0709	0.3796	0.6230	0.5873	0.2084

$\hat{\delta} \simeq 0.1$; No tax system welfare level: $^oC = 0.1688$

TABLE 3.2(b)

$\nu = 0.97$, $\sigma = \frac{1}{2}$

Optimum Lump-Sum Taxation with Errors

δ	t	G_N	G_S	W_N	W_S	Y	^oC
0	0	0.1221	-0.1221	0.5053	0.5224	0.5991	0.2114
0.1	0.0027	0.1520	-0.1504	0.5032	0.5239	0.5986	0.2112
0.2	0.0088	0.2001	-0.1948	0.4985	0.5273	0.5974	0.2110
0.3	0.0279	0.2862	-0.2697	0.4851	0.5371	0.5938	0.2102
0.4	0.1397	0.4085	-0.3285	0.4226	0.5859	0.5728	0.2070
0.5	0.3262	0.0853	0.0877	0.2989	0.7023	0.5303	0.2001

Optimum Non-Linear Income Taxation

$(1-MTR_N)$	$(1-MTR_S)$	G_N	G_S	W_N	W_S	Y	^oC
0.7679	1.0010	0.0929	-0.0467	0.3727	0.6292	0.5721	0.2064

Optimum Linear Income Taxation

t	G	W_N	W_S	Y	^oC
0.3250	0.0862	0.2987	0.7025	0.5305	0.2001

First-Best Maxi-Min

G_N	G_S	W_N	W_S	Y	^oC
0.0709	-0.0709	0.3796	0.6230	0.5873	0.2084

$\hat{\delta} \approx 0.409$; No tax system welfare level: $^oC = 0.1947$

TABLE 3.3

Optimum Lump-Sum Taxation with Errors

The Base Run $\nu = -1$ $\sigma = 4$

$\delta_N = 0$

δ_S	t	G_N	G_S	W_N	W_S	Y	°C
0	0	0.1235	-0.1235	0.3443	0.6573	0.5896	0.2100
0.1	0.0603	0.1195	-0.1072	0.3420	0.6592	0.5807	0.2081
0.2	0.1137	0.1161	-0.0928	0.3398	0.6610	0.5724	0.2063
0.3	0.1611	0.1131	-0.0801	0.3378	0.6627	0.5645	0.2047
0.4	0.1977	0.1097	-0.0703	0.3360	0.6644	0.5578	0.2032
0.5	0.2395	0.1076	-0.0592	0.3343	0.6659	0.5506	0.2017

$\hat{\delta}_S \simeq 0.262$

$\nu = -1$ $\sigma = \frac{1}{2}$

$\delta_N = 0$

δ_S	t	G_N	G_S	W_N	W_S	Y	°C
0	0	0.0803	-0.0803	0.4011	0.6041	0.5895	0.2089
0.1	0.0439	0.0858	-0.0764	0.3957	0.6088	0.5834	0.2080
0.2	0.0992	0.0927	-0.0678	0.3894	0.6143	0.5754	0.2069
0.3	0.1341	0.0958	-0.0689	0.3831	0.6199	0.5699	0.2057
0.4	0.1988	0.1035	-0.0560	0.3765	0.6258	0.5596	0.2044
0.5	0.2374	0.1057	-0.0548	0.3681	0.6334	0.5526	0.2030

$\hat{\delta}_S \simeq 0.300$

TABLE 3.4

Optimum Lump-Sum Taxation with Errors

The Base Run $\delta_S - \delta_N = 0.1$, $\nu = -1$, $\sigma = 4$

δ_S	δ_N	t	G_N	G_S	W_N	W_S	Y	^oC
0.1	0	0.0603	0.1195	-0.1072	0.3420	0.6592	0.5807	0.2081
0.2	0.1	0.2616	0.1317	-0.0018	0.3361	0.6642	0.5478	0.2022
0.3	0.2	0.3290	0.1263	0.0408	0.3318	0.6682	0.5338	0.1989
0.4	0.3	0.3632	0.1173	0.0689	0.3291	0.6709	0.5260	0.1969
0.5	0.4	0.3744	0.1048	0.0895	0.3277	0.6724	0.5231	0.1960

$\delta_S - \delta_N = 0.1$, $\nu = -1$, $\sigma = \frac{1}{2}$

δ_S	δ_N	t	G_N	G_S	W_N	W_S	Y	^oC
0.1	0	0.0439	0.0858	-0.0764	0.3957	0.6088	0.5834	0.2080
0.2	0.1	0.2206	0.1240	-0.0150	0.3838	0.6193	0.5568	0.2042
0.3	0.2	0.3488	0.1405	0.0347	0.3628	0.6384	0.5324	0.2003
0.4	0.3	0.4235	0.1383	0.0734	0.3403	0.6599	0.5150	0.1971
0.5	0.4	0.4601	0.1255	0.1051	0.3275	0.6725	0.5056	0.1953

3.4 The Redistributive Impact of Relative Wages

It has already been stated that the Mirrlees (1971) incentive model has the capacity to redistribute income through the Exchequer, the fiscal effect in Allen's (1982) terminology, which essentially involves a redistribution of purchasing power based solely on the shape of the optimum tax schedule. Within this same framework Sheshinski (1972) proved that with positive labour supply elasticities redistribution should take place from rich to poor and yield lump-sum grants coupled with positive marginal tax rates.

Allen (1982) has demonstrated that for the present Feldstein type framework redistribution can also take place through the production function. Here the interdependence of labour supplies will involve general equilibrium effects on wage rates. Such an adjustment process he labels the production effect. An analytical appraisal of a linear tax model enables him to show that, for the Feldstein computations with a Cobb-Douglas production function, both the fiscal and production effects work together to redistribute from rich to poor, which effectively maintains the conventional linear tax schedule with positive intercept and positive slope. All previous computational work would appear to have reached similar conclusions.

Nevertheless Allen's (1982) two linear tax theorems indicate that where the production effect works in the opposite direction to the fiscal effect and dominates, we should expect a negative optimum marginal tax rate in conjunction with a lump-sum tax. The Exchequer may be redistributing from poor to rich but the attendant labour supply adjustment leads to a relative improvement in unskilled wages which overall makes them better off. His elasticity analysis posits that a likely candidate for this outcome arises when skilled individuals have negative, and unskilled positive, labour supply elasticities along with a low degree of substitution between labour types (low σ). To try and imitate such conditions it is necessary to work with non-identical utility functions, given that labour supply elasticities are endogenous. Essentially we allow individuals to have different consumption-leisure substitution elasticities so ε $(= \varepsilon_S = \varepsilon_N)$ no longer holds.

This requires a redefinition of the calibration of social welfare in consumption units. We propose to use the relationship

$$\beta U_S{}^{\nu}({}^{o}C,\ 0) + (2-\beta)U_N{}^{\nu}({}^{o}C,\ 0) = \nu W_{\nu}$$

which we again wish to solve for ${}^{o}C$. An explicit formula is not readily generated, so it was easier to let the computer find a numerical solution. The NAG routine C05AZF was convenient for this purpose. ${}^{o}C$ has of course a similar interpretation to before.

It is also of interest to examine the Allen arguments in terms of the optimum tax with errors framework above, rather than simply linear income taxation. This will provide a further test of the robustness of Stern's results. However the number of tabulations will be kept to a minimum.

Again it is helpful to define a base run set of parameter values which will be fixed throughout:

$$\nu = -1,\ \varepsilon_S = 0.5,\ \alpha = 0.5,\ R = 0,\ \gamma = 1,\ \theta = 0.67,\ \beta = 1.$$

We are only concerned with the influence of σ and ε_N, which can allow us to generate the case where production substitutability is low, the skilled have a negative, and the unskilled a positive, labour supply elasticity. Empirical evidence on the non-monotonicity of labour supply schedules is by no means clear cut. The closest distinction is that between low and high pay (rather than skill). Hall (1973) and

Metcalf, Nickell and Richardson (1976) find some support for a supply schedule shaped as a right hand side parenthesis - the Allen case above. However Atkinson, Stern and Gomulka (1980) support a left hand side parenthesis shape. It will become apparent that this is an important consideration for the shape of optimum-tax schedules derived from variable wage models.

Tables 3.5(a) and 3.6(a) have the same format as earlier except that the maximin outcome is no longer evaluated. Tables 3.5(b) and 3.6(b) provide the values of the labour supply elasticities for the solutions of Tables 3.5(a) and 3.6(a). With σ = $^1/2$ and $^1/5$ and, ε_N = 1.6 in both, Tables 3.5(b) and 3.6(b) show that we are able to simulate the circumstances where Allen suggests the production effect will dominate.

Table 3.5(a) indicates that the welfare levels are almost invariant with respect to δ for the lump-sum case. Moreover since the no tax system welfare level is ^oC = 16900, then the welfare gains from any form of taxation are negligible. The production effect has brought W_N/W_S close to unity, and while the marginal tax rates are negative they are rather small. This is hardly surprising given the welfare invariance to taxation. Similarly for linear and non-linear income taxation the marginal tax rates are close to zero. $\hat{\delta}$ = 0.1 is in line with earlier results.

Table 3.6(a) is even more interesting. With σ = $^1/5$ the optimum marginal tax rates for lump-sum are substantially negative. Discrimination is required to be fairly accurate, approximately 96% ($\hat{\delta}$ = 0.037) of individuals require correct classification before lump-sum is superior to income taxation. However societies distributional values do not have such a strong influence: with ν = 0.97 $\hat{\delta}$ = 0.1 ($\hat{\delta}$ = 0.2 for Table 3.5(a)). Nevertheless the movement is in the expected direction. Utilitarianism requires less accuracy in classification to favour a personalised tax regime (lump-sum). The production effect has $W_N/W_S > 1$ for $\delta \leqslant 0.3$ and fairly close to unity otherwise. The optimum non-linear income tax now has a marginal wage subsidy for the unskilled and a positive marginal tax rate for the skilled. Here, of course, the unskilled face a lump-sum tax with $G_N < 0$. Remember also that there is no scope for unemployment in the present framework. This possibility stands in stark contrast to the actual tax and welfare system, which is often criticised for having effectively 100% marginal tax rates for individuals at the bottom of the income distribution, who have to rely on welfare payments. With the reduction in σ to $^1/5$ the welfare gains from redistributive taxation are not inconsequential, like Table 3.5(a). The first-best outcome yields a gain of 19.4% in consumption units, non-linear income tax 19.1% and linear income tax 17.1%. The gain in going from income tax to first-best is rather small.

Table 3.7(a) explores the influence of the production effect in greater detail, but only with respect to income taxation. Table 3.7(b) provides the respective labour supply elasticities. The influence of ε, consumption-leisure substitution possibilities, is particularly striking. With a linear income tax system, when $\sigma = {}^1/5$, a move from $\varepsilon_S = \varepsilon_N = 0.5$ to $\varepsilon_S = 0.5$, $\varepsilon_N = 1.2$ changes the optimum marginal tax rate by 70% from 51% to -19%. The labour supply elasticity switch is also clear from Table 3.7(b). Not surprisingly we witness a very large relative wage effect with W_N rising 47% and W_S falling 23%. The same parameter changes give a considerable jolt to the optimum non-linear tax solutions. The endpoint conditions switch round because the unskilled now appear to have a higher social marginal utility of consumption. Hence constraint (3.14b) was appropriate to evaluate this outcome. Finally notice that we found an optimum linear marginal tax rate as low as -51%. It serves to underline the rather important interactions between Allen's fiscal and production effects, and casts doubt on Feldstein's claim that variable wage tax models have little effect on the structure of optimum taxation.

TABLE 3.5(a)[$]

Base run with $\sigma = \frac{1}{2}$ and $\varepsilon_N = 1.6$

Optimum Lump-Sum Taxation with Errors

δ	t	G_N	G_S	W_N	W_S	Y	^oC
0.0	0	-0.0054	0.0058	0.5120	0.5177	0.5072	0.16916
0.1	-0.0003	-0.0064	0.0062	0.5127	0.5172	0.5075	0.16915
0.2	-0.0190	-0.0086	-0.0011	0.5102	0.5190	0.5114	0.16914
0.3	-0.0266	-0.0094	-0.0043	0.5094	0.5196	0.5130	0.16913
0.4	-0.0304	-0.0092	-0.0064	0.5092	0.5197	0.5138	0.16913
0.5	-0.0326	-0.0084	-0.0084	0.5088	0.5200	0.5142	0.16912

Optimum Non-Linear Income Taxation*

$(1-MTR_N)$	$(1-MTR_S)$	G_N	G_S	W_N	W_S	Y	^oC
0.9666	1.0196	-0.0027	0.0039	0.5208	0.5116	0.5046	0.16915

* required constraint (3.14b) and likewise for Table 3.6(a).

Optimum Linear Income Taxation

t	G	W_N	W_S	Y	^oC
-0.0323	-0.0083	0.5089	0.5199	0.5142	0.16912

$\hat{\delta} = 0.10$ No tax system welfare level: $^oC = 0.16900$

$ see the notes to Table 3.1(a). All solutions for this section of Tables were generated by NAG routine E04UAF

TABLE 3.5(b) LABOUR SUPPLY ELASTICITIES$

OPTIMUM LUMP SUM TAXATION WITH ERRORS

δ	ELS^o	ELS^1	ELN^o	ELN^1
0.0	−0.21	−0.21	0.33	0.39
0.1	−0.21	−0.21	0.33	0.37
0.2	−0.21	−0.22	0.32	0.35
0.3	−0.21	−0.22	0.32	0.34
0.4	−0.22	−0.22	0.32	0.32
0.5	−0.22	−0.22	0.32	0.32

OPTIMUM INCOME TAXATION

ELS	ELN
−0.21	0.35

OPTIMUM LINEAR TAXATION

ELS	ELN
−0.22	0.32

$ ELS ≡ skilled elasticity of labour supply

ELN ≡ unskilled elasticity of labour supply superscript o refers to individuals correctly classified, 1 to those misclassified.

TABLE 3.6(a)

Base Run with $\sigma = {}^{1}/5$ and $\varepsilon_N = 1.6$

Optimum Lump Sum Taxation with Errors

δ	t	G_N	G_S	W_N	W_S	Y	^oC
0.0	0	-0.0465	0.0465	0.5618	0.4589	0.5147	0.16628
0.1	-0.1006	-0.0712	0.0173	0.5466	0.4715	0.5354	0.16539
0.2	-0.2158	-0.0967	-0.0233	0.5289	0.4866	0.5564	0.16457
0.3	-0.3135	-0.1148	-0.0645	0.5130	0.5002	0.5721	0.16396
0.4	-0.3724	-0.1203	-0.0961	0.5042	0.5078	0.5809	0.16361
0.5	-0.3908	-0.1147	-0.1133	0.5015	0.5102	0.5836	0.16350

Optimum Non-Linear Income Taxation

$(1-MTR_N)$	$(1-MTR_S)$	G_N	G_S	W_N	W_S	Y	^oC
1.2192	0.8383	-0.0628	0.0576	0.4870	0.5229	0.5227	0.16595

Optimum Linear Income Taxation

t	G	W_N	W_S	Y	^oC
-0.3905	-0.1139	0.5018	0.5099	0.5836	0.16350

$\hat{\delta} = 0.037$ No tax system welfare level: $^oC = 0.13930$

TABLE 3.6(b) LABOUR SUPPLY ELASTICITIES

OPTIMUM LUMP SUM TAXATION WITH ERRORS

δ	ELS^o	ELS^1	ELN^o	ELN^1
0.0	-0.17	-0.23	0.18	0.56
0.1	-0.20	-0.25	0.12	0.41
0.2	-0.23	-0.27	0.08	0.27
0.3	-0.26	-0.29	0.05	0.16
0.4	-0.28	-0.29	0.05	0.06
0.5	-0.29	-0.29	0.06	0.06

OPTIMUM INCOME TAXATION

ELS	ELN
-0.15	0.14

OPTIMUM LINEAR TAXATION

ELS	ELN
-0.29	0.06

TABLE 3.7(a)

Optimum Linear Income Taxation

t	$G_S=G_N$	W_N	W_S	Y	^{o}C	ε_N**	σ
0.3055	0.0686	0.3709	0.6403	0.4490	0.1715	1.2	4.0
0.2635	0.0556	0.3994	0.6280	0.4218	0.1718	1.6	4.0
0.2232	0.0451	0.4257	0.6198	0.4043	0.1738	2.0	4.0
0.1157	0.0294	0.4825	0.5391	0.5080	0.1734	1.2	0.5
-0.1439	-0.0373	0.5216	0.5110	0.5191	0.1666	2.0	0.5
0.5105	0.1262	0.3420	0.6580	0.4943	0.1952	0.5	0.2
-0.1976	-0.0563	0.5031	0.5088	0.5699	0.1710	1.2	0.2
-0.5080	-0.1500	0.5006	0.5110	0.5907	0.1591	2.0	0.2

Optimum Non-Linear Income Taxation***

$(1-MTR_N)$	$(1-MTR_S)$	G_N	G_S	W_N	W_S	Y	^{o}C	ε_N**	σ
1.0038	1.0000	0.0942	-0.0947	0.3735	0.6390	0.514	0.1787	1.2	4.0
1.1734	0.9990	0.0836	-0.1012	0.3855	0.6334	0.5014	0.1754	1.6	4.0
1.3173	0.99861	0.0747	-0.1048	0.3930	0.6304	0.4946	0.1738	2.0	4.0
1.0516	0.9859	0.0159	-0.0222	0.4730	0.5461	0.5365	0.1739	1.2	0.5
1.0867	0.9517	-0.0321	0.0284	0.5156	0.5151	0.4948	0.1672	2.0	0.5
0.6912	1.3066	0.0888	-0.0952	0.5277	0.4876	0.5841	0.2075	0.5	0.2
1.0866	0.9322	-0.0277	0.0261	0.4988	0.5126	0.5393	0.1717	1.2	0.2
1.3223	0.7724	-0.0884	0.0801	0.4775	0.5313	0.5091	0.1631	2.0	0.2

** The other parameters are defined by the base run.

*** Non-linear optima for σ = 0.5 and 0.2 required the use of constraint (3.14b).

TABLE 3.7(b) LABOUR SUPPLY ELASTICITIES

OPTIMUM LINEAR TAXATION

ELS	ELN	ε_N	σ
-0.14	0.76	1.2	4.0
-0.16	1.44	1.6	4.0
-0.17	2.15	2.0	4.0
-0.18	0.22	1.2	0.5
-0.24	0.44	2.0	0.5
-0.04	-0.08	0.5	0.2
-0.25	0.01	1.2	0.2
-0.31	0.15	2.0	0.2

OPTIMUM INCOME TAXATION

ELS	ELN	ε_N	σ
-0.28	0.64	1.2	4.0
-0.28	1.05	1.6	4.0
-0.28	1.43	2.0	4.0
-0.24	0.25	1.2	0.5
-0.18	0.46	2.0	0.5
-0.28	-0.10	0.5	0.2
-0.18	0.04	1.2	0.2
-0.12	0.25	2.0	0.2

3.5 Conclusion

The incorporation of a CES production function into the optimum variable wage tax model has provided some interesting results. First is the increasing welfare gains from having a tax system when the degree of substitutability between labour types fell. With $\sigma = \frac{1}{2}$ (Table 3.2) first-best lump-sum taxation improved welfare by almost 20% in consumption units. Second is the important role for society's views on inequality. Under utilitarianism ($\nu = 0.97$) the government classification scheme could be highly inaccurate but still be preferable to income taxation. Moreover when $\sigma = 4$, a situation with the unskilled incorrectly classified working full time for zero consumption was still preferable to income taxation, (Table 3.2(b)). This reflects a weakness in the approach where it is impossible for any workers to remain idle. One might also criticise the fact that governments make mistakes, yet firms have no problem in classifying the workers.

Under the Allen hypotheses Table 3.5 raised the possibility that the welfare gains from having a tax system may be small. This and the above remarks indicate an extension of the range of outcomes found by Stern (1982), rather than any contradiction. It is also apparent that the neglect of Allen's production effect has been of significance. In particular the shape of potential optimum-tax schedules has been given much wider license than previously thought from the Mirrlees and Seade theorems. The key features are the relationships between labour supply elasticities and the degree of substitution in production between labour types. Tables 3.6 and 3.7 provide optimum tax structures with substantial negative marginal tax rates. Intuitively with high ε_N we have large quantity responses in unskilled labour with respect to wage rates, and low σ gives big wage responses to changes in the quantities of labour. Here lies the power of the production effect for variable wage tax models. The Feldstein (1973) claim that endogenous wages have little effect on optimum tax rates is no longer tenable.

The policy implications of wage subsidies provide the reverse reaction to the real world system. There the game is to under-report income (if possible!). Here the incentive would be to inflate declared earnings. At least our present system has a lower bound.

Finally the NAG library has been put to extensive use in this chapter; in particular the routine E04UAF was very effective for this type of optimum-tax problem. I should recommend it to anyone working with the Feldstein framework.

Chapter 4: A Two Sector General Equilibrium Model of the Determination of Union and Non-Union Wage Rates in a Closed Economy

4.1 Introduction

The concept of equilibrium can be handled relatively easily for a two sector model using computational techniques. When the work was carried out the equations were solved by an algorithm due to Powell (1970), which was available through the NAG C05NAF.

It would seem that much of the union/non-union literature, for example, Johnson and Meiszkowski (1970), Jones (1971) and Magee (1971), (1973) have used the conventional two sector model of general equilibrium analysis to examine the effects on factor prices of a rise in the wage differential earned by unionised workers. Another interesting question for a partially unionised economy, where union workers force up their absolute wage rate, is how this will affect the wage paid in the non-union sector. It turns out that for a closed economy unambiguous theoretical answers are impossible to obtain, so numerical techniques are employed to try and isolate different possibilities.

Carruth and Oswald (1981) also examine an open economy framework which yields clear results, essentially that in a small open economy, a rise in the union wage rate will always increase the non-union wage rate and decrease the rental rate on capital. It can also be shown that we require the unionised sector to be capital intensive.

An examination of the determination of input prices can be simplified by an appeal to the minimum cost functions of duality theory. It is implicitly assumed that the trade union is a rational agent and acts as if maximising a utility function subject to constraints. Imagine an economy producing two types of output, using two factors of production, capital and labour. Assume that labour in one sector is unionised and that labour in the other sector has its wage determined on a competitive market. Let w be the union wage and n be the non-union wage. Assume that both factors are in fixed supply in the economy as a whole and denote the capital and labour endowments by K and L respectively. Let both unionised output, x, and non-unionised output, y, be produced under conditions of constant returns to scale; and assume, taking the price of x as the numeraire, that p is the relative price of output from the non-unionised sector.[1]

With constant returns to scale the unit minimum cost of production in the union sector is $c(w, r)$, where r is the rental on capital, and the unit minimum cost of production in the non-union sector is $\phi(n, r)$. By standard theorems we know that

(i) $c(.\ .)$, $\phi(.\ .)$ are increasing, concave and homogeneous of degree 1

(ii) unit input demands are given by appropriate partial derivatives of $c(.\ .)$, $\phi(.\ .)$.

It is also assumed that both minimum cost functions are twice differentiable.

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1. In an open economy framework p would be determined on world markets, and so would be exogenous. Note also that wage in terms of own production is being treated as numèraire.

4.2 Wages in a Closed Economy with Endogenous Prices

We follow convention - see Jones (1971), for example - and assume that all consumers have identical, homothetic preferences. Thus, in equilibrium we may write[2]

$$y = x.f(p) \qquad f'(p) < 0 \tag{4.1}$$

where y is output from the non-union sector, x is output from the union sector, and f(p) is a negatively sloped (relative) demand curve. The full model is given by the following equations

$$p = \phi(n, r) \tag{4.2}$$

$$1 = c(w, r) \tag{4.3}$$

$$L = xc_w + y\phi_n \tag{4.4}$$

$$K = xc_r + y\phi_r \tag{4.5}$$

$$y = x\, f(p) \tag{4.6}$$

where L and K are constants, and w is treated as a parameter set by the union, the details of the process being unimportant for the analysis to come. Equations (4.4) and (4.5), the input demands, are kept relatively simple through the power of the cost function approach.

It is helpful to eliminate some equations. Using (4.4) and (4.5) it is easy to show that

$$y/x = \frac{Lc_r - Kc_w}{K\phi_n - L\phi_r} \tag{4.7}$$

Thus equation (4.6) becomes

$$Lc_r - Kc_w = -\{L\phi_r - K\phi_n\}f(p) \tag{4.8}$$

---

2. A prime denotes a derivative and a subscript a partial derivative.

We now have a 3 equation system - equations (4.2) and (4.3), and equation (4.8) above. Totally differentiating, we may write down the following matrix system:

$$\begin{bmatrix} \phi_n & \phi_r & -1 \\ 0 & c_r & 0 \\ A & B & C \end{bmatrix} \begin{bmatrix} dn/dw \\ dr/dw \\ dp/dw \end{bmatrix} = \begin{bmatrix} 0 \\ -c_w \\ D \end{bmatrix} \qquad (4.9)$$

where

$$A = \{L\phi_{nr} - K\phi_{nn}\}\ f(p) > 0 \qquad (4.10)$$

$$B = Lc_{rr} - Kc_{wr} + \{L\phi_{rr} - K\phi_{nr}\}\ f(p) < 0 \qquad (4.11)$$

$$C = f'(p)\ \{L\phi_r - K\phi_n\} \gtrless 0 \qquad (4.12)$$

$$D = -\{Lc_{wr} - Kc_{ww}\} < 0 \qquad (4.13)$$

It is now easy to show how a rise in the union wage affects the non-union wage, the rental rate and the relative commodity price. By standard methods we find

$$dn/dw = \frac{1}{J}\{\phi_r c_w C + c_w B + c_r D\} \qquad (4.14)$$

$$dr/dw = \frac{-1}{J}\{\phi_n c_w C + c_w A\} \qquad (4.15)$$

$$dp/dw = \frac{1}{J}\{\phi_n(c_r D + Bc_w) - \phi_r A c_w\} \qquad (4.16)$$

where the determinant of the matrix in equation (4.9) is

$$J = c_r(A + C\phi_n) \qquad (4.17)$$

Now $J \gtrless 0$ as $A + C\phi_n \gtrless 0$. But as $A > 0$, a sufficient condition for $J > 0$ is $C > 0$, whilst a necessary condition for $J < 0$ is $C < 0$.

Moreover, as

$$C = f'(p) \{L\phi_r - K\phi_n\}, \tag{4.18}$$

C takes the opposite sign of the term in curly brackets, which itself is obviously positive (negative) when the union (non-union) sector is labour-intensive. Put differently, C is greater than or less than zero as the union sector's degree of capital-intensitivity is greater than or less than the non-union sector's degree of capital-intensivitity.

Equations (4.14) - (4.16) cannot be signed unambiguously, and this should not surprise us. Nevertheless, it seems reasonable to expect that a little progress can be made by contrasting the case in which the union sector is capital-intensive with that in which it is labour-intensive. Now if the unionised sector is capital intensive, $C > 0$ and $J > 0$, so by equation (4.14) - (4.16) we have

$$\text{sign } (dn/dw) = \text{sign } (\phi_r c_w C + c_w B + c_r D) \gtrless 0 \tag{4.19}$$

$$\text{sign } (dr/dw) = \text{sign } - (\phi_n c_w C + c_w A) < 0 \tag{4.20}$$

$$\text{sign } (dp/dw) = \text{sign } (\phi_n c_r D + Bc_w \phi_n - \phi_r A c_w) < 0 \tag{4.21}$$

If the unionised sector is labour-intensive, $C < 0$ and $J \gtrless 0$. Hence we find that

$$\text{sign } (dn/dw) = \text{sign } (-J) \gtrless 0 \tag{4.22}$$

$$\text{sign } (dr/dw) = \text{sign } (-1/J) \{\phi_n c_w C + c_w A\}) \gtrless 0 \tag{4.23}$$

$$\text{sign } (dp/dw) = \text{sign } (-J) \gtrless 0 \tag{4.24}$$

Although most of the ambiguity still remains, two definite results have emerged: when the union sector is capital-intensive a rise in the union wage rate lowers both (i) the rental rate on capital and (ii) the relative price of non-union output. But the relationship between the wage rates, the principal concern of this chapter, is still not clear.

Carruth and Oswald (1981) also examine some special cases, and how the sign of dn/dw is affected by varying some key elasticities. Here we want to confront the ambiguity directly by numerical methods, and study how the results are influenced by particular functional forms and parameter values.

4.3 A Numerical Evaluation of the Union-Non-Union Wage Partial Derivative

In a closed two-sector economy a rise in the union wage rate may raise or reduce the non-union wage rate. The results of the previous section do not allow us to be more specific, so we cannot say anything about which is most likely in the real world. To try and overcome such a problem the numerical approach may be of service.

We experimented with Cobb-Douglas and Constant Elasticity of Substitution (C.E.S.) production functions, and with constant elasticity and linear demand schedules. Taking the non-union sector as an example, the unit cost functions for the above two production functions are respectively

$$\phi(n,\ r) = \frac{1}{A}\,(r/a_1)^{a_1}\cdot(n/a_2)^{a_2} \tag{4.25}$$

$$\phi(n,\ r) = \frac{1}{A}\,\{(a_3 r^{\rho})^{1/(1+\rho)} + (a_4 n^{\rho})^{1/(1+\rho)}\}^{1/(1+\rho)} \tag{4.26}$$

The usual interpretation attaches to the different parameters of the Cobb-Douglas and C.E.S. functions. Equivalent functions were adopted for the union sector but with parameters labelled B, b_1, b_2, b_3, b_4. We also used linear and iso-elastic demand functions, defined respectively as

$$\begin{aligned} f(p) &= g - hp \\ f(p) &= gp^{-e} \end{aligned} \tag{4.27}$$

where e is the elasticity of demand and g and h are constants.

A <u>base run</u> parameter set was defined as follows (where K and L are capital and labour supplies)

A	B	$a_{1,3}$	$a_{2,4}$	$b_{1,3}$	$b_{2,4}$	K	L	g	e
1	1	.25	.75	.25	.75	10	10	1	.5

For the C.E.S. case values of $\rho = 1$ or -0.75[3] were adopted, except for the mixed model where ρ was systematically varied in the C.E.S. sector. The returns to scale parameters were set roughly in line with empirical estimates, see Nerlove (1967), among others. There is also a certain amount of information on likely values of the demand elasticity, and sensitivity tests were conducted for e in the range 0.25 to 1.5.

Given a parameter set it is desirable to search over a range of values for w to determine how the equilibrium configuration, particularly the derivative $\frac{dn}{dw}$, adjusts. Empirical evidence suggests a range of values for w between 0 per cent and 30 per cent above n. We shall use the 0 per cent as a lower bound but set an upper bound a good deal higher than the 30 per cent mark-up. One intermediate solution will also be provided in each case. As a matter of course we solved each problem over a larger range and finer grid, including a wide variety of sensitivity tests. To present all this in the main body of the text would be burdensome, particularly when a very small subset of results illustrates the main points.[4] Furthermore, the Cobb-Douglas results can be approximately obtained from the C.E.S. programme through the adoption of a small ρ value (equal to 0.0001). This provides a good check on the consistency of the computer programmes, which were written independently. As such the Cobb-Douglas function is restricted to the mixed model of Table 4.2 where the union sector is Cobb-Douglas and the non-union sector C.E.S. Table 4.1 presents solutions with both sectors adopting the C.E.S. form.

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3. In other words $\sigma = \frac{1}{2}$ or 4, as in the previous chapter on optimum taxation.

4. Additional Tables of sensitivity tests are given in an appendix to the chapter.

Table 4.1 illustrates two cases with $\rho = -0.75$ and $\rho = 1.00$. Demand is iso-elastic. Table 4.2 has ρ changing systematically for the mixed model plus a linear relative commodity demand schedule. The algorithm usually converged fairly rapidly - much of course depending on the starting values. It is also worth remembering that the numerical routines can only evaluate local solutions.

The striking thing about the results is their simplicity and similarity: in all cases the derivative dn/dw was negative and declined in absolute value under an increasing mark-up of union over non-union wages. It is obvious that we cannot lay too heavy an emphasis upon this sort of test. However, the parameters chosen were within the bounds indicated by empirical estimates, and the computations were remarkably consistent,[5] so that in practice a rise in the union wage is likely to depress the non-union wage rate of a closed economy. Additional Tables of results are confined to an appendix. Their inclusion is solely to back up the consistent appearance of this inverse relationship between union and non-union wage rates.

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5. The Tables in the appendix provide additional support for this view.

TABLE 4.1[a]

C.E.S. Results

w	p	x	y	n	r	ρ	$\frac{dn}{dw}$
0.750	1.00	5.00	5.00	0.75	0.25	-0.75	-7.00
0.825	0.57	4.16	5.48	0.40	0.21	-0.75	-3.28
0.900	0.32	3.30	5.87	0.22	0.19	-0.75	-1.69
0.750	1.00	5.00	5.00	0.75	0.25	1.00	-7.00
0.825	0.57	4.27	5.67	0.39	0.18	1.00	-3.25
0.900	0.33	3.56	6.20	0.21	0.13	1.00	-1.71

[a] "base-run" parameter set as specified in the text.

w = union wage; p = relative price of non-union output; x = union output; y = non-union output; n = non-union wage; r = rental rate.

The algorithm used was that specified by NAG routine C05NAF which finds a solution to N nonlinear equations in N variables, and was suggested by Powell (1970).

TABLE 4.2[b]

Mixed Model Results

w	p	x	y	n	r	ρ	$\frac{dn}{dw}$
0.750	1.00	3.33	6.67	0.75	0.25	-0.65	-3.70
0.825	0.70	3.01	6.93	0.51	0.19	-0.50	-2.52
0.900	0.48	2.78	7.01	0.34	0.14	-0.35	-1.74

[b] Union Sector: Cobb Douglas; Non-Union Sector: C.E.S.; Linear demand Schedule; Variable ρ; Parameter Values: "base-run" except for linear demand where g=3 and h=1. C05NAF was also used to solve for this model.

4.4 Conclusions

This chapter has tried to explore the way in which wages are determined in a partially unionised, closed economy. We have been concerned, in particular, with the mechanisms by which the wage paid to non-unionised workers is affected by a rise in the wage received by union men. The ambiguity which arose in this relationship for the closed economy made us fall back on computational procedures. Therefore a tentative conclusion would be that in a closed economy (or one with some monopoly power in world trade) a rise in the union wage rate is likely to depress the non-union wage rate.

Finally it seems sensible to think of this result as complementing, rather than competing with the Johnson and Meiskowski (1970), Jones (1971) and others, literature. Carruth and Oswald (1981) essentially show that the key empirical question for the absolute wage approach is whether the economy approximates well or badly to the 'small open country' assumption of economic theory.

Appendix to Chapter 4

The following set of tabulations provides sensitivity tests for the CES cost function model (in both sectors). The value of $\rho = -0.75$ implies $\sigma = 4$. The case $\sigma = \frac{1}{2}$ with identical tests yields exactly the same sign for dn/dw in all computations. It is helpful to list the base run parameter set and then only note the value of the parameter which has changed for each sensitivity test. It should also be stated that similar results were found for the mixed model. In fact over a considerable number of computations dn/dw was never positive.

Clearly the number of parameter permutations are vast especially if more than one parameter is varied at once. Often there is an element of offset, so only single changes are presented, except for the values of the factor share parameters (a_1, a_2, b_1, b_2).

Sensitivity Tests

CES Cost function in both sectors and iso-elastic demand operate in all the results to follow. Similarly $\rho = -0.75$ ($\Rightarrow \sigma = 4$) holds in all cases. Remember W is the exogenous variable.

Base run:	A	B	A1	A2	B1	B2	K	L	G	H
	1	1	0.25	0.75	0.25	0.75	10	10	1	0.5

W	P	X	Y	N	R	dn/dw
0.700	1.79	5.50	4.11	1.48	0.37	-36.34
0.725	1.25	5.24	4.68	0.98	0.28	-12.07
0.750	1.00	5.00	5.00	0.75	0.25	-7.00
0.775	0.83	4.74	5.20	0.60	0.23	-4.98
0.800	0.69	4.45	5.35	0.49	0.22	-3.93
0.825	0.57	4.16	5.48	0.40	0.21	-3.28
0.850	0.47	3.85	5.60	0.33	0.20	-2.75
0.875	0.39	3.56	5.73	0.27	0.19	-2.21
0.900	0.32	3.30	5.87	0.22	0.19	-1.69
0.925	0.26	3.08	5.99	0.18	0.19	-1.26
0.950	0.22	2.89	6.11	0.15	0.18	-0.93
0.975	0.19	2.74	6.20	0.13	0.18	-0.70
1.000	0.17	2.61	6.28	0.12	0.18	-0.54

W ≡ UNION WAGE

P ≡ NON-UNION RELATIVE PRICE OF OUTPUT

X ≡ UNION OUTPUT

Y ≡ NON-UNION OUTPUT

N ≡ NON-UNION WAGE

R ≡ RENTAL RATE

H	/ W	P	X	Y	N	R	dn/dw
0.25	0.700	1.75	5.16	4.49	1.43	0.37	-32.10
	0.725	1.25	5.10	4.82	0.97	0.28	-11.43
	0.750	1.00	5.00	5.00	0.75	0.25	-7.00
	0.775	0.82	4.85	5.09	0.60	0.23	-5.32
	0.800	0.67	4.65	5.14	0.48	0.22	-4.70
	0.825	0.52	4.39	5.18	0.36	0.21	-4.71
	0.850	0.35	4.06	5.28	0.24	0.20	-4.72
	0.875	0.20	3.68	5.49	0.14	0.19	-3.11
	0.900	0.12	3.36	5.72	0.08	0.19	-1.58
	0.925	0.08	3.11	5.90	0.05	0.19	-0.84
	0.950	0.05	2.91	6.04	0.04	0.18	-0.48
	0.975	0.04	2.75	6.16	0.03	0.18	-0.30
	1.000	0.03	2.62	6.25	0.02	0.18	-0.20

H	/ W	P	X	Y	N	R	dn/dw
0.75	0.700	1.83	5.84	3.71	1.55	0.37	-42.40
	0.725	1.26	5.39	4.53	0.98	0.28	-12.80
	0.750	1.00	5.00	5.00	0.75	0.25	-7.00
	0.775	0.83	4.63	5.31	0.61	0.23	-4.68
	0.800	0.71	4.28	5.54	0.51	0.22	-3.44
	0.825	0.61	3.96	5.72	0.43	0.21	-2.65
	0.850	0.53	3.66	5.87	0.37	0.20	-2.09
	0.875	0.47	3.40	6.00	0.33	0.19	-1.65
	0.900	0.42	3.18	6.10	0.29	0.19	-1.31
	0.925	0.38	2.98	6.20	0.26	0.19	-1.04
	0.950	0.34	2.82	6.28	0.24	0.19	-0.84
	0.975	0.32	2.68	6.35	0.22	0.18	-0.68
	1.000	0.29	2.57	6.41	0.20	0.18	-0.55

H	W	P	X	Y	N	R	dn/dw
1.0	0.700	1.88	6.20	3.29	1.64	0.37	-51.74
	0.725	1.26	5.54	4.37	0.99	0.28	-13.66
	0.750	1.00	5.00	5.00	0.75	0.25	-7.00
	0.775	0.84	4.53	5.42	0.61	0.23	-4.43
	0.800	0.72	4.13	5.71	0.52	0.22	-3.10
	0.825	0.64	3.78	5.93	0.45	0.21	-2.29
	0.850	0.57	3.48	6.10	0.40	0.20	-1.74
	0.875	0.52	3.24	6.23	0.36	0.19	-1.36
	0.900	0.48	3.03	6.34	0.33	0.19	-1.08
	0.925	0.44	2.86	6.42	0.31	0.19	-0.87
	0.950	0.42	2.71	6.49	0.29	0.18	-0.71
	0.975	0.39	2.59	6.55	0.27	0.18	-0.59
	1.000	0.38	2.48	6.60	0.26	0.18	-0.49

H	W	P	X	Y	N	R	dn/dw
1.25	0.700	1.95	6.57	2.85	1.76	0.37	-67.62
	0.725	1.27	5.69	4.21	1.00	0.28	-14.68
	0.750	1.00	5.00	5.00	0.75	0.25	-7.00
	0.775	0.84	4.44	5.51	0.61	0.23	-4.22
	0.800	0.73	3.98	5.87	0.53	0.22	-2.84
	0.825	0.66	3.62	6.12	0.47	0.21	-2.04
	0.850	0.60	3.32	6.30	0.42	0.20	-1.52
	0.875	0.55	3.08	6.44	0.39	0.19	-1.18
	0.900	0.52	2.89	6.55	0.36	0.19	-0.93
	0.925	0.49	2.72	6.63	0.34	0.19	-0.75
	0.950	0.47	2.59	6.70	0.32	0.18	-0.62
	0.975	0.45	2.48	6.75	0.31	0.18	-0.51
	1.000	0.43	2.38	6.79	0.30	0.18	-0.43

G	/ W	P	X	Y	N	R	dn/dw
2.0	0.700	1.65	3.82	5.96	1.29	0.37	-27.20
	0.725	1.20	3.52	6.43	0.92	0.28	-8.91
	0.750	1.00	3.33	6.67	0.75	0.25	-5.00
	0.775	0.88	3.18	6.79	0.65	0.23	-3.38
	0.800	0.79	3.04	6.68	0.57	0.22	-2.52
	0.825	0.72	2.92	6.90	0.52	0.21	-1.98
	0.850	0.66	2.81	6.91	0.47	0.20	-1.62
	0.875	0.62	2.71	6.92	0.44	0.19	-1.36
	0.900	0.57	2.62	6.92	0.40	0.19	-1.16
	0.925	0.54	2.53	6.91	0.38	0.19	-1.00
	0.950	0.51	2.46	6.91	0.35	0.18	-0.88
	0.975	0.48	2.39	6.90	0.33	0.18	-0.77
	1.000	0.45	2.32	6.89	0.32	0.18	-0.68

G	/ W	P	X	Y	N	R	dn/dw
0.5	0.700	2.02	6.90	2.43	1.94	0.37	-64.88
	0.725	1.35	6.89	2.96	1.10	0.28	-18.87
	0.750	1.00	6.67	3.33	0.75	0.25	-11.00
	0.775	0.72	6.21	3.67	0.51	0.23	-8.64
	0.800	0.45	5.52	4.10	0.31	0.22	-7.02
	0.825	0.26	4.74	4.66	0.18	0.21	-3.74
	0.850	0.16	4.13	5.12	0.11	0.20	-1.79
	0.875	0.11	3.69	5.46	0.08	0.19	-0.97
	0.900	0.09	3.36	5.71	0.06	0.19	-0.58
	0.925	0.07	3.11	5.90	0.05	0.19	-0.38
	0.950	0.06	2.91	6.04	0.04	0.18	-0.26
	0.975	0.05	2.75	6.16	0.03	0.18	-0.19
	1.000	0.04	2.61	6.25	0.03	0.18	-0.14

K	/ W	P	X	Y	N	R	dn/dw
5.0	0.700	1.49	4.70	3.85	1.12	0.37	-27.92
	0.725	0.98	4.32	4.35	0.71	0.28	-10.79
	0.750	0.70	3.90	4.67	0.49	0.25	-7.62
	0.775	0.46	3.38	5.00	0.31	0.23	-6.30
	0.800	0.27	2.81	5.38	0.18	0.22	-3.82
	0.825	0.17	2.37	5.73	0.12	0.21	-1.91
	0.850	0.12	2.07	5.97	0.08	0.20	-1.03
	0.875	0.09	1.85	6-14	0.06	0.19	-0.62
	0.900	0.07	1.68	6.26	0.05	0.19	-0.40
	0.925	0.06	1.55	6.36	0.04	0.19	-0.28
	0.950	0.05	1.46	6.43	0.03	0.18	-0.20
	0.975	0.04	1.37	6.49	0.03	0.18	-0.15
	1.000	0.04	1.31	6.53	0.03	0.18	-0.11

L	/ W	P	X	Y	N	R	dn/dw
5.0	0.700	2.10	3.32	2.29	2.17	0.37	-64.51
	0.725	1.50	3.26	2.67	1.36	0.28	-17.48
	0.750	1.24	3.21	2.88	1.05	0.25	-8.95
	0.775	1.08	3.14	3.02	0.87	0.23	-5.69
	0.800	0.96	3.05	3.11	0.75	0.22	-4.04
	0.825	0.87	2.96	3.18	0.66	0.21	-3.08
	0.850	0.79	2.86	3.22	0.59	0.20	-2.45
	0.875	0.73	2.77	3.24	0.54	0.19	-2.01
	0.900	0.68	2.68	3.26	0.49	0.19	-1.69
	0.925	0.63	2.60	3.28	0.45	0.19	-1.44
	0.950	0.59	2.52	3.29	0.42	0.18	-1.24
	0.975	0.55	2.44	3.30	0.39	0.18	-1.08
	1.000	0.52	2.38	3.30	0.36	0.18	-0.94

A	/ W	P	X	Y	N	R	dn/dw
2.0	0.700	0.95	6.26	6.43	1.66	0.37	-45.64
	0.725	0.65	6.10	7.58	1.03	0.28	-14.77
	0.750	0.50	5.86	8.28	0.75	0.25	-8.66
	0.775	0.39	5.52	8.80	0.56	0.23	-6.40
	0.800	0.30	5.08	9.28	0.42	0.22	-5.32
	0.825	0.22	4.58	9.83	0.30	0.21	-4.28
	0.850	0.15	4.09	10.44	0.21	0.20	-2.89
	0.875	0.11	3.68	11.00	0.15	0.19	-1.78
	0.900	0.08	3.36	11.46	0.12	0.19	-1.12
	0.925	0.07	3.11	11.82	0.09	0.19	-0.74
	0.950	0.06	2.91	12.10	0.08	0.18	-0.52
	0.975	0.05	2.75	12.32	0.07	0.18	-0.38
	1.000	0.04	2.61	12.51	0.06	0.18	-0.28

B	/ W	P	X	Y	N	R	dn/dw
2.0	1.450	2.44	8.72	5.58	1.88	0.57	-10.21
	1.475	2.19	8.49	5.74	1.66	0.53	-7.45
	1.500	2.00	8.28	5.86	1.50	0.50	-5.83
	1.525	1.84	8.08	5.95	1.37	0.48	-4.77
	1.550	1.71	7.88	6.02	1.26	0.46	-4.03
	1.575	1.60	7.96	6.08	1.16	0.45	-3.49
	1.600	1.50	7.49	6.12	1.08	0.43	-3.07
	1.625	1.41	7.30	6.15	1.01	0.42	-2.75
	1.650	1.33	7.12	6.18	0.94	0.42	-2.48
	1.675	1.25	6.94	6.21	0.89	0.41	-2.26
	1.700	1.18	6.76	6.23	0.83	0.40	-2.08
	1.725	1.11	6.59	6.25	0.78	0.40	-1.92
	1.750	1.05	6.43	6.26	0.74	0.39	-1.77

A1	A2	B1	B2	/	W	P	X	Y	N	R	dn/dw
0.5	0.5	0.5	0.5		0.450	1.31	5.16	4.50	0.79	0.58	-11.72
					0.475	1.12	5.11	4.82	0.60	0.53	-5.17
					0.500	1.00	5.00	5.00	0.50	0.50	-3.00
					0.525	0.91	4.87	5.09	0.44	0.48	-1.98
					0.550	0.85	4.74	5.14	0.40	0.46	-1.41
					0.575	0.80	4.62	5.15	0.37	0.45	-1.05
					0.600	0.76	4.51	5.16	0.34	0.44	-0.81
					0.625	0.73	4.41	5.15	0.32	0.44	-0.64
					0.650	0.71	4.32	5.14	0.31	0.43	-0.52
					0.675	0.68	4.24	5.13	0.30	0.43	-0.42
					0.700	0.67	4.18	5.12	0.29	0.42	-0.35
					0.725	0.65	4.12	5.10	0.28	0.42	-0.30
					0.750	0.64	4.07	5.09	0.28	0.42	-0.25

A1	A2	B1	B2	/	W	P	X	Y	N	R	dn/dw
0.75	0.25	0.75	0.25		0.225	1.09	5.05	4.83	0.33	0.78	-6.16
					0.250	1.00	5.00	5.00	0.25	0.75	-1.67
					0.275	0.95	4.91	5.04	0.22	0.73	-0.79
					0.300	0.92	4.83	5.05	0.21	0.72	-0.45
					0.325	0.90	4.77	5.04	0.20	0.71	-0.28
					0.350	0.88	4.72	5.03	0.19	0.70	-0.19
					0.375	0.87	4.68	5.01	0.19	0.70	-0.13
					0.400	0.86	4.64	5.00	0.18	0.69	-0.10
					0.425	0.86	4.62	4.99	0.18	0.69	-0.07
					0.450	0.85	4.60	4.98	0.18	0.69	-0.06
					0.475	0.85	4.58	4.97	0.18	0.69	-0.04
					0.500	0.84	4.57	4.96	0.18	0.69	-0.03
					0.525	0.84	4.55	4.96	0.18	0.69	-0.03

Chapter 5 The Reappraisal of Chamberlin's Welfare Ideal: A Trade-Off Between Scale Economies and Product Variety

5.1 Introduction*

This chapter extends the numerical computations of Spence (1976a) on the welfare trade-off between product diversity and scale economies to include a second-best solution and U-shaped cost curves. A trade-off between optimum scale and heterogeneous products would appear to be submerged in Chamberlin's 'welfare ideal'; however early commentators overlooked the significance of this statement. Spence's (1976b) recent reappraisal has shown inefficiency to be no longer just a matter of non-marginal cost pricing, for the actual number of commodities and the product mix are important considerations for any welfare analysis of product differentiation.

It is intuitive that a diversity of products is desirable from the consumers' point of view, if the products in question are not close substitutes for one another; but, variety may be costly if production economies are great. This chapter will demonstrate that such considerations are crucial in determining the optimal variety of products. Work by Lancaster (1975), Dixit and Stiglitz (1977) and Meade (1974) provide similar theoretical insights but from rather different approaches to that of Spence (1976a, 1976b). Ireland (1982) has recently introduced uncertainty into a Spence type framework. He shows that output per firm falls with the imposition of uncertainty, but the number of firms (products) may increase or decrease.

The analytical framework of Spence's (1976a) paper is a monopolistic competition model specified by linear demand and cost

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* This chapter is based on Carruth (1979).

functions from which he derives a monopolistic equilibrium and two welfare outcomes, the social optimum and a marginal cost pricing interpretation of market equilibrium. By using a quadratic cost function we can easily approximate linear costs, and still have greater freedom to consider Chamberlin's (1951) welfare ideal which was a feature of the so-called excess capacity debate. Moreover a second-best solution is also a zero profit welfare outcome which is an interesting contrast with the zero profit market equilibrium. Finally a computer graphic presentation of a few outcomes helps to illustrate the numerical results.

The next section spells out the market implications of product selection, and is followed by a presentation of Spence's analytical framework through his missing equations, but including our extensions. Section 5.4 provides the numerical results and graphical illustrations. A short concluding section ends the chapter.

5.2 Product Selection

It is well known that an important function of the price mechanism is the choice of products to be placed on the market. However Spence suggests:

> "The full range of products may be neither feasible nor desirable due to the presence of increasing returns to scale." (1976a; pp 407)

The reason is simply that most production units incur fixed costs which by definition are independent of output. It is these fixed costs which are instrumental in imperfectly competitive pricing and profitability ramifications.

In a separate theoretical paper Spence (1976b) focusses attention on the influence of fixed costs for product selection within a Chamberlin group framework.[1] Casual observation indicates that products which exist must be capable of extracting revenues sufficient to cover fixed and variable costs. However revenues do not provide an adequate measure of the social benefits derived from products, evidenced by the economics of consumer surplus. Only a perfectly price discriminating monopolist can extract all consumer surplus.[2] In this rather special case the welfare aspects of the product choice problem are eliminated simply by the rationale of revenues reflecting the true social benefits obtained from a product. The inability of firms to extract the true social benefits of their products is a market force working against product existence. In effect it reflects a tendency to reduce variety, and is symptomatic of market failure.

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1. No attempt will be made to question the existence of equilibrium in the face of scale economies, except in the sense of the tangency solution.

2. A potential virtue in terms of no efficiency loss. Of course the distributive implications would need to be taken into account.

The actual degree to which firms can capture consumer surplus depends to a certain extent on the properties of individual firm demand functions. The following summarises an argument due to Spence (1976a). Suppose demand functions exist and have constant elasticities. Let r be the ratio of total revenue to gross consumer surplus. Therefore

$$r = \frac{\int MR(x)\, dx}{\int p(x)\, dx} \qquad (5.1)$$

Where p(x) is the inverse demand function, MR(x) is the marginal revenue function and x is output. Equation (5.1) can be manipulated to yield

$$r = (1 - \frac{1}{e}) \qquad (5.2)$$

where e is the constant own price elasticity of demand.[3]

As e rises the ratio of total revenue to gross surplus, r, rises. The implication is a product selection bias whereby it is possible for a product with a low price elasticity to have a higher net welfare surplus but lower profit than a product with a high own price elasticity. Hence there may be a greater tendency to lose low elasticity products, particularly in the light of fixed costs. Moreover an implicit welfare bias may also be distinguished as low elasticity products are often attached greater welfare weights. Subsequently the incentive for sellers to price discriminate will be greater for low elasticity products. It turns out, Spence (1976b), that it is not just elasticity that matters, but what fraction of net potential surplus for a product is capturable by a selling firm. This will involve both demand and costs. As a market force selection bias will tend to eliminate products.

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3. Note that net welfare surplus as a percentage will be equal to (1 - r).100.

Likewise market interdependence may lead to non-optimal degrees of variety. Consider the case where products are imperfect substitutes. When a new product is introduced it affects other firms' market positions by reducing their demand which leads to a contraction of output for the existing set of firms. Gains arise from the profit and consumer surplus of the new product but losses are incurred on the profit and surplus of the existing set of products. When products are fairly close substitutes losses may outweigh gains. However the entering firm does not take account of such interactions: it may enter when it is not generating a net social benefit. This is a market force tending to generate too many products. On the other hand if products are complements then the monopolistic equilibrium by reducing output and raising price above marginal cost lowers the demand for other complementary products. This induces further quantity cutbacks and possibly the exit of products from the market. The process reinforces itself and leads to an equilibrium where all outputs are below the optimum and some of the products in the optimal set are not produced at all.

Profitability, therefore, is to be considered a fairly weak criterion for product selection. However, it is the only benchmark available and as Spence (1976a) points out

> "One can reasonably accept profitability as a constraint and pose the problem of product selection as that of determining the right set of products subject to that constraint. The solution to the problem includes specification of not only the products but also the prices. The prices will typically be above marginal cost, since that may be required to increase the profitability of products to permit the entry of products that are not profitable under marginal cost pricing. In short the solution to the second-best problem will include a trade-off between numbers of products on the one hand and the inefficiency due to non-marginal cost pricing on the other." (1976a; pp. 411).

Spence goes on to suggest that the monopolistic equilibrium has the qualitative features of the constrained optimum as both problems involve the trade-off between product variety and inefficiency through non-marginal cost pricing. However it is apparent that both outcomes have a different objective function and price-output configuration. Extension of Spence's (1976a) paper to include the 'true second-best' solution facilitates a comparison of the implications arising from these distinctions. The analytical framework employed by Spence (1976a) and extended in this chapter is set out in the next section.

5.3 A Framework for Welfare Computations

The essence of the approach is a computational comparison of the aforementioned market outcomes and welfare optima. Welfare is measured by the multiproduct net surplus which is simply the sum of producer and consumer surplus. Income effects are to be ignored. Recent work in this area by Dixit and Weller (1977), Seade (1978) and Willig (1976) indicates that this type of assumption may be less restrictive for welfare analysis than was once thought. Nevertheless it should be borne in mind that it underlies all the subsequent analysis of this chapter. A product's marginal contribution to total surplus is defined to be the area under the inverse demand function minus the costs of production for that particular product.[4]

The numerical results depend upon the following framework. The quantity of the i^{th} product is x_i. The inverse demand for the i^{th} product is

$$p_i = a - 2bx_i - 2d \sum_{i \neq j} x_j \quad (5.3)$$

[5]

The cost function[6] of the i^{th} firm is

$$TC = f + kx_i + gx_i^2 \quad (5.4)$$

where f is the fixed cost, d is the interaction effect, and a and b are the intercept and slope of the inverse demand for each product when there are no other products. Equations (5.3) and (5.4) plus the final form of the total surplus function are all the information provided by Spence (1976a) before he proceeds to his table of numerical

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4. Note that the total net surplus is not exactly equal to the sum of these marginal surpluses as account must be taken of entry repercussions on existing members of the 'group'.

5. The analysis is restricted to linear demand functions.

6. We include the quadratic term at this stage. The Spence "missing" equations are easily obtained by setting g = 0.

results for the different market outcomes and welfare optima (1976a; pp. 412, Table I). However it aids understanding if the analytical framework is spelled out in detail.

From (5.3) we know that the marginal revenue function has twice the slope of the inverse demand function, that is

$$MR_i = a - 4bx_i - 2d \sum_{i \neq j} x_j \qquad (5.5)$$

Under the symmetry and uniformity assumptions of the Chamberlin 'group' the inverse demand function, equation (5.3), can be rewritten as

$$p_i = a - 2bx_i - 2d(n - 1)x_i \qquad (5.6)$$

Now gross surplus (G.S.) is defined to be the area under the inverse demand function. Integrating[7] (5.6) with respect to x_i gives

$$G.S. = a\,x_i - b\,x_i^2 - d(n - 1)x_i^2 \qquad (5.7)$$

Net surplus is simply G.S. minus production costs. Using equation (5.4) we obtain

$$N.S. = a\,x_i - b\,x_i^2 - d(n - 1)x_i^2 - f - k\,x_i - g\,x_i^2 \qquad (5.8)$$

Total net surplus can then be derived from (5.8) by summing over the n firms and dropping labels (subscripts) due again to symmetry and uniformity. This gives[8]

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7. Analytically symmetry means that $\int p_i \, dx_i$ is well defined with no 'path of integration' difficulties. This is an important property of compensated demand functions.

8. Equation (5.9) corresponds to the surplus equation presented by Spence (1976a, in a footnote, pp. 412), having set $g = 0$.

$$T(n, x) = n(ax - bx^2) - dn(n - 1)x^2 - nf - nkx - ngx^2 \qquad (5.9)$$

which means that any outcome must be completely described by x, the output per firm and n, the number of firms. This is clearly restrictive, for in this type of problem we often have product ordering along a spectrum and two products closer together on this spectrum will be closer substitutes than two products further apart. With asymmetry the actual product labels will become important and (5.9) will not be valid. To aid tractable results symmetry will remain an integral assumption of this chapter. Dixit and Stiglitz (1977) have provided some analysis of the asymmetry problem.

A profit function can also be derived from demand and cost conditions in a similar manner to the surplus function. Multiply (5.6) through by x_i to yield total revenue and then subtract the total cost function, equation (5.4). With the symmetry condition we can drop labels and so obtain

$$\pi(n, x) = ax - 2bx^2 - 2d(n - 1)x^2 - f - kx - gx^2 \qquad (5.10)$$

With entry in monopolistic competition the zero profit condition is simply equivalent to average revenue equal to average cost, that is, AR = AC. Therefore, ignoring subscripts under symmetry, equation (5.6) and equation (5.4) can be used to obtain

$$a - 2bx - 2d(n - 1)x = k + \frac{f}{x} + gx \qquad (5.11)$$

This helps to simplify the derivation of the monopolistic equilibrium. Equations (5.3) to (5.11) provide all the necessary information for deriving the different outcomes in terms of our control variables n and x.

The Optimum (O.)

In this case we wish to

$$\text{Max.} \quad T(n, x) = n(ax - bx^2) - dn(n-1)x^2 - nf - nkx - ngx^2$$

First order conditions (F.O.C.) are

$$T_n = ax - bx^2 - (2n-1)dx^2 - f - kx - gx^2 = 0 \tag{5.12}$$

$$T_x = n(a - 2bx) - dn(n-1)2x - nk - 2ngx = 0 \tag{5.13}$$

where subscripts are used to denote partial derivatives. Rearranging (5.13) we can derive

$$x = \frac{a - k}{2b + 2d(n-1) + 2g} \tag{5.14}$$

and by substitution for x in (5.12) a little manipulation yields

$$n = 1 + \left[\frac{(b + g - d)(a - k)^2}{4d^2 f} \right]^{\frac{1}{2}} - \frac{b + g}{d} \tag{5.15}$$

(5.14) and (5.15) enable the isolation of x and n for different initial values of our parameters which will consistently describe this welfare outcome and allow us to calculate total surplus, profit/loss, prices, revenues and costs.

Market Equilibrium (E.)

Within the monopolistic competition model this solution is described by the conditions that marginal revenue equals marginal cost and under free entry average revenue equals average cost. Now from (5.6), (5.5) and (5.4) MR = MC gives

$$a - 4bx - 2d(n - 1)x = k + 2gx \tag{5.16}$$

and from (5.11) AR = AC implies

$$a - 2bx - 2d(n - 1)x = k + \frac{f}{x} + gx \tag{5.17}$$

(5.16) and (5.17) depict two simultaneous equations in n and x. Now by simply subtracting we obtain

$$x = (f/(g + 2b))^{\frac{1}{2}} \tag{5.18}$$

Substitution for x yields

$$n = 1 + \left[\frac{(a - k)^2(g + 2b)}{4d^2f} \right]^{\frac{1}{2}} - \frac{(2b + g)}{d} \tag{5.19}$$

Hence equations (5.18) and (5.19) capture the monopolistic market equilibrium.

Equilibrium Number of Firms with MC Pricing (M.)

In this example we know that supply price has to equal marginal cost and that n is constrained to equal (5.19). From equations (5.4) and (5.6), ignoring subscripts, we have

$$a - 2bx - 2d(n - 1)x = k + 2gx$$

and

$$x = \frac{a - k}{2b + 2d(n - 1) + 2g} \tag{5.20}$$

Equations (5.19) and (5.20) model this outcome. Notice that equations (5.14) and (5.20) are identical which simply reflects first-best efficiency with no thought to loss-offset problems.

'Second-Best' Solution[9]

The analysis of this welfare outcome involves a fairly straightforward constrained optimisation problem. It can be expressed as

$$\text{Max } T(n, x) \qquad (5.21)$$

$$\text{s.t. } \pi(n, x) = 0$$

In Lagrangean form (5.21) becomes

$$L = T(n, x) + \lambda\, \pi(n, x) \qquad (5.22)$$

F.O.C.

$$L_x = T_x + \lambda \pi_x = 0 \qquad (5.23)$$

$$L_n = T_n + \lambda \pi_n = 0 \qquad (5.24)$$

$$L_\lambda = \pi(n, x) = 0 \qquad (5.25)$$

From (5.9) and (5.10) we can substitute for the partials T_x, π_x, T_n, π_n to obtain a system of three nonlinear simultaneous equations in the three unknowns n, x and λ, the Lagrange multiplier. This gives

$$L_x = na - 2nbx - 2dn(n-1)x - nk - 2ngx + \lambda(a-4bx-4d(n-1)x-k-2gx) = 0 \quad (5.26)$$

$$L_n = ax - bx^2 -(2n-1)dx^2 - f - kx - gx^2 - \lambda 2dx^2 = 0 \qquad (5.27)$$

$$L_\lambda = ax - 2bx^2 - 2d(n-1)x^2 - f - kx - gx^2 = 0 \qquad (5.28)$$

9. This welfare outcome was not considered by Spence (1976a), although it was discussed in (1976b). It can therefore be viewed as an extension of his (1976a) simulation exercise.

This system ((5.26) to (5.28)) did not manipulate into manageable formulas in x and n as in the previous solutions; however again we can easily appeal to the NAG library and the Powell (1970) routine CO5NAF. On the other hand we could just as easily have returned to the initial constrained optimisation problem [Max $T(n, x)$ s.t. $\pi(n, x) = 0$] and used a routine like EO4UAF from the optimum tax chapter. It turns out that CO5NAF is rather simpler to use and involves less computational resources, so we remained with the solution to the first-order conditions. Therefore for any given set of values placed on a, b, d, f, k, and g, we can solve for n, x and λ.

5.4 Welfare Comparisons

The first set of numerical results are presented in Table 5.1. Some columns are replications of Spence's Table 1 (1976a, pp.412) given that the chosen parameter sets are identical.[10] It is pleasing that in these cases the results both coincide.

Table 5.1 is extended compared to Spence's Table 1 to include the exact second-best solution, equilibrium and second best prices plus the Lagrange multiplier. T stands for total net surplus and subscripts O, E, M, S are the optimum, market equilibrium, marginal cost pricing with equilibrium firms and second-best respectively. ΔT_i represents the difference between the total net surplus and the surplus pertaining to the relevant subscript, i, that is, welfare losses for i = E, M and S. N_j and X_j are the number of firms and output per firm respectively for the different outcomes with j = O, E, M and S. P_E is the market equilibrium price and P_S is the second-best solution price. Finally λ_S is the Lagrange multiplier for the second-best outcome.

Spence discussed columns ΔT_E and ΔT_M of Table 5.1, which essentially illustrates that in some cases the welfare loss for non-marginal cost pricing is sometimes less than half the total welfare loss. In other words a move from market equilibrium to marginal cost pricing with the equilibrium number of firms is <u>not enough</u> to remove the welfare loss: ΔT_M is unlikely to be zero.[11] Therefore the optimal price-output decision also common to the excess capacity debate neglects the optimal number of products (firms). This may be seen as

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10. ie. we set g = 0.

11. It is zero in a single case, row 6 of Table 5.1.

Table 5.1[$]

GROUP	CASE	a	b	d	f	T_O	T_S	ΔT_E	ΔT_M	ΔT_S	N_O	N_E	N_S	X_O	X_E	X_M	X_S	P_E	P_S	λ_S
I	1	10	1	0.5	1	28.77	28.53	2.7	1.8	0.24	5.4	9.7	5.1	1.4	0.7	0.8	1.4	2.3	1.7	0.5
	2				2	24.50	24.04	3.5	1.8	0.46	3.5	6.0	3.2	2.0	1.0	1.3	1.9	3.0	2.1	0.5
	3				4	19.04	18.14	4.3	1.2	0.90	2.2	3.4	1.9	2.8	1.4	2.1	2.6	3.6	2.6	0.5
	4				6	15.32	14.00	4.8	0.6	1.32	1.6	2.2	1.3	3.5	1.7	2.8	3.1	4.5	3.0	0.5
	5				8	12.50	10.79	5.0	0.2	1.71	1.3	1.5	0.9	4.0	2.0	3.6	3.4	5.0	3.3	0.5
	6				10	10.25	8.18	5.1	0	2.07	1.0	1.0	0.7	4.5	2.2	4.4	3.7	5.5	3.7	0.5
II	7	10	2	1.5	2	8.20	8.01	2.6	1.4	0.19	1.2	2.6	1.1	2.0	0.7	1.0	1.9	3.8	2.1	0.17
	8				4	6.35	6.05	3.0	0.8	0.30	0.7	1.3	0.6	2.8	1.0	1.8	2.6	5.0	2.6	0.17
	9				6	5.11	4.67	3.1	0.3	0.44	0.5	0.8	0.4	3.4	1.2	2.7	3.1	5.9	3.0	0.17
III	10	10	1	0.1	2	99.75	99.65	8.7	1.3	4.10	21.2	26.0	18.6	1.5	1.0	1.3	1.4	3.0	2.5	4.5
	11				5	56.58	47.12	12.9	0.1	9.46	10.1	9.5	7.2	2.4	1.6	2.4	2.0	4.2	3.5	4.5
	12				7	39.60	27.07	14.7	2.2	12.53	7.1	5.0	4.1	2.8	1.9	3.2	2.3	4.7	4.1	4.5
	13				9	27.36	12.21	16.2	7.4	15.15	5.2	2.2	1.9	3.2	2.1	4.0	2.4	5.2	4.7	4.5
	14				10	22.50	6.26	16.8	11.3	16.24	4.5	1.1	1.0	3.3	2.2	4.4	2.5	5.5	5.0	4.5
IV	15	10	0.7	0.5	2	29.92	29.72	4.8	3.5	0.20	2.4	5.7	2.3	3.2	1.2	1.5	3.1	2.7	1.7	0.2
	16				4	26.00	25.62	6.2	3.7	0.38	1.6	3.5	1.5	4.5	1.7	2.3	4.2	3.4	1.9	0.2
	17				6	23.18	22.62	7.1	3.5	0.56	1.2	2.6	1.1	5.5	2.1	3.1	5.1	3.9	2.2	0.2
	18				8	20.93	20.19	7.6	3.0	0.74	1.0	2.0	0.9	6.3	2.4	3.8	5.8	4.3	2.4	0.2
V	19	10	0.3	0.2	2	82.13	81.88	7.6	6.2	0.25	4.5	10.3	4.4	4.8	1.8	2.1	4.4	2.1	1.5	0.25
	20				6	69.39	68.67	12.0	7.9	0.72	2.4	5.1	2.3	7.7	3.2	4.0	7.4	2.9	1.8	0.25
	21				10	61.25	60.08	14.3	7.7	1.17	1.8	3.5	1.6	10.0	4.1	5.6	9.4	3.4	2.1	0.25
	22				15	53.64	51.91	15.1	6.7	1.73	1.3	2.5	1.2	12.2	5.0	7.5	11.3	4.0	2.3	0.25
VI	23	10	0.3	0.05	10	170.39	159.24	22.7	2.1	11.15	9.2	11.0	7.7	6.3	4.1	5.6	5.7	3.4	2.8	2.5
	24				20	102.51	81.81	28.2	0.5	20.70	5.1	4.6	3.4	8.9	5.8	9.4	7.5	4.5	3.7	2.5
	25				30	62.05	33.34	32.9	9.8	28.71	3.2	1.7	1.4	10.9	7.1	13.6	8.6	5.2	4.5	2.5

[$] N.B. $k = 1$ and $g = 0$ for all cases.

the essence of Chamberlin's welfare ideal.

The actual cases, where the welfare costs from having the wrong number of products are significant, vary in respect of elasticities and fixed costs. When cross elasticities are high they occur when fixed costs are low, for example, column ΔT_M, groups I and II. Here the substitution[12] effect dominates and results in a larger number of products than is socially desirable. This supports a proposition advanced in section 5.2. Too many products also tend to occur when cross elasticities are high relative to own elasticities and fixed costs are low: compare columns N_O and N_E for groups I, II, IV and V. Variety is costly with low cross elasticities and high fixed costs, groups III and VI; and given the earlier arguments on product selection it is not surprising that here we find a situation of <u>too few products</u> in equilibrium. This also contrasts markedly with the excess capacity debate, where the concentration on scale effects alone came down in favour of too many products all the time.

Spence's treatment of profitability and product selection used a zero profit market equilibrium as the second-best welfare approximation. This was not unreasonable as regards the theory of the firm; and even though not such a close welfare approximation as Spence may have imagined, it in effect gives fairly similar qualitative predictions to the true second best solution in terms of the restricted role of marginal cost pricing. However the price-output configuration is substantially different between the market equilibrium and second-best. Second-best has a higher output and lower price (c.f. columns X_E and X_S, P_E and P_S). Moreover the second-best outcome involves <u>too few products</u> in all cases. This reflects an "optimum"

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12. ie. products as substitutes.

amount of variety and output being traded to cover the first-best loss, rather than firms entering until profits are driven to zero.

Figures 5.1 and 5.2 have taken two cases from Table 5.1, and, using computer graphic software, have provided a pictorial illustration of two key outcomes. Figure 5.1 corresponds to group I with f = 6 and presents the case where there are too many products in equilibrium. The notation is similar to the text, although the marginal cost pricing contour which passes through the optimum, O, and M, is not annotated. The full contours represent the net surplus function, T(n, x). The axis scales are not significant, and simply represent the original size of the diagrams which was 20 cm on each axis. Figure 5.2 presents the group III case with f = 7. Here we have too few products in equilibrium, although the welfare level in equilibrium is much closer to second-best, S.[13]

It is also notable that the Lagrange multiplier values, λ_S, are constant within groups but vary between groups. It can be shown that[14]

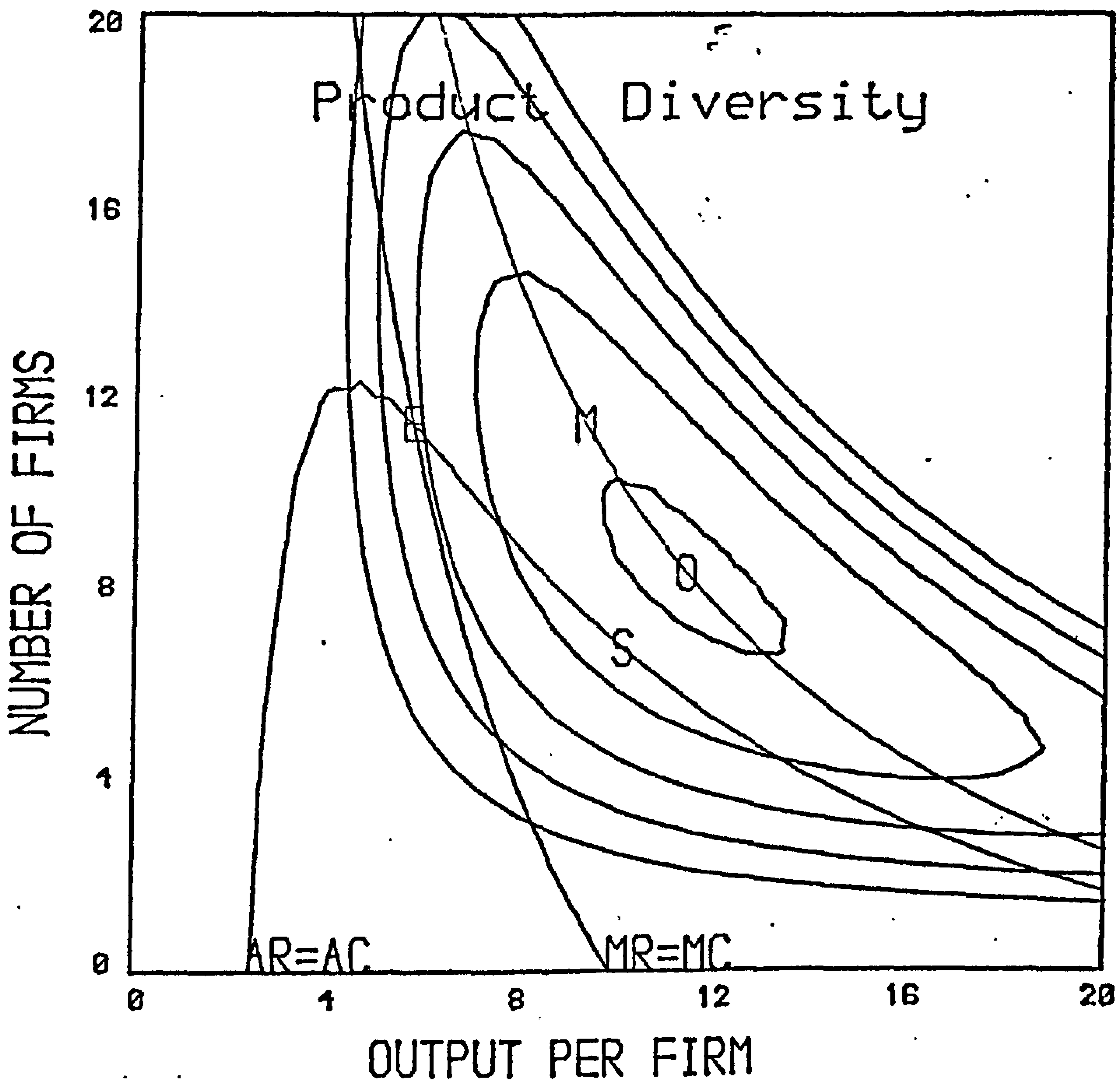
$$\lambda_S = (b - d)/2d \qquad (5.29)$$

which means that λ is only affected by the demand slope and interaction parameters. Changes in the demand intercept, fixed and variable cost will have no effect on the contribution of an additional unit of the profit constraint to the net surplus. This concentration on the demand side is reflected in the "too few products" result of the second best outcome.

13. The second-best equations (5.23), (5.24) and (5.25) can be used to generate the equality $T_x/T_n = \pi_x/\pi_n$, which just states that the slope of the surplus contour must be the same as the slope of the profit constraint at an interior optimum: so S depicts this tangency position.

14. see appendix.

Figure 5.1[$]



Case: Group I of Table 5.1, f = 6

[$] Notation:

O = Social Optimum, so marginal cost pricing passes through O, and M.

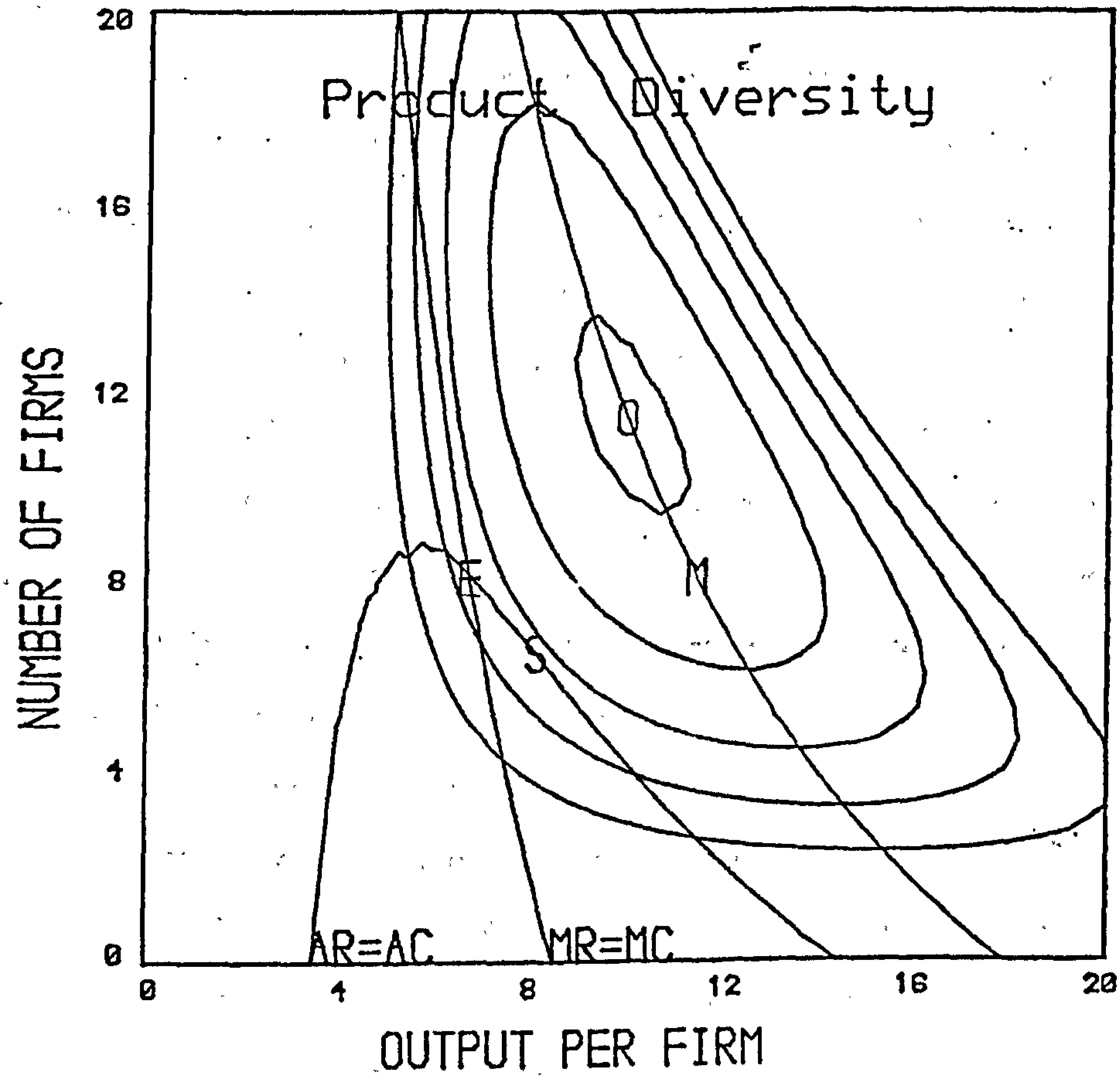
M = Marginal cost pricing with the equilibrium set of firms

E = Market equilibrium

S = second-best

The contours were drawn using GINO-SURF (1980), the rest of the figure by GINO-F (1976).

Figure 5.2 $



Case: Group III from Table 5.1, f = 7

$ Notation:

O = Social Optimum, so marginal cost pricing passes through O, and M.

M = Marginal cost pricing with the equilibrium set of firms

E = Market equilibrium

S = second-best

The contours were drawn using GINO-SURF (1980), the rest of the figure by GINO-F (1976).

The more general case of a quadratic cost curve returns us to the arguments expressed in the appendix to Chapter 2 concerning the extent of cost penalties for firms operating at 50% of minimum efficient scale (mes). It is worthwhile adopting the demand parameter values of Figures 5.1 and 5.2 to examine the influence of the quadratic cost function. For the sake of illustration we have taken a unit cost penalty of 12% to operate at output rates 50% below mes. This gives f = 2.0, k = 3.0 and g = 1.0. Figure 5.3 corresponds closely to Figure 5.1 and similarly for Figure 5.4 with 5.2; however, the movement away from the optimum by the other outcomes is less pronounced. Note also that the zero profit condition, AR = AC, now cuts the marginal cost pricing contour for high output rates and few firms. With U-shaped scale curves it is possible for marginal cost pricing to make profits. Nevertheless the possibility, and implications for the excess capacity debate, of having too few or too many firms still remains.

Finally with U-shaped scale curves we can consider to what extent the social optimum will correspond to optimum scale. The excess capacity debate often criticised monopolistic equilibrium for this failure. Table 5.2 presents results with respect to the parameter values of Figures 5.3 and 5.4.

It is apparent that the social optimum output rate does not correspond to optimum scale. Hence the use of X* as a benchmark in the excess capacity debate was inappropriate.

A final empirical comment on the above results would reflect on the implications for a regulatory body attempting to bring about a certain welfare outcome. For simplicity take the market equilibrium as a benchmark where no intervention is apparent, and that regulation can lead to welfare improving adjustments. The optimum will require control of the number of firms, n, and the output per firm, x; and, losses will have to be covered in some way. The second-best outcome requires the manipulation of n and x, but there is no problem of loss offset. Imposing marginal cost pricing on the set of monopolistic products in existence will also involve the regulatory body in loss offset provision. Needless to say it the information problem will be considerable in attempting any such welfare improvements, especially as it is no longer a foregone conclusion that equilibrium is simply a case of having too many products. Moreover any costs of regulation are unaccounted for. Therefore monopolistic market behaviour may be less of an inefficiency problem than was once feared, especially as it was expressed through the excess capacity debate.

Table 5.2[a]

a	b	d	f	k	g	X*	X_O	X_E
10	1	0.5	2.0	3.0	1.0	1.41	1.15	0.82
10	1	0.1	2.0	3.0	1.0	1.41	1.03	0.82

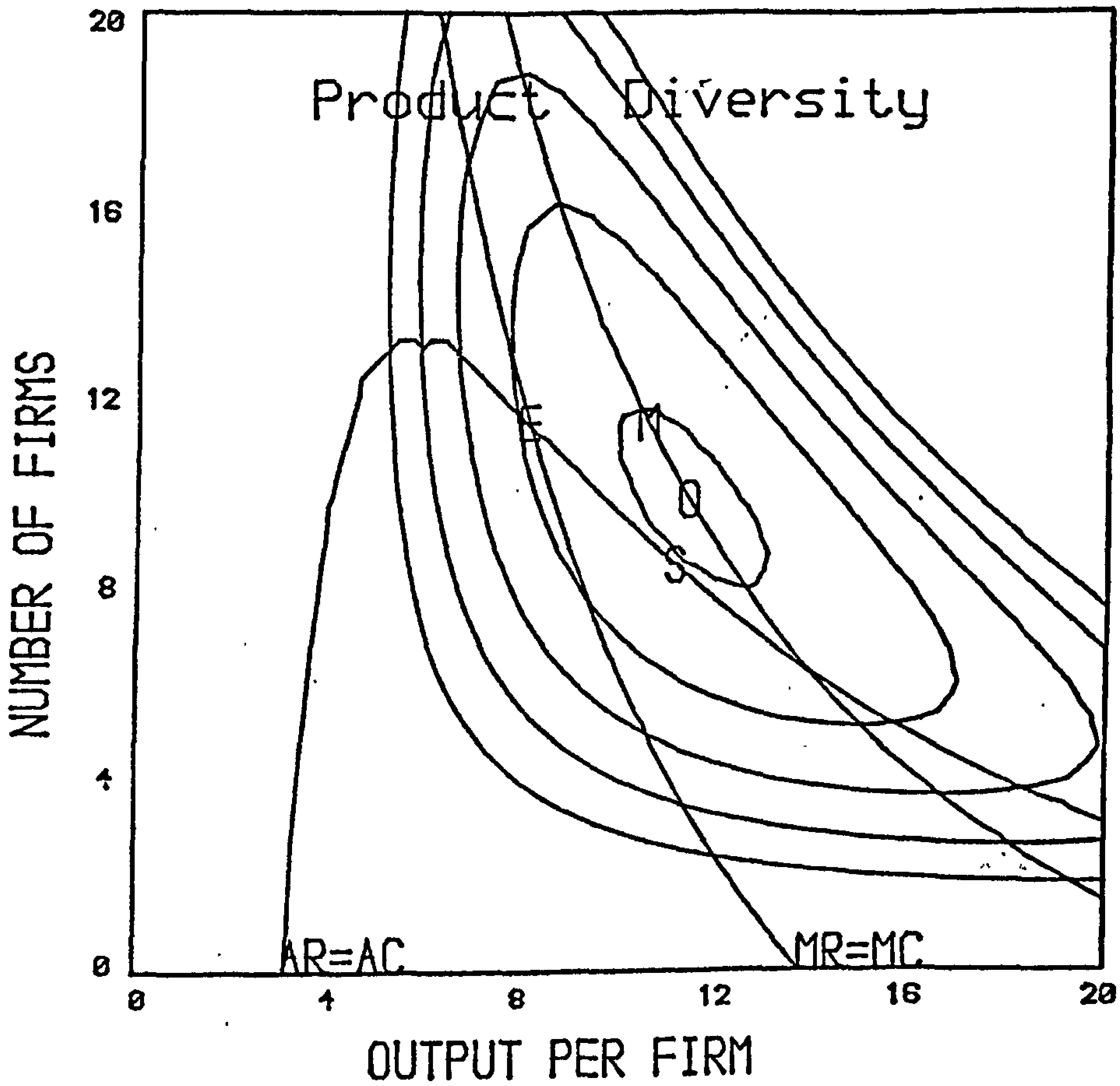
$X^* = (f/g)^{\frac{1}{2}}$ ≡ optimum scale output rate

$X_O = (a - k)/(2b + 2d(n - 1) + 2g)$ ≡ social optimum output rate

$X_E = (f/(g + 2b)^{\frac{1}{2}}$ ≡ market equilibrium output rate

a. Notice that when perceived demand is horizontal ($b = 0$) X_E and X^* coincide, but not otherwise.

Figure 5.3 $



Case: Group I demand, a = 10, b = 1, d = 0.5, and costs
f = 2, k = 3, g = 1.

$ Notation:

O = Social Optimum, so marginal cost pricing passes through O, and M.

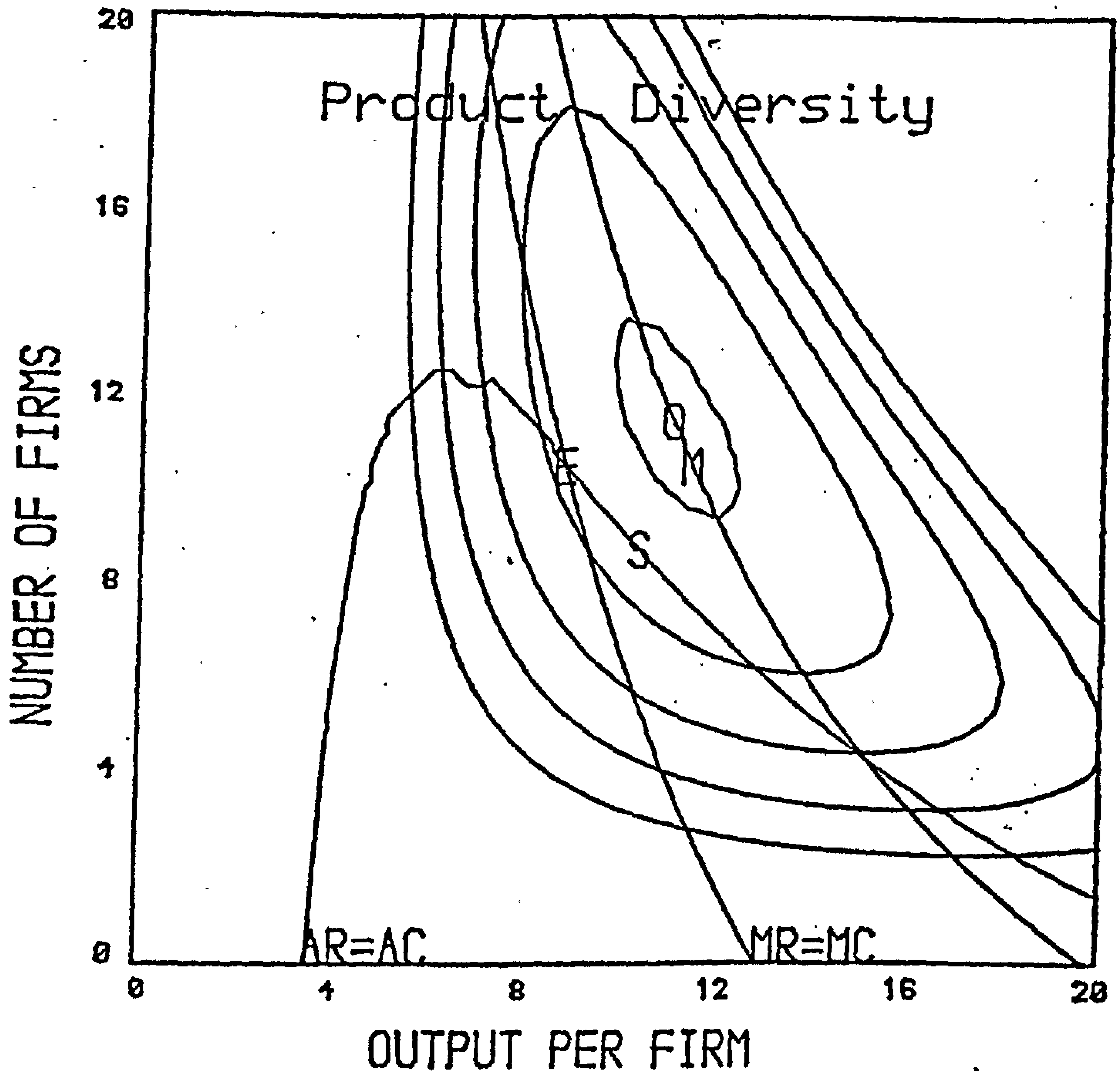
M = Marginal cost pricing with the equilibrium set of firms

E = Market equilibrium

S = second-best

The contours were drawn using GINO-SURF (1980), the rest of the figure by GINO-F (1976).

Figure 5.4 $



Case: Group III demand, a = 10, b = 1, d = 0.1
and costs f = 2, k = 3, g = 1

$ Notation:

O = Social Optimum, so marginal cost pricing passes through O, and M.

M = Marginal cost pricing with the equilibrium set of firms

E = Market equilibrium

S = second-best

The contours were drawn using GINO-SURF (1980), the rest of the figure by GINO-F (1976).

5.5 Conclusion

This chapter can be summarised by the following points. First is the important observation that variety is costly if scale economies are considerable; yet consumers will prefer variety when products are not close substitutes for one another. It is this trade-off which is pertinent to a proper analysis of the welfare implications of heterogeneous products versus efficiency.

Second is the implication that inefficiency can arise from an undesired product mix, too little or too many products, as well as incorrect output rates. The preceding discussion focussed on the trade-off between output levels and product numbers under a particular set of restrictions. It was evident that the degree of competition and the extent of scale economies had important bearings on where welfare losses could be attributed - Tables 5.1 and 5.2 and Figures 5.1 to 5.4. It was clear that welfare losses were not simply a result of non-marginal cost pricing.

The excess capacity debate must now be seen in a rather different perspective.

Appendix to Chapter 5

The second-best solution is

$$\text{Max} \quad T(n, x)$$

$$\text{s.t.} \quad \pi(n, x) = 0$$

with F.O.C.

$$T_n + \lambda\pi_n = 0 \qquad (5.30)$$

$$T_x + \lambda\pi_x = 0 \qquad (5.31)$$

From (5.9) and (5.10) we have

$$T(n,x) = n[ax - bx^2 - d(n-1)x^2 - f - kx - gx^2] \qquad (5.32)$$

$$\pi(n,x) = ax - 2bx^2 - 2d(n-1)x^2 - f - kx - gx^2 \qquad (5.33)$$

Therefore

$$T_n = [ax - bx^2 - d(n-1)x^2 - f - kx - gx^2] - ndx^2$$

$$= [bx^2 + d(n-1)x^2] - ndx^2 \qquad \text{when } \pi = 0 \text{ (using (5.33))}$$

$$= (b - d)x^2$$

$$\pi_n = - 2dx^2$$

From (5.30) $\lambda_S = - T_n/\pi_n$ at $\pi = 0$

Hence $\lambda_S = (b - d)/2d$

Chapter 6 A Quantity-Setting Duopoly Analysis of the Structure-Conduct-Performance Paradigm

6.1 Introduction*

Since the pioneering work of Harberger (1954) welfare losses due to monopoly have received much attention in the literature. Recently published estimates put these at 7-13 per cent of gross corporate product in the U.S. and 3-7 per cent in the U.K. (Cowling and Mueller, 1978). These are much larger numbers than Harberger's 'less than one-tenth of one per cent of GNP', though the whole issue remains controversial, for example, see the exchange between Littlechild (1981) and Cowling and Mueller (1981). The empirical analyses typically assume linear demand and constant costs. On these assumptions it can easily be shown that the monopoly loss will be exactly 25 per cent of the level of welfare (net surplus) obtaining under a social optimum characterised by zero profit and marginal-cost pricing, irrespective of demand and cost conditions.[1] This is a maximum figure in that it assumes monopoly pricing behaviour, whereas many previous studies have assumed limit pricing. On the other hand it takes no account of the costs of securing monopoly positions, analysed by Posner (1975) and others, or of the possibility of reduced technical efficiency in markets where competitive pressure is lacking; Leibenstein's (1966) X-inefficiency.

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* This chapter is based on joint work - see the Preface to the thesis for full details of responsibilities.

1. Whatever the level of costs and slope of the demand curve, monopoly output is half the competitive level and the monopoly profit rectangle is twice as large as (a) the consumers' surplus under monopoly and (b) the monopoly deadweight loss triangle. Under competition net surplus is defined to be the sum of these areas.

This chapter is concerned with the extent to which welfare losses may be attenuated by inter-firm rivalry in oligopolistic markets. It would appear that this has not been considered in the literature to date, although Cowling (1976) and Cowling and Waterson (1976) have derived the relation

$$(p - c)/p = H(1 + \lambda)/e_p$$

where p = price, c = marginal and average cost, H is the Hirschman-Herfindahl index of seller concentration, e_p = the industry price elasticity of demand, and they interpret $\lambda = dx/dx_i$ as summarising firms expectations concerning the response of rivals to their own output decisions. Thus they relate the Lerner index of monopoly power $(p - c)/p$ to the degree of oligopolistic interaction λ. However alternative oligopoly solutions other than Cournot are not considered explicitly, and the econometric results suffer from a lack of industry elasticity data plus the difficulty of estimating an equilibrium condition.

Another approach is to postulate a specific welfare function and solve for the level of welfare (net surplus) under various combinations of conduct and structure. By 'conduct' we mean alternative conjectural variations determining the way in which the oligopoly game is played. 'Structure' embraces both the number of firms (which is fixed at two throughout the present analysis) and also consumer preferences and production technology, as summarised in the parameters of the relevant demand and cost functions. Such definitions neglect the question of entry.

The analysis is based on quantity-setting duopoly and employs computational techniques plus computer graphical illustrations to examine the welfare losses both for given modes of conduct across alternative plausible structures, and given structure under different patterns of conduct, all relative to the social optimum.

This approach departs from a long standing tradition in industrial economics in which performance (profit) is explained in terms of structural characteristics, notably the level of seller concentration: a good example is Holterman (1973). For, we look directly at welfare and, in addition, concentration is found jointly with prices, profits and outputs, as part of an equilibrium determined by preferences, behaviour and technology. The traditional framework overlooks this endogeneity, and so causal relationships are inferred from equilibrium conditions, such as the Cowling-Waterson relationship. A systematic relationship between concentration and welfare, if it exists, may nevertheless be important to know, not least as a practical aid in the determination of priorities for antitrust agencies like the Monopolies Commission. The present analysis permits the observation of such a structure performance mapping, or alternatively will show whether a given structure (concentration level) may correspond to several states of conduct-performance.

Section 6.2 draws on recent work by Bramness (1979), Dixit (1979), Ulph (1980) and others providing a unifying framework within which alternative conjectural-variations equilibria may be compared with each other and with the social optimum. It should be mentioned at this point that we are not concerned with the relative merits of the alternative models as oligopoly solution concepts: the 'arbitrariness' or 'correctness' of firms' conjectures and so forth. Rather we focus on the social value of the outcomes produced by alternative behavioural postulates which exist in the literature. Section 6.3 reports the results of the numerical computations, showing indices of social welfare for different types of oligopolistic interaction under variation in the degree of product homogeneity, and cost and demand asymmetries. The implications of the results for antitrust policy are spelled out along with the conclusions in section 6.4.

The computations for this chapter are simple enough not to require any appeal to NAG software; however a number of diagrams do rely on the graphic software GINO-F and GINO-SURF, discussed in Chapter 1.

6.2 Alternative Conjectural Variations Equilibria

Let us consider a quantity-setting duopoly and assume constant marginal costs c_1, c_2. The utility function is assumed quadratic, following Dixit (1979):

$$U = x_o + \alpha_1 x_1 + \alpha_2 x_2 - \frac{1}{2}(\beta_1 x_1^2 + 2\gamma x_1 x_2 + \beta_2 x_2^2) \qquad (6.1)$$

with $\quad \alpha_i, \beta_i, \gamma > 0$ and $\beta_1 \cdot \beta_2 \geqslant \gamma^2$.

x_1, x_2 are the duopolists' outputs and x_o is the composite output of the rest of the economy, assumed competitive.

First order conditions for consumers' equilibrium yield linear inverse demands:

$$\begin{aligned} p_1 &= \alpha_1 - \beta_1 x_1 - \gamma x_2 \\ p_2 &= \alpha_2 - \beta_2 x_2 - \gamma x_1 \end{aligned} \qquad (6.2)$$

γ captures cross-price effects between the competing firms and may be interpreted as a measure of product differentiation. By definition we require that x_1, x_2 are substitutes in an oligopoly, hence $\gamma > 0$. The products are perfect substitutes when both $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 = \gamma$. In the numerical computations the homogeneous product case is for convenience treated as $\beta_1 = \beta_2 = \gamma = 1$. Absolute demand advantages for either firm may be captured in a higher value of α_i. Writing $\alpha_i - c_i = \theta_i$, the duopolists' profit functions are respectively

$$\pi_1 = \theta_1 x_1 - \beta_1 x_1^2 - \gamma x_1 x_2$$

(6.3)

and $$\pi_2 = \theta_2 x_2 - \beta_2 x_2^2 - \gamma x_1 x_2$$

Equilibrium conditions for the social optimum, pure monopoly and various oligopoly solutions are set out in equations (6.4)-(6.10) in Table 6.1. The social optimum maximises net surplus: $U - (c_1 x_1 + c_2 x_2)$. Equilibrium is characterised by marginal-cost pricing by both firms, and is the benchmark for subsequent welfare comparisons. In the absence of fixed costs equilibrium will result in zero profits being earned. The Cournot, Bertrand, and Stackelberg solutions are familiar and require no comment.

Market share maximisation, or the maintenance of a given market share, has been proposed as the way oligopolists will formulate their strategy in practice, and casual empiricism lends this view plausibility. A true equilibrium must ensure that compatible market shares are chosen; i.e. must simultaneously satisfy both the reaction functions (6.7). With symmetric cross-price effects, implicit in our specification of the utility function, it turns out that market share equilibrium coincides with collusive behaviour leading to joint profit maximisation.[2] Thus, (6.7) are also first-order conditions for a maximum of industry profits:

$$\pi_1 + \pi_2 = \theta_1 x_1 + \theta_2 x_2 - \beta_1 x_1^2 - \beta_2 x_2^2 - 2\gamma x_1 x_2.$$

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2. See also Bramness (1979).

With a quadratic welfare function the market share/collusion equilibrium will, like pure monopoly, always generate exactly half the output rate and three quarters the net surplus obtaining at the social optimum, irrespective of demand and cost parameter values. Thus, denoting the social optimum outputs by (x_1^o, x_2^o) and comparing (6.4) and (6.7), it is obvious that $(\frac{1}{2}x_1^o, \frac{1}{2}x_2^o)$ solve (6.7). Now at the social optimum

$$U^o = \theta_1 x_1^o + \theta_2 x_2^o - \frac{1}{2}(\beta_1 (x_1^o)^2 + 2\gamma x_1^o x_2^o + \beta_2 (x_2^o)^2).$$

Recognising that $\pi_1^o = \pi_2^o = 0$ in equilibrium and substituting $\theta_1 x_1^o = \beta_1 (x_1^o)^2 + \gamma x_1 x_2$, $\theta_2 x_2^o = \beta_2 (x_2^o)^2 + \gamma x_1 x_2$ from (6.3) yields

$$U^o = \theta_1 x_1^o + \theta_2 x_2^o - \frac{1}{2}(\theta_1 x_1^o + \theta_2 x_2^o) \qquad (6.14)$$

Obviously

$$U^o = \frac{1}{2}(\theta_1 x_1^o + \theta_2 x_2^o)$$

whereas, substituting for x_1^{ms} , x_2^{ms} from (6.12) and (6.13),

$$U^{ms} = \theta_1 \frac{x_1^o}{2} + \theta_2 \frac{x_2^o}{2} - \frac{1}{2}(\theta_1 \frac{x_1^o}{2} + \theta_2 \frac{x_2^o}{2}) \qquad (6.15)$$

$$= (1 - \frac{1}{4})U^o$$

The concept of <u>rational conjectures equilibria</u> (RCE) has recently been discussed by Ulph (1980), Perry (1982), Bresnahan (1981) and others. The essential requirement is that, for a rational conjectural equilibrium, each firm's conjectures concerning the rival's reactions are correct. In this framework a local RCE implies that each firm has effectively predicted the slope of its rival's reaction function in the neighbourhood of equilibrium.

To capture local RCE completely we first obtain general reaction functions by differentiating (6.4) and setting to zero:

$$\begin{aligned} \partial\pi_1/\partial x_1 &= \theta_1 - 2\beta_1 x_1 - \gamma x_2 - \gamma x_1 k_1 = 0 \\ \partial\pi_2/\partial x_2 &= \theta_2 - 2\beta_2 x_2 - \gamma x_1 - \gamma x_2 k_2 = 0 \end{aligned} \tag{6.16}$$

where $k_i = dx_j/dx_i$ as conjectured by firm i. The equilibrium so defined uses the notion of 'correct' conjectures to provide uniqueness. Suppose firm 2 changes output from its equilibrium value by an infinitesimal amount dx_2. Then dx_1 is found from (6.16):

$$dx_1 = -\gamma/(2\beta_1 + \gamma k_1)dx_2$$

Therefore, if firm 2's conjecture is to be correct, we require

$$k_2 = -\gamma/(2\beta_1 + \gamma k_1) \tag{6.17}$$

Similarly, firm 1's conjecture must be

$$k_1 = -\gamma/(2\beta_2 + \gamma k_2) \tag{6.18}$$

Equations (6.16) - (6.18) are four equations in four unknowns, (x_1, x_2, k_1, k_2), so that it is possible to solve for the equilibrium conjectures and output. Ulph considers both interior and boundary optima; the equilibrium conditions (6.8) in Table 6.1 are for an interior solution with positive profit.[3]

Our <u>dominant firm</u> solution may be thought of as an extension of Stackelberg's follower-leader solution concept. Follower j is thought of as the aggregate of a competitive fringe of sellers. Leader i maximises π_i over residual demand (market demand minus fringe supply) and costs. Fringe supply is governed by marginal-cost pricing hence π_i is maximised subject to $dx_j/dx_i = -\gamma/\beta_j$. The dominant firm is distinguished by a Bertrand reaction function, the fringe by a social optimum. The outcome may be regarded as <u>quasi</u> rational conjectural equilibrium, in that the dominant firm's conjecture is correct while the individual fringe suppliers are price takers.[4]

Inspection of Table 6.1 shows that the extent of product differentiation, captured in the parameter γ, bears importantly on the equilibrium outcomes. Thus with homogeneous products ($\gamma = \beta_1 = \beta_2 = 1$ in our case) the Bertrand, RCE and dominant-firm equilibria all converge on the social optimum, since $(2 - \gamma^2/\beta_1\beta_2) = 1$ (Bertrand, dominant-firm), and $\delta = 0$ (RCE) respectively. Conversely, as γ goes to zero and there are no cross-price effects (i.e. complete product differentiation) the Cournot, Bertrand, RCE and Stackelberg outcomes converge on the market share/collusion position; in all cases the reaction functions reduce to

---

3. For details see Ulph (1980).

4. Fringe firms assume $dp_j/dx_i = 0$ with $p_j = p_i$, where j is the dominant firm and i = 1, 2, ... n is a member of an n firm fringe.

$$\theta_1 = 2\beta_1 x_1$$

and $$\theta_2 = 2\beta_2 x_2$$

In effect we are no longer dealing with a duopoly; interaction vanishes as the firms are now monopolists serving disjoint demands. Notice, however, that x_1 and x_2 are both positive and the outcome differs from the pure monopoly case in table 6.1. This refers to the homogeneous products case where only one firm exists. Hence π_i is maximised subject to $x_j = 0$. Clearly, constraining x_j to zero under complete product differentiation would involve more than merely that firm i has a monopoly.

Each of the foregoing equilibria is depicted in Figure 6.1. R_1M_1, R_2M_2 are the familiar Cournot reaction functions. Along each firm's equilibrium locus marginal cost equals perceived marginal revenue and a stable equilibrium exists at C. R_1N_1, R_2N_2 are the social-optimum 'reaction functions', where respectively firm 1's, 2's marginal cost equals price. Market share reaction functions are the loci M_1Q_1, M_2Q_2. The market share equilibrium MS at their intersection necessarily belongs to the set of efficient profit points: the curve M_1M_2, which is the locus of points of tangency of the duopolists' iso-profit curves. As we have seen, MS is also the joint-profit-maximising equilibrium. The Stackelberg outcomes S1, S2 occur at points of tangency between i's iso-profit curve and j's (Cournot) reaction function. Likewise, the dominant-firm equilibria may be found as points of tangency between the dominant-firm's iso-profit curve and the fringe's socially-optimal reaction-function, D1 (D2). In the homogeneous products case we could legitimately identify the end-points M1, M2 of the Cournot reaction functions as the pure monopoly outcomes for firms 1 and 2 respectively; marginal revenue equals marginal cost with x_2, x_1 respectively equal to zero.[5]

Bramness (1979) has delimited the area where kinked-demand-curve equilibria can arise within the framework we are using. Firm i believes that if x_i is increased x_j will increase equiproportionately, but that if x_i is decreased, x_j will stay unchanged. Then its 'equilibrium locus' is the whole zone between its Cournot and market-share reaction functions. The intersection of these zones for firms 1 and 2 covers the whole area where kinked-demand-curve equilibria can arise and is the shaded area in figure 6.1.

5. Figure 6.1 is not drawn for this case where, as was seen, B, RCE, D1 and D2 would converge on SO.

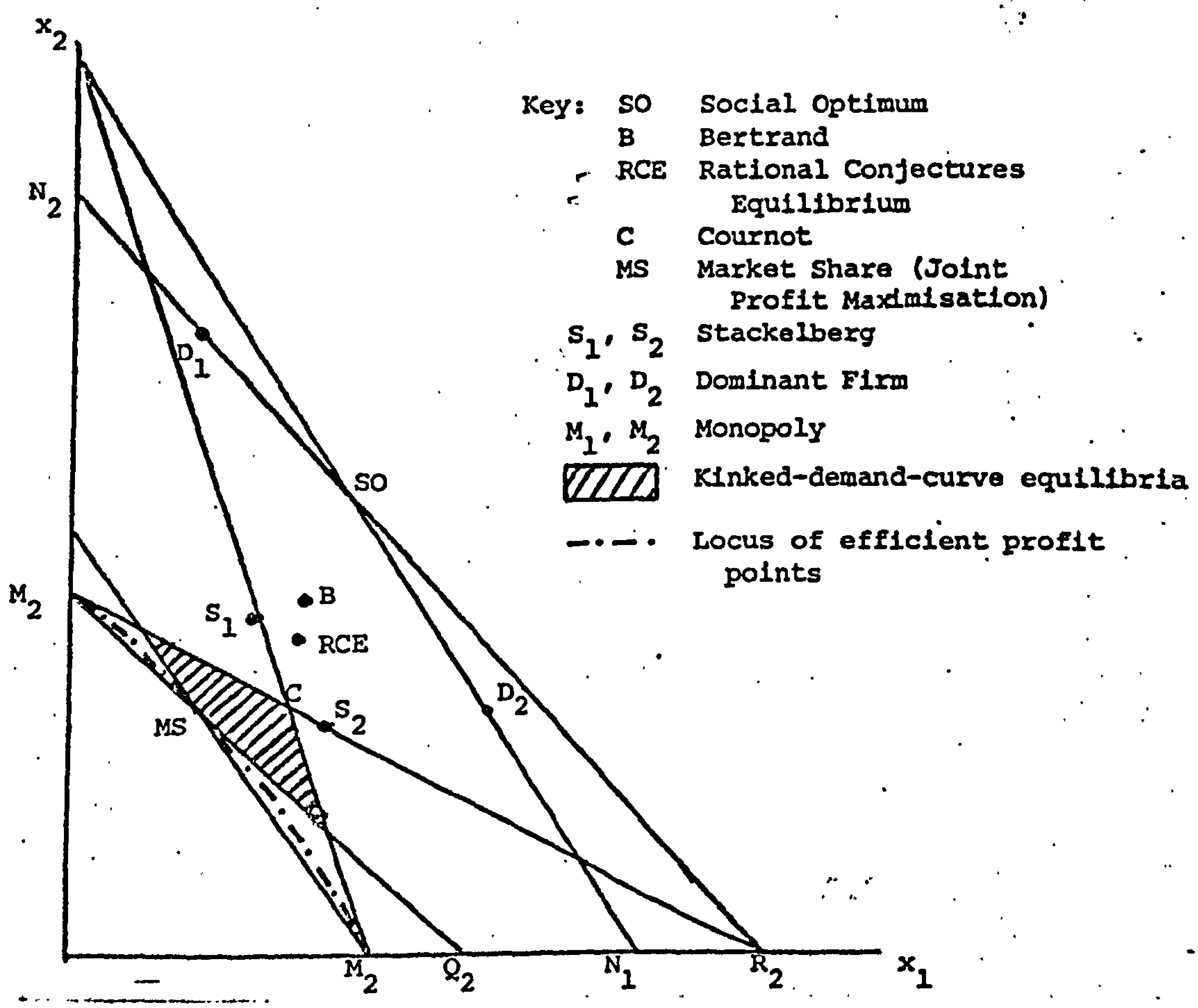
Table 6.1: Alternative Equilibria

Model	Maximand/Conjectural Variations	Equilibrium Conditions ("Reaction Functions")	
Social Optimum (SO)	$\text{Max}\{U - (c_1x_1 + c_2x_2)\}$	$\theta_1 = \beta_1x_1 + \gamma x_2$ $\theta_2 = \beta_2x_2 + \gamma x_1$	(6.4)
Cournot (C)	$dx_j/dx_i = 0$ $(i \neq j;\ i, j = 1,2)$	$\theta_1 = 2\beta_1x_1 + \gamma x_2$ $\theta_2 = 2\beta_2x_2 + \gamma x_1$	(6.5)
Bertrand (B)	$dx_j/dx_i = -\gamma/\beta_j$ (i.e. firm i chooses x_i assuming x_j changes such that p_j is constant)	$\theta_1 = (2 - \frac{\gamma^2}{\beta_1\beta_2})\beta_1x_1 + \gamma x_2$ $\theta_2 = (2 - \frac{\gamma^2}{\beta_1\beta_2})\beta_2x_2 + \gamma x_1$	(6.6)
Market Share (MS)	$dx_j/dx_i = x_j/x_i$ (i.e. firm i chooses x_i assuming x_j changes proportionately)	$\theta_1 = 2\beta_1x_1 + 2\gamma x_2$ $\theta_2 = 2\beta_2x_2 + 2\gamma x_1$	(6.7)
Collusion	$\text{Max}\{\Pi_1 + \Pi_2\}$		
Rational Conjectures (RCE)	Conjectural derivatives are endogenous.	$dx_2/dx_1 = -\beta_1(1 - \delta)/\gamma$ $dx_1/dx_2 = -\beta_2(1 - \delta)/\gamma$ $\theta_1 = \beta_1(1 + \delta)x_1 + \gamma x_2$ $\theta_2 = \beta_2(1 + \delta)x_2 + \gamma x_1$ with $\delta = \sqrt{1 - \gamma^2/\beta_1\beta_2}$	(6.8)
Stackelberg(i) (S1,S2)	$\text{Max } \Pi_i \text{ s.t. } dx_j/dx_i = -\gamma/2\beta_j$ (i.e. Cournot reaction)	$\theta_1 = (2-\gamma^2/2\beta_1\beta_2)\beta_1x_1+\gamma x_2$ $\theta_2 = 2\beta_2x_2 + \gamma x_1$	(6.9)
Dominant Firm(i) (D1,D2)	$\text{Max } \Pi_i \text{ s.t. } dx_j/dx_i = -\gamma/\beta_j$ (i.e. 'fringe' supply priced at marginal cost)	$\theta_1 = (2-\gamma^2/\beta_1\beta_2)\beta_1x_1+\gamma x_2$ $\theta_2 = \beta_2x_2 + \gamma x_1$	(6.10)
Monopoly(i)(ii) (M1,M2)	$\text{Max } \Pi_i \text{ s.t. } x_j = 0$	$x_i = \theta_i/2\beta_i$	

Notes: (i) Equilibrium condition assumes firm 1 'leads'. Similarly for firm 2.

(ii) Strictly, applies only where products are homogeneous $(\alpha_1 = \alpha_2,\ \beta_1 = \beta_2 = \gamma)$.

Figure 6.1 $



$ x_1 = output of firm 1

x_2 = output of firm 2

The outcomes reflect a mildly asymmetric case:- see Figure 6.4.

6.3 Welfare Comparisons

It is now possible to examine the extent of welfare losses in duopolistic markets, and how these vary according to

(i) alternative modes of interaction (conduct), and

(ii) alternative competitive states (structure) as captured in the underlying cost and demand parameters.

In respect to (ii), and with ultimate antitrust-policy implications in mind, we focus in particular on the way welfare losses behave as the degree of product differentiation increases (γ falls), and as one firm enjoys progressively larger cost- or demand-advantages (e.g. c_i/c_j falls or α_i/α_j increases). As any of these happen, competition is reduced in some sense, and we would expect an increase in the shortfall in welfare from the socially-optimal level. Our interest is in the gradient of the relationship between welfare and competition; in whether this relationship dominates or is dwarfed by the impact of alternative modes of conduct on welfare, for a given competitive state; and in whether variations in competitiveness bear on different behaviour patterns uniformly or differentially, i.e. whether the welfare ranking of alternative behavioural outcomes is preserved as the degree of competitiveness varies.

These questions are tackled with the aid of numerical computations. For specified parameter values we solve for equilibrium prices, outputs, profit, implied elasticities, net surplus (absolute and relative to the social optimum) and level of concentration (as measured by the Herfindahl index.[6] This output is also available in graphic form and figures 6.2

6. Since $H = 1/n + \sigma_n^2$ where n is the number of firms in the industry, the minimum H value in our case is 0.5 (except under dominant firm equilibrium), obtained whenever $x_1 = x_2$.

to 6.4 are examples. The contours of our social welfare function are elipses centred on SO, with zero or infinite slope as they intersect the social-optimum reaction functions. All three diagrams have $\gamma = 0.75$.[7] Figure 6.2 is for the symmetric case, and so MS, C, RCE, B and SO all lie along a ray through the origin with a similar ascending order welfare ranking. Figure 6.3 has firm 2 enjoying a 50 per cent cost advantage. Figure 6.4 features mildly asymmetric demand and costs. Since products are not homogeneous M_1, M_2 cannot be considered pure monopoly outcomes. Otherwise, as we should expect, market share (joint profit maximisation) generates least net surplus. Dominant-firm equilibria cause least reduction in welfare from the social optimum, other outcomes tending to cluster in-between.

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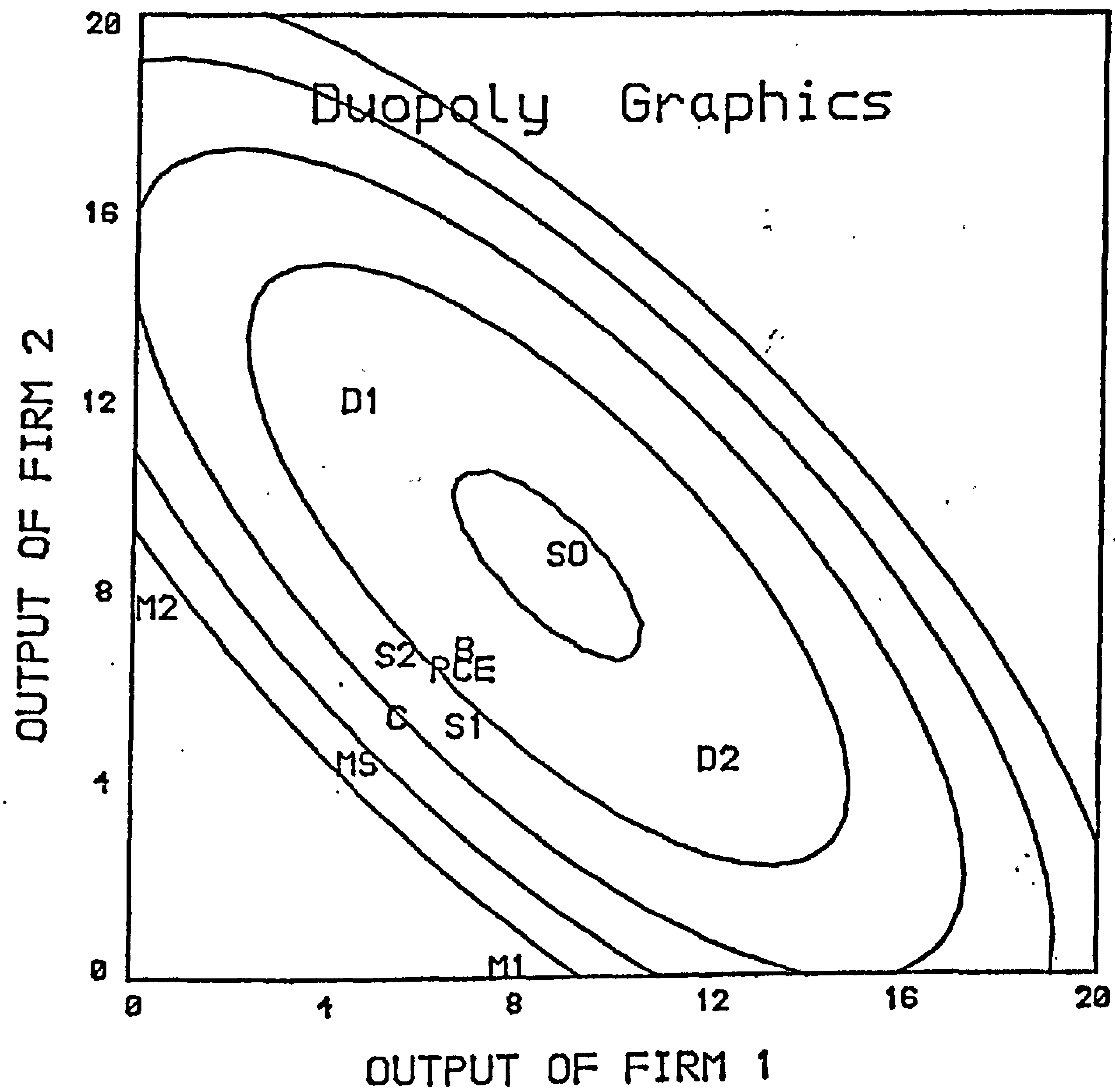
7. Parameter values are:

	α_1	α_2	β_1	β_2	γ	c_1	c_2
Fig. 6.2	20	20	1	1	0.75	6	6
Fig. 6.3	20	20	1	1	0.75	6	3
Fig. 6.4	20	22	1	1	0.75	6	6.6

The reaction function intersections are depicted by the label for each outcome. The figures become unduly cluttered if the reaction functions themselves are drawn.

Figure 6.2 $



$ Contours were again drawn by GINO-SURF (1980), the rest of the figure by GINO-F.

α_1	α_2	β_1	β_2	γ	c_1	c_2
20	20	1	1	0.75	6	6

Additional details about the computer drawn diagrams are given on the next page.

Notes for Figures 6.2 - 6.4

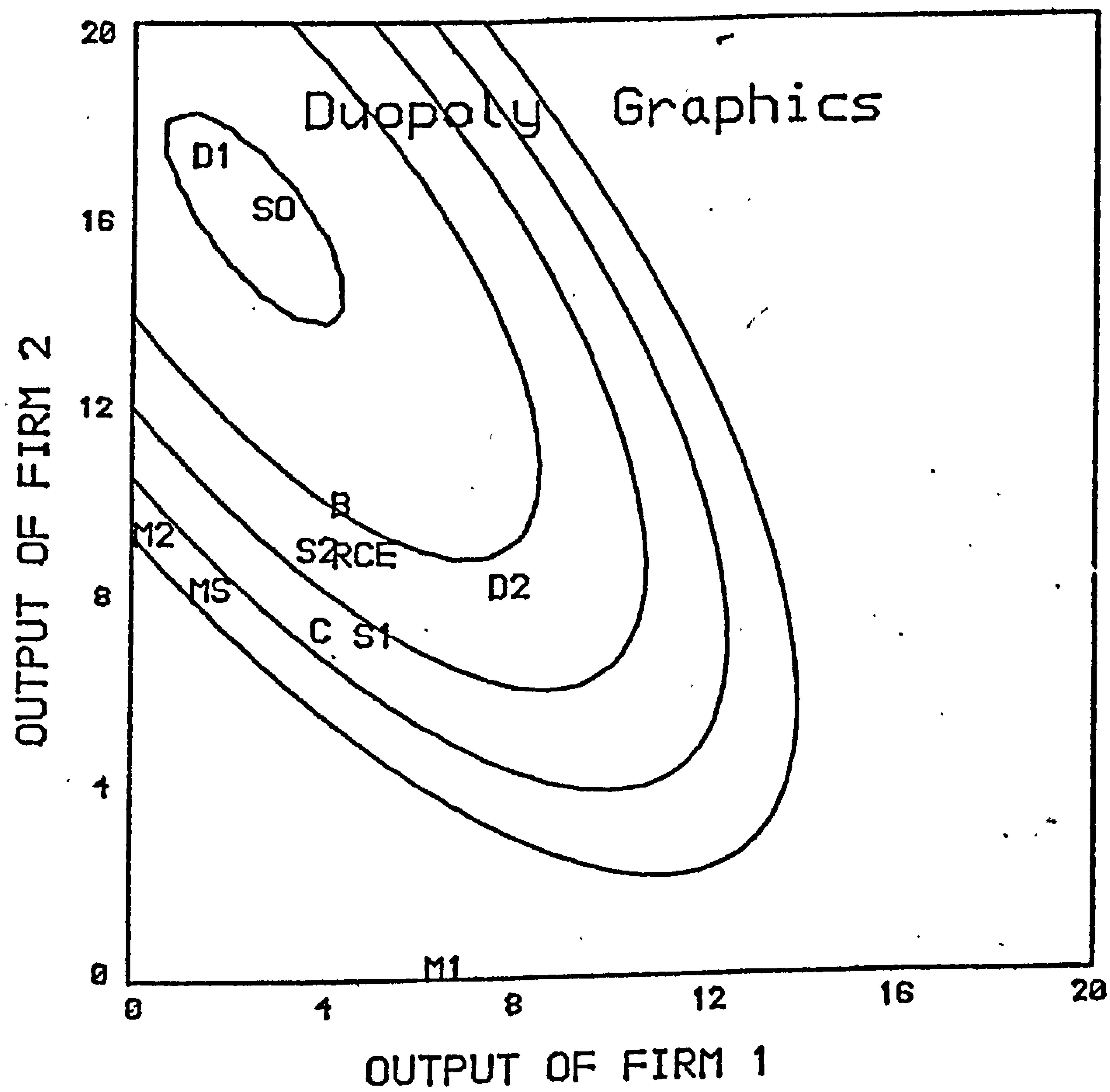
1. The exact coordinates of the conjectural equilibria are to be found at the bottom left hand corner of the first character in the label: except for single character labels like B where the reaction function intersection is central to the space occupied by the character. This is the reason why the SO does not appear central to the highest contour.

2. As in Chapter 5 the axis scales are not important. They simply represent roughly the original length of each axis at 20 cm. Moreover the heights of the contours do not feature because the discussion focusses on welfare losses relative to the social optimum.

3. A listing of the program which drew the figures for this Chapter is presented in an appendix.

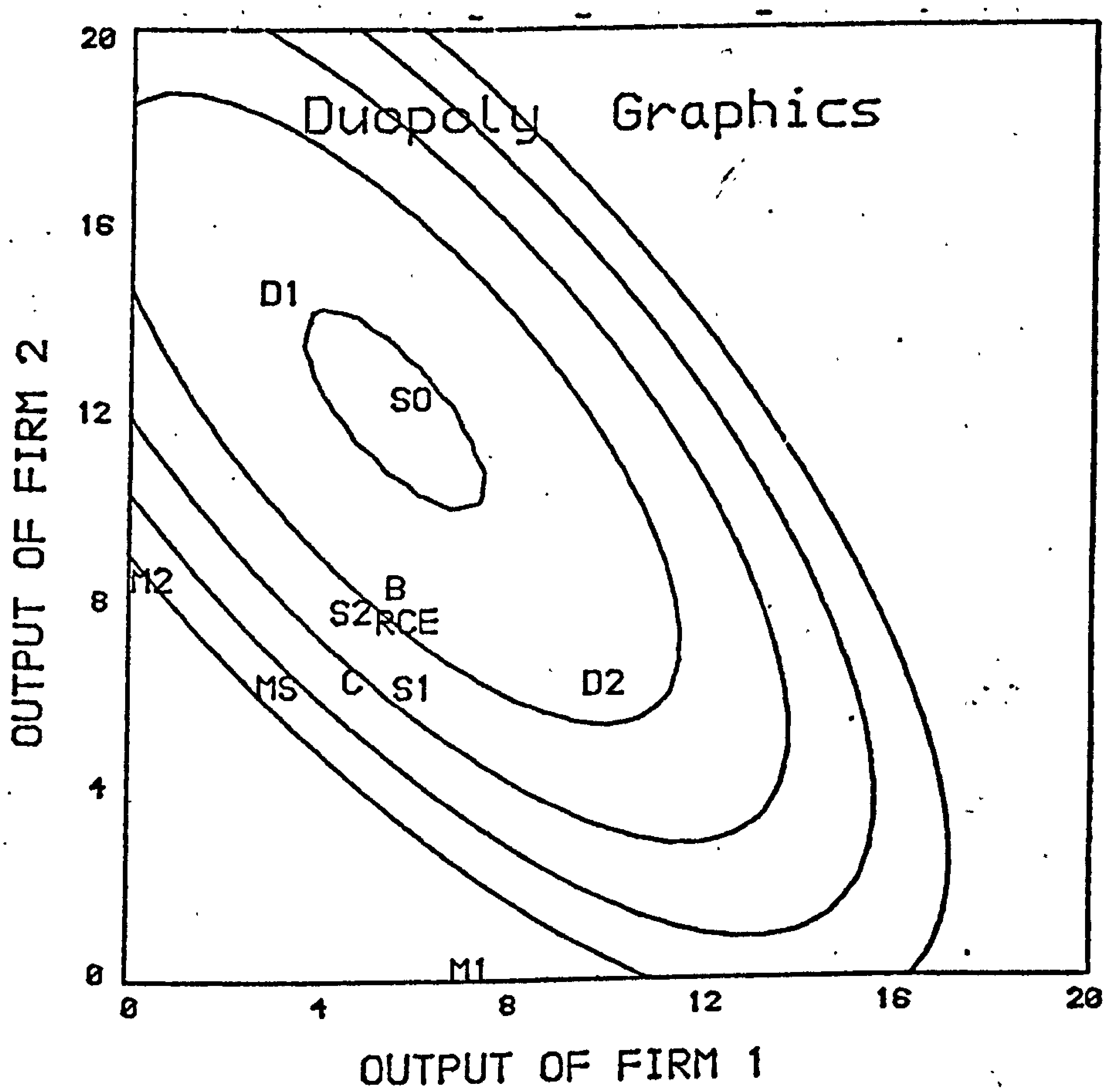
4. The graphics software employed was GINO-F and GINO-SURF.

Figure 6.3



α_1	α_2	β_1	β_2	γ	c_1	c_2
20	20	1	1	0.75	6	3

Figure 6.4



α_1	α_2	β_1	β_2	γ	c_1	c_2
20	22	1	1	0.75	6	6.6

With eight solution concepts and seven parameters to consider the number of possible permutations is large. However, not all make economic sense. Thus, product homogeneity is implausible where there are cost or demand asymmetries, and vice versa. For this would imply non-optimising behaviour by at least one firm or by consumers; if costs are asymmetric one or both firms must be inefficient, and if demands differ consumer preferences are irrational. On the other hand, where there is product differentiation, costs and demand may be either symmetric or asymmetric. For differentiation can arise either from both firms incurring extra costs to secure customer allegiance, or from one firm doing so. Finally, it could be argued that leader-follower behaviour, as under Stackelberg and dominant-firm equilibrium, is plausible only where one firm has a cost or demand advantage. Otherwise the assumption of leader-follower roles is arbitrary. In addition, we may discount Stackelberg and dominant-firm equilibria under which the leader is at a cost or demand disadvantage on grounds of total implausibility.

In presenting the results we first consider the special case of homogeneous products. We then examine separately the impact on the welfare rankings of variations in the degree of differentiation (γ) cost asymmetry (c_1/c_2) and demand asymmetry (α_1/α_2). Although, as we have said, not all of the implied combinations make sense, it is helpful to get some feel for the "partials" of welfare with respect to the parameters in this way. Next we consider joint variations in the parameters, focussing our attention on what we consider to be the most plausible or interesting combinations. In particular, we consider cases of low, medium and high product differentiation in conjunction with correspondingly low, medium and high degrees of cost disadvantage for one firm (firm 2) accompanied by a concomitant demand advantage. This is tantamount to extending our analysis to incorporate product quality and selling effort as decision variables to the firm, albeit for the special case where an x% cost differential secures the same percentage absolute demand advantage.

Homogeneous Products

Table 6.2 confirms the earlier analytical result that with homogeneous products socially optimal behaviour results not only from explicit marginal-cost pricing, but also under Bertrand interaction and RCE.[8] At the other extreme market share behaviour coincides with pure monopoly, as we expect from the theory, generating only half the socially optimal output at more than twice the competitive price, and the expected monopoly welfare loss of 25 per cent. Cournot interaction cuts the monopoly loss to only 11.1 per cent and Stackelberg behaviour to 6.2 per cent.

Note the perverse relationship between 'market structure' and welfare loss. Thus the Herfindahl index fails to distinguish the social optimum, Bertrand and RCE, on the one hand, and the market share equilibrium on the other, whereas these lie at extreme ends of the range of variation in welfare! Furthermore the intermediate Cournot and Stackelberg cases are ranked perversely, the latter generating little over half the welfare loss of the former, despite a more concentrated market structure. Similarly non-discriminating or perverse results occurred throughout our analysis. We conclude that evidently, and contrary to a strong tradition in industrial organisation, conduct matters.

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8. Dominant-firm equilibrium is not reported in Table 6.2 for the reasons given.

Product Differentiation

Table 6.3 confirms the convergence of the Cournot, Bertrand, RCE and Stackelberg equilibria on the market share/collusion outcome. Notice that the convergence proceeds quite rapidly as the degree of product differentiation increases and γ falls from unity. Thus, despite their differing starting levels, the Cournot, Bertrand, RCE and Stackelberg welfare losses all lie between 18 and 20 per cent of the social optimum when $\gamma = 0.25$. At this point the average increase in welfare loss for these solutions compared with the homogeneous product case is 15.5 per cent. We conclude that the gradient of the competitiveness - welfare relationship in this plane is quite steep. However, the welfare ranking of the alternative equilibria is preserved. Meanwhile the dominant-firm welfare loss also increases as γ falls, to 12.5 per cent - one half that of other solutions. This simply registers the fact that half the total output produced is subject to pure monopoly pricing and half is priced competitively. However this result is little more than a curiosum, in the absence of the cost asymmetry needed to render the dominant-firm solution concept plausible.

Cost Asymmetry

As expected, where rivalry is reduced due to one firm having lower costs (firm 2 in our examples), relative welfare losses increase with the degree of cost advantage (table 6.4). At high levels of product differentiation (low γ) the effect is barely perceptible (table 6.4(b)). It remains small even where products are relatively homogeneous; where $\gamma = 0.75$ (table 6.4(a)) the average percentage welfare loss for five meaningful cases (i.e. omitting the social optimum,

Table 6.2: Price, Output and Welfare: Homogenous Products [(i)]

Model	X_1	X_2	P_1	P_2	η_1	η_2	Welfare Index (S=100)	HERF
Social optimum	7.0		6.0		0.9		100.0	0.50
Cournot	4.7		10.7		2.3		88.9	0.50
Bertrand	7.0		6.0		0.9		100.0	0.50
Market Share	3.5		13.0		3.7		75.0	0.50
RCE	7.0		6.0		0.9		100.0	0.50
Stackelberg(1)	7.0	3.5	9.5		1.4	2.7	93.8	0.56
Stackelberg(2)	3.5	7.0	9.5		2.7	1.4	93.8	0.56
Monopoly(1)	7.0'	-	13.0	-	1.9	-	75.0	1.00
Monopoly(2)	-	7.0	-	13.0	-	1.9	75.0	1.00

Note (i) Assumes $\alpha_1, \alpha_2 = 20.0;\ \beta_1, \beta_2 = 1.0;\ c_1, c_2 = 6.0$

Table 6.3: Welfare Indices and the Degree of Product Differentiation (γ)

Model	$\gamma \to 1$	$\gamma = 0.75$	$\gamma = 0.5$	$\gamma = 0.25$	$\gamma \to 0$
SO	100.0	100.0	100.0	100.0	100.0
C	88.9	86.8	84.0	80.2	75.0
B	100.0	96.0	88.9	81.6	75.0
M	75.0	75.0	75.0	75.0	75.0
RCE	100.0	92.5	86.6	81.0	75.0
S1, S2	93.8	89.3	85.2	80.6	75.0
D1, D2	(100.0)	96.9	93.7	90.6	87.5
M1, M2	75.0	(-)	(-)	(-)	(-)

Table 6.4: Welfare Indices under Cost Asymmetry

Model \ $c_1=6.0; c_2=$	6.0	5.0	4.0	3.0
(a) $\gamma = 0.75$				
SO	100.0	100.0	100.0	100.0
C	86.8	86.4	85.2	83.6
B	96.0	95.7	94.9	93.8
M	75.0	75.0	75.0	75.0
RCE	92.5	92.1	91.1	89.7
S2	89.3	89.6	89.1	88.0
D2	96.9	95.2	93.5	91.7
(b) $\gamma = 0.25$				
SO	100.0	100.0	100.0	100.0
C	80.2	80.2	80.2	80.0
B	81.6	81.6	81.5	81.4
M	75.0	75.0	75.0	75.0
RCE	81.0	80.9	80.6	80.8
S2	80.6	80.6	80.6	80.5
D2	90.6	89.5	88.5	87.6

Note: Demand symmetric: $\alpha_1, \alpha_2 = 20$; $\beta_1, \beta_2 = 1$; $c_1 = 6.0$

M S1 and D1) is 10.6 per cent with a 50 per cent cost differential, compared with 7.7 per cent where there is none. Over this range the welfare ranking is substantially unaffected, only Bertrand and D2 interchanging places, these being very close in the original, symmetric cost case. In practice we would expect differential costs and diverse products to go together; if both firms have access to the same technology and are cost minimisers, inter-firm cost differences are most likely to be product-related. Hence, where cost asymmetries are most likely to be found, their impact on welfare, though adverse, is very slight.

Demand Asymmetry

A similar conclusion applies in the case of demand asymmetry. Thus, where products are relatively homogeneous there is a sharp increase in relative welfare loss in all cases (except, of course, market share) as firm 2's demand advantage is increased (table 6.5(a)). But this is an unlikely state of affairs. More plausible is that a marked demand advantage will be associated with highly differential products. In this case the impact of demand asymmetry on the indices of welfare loss for different types of equilibrium is, with one exception, minimal (table 6.5(b)). The exception is dominant firm equilibrium, where relative welfare loss almost doubles from 9.9 per cent where there is no asymmetry to 17.4 per cent where the dominant firm has a 50 per cent absolute demand advantage. Thus the welfare-enhancing effect of a competitive fringe, it appears, is much reduced where it supplies an inferior product.

Joint Variation

Table 6.6 shows what happens when the degree of product

Table 6.5: Welfare Indices under Demand Asymmetry ($\alpha_1 \neq \alpha_2$; $\beta_1 = \beta_2$)

Model \ α_1=20; α_2=	20	22	24	26	28	30
(a) $\gamma = 0.75$						
SO	100.0	100.0	100.0	100.0	100.0	100.0
C	86.8	85.3	81.8	77.7	73.7	70.2
B	96.0	94.9	92.4	89.5	86.6	84.1
M	75.0	75.0	75.0	75.0	75.0	75.0
RCE	92.5	91.1	88.0	84.4	80.9	77.7
S2	89.4	89.1	86.6	83.3	79.8	76.6
D2	96.9	93.5	90.1	87.2	84.8	83.0
(b) $\gamma = 0.25$						
SO	100.0	100.0	100.0	100.0	100.0	100.0
C	80.3	80.2	79.9	79.6	79.3	78.9
B	81.6	81.5	81.3	81.0	80.7	80.3
M	75.0	75.0	75.0	75.0	75.0	75.0
RCE	81.0	80.9	80.6	80.3	80.0	79.6
S2	80.6	80.6	80.4	80.1	79.8	79.5
D2	90.6	88.5	86.7	85.1	83.7	82.6

differentiation varies in the presence of simultaneous, offsetting asymmetries in both cost and demand. We focus only on plausible combinations: e.g. 'mild' product heterogeneity accompanied by 'modest' additional costs and demand advantage, etc. The results in general confirm previous conclusions. Thus, scanning any column, we see that the type of interactive behaviour in force makes a substantial difference to welfare. Average percentage losses over the nine reported cases are Cournot 16.8; Bertrand 11.5; Market Share 25.0; RCE 13.7; Stackelberg 15.4; and Dominant-Firm 6.4. Similarly, welfare losses are much affected by the degree of product differentiation. Average percentage losses across all types of equilibria in 6(a) 6(b) and 6(c) are 10.0, 15.8, and 17.7 respectively. However cost and demand asymmetries, in this case across a range of variation appropriate to the degree of product heterogeneity, make very little difference, again with the exception of the dominant firm case.

Table 6.6: Joint Variation

	(a) Mild Differentiation (γ = .75)		(b) Medium Differentiation (γ = .5)		(c) High Differentiation (γ = .25)				
	a(i)	a(ii)	b(i)	b(ii)	c(i)	c(ii)	c(iii)	c(iv)	c(v)
S	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
C	86.6	86.0	83.8	83.3	80.2	80.1	79.9	79.7	79.4
B	95.9	95.4	88.7	88.3	81.6	81.5	81.3	81.1	80.9
M	75.0	75.0	75.0	75.0	75.0	75.0	75.0	75.0	75.0
RCE	92.3	91.8	86.4	86.0	80.9	80.8	80.6	80.4	80.2
S1	88.7	87.7	84.8	84.1	80.5	80.3	80.1	79.9	79.6
S2	89.6	89.5	85.3	85.1	80.6	80.5	80.4	80.2	80.0
D1	97.9	98.6	95.4	96.8	92.1	93.3	94.4	95.3	96.1
D2	95.7	94.5	91.9	90.1	89.1	87.7	86.5	85.4	84.4
Parameter Values:									
α_1	20	20	20	20	20	20	20	20	20
α_2	21	22	22	24	22	24	26	28	30
β_1	1	1	1	1	1	1	1	1	1
β_2	1	1	1	1	1	1	1	1	1
c_1	6.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0
c_2	6.3	6.6	6.6	7.2	6.6	7.2	7.8	8.4	9.0

6.4 Conclusions

Four principal conclusions emerge from the analysis, subject of course to the assumptions underlying this approach: static duopoly equilibria with no entry and a specific utility function which rules out income effects.

Firstly, under duopolistic rivalry the particular form of oligopolistic interaction exerts a major influence on the level of welfare. Conduct matters! In general Dominant Firm equilibria involve least welfare loss, usually around one third of the maximum, Market Share - Collusion loss level. The intermediate cases are consistently ranked Bertrand, RCE, Stackelberg and Cournot, in ascending order of welfare loss, in the range one half to somewhat over two thirds of the maximum.[9] As we have seen, kinked demand curve equilibria will lie between the Cournot and Market Share values. It follows that the design and execution of antitrust policies should not focus wholly or primarily on structural conditions. Two cases merit special attention.

First, we have seen that Market Share behaviour coincides with joint-profit-maximisation and produces the largest welfare loss: 25 per cent in the case of linear demand.[10] Under competition law in most countries where such policy exists, overt collusion is proscribed. However non-competitive, adaptive behaviour does not infringe the law unless an agreement can be inferred. The analysis shows that, where non-cooperative interaction takes the form of mutual market share maximisation, precisely the same outcome will be reached. It thus calls into question the existence of a clear distinction in law between

---

9. This ranking is also discernible from the computer drawn diagrams.

10. This is, of course, dependent upon the symmetry of cross price effects.

the two cases. In countries like the U.K. and West Germany, where competition policy provides for the application of a test of the public interest on a case-by-case basis, the analysis suggests that evidence of Market Share interaction should invariably lead to a negative finding whether or not an implicit agreement can be inferred.

Secondly, the computations draw attention to the welfare enhancing effect of competition from a competitive fringe. This almost invariably produces less welfare loss than any other form of rivalry, and in many cases the losses amount to only a few percentage points. However, the constraining influence of competition from the fringe is much weakened where products are heterogeneous. When evaluating dominant-firm cases antitrust agencies should therefore pay close attention to the cross elasticities of demand between the fringe and dominant firm's products. Needless to say they should also be careful to ensure that the fringe prices at marginal cost and earns zero profit.

The second conclusion argues that the power of inter-firm rivalry to further social welfare is highly sensitive to the degree of product differentiation in the market. Where products are homogeneous three types of interactive behaviour generate welfare levels equal to the social optimum, whereas all but the dominant firm case lead to maximum, market share (collusion) losses if there is complete differentiation of products. Furthermore, welfare losses increase rapidly as product heterogeneity enters.[11] Antitrust policy and agencies should therefore pay close attention to the cross-elasticities of demand between rival's products in all cases.

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11. Focussing on <u>relative</u> welfare losses we ignore improvements in welfare at the social optimum through increased product differentiation and the resource costs of securing them.

Thirdly, over broad ranges, asymmetric cost and demand conditions as between rivals generally have little effect on the size of welfare losses. The one (dominant firm) exception has already been discussed. Finally, measures of market structure are an unreliable guide to the level of welfare in duopoly markets is essentially a corollary of the first conclusion. Because conduct matters it cannot be assumed that there is a unique or even close relationship between particular structural conditions and performance. In particular, measures of seller concentration such as the Herfindahl index may either fail to distinguish different social outcomes or even rank them perversely.

Appendix to Chapter 6

The use of computer software often requires the knowledge of a high level language. In the case of NAG a working knowledge of Fortran or Algol is necessary. In the present case the many programs written to evaluate the diversity of problems used Fortran. It would be tedious to provide a listing of all the programs; so it was decided to present only the program used in Chapter 6. A listing is presented below, and is based on ICL Fortran, 1966 standard.

```
      PROGRAM DUOPLY
      REAL BETA(11,2),GAMMA(11,2),X(11),Y(11),RFL(4),UHAT(11),UCON(6)
      REAL P2(2),BETA1,BETA2,ALPHA1,ALPHA2,GAMM,C1,C2,DT,THETA1,THETA2
      REAL XMAX,YMAX,YP,XP1,XP2,XT(11),YT(11),P1(2)
      INTEGER I,J,K,L,HEADIN(6)
      LOGICAL JUMP
      COMMON/PARAM/ ALPHA1,ALPHA2,BETA1,BETA2,GAMM,C1,C2,SCALX,SCALY
      EXTERNAL A,PI1,PI2
      DATA HEADIN(1)/4HD*LU/
      DATA HEADIN(2)/4HOPOL/
      DATA HEADIN(3)/4HY  */
      DATA HEADIN(4)/4HUG*L/
      DATA HEADIN(5)/4HRAPH/
      DATA HEADIN(6)/4HICS /
      JUMP=.TRUE.
C
C THIS PROGRAM  EXAMINES THE WELFARE RANKINGS OF DIFFERENT
C DUOPOLY OUTCOMES.
C
      CALL CC936N
      CALL UNITS(10.0)
    1 READ(5,*,END=160)ALPHA1,ALPHA2,BETA1,BETA2,GAMM,C1,C2
      WRITE(6,150)ALPHA1,ALPHA2,BETA1,BETA2,GAMM,C1,C2
  150 FORMAT(1H0,7HALPHA1=,F5.2,7HALPHA2=,F5.2,6HBETA1=,F5.2,
     16BETA2=,F5.2,5HGAMM=,F5.2,3HC1=,F5.2,3HC2=,F5.2,//)
      THETA1=ALPHA1-C1
      THETA2=ALPHA2-C2
      RFL(1)=THETA1/BETA1
      RFL(2)=THETA2/GAMM
      RFL(3)=THETA1/GAMM
      RFL(4)=THETA2/BETA2
      XMAX=0.0
      YMAX=0.0
      DT=SQRT(1.0-(GAMM*GAMM)/(BETA1*BETA2))
      DO 120 I=1,2
      IF(RFL(I).GT.XMAX)XMAX=RFL(I)
  120 CONTINUE
      DO 140 I=3,4
      IF(RFL(I).GT.YMAX)YMAX=RFL(I)
  140 CONTINUE
      WRITE(6,800)XMAX,YMAX
  800 FORMAT(2F14.6)
      XPHYS=((XMAX-0.0)/(XMAX-0.0))*14.0
      YPHYS=((YMAX-0.0)/(YMAX-0.0))*14.0
      WRITE(6,825)XPHYS,YPHYS
  825 FORMAT(2F14.6)
      SCALX=14.0/XMAX
      SCALY=14.0/YMAX
C
C     SOCIAL OPTIMUM
C
      BETA(1,1)=BETA1
      BETA(1,2)=BETA2
      GAMMA(1,1)=GAMM
      GAMMA(1,2)=GAMM
```

```
C
C     COURNOT
C
      BETA(2,1)=2.0*BETA1
      BETA(2,2)=2.0*BETA2
      GAMMA(2,1)=GAMM
      GAMMA(2,2)=GAMM
C
C     BERTRAND
C
      BETA(3,1)=(2.0-GAMM*GAMM/(BETA1*BETA2))*BETA1
      BETA(3,2)=(2.0-GAMM*GAMM/(BETA1*BETA2))*BETA2
      GAMMA(3,1)=GAMM
      GAMMA(3,2)=GAMM
C
C     MARKET SHARE
C
      BETA(4,1)=BETA(2,1)
      BETA(4,2)=BETA(2,2)
      GAMMA(4,1)=2.0*GAMM
      GAMMA(4,2)=2.0*GAMM
C
C     RATIONAL EXPECTATIONS
      BETA(5,1)=2.0*BETA1-BETA1*(1.0-DT)
      BETA(5,2)=2.0*BETA2-BETA2*(1.0-DT)
      GAMMA(5,1)=GAMM
      GAMMA(5,2)=GAMM
C
C     STACKELBERG
C
C     FIRM 1 LEADER
C
      BETA(6,1)=(2.0-GAMM*GAMM/(2.0*BETA1*BETA2))*BETA1
      BETA(6,2)=2.0*BETA2
      GAMMA(6,1)=GAMM
      GAMMA(6,2)=GAMM
C
C     FIRM 2 LEADER
C
      BETA(7,1)=2.0*BETA1
      BETA(7,2)=(2.0-GAMM*GAMM/(2.0*BETA1*BETA2))*BETA2
      GAMMA(7,1)=GAMM
      GAMMA(7,2)=GAMM
C
C     DOMINANT FIRM/COMPETITIVE FRINGE
C
C     FIRM 1 LEADER
C
      BETA(8,1)=(2.0-GAMM*GAMM/(BETA1*BETA2))*BETA1
      BETA(8,2)=BETA2
      GAMMA(8,1)=GAMM
      GAMMA(8,2)=GAMM
C
C     FIRM 2 LEADER
C
      BETA(9,1)=BETA1
      BETA(9,2)=(2.0-GAMM*GAMM/(BETA1*BETA2))*BETA2
      GAMMA(9,1)=GAMM
      GAMMA(9,2)=GAMM
```

```
C
C     SET UP GRAPHICAL SPACE
C
      CALL PICCLE
      CALL WINDO2(0.0,15.0,0.0,15.0)
C
C     EVALUATE INTERSECTIONS AND CONTOUR HEIGHTS
C
      X(10)=THETA1/(2.0*BETA1)
      Y(10)=0.0
      X(11)=0.1
      Y(11)=THETA2/(2.0*BETA2)
      DO 80 I=1,11
      IF(I.EQ.10.OR.I.EQ.11) GO TO 17
      Y(I)=(GAMMA(I,2)*THETA1-BETA(I,1)*THETA2)/(GAMMA(I,1)*
     1GAMMA(I,2)-BETA(I,1)*BETA(I,2))
      X(I)=(THETA1-GAMMA(I,1)+Y(I))/BETA(I,1)
   17 UHAT(I)=ALPHA1*X(I)+ALPHA2*Y(I)-0.5*(BETA1*X(I)*X(I)+2.0*GAMM*
     1X(I)*Y(I)+BETA2*Y(I)*Y(I))-(C1*X(I)+C2*Y(I)
      IF(I.EQ.6)P1(2)=THETA1*X(I)-BETA1*X(I)*X(I)-GAMM*X(I)*Y(I)
      IF(I.EQ.7)P2(2)=THETA2*Y(I)-BETA2*Y(I)*Y(I)-GAMM*X(I)*Y(I)
      IF(I.EQ.8)P1(1)=THETA1*X(I)-BETA1*X(I)*X(I)-GAMM*X(I)*Y(I)
      IF(I.EQ.9)P2(1)=THETA2*Y(I)-BETA2*Y(I)*Y(I)-GAMM*X(I)*Y(I)
      Y(I)=Y(I)*SCALY
      X(I)=X(I)*SCALY
      WRITE(6,500)X(I),Y(I),UHAT(I),I
  500 FORMAT(1HO,4HX1 =,F8.4,3X,4HX2 =,F8.4,3X,6HUHAT =,F8.4,4X,2HI=,12)
   80 CONTINUE
      WRITE(6,550) (P1(I),P2(I),I=1,2)
  550 FORMAT(1HO,4HP1 =,F8.4,3X,4HP2 =,F8.4)
C
C     EVALUATE CONTOUR HEIGHTS FOR FUNCON
C
      CONTU=0.993*UHAT(1)
      CONTL=0.75*UHAT(1)
C
C     PLOT SURPLUS CONTOURS
C
      CALL SETFRA(1)
      CALL LEVELS(CONTL,CONTU)
      CALL FUNCON(0.0,XPHYS,0.0,YPHYS,A,5,1)
C
C     LABEL REACTION FUNCTION INTERSECTIONS
C
      CALL CHASIZ(0.25,0.25)
      DO 90 I=1,11
      CALL CONSPA(X(I),Y(I),XT(I),YT(I))
      CALL MOVTO2(XT(I),YT(I))
      IF(I.EQ.1)CALL CHAHOL(4HSO*.)
      IF(I.EQ.2)CALL CHACEN(1HC)
      IF(I.EQ.3)CALL CHACEN(1HB)
      IF(I.EQ.4)CALL CHAHOL(4HMS*.)
      IF(I.EQ.5)CALL CHAHOL(5HRCE*.)
      IF(I.EQ.6)CALL CHAHOL(4HS1*.)
      IF(I.EQ.7)CALL CHAHOL(4HS2*.)
      IF(I.EQ.8)CALL CHAHOL(4HD1*.)
      IF(I.EQ.9)CALL CHAHOL(4HD2*.)
      IF(I.EQ.10)CALL CHAHOL(4HM1*.)
      IF(I.EQ.11)CALL CHAHOL(4HM2*.)
   90 CONTINUE
```

```
C
C     PLOT REACTION FUNCTIONS IF JUMP IS FALSE
C
      IF(JUMP) GO TO 75
      J=1
      DO 40 I=1,5
      X(1)=0.0
      Y(1)=THETA1/GAMMA(I,J)*SCALY
      X(2)=THETA1/BETA(I,J)*SCALX
      Y(2)=0.0
      CALL CONSPA(X(1),Y(1),XT(1),YT(1))
      CALL MOVTO2(XT(1),YT(1))
      CALL CONSPA(X(2),Y(2),XT(2),YT(2))
      CALL LINTO2(XT(2),YT(2))
   40 CONTINUE
      J=2
      DO 50 I=1,5
      X(1)=0.0
      Y(1)=THETA2/BETA(I,J)*SCALY
      X(2)=THETA2/GAMMA(I,J)*SCALX
      Y(2)=0.0
      CALL CONSPA(X(1),Y(1), XT(1),YT(1))
      CALL MOVTO2(XT(1),YT(1))
      CALL CONSPA(X(2),Y(2),XT(2),YT(2))
      CALL LINTO2(XT(2),YT(2))
   50 CONTINUE
   75 CONTINUE
C
C     LABEL AXES AND ADD TITLE
C
      CALL CHASIZ(0.30,0.30)
      CALL CONSPA(3.5,-1.5,X1,Y1)
      CALL MOVTO2(X1,Y1)
      CALL CHAHOL(18HOUTPUT OF FIRM 1*.)
      CALL CONSPA(-1.25,3.5,X2,Y2)
      CALL MOVTO2(X2,Y2)
      CALL CHAANG(90.0)
      CALL CHAHOL(18HOUTPUT OF FIRM 2*.)
      CALL CHAANG(0.0)
      CALL CHASIZ(0.40,0.40)
      CALL CONSPA(2.5,12.5,X3,Y3)
      CALL MOVTO2(X3,Y3)
      CALL CHAARR(HEADIN,6,4)
      CALL CHASIZ(0.2,0.2)
      XA=0.0
      YA=0.50
      DO 45 I=1,12
      IF(I.LT.7)CALL CONSPA(XA,YA,A1,B1)
      IF(I.EQ.7)XA=0.0
      IF(I.EQ.10)YA=0.75
      IF(I.GE.7)CALL CONSPA(YA,XA,A1,B1)
      CALL MOVTO2(A1,B1)
      IF(I.EQ.1.OR.I.EQ.7)CALL CHAHOL(3H0*.)
      IF(I.EQ.2.OR.I.EQ.8)CALL CHAHOL(3H4*.)
      IF(I.EQ.3.OR.I.EQ.9)CALL CHAHOL(3H8*.)
      IF(I.EQ.4.OR.I.EQ.10)CALL CHAHOL(4H12*.)
      IF(I.EQ.5.OR.I.EQ.11)CALL CHAHOL(4H16*.)
      IF(I.EQ.6.OR.I.EQ.12)CALL CHAHOL(4H20*.)
      XA=XA+2.75
```

contd/

contd/

```
 45 CONTINUE
    GO TO 1
160 CONTINUE
    CALL DEVEND
    STOP
    END
    REAL FUNCTION A(X,Y)
    COMMON/PARAM/ ALPHA1,ALPHA2,BETA1,BETA2,GAMM,C1,C2,SCALX,SCALY
    X=X/SCALX
    Y=Y/SCALY
    A=ALPHA1*X+ALPHA2*Y-0.5*(BETA1*X*X+2.0*GAMM*X*Y+BETA2*Y*Y)
   1-(C1*X+C2*Y)
    RETURN
    END
    REAL FUNCTION PI1(X,Y)
    COMMON/PARAM/ ALPHA1,ALPHA2,BETA1,BETA2,GAMM,C1,C2,SCALX,SCALY
    X=X/SCALX
    Y=Y/SCALY
    PI1=THETA1*X-BETA1*X*X-GAMM*X*Y
    RETURN
    END
    REAL FUNCTION PI2(X,Y)
    COMMON/PARAM/ ALPHA1,ALPHA2,BETA1,BETA2,GAMM,C1,C2,SCALX,SCALY
    X=X/SCALX
    Y=Y/SCALY
    PI2=THETA2*Y-BETA2*Y*Y-GAMM*X*Y
    RETURN
    END
```

Conclusion*

The main aim of this thesis has been to demonstrate the scope of an alternative method to quantify relevant features of microeconomic models, which were not suitable for conventional statistical approaches. The methodology reflects numerical analysis procedures, which have been given an 'easy to use' status through the growth in computer software, in particular, the advance of the Numerical Algorithms Group (NAG, 1981) library of programme subroutines. The NAG development was conscious of the importance of portability, so the library is available at most university computer sites.

The demonstration took the form of a number of alternative applications across a wide spectrum of microeconomic models. The lack of a statistical basis made the interpretation of the results rather more tentative, but in no way invalidated them. A fairly general synopsis may help to justify this position. The second chapter explored the likelihood of single price equilibria in a world of imperfect information. It would appear that the single price outcome is the exception rather than the rule. For example, in a framework of monopolistic competition where no individuals have zero search costs, the proportion of the market induced to search must be substantially uninfluenced by considerable price dispersion. The third chapter advertised the versatility of the NAG routine EO4UAF in solving complex optimum tax problems. The shape of optimum tax schedules took on new dimensions where optimum negative marginal tax rates were found to be no longer a curiosity. Of key significance are the general equilibrium

* The diversity of economic subject matter warranted a concluding section to each chapter. The reader is referred back for specific details of the different analyses.

interrelationships of labour supplies and wage rates when the latter are endogenous.

The fourth chapter examined a partially unionised labour market model of a closed economy which yielded ambiguous theoretical results about the link between union and non-union wage rates. Numerical computations suggested that an inverse relationship would receive most support. The fifth chapter discussed some aspects of the excess capacity debate peculiar to the monopolistic competition literature. A computational appraisal made it clear that the issue of the optimum number of products depended upon the complex interactions of firms' demands and costs. There is no longer support for the view that excess capacity is simply too many products. Cases with too few products in relation to the social optimum were not hard to compute for reasonable demand and cost conditions. The final chapter suggested a framework for including all manner of duopoly equilibria under a specific social welfare function. This enabled the examination of the structure-conduct-performance paradigm where structure is endogenous and the relative welfare position of each duopoly outcome can be assessed. It was apparent that the oft postulated structure-performance relationship was tenuous: conduct does matter. Chapters 5 and 6 also illustrated the scope for computer graphics in the presentation of results. This may be of more pedagogic value.

Finally, it should be stressed that the NAG library circumvents any need for a specialism in either numerical analysis or computer programming. However any use of NAG in its present form does require knowledge of a high-level computer language like Fortran or Algol 60. This is not so difficult, as there are many ways to write a programme. An experienced programmer would have written a rather different version of the listing presented in the appendix to Chapter 6. This is of little consequence when most of the programme run time is spent within the graphics software.

It may now be appropriate to indicate my belief that computational methods provide an additional quantitative tool for the microeconomist and can offer useful insights into microeconomic problems.

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