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Abstract

In this paper, in an exchange economy with atoms and an atomless part, we analyze the relationship between the set of the Cournot-Nash equilibrium allocations of a strategic market game and the set of the Walras equilibrium allocations of the exchange economy with which it is associated. In an example, we show that, even when atoms are countably infinite, Cournot-Nash equilibria yield different allocations from the Walras equilibrium allocations of the underlying exchange economy. We partially replicate the exchange economy by increasing the number of atoms without affecting the atomless part while ensuring that the measure space of agents remains finite. We show that any sequence of Cournot-Nash equilibrium allocations of the strategic market game associated with the partially replicated exchange economies approximates a Walras equilibrium allocation of the original exchange economy.

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1 Introduction

Okuno at al. (1980) proposed an approach to modeling oligopoly in general equilibrium where "[...] either perfectly or imperfectly competitive behavior may emerge endogenously [...], depending on the characteristics of the

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agent and his place in the economy" (see p. 22). Okuno et al. (1980) proposed to use a mixed measure space of traders à la Shitovitz (see Shitovitz (1973)) - that is a measure space consisting of atoms, which represent the large traders, and an atomless part, which represents the small traders - in the framework of strategic market games (see Shapley and Shubik (1977), Dubey and Shubik (1978), Mas-Colell (1982), Sahi and Yao (1989), Amir et al. (1990), Peck et al. (1992), Dubey and Shapley (1994), among others). Busetto et al. (2011), in a generalization of Okuno et al. (1980), considered a model of noncooperative exchange proposed by Lloyd S. Shapley (and further analyzed by Sahi and Yao (1989)): they proved the existence of a Cournot-Nash equilibrium for this model. The approach adopted by Okuno et al. (1980) contrasts to an approach to noncooperative oligopoly in general equilibrium proposed by Gabszewicz and Vial (1972) (see also Roberts and Sonnenschein (1977), Roberts (1980), Mas-Colell (1982), Dierker and Grodal (1986), Codognato and Gabszewicz (1993), d'Aspremont et al. (1997), Gabszewicz and Michel (1997), Shitovitz (1997), among others). In this approach, it is explicitly assumed that some agents behave competitively while others behave noncompetitively.

In this paper, using the same framework as Busetto et al. (2011), without assuming that some agents behave competitively while others behave noncompetitively, we analyze the relationship between the set of the Cournot-Nash equilibrium allocations of the strategic market game and set of the Walras equilibrium allocations of the exchange economy with which it is associated. Since the mixed measure space we are considering in this paper may contain countably infinite atoms, the question arises whether an equivalence result could hold in this case. We provide an example of an exchange economy with countably infinite atoms and an atomless part which shows that any Cournot-Nash equilibrium allocation of the strategic market game is not a Walras equilibrium allocation of the exchange economy with which it is associated. Our example contrasts with the counterintuitive possibility that the core allocations are competitive despite the presence of atoms as in Shitovitz (1973).

The nonequivalence result provided by the example leads to consider the question of a possible asymptotic relationship between appropriately defined sequences of Cournot-Nash equilibrium allocations of the strategic market game and the Walras equilibrium allocations of the exchange economy with which it is associated. To this end, we partially replicate the exchange economy by increasing the number of atoms, while making them asymptotically negligible, without affecting the atomless part. The replication of atoms used in this paper ensures that the mixed measure space remains finite: without this feature required, the model of oligopoly in general equilibrium we study would not be well-defined. Then, we show that any sequence of Cournot-Nash equilibrium allocations of the strategic market game associated with the partially replicated exchange economies approximates a Walras equilibrium allocation of the original exchange economy.

Sahi and Yao (1989) showed the convergence of sequences of Cournot-Nash equilibrium allocations to a Walras equilibrium allocation starting from a finite exchange economy by replicating traders. Codognato and Ghosal (2000) showed the equivalence between Cournot-Nash and Walras equilibrium allocations in exchange economies with an atomless continuum of traders. While our convergence result synthesizes elements of both these papers, there are two new elements. First, the replication of atoms we use is different form the one used by Sahi and Yao (1989) for a finite number of traders: our replication ensures that the mixed measure space remains finite and, further, the convergence proof relies in an essential way on it. Second, we have to solve a new technical issue: in order to ensure that a sequence of Cournot-Nash equilibrium allocations approximates a Walras equilibrium allocation, we have to use a version of Fatou's Lemma in several dimensions proved by Artstein (1979).

Some limit results have already been proved in the approach adopted by Gabszewicz and Vial (1972), where competitive or noncompetitive traders' behavior is explicitly assumed (see Roberts (1980), Mas-Colell (1983), Novshek and Sonnenschein ((1983), (1987)), among others). Here, consistently with the Okuno et al. (1980)'s approach we have adopted, we prove a limit result à la Cournot (see Cournot (1838)) without making any further behavioral assumption and preserving the feature that the mixed measure space remains finite.

2 The mathematical model

We consider a pure exchange economy, E, with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_1 denote the set of atoms and $T_0 = T \setminus T_1$ the atomless part of T. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "all" traders, or "each" trader, or "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are l different commodities. A commodity bundle is a point in R^l_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^l_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. $\mathbf{w}(t) > 0$, *for each* $t \in T$.

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^l_+ \to R$, satisfying the following assumptions.

Assumption 2. $u_t : R^l_+ \to R$ *is continuous, strongly monotone in* R^l_{++}, *quasi-concave, for each* $t \in T_0$, *and concave, for each* $t \in T_1$.

Let $\mathcal{B}(R^l_+)$ denote the Borel σ-algebra of R^l_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u : T \times R^l_+ \to R$ *given by* $u(t, x) = u_t(x)$, *for each* $t \in T$ *and for each* $x \in R^l_+$, *is* $\mathcal{T} \bigotimes \mathcal{B}$ *measurable.*

We also need the following assumption (see Sahi and Yao (1989)).

Assumption 4. *There are at least two traders in* T_1 *for whom* $\mathbf{w}(t) \gg 0$; u_t *is continuously differentiable in* R^l_{++}; $\{x \in R^l_+ : u_t(x) = u_t(\mathbf{w}(t))\} \subset R^l_{++}$.

A price vector is a vector $p \in R^l_+$. A Walras equilibrium of E is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector p^* and an allocation $\mathbf{x}^*$, such that, for each $t \in T$, $u_t(\mathbf{x}^*(t)) \geq u_t(y)$, for all $y \in \{x \in R^l_+ : p^* x = p^* \mathbf{w}(t)\}$.

We introduce now the strategic market game Γ associated with E. Let $b \in R^{l^2}_+$ be a vector such that $b = (b_{11}, b_{12}, \ldots, b_{ll-1}, b_{ll})$. A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^{l^2}_+)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{b \in R^{l^2}_+ : \sum_{j=1}^{l} b_{ij} \leq \mathbf{w}^i(t),\ i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{b} : T \to R^{l^2}$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. For each $t \in T$, $\mathbf{b}_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ a strategy selection obtained by replacing $\mathbf{b}(t)$ in $\mathbf{b}$ with $b \in \mathbf{B}(t)$. With some abuse of notation, $\mathbf{b} \setminus b(t)$ will also denote the value of the strategy selection $\mathbf{b} \setminus b(t)$ at t.

Then, we introduce two further definitions (see Sahi and Yao (1989)).

Definition 1. *A nonnegative square matrix A is said to be irreducible if, for every pair i, j, with $i \neq j$, there is a positive integer $k = k(i,j)$ such that $a_{ij}^{(k)} > 0$, where $a_{ij}^{(k)}$ denotes the ij-th entry of the k-th power A^k of A.*

Definition 2. *Given a strategy selection $\mathbf{b}$, a price vector p is market clearing if*

$$p \in R^l_{++}, \sum_{i=1}^{l} p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^{l} \bar{\mathbf{b}}_{ji}),\ j = 1, \ldots, l. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector $p \in R^l_{++}$ satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Given a strategy selection $\mathbf{b}$, let $p(\mathbf{b})$ be a function such that $p(\mathbf{b})$ is the unique, up to a scalar multiple, price vector $p \in R^l_{++}$ satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, $p(\mathbf{b}) = 0$, otherwise.

Given a strategy selection $\mathbf{b}$ and a price vector $p \in R^l_{+}$, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{b}_{ji}(t) + \sum_{i=1}^{l} \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in R^l_{++}, \\
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,}
\end{aligned}$$

for each $t \in T$, $j = 1, \ldots, l$. Given a strategy selection $\mathbf{b}$, the traders' final holdings are given by $\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b}))$, for each $t \in T$.

It is straightforward to show that the assignment corresponding to the final holdings is an allocation. This reformulation of the Shapley's model allows us to define the following concept of Cournot-Nash equilibrium for exchange economies with an atomless part (see Codognato and Ghosal (2000)).

Definition 3. *A strategy selection $\hat{\mathbf{b}}$ such that $\bar{\hat{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium of Γ if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for all $b \in \mathbf{B}(t)$ and for each $t \in T$.

3 The existence of a δ-positive Cournot-Nash equilibrium of Γ

We introduce now the notion of a δ-positive strategy correspondence which was used by Sahi and Yao (1989) to prove their existence and limit theorems and which we shall use here for the same purposes. Let $\bar{T}_1 \subset T_1$ be a set consisting of two traders in T_1 for whom Assumption 4 holds. Moreover, let $\delta = \min_{t \in \bar{T}_1}\{\frac{1}{l}\min\{\mathbf{w}^1(t), \ldots, \mathbf{w}^l(t)\}\}$. We say that the correspondence $\mathbf{B}^\delta : T \to R^{l^2}$ is a δ-positive strategy correspondence if $\mathbf{B}^\delta(t) = \mathbf{B}(t) \cap \{b \in R^{l^2} : \sum_{i \notin J}\sum_{j \in J}(b_{ij} + b_{ji}) \geq \delta,$ for each $J \subseteq \{1, \ldots, l\}\}$, for each $t \in \bar{T}_1$; $\mathbf{B}^\delta(t) = \mathbf{B}(t)$, for the remaining traders $t \in T$. We say that a strategy selection $\mathbf{b}$ is δ-positive if $\mathbf{b}(t) \in \mathbf{B}^\delta(t)$, for each $t \in T$. Moreover, we say that a Cournot-Nash equilibrium $\hat{\mathbf{b}}$ of Γ is δ-positive if $\hat{\mathbf{b}}$ is a δ-positive strategy selection. The following theorem, which follows straightforwardly from the existence theorem in Busetto et al. (2011), shows the existence of a δ-positive Cournot-Nash equilibrium of Γ.

Theorem 1. *Under Assumptions 1, 2, 3, and 4, there exists a δ-positive Cournot-Nash equilibrium of* Γ, $\hat{\mathbf{b}}$

Proof. Busetto et al. (2011) showed that, under Assumptions 1, 2, 3, and 4, there exists a Cournot-Nash equilibrium of Γ, $\hat{\mathbf{b}}$, such that, for each $t \in T$, $\hat{\mathbf{b}}(t) \in \mathbf{B}^\delta(t)$. This implies that $\hat{\mathbf{b}}$ is a δ-positive Cournot-Nash equilibrium of Γ. ■

4 An example

Codognato and Ghosal (2000) analyzed the Sahi and Yao (1989)'s model in exchange economies with an atomless continuum of traders. In this framework, they showed an equivalence result à la Aumann (see Aumann (1964)) between the set of the Cournot-Nash equilibrium allocations of Γ and the set of the Walras equilibrium allocations of E. The mixed measure space we are considering here may contain countably infinite atoms. This raises the question whether an equivalence result à la Aumann could hold in this case. The following example considers an exchange economy E with countably infinite atoms and it shows that any Cournot-Nash equilibrium allocation of the strategic market game Γ associated with E is not a Walras equilibrium allocation of E.

Example. *Consider an exchange economy E where $l = 2$, $T_1 = T_1' \cup T_1''$, $T_1' = \{2, 3\}$, $T_1'' = \{4, 5, \ldots\}$, $T_0 = [0, 1]$, $\mathbf{w}(2) = \mathbf{w}(3) \gg 0$, $\mathbf{w}(t) = (0, 1)$, for each $t \in T_1'' \cup T_0$, $u_t(\cdot)$ satisfies Assumptions 2 and 3, for each $t \in T$, $u_2(\cdot)$ and $u_3(\cdot)$ satisfy Assumption 4, $u_2(x) = u_3(x)$, $u_t(x) > u_t(y)$, whenever $x \in R^l_{++}$ and $y \in (R^l_+ \setminus R^l_{++})$, for each $t \in T_1'' \cup T_0$, μ is the Lebesgue measure, when restricted to T_0, and $\mu(t) = (\frac{1}{2})^t$, for each $t \in T_1$. Then, if $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium of Γ, the pair $(\hat{p}, \hat{\mathbf{x}})$ such that $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$, for each $t \in T$, is not a Walras equilibrium of E.*

Proof. Suppose that $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium of Γ and that the pair $(\hat{p}, \hat{\mathbf{x}})$ such that $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$, for each $t \in T$, is a Walras equilibrium of E. Clearly, $\hat{\mathbf{b}}_{21}(t) > 0$, for each $t \in T_1'' \cup T_0$. Let $h = \int_{T_1'' \cup T_0} \hat{\mathbf{b}}_{21}(t)\, d\mu$. Since, for each $t \in T_1'$, at a Cournot-Nash equilibrium, the marginal price (see Okuno et al. (1980)) must be equal to the marginal rate of substitution which, in turn, at a Walras equilibrium, must be equal to the relative price of commodity 1 in terms of commodity 2, we must have

$$\frac{dx_2}{dx_1} = -\hat{p}^2 \frac{\hat{\mathbf{b}}_{12}(t)}{\hat{\mathbf{b}}_{21}(t) + h} = -\hat{p},$$

for each $t \in T_1'$. But then, we must also have

$$\frac{\hat{\mathbf{b}}_{21}(2) + h}{\hat{\mathbf{b}}_{12}(2)} = \frac{\hat{\mathbf{b}}_{21}(2) + \hat{\mathbf{b}}_{21}(3) + h}{\hat{\mathbf{b}}_{12}(2) + \hat{\mathbf{b}}_{12}(3)} = \frac{\hat{\mathbf{b}}_{21}(3) + h}{\hat{\mathbf{b}}_{12}(3)}. \tag{2}$$

The last equality in (2) holds if and only if $\hat{\mathbf{b}}_{21}(2) = k(\hat{\mathbf{b}}_{21}(3) + h)$ and $\hat{\mathbf{b}}_{12}(2) = k\hat{\mathbf{b}}_{12}(3)$, with $k > 0$. But then, the first and the last members of (2) cannot be equal because

$$\frac{k(\hat{\mathbf{b}}_{21}(3) + h) + h}{k\hat{\mathbf{b}}_{12}(3)} \neq \frac{\hat{\mathbf{b}}_{21}(3) + h}{\hat{\mathbf{b}}_{12}(3)}.$$

This implies that the pair $(\hat{p}, \hat{\mathbf{x}})$ such that $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$, for each $t \in T$, cannot be a Walras equilibrium of E. ■

The example shows that the condition that E contains a countably infinite number of atoms is not sufficient to guarantee that the set of the Cournot-Nash equilibrium allocations of Γ coincides with the set of the Walras equilibrium allocations of E. This leads us to consider the question whether partially replicating E à la Cournot (see Cournot (1838) - that is increasing the number of atoms, while making them asymptotically negligible, without affecting the atomless part - would generate sequences of Cournot-Nash equilibrium allocations which approximate, in some way, a Walras equilibrium allocation of E. We shall address this question in the following sections.

5 The replication à la Cournot of E

We consider the replication à la Cournot of E which, by analogy with the replication proposed by Cournot (1838) in a partial equilibrium framework, consists in replicating only the atoms of E, while making them asymptotically negligible. Let E^n be an exchange economy characterized as in Section 2 where each atom is replicated n times. For each $t \in T_1$, let tr denote r-th element of the n-fold replication of t. We assume that $\mathbf{w}(tr) = \mathbf{w}(ts) = \mathbf{w}(t)$, $u_{tr}(\cdot) = u_{ts}(\cdot) = u_t(\cdot)$, $r, s = 1, \ldots, n$, $\mu(tr) = \frac{\mu(t)}{n}$, $r = 1, \ldots, n$, for each $t \in T_1$. Clearly, E coincides with E^1.

The strategic market game Γ^n associated with E^n can then be characterized, *mutatis mutandis*, as in Section 2. Clearly, Γ coincides with Γ^1. A strategy selection $\mathbf{b}$ of Γ^n is atom-type-symmetric if $\mathbf{b}^n(tr) = \mathbf{b}^n(ts)$, $r, s = 1, \ldots, n$, for each $t \in T_1$. We can now provide the definition of an atom-type-symmetric Cournot-Nash equilibrium of Γ^n.

Definition 4. *A strategy selection $\hat{\mathbf{b}}$ such that $\bar{\hat{\mathbf{B}}}$ is irreducible is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n if $\hat{\mathbf{b}}$ is atom-type-symmetric and*

$$u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}}(tr), p(\hat{\mathbf{b}}^n))) \geq u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}} \setminus b(tr), p(\hat{\mathbf{b}} \setminus b(tr)))),$$

for all $b \in \mathbf{B}(tr)$, $r = 1, \ldots, n$, and for each $t \in T_1$;

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for all $b \in \mathbf{B}(t)$ and for each $t \in T_0$.

6 The existence of a δ-positive atom-type-symmetric Cournot-Nash equilibrium of Γ^n

Let δ be determined as in Section 3 and define the δ-positive strategy correspondence $\mathbf{B}^\delta$, *mutatis mutandis*, as in Section 3. Notice that $\mathbf{B}^\delta(tr) = \mathbf{B}^\delta(ts)$, $r, s = 1, \ldots, n$, for each $t \in T_1$. We say that a strategy selection $\mathbf{b}$ is

δ-positive if $\mathbf{b}(tr) \in \mathbf{B}^{\delta}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, $\mathbf{b}(t) \in \mathbf{B}^{\delta}(t)$, for each $t \in T_0$. Moreover, we say that an atom-type-symmetric Cournot-Nash equilibrium $\hat{\mathbf{b}}$ of Γ^n is δ-positive if $\hat{\mathbf{b}}$ is a δ-positive strategy selection. The following theorem shows the existence of an atom-type-symmetric δ-positive Cournot-Nash equilibrium of Γ^n.

Theorem 2. *Under Assumptions 1, 2, 3, and 4, there exists a δ-positive atom-type-symmetric Cournot-Nash equilibrium of Γ^n, $\hat{\mathbf{b}}$.*

Proof. Following Sahi and Yao (1989) and Busetto et al. (2011), we define the game $\Gamma^n(\epsilon)$. Given $\epsilon > 0$ and a strategy selection $\mathbf{b}$, we define the aggregate bid matrix $\bar{\mathbf{B}}^{\epsilon} = (\bar{\mathbf{b}}_{ij} + \epsilon)$. Clearly, the matrix $\bar{\mathbf{B}}^{\epsilon}$ is irreducible. The interpretation is that an outside agency places fixed bids of ϵ for each pair of commodities (i, j). Given $\epsilon > 0$, we denote by $p^{\epsilon}(\mathbf{b})$ the function which associates, with each strategy selection $\mathbf{b}$, the unique, up to a scalar multiple, price vector which satisfies

$$\sum_{i=1}^{l} p^i(\bar{\mathbf{b}}_{ij} + \epsilon) = p^j(\sum_{i=1}^{l}(\bar{\mathbf{b}}_{ji} + \epsilon),\ j = 1, \ldots, l. \quad (3)$$

Definition 5. *Given $\epsilon > 0$, a strategy selection $\hat{\mathbf{b}}^{\epsilon}$ is an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$ if $\hat{\mathbf{b}}^{\epsilon}$ is atom-type-symmetric and*

$$u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}}^{\epsilon}(tr), p^{\epsilon}(\hat{\mathbf{b}}^{\epsilon}))) \geq u_{tr}(tr, \hat{\mathbf{b}}^{\epsilon} \setminus b(tr), p^{\epsilon}(\hat{\mathbf{b}}^{\epsilon} \setminus b(tr)))),$$

for all $b \in \mathbf{B}(tr)$, $r = 1, \ldots, n$, and for each $t \in T_1$;

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}^{\epsilon}(t), p^{\epsilon}(\hat{\mathbf{b}}^{\epsilon}))) \geq u_t(t, \hat{\mathbf{b}}^{\epsilon} \setminus b(t), p^{\epsilon}(\hat{\mathbf{b}}^{\epsilon} \setminus b(t)))),$$

for all $b \in \mathbf{B}(t)$ and for each $t \in T_0$.

We neglect, as usual, the distinction between integrable functions and equivalence classes of such functions. The locally convex Hausdorff space we shall be working in is $L_1(\mu, R^{l^2})$, endowed with its weak topology. Denote by $L_1(\mu, R^{l^2})$ the set of integrable functions taking values in R^{l^2}, by $L_1(\mu, \mathbf{B}(\cdot))$ the set of strategy selections, and by $L_1(\mu, \mathbf{B}^*(\cdot))$ the set of atom-type-symmetric strategy selections.

The first lemma has been proved by Busetto et al. (2011).

Lemma 1. *The set $L_1(\mu, \mathbf{B}(\cdot))$ is nonempty, convex and weakly compact.*

The next lemma provides us with the properties of $L_1(\mu, \mathbf{B}^*(\cdot))$ required for the application of the Kakutani-Fan-Glicksberg Theorem (see Theorem 17.55 in Aliprantis and Border (2006), p. 583).

Lemma 2. *The set* $L_1(\mu, \mathbf{B}^*(\cdot))$ *is nonempty, convex and weakly compact.*

Proof. $L_1(\mu, \mathbf{B}^*(\cdot))$ is nonempty, convex and it has a weakly compact closure by the same argument used by Busetto et al. (2011). Now, let $\{\mathbf{b}^m\}$ be a convergent sequence of $L_1(\mu, \mathbf{B}^*(\cdot))$. Since $L_1(\mu, R^{l^2})$ is complete, $\{\mathbf{b}^m\}$ converges in the mean to an integrable function $\mathbf{b}$. But then, there exists a subsequence $\{\mathbf{b}^{k_m}\}$ of $\{\mathbf{b}^m\}$ such that $\mathbf{b}^{k_m}(tr)$ converges to $\mathbf{b}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, and $\mathbf{b}^{k_m}(t)$ converges to $\mathbf{b}(t)$, for each $t \in T_0$ (see Theorem 25.5 in Aliprantis and Burkinshaw (1998), p. 203). The compactness of $\mathbf{B}(t)$, for each $t \in T$, and the fact that $\mathbf{b}^{k_m}(tr) = \mathbf{b}^{k_m}(ts)$, $r, s = 1, \ldots, n$, for each $t \in T_1$, implies that $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. Hence $L_1(\mu, \mathbf{B}^*(\cdot))$ is norm closed and, since it is also convex, it is weakly closed (see Corollary 4 in Diestel (1984), p. 12). ■

Now, given $\epsilon > 0$, let $\alpha^\epsilon_{tr} : L_1(\mu, \mathbf{B}^*(\cdot)) \to \mathbf{B}(tr)$ be a correspondence such that $\alpha^\epsilon_{tr}(\mathbf{b}) = \operatorname{argmax}\{u_{tr}(\mathbf{x}(t, \mathbf{b} \setminus b(tr), p^\epsilon(\mathbf{b} \setminus b(tr)))) : b \in \mathbf{B}(tr)\}$, $r = 1, \ldots, n$, for each $t \in T_1$, and let $\alpha^\epsilon_t : L_1(\mu, \mathbf{B}(\cdot)) \to \mathbf{B}(t)$ be a correspondence such that $\alpha^\epsilon_t(\mathbf{b}) = \operatorname{argmax}\{u_t(\mathbf{x}(t, \mathbf{b} \setminus b(t), p^\epsilon(\mathbf{b} \setminus b(t)))) : b \in \mathbf{B}(t)\}$, for each $t \in T_0$. Let $\alpha^\epsilon : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}(\cdot))$ be a correspondence such that $\alpha^\epsilon(\mathbf{b}) = \{\mathbf{b} \in L_1(\mu, \mathbf{B}(\cdot)) : \mathbf{b}(tr) \in \alpha^\epsilon_{tr}(\mathbf{b}),\ r = 1, \ldots, n,$ for each $t \in T_1$, and $\mathbf{b}(t) \in \alpha^\epsilon_t(\mathbf{b})$, for each $t \in T_0\}$.

The following lemma can be proved by the same argument used to show Lemma 2 in Busetto et al. (2011).

Lemma 3. *Given* $\epsilon > 0$, *the correspondence* $\alpha^\epsilon : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}(\cdot))$ *is such that the set* $\alpha^\epsilon(\mathbf{b})$ *is nonempty and convex, for all* $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, *and it has a weakly closed graph.*

We say that an atom-type-symmetric ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^\epsilon$ of $\Gamma^n(\epsilon)$ is δ-positive if $\hat{\mathbf{b}}^\epsilon$ is a δ-positive strategy selection.

Now, given $\epsilon > 0$, let $\alpha^{\epsilon\delta}_{tr} : L_1(\mu, \mathbf{B}^*(\cdot)) \to \mathbf{B}(tr)$ be a correspondence such that $\alpha^{\epsilon\delta}_{tr}(\mathbf{b}) = \alpha^\epsilon_{tr}(\mathbf{b}) \cap \mathbf{B}^\delta(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, and let $\alpha^\epsilon_t : L_1(\mu, \mathbf{B}(\cdot)) \to \mathbf{B}(t)$ be a correspondence such that $\alpha^{\epsilon\delta}_t(\mathbf{b}) = \alpha^\epsilon_t(\mathbf{b}) \cap \mathbf{B}^\delta(t)$, for each $t \in T_0$. Let $\alpha^{\epsilon\delta} : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}(\cdot))$ be a correspondence such that $\alpha^{\epsilon\delta}(\mathbf{b}) = \{\mathbf{b} \in L_1(\mu, \mathbf{B}(\cdot)) : \mathbf{b}(tr) \in \alpha^{\epsilon\delta}_{tr}(\mathbf{b}),\ r = 1, \ldots, n,$ for each $t \in T_1$, and $\mathbf{b}(t) \in \alpha^{\epsilon\delta}_t(\mathbf{b})$, for each $t \in T_0\}$.

Moreover, given $\epsilon > 0$, let $\alpha^{\epsilon\delta *} : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}^*(\cdot))$ be a correspondence such that $\alpha^{\epsilon\delta *}(\mathbf{b}) = \alpha^{\epsilon\delta}(\mathbf{b}) \cap L_1(\mu, \mathbf{B}^*(\cdot))$. Now, we are ready to prove the existence of a δ-positive atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$.

Lemma 4. *Given $\epsilon > 0$, there exists a δ-positive atom-type-symmetric ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^\epsilon$ of $\Gamma^n(\epsilon)$.*

Proof. Let $\epsilon > 0$ be given. By Lemma 6 in Sahi and Yao (1989), we know that, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, $\alpha_{tr}^{\epsilon\delta}(\mathbf{b})$ is nonempty, $r = 1, \ldots, n$, for each $t \in \bar{T}_1$. Moreover, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$ and for each $t \in T_1$, there exists $\bar{b} \in \mathbf{B}(t)$ such that $\bar{b} \in \alpha_{tr}^{\epsilon\delta}(\mathbf{b})$, $r = 1, \ldots, n$ as $\mathbf{b}$ is an atom-type-symmetric strategy profile. But then, by the same argument of Lemma 2 in Busetto et al. (2011), $\alpha^{\epsilon\delta*}(\mathbf{b})$ is nonempty. The convexity of $\alpha^{\epsilon\delta}(\mathbf{b})$, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, is a straightforward consequence of the convexity of $\alpha_{tr}^{\epsilon}(\mathbf{b})$ and $\mathbf{B}^\delta(t)$, $r = 1, \ldots, n$, for each $t \in T_1$, and of $\alpha_t^\epsilon(\mathbf{b})$ and $\mathbf{B}^\delta(t)$, for each $t \in T_0$. But then, $\alpha^{\epsilon\delta*}$ is convex valued as $L_1(\mu, \mathbf{B}^*(\cdot))$ is convex. $\alpha_{tr}^{\epsilon\delta}$ is upper hemicontinuous and compact valued, $r = 1, \ldots, n$, for each $t \in T_1$, as it is the intersection of the correspondence α_{tr}^ϵ, which is upper hemicontinuous and compact valued by Lemma 2 in Busetto et al. (2011), and the continuous and compact valued correspondence which assigns to each strategy selection $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$ the strategy set $\mathbf{B}^\delta(tr)$ (see Theorem 17.25 in Aliprantis and Border (2006), p. 567). Moreover, $\alpha_t^{\epsilon\delta}$ is upper hemicontinuous and compact valued, for each $t \in T_0$, using the same argument. Therefore, $\alpha^{\epsilon\delta}$ has a weakly closed graph, by the same argument used in the proof of Lemma 3. Finally, $\alpha^{\epsilon\delta*}$ has a weakly closed graph as it is the intersection of the correspondence $\alpha^{\epsilon\delta}$ and the continuous correspondence which assigns to each strategy selection $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$ the weakly closed set $L_1(\mu, \mathbf{B}^*(\cdot))$ which, by the Closed Graph Theorem (see Theorem 17.11 in Aliprantis and Border (2006), p. 561), has a weakly closed graph (see Theorem 17.25 in Aliprantis and Border (2006), p. 567). But then, by the Kakutani-Fan-Glicksberg Theorem (see Theorem 17.55 in Aliprantis and Border (2006), p. 583), there exists a fixed point $\hat{\mathbf{b}}^\epsilon$ of the correspondence $\alpha^{\epsilon\delta*}$ and hence a δ-positive atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$. ■

Let $\epsilon_m = \frac{1}{m}$, $m = 1, 2, \ldots$. By Lemma 4, for each $m = 1, 2, \ldots$, there is a δ-positive atom-type-symmetric ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon_m}$. The fact that the sequence $\{\overline{\hat{\mathbf{B}}^{\epsilon_m}}\}$ belongs to the compact set $\{b_{ij} \in R^{l^2} : b_{ij} \leq n \int_{T_1} \mathbf{w}^i(t)\, d\mu + \int_{T_0} \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \ldots, l,\ \sum_{i \notin J} \sum_{j \in J}(b_{ij} + b_{ji}) \geq n \int_{\bar{T}_1} \delta\, d\mu,$ for each $J \subseteq \{1, \ldots, l\}\}$, the sequence $\{\hat{\mathbf{b}}^{\epsilon_m}(tr)\}$ belongs to the compact set $\mathbf{B}^\delta(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, and the sequence $\{\hat{p}^{\epsilon_m}\}$, where $\hat{p}^{\epsilon_m} = p(\hat{\mathbf{b}}^{\epsilon_m})$, for each $m = 1, 2, \ldots$, belongs, by Lemma 9 in Sahi and Yao (1989), to a compact set P, implies that there is a subsequence $\{\overline{\hat{\mathbf{B}}^{\epsilon_{k_m}}}\}$ of the sequence $\{\overline{\hat{\mathbf{B}}^{\epsilon_m}}\}$ which converges to an element of the set $\{b_{ij} \in R^{l^2} :$

$b_{ij} \le n \int_{T_1} \mathbf{w}^i(t)\, d\mu + \int_{T_0} \mathbf{w}^i(t)\, d\mu$, $i, j = 1, \dots, l$, $\sum_{i \notin J} \sum_{j \in J} (b_{ij} + b_{ji}) \ge n \int_{\bar{T}_1} \delta\, d\mu$, for each $J \subseteq \{1, \dots, l\}\}$, a subsequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$ of the sequence $\{\hat{\mathbf{b}}^{\epsilon_m}(tr)\}$ which converges to an element of the set $\mathbf{B}^{\delta}(tr)$, $r = 1, \dots, n$, for each $t \in T_1$, and a subsequence $\{\hat{p}^{\epsilon_{k_m}}\}$ of the sequence $\{\hat{p}^{\epsilon_m}\}$ which converges to an element of the set P. Since the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}\}$ satisfies the assumptions of Theorem A in Artstein (1979), there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(tr)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$, $r = 1, \dots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(t)\}$, for each $t \in T_0$, and such that the sequence $\{\overline{\hat{\mathbf{B}}^{\epsilon_{k_m}}}\}$ converges to $\bar{\hat{\mathbf{B}}}$. Then, $\hat{\mathbf{b}}(tr) = \hat{\mathbf{b}}(ts)$ as $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\} = \{\hat{\mathbf{b}}^{\epsilon_{k_m}}(ts)\}$, $r, s = 1, \dots, n$, for each $t \in T_1$, and $\hat{\mathbf{b}}(tr)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$, $r = 1, \dots, n$, for each $t \in T_1$. Hence, it can be proved, by the same argument used by Busetto et al. (2011) to show their existence theorem, that $\hat{\mathbf{b}}$ is a δ-positive atom-type-symmetric Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$. ∎

7 The limit theorem

The following theorem shows that the sequences of Cournot-Nash equilibrium allocations generated by the replication à la Cournot of E approximate a Walras equilibrium allocation of E. In particular, the next theorem shows that given a sequence of atom-type-symmetric Cournot-Nash equilibrium allocations of Γ^n, there exists a Walras equilibrium allocation of E such that, for each trader, his final holding at this Walras equilibrium is a limit point of the sequence of his final holdings at the sequence of atom-type-symmetric Cournot-Nash equilibrium allocations of Γ^n.

We now state and prove the limit theorem.

Theorem 3. *Under Assumptions 1, 2, 3, and 4, let $\{\hat{\mathbf{b}}^n\}$ be a sequence of strategy selections of Γ and let $\{\hat{p}^n\}$ be a sequence of prices such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1, \dots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, and $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, where $\hat{\mathbf{b}}^{\Gamma^n}$ is a δ-positive atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for $n = 1, 2, \dots$. Then, (i) there exists a subsequence $\{\hat{\mathbf{b}}^{k_n}\}$ of the sequence $\{\hat{\mathbf{b}}^n\}$, a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$, a strategy selection $\hat{\mathbf{b}}$ of Γ, and a price vector $\hat{p}$, with $p \gg 0$, such that $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, the sequence $\{\overline{\hat{\mathbf{B}}^{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$; (ii) $\hat{\mathbf{x}}(t)$ is the limit of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_1$, and $\hat{\mathbf{x}}(t)$ is a limit point of the sequence*

$\{\hat{\mathbf{x}}^{k_n}(t)\}$, *for each* $t \in T_0$, *where* $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$ *for each* $t \in T$, $\hat{\mathbf{x}}^{k_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{k_n}(t), \hat{p}^{k_n})$, *for each* $t \in T$, *and for* $n = 1, 2, \ldots$; *(iii) The pair* $(\hat{p}, \hat{\mathbf{x}})$ *is a Walras equilibrium of* E.

Proof (i) Let $\{\hat{\mathbf{b}}^n\}$ be a sequence of strategy selections of Γ and let $\{\hat{p}^n\}$ be a sequence of prices such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, and $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, where $\hat{\mathbf{b}}^{\Gamma^n}$ is a δ-positive atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for $n = 1, 2, \ldots$. The fact that the sequence $\{\overline{\hat{\mathbf{B}}^n}\}$ belongs to the compact set $\{b_{ij} \in R^{l^2} : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \ldots, l,\ \sum_{i \notin J} \sum_{j \in J}(b_{ij} + b_{ji}) \geq \int_{\bar{T}_1} \delta\, d\mu,$ for each $J \subseteq \{1, \ldots, l\}\}$, the sequence $\{\hat{\mathbf{b}}^n(t)\}$ belongs to the compact set $\mathbf{B}^\delta(t)$, for each $t \in T_1$, and the sequence $\{\hat{p}^n\}$, belongs, by Lemma 9 in Sahi and Yao, to a compact set P, implies that there is a subsequence $\{\overline{\hat{\mathbf{B}}^{k_n}}\}$ of the sequence $\{\overline{\hat{\mathbf{B}}^n}\}$ which converges to an element of the set $\{b_{ij} \in R^{l^2} : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \ldots, l,\ \sum_{i \notin J} \sum_{j \in J}(b_{ij} + b_{ji}) \geq \int_{\bar{T}_1} \delta\, d\mu,$ for each $J \subseteq \{1, \ldots, l\}\}$, a subsequence $\{\hat{\mathbf{b}}^{k_m}(t)\}$ of the sequence $\{\hat{\mathbf{b}}^n(t)\}$ which converges to an element of the set $\mathbf{B}^\delta(t)$, for each $t \in T_1$, and a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$ which converges to an element $\hat{p}$ of the set P. Moreover, by Lemma 9 in Sahi and Yao, $\hat{p} \gg 0$. Since the sequence $\{\hat{\mathbf{b}}^{k_n}\}$ satisfies the assumptions of Theorem A in Artstein (1979), there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, and such that the sequence $\{\overline{\hat{\mathbf{B}}^{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$. (ii) Let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$ for each $t \in T$, $\hat{\mathbf{x}}^{k_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{k_n}(t), \hat{p}^{k_n})$, for each $t \in T$, and for $n = 1, 2, \ldots$. Then, $\hat{\mathbf{x}}(t)$ is the limit of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_1$, as $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$, $\hat{\mathbf{x}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_0$, as $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$. (iii) $\overline{\hat{\mathbf{B}}^{\Gamma^n}} = \overline{\hat{\mathbf{B}}^n}$ as $\overline{\hat{\mathbf{b}}_{ij}^{\Gamma^n}} = \sum_{t \in T_1} \sum_{r=1}^n \hat{\mathbf{b}}_{ij}^{\Gamma^n}(tr)\mu(tr) + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^{\Gamma^n}(t) = \sum_{t \in T_1} n \hat{\mathbf{b}}_{ij}^n(t) \frac{\mu(t)}{n} + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^n(t) = \sum_{t \in T_1} \hat{\mathbf{b}}_{ij}^n(t)\mu(t) + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^n(t) = \overline{\hat{\mathbf{b}}_{ij}^n}$, $i, j = 1, \ldots, l$, for $n = 1, 2, \ldots$. Then, $\hat{p}^n = p(\hat{\mathbf{b}}^n)$ as $\hat{p}^n$ and $\hat{\mathbf{b}}^n$ satisfy (1), for $n = 1, 2, \ldots$. But then, by continuity, $\hat{p}$ and $\hat{\mathbf{b}}$ must satisfy (1). Therefore, Lemma 1 in Sahi and Yao implies that $\bar{\hat{\mathbf{B}}}$ is completely reducible as $\hat{p} \gg 0$. Moreover, $\hat{\mathbf{b}}(t) \in \mathbf{B}^\delta(t)$ since $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for all $t \in T$. Then, $\hat{\mathbf{b}}$ is δ-positive. But then, by Remark 3 in Sahi and Yao, $\bar{\hat{\mathbf{B}}}$ must be irreducible. Consider the pair $(\hat{p}, \hat{\mathbf{x}})$. It is straightforward to show that the assignment $\hat{\mathbf{x}}$ is an allocation as $\hat{p}$ and $\hat{\mathbf{b}}$ satisfy (1) and that $\hat{\mathbf{x}}(t) \in \{x \in R^l_+ : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for all $t \in T$. Suppose that $(\hat{p}, \hat{\mathbf{x}})$ is not a Walras equilibrium of E. Then, there exists a trader $\tau \in T$ and a commodity bundle $\tilde{x} \in \{x \in R^l_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$ such that $u_\tau(\tilde{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$. By Lemma 5 in Codognato and Ghosal (2000), there exist $\tilde{\lambda}^j \geq 0$, $\sum_{j=1}^l \tilde{\lambda}^j = 1$, such that

$$\tilde{x}^j = \tilde{\lambda}^j \frac{\sum_{i=1}^l \hat{p}^i \mathbf{w}^i(\tau)}{\hat{p}^j}, \; j = 1, \ldots, l.$$

Let $\tilde{b}_{ij} = \mathbf{w}^i(\tau)\tilde{\lambda}^j$, $i, j = 1, \ldots, l$. Then, it is straightforward to verify that

$$\tilde{x}^j = \mathbf{w}^j(\tau) - \sum_{i=1}^l \tilde{b}_{ji} + \sum_{i=1}^l \tilde{b}_{ij} \frac{\hat{p}^i}{\hat{p}^j},$$

for each $j = 1, \ldots, l$. Consider the following cases.

Case 1. $\tau \in T_1$. Let $\{h_n\}$ denote a sequence such that $h_n = k_n$, if $k_1 > 1$, $h_n = k_{n+1}$, otherwise, for $n = 1, 2, \ldots$. Let ρ denote the k_1-th element of the n-fold replication of E and let $\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)$ denote the aggregate matrix corresponding to the strategy selection $\overline{\hat{\mathbf{b}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)$, where $\tilde{b}(\tau\rho) = \tilde{b}$, for $n = 1, 2, \ldots$. Let $\Delta\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}}$, $\Delta\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)$, and $\Delta\overline{\hat{\mathbf{B}}}^{h_n}$ denote the matrices of row sums of, respectively, $\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}}$, $\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)$, and $\overline{\hat{\mathbf{B}}}^{h_n}$, for $n = 1, 2, \ldots$. Moreover, let $q^{\Gamma^{h_n}}$, $q^{\Gamma^{h_n}}_{\tau\rho}$, and q^{h_n} denote the vectors of the cofactors of the first column of, respectively, $\Delta\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} - \overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}}$, $\Delta\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho) - \overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)$, and $\Delta\overline{\hat{\mathbf{B}}}^{h_n} - \overline{\hat{\mathbf{B}}}^{h_n}$, for $n = 1, 2, \ldots$. Clearly, $q^{\Gamma^{h_n}} = q^{h_n}$ as $\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} = \overline{\hat{\mathbf{B}}}^{h_n}$, for $n = 1, 2, \ldots$. Let $\Delta\bar{\hat{\mathbf{B}}}$ be the matrix of row sums of $\bar{\hat{\mathbf{B}}}$ and q be the cofactors of the first column of $\Delta\bar{\hat{\mathbf{B}}} - \bar{\hat{\mathbf{B}}}$. The sequences $\{q^{\Gamma^{h_n}}\}$ and $\{q^{h_n}\}$ converge to q as the sequence $\overline{\hat{\mathbf{B}}}^{h_n}$ converges to $\bar{\hat{\mathbf{B}}}$ and $q^{\Gamma^{h_n}} = q^{h_n}$, for $n = 1, 2, \ldots$. Let $\bar{w} = \max\{\mathbf{w}^1(\tau), \ldots, \mathbf{w}^l(\tau)\}$. Consider the matrix $\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} - \overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)$, for $n = 1, 2, \ldots$. Then, $\overline{\hat{\mathbf{b}}}^{\Gamma^{h_n}}_{ij} - \overline{\hat{\mathbf{b}}}^{\Gamma^{h_n}}_{ij} \setminus \tilde{b}_{ij}(\tau\rho) = (\frac{1}{n}\hat{\mathbf{b}}^{\Gamma^{h_n}}_{ij}(\tau\rho) - \frac{1}{n}\tilde{b}_{ij}(\tau\rho))$, $i, j = 1, \ldots, l$, for $n = 1, 2, \ldots$. But then, the sequence of Euclidean differences $\{\|\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} - \overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)\|\}$ converges to 0 as $|\frac{1}{n}\hat{\mathbf{b}}^{\Gamma^{h_n}}_{ij}(\tau\rho) - \frac{1}{n}\tilde{b}_{ij}(\tau\rho)| = \frac{1}{n}|\hat{\mathbf{b}}^{\Gamma^{h_n}}_{ij}(\tau\rho) - \tilde{b}_{ij}(\tau\rho)| \leq \frac{1}{n}\bar{w}$, $i, j = 1, \ldots, l$, $n = 1, 2, \ldots$. The sequence $\{\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)\}$ converges to $\bar{\hat{\mathbf{B}}}$ as $\|\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)\} - \bar{\hat{\mathbf{B}}}\| \leq \|\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} - \overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)\| + \|\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} - \bar{\hat{\mathbf{B}}}\| = \|\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} - \overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)\| + \|\overline{\hat{\mathbf{B}}}^{h_n} - \bar{\hat{\mathbf{B}}}\|$, for $n = 1, 2, \ldots$, the sequence $\{\|\overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} - \overline{\hat{\mathbf{B}}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)\|\}$ converges to 0, and the sequence $\{\|\overline{\hat{\mathbf{B}}}^{h_n} - \bar{\hat{\mathbf{B}}}\|\}$ converges to 0. Then, the sequence $\{q^{\Gamma^{h_n}}_{\tau\rho}\}$ converges to

q. $u_{\tau\rho}(\mathbf{x}(\tau\rho, \hat{\mathbf{b}}^{\Gamma^{h_n}}(\tau\rho), p(\hat{\mathbf{b}}^{\Gamma^{h_n}}))) \geq u_{\tau\rho}(\mathbf{x}(\tau\rho, \hat{\mathbf{b}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho), p(\hat{\mathbf{b}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho))))$ as $\hat{\mathbf{b}}^{\Gamma^{h_n}}$ is a δ-positive atom-type-symmetric Cournot-Nash equilibrium of Γ^{h_n}, for $n = 1, 2, \ldots$. Let $\hat{\mathbf{b}}^{h_n} \setminus \tilde{b}(\tau)$ be a strategy selection obtained by replacing $\hat{\mathbf{b}}^{h_n}(\tau)$ in $\hat{\mathbf{b}}^{h_n}$ with $\tilde{b}$, for $n = 1, 2, \ldots$. Then, $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{h_n}(\tau), q^{\Gamma^{h_n}})) \geq u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{h_n} \setminus \tilde{b}(\tau), q^{\Gamma^{h_n}}_{\tau\rho}))$ as $\hat{\mathbf{b}}^{h_n}(\tau) = \hat{\mathbf{b}}^{\Gamma^{h_n}}(\tau\rho)$, $p(\hat{\mathbf{b}}^{\Gamma^{h_n}}) = \alpha_n q^{\Gamma^{h_n}}$, with $\alpha_n > 0$, by Lemma 2 in Sahi and Yao, $\hat{\mathbf{b}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho) = \hat{\mathbf{b}}^{h_n} \setminus \tilde{b}(\tau)$, and $p(\hat{\mathbf{b}}^{\Gamma^{h_n}} \setminus \tilde{b}(\tau\rho)) = \beta_n q^{\Gamma^{h_n}}_{\tau\rho}$, with $\beta_n > 0$, by Lemma 2 in Sahi and Yao, for $n = 1, 2, \ldots$. But then, $u_\tau(\hat{\mathbf{x}}(\tau)) \geq u_\tau(\tilde{x})$, by Assumption 2, as the sequence $\{\hat{\mathbf{b}}^{h_n}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$, the sequence $\{q^{\Gamma^{h_n}}\}$ converges to q, the sequence $\{q^{\Gamma^{h_n}}_{\tau\rho}\}$ converges to q, and $\hat{p} = \theta q$, with $\theta > 0$, by Lemma 2 in Sahi and Yao, a contradiction.
Case 2. $\tau \in T_0$. Let $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ be a subsequence of the sequence $\{\hat{\mathbf{b}}^{k_n}(\tau)\}$ which converges to $\hat{\mathbf{b}}(\tau)$. Moreover, let $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau)$ be a strategy selection obtained by replacing $\hat{\mathbf{b}}^{h_{k_n}}(\tau)$ in $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}$ with $\tilde{b}$, for $n = 1, 2, \ldots$. $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}(\tau), p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}))) \geq u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau), p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau))))$ as $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}$ is a δ-positive atom-type-symmetric Cournot-Nash equilibrium of $\Gamma^{h_{k_n}}$, for $n = 1, 2, \ldots$. Let $\hat{\mathbf{b}}^{h_{k_n}} \setminus \tilde{b}(\tau)$ be a strategy selection obtained by replacing $\hat{\mathbf{b}}^{h_{k_n}}(\tau)$ in $\hat{\mathbf{b}}^{h_{k_n}}$ with $\tilde{b}$, for $n = 1, 2, \ldots$.. Then, $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{h_{k_n}}(\tau), \hat{p}^{h_{k_n}})) \geq u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{h_{k_n}} \setminus \tilde{b}(\tau), \hat{p}^{h_{k_n}}))$ as $\hat{\mathbf{b}}^{h_{k_n}}(\tau) = \hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}(\tau)$, $\hat{p}^{h_{k_n}} = p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}})$, $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau) = \hat{\mathbf{b}}^{h_{k_n}} \setminus \tilde{b}(\tau)$, and $\hat{p}^{h_{k_n}} = p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau))$. But then, $u_\tau(\hat{\mathbf{x}}(\tau)) \geq u_\tau(\tilde{x})$, by Assumption 2, as the sequence $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$ and the sequence $\{p^{h_{k_n}}\}$ converges to $\hat{p}$, a contradiction.

Hence, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium of E. ■

References

[1] Aliprantis C.D., Border K.C. (2006), *Infinite dimensional analysis*, Springer, New York.

[2] Aliprantis C.D., Burkinshaw O. (1998), *Principles of real analysis*, Academic Press, Boston.

[3] Amir R., Sahi S., Shubik M., Yao S. (1990), "A strategic market game with complete markets," *Journal of Economic Theory* **51**, 126-143.

[4] Artstein Z. (1979), "A note on Fatou's lemma in several dimensions," *Journal of Mathematical Economics* **6**, 277-282.

[5] Aumann R.J. (1964), "Markets with a continuum of traders," *Econometrica* **32**, 39-50.

[6] Busetto F., Codognato G., Ghosal S. (2011), "Noncooperative oligopoly in markets with a continuum of traders," *Games and Economic Behavior*, **72**, 38-45.

[7] Codognato G., Gabszewicz J.J. (1993), "Cournot-Walras equilibria in markets with a continuum of traders," *Economic Theory* **3**, 453-464.

[8] Codognato G., Ghosal S. (2000), "Cournot-Nash equilibria in limit exchange economies with complete markets and consistent prices," *Journal of Mathematical Economics* **34**, 39-53.

[9] Cournot A. (1838), *Recherches sur les principes mathématiques de la théorie des richesses*, Hachette, Paris.

[10] d'Aspremont C., Dos Santos Ferreira R., Gérard-Varet L.-A. (1997), "General equilibrium concepts under imperfect competition: a Cournotian approach," *Journal of Economic Theory* **73**, 199-230.

[11] Dierker H., Grodal B. (1986), "Nonexistence of Cournot-Walras equilibrium in a general equilibrium model with two oligopolists," in Hildenbrand W., Mas-Colell A. (eds), *Contributions to mathematical economics in honor of Gérard Debreu*, North-Holland, Amsterdam.

[12] Diestel J. (1984), *Sequences and series in Banach spaces*, Springer, New York.

[13] Dubey P., Shapley L.S. (1994), "Noncooperative general exchange with a continuum of traders: two models, " *Journal of Mathematical Economics* **23**, 253-293.

[14] Dubey P., Shubik M. (1978), "The noncooperative equilibria of a closed trading economy with market supply and bidding strategies," *Journal of Economic Theory* **17**, 1-20.

[15] Gabszewicz J.J., Michel P. (1997), "Oligopoly equilibrium in exchange economies," in Eaton B.C., Harris R. G. (eds), *Trade, technology and economics. Essays in honour of Richard G. Lipsey*, Edward Elgar, Cheltenham.

[16] Gabszewicz J.J., Vial J.-P. (1972), "Oligopoly 'à la Cournot-Walras' in a general equilibrium analysis," *Journal of Economic Theory* **4**, 381-400.

[17] Okuno M., Postlewaite A., Roberts J. (1980), "Oligopoly and competition in large markets," *American Economic Review* **70**, 22-31.

[18] Mas-Colell A. (1982), "The Cournotian foundations of Walrasian equilibrium theory," in Hildenbrand W. (ed), *Advances in economic theory*, Cambridge University Press, Cambridge.

[19] Mas-Colell A. (1983), "Walrasian equilibria as limits of noncooperative equilbria. Part I: mixed strategies," *Journal of Economic Theory* **30**, 153-170.

[20] Novshek W., Sonnenschein H. (1983), "Walrasian equilibria as limits of noncooperative equilibria. Part I: pure strategies," *Journal of Economic Theory* **30**, 171-187.

[21] Novshek W., Sonnenschein H. (1987), "General equilibrium with free entry: a synthetic approach to the theory of perfect competition," *Journal of Economic Literature* **25**, 1281-1306.

[22] Peck J., Shell K., Spear S.E. (1992), "The market game: existence and structure of equilibrium," *Journal of Mathematical Economics* **21**, 271-299.

[23] Roberts D.J., Sonnenschein H. (1977), "On the foundations of the theory of monopolistic competition," *Econometrica* **45**, 101-114.

[24] Roberts K. (1980), "The limit points of monopolistic competition," *Journal of Economic Theory* **22**, 256-278.

[25] Sahi S., Yao S. (1989), "The noncooperative equilibria of a trading economy with complete markets and consistent prices," *Journal of Mathematical Economics* **18**, 325-346.

[26] Shapley L.S., Shubik M. (1977), "Trade using one commodity as a means of payment," *Journal of Political Economy* **85**, 937-968.

[27] Shitovitz B. (1973), "Oligopoly in markets with a continuum of traders," *Econometrica* **41**, 467-501.

[28] Shitovitz B. (1997), "A comparison between the core and the monopoly solutions in a mixed exchange economy," *Economic Theory* **10**, 559-563.