TRANSFER FOR COMPACT LIE GROUPS, INDUCED REPRESENTATIONS, AND BRAID RELATIONS

by

Sam Evens

B.A. Haverford College
Haverford, PA
May, 1984

Submitted to the
Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology
May, 1988

Signature of author ______________________

Certified by ______________________
Professor Bertram Kostant
Thesis Supervisor

Accepted by ______________________
Professor Sigurdur Helgason, Chairman
Departmental Graduate Committee
Department of Mathematics

DEDICATION

I dedicate this thesis to the memory of Max, and to Meridian for eating most of the magazines in my room so I had to work on math, and for sitting on my back so I had to finish writing this thesis.

TABLE OF CONTENTS

TRANSFER FOR COMPACT LIE GROUPS,

INDUCED REPRESENTATIONS AND BRAID RELATIONS

by

Sam Evens

Submitted to the Department of Mathematics
on April 29, 1988 in partial fulfillment of the
requirements for the Degree of Doctor of Philosophy
in Mathematics

ABSTRACT

Let $T \subset G$ be a maximal torus of a compact connected Lie group and $\pi : BT \to BG$ the induced G/T fiber bundle. Suppose h^* is a cohomology theory with characteristic classes for complex vector bundles and let $\pi_* : h^*(BT) \to h^*(BG)$ be the resulting *umkehr* homomorphism. A formula for the composition $\pi^* \circ \pi_*$ is given. For the special case when h^* is K-theory, this formula is interpreted as the Weyl character formula. A generalization of the operation $\pi^* \circ \pi_*$ is then used to construct irreducible subquotients for certain Verma modules.

Using the formula described above, certain operators D_i on $h^*(BT)$ are constructed which generalize the BGG operators and the Demazure operators. We prove that ordinary cohomology and K-theory are essentially the only theories in which the D_i satisfy braid relations. Moreover, this fact is given a geometric interpretation which is important for the study of $h^*(G/T)$.

Thesis Supervisor: Professor Bertram Kostant
Title: Professor of Mathematics

ACKNOWLEDGEMENTS

My thesis advisor Bertram Kostant has been a great aid to me throughout my time at MIT. He taught me to think creatively about mathematics, encouraged me, and shared a lot of ideas with me, both by directing me to his papers and in conversations. I hope this thesis, which represents only a small fraction of those ideas, is in some sense indicative of what he has taught me. I would like to thank him for all the help he has given me.

Haynes Miller has explained a tremendous amount of topology to me and helped me clarify a number of ideas in this thesis. Moreover, he has helped me learn to think about topology. I suspect this thesis would be much improved if I understood more of the topology he has explained to me.

About my collaborator Paul Bressler, the least I can say is that this thesis might not exist were it not for him. Paul is one in a million, which means there are two more of him walking around the Boston area. On that somewhat sobering note, I would like to thank him for the time we have spent working together.

My father Leonard Evens has encouraged me throughout my mathematical odyssey. He helped me understand a number of ideas and directed me toward people who have helped shape ideas in this thesis. Also, he has been an invaluable TeXconsultant.

I would like to thank a number of other people for helping me learn mathematics. Dale Husemoller taught me a great deal of mathematics while I was an undergraduate and pointed me toward good areas of mathematics, both in a geographical and an intellectual sense. He also told me never to work on mathematics after eight P.M. Unfortunately, I have the most part inserted "before" for "after". Jesper Bang-Jensen, Dr. Monty McGovern, Susana Salamanca, and Jim Schwartz gave a lot of good advice in their roles as older graduate students in Lie theory (and also in other roles). I have had a number of enlightening discussions with fellow students Bill Graham, Jing-song Huang, Yi Ching, and Hisayoshi Matumoto. David Blanc and Hal Sadofsky have tirelessly (or so it seemed) explained a great deal of algebraic topology to me, and Angelo Vistoli and Valery Lunts have explained "other stuff" to me related to this thesis. I have learned a great deal of geometry in conversations with Eugene Lerman and Paul van Mulbregt. Martin Hardy and Briar helped me a great deal when I was still trying to figure out what a Lie group is. Amnon and David Yavin deserve credit for something, although I am not sure exactly what. Roger Zierau and Kari Vilonen taught me a great deal and also provided a great deal of entertainment, especially when they were together. I would also like to thank Luis Casian for not laughing aloud at several things I said to him. I would never have managed to register for my first semester at MIT without the help of Phyllis Ruby, and I would like to thank her. David Vogan has patiently explained a number of things to me, both individually and in groups. I would like to thank Victor Kač , Shrawan Kumar, and Joseph Bernstein for some enlightening discussions. I also profited immensely from courses given by George Lusztig and Wilfried Schmid. Stewart Priddy and Mark Feshbach gave me some good advice which I hope is reflected in chapter twelve. I would like to thank Eugene Gutkin for expressing interest in my work and for proving a theorem which made this thesis more complete.

Many of the above people have given me a great deal of personal support over the past four years. I would also like to thank Amy, Mike, and Nina. Adam has told me most of the jokes I have heard in the last few years, and also helped me develop the philosophical system underlying this work. I would like to thank my sisters Sally and Nancy. I thank my mother for everything, especially for giving me an early introduction to Lie theory.

I would like to thank the National Science Foundation and the Sloan Foundation for financial support, and for having the good sense not to inflict me on MIT undergraduates.

Introduction

Much work in Lie theory in the past thirty years has centered on geometric properties of the flag manifold. For example, the work of Borel, Kostant, Bernstein-Gel'fand-Gel'fand, and Kostant-Kumar has given an explicit determination of the generalized Schubert calculus, even for the case of a Kač-Moody group. From the point of view of representation theory, the work of Borel and Weil gave an explicit realization of the finite dimensional irreducible representations of a compact Lie group G as the space of sections of a line bundle over the flag manifold. Later work by Bott (see also Kostant and Demazure) implies that this representation may be interpreted as a class in K-theory of the classifying space of the group G. More recently, a spectacular conjecture by Kazhdan and Lusztig led to a realization of infinite-dimensional representations of the Lie algebra of G as complexes of sheaves on the flag variety. Beilinson and Bernstein's proof of the Kazhdan-Lusztig conjecture gave in addition a realization of the irreducible representations of an associated real semisimple Lie group.

Almost all of this work used the differential or algebraic structure of the flag variety. As such, it has largely ignored the machinery of modern algebraic topology. This thesis is perhaps best described as an attempt to use algebraic topology to study the flag variety. Of course, I have not managed to recover all, nor even very many of the results described above. However, this work has the advantage that because the techniques are not specific to cohomology or K-theory, they generalize immediately to a wide variety of cohomology theories. Aside from cohomology and K-theory, the most important of these is complex cobordism, although the new theory of elliptic cohomology may eventually prove to be much more interesting.

The class of cohomology theories which we study here are the complex orientable theories. These are are theories with a reasonable theory of characteristic classes for complex vector bundles. This condition is necessary for the following reason. Almost all of the work cited above involves some form of a push-forward operation. That is, suppose we have some contravariant functor F from some category of topological spaces to rings. Then for any morphism $\pi : X \to Y$ we have a natural map $\pi^* : F(Y) \to F(X)$. A push-forward operation is a functor $\pi_* : F(X) \to F(Y)$. Then if F is a cohomology theory h^*, the assumption that h^* be complex orientable implies the existence of an *umkehr* homomorphism $\pi_* : h^*(X) \to h^*(Y)$, given certain assumptions on the map π.

The solution to the generalized Schubert calculus provides a good illustration of the above idea. First, we need a short introduction to the problem. Let G be a compact connected Lie group with maximal torus T. Then the flag manifold G/T has a complex structure, given by the identification $G/T \cong G_{\mathbf{C}}/B$, where $G_{\mathbf{C}}$ is the complexification of G and B is a Borel subgroup of $G_{\mathbf{C}}$. Then G/T has a cell decomposition into Schubert cells $\bar{X}_w$, parameterized by elements w of the Weyl group W. These cells are $2l(w)$−dimensional, where $l(w)$ is the length of w. As a consequence,

$$H^*(G/T, \mathbf{Q}) \cong \oplus \mathbf{Q} P^w,$$

where

$$P^w \in H^{2l(w)}(G/T, \mathbf{Q})$$

is the cohomology class dual to the cell $\bar{X}_w$. Then for the classes P^w and P^v, the cup product

$$P^w \cup P^v = \sum_{y \in W} a_y P^y$$

The problem the generalized Schubert calculus solves is the determination of the numbers a_y.

The solution is as follows. Since $G \to G/T$ is a T-principal bundle, there is a classifying map $\theta : G/T \to BT$, the classifying space of T. The induced map $\theta^* : H^*(BT, \mathbf{Q}) \to H^*(G/T, \mathbf{Q})$ is surjective. $H^*(BT, \mathbf{Q})$ is a polynomial algebra $\mathbf{Q}[\alpha_1, \cdots, \alpha_l]$, generated by the Chern classes of line bundles associated to simple roots. Moreover, the unique top dimensional class P^{w_0} of $H^*(G/T, \mathbf{Q})$, dual to G/T, is easily seen to be the image of the product $\prod \alpha$, where the product is over the full set of positive roots R^+. To express other cohomology classes P^w in terms of characteristic classes, Bernstein-Gel'fand-Gel'fand introduced certain operators

$$A_i = (1 + s_i)\frac{1}{\alpha_i} : H^k(G/T) \to H^{k-2}(G/T)$$

(s_i is the simple reflection associated to the simple root α_i)[BGG]. The A_i have the property that if $l(ws_i) < l(w)$, then $A_i(P^w) = P^{ws_i}$. Thus, if $w = w_0 s_1 \cdots s_k$, where $s_1 \cdots s_k$ is a minimal decomposition of $w_0^{-1}w$, then

$$P^w = A_k \cdots A_1(P^{w_0})$$

Since the operators A_i are already expressed in terms of characteristic classes and $P^{w_0} = \prod \alpha$, we may express P^w and P^v as classes in $H^*(BT, \mathbf{Q})$ and multiply them there. Then we evaluate the product on $H^*(G/T, \mathbf{Q})$ to determine the numbers a_y.

Thus, the operators A_i play a crucial role. They are constructed as follows. Let H_i be a rank one subgroup of G corresponding to the root α_i. Then the induced map $\pi_i : BT \to BH_i$ is a $\mathbf{C}P(1)$ fiber bundle. Let

$$\pi_{i_*} : H^k(BT) \to H^{k-2}(BH_i)$$

be the standard integration over the fiber map, used by Borel and Hirzebruch [BH]. Then $\pi_i^* \circ \pi_{i_*} \circ \theta^* = A_i$.

It is clear that the above operation may be carried out whenever π_{i_*} is defined. So suppose h^* is complex orientable and that the coefficients $h^*(pt)$ have no 2-torsion. Moreover, the situation is no more difficult if we replace H_i with G so we consider $\pi : BT \to BG$. We have the following formula, whose proof uses the Becker-Gottlieb transfer.

Theorem 11.2 : (Bressler-Evens) Let $\chi(-\alpha)$ be the Euler class of the line bundle over BT induced by the character $e^{-\alpha_i}$. Then for $x \in h^*(BT)$,

$$\pi^* \circ \pi_*(x) = \sum_{w \in W} w \frac{1}{\prod_{\alpha \in R^+} \chi(-\alpha)} x$$

For the case when $\pi = \pi_i$,

$$D_i(x) = \pi_i^* \pi_{i_*}(x) = (1 + s_i) \frac{1}{\chi(-\alpha_i)} x$$

A striking fact about the above formulas is that there is no purely algebraic reason why the expressions on the right hand side should lie in $h^*(BT)$ rather than in its fraction field. This fact is proved via topology.

One consequence is the following theorem. A crucial fact for BGG (and for Demazure in his investigations of K-theory) is that the operators D_i satisfy the so-called braid relations. That is, if s_i and s_j are two distinct roots with

$$(s_i s_j)^{m_{ij}} = 1$$

then

$$D_i D_j D_i \cdots = D_j D_i D_j \cdots \quad (m_{ij} \text{ terms on each side})$$

One can then prove the following surprising theorem.

Theorem 13.9 : (Bressler-Evens) Suppose the D_i satisfy braid relations. Then if h^* is torsion-free, h^* is essentially either cohomology or complex K-theory. More precisely, its formal group law must be that of ordinary cohomology or complex K-theory.

Moreover, using Quillen's construction of cobordism, this fact is given a geometric interpretation in chapter fourteen, which I hope will eventually lead to an understanding of the ring structure of $h^*(G/T)$. One can use this geometric interpretation to recover many of the standard facts about the Schubert calculus and its generalization to K-theory. Another interpretation of Theorem 13.9 is that the failure to satisfy braid relations gives geometric information about the flag variety which cohomology and K-theory do not detect.

A second interpretation of Theorem 11.2 is more closely related to representation theory. Suppose h^* is K-theory. In K-theory the Euler class of a line bundle L may be taken to be $[1] - [L^*]$, where 1 is the trivial line bundle and $[L]$ is the K-theory class of the dual of L (We use the dual L^* instead of L in order to make topological push-forwards correspond to sheaf theoretic push-forwards). Moreover, $K(BT)$ is a completion of $R(T)$, the representation ring of T. In this correspondence $\chi(-\alpha_i) = 1 - e^{\alpha_i}$. Then 11.2 reduces to the Weyl character formula. By an argument due to Atiyah and Bott[AB], the Grothendieck-Riemann-Roch Theorem and the Borel-Weil-Bott Theorem imply that $\pi^*\pi_*(e^\lambda)$ is the irreducible representation of lowest weight λ when λ is anti-dominant. Thus, one can compute the character of a finite dimensional representation via very general topological machinery.

Of course, one is also interested in infinite-dimensional representations. The expression

$$\frac{e^{\lambda}}{\prod(1-e^{\alpha})}$$

can be interpreted as the formal character of a Verma module, which is a Lie algebra induced representation. On the other hand, it may also be interpreted as an element of K-theory of the Thom space of a certain vector bundle over BT. Then a variation on the argument in the proof of Theorem 11.2 gives the irreducible subquotient for Verma modules which correspond to Weyl group elements corresponding to nonsingular Schubert varieties. It seems unlikely that arbitrary irreducible subquotients of Verma modules can be produced in this manner. Nevertheless, it would be interesting to try, especially since the Thom space associated to BT is important in the Segal conjecture on cohomotopy [May].

Some comments about the structure of this thesis are in order. It is written for a reader with a first course in algebraic topology and some understanding of the structure of compact Lie groups and their representations, plus some mathematical maturity. Since additional material from algebraic topology is necessary, the first eight chapters are an exposition on the necessary subjects. Few proofs are included, and those that are included are to illustrate ideas, rather than for rigor. Many readers may prefer to skip these chapters entirely, or to use them only for reference. The chapters are kept short to facilitate reference use. Topologists will note that we have avoided use of the stable category. We have not included any formal expositions on Lie theory. For the most part, we have not used any definitions from Lie theory which would mystify a reader with a basic knowledge of the representation theory of compact Lie groups. The notion of a Verma module is explained when it arises. Some good general references are Adams [Ad] for complex oriented cohomology theories, Dyer [Dy] for the *umkehr* homomorphism, and Humphreys [Hum] for Lie theory.

The organization of the first eight chapters is as follows. The first chapter makes some preliminary remarks on generalized cohomology and constructs the Atiyah-Hirzebruch spectral sequence. Chapter two defines the Becker-Gottlieb transfer, which is the key tool in the proof of 11.2. Chapter three is a brief exposition on classifying spaces. In chapter four, we discuss a theorem of Brumfiel and Madsen which computes the Becker-

Gottlieb transfer in the case relevant to 11.2. Chapter five defines the notoion of a complex orientable cohomology theory. Chapter six is an exposition on the role of formal group laws in algebraic topology, which describes the behavior of characteristic classes of tensor products of line bundles. Chapter seven discusses the complex orientable theories K-theory, complex cobordism, and elliptic cohomology. In chapter eight, we define the *umkehr* homomorphism π_* and relates it to the Becker-Gottlieb transfer.

The remainder of this thesis is devoted to results described earlier in the introduction. Chapters nine and ten are preparation for 11.2, which is proved in chapter eleven. Chapter twelves explains the application to induced representations. Chapters thirteen, fourteen, and fifteen discuss the operators $D_i = \pi_i^* \pi_{i*}$ and their application to the study of $h^*(G/T)$. Chapter sixteen is a collection of some results on the algebra generated by the operators D_i.

It will be clear to the reader that certain results are joint work with Paul Bressler. In particular, the torsion-free version of 11.2, and most of the results in chapters thirteen through fifteen are joint. Many other results owe some debt to him. It goes without saying that many of the ideas discussed here were suggested by my thesis advisor, Bertram Kostant. The same statement is true for Haynes Miller.

Chapter 1

Generalities on generalized cohomology

A generalized cohomology theory h^* is a contravariant functor from topological spaces to abelian groups which satisfies all the Eilenberg-Steenrod axioms except the dimension axiom. That is, we do not assume that the coefficients $h^* = h^*(pt)$ are concentrated in a single degree. We will always assume h^* is multiplicative, and that the associated ring structure is commutative in the graded sense. The first example is ordinary cohomology with coefficients in $\mathbf{Z}$. To fix ideas, we take $H^i(X) = H^i(X, \mathbf{Z}) = [X, K(\mathbf{Z}, i)]$, where $K(\mathbf{Z}, i)$ is an Eilenberg-MacLane space, and $[X, Y]$ denotes homotopy classes of maps from X to Y.

For a generalized theory h^* there is a spectral sequence which computes $h^*(X)$ in terms of $H^*(X)$ and the coefficients h^*. This spectral sequence was invented by Whitehead and is therefore called the Atiyah-Hirzebruch spectral sequence. It is constructed as follows [Ad, p. 215]. First we construct an exact couple. Let X be a CW-complex with a finite filtration by subcomplexes,

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X.$$

Then for the pair (X_p, X_{p-1}), we have the long exact cohomology sequence given by

$$\begin{array}{ccc} h^*(X_p) & \xrightarrow{i^*} & h^*(X_{p-1}) \\ j^* \nwarrow & & \swarrow \partial \\ & h^*(X_p/X_{p-1}) & \end{array}$$

where ∂ increases degrees by 1. Summing over p, we get,

$$\begin{array}{ccc} \sum_p h^*(X_p) & \xrightarrow{i^*} & \sum_p h^*(X_{p-1}) \\ j^* \nwarrow & & \swarrow \partial \\ & \sum_p h^*(X_p/X_{p-1}) & \end{array}$$

Thus, we have a triangle of the form,

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ j \nwarrow & & \swarrow g \\ & C & \end{array}$$

and from this exact couple we produce a spectral sequence in the standard way [HS, pp. 256-261]. The E_1 term is $E_1 = H^*(\sum_p h^*(X_p, X_{p-1}), d_1)$ with the differential $d_1 = \partial \circ j^*$. Then $E_1^{p,q} = h^{p+q}(X_p, X_{p-1})$. But $X_p/X_{p-1} \cong \vee S^p$, so $E_1^{p,q} \cong C^p(X, h^q(pt))$. Then $E_2^{p,q} \cong H^p(X, h^q(pt))$. As usual, $E_\infty^{p,q} \cong G_p h^{p+q}(X)$. To summarize,

Theorem 1.1 There is a spectral sequence with E_2 term

$$H^p(X, h^q(pt)) \Longrightarrow h^{p+q}(X).$$

The differentials d_r are of degree $(r, -(r-1))$.

Corollary 1.2 Suppose X has no odd dimensional cells and $h^q(pt) = 0$ for q odd. Then the Atiyah-Hirzebruch spectral sequence degenerates at the E_2 term.

Reduced cohomology is defined as follows. Let $i : pt \to X$ be the inclusion of a point and $\pi : X \to pt$ be the collapsing map. Then $\pi \circ i = id$, so $i^* \circ \pi^* = id$ on $h^*(pt)$. Let $\tilde{h}(X) = \ker i^*$ be the reduced cohomology of X. Then $h^*(X) \cong \tilde{h}^*(X) \oplus h^*(pt)$. Indeed, π^* gives the required splitting.

Chapter 2

The transfer map

Let F be a compact differentiable G-manifold for a compact Lie group G and let $\pi : E \to B$ be a fiber bundle with fiber F. For a cohomology theory h^*, we have a natural induced map $\pi^* : h^*(B) \to h^*(E)$. A transfer map is a backwards map $h^*(E) \to h^*(B)$. Of course, one way to produce a transfer would be to produce a section $s : B \to E$. However, such a section usually does not exist. The technique for producing a transfer will instead be to produce a map $\tau(\pi) : \Sigma^i B^+ \to \Sigma^i E^+$, and to use the suspension axiom to get the desired backwards map.

As a preliminary, we need the notion of the Thom space of a vector bundle $V \to B$. If we suppose B is paracompact, we may assume V has a metric, and we can define the Thom space as follows. Let $D(V) = (v \in V : |v| \leq 1)$ and let $S(V) = (v \in V : |v| = 1)$. Then the Thom space $B^V = D(V)/S(V)$. If V is the trivial n-dimensional vector bundle, then $B^V = \Sigma^n B^+$, where B^+ denotes $B \sqcup$ point. We will generally regard B^V as the pair $(D(V), S(V)$ and $h^*(B^V)$ will refer to $h^*(D(V), S(V))$. If ξ and η are two vector bundles over B, then

$$B^\xi \wedge B^\eta \cong B^{\xi \oplus \eta}.$$

Here if (X, A) and (Y, B) are two pairs of spaces, $X \wedge Y$ refers to the smash product of X and Y.

We are now ready to define the transfer. There are a number of different definitions. We will give one of the simplest. While the basic properties of the transfer follow relatively easily from this definition, the method of computing the transfer involves a different definition. Since we are only interested in applying formulas for the transfer, we present only this relatively simple definition.

We first define the transfer for the map from F to a point. We may embed F equivariantly into a real G-representation V of dimension r. Assume

$r \gg \dim F$. Let $N \to F$ be the normal bundle to the embedding. N may be identified with a tubular neighborhood U of F by a diffeomorphism ϕ. Let S^V denote the one point compactification of V. Then there is an associated Pontryagin-Thom collapsing map $c : S^V \to F^N$ defined as follows.

$$c(x) = \text{ base pt of } F^N \text{ if } x \notin U$$

$$c(x) = \phi(x) \text{ if } x \in U.$$

Let $T(F)$ be the tangent bundle to F. Then $T(F) \oplus N$ may be identified with the G-equivariant bundle $F \times V$. There is a corresponding inclusion

$$i : N \to N \oplus T(F) \cong F \times V,$$

and hence an inclusion of Thom spaces

$$i : F^N \to S^V \wedge F^+.$$

Definition 2.1: The transfer τ to a point is the composition $\tau = i \circ c$.

Remark 2.2: Clearly, the transfer depends on the embedding. However, the cohomology class τ^* is independent of the embedding for any cohomology theory. The proof [BG, p. 5] follows from standard arguments and in fact asserts that the stable homotopy class of τ is independent of the embedding. By construction, the transfer is equivariant.

Let $\pi : E \to B$ be a fiber bundle with fiber F whose structure group reduces to G. Then there is a principal G-bundle $p : P \to B$ associated to π. The above definition of transfer to a point gives a map

(2.3). $$id \times \tau : P \times_G S^V \to P \times_G (F \times V)^+$$

When we collapse the section at ∞ to a point, which is equivalent to taking Thom spaces, we get a map

$$t : B^\xi \to E^{\pi^*(\xi)}$$

where ξ is the vector bundle associated to the representation V. Let $\bar{\xi}$ be the inverse bundle so $\xi \oplus \bar{\xi} = [m]$, the m-dimensional trivial bundle. Then there is a map

$$t \wedge 1 : B^\xi \wedge B^{\bar{\xi}} \to E^{\pi^*(\xi)} \wedge B^{\bar{\xi}}.$$

Restricting to the diagonal in $B \times B$, we have a map

$$\tau(\pi) : \Sigma^m B^+ \to \Sigma^m E^+.$$

Definition 2.4: The transfer associated to the fiber bundle $\pi : E \to B$ is the map $\tau(\pi)$.

Remark 2.5: The transfer need not be defined in an equivariant setting. For some equivalent non-equivariant definitions see my preprint with Bressler [BE2], or for a more complete version Brumfiel-Madsen [BM], Dold [Do], or Feshbach [Fe1]. For a more complete version of the equivariant transfer, see the original paper of Becker-Gottlieb [BG], Brumfiel-Madsen, or for the most general version Lewis-May-Steinberger [LMS]. The preprint of Mitchell and Priddy [MP] is also illuminating. The proof that all these versions are equivalent may be found in [LMS].

Remark 2.6: The formula for transfer gives a map

$$\tau(\pi) : \Sigma^i B^+ \to \Sigma^i E^+.$$

For $i \geq 1$, the set of homotopy classes of maps $[\Sigma^i B^+, \Sigma^i E^+]$ forms a group. We let $g, f : S^i \wedge B^+ \to S^i \wedge E^+$. We can multiply $g \cdot f$ as follows. We may regard S^i as $I^i/\partial I^i$, where I^i is the i-fold cartesian product of the unit interval. Then

$$g \cdot f(t_1, \cdots, t_i, x) = g(2t_1, t_2, \cdots, t_i, x) \text{ if } 0 \leq t_1 \leq 1/2$$

$$g \cdot f(t_1, \cdots, t_i, x) = f(2t_1 - 1, t_2, \cdots, t_i, x) \text{ if } 1/2 \leq t_1 \leq 1$$

as in defining the group structure on homotopy groups. If $i \geq 2$, then the group structure is abelian. Thus, a formula

$$\tau(\pi) = \sum f_j$$

means

$$\tau(\pi) = \Sigma^{-i}(\sum f_j \Sigma^i)$$

for some i.

Chapter 3

Classifying spaces

In this chapter we summarize some basic facts about classifying spaces of groups. Let G be a compact Lie group. Then there is a universal space EG with a free G-action and $\pi_i(EG) = 0$ for all $i > 0$. Moreover, EG is a limit of manifolds with the inductive limit topology. For example, for $G = U(n)$, the unitary group,

$$EU(n) = lim_{m \to \infty} V_n(\mathbf{C}^{n+m}),$$

where the Stiefel manifold $V_n(\mathbf{C}^{n+m}) = U(n+m)/U(m)$. Any compact Lie group G embeds into $U(n)$ for sufficiently large n so EG may always be given as $EU(n)$. The classifying space BG is then defined as EG/G. For $G = U(n)$, $BG \cong lim_{m \to \infty} G_n(n+m)$, the infinite Grassmannian of n-planes. BG has the following universal property. Let $P \to B$ be a numerable G-principal bundle. Then there exists a unique (up to homotopy) classifying map $\theta : B \to BG$ such that

$$\theta^{-1}(EG) \cong P$$

as G-principal bundles over B. As a consequence, BG is well-defined up to homotopy-type.

BG also classifies induced vector bundles. Let $P \to B$ be our G-principal bundle. Then if V is a finite dimensional representation of G, $E \times_G V$ is the associated vector bundle over B with structure group G. Then E induces a classifying map $\theta : B \to BG$ with

$$\theta^{-1}(EG \times_G V) \cong E$$

as G-vector bundles over B.

The special case of the classifying space for complex line bundles will be important. Here the appropriate structure group is $U(1)$ so the appropriate classifying space is $BU(1)$. By the above construction,

$$BU(1) = lim_{m\to\infty}\mathbf{C}P(m) = \mathbf{C}P(\infty).$$

As a consequence,

$$H^*(BU(1),\mathbf{Z}) = lim_{m\to\infty}H^*(\mathbf{C}P(m),\mathbf{Z}) \cong \mathbf{Z}[[t]],$$

where t occurs in degree 2, and $R[[t_1,\cdots,t_l]]$ denotes the ring of formal power series in l variables with coefficients in the ring R. Of course, it is more standard to think of $H^*(BU(1))$ as a polynomial algebra. Because H^* is graded, the distinction is not important here. The power series formulation is more consistent with other cohomology theories. Now let $T = \prod_{i=1}^{l} U(1)$ be a torus. Then we may take $ET = \prod_{i=1}^{l} EU(1)$. Therefore, $BT = \prod_{i=1}^{l} BU(1)$, and

$$H^*(BT,\mathbf{Z}) \cong \hat{\otimes}_{i=1}^{l} H^*(BU(1),\mathbf{Z}) \cong \mathbf{Z}[[t_1,\cdots,t_l.]].$$

There are no Ext groups since $H^*(BU(1))$ is torsion-free.

Chapter 4

The Brumfiel-Madsen formula for transfer

We give a formula for $\tau(\pi)^*$ due to Brumfiel and Madsen. Let G be a compact connected Lie group with maximal torus T. Let H be a closed connected subgroup of G containing T. Let W_G and W_H be the Weyl groups of G and H. Suppose $P \to B$ is a principal G-bundle. We have associated bundles $E_1 = P \times_G G/T$ and $E_2 = P \times_G G/H$ over B. There are fibrations

$$\begin{array}{ccc} E_1 & \xrightarrow{\pi} & E_2 \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & B & \end{array}$$

The Weyl group W_G acts on G/T by $w \cdot gT = gwT$. Hence, W_G acts on E_1 over B. The fibers of π are H/T. Hence, the Weyl group W_H of H acts on E_1 over E_2. Thus, cosets $w \in W_G/W_H$ define maps $\pi \circ w$ on E_1 over E_2. We have

Theorem 4.1(Brumfiel-Madsen[BM]):

$$\pi_1^* \circ \tau(\pi_2)^* = \sum_{w \in W_G/W_H} w \circ \pi^*.$$

Corollary 4.2: If $H = T$ so $\pi_1 = \pi_2$,

$$\pi_1^* \circ \tau(\pi_1)^* = \sum_{w \in W_G} w.$$

Remark 4.3: Although Brumfiel and Madsen were the first to assert that this theorem is true, their proof is wrong. Indeed, Mitchell and Priddy have constructed a counterexample to Theorem 2.10 in Brumfiel-Madsen, which is used in the proof of the above theorem. However, Feshbach [Fe1] and Lewis-May-Steinberger [LMS] have given independent proofs of Theorem 4.1.

Now let $P = EG$. Since EG is a universal space for T, we have

Theorem 4.4: Let $\pi : BT \to BG$ be the associated fiber bundle. Then

$$\pi^* \circ \tau(\pi)^* = \sum_{w \in W_G} w.$$

We will need the following series of theorems, due to Feshbach [Fe2]. Recall that we are assuming h^* is multiplicative and commutative.

Theorem 4.5: Let B be locally compact. Let $\pi : E \to B$ be a fiber bundle with fiber F a compact G-manifold. Then

$$\tau(\pi)^* \circ \pi^*(x) = \chi(F)x + ux,$$

where $u \in \tilde{h}^0(B)$ is nilpotent, and $\chi(F)$ is the usual Euler characteristic of F.

Corollary 4.6: If $\chi(F)$ is a unit, then π^* is injective.

Corollary 4.7: $\pi^* : h^*(BG) \to h^*(BT)$ is injective if $|W|$ is a unit in $h^*(pt)$.

Proof: It is well-known that $\chi(G/T) = |W|$.

Proposition 4.8: $\pi^*(h^*(BG)) = h^*(BT)^W \otimes \mathbf{Z}[\frac{1}{|W|}]$.

Proof: First we show $\subset$. Any group G acts on its classifying space by conjugation of the group. This action is always homotopic to the identity. The Weyl group $W_G \cong N(T)/T$ acts on BT by the conjugation action of $N(T)$. Moreover, the map $\pi : BT \to BG$ is $N(T)$-equivariant. Therefore, for $x \in h^*(BG)$, $w\pi^*(x) = \pi^*(wx) = \pi^*(x)$. So $\subset$ is clear. $\supset$ follows from Theorem 4.4.

Chapter 5

Complex orientable cohomology theories

For the results of the previous section to be applicable we need a class of cohomology theories for which $h^*(BT)$ is easy to understand. For these theories, we will have

$$h^*(BT) \cong \hat{\otimes}_{i=1}^{l} h^*(BU(1)).$$

The crucial point will be that $h^*(BU(1))$ has an orientation class.

Definition 5.1: Let $i : \mathbf{C}P(1) \to \mathbf{C}P(\infty) = BU(1)$ be the obvious inclusion. We say h^* is complex orientable if there exists a class $x \in \tilde{h}^*(\mathbf{C}P(\infty))$ such that $i^*(x)$ is a generator of $\tilde{h}^*(\mathbf{C}P(1))$ over $h^*(pt)$.

Remark: By the suspension axiom, $\tilde{h}^*(\mathbf{C}P(1)) \cong \tilde{h}^*(S^2)$ is generated by one element over $h^*(pt)$.

Example: If $h^* = H^*$, x may be taken as a ring generator of $H^*(\mathbf{C}P(1), \mathbf{Z})$, so $x \in H^2(\mathbf{C}P(\infty), \mathbf{Z})$.

$\mathbf{C}P(\infty)$ has a universal line bundle $L(\lambda)$ given as follows. Let e^λ be the one-dimensional representation $\mathbf{C}_\lambda$ of $S^1 = U(1)$ given by $e^\lambda(e^{i\theta}) \cdot v = e^{i\theta} \cdot v$. Then by definition, for a complex orientable theory h^* with orientation given by x, $x = c_1(L(\lambda))$, the h^* first Chern class. Let T be an l-dimensional torus.

Theorem 5.2[Ad, p. 39]: In a complex orientable theory h^*,

$$(1) h^*(\mathbf{C}P(\infty)) \cong h^*(pt)[[x]]$$

$$(2) h^*(BT) \cong h^*(pt)[[x_1, \cdots, x_l]]$$

$$(3) h^*(\mathbf{C}P(n)) \cong h^*(pt)[x]/(x^{n+1})$$

$$(4) h^*(\prod_{i=1}^{l} \mathbf{C}P(n_i)) \cong h^*(pt)[x_1, \cdots, x_l]/(x_1^{n_1}, \cdots, x_l^{n_l})$$

Proof: We have an Atiyah-Hirzebruch spectral sequence with E_2 term

$$E_2^{p,q} \cong H^p(\mathbf{C}P(\infty), h^q(pt))$$

and

$$H^*(\mathbf{C}P(\infty), h^*(pt)) \cong h^*(pt)[[x]].$$

There is also an Atiyah-Hirzebruch spectral sequence for $\mathbf{C}P(1)$ with

$$E_2'^{p,q} \cong H^p(\mathbf{C}P(1), h^q(pt)),$$

which is isomorphic to $h^*(pt)[x']/(x'^2)$. Moreover, by the naturality of the construction of the Atiyah-Hirzebruch spectral sequence, we may assume that $i^*(x) = x'$. The E' spectral sequence degenerates at the $E_2{}'$ level since we know $h^*(\mathbf{C}P(1)) \cong$ the $E_2{}'$ term by the suspension axiom. Thus, for $y \in E_2$, $i^*(y) = 0$ in $E_2{}'$ if and only if $i^*(y) = 0$ in $h^*(\mathbf{C}P(1))$. Any element $y \in h^*(\mathbf{C}P(\infty))$, comes from an element of the form $x^k \otimes a$, where $a \in h^*(pt)$. Suppose y defines the complex orientation of h^*, so $i^*(y) = ux'$, where u is a unit. Then also $i^*(x^k \otimes a) = ux'$. It follows that $k = 1$ and $a = u$. Thus, there exists an element $x \otimes u$ which survives in the spectral sequence, so $d_2(x \otimes u) = 0$. But d_2 is h^*-linear, so $ud_2(x) = 0$ so $d_2(x) = 0$. Also, d_2 is a derivation so $d_2(x^j \otimes a) = 0$ for all j and all a. Similarly, all d_r vanish. Hence, the $\mathbf{C}P(\infty)$ spectral sequence collapses at the E_2 term. This proves (1). The proofs of (2), (3), and (4) follow by analogous arguments.

Now let $\pi : L \to X$ be a line bundle over X. Then L induces a classifying map $\theta : X \to \mathbf{C}P(\infty)$.

Definition 5.3: The first Chern class of L, $c_1(L) = \theta^*(x)$.

Remark: Let $f^{-1}(L) \to Y$ be the pulback of L under a map $f : Y \to X$. Then $f^{-1}(L)$ induces a classifying map $\theta' : Y \to \mathbf{C}P(\infty)$. Then $\theta' = \theta \circ f$ so

$$c_1(f^{-1}(L)) = \theta'^*(x) = f^*\theta^*(x) = f^*(c_1(L)).$$

One can go on to define Chern classes of vector bundles following the procedure of Grothendieck[CF, pp. 48-52]. We will only need the top Chern class of a vector bundle, which can be defined as follows. Let $\pi : E \to X$ be a vector bundle. Then there is a space $Fl(E)$ and a map $f : Fl(E) \to X$ such that $f^{-1}(E) \cong \oplus L_i$, where the L_i are line bundles. Moreover, f^* is injective. Then $c_n(E)$ is given by the formula, $f^* c_n(E) = \prod c_1(L_i)$.

Remark: We will often refer to the top Chern class $c_n(E)$ as the Euler class $\chi(E)$. Note that this class lies in $h^*(B)$. It is the usual Euler class when $h^* = H^*$.

Chapter 6

Formal group laws

In ordinary cohomology, $c_1(L\otimes M) = c_1(L)+c_1(M)$, when L and M are line bundles. In a complex oriented theory, $c_1(L \otimes M) = F(c_1(L), c_1(M))$ where F is a power series. F is called the formal group law of the theory. In this section, we explain this notion. A reference for this section is [Ad, pp. 36-46].

A line bundle L over a space X is equivalent to a homotopy class of maps $f_L : X \to \mathbf{C}P(\infty)$. Suppose L and M are two line bundles. Then we have $f_L \times f_M : X \to \mathbf{C}P(\infty) \times \mathbf{C}P(\infty)$. $\mathbf{C}P(\infty)$ has an H-space structure $m : \mathbf{C}P(\infty)\times\mathbf{C}P(\infty) \to \mathbf{C}P(\infty)$. Indeed, since $U(1)$ is abelian, the product map $\tilde{m} : U(1) \times U(1) \to U(1)$ is a group homomorphism. Thus, there is a map

$$m : EU(1) \times EU(1)/U(1) \times U(1) \to EU(1) \times EU(1)/U(1).$$

Thus, we have $m : \mathbf{C}P(\infty) \times \mathbf{C}P(\infty) \to \mathbf{C}P(\infty)$. The homotopy class of $m \circ (f_L \times f_M)$ is then equivalent to the tensor product $L \otimes M$. There is an induced map,

$$m^* : h^*(\mathbf{C}P(\infty)) \to h^*(\mathbf{C}P(\infty) \times \mathbf{C}P(\infty)).$$

Since,

$$h^*(\mathbf{C}P(\infty)) \cong h^*(pt)[[x]] \text{ and } h^*(\mathbf{C}P(\infty) \times \mathbf{C}P(\infty)) \cong h^*(pt)[[x_1, x_2]],$$

m^* has the form,

$$m^*(x) = \sum a_{ij} x_1^i x_2^j = F(x_1, x_2).$$

Then

$$c_1(L \otimes M) = F(c_1(L), c_1(M)).$$

Proposition 6.1:

$$(1) a_{ij} = a_{ji}$$
$$(2) F(F(x_1, x_2), x_3) = F(x_1, F(x_2, x_3))$$
$$(3) a_{i0} = a_{0i} = 0 \text{ if } i \geq 2.$$
$$(4) a_{00} = 0$$

Proof: (1) and (2) follows from the isomorphisms $L \otimes M \cong M \otimes L$ and $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$ (N a line bundle). Suppose L is trivial. L induces a classifying map $\theta : X \longrightarrow \mathbf{C}P(\infty)$. $L = \pi^{-1}[1]$ where π is the map from X to a point and [1] is the line bundle over a point. Let $\theta' : pt \longrightarrow \mathbf{C}P(\infty)$ be the map induced by [1]. Then

$$c_1(L) = \theta^*(x) = \pi^* \circ \theta'^*(x).$$

Since

$$\theta'^*(x) \in \tilde{h}^*(pt) = 0, \theta'^*(x) = 0.$$

Hence, $c_1(L) = 0$. Now, let M be arbitrary. Then

$$x_2 = c_1(M) = c_1(L \otimes M) = F(0, x_2) = \sum a_{0i} x_2^i.$$

(1) and (2) follow easily.

Remark: The requirement that a formal power series F satisfy (1) through (4) is a strict requirement. Indeed, it is easy to show that if F is polynomial then $F(X,Y) = X + Y$ or $F(X,Y) = X + Y + a_{11}XY$, provided the coefficients a_{ij} are not nilpotent [Ha].

Chapter 7

Some examples of complex orientable cohomology theories

In this section we present our main examples which are (1) ordinary cohomology, (2) complex K-theory, (3) complex cobordism, and (4) elliptic cohomology. We have already discussed ordinary cohomology. As stated, ordinary cohomology $H^*(X)$ refers to $H^*(X, \mathbf{Z})$.

(2) Complex K-theory [Huse]. The familiar notion of complex K-theory as the Grothendieck group of vector bundles is inadequate for our purposes. It is appropriate only for locally compact spaces. For a classifying space one needs a homotopy theoretic definition. The space BU is defined as follows. The Grassmannian of n-planes in 2n-space

$$G_n(\mathbf{C}^{2n}) \cong U(2n)/U(n) \times U(n).$$

There is an inclusion to the middle 2n-coordinates $U(2n) \hookrightarrow U(2m)$ for $m \geq n$. Under this inclusion, $U(n) \times U(n) \to U(m) \times U(m)$. Hence, there is an inclusion

$$G_n(\mathbf{C}^{2n}) \hookrightarrow G_m(\mathbf{C}^{2m}).$$

Then

$$BU = lim_{n \to \infty} G_n(\mathbf{C}^{2n}),$$

topologized via the inductive limit topology. By definition, for n even

$$K^n(X) = [X, BU \times \mathbf{Z}],$$

where $[X, Y]$ denotes homotopy classes of maps from X to Y. For n odd

$$K^n(X) = [X, \Omega(BU \times \mathbf{Z})]$$

where ΩX denotes the loop space of X. If $g \in K^0(X)$ and $f : Y \to X$ is a map, then $f^*(g) = g \circ f$. It is well-known that for a finite complex X, $K^0(X)$ is the Grothendieck group of complex vector bundles and that the above definition of f^* coincides with pullback of vector bundles. Suppose $X = BG$ is the classifying space for a group G. Then any representation (π, V) of G induces a vector bundle over BG and a class in $K^0(BG)$ via finite approximations. Indeed,

$$K^0(BG) = \varprojlim K^0(X_n)$$

where X_n is a finite approximation of BG [AS, Prop. 4.1].

Implicit in this definition of K-theory is the periodicity isomorphism

$$K^n(X) \cong K^{n-2}(X),$$

which is given by the mapping

$$y \mapsto \omega \cdot y,$$

where $\omega \in K^{-2}(pt) \cong \tilde{K}^0(S^2)$ is the Bott class, which represents the univeral line bundle over $S^2 \cong \mathbf{C}P(1)$. As defined, K-theory is a generalized cohomology theory.

Complex K-theory is complex orientable. The complex orientation is given as follows [Ad, p. 38]. Let V^* be the dual bundle to a bundle V. Let L be the canonical line bundle over $\mathbf{C}P(\infty)$. Then $[1] - [L^*]$ represents a class in $\tilde{K}^0(\mathbf{C}P(\infty))$. Moreover, restricted to $\mathbf{C}P(1)$, $[1]-[L^*]$ is a generator of $\tilde{K}^0(\mathbf{C}P(1))$ so we set ${c_1}^K(M) = [1] - [M^*]$ for a line bundle M. We use $[M^*]$ instead of $[M]$ in order to make topological constructions correspond to sheaf-theoretic constructions. The formal group law is easy to compute. It is

$$F(X,Y) = X + Y - XY.$$

We can, and sometimes will, regard $x = c_1(L)$ as $[1] - [L^*] \in \tilde{K}^2(\mathbf{C}P(\infty))$ by the periodicity isomorphism. In that case,

$$F(X,Y) = X + Y - \omega XY.$$

By Theorem 5.2, $K^0(BT) \cong \mathbf{Z}[[t_1, \cdots t_n]]$. However, $K^0(BG)$ for any compact Lie group G has a more natural realization. Let $R(G)$ be the representation ring of G. Then there is a map $i^* : R(G) \to K^0(BG)$ given as follows. Let (π, V) be a representation of G. Then

$$EG \times_G V \cong \tilde{V}$$

defines a class in $K^0(BG)$. Let

$$I(G) = \ker(\epsilon : R(G) \to \mathbf{Z})$$

where $\epsilon(\tilde{V}) = dimV$. Then we can complete $R(G)$ with respect to the ideal $I(G)$ to get $\hat{R}(G)$.

Theorem 7.2: ([AH],[AS]) $K^0(BG) \cong \hat{R}(G)$. Moreover, if G is connected, the natural map $R(G) \to \hat{R}(G)$ is an inclusion.

For the case where $G = T$ is a torus then $R(T)$ is generated by one-dimensional representations of T, and $I(T)$ is generated by the Chern classes $[1] - [L^*]$ of the associated line bundles.

(3) Complex cobordism. Complex cobordism is the universal complex orientable theory, in a sense we will describe. As with K-theory, there is a geometric definition valid for manifolds, and a homotopy theoretic definition valid in general. We will give both definitions because they both will be useful.

We begin with the geometrical definition due to Quillen [Qu]. The first step is to define a complex-orientable map. Let $f : Z \to X$ be a proper map of manifolds where Z is compact. Suppose

$$\dim Z_x - \dim X_{f(x)}$$

is even. Then a complex orientation of (Z, f) is given by a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & E \\ f\searrow & & \swarrow p \\ & X & \end{array}$$

where i is an embedding into a complex vector bundle E over X, together with a complex structure on the normal bundle to i. We say

$$\begin{array}{ccc} Z & \xrightarrow{i} & E \\ f\searrow & & \swarrow p \\ & X & \end{array}$$

is equivalent to

$$\begin{array}{ccc} Z & \xrightarrow{i'} & E' \\ f \searrow & & \swarrow p' \\ & X & \end{array}$$

if E and E' embed into a vector bundle $E'' \to X$ such that i and i' are isotopic in E'' compatibly with the complex structure on the normal bundle. If Z and X are complex manifolds, and f is holomorphic then (Z, f) is complex orientable.

Now suppose $\dim Z_x - \dim X_{f(x)}$ is odd. Then a complex orientation for $f : Z \to X$ will be given by a complex orientation for $(f, \epsilon) : Z \to X \times \mathbf{R}$ with $\epsilon(z) = 0$. For a general map, let $Z = Z' \sqcup Z''$, where

$$\dim Z'_x - \dim X_{f(x)}$$

is even, and

$$\dim Z''_x - \dim X_{f(x)}$$

is odd. Then f is complex-orientable if it is complex orientable on each piece.

We will define $T^q(X)$ to be equivalence classes of complex oriented maps (Z, f) $f : Z \to X$, where dim X -dim Z =q. T^q is contravariant with respect to maps in the following sense. Let $g : Y \to X$ be a map transverse to f. Then the induced map g^* is defined by

$$g^*(Z, f) = p_1 : Y \times_X Z \to Y$$

which has a complex orientation given by the pullback of the bundle defining the complex orientation on f. The transversality condition implies $Y \times_X Z$ is a manifold. Since every map $\tilde{g} : Y \to X$ is homotopic to a map $g : Y \to X$ tranverse to f, we can define $\tilde{g}^*(Z, f) = g^*(Z, f)$ in a homotopy invariant way. Thus, T^q is a contravariant functor from spaces to abelian groups.

Complex cobordism, $MU^q(X)$ is $T^q(X)$ modulo an equivalence relation given as follows. Let

$$f_1 : Z_1 \to X, f_2 : Z_2 \to X$$

be two proper complex-oriented maps. Let

$$\epsilon_1, \epsilon_2 : X \to X \times \mathbf{R}$$

be given by $\epsilon_i(x) = (x, i)$. We say (Z_1, f_1) is cobordant to (Z_2, f_2) if there is a proper complex-oriented map $b : W \to X \times \mathbf{R}$ such that b is transversal to ϵ_i and $\epsilon_i^*(b)$ is equivalent to f_i. Then cobordism is an equivalence relation $\sim$ and by definition, $MU^q(X) \cong T^q(X)/\sim$. To review,

Definition 7.3: $MU^q(X)$ is the set of cobordism classes of proper complex-oriented maps of codimension q.

$MU^*(X)$ has a ring structure given as follows. If $f_1 : Z_1 \to X$ and $f_2 : Z_2 \to X$ are two cobordism classes, (Z_1, f_1) and (Z_2, f_2) then

$$(Z_1, f_1) + (Z_2, f_2)$$

is the class of the map

$$f_1 \sqcup f_2 : Z_1 \sqcup Z_2 \to X.$$

The negative of (Z, f) is (Z, f) with the opposite orientation on the normal bundle. The product structure on $MU^*(X)$ is given by products of manifolds. That is, if $(Z_1, f_1) \in MU^q(X)$ and $(Z_2, f_2) \in MU^r(X)$ are as before, then

$$(f_1, f_2) : Z_1 \times Z_2 \to X \times X \in MU^{q+r}(X \times X).$$

Then

$$(Z_1, f_1) \cdot (Z_2, f_2) = \Delta^*((Z_1, Z_2), (f_1, f_2))$$

($\Delta : X \to X \times X$ is the diagonal embedding). The unit element $1 \in MU^0(X)$ is given by the identity map $X \to X$.

Now we sketch the homotopy theoretic definition [Ad, p. 5]. Let $E(n) \to BU(n)$ be the universal rank n vector bundle over $BU(n)$. Let $D(E)$ and $S(E)$ be its disc and sphere bundles. Then $D(E)/S(E) = MU(n)$. Moreover, if $E(n+1) \to BU(n+1)$ is the universal bundle, and $i : BU(n) \to BU(n+1)$ is the map induced by the inclusion $U(n) \to U(n+1)$,

$$i^{-1}E(n+1) \cong E(n) \oplus \mathbf{C}.$$

Then the Thom space

$$BU(n)^{E(n) \oplus \mathbf{C}} \cong S^2 \wedge MU(n)$$

maps to $MU(n+1)$ in the obvious way. Thus, we have a sequence of spaces $\Sigma_{2n} = MU(n)$ and maps

$$S^2 \wedge \Sigma_{2n} \to \Sigma_{2n+2}.$$

In other words, the above sequence of spaces defines a spectrum denoted MU.. By standard reasoning, the group of maps $\tilde{MU}(X) = [\Sigma^\infty X, MU]$ defines a cohomology theory [Huss, pp. 103-111]. Moreover, if X is a manifold, $\tilde{MU}(X) = MU(X)$ [Qu]. For applications, we will generally use the geometric description of $MU^*(X)$.

The complex orientation of MU^* is given as follows. By construction, $MU(1) = D(E)/S(E)$ where E is the canonical line bundle over $BU(1)$. We have the principal $U(1)$ bundle $S(E) \to BU(1)$, which realizes $S(E)$ as the universal $U(1)$ principal bundle. In particular, $S(E)$ is contractible, so $MU(1) \simeq D(E) \simeq BU(1)$. Hence, we have a homotopy equivalence

$$\theta : BU(1) \to MU(1)$$

given by the zero section. By definition, θ defines a class $x \in \tilde{MU}^2(\mathbf{C}P(\infty))$. By construction, $i^*(x)$ is given by the inclusion

$$\theta \circ i : \mathbf{C}P(1) \to MU(1).$$

It is trivial to check that $\theta \circ i$ is equivalent to the image of $1 \in MU(pt)$ under the suspension isomorphism, $MU^0(pt) \cong \tilde{MU}^2(\mathbf{C}P(1))$. Thus, x defines the complex orientability of MU^*.

The geometric definition of $MU^*(X)$ has the virtue that a push-forward map is easy to construct [Qu]. Indeed, let $f : X \to Y$ be a proper complex-oriented map of codimension $-d$. Let $g : Z \to X$ define a class $(Z, g) \in MU^q(X)$. Then since g and f are proper and complex-oriented, so is $f \circ g$ [Dy, p. 57], so we have a new class

$$(Z, f \circ g) \in MU^{q-d}(X).$$

Definition 7.4: The push-forward $f_*(Z, g) = (Z, f \circ g)$.

The universality of MU^* can then be described as follows. Let h^* be a generalized cohomology theory with a push-forward, i.e., for any proper complex-oriented map $g : Z \to X$, there is a map $g_* : h^*(Z) \to h^*(X)$. Suppose moreover that push-forward satisfies the following two properties.

(1):Base change. Let

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Z \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}$$

be a Cartesian square of manifolds, with g transversal to f, and suppose f is proper and complex-oriented and give f' the pullback of the complex orientation of f. Then,

$$g^* \circ f_* = f'_* \circ g'^* : h^*(Z) \to h^*(Y).$$

(2):Functoriality. If $f : Z \to X$ and $g : X \to Y$ are proper complex-oriented maps, and if $g \circ f$ is given the composition of the complex orientations, then $(g \circ f)_* = g_* \circ f_*$.

Proposition 7.5: (Quillen) Given an element a of $h^*(pt)$, there is a unique morphism θ of functors commuting with pushforwards such that $\theta 1 = a$, where $1 \in MU(pt)$ is the cobordism class of the identity.

Proof: Let $(Z, f) \in MU^*(X)$ be a cobordism class. Let $1_Z \in MU^*(Z)$ be the cobordism class of the identity. Then it follows from the definitions that $f_*(1_Z) = (Z, f)$. Let $\pi_Z : Z \to pt$ be the projection to a point. $1_Z = {\pi_Z}^*(1)$. Then,

$$\theta(Z, f) = \theta f_* {\pi_Z}^*(1) = f_* {\pi_Z}^* \theta(1) = f_* {\pi_Z}^*(a)$$

Thus, uniqueness of clear. Existence of θ amounts to showing that $\theta(Z, f)$ depends only on the cobordism class of (Z, f). This follows from the homotopy invariance of h^*.

Thus, MU^* is universal with respect to push-forwards. Quillen uses this fact to prove the following result.

Theorem 7.6: (Quillen, see also Adams) Let h^* be a complex-orientable cohomology theory. Then there is a multiplicative natural transformation $\theta : MU^* \to h^*$, and

$$\theta F_{MU}(X, Y) = F_h(\theta(X), \theta(Y))$$

(F_h refers to the formal group law in h^*).

As a consequence, to prove certain relations about complex orientable theories it suffices to prove statements about MU. As a point of information,

$$MU^*(pt) \cong \mathbf{Z}[x_1, x_2, \cdots],$$

where x_n represents the map from a particular $2n-$ dimensional manifold $M^{2n} \to pt$ and so x_n occurs in degree $-2n$ [Ad, p. 75]. If we work over the rationals, we may take $M^{2n} = \mathbf{C}P(n)$. The formal group law for MU is the universal one-dimensional formal group law introduced by Lazard.

(4): Elliptic cohomology. One can produce cohomology theories from MU^* as follows [La]. $MU^*(X)$ is a module over $MU^*(pt)$. Suppose R is a ring and we have some ring homomorphism $\phi : MU(pt) \to R$. Then we can define a functor

$$h_\phi{}^*(X) = MU(X) \otimes_{MU(pt)} R,$$

where R is a $MU(pt)-$module under the ring homomorphism ϕ. Of course, $h_\phi{}^*$ need not be a cohomology theory, since R need not be flat over $MU^*(pt)$. However, there is an exact functor theorem which gives a criterion for $h_\phi{}^*$ to be a cohomology theory. For example, if ϕ is given by the classical Todd genus, Conner and Floyd proved $h_\phi{}^* = K-$ theory.

A cohomology theory defined as above is said to be elliptic in the following situation. Let δ and ϵ be 2 algebraically independent elements over $\mathbf{Z}[1/2]$ of degrees -4 and -8 respectively. Let $R = \mathbf{Z}[1/2][\delta, \epsilon, \epsilon^{-1}]$. We construct a ring homomorphism $\phi : MU \to R$. $\mathbf{C}P(n)$ defines a class y_n in MU. Let

$$R(x) = 1 - 2\delta x^2 + \epsilon x^4.$$

Then $R(t)^{-1/2}$ has a power series expansion

$$R(t) = \sum_n a_n t^n.$$

Then ϕ is said to be elliptic if $\phi(y_n) = a_n$. As a consequence, $\phi(y_n) = 0$ if n is odd for an elliptic genus.

Proposition-Definition 7.7: Let ϕ be elliptic. The functor

$$h_\phi{}^*(X) = MU^*(X) \otimes_{MU(pt)} R$$

defined by $\phi : MU(pt) \rightarrow R$ is a cohomology theory $Ell^*(X)$, called elliptic cohomology. It is complex orientable and its formal group law is given by the expression,

$$F(x, y) = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - \epsilon x^2 y^2}.$$

The complex orientation is given by $x \rightarrow x \otimes 1$ in $Ell^*(\mathbf{C}P(\infty))$, where x gives the complex orientation of $MU^*(X)$.

Chapter 8

The *umkehr* homomorphism

We saw in the previous section that complex cobordism was universal with respect to cohomology theories with push-forwards. In this section we construct a push-forward π_* for complex orientable generalized cohomology theories. Here, $\pi : E \to B$ is a fiber bundle with fiber a compact complex manifold associated to a $G-$principal bundle. π_* is closely related to transfer. Our discussion follows Becker-Gottlieb [BG].

First, we need a Thom isomorphism theorem for a complex orientable theory h^*. Let $\pi : M \to X$ be a complex line bundle. There is an induced map $\theta : X \to \mathbf{C}P(\infty)$ and an induced map of line bundles $M \to L$, where L is the universal line bundle over $\mathbf{C}P(\infty)$. Then there is also an induced map of Thom spaces $\tilde{\theta} : X^M \to \mathbf{C}P(\infty)^L$. Let $D(L)$ and $S(L)$ be the disc and sphere bundles of L. Then

$$\mathbf{C}P(\infty)^L = D(L)/S(L) \simeq D(L)$$

since $S(L)$ is contractible. But $D(L) \simeq \mathbf{C}P(\infty)$. Hence $\mathbf{C}P(\infty)^L \simeq \mathbf{C}P(\infty)$. As a consequence, we can define $u \in h^*(\mathbf{C}P(\infty)^L)$ to be the class corresponding to $c_1(L)$ under the isomorphism

$$h^*(\mathbf{C}P(\infty)^L) \cong h^*(\mathbf{C}P(\infty)).$$

Let

$$u_M = \tilde{\theta}^*(u) \in h^*(X^M).$$

Recall that $h^*(X^M)$ is a $h^*(X)$ − module.

Theorem 8.1: (Dold [Dy, p. 45]) $\Phi : h^*(X) \to h^*(X^M)$ given by $\Phi(x) = u_M \cdot \pi^*(x)$ is an isomorphism.

More generally, if E is any vector bundle, there is an associated Thom class $u \in h^*(X^E)$. We have,

Theorem 8.2(Dold) : $\Phi : h^*(X) \to h^*(X^E)$ given by $\Phi(x) = u \cdot \pi^*(x)$ is an isomorphism.

Let $\pi : E \to B$ be a fiber bundle with compact smooth f-dimensional fiber F associated to a G-principal bundle $P \to B$. As in chapter 2, let $i : F \to V$ be an embedding with normal bundle N. Let $c : S^V \to F^N$ be the associated Pontryagin-Thom collapsing map. Let ξ be the vector bundle over B associated to the representation V and let $\bar{\xi}$ be its inverse bundle. The Pontryagin-Thom map induces a map

$$1 \times_G c : P \times_G S^V \to P \times_G F^N$$

and hence a map,

$$(1 \times_G c) \wedge 1 : P \times_G S^V \wedge_B \bar{\xi} \to P \times_G F^N \times_B \bar{\xi}.$$

Identifying B to a point, we get a map

$$\tilde{t} : B^{\xi \oplus \bar{\xi}} = \Sigma^s(B^+) \to E^{N \oplus p^{-1}(\bar{\xi})}.$$

Now, suppose $T(F)$, the tangent bundle to F, is a complex vector bundle. Then its inverse bundle

$$N \oplus p^{-1}(\bar{\xi}) = \beta$$

may be chosen to have a complex structure so it has a Thom class $U \in h^{s-f}(E^\beta)$ and an associated Thom isomorphism

$$\Phi : h^k(E^+) \to h^{k+s-f}(E^\beta).$$

Let

$$\sigma : h^i(\Sigma^s B^+) \to h^{i-s}(B^+)$$

be the suspension isomorphism.

Definition 8.3: The *umkehr* homomorphism

$$\pi_* = \sigma \circ c^* \circ \Phi : h^k(E^+) \to h^{k+s-f}(E^{N \oplus p^{-1}(\bar{\xi})}) \to h^{k+s-f}(\Sigma^s B^+) \to h^{k-f}(B^+).$$

Since the tangent bundle $T(F)$ has a complex structure so does T_π. Hence, in our complex orientable theory h^*, T_π has an Euler class,

$$\chi(T_\pi) = c_n(T_\pi).$$

Theorem 8.4: (BG, Thm 4.3) The transfer

$$\tau(\pi)^* : h^k(E^+) \to h^k(B^+)$$

is given by

$$\tau(\pi)^*(x) = \pi_*(x \cdot \chi(T_\pi)).$$

Proof: The following diagram is commutative.

$$\begin{array}{ccccc} h^k(E^+) & \xrightarrow{\cdot\chi(T_\pi)} & h^{k+f}(E^+) & \xrightarrow{\pi_*} & h^k(B^+) \\ \sigma\downarrow & & \downarrow\Phi & & \uparrow\sigma \\ h^{k+n}(\Sigma^n E^+) & \xrightarrow{i^*} & h^{k+n}(E^\beta) & \xrightarrow{c^*} & h^{k+n}(\Sigma^n B^+) \end{array}$$

Here the map i^* is induced from the inclusion of β into the trivial rank n vector bundle. The commutativity of the left hand square follows from the naturality properties of the Thom isomorphism. The commutativity of the right hand square is just the definition of π^*. The lower line of the diagram is the transfer. The result follows by naturality of the suspension isomorphism.

The map π_* satisfies the universal properties for push-forwards described in chapter 7 .

(1): If the following is a Cartesian square of fiber bundles

$$\begin{array}{ccc} E & \xrightarrow{\theta} & E' \\ \pi\downarrow & & \downarrow\pi' \\ B & \xrightarrow{\tilde{\theta}} & B' \end{array}$$

then

$$\tilde{\theta}^* \circ {\pi'}_* = \pi_* \circ \theta^*.$$

(2):functoriality.

As a consequence of (1), we have

Proposition 8.5 :

$$\theta^* \circ \pi'^* \circ \pi'_* = \pi^* \circ \pi_* \circ \theta^*.$$

Thus, θ^* is an intertwining operator between $\pi^* \circ \pi_*$ and $\pi'^* \circ \pi'_*$.

Chapter 9

The Weyl group invariants on the cohomology of the classifying space

By the results of chapters 3 and 4, we know that rationally the generalized cohomology of the classifying space of a compact connected Lie group is given by the Weyl group invariants in the cohomology of the classifying space of a maximal torus. This fact makes the Weyl group action on the cohomology of the classifying space of a maximal torus very important. In this chapter we study that action. The main result is that this action is equivalent to the action on the ordinary cohomology. Most of these results will be proved under the assumption that $\mid W \mid$ is invertible in our cohomology theory h^*.

Let W be the Weyl group and let

$$\hat{h}^* = h^*(\frac{1}{\mid W \mid}).$$

The W-action on BT is induced from the W-action on T and on $R(T)$. Let e^{λ} be a one-dimensional representation of T and let $L(\lambda) \rightarrow BT$ be the corresponding line bundle. Let $\chi(\lambda) = c_1(L(\lambda))$, where c_1 denotes the h^* Chern class. Then w induces an automorphism ϕ_w of BT, so

$$\phi_w^* \chi(\lambda) = \chi(\phi_w^{-1} L(\lambda))$$

by naturality. Since

$$\phi_w^{-1} L(\lambda) = L(w\lambda), \phi_w^* \chi(\lambda) = \chi(w(\lambda)).$$

Let s_α be a reflection. Then

$$s_\alpha(\lambda) = \lambda - < \lambda, \alpha > \alpha,$$

where $< \lambda, \alpha > \in \mathbf{Z}$ is the Cartan pairing

$$\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$$

Thus,

$\chi(s_\alpha(\lambda)) = \chi(\lambda - <\lambda, \alpha> \alpha) = \chi(\lambda) - <\lambda, \alpha> \chi(\alpha) +$ higher order terms

because the formal group law is of the form

$$F(X,Y) = X + Y + \text{ higher order terms.}$$

If we filter $\hat{h}^*(BT)$ as in the Atiyah-Hirzebruch spectral sequence, then the s_α-action on $Gr\hat{h}^*(BT)$ is identical with the s_α-action on

$$H^*(BT, \hat{h}^*(pt)).$$

Since the Weyl group is generated by reflections, the same statement is true for the Weyl group. Of course, this fact follows from generalities about the Atiyah-Hirzebruch spectral sequence but it seems more instructive to give an explicit proof.

Proposition 9.1 : Assume h^* has no 2-torsion. Then the Weyl group acts faithfully on $h^*(BT)$.
Proof: It is well-known that W acts faithfully on $H^*(BT, \mathbf{Z}_p)$ for $p \neq 2$. Thus, W acts faithfully on $Grh^*(BT)$ so it acts faithfully on $h^*(BT)$.

Theorem 9.2 :

$$\hat{h}^*(BT) \cong H^*(BT, \hat{h}^*(pt))$$

as W-modules and as $\hat{h}^*(pt)$ – modules.
Proof: The E_2 term of the Atiyah-Hirzebruch spectral sequence is

$$H^*(BT, \hat{h}^*(pt)).$$

The spectral sequence degenerates at the E_2 term so

$$gr\hat{h}^{p+q}(BT) \cong H^p(BT, \hat{h}^q(pt))$$

where the grading is on the p-index. Since BT is simply connected,

$$H^*(BT, \hat{h}^*(pt)) \cong H^*(BT) \otimes \hat{h}^*(pt).$$

Moreover, the same considerations are valid for $X_n = \prod_{i=1}^{l} \mathbf{C}P(n)$ where l=dim T.

First we construct a linear map θ from

$$H^*(X_n, \hat{h}^*(pt)) \to \hat{h}^*(X_n).$$

Choose an algebraically independent set of ring generators $e^{\lambda_1}, \cdots, e^{\lambda_l}$ for $R(T)$. Let $x_i = c_1{}^H(L(\lambda))$, the Chern class in ordinary cohomology. Then the x_i are ring generators for $H^*(X_n, \hat{h}^*(pt))$ over $\hat{h}^*(pt)$. Define $\tilde{\theta}(x_i) = \chi(\lambda_i)$. Extend $\tilde{\theta}$ linearly and multiplicatively to $H^*(X_n, \hat{h}^*(pt))$. Thus, for example, $\tilde{\theta}(-x_i) = -\chi(\lambda_i)$. Define θ by averaging over the Weyl group so

$$\theta(x) = \frac{1}{\mid W \mid} \sum_{w \in W} \pi_h(w)\tilde{\theta}(\pi_H(w^{-1})x).$$

Here π_H denotes the representation of W on

$$H^*(X_n, \hat{h}^*(pt))$$

and π_h denotes the representation of W on $h^*(X_n)$. By construction, θ is W-equivariant. Since the representation of W on $Gr\hat{h}^*(X_n)$ is isomorphic with the representation of W on

$$H^*(X_n, \hat{h}^*(pt)),$$

$\theta(x) = x +$ higher order terms.

$$H^*(X_n, \hat{h}^*(pt))$$

and $h^*(X_n)$ are finitely generated $\hat{h}^*(pt)-$modules. Thus. as a finite matrix, θ is upper triangular with 1's on the diagonal and coefficients in $\hat{h}^*(pt)$. Hence it has an inverse θ^{-1} which is automatically W-equivariant. Denote the homomorphism θ by $\theta(n)$. Then $\theta(n)$ respects inclusions $i : X_n \to X_m$ for $m \geq n$. That is, $\theta(n) \circ i^* = i^* \circ \theta(m)$. Thus, $\theta(n)$ and its inverse $\theta(n)^{-1}$ can be extended to

$$\theta : H^*(BT, \hat{h}^*(pt)) \to \hat{h}^*(BT)$$

and

$$\theta^{-1} : \hat{h}^*(BT) \to H^*(BT, \hat{h}^*(pt))$$

and are still W-equivariant. The result follows.

Chapter 10

Nonvanishing of some characteristic classes

Suppose $L(\alpha)$ is the line bundle on BT associated to a character e^α where α is a root. We want to determine when its characteristic classes are not zero divisors. This will be important later when we seek to invert characteristic classes. The main difficulty is that the characters e^α do not usually generate $R(T)$.

Let

$$\mathbf{g} = Lie(G) \otimes_{\mathbf{R}} \mathbf{C}$$

be the complexified Lie algebra of G so

$$\mathbf{t} = Lie(T) \otimes_{\mathbf{R}} \mathbf{C}$$

is a Cartan subalgebra of $\mathbf{g}$. We begin by introducing a particular set of generators for $R(T)$. Let R be the set of roots and let R^+ be a system of positive roots with corresponding simple roots $\Delta = (\alpha_1, \cdots, \alpha_l)$. For each simple root α_i, there is a corresponding subgroup

$$S_i = PSU(2) \text{ or } SU(2) \subset G$$

with $Lie(S_i) \otimes_{\mathbf{R}} \mathbf{C}$ having roots α_i and $-\alpha_i$. Let $T_i \subset T$ be a maximal torus of this S_i. Let e^{λ_i} be a generator of $R(T_i)$ such that $e^{\alpha_i} = e^{\lambda_i}$ or $e^{2\lambda_i}$ on T_i. Extend e^{λ_i} to $R(T)$ by setting $e^{\lambda_i} = 1$ on T_j for $i \neq j$. Then the e^{λ_i} are a set of generators for $R(T)$. As a consequence, Proposition 5.2 asserts that

$$h^*(BT) \cong h^*(pt)[[\chi(\lambda_i), \cdots, \chi(\lambda_l)]].$$

Thus, the $\chi(\lambda_i)$ are not zero-divisors in $h^*(BT)$. Suppose

$$e^\lambda \in R(T).$$

Then in $\mathbf{t}^*$,

$$\lambda = \sum_i n_i \lambda_i.$$

Thus,

$$\chi(\lambda) = \sum_i n_i \chi(\lambda_i) + \text{higher order terms}$$

If some n_i is not a zero-divisor in $h^*(pt)$, then $\chi(\lambda)$ is not a zero-divisor in $h^*(BT)$. Indeed, $\chi(\lambda)$ is then a power series with first term not a zero-divisor.

We want to apply this result to the case where λ is a root. Each

$$\alpha_i = \sum_i n_i \lambda_i, n_i \in \mathbf{Z}.$$

Proposition 10.1: If $p \geq 3$ is a prime, there is some n_i such that p does not divide n_i.
Proof: This follows from the classification of complex semisimple Lie algebras. If p divides n_i for all i, then p divides all entries in the corresponding row of the Cartan matrix. By examination of Cartan matrices, one sees that $p = 2$.

Corollary 10.2: If $h^*(pt)$ has no 2-torsion, then $\chi(\alpha_i)$ is not a zero-divisor for any simple root α_i.

Proposition 10.3: If $h^*(pt)$ has no 2-torsion, $\chi(\alpha)$ is not a zero-divisor for any root α.
Proof: Every root α is Weyl group conjugate to a simple root α_i. Since the Weyl group acts by automorphism on $h^*(BT)$ and $\chi(\alpha)$ is W-conjugate to $\chi(\alpha_i)$, the result follows from the above corollary.

Chapter 11

The formula for the *umkehr* homomorphism

In this section, we combine the Brumfiel-Madsen formula from chapter 4 with the results of the previous section to prove a formula for the *umkehr* homomorphism. In the next few sections, we will interpret this formula in various ways.

Again, G is a compact connected Lie group with maximal torus T. Suppose h^* is a complex-orientable cohomology theory in which $\chi(\alpha)$ is not a zero-divisor for any root α. Let

$$\pi : BT \to BG$$

be the fiber bundle with fiber G/T induced by the inclusion of T in G. To define the *umkehr* homomorphism

$$\pi_* : h^*(BT) \to h^*(BG)$$

we need a complex structure on G/T.

Since G is compact, it embeds into $GL(n, \mathbf{C})$ for some n. As before $\mathfrak{g}$ is the complexified Lie algebra of G. Let $G_{\mathbf{C}}$ denote the complex Lie group in $GL(n, \mathbf{C})$ which corresponds to $\mathfrak{g}$. Let B be a Borel subgroup of $G_{\mathbf{C}}$. Then $G_{\mathbf{C}}/B \cong G/T$. We may assume $T \subset B$. In this way, a choice of a Borel subgroup determines a complex structure on G/T[BH].

We want to compute the tangent bundle to the fibers T_π in terms of Lie theoretic data. Let R^+ be the positive root system determined by B, so $\alpha \in R^+$ means α is a weight of the action of $\mathfrak{t}$ on $Lie(B) = \mathfrak{b}$. We have the identification of $T(G_{\mathbf{C}}/B)$ with $G \times_T \mathfrak{g}/\mathfrak{b}$. As a T-representation space,

$$\mathfrak{g} = \mathfrak{b} \oplus (\oplus_{\alpha \in R^+} \mathbf{C}_{-\alpha})$$

where $\mathbf{C}_{-\alpha}$ denotes the character of T defined by the root $-\alpha$. As a consequence,

$$\mathfrak{g}/\mathfrak{b} \cong \oplus_{\alpha \in R^+} \mathbf{C}_{-\alpha},$$

Then the tangent bundle along the fibers is

$$T_\pi = EG \times_T \mathfrak{g}/\mathfrak{b} \cong EG \times_T \oplus_{\alpha \in R^+} \mathbf{C}_{-\alpha} = \oplus L(-\alpha),$$

where $L(-\alpha)$ is the line bundle associated to the root $-\alpha$. In any complex-orientable theory h^*,

$$\chi(T_\pi) = \prod_{\alpha \in R^+} \chi(-\alpha) \tag{11.1}$$

Now assume h^* has coefficients with no 2-torsion. Theorem 8.4 asserts that for $x \in h^*(BT)$,

$$\pi^* \circ \tau(\pi)^*(x) = \pi^* \circ \pi_*(\chi(T_\pi) \cdot x)$$

By 10.3, $\chi(T_\pi)$ is a product of non-zero-divisors in $h^*(BT)$, so we may formally invert it in a localization

$$Q = h^*(BT)[\frac{1}{\chi(T_\pi)}],$$

with $h^*(BT) \hookrightarrow Q$. Thus, using the Brumfiel-Madsen formula 4.4, we have the identity in Q,

$$\pi^* \circ \pi_*(x) = \pi^* \circ \tau(\pi)^* \frac{1}{\chi(T_\pi)} x = \sum_{w \in W} w \frac{1}{\prod_{\alpha \in R^+} \chi(-\alpha)} x$$

As written, this is an identity on Q but since the left hand side preserves the subring $h^*(BT)$, it may be regarded as an identity on $h^*(BT)$.

Theorem 11.2 : (Bressler-Evens) Suppose h^* has no 2-torsion, then for $x \in h^*(BT)$ we have the formula

$$\pi^* \circ \pi_*(x) = \sum_{w \in W} w \frac{1}{\prod \chi(-\alpha)} x$$

Remark 11.3 : The only place where we used the absence of 2-torsion was when we asserted that $\chi(\alpha)$ was not a zero-divisor. If we have some other reason for knowing $\chi(\alpha)$ is not a zero-divisor, then the result is still true. For example, if the formal group law of h^* is of the form

$$X + Y + a_{11}XY + \cdots$$

and a_{11} is a unit, then for $G = SU(2)$, $\chi(\alpha)$ is not a zero-divisor. In any case when $G = SO(3)$, $\chi(\alpha)$ is not a zero-divisor because the associated character generates $R(T)$.

Chapter 12

The Weyl character formula and Verma modules

In this section we interpret Theorem 11.2 for the case of complex K-theory. As a first application, we show that 11.2 is just the Weyl character formula. Then we will interpret a part of the formula as a topological construction of a Verma module. A modification of the Brumfiel-Madsen formula will then give irreducible quotients of Verma modules associated to nonsingular Schubert varieties.

We use the notation of the previous section. Also, for all applications in this section $K^1 = 0$ so by Bott periodicity we may regard $K^0 = K$. Using this convention, we have the identifications $K(BT) \cong \hat{R}(T)$ and $K(BG) \cong \hat{R}(G)$ (Theorem 7.2). In these identifications, a representation of a compact group is identified with the associated vector bundle over the classifying space of the group. Thus, if e^λ is a representation of a torus T, then $[L(\lambda)]$ is the associated class in $K(BT)$. Using our convention that

$$\chi(L) = [1] - [L^*]$$

$$\chi(-\alpha_i) = 1 - e^{\alpha_i}$$

as an element of $R(T)$. With this identification, 11.2 reads

$$\pi^* \circ \pi_*(x) = \sum_{w \in W} w \frac{1}{\prod(1 - e^\alpha)} x.$$

We apply this formula to the class of the line bundle $[L(\lambda)] = e^\lambda$. Thus we have,

$$\pi^* \circ \pi_*(e^\lambda) = \sum_{w \in W} w \frac{1}{\prod_{\alpha \in R^+}(1 - e^\alpha)} e^\lambda.$$

The action of W induced on $K(BT)$ agrees with its action on roots. Hence, we may compute as follows. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

Let $l(w)$ be the length of w, which is equal to the number of positive roots whose sign is changed by w. Then,

$$\sum_{w\in W} w\frac{1}{\prod_{\alpha\in R^+}(1-e^{\alpha})}e^{\lambda} = \sum_{w\in W} w\frac{1}{\prod_{\alpha\in R^+}(e^{-\alpha/2}-e^{\alpha/2})}e^{\lambda-\rho} =$$

$$\frac{1}{\prod_{\alpha\in R^+}(e^{-\alpha/2}-e^{\alpha/2})}\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda-\rho)},$$

which is the ordinary form of the Weyl character formula [Hum, p. 139].

Of course, there is a theorem attached to the above formula which asserts that the right hand side is the restriction to T of a finite dimensional irreducible representation. The proof of this theorem will require some additional results. We need to interpret $\pi^* \circ \pi_*$ in another way. First, observe that

$$\pi_* : K(BT) \to K(BG)$$

can be interpreted as a map

$$\pi_* : \hat{R}(T) \to \hat{R}(G).$$

In this guise, π_* is called holomorphic induction. π^* is naturally identified as the restriction map $\hat{R}(G) \to \hat{R}(T)$. We can interpret π_* as follows. Let $L(\lambda)$ (resp. $\mathcal{L}(\lambda)$ be the line bundle over BT (resp. G/T) associated to a character of T. Let $H^i(G/T, \mathcal{L}(\lambda))$ be the Cěch cohomology groups defined with respect to holomorphic sections of $\mathcal{L}(\lambda)$. These cohomology groups are G-representations since $\mathcal{L}(\lambda)$ is G-equivariant, and finite dimensional since G/T is Kaehler. Let $\mathcal{H}^i$ be the associated vector bundle over BG. Then define,

$$\pi_!([L(\lambda)]) = \sum(-1)^i\mathcal{H}^i.$$

The Atiyah-Hirzebruch version of the Grothendieck-Riemann-Roch theorem would asserts that $\pi_* = \pi_!$ for the case where BT and BG are replaced by finite approximations [AH2]. The generalization of the above result to the classifying spaces if fairly easy, but lacking a reference, we will give it.

Lemma : $\pi_* = \pi_!$.

Proof: Let X_n give a finite approximation of BT and let Y_n give a finite approximation of BG, so that $X_n \to Y_n$ is a G/T fiber bundle. Then there is a Cartesian square

$$\begin{array}{ccc} X_n & \longrightarrow & BT \\ p_n \downarrow & & \downarrow \pi \\ Y_n & \longrightarrow & BG \end{array}$$

which means the induced diagram commutes

$$\begin{array}{ccc} K(BT) & \longrightarrow & K(X_n) \\ \pi_* \downarrow & & \downarrow p_{n*} \\ K(BG) & \longrightarrow & K(Y_n) \end{array}$$

Applying limits and using the fact that $K(BG) = \lim K(Y_n)$ for any compact group G, the following diagram commutes.

$$\begin{array}{ccc} K(BT) & \xrightarrow{\cong} & K(BT) \\ \pi_* \downarrow & & \downarrow \lim p_{n*} \\ K(BG) & \xrightarrow{\cong} & K(BG) \end{array}$$

By the similar reasoning, the following diagram commutes.

$$\begin{array}{ccc} R(T) & \longrightarrow & K(BT) \\ \pi_! \downarrow & & \downarrow \lim p_{n*} \\ R(G) & \longrightarrow & K(BG) \end{array}$$

The lemma follows.

Returning to the Weyl Character formula, by the Borel-Weil-Bott theorem, all but one cohomology group is nonzero [Bott]. Moreover, if λ is anti-dominant with respect to R^+, then $\mathcal{H}^0$ is the irreducible representation $V(\lambda)$ of lowest weight λ. Let $chV(\lambda)$ denote its decomposition as a representation of T. Thus, we have

Theorem 12.1 (Weyl character formula): The representation of T given by the finite dimensional G-representation $V(\lambda)$ of lowest weight λ, is given by the formula,

$$chV(\lambda) = \sum_{w \in W} w \frac{1}{\prod_{\alpha \in R^+}(1 - e^{\alpha})} e^{\lambda}.$$

There are several points worth making about this argument. First, it uses two relatively difficult facts : the identification of π_* with $\pi_!$, and the Borel-Weil-Bott theorem. The logic in the argument is to some extent reversible.

That is, given our results computing $\pi^* \circ \pi_*$ and the vanishing part of the Borel-Weil-Bott theorem, we have the following result.

Theorem 12.2 : Any 2 of the following imply the third.

$$(1) ch(V(\lambda)) = \sum_{w \in W} w \frac{1}{\prod_{\alpha \in R^+}(1 - e^\alpha)} e^\lambda$$

$$(2) \pi_* : K(BT) \to K(BG) \text{ and } \pi_! : K(BT) \to K(BG) \text{ coincide.}$$

$$(3) H^0(G_{\mathbf{C}}/B, L(\lambda)) = V(\lambda).$$

Proof: We have already shown that (2) and (3) imply (1). Now suppose we are given (1) and (3). Then computing $\pi^* \circ \pi_*(e^\lambda)$ we get $chV(\lambda)$ by (1) and Theorem 12.1. By (3), computing $\pi^* \circ \pi_!(e^\lambda)$ we get $chV(\lambda)$, since applying π^* amounts to taking the character. Therefore,

$$\pi^* \circ \pi_* = \pi^* \circ \pi_!$$

on line bundles over BT. Since $K(BT)$ is additively generated by line bundles, they are equal on $K(BT)$. But π^* is injective (Corollary 4.7), so $\pi_* = \pi_!$. Hence, we have (2). The fact that (1) and (2) imply (3) follows in a similar way.

The consequence that (1) and (2) imply (3) is not so interesting since the Borel-Weil theorem admits several much more illuminating proofs [Se] [BB]. However, (2) is a fairly difficult analytic fact and is in fact reasonably far-reaching. Indeed, suppose X is any space with a free G-action. Consider the fiber bundle $X/T \to X/G$. Then we have a classifying map $\theta : X/T \to BT$ and an induced map

$$\theta^* : K(BT) \to K(X/T).$$

Then the diagram,

$$\begin{array}{ccc} X/T & \xrightarrow{\theta} & BT \\ p \downarrow & & \downarrow \pi \\ X/G & \xrightarrow{\theta_G} & BG \end{array}$$

is a pullback diagram. By base change, it follows that

$$\theta^* \circ \pi^* \circ \pi_* = p^* \circ \theta_G^* \circ \pi_* = p^* \circ p_* \circ \theta^*.$$

But the left-hand side of the above equation is

$$\theta^* \circ \pi^* \pi_!.$$

If X/T and X/G are complex projective manifolds (or lie in some other category of spaces admitting base change), the right-hand side is

$$p^* \circ p_! \circ \theta^*.$$

Therefore, if θ^* is surjective,

$$p^* \circ p_* = p^* \circ p_!.$$

If we assume further that $\mid W \mid$ is invertible in $K(X/G)$, then $p_* = p_!$ since p^* is injective (4.6). This fact for general projective complex manifolds requires some fairly difficult analysis. It seems useful to have a fairly simple proof of this fact for the special case described above. Note that fibrations of the form $G/T \to G/H$, where T is a subgroup of H, satisfy the hypotheses, i.e. they are complex projective manifolds [Se] and the classifying map is surjective [St]. Moreover, the same result is true for maps of the form $G/H \to G/K$, with $T \subset H \subset K$.

Of course, the Weyl character formula is not a difficult theorem to prove by standard algebraic machinery. However, the above proof has the virtue that the terms have some intrinsic meaning. This meaning is very suggestive for representation theory. Indeed, the Weyl character formula admits the following interpretation. Let $U(\mathfrak{g})$ and $U(\mathfrak{b})$ denote the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{b}$ respectively. Let $\mathfrak{b}_-$ denote the opposite Borel subalgebra to $\mathfrak{b}$. As before, $\mathfrak{t} \subset \mathfrak{b}_-$ is a Cartan subalgebra of $\mathfrak{g}$. If we write

$$\frac{1}{(1-e^\alpha)} \text{ as } (1 + e^\alpha + e^{2\alpha} + \cdots),$$

the expression

$$\frac{1}{\prod_{\alpha \in R^+}(1-e^\alpha)} e^\lambda$$

is the formal character of the Verma module $M(\lambda)$ of lowest weight λ [Hum, p. 136]. Thus, the formal character $chM(\lambda)$ is given by dividing the class of the line bundle $[L(\lambda)]$ by the Euler class of the tangent bundle to $G_{\mathbf{C}}/B$. $\mathfrak{b}_-$ decomposes as a Lie algebra direct sum $\mathfrak{b}_- = \mathfrak{t} \oplus \mathfrak{n}_-$ as a Lie algebra direct sum, where $\mathfrak{n}_-$ is the nilradical of $\mathfrak{b}_-$. Thus a linear functional λ of $\mathfrak{t}$ can be extended to $\mathfrak{b}_-$ by setting λ trivial on $\mathfrak{n}_-$. Call the associated one-dimensional representation space $\mathbf{C}_\lambda$. $M(\lambda)$ is defined as the induced Lie algebra representation

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_-)} \mathbf{C}_\lambda.$$

Thus, as others have no doubt noticed, $chV(\lambda)$ is obtained by formally averaging $chM(\lambda)$ when λ is anti-dominant. If λ is not anti-dominant, we still get a virtual finite dimensional representation by averaging, which is actually an irreducible representation up to a factor of -1.

Readers familiar with representation theory will realize that Verma modules are usually highest weight modules rather than lowest weight modules. We are taking Verma modules to be lowest weight modules in order to identify a topological construction with a sheaf-theoretic construction. Our treatment is of course completely equivalent to the standard version.

We want to give a topological interpretation to $chM(\lambda)$ and to certain associated objects. None of the results discussed here are new results in representation theory. They only give topological realizations to familiar objects. Again, we let $\pi : BT \to BG$ be the fiber bundle with fiber $G_{\mathbf{C}}/B$ of real dimension n. Let $\chi(T_\pi)$ be the Euler class of the tangent bundle to the fiber. Let β be a $s-n$-dimensional complementary vector bundle over BT with associated Thom isomorphism Φ. The relation between π_* and $\tau(\pi)^*$ was derived from the commutativity of the following diagram.

$$\begin{array}{ccc} h^k(BT^+) & \xrightarrow{\cdot\chi(T_\pi)} & h^{k+n}(BT^+) \\ \sigma\downarrow & & \downarrow\Phi \\ h^{k+s}(\Sigma^s(BT^+)) & \xrightarrow{i^*} & h^{k+s}(BT^\beta) \end{array}$$

We want to apply this diagram to K-theory. We may assume all dimensions are even since n is even, so we work with $K = K^0$. Then the above diagram reads,

$$\begin{array}{ccc} K(BT^+) & \xrightarrow{\cdot\chi(T_\pi)} & K(BT^+ \\ \sigma\downarrow & & \downarrow\Phi \\ K(\Sigma^s(BT^+) & \xrightarrow{i^*} & K(BT^\beta \end{array}$$

For a vector bundle $\pi : E \to B$ with Thom class U we will write the

Thom isomorphism

$$h^*(B) \to h^*(B^E)$$
$$\text{as } x \to x \cdot U$$
$$\text{instead of as } \pi^*(x) \cdot U$$

in order to simplify notation. Let

$$\pi_{[s]} : BT \times \mathbf{R}^s = T_\pi \oplus \beta \to BT$$

and

$$\pi_\beta : \beta \to BT$$

be the associated vector bundles with respective Thom classes

$$\Theta \in K(\Sigma^s(BT^+))$$

and

$$U \in K(BT^\beta).$$

Then for $x \in K(BT^+)$, the suspension isomorphism σ is

$$\sigma(x) = x \cdot \Theta.$$

Moreover, Θ is invariant under any automorphism of BT, in particular under the Weyl group. The inclusion $\beta \to \mathbf{R}^s$ induces the inclusion

$$i : BT^\beta \to \Sigma^s(BT^+).$$

Thus, in K-theory, we have an induced map

$$i^* : \sigma(x) \mapsto U \cdot \chi(T_\pi) \cdot x.$$

$\chi(T_\pi)$ is not a zero-divisor in K-theory so adjoining its inverse, we have a map

$$i^* : \sigma(\frac{x}{\chi(T_\pi)}) \mapsto U \cdot x.$$

Thus, multiplication by U is equivariantly division by $\chi(T_\pi)$. But $\chi(T_\pi) = \prod_{\alpha \in R^+}(1 - e^\alpha)$ so the Thom isomorphism $\Phi : K(BT) \to K(BT^\beta)$ is

$$\Phi(x) = U \cdot x = i^*\sigma(\frac{1}{\prod_{\alpha \in R^+}(1 - e^\alpha)}x)$$

Thus, we have

Theorem 12.4 :

$$\Phi(e^\lambda) = i^*\sigma(chM(\lambda)),$$

where e^λ denotes the line bundle on BT corresponding to the character e^λ of T.

In the sequel we will regard $chM(\lambda)$ as an element of $K(BT^\beta)$ via the above map.

Recall that a Verma module has a central character. That is, if $z \in Z(\mathbf{g})$, the center of the universal enveloping algebra, then if $v \in M(\lambda)$,

$$z \cdot v = \chi_\lambda(z) \cdot v,$$

where

$$\chi_\lambda : Z(\mathbf{g}) \to \mathbf{C}$$

is an algebra homomorphism independent of v. Moreover, we have

$$\chi_\lambda = \chi_{w(\lambda-\rho)+\rho}$$

for $w \in W$ [Hum, p. 130] Regarded as elements of $K(BT^\beta)$, characters of Verma modules admit a Weyl group action.

Proposition 12.5 :

$$wchM(\lambda) = (-1)^{l(w)}chM(w(\lambda - \rho) + \rho).$$

Proof:

$$w \cdot \frac{1}{\prod_{\alpha\in R^+}(1-e^\alpha)}e^\lambda = w \cdot \frac{1}{\prod_{\alpha\in R^+}(e^{-\alpha/2}-e^{\alpha/2})}e^{\lambda-\rho} =$$

$$(-1)^{l(w)}\frac{1}{\prod_{\alpha\in R^+}(e^{-\alpha/2}-e^{\alpha/2})}e^{w(\lambda-\rho)} = (-1)^{l(w)}\frac{1}{\prod_{\alpha\in R^+}(1-e^\alpha)}e^{w(\lambda-\rho)+\rho}.$$

Corollary 12.6 : Weyl group orbits on $K(BT^\beta)$ correspond to Verma modules with the same infinitessimal character.

The motivation behind the previous result is that we have constructed the Verma modules (or at least their formal characters) without defining a $\mathbf{g}$ action on them. Thus, our construction of Verma modules is lacking a crucial structure. If $M(\lambda)$ and $M(\lambda')$ are Verma modules, there can be Lie algebra homomorphisms

$$\phi : M(\lambda) \to M(\lambda')$$

only if $\chi_\lambda = \chi_{\lambda'}$. Therefore, the category of modules with central character generated by Verma modules breaks up into a direct sum of subcategories. The corollary asserts that these subcategories correspond to Weyl group orbits. As a result, our construction can see some of the $\mathbf{g}-$ module structure.

Definition 12.7 : Let $P(\chi)$ denote the category of modules with central character χ generated by Verma modules $M(\chi)$ with $\chi_\lambda = \chi$.

We want to discuss irreducible objects in the category $P(\chi)$. We recall some basic facts. $M(\lambda)$ has a unique irreducible quotient, which we denote $L(\lambda)$. $L(\lambda)$ has the property that the weight λ occurs exactly once and every other weight λ' which occurs has

$$\lambda - \lambda' = \sum_{\alpha \in R^+} n_\alpha \alpha,$$

with $n_\alpha \leq 0$. Such modules are called lowest weight modules. $M(\lambda)$ has a composition series of finite length, and each composition factor is of the form $L(\mu)$ with $\chi_\mu = \chi$. Thus, we have the expression,

$$chM(\lambda) = \sum_{\mu : \chi_\mu = \chi_\lambda} a(\lambda, \mu) chL(\mu)$$

where $a(\lambda, \mu)$ is the number of composition factors of the form $L(\mu)$ which occur in $M(\lambda)$. Then $a(\lambda, \lambda) = 1$. Furthermore, if μ appears in the sum,

$$\mu = w(\lambda - \rho) + \rho.$$

There exists a unique $\mu = \mu_0$ such that

$$< \mu_0 - \rho, \alpha > \geq 0$$

for all $\alpha \in R^+$. Let

$$M(w) = M(w(\mu_0 - \rho) + \rho).$$

We can rewrite the above expression as

$$chM(w) = \sum_{y \in W} a(w,y) chL(y).$$

Assume further that $\lambda - \rho$ is regular, so

$$< \lambda - \rho, \alpha > \neq 0 \text{ for } \alpha \in R.$$

For example, it is usual to take

$$\mu_0 = 2\rho,$$

so $Z(\mathbf{g})$ acts trivially on $M(\mu_0) = M(e)$. Then $a(w,y) \neq 0$ if and only if $y \leq w$ in the Bruhat order [Vo, p. 95]. This fact, combined with the fact that $a(w,w) = 1$ enables us to invert the matrix $A = a(w,y)$ giving,

$$chL(w) = \sum_{y \leq w} (-1)^{l(y)-l(w)} b(w,y) chM(y),$$

with $b(w,y) \in \mathbf{Z}$.

We recall some statements about Schubert cells. The flag variety

$$G_{\mathbf{C}}/B = \cup_{w \in W} X_w$$

with the B-orbit $BwB = X_w$. Also X_w is isomorphic to $l(w)$ – dimensional complex affine space. The closures

$$\bar{X}_w = \cup_{y \leq w} X_y$$

are the Schubert cells. Since each X_y is a B-orbit it is in particular T-stable. Hence we can form the space

$$EG \times_T \bar{X}_w,$$

which is a fiber bundle over BT with fiber $\bar{X}_w$.

The proof of the Brumfiel-Madsen formula goes as follows [Fe1]. We have $\pi : BT \to BG$ and we want to compute $\pi^* \circ \tau(\pi)^*$. We have the Cartesian diagram,

$$\begin{array}{ccc} EG \times_T G/T & \xrightarrow{\pi_2} & BT \\ \pi_1 \downarrow & & \downarrow \pi \\ BT & \xrightarrow{\pi} & BG \end{array}$$

with

$$\pi_1(x,g) = x, \quad \pi_2(x,g) = xg,$$

for $x \in EG$, and $g = gT \in G/T$. Then by naturality of the transfer, we have

$$\pi^* \circ \tau(\pi)^* = \tau(\pi_1)^* \circ \pi_2{}^*.$$

The proof of the Brumfiel-Madsen formula is then a computation of $\tau(\pi_1)^*$. Now there is no way to fill in the lower right-hand diagonal in the above Cartesian square to produce an operator corresponding to an arbitrary Schubert variety. However, there is an obvious way to fill in the upper left-hand part of the square. We just replace $EG \times_T G/T$ with $EG \times_T \bar{X}_w$. The maps π_1 and π_2 are the same as with G/T. Thus, we can form the operator

$$\phi_w = \tau(\pi_1)^* \circ \pi_2{}^* \text{ on } h^*(BT).$$

Theorem 12.8 :$\phi_w = \sum_{y \leq w} y$.

Proof. This fact is essentially Theorem 1.4 in Mitchell-Priddy. First, we give $\bar{X}_w$ a finite bundle decomposition. This means,

$$\bar{X}_w = \sqcup X_i,$$

where each X_i is homeomorphic to an open disc bundle D_i over a finite complex F_i satisfying the following two conditions: (1) $D_i \rightarrow X_i$ extends to a continuous map of the closed disc bundle $\bar{D}_i$ into $\bar{X}_w$, (2) $\bar{X}_i - X_i = \bigcup X_j$, where the fiber dimension of X_j is less than the fiber dimension of X_i. For $\bar{X}_w$, $X_i = X_y$, the open Schubert cell. Of course, X_y is diffeomorphic to a vector bundle over a point. Indeed, let N be the unipotent radical of B and let N_- be the unipotent radical of the opposite Borel subgroup. Then $N_y = yN_-y^{-1} \bigcap N$ is a complex connected unipotent subgroup of real dimension $2l(y)$. The map $N_y \rightarrow X_y$ given by $n \rightarrow nyB$ is a diffeomorphism [BGG, Theorem 1.1]. Of course, any complex connected unipotent group is a vector space. Thus, conditions (1) and (2) are satisfied.

This bundle decomposition is T-equivariant. This means that the vector bundles X_y and the attaching maps are T-equivariant. To see that X_y is T-equivariant, we show that the T-action on X_y is equivalent to the T-action by conjugation on N_y, which is equivariantly diffeomorphic to the T-action on $Lie(N_y)$ via the exponential map. Indeed, the above identification maps

$$tnt^{-1} \to tnt^{-1}wB = tnw\tilde{t}B = tnwB.$$

There is a transfer τ_y for the vector bundle

$$EG \times_T X_y \to BT.$$

Then Mitchell and Priddy have proved,

Theorem 12.9 : Let $P \times_G F \to B$ be a fiber bundle where the fiber F has an equivariant bundle decomposition. Suppose the fiber F decomposes into complex vector bundles X_i with associated transfers τ_i. Then $\tau^* = \sum \tau_i{}^*$.

In our situation, the bundles X_y have a unique T-fixed point at the origin. By standard arguments about the transfer, τ_y is just the inclusion of the zero-section. In our situation, the zero-section corresponds to the map $x \to (x, yT), y \in W$. Thus,

$$\tau(\pi_1)^* \circ \pi_2{}^* = \sum_{y \leq w} y.$$

The operator ϕ_w can be lifted to an operator on $K(BT^\beta)$ via the Thom isomorphism.

Corollary 12.10: Suppose $\bar{X}_w$ is smooth. Then as an operator on $K(BT^\beta)$,

$$(-1)^{l(w)}\phi_w(chM(e)) = chL(w).$$

Proof: Proposition 12.5 asserts that

$$y(chM(e)) = (-1)^{l(y)}chM(y).$$

Thus,

$$(-1)^{l(w)}\phi_w = \sum_{y \leq w}(-1)^{l(w)-l(y)}chM(y).$$

By a famous result of Kazhdan and Lusztig, the integers $b(w, y) = 1$ when $\bar{X}_w$ is smooth [KL1].

So we have a formula for $\pi^* \circ \pi_*$, which when applied to K-theory gives the Weyl character formula. Moreover, a variation of $\pi^* \circ \pi_*$, still applied to K-theory, gives certain irreducible quotients of Verma modules. The obvious generalization is to apply these operations to other cohomology theories and to try to interpret the results. The main question here is to which cohomology classes one should apply these operators. In K-theory we applied them to $e^\lambda = [L(\lambda)]$. In an arbitrary theory we do not have an analogous notion of a class associated to a line bundle. The notion of a characteristic class is not a good generalization, since applying our operator to it does not give a representation. The philosophical point here is that the fundamental objects of study in representation theory are not really natural from the point of view of algebraic topology. Representations give a basis in $K(BT)$, but a particular basis is not a natural object to generalize to other cohomology theories. There are several approaches which might be useful in trying to interpret the Weyl character formula in other cohomology theories. First, many of these other cohomology theories have geometric interpretations. For example, elliptic cohomology may be interpreted as a character-valued index for a certain operator associated to a spin structure [Wi]. Thus, the meaning of our Weyl character formula should have some importance in this context. Second, Adams operations provide a way of picking out the actual vector bundles from the virtual vector bundles in K-theory. An Adams operation is just an example of a cohomology operation in a generalized cohomology theory. Thus, it should be interesting to see if cohomology operations can provide a way of picking out distinguished classes in generalized cohomology theories which play the same role as vector bundles in K-theory. I hope to do some later work on these topics.

Chapter 13

Braid relations and induction

In this section, we discuss some work of Bressler and myself and reinterpret in terms of induction. The basic idea is to introduce an operator on $h^*(BT)$ for each simple root. Our main theorem is that the operators attached to pairs of simple roots satisfy the same relations as the corresponding simple reflections in the Weyl group if and only if the formal group law of the cohomology theory is that of ordinary cohomology or of K-theory.

Again let $T \subset G$ be a maximal torus of a compact Lie group. The braid relations are defined as follows. The Weyl group is a Coxeter group. As such, it is a free group on l generators $s_1, \cdots, s_l$ modulo certain relations. First,

$$s_i^2 = 1.$$

We will for the most part ignore these relations. Second, for $i \neq j$ there exist integers m_{ij} such that

$$(s_i s_j)^{m_{ij}} = 1.$$

Therefore,

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots, \text{ (m}_{\text{ij}}\text{ factors on each side).}$$

These are called the braid relations because for the special case when the Weyl group is the symmetric group, they give the defining relations for Artin's braid group. It is well-known that the possible values for m_{ij} are $2, 3, 4,$ and 6.

Definition 13.1 : Let $D_i, i = 1, \cdots l$ be linear operators associated to each simple root α_i. Then we say the D_i satisfy braid relations if

$$D_i D_j D_i \cdots = D_j D_i D_j \cdots \text{ (m}_{\text{ij}}\text{ factors on each side)}$$

for all pairs i and j.

Remark 13.2 :If R is an algebra generated by operators D_i which satisfy braid relations we can define an operator D_w for $w \in W$ as follows[Bour, Prop. 5, p. 16]. We take a minimal decomposition of $w = s_{i_1} \cdots s_{i_k}$ and then we set $D_w = D_{i_1} \cdots D_{i_k}$. By a theorem of Matsumoto, D_w does not depend on the choice of a minimal decompostion.

We define operators D_i on $h^*(BT)$ as follows. Any simple root α_i (in fact any root) defines a subgroup

$$H_i \cong SU(2) \times T \subset G.$$

Lie$H_i \otimes_{\mathbf{R}} \mathbf{C} = \mathsf{h}_i$ has as its root spaces $\mathbf{g}_{\alpha_i}$ and $\mathbf{g}_{-\alpha_i}$ [Hum, p. 37]. The induced fiber bundle $\pi_i : BT \to BH_i$ has fiber $H_i/T \cong \mathbf{C}P(1)$. Then for any complex oriented cohomology theory h^*, we have operators $\pi_i{}^* \circ \pi_{i_*}$ on $h^*(BT)$.

Definition 13.3 : $D_i = \pi_i{}^* \circ \pi_{i_*}$.

Operators of this form were first introduced by Bernstein, Gel'fand, and Gel'fand on the cohomology of the flag variety [BGG] and by Demazure on K-theory and on the Chow ring of the flag variety [De]. They were used to prove statements about the ring structures of these objects. Later, Kač and Peterson gave the purely topological description described above [KP]. Kostant and Kumar extended many of the results of BGG and Demazure to the Kač-Moody situation [KK1,KK2]. I should also mention the work of Arabia [Ar] and Gutkin-Slodowy [GS] on related questions. Kač has used these operators to prove some striking theorems on torsion in the cohomology of Lie groups [Kač]. Perhaps most importantly, Kazhdan and Lusztig have used a mild generalization of this construction to give a topological construction of Hecke algebras [Lu] and to prove the Deligne-Langlands conjecture [KL2,KL3].

Proposition 13.4 :

$$D_i(x) = (1 + s_i)\frac{1}{\chi(-\alpha_i)}x.$$

Proof: This is just Theorem 11.2.

There is an isomorphism $\theta : \mathbf{t}^* \to H^2(BT, \mathbf{C})$ given by $\lambda \to \chi(\lambda)$. θ extends to an inclusion of the symmetric algebra $S(\mathbf{t}^*)$ into $H^*(BT, \mathbf{C})$. Given this identification the previous theorem asserts that

$$D_i = (1 + s_i)\frac{1}{-\alpha_i}$$

Propositon 13.5 :

$$D_i^2(x) = D_i(1)D_i(x) = [\frac{1}{\chi(-\alpha_i)} + \frac{1}{\chi(\alpha_i)}]D_i(x).$$

Proof: There is a projection formula

$$\pi_{i*}(\pi_i{}^*(y) \cdot z) = y \cdot \pi_{i*}(z) \text{ [Dy, p.54]}.$$

Let $y = \pi_{i*}(x)$ and let $z = 1$. Then we have,

$$\pi_i{}^*\pi_{i*}\pi_i{}^*\pi_{i*}(x) = \pi_i{}^*\pi_{i*}(x) \cdot \pi_i{}^*\pi_{i*}(1) = D_i(x)D_i(1).$$

But,

$$D_i(1) = (1 + s_i)\frac{1}{\chi(-\alpha_i)} = \frac{1}{\chi(-\alpha_i)} + \frac{1}{\chi(\alpha_i)}.$$

Corollary 13.6 :Let L^* be the dual bundle to L. If $\chi(L^*) = -\chi(L)$, then $D_i^2 = 0$.

As a consequence, $D_i^2 = 0$ in ordinary cohomology and in elliptic cohomology. For ordinary cohomology this relation is well-known. In K-theory the formal group law implies

$$\chi(-\alpha_i) = \frac{-\chi(\alpha_i)}{\omega \cdot \chi(\alpha_i) + 1}$$

where ω is the Bott class. Therefore, $D_i^2 = \omega D_i$. If we use the periodicity isomorphism, we get $D_i^2 = D_i$. This is the well-known formula for the Demazure operator [Lu]. Thus, we can recover some of the standard facts about these operators solely in terms of a formal group laws.

Suppose h^* is a complex oriented theory satisfying the assumptions of 11.2. Consider the extension

$$Q = h^*(BT)[\frac{1}{\chi(-\alpha_i)}, \frac{1}{\chi(\alpha_i)}].$$

Proposition 13.7 : The elements $w \in W$ are linearly independent over Q as operators on Q.

Proof: If Q were an integral domain, this fact would follow from a well-known theorem of Artin [Art, p. 35]. In our four main examples this is the case. However, Q is not a domain in general, but we can modify Artin's argument for our purposes. Suppose there is some relation

$$\sum_{w \in W} a_w w = 0 \tag{13.8}$$

We can choose such a relation with a minimal number of nonzero a_w. Suppose a_{w_1} and a_{w_2} are nonzero. The argument that W acts faithfully on $h^*(BT)$ implies there is some characteristic class $\chi(\lambda)$ on which w_1 and w_2 differ. Apply (13.8) to $\chi(\lambda) \cdot x$. Then,

$$\sum_{w \in W} a_w \chi(w\lambda) \cdot wx = 0$$

and also,

$$\sum_{w \in W} a_w \chi(w_1\lambda) wx = 0$$

Subtracting, we get

$$\sum_{w \in W} a_w(\chi(w\lambda) - \chi(w_1\lambda))w(x) = 0$$

and there is no a_{w_1} term in the above sum. Hence, all terms are zero since we took a minimal relation. But

$$\chi(w_2\lambda) - \chi(w_1\lambda) = \chi(w_2(\lambda) - w_1(\lambda)) + \text{ higher order terms}$$

If we can show that the above expression is not a zero-divisor, then we have a contradiction. We may as well assume that $w_2 = e$ and λ is a simple root α_i. Then it is not difficult to check that we can choose α_i so that

$$w_1(\alpha_i) = \sum m_j \alpha_j,$$

with some $m_j \leq | 2 |$ and $\neq 0$. It follows that

$$\chi(w\alpha_i - \alpha_i) = \sum m_j \chi(\alpha_j) + \text{ higher order terms}$$

Since there is no 2-torsion and $\chi(\alpha_j)$ is not a zero-divisor then the above expression is not a zero-divisor. Hence, we have the desired contradiction.

Theorem 13.9 : Let G be a compact connected Lie group with at least two nonorthogonal roots and let h^* be a complex oriented cohomology theory with no torsion. Let the D_i be the operators defined above. Then the D_i satisfy braid relations if and only if the formal group law is that of cohomology or K-theory.

Proof: There are three cases to consider. These cases are when the two nonorthogonal roots α_i and α_j have $m_{ij} = 3, 4,$ or 6. ($m_{ij} = 2$ implies the roots are orthogonal.) We first study the case $m_{ij} = 3$.

We need to evaluate an identity of the form

$$D_1 D_2 D_1 = D_2 D_1 D_2.$$

By Proposition 13.6 we can check the above identity by expanding each side and equating coefficients of a Weyl group element on each side. When we expand each side we have to interchange characteristic classes $\chi(\alpha)$ with Weyl group elements. for $x \in h^*(BT)$,

$$w(\chi(\alpha)x) = \chi(w\alpha)w(x)$$

so as operators,

$$w\chi(\alpha) = \chi(w\alpha)w.$$

Using this rule, when we examine terms on each side with coefficients s_i, we get

$$\left(\frac{1}{\chi(\alpha_i)}\right)\left(\frac{1}{\chi(-\alpha_i - \alpha_j)\chi(\alpha_i)} + \frac{1}{\chi(-\alpha_i)\chi(-\alpha_j)}\right)s_i =$$
$$\left(\frac{1}{\chi(\alpha_i)}\right)\left(\frac{1}{\chi(-\alpha_j)\chi(-\alpha_i - \alpha_j)}\right)s_i$$

since $s_i(\alpha_j) = \alpha_i + \alpha_j$ and $s_i(\alpha_i) = -\alpha_i$. Therefore, when we equate coefficients, multiply through by $\chi(\alpha_i)$, and reduce denominators, we get the expression

$$\chi(-\alpha_i)\chi(-\alpha_j) + \chi(\alpha_i)\chi(-\alpha_1 - \alpha_2) = \chi(-\alpha_i)\chi(\alpha_i)$$

By our discussion on formal group laws,

$$\chi(-\alpha_i - \alpha_j) = \chi(-\alpha_i) + \chi(-\alpha_j) + \sum_{k,l\geq 1} a_{kl}\chi(-\alpha_i)^k\chi(-\alpha_j)^l$$

It is easy to see that

$$\chi(\alpha_i) = \sum_{k\geq 1} b_k\chi(-\alpha_i)^k,$$

with $b_1 = -1$ [Ad, p. 45]. Simplifying further and replacing $\chi(-\alpha_i)$ by X and $\chi(-\alpha_j)$ by Y, we get

$$XY = -(\sum_{k\geq 1} b_k X^k)(Y + \sum_{k,l\geq 1} a_{kl}X^kY^l$$

Since the left hand side has no expressions with powers of $Y > 1$, neither does the right hand side. The expression,

$$\chi(\alpha_i) = \sum_{k\geq 1} b_k\chi(-\alpha_i)^k,$$

is not a zero-divisor so $a_{kl} = 0$ for $l > 1$. But $a_{kl} = a_{lk}$, so the formal group law has the form

$$F(X,Y) = X + Y + a_{11}XY$$

If $a_{11} = 0$, h^* has the formal group law of cohomology. If $a_{11} \neq 0$, h^* has the formal group law of complex K-theory, with the ambiguity in the choice of a_{11} corresponding to the ambiguity in choices of Chern classes. It is easy to check that the operators for cohomology and K-theory satisfy braid relations.

For the cases $m_{ij} = 4$ and 6 a similar, but much more involved, argument works. However, the calculations are burdensome and we will not present them here. Instead we will explain how our earlier paper combined with Gutkin's work solves the problem. Since there is no torsion, we can replace $\chi(\alpha_i)$ by a formal power series $g(\alpha_i)$ in $L(\alpha_i)$ called the logarithm. g has the property that

$$g(L \otimes M) = F(g(L), g(M))$$

Gutkin [G3] has established necessary conditions for when an operator of the form $D_i = (f(\alpha_i) + s_i g(\alpha_i))$ satisfies braid relations for the cases $m_{ij} = 4$ or 6 (Bressler and I established the necessity for $m_{ij} = 3$ and sufficiency for the general case [BE1]). In particular, taking $f = g$, braid relations imply that the formal group law is either that of cohomology or complex K-theory.

Remark 13.10: The argument given for the case $m_{ij} = 3$ requires only that there is no 2-torsion. However, Gutkin's argument requires that h^* be torsion-free. It seems likely that 13.9 can be proved for the case with no 2-torsion.

Remark 13.11: One sees easily that the argument is valid for Affine Kač-Moody groups also.

Remark 13.12 : As the proof implies, the assertion in 13.9 is actually slightly stronger working with complex coefficients. Suppose any operator D_i of the form $f(\alpha_i) - g(\alpha_i)s_i$ satisfies braid relations, where f and g are meromorphic functions, not identically zero. Then

$$f(\alpha_i) = \frac{1}{\alpha_i} \text{ or } \frac{1}{1 - e^{\alpha_i}}$$

(up to some obvious identifications) and

$$g(\alpha_i)g(-\alpha_i) - f(\alpha_i)f(-\alpha_i) \text{ is constant}$$

This result was proved by Bressler and myself for the root system A_2 and by Gutkin for the other cases. Moreover, Bressler and I proved that these results are sufficient to imply braid relations for any case. This result may be interpreted as giving a new distinguishing property for the Todd series. To see this, let $f = g$. Given that the function $f(\alpha_i) = \frac{1}{\alpha_i}$ defines an operator which satisfies braid relations, one can ask when a new function $h(\alpha_i) = f(\alpha_i) \cdot j(\alpha_i)$ defines an operator which satisfies braid relations. Our theorem asserts that up to various choices of constants, the only non-constant choice for j is the Todd series

$$\frac{\alpha_i}{1 - e^{-\alpha_i}}.$$

In the case of K-theory, we can interpret the D_i as a sequence of induced representations. On the other hand, we could also apply

$$\pi^* \circ \pi_* : h^*(BT) \to h^*(BT)$$

where $\pi : BT \to BG$ is the induced G/T fiber bundle.A consequence of the above theorem is that the sequence $D_iD_jD_i \cdots$ is the same as $\pi^* \circ \pi_*$ only for cohomology and K-theory.

Proposition 13.13 : Let G be a simple rank 2 compact connected Lie group. Let $s_i s_j s_i \cdots = w_0$ be a reduced decomposition of the long element of the Weyl group. Then $D_i D_j D_i \cdots = \pi^* \circ \pi_*$ if and only if the formal group law of h^* is that of ordinary cohomology or K-theory.

Proof: If $D_i D_j D_i \cdots = \pi^* \circ \pi_*$ then since the right hand side is independent of order, braid relations are satisfied, so the group law is that of cohomology or K-theory. On the other hand, one can check that the required relation is true in those two cases.

Chapter 14

Some comments about the flag variety

Up to this point, we have concentrated our attention on BT. However, most of the interesting applications for the operators D_i are for the generalized flag variety G/T. In this chapter, we explain how the results of the previous chapters apply to G/T.

First we need some basic facts about G/T. It is a finite CW-complex with cells only in even degrees. As a result, its integral cohomology is completely given by its cell complex decomposition.

Proposition 14.1 :Let h^* be any complex oriented cohomology theory. Then the Atiyah-Hirzebruch spectral sequence for G/T degenerates at the E_2 term.

Proof. MU^* has coefficients only in even degrees. As a consequence, the MU^* spectral sequence degenerates at the E_2 – term (Corollary 1.2). For any complex oriented theory, there is a natural transformation $t : MU^* \longrightarrow h^*$ such that $t(1) = 1$. By naturality of the Atiyah-Hirzebruch spectral sequence, $t{d_r}^{MU} = {d_r}^h t$. Hence ${d_r}^h = 0$ on the image of t. But at the E_2 – level, the image of t generates the spectral sequence as a module over $h^*(pt)$. Since differentials commute with $h^*(pt)$, ${d_2}^h = 0$ on all of ${E_2}^h$. By similar reasoning, all higher ${d_r}^h = 0$.

We have considered the fiber bundle $\pi_i : BT \longrightarrow BH_i$. Since G/T is a T-principal bundle, there is a classifying map $\theta : G/T \longrightarrow BT$. Similarly, there is a classifying map $\theta_i : G/H_i \longrightarrow BH_i$. The diagram,

$$\begin{array}{ccc} G/T & \xrightarrow{\theta} & BT \\ p_i \downarrow & & \downarrow \pi_i \\ G/H_i & \xrightarrow{\theta_i} & BH_i \end{array}$$

is a Cartesian square. Let $C_i = {p_i}^* \circ {p_i}_*$. By Proposition 8.5, $\theta^* \circ D_i = C_i \circ \theta^*$.

The following theorem implies that rationally the D_i determine the C_i.

Theorem 14.2 : Let h^* be a complex oriented cohomology theory with $h^*(pt)$ containing the rationals $\mathbf{Q}$. Then θ^* is surjective. Moreover,

$$\ker \theta^* \supset < \pi^*(\tilde{h}^*(BG)) >,$$

the ideal of $h^*(BT)$ generated by the reduced cohomology of BG.

Proof: It is well-known that θ^* is surjective for ordinary cohomology [Bor]. Thus,

$$\theta^* : E_2(BT) \to E_2(G/T)$$

is surjective. Since each spectral sequence is trivial,

$$\theta^* : h^*(BT) \to h^*(G/T)$$

is surjective. The second assertion is trivial.

Remark 14.3 : To understand the operators C_i, we did not really need to work over $\mathbf{Q}$. It suffices to work with torsion-free coefficients. Indeed, in this case $h^*(G/T)$ embeds in $h^*(G/T) \otimes Q$ so we may consider C_i on $h^*(G/T)$.

Theorem 14.2 implies that the operators C_i satisfy braid relations for ordinary cohomology and for K-theory. The C_i for these two cases are the original operators considered by BGG, Demazure, Kač-Peterson, et al. The basic difference between our treatment and theirs is that we work with the classifying space explicitly while they work with it only implicitly, by introducing operators on its cohomology. The advantage of our approach is that on the classifying space it is natural to invert characteristic classes, while such an operation makes no sense on the flag variety, where characteristic classes are nilpotent.

It would be desirable to extend Theorem 13.9 to $h^*(G/T)$. However, we have not succeeded in overcoming the technical obstacles involved in this problem. We do have the following result.

Proposition 14.4 : The operator C_i do not satisfy the braid relations for elliptic cohomology or for complex cobordism in the case $m_{ij} = 3$.

Proof: First we remark that if the C_i satisfy braid relations for complex cobordism, they satisfy braid relations for elliptic cohomology. There is a natural transformation $t : MU^* \to Ell^*$ commuting with π_{i*} and $\pi_i{}^*$. Thus, if

$$C_iC_jC_i\cdots = C_jC_iC_j\cdots$$

in complex cobordism, the same relation is satisfied in elliptic cohomology.

Thus, it suffices to prove the claim for elliptic cohomology. We apply,

$$C_iC_jC_i \prod_{\alpha\in R^+} \chi(\alpha).$$

Using the fact that $\chi(-\alpha) = -\chi(\alpha)$, we get

$$4 - 2[\frac{\chi(\alpha_i + \alpha_j) - \chi(\alpha_j)}{\chi(\alpha_i)}]$$

We replace $\chi(\alpha_i)$ with X and $\chi(\alpha_j)$ with Y. In elliptic cohomology, the group law is

$$F(X,Y) = X + Y - \delta(XY^2 + X^2Y) + \text{ higher order terms.}$$

Thus, the expression,

$$\frac{F(X,Y) - Y}{X} = 1 - \delta(XY + X^2) + \text{ higher order terms} \tag{14.6}$$

We need to check if it equals,

$$\frac{F(Y,X) - X}{Y} \tag{14.7}$$

The difference between (14.6) and (14.7) is

$$\delta(X^2 - Y^2) + \text{ higher order terms.}$$

Moreover, these higher order terms all involve products of at least four characteristic classes. But a product of four characteristic classes lies in $Ell^8(G/T)$, which is zero since dim $G/T = 6$. Since the Atiyah-Hirzebruch spectral sequence collapses, δ is not a zero-divisor in $Ell^*(G/T)$ so it suffices to prove that $X^2 - Y^2$ does not map to 0. There is a natural transformation from elliptic cohomology to rational cohomology and the image under this natural transformation of $\chi_{Ell}(\alpha)$ is $\chi_H(\alpha) = \alpha$ (see remark after 13.4). Thus, it suffices to show that ${\alpha_2}^2 - {\alpha_1}^2$ does not vanish in ordinary cohomology. A simple calculation reveals that it does not.

Remark 14.5 : It presumably possible to check in a similar fashion that the above result is true for $m_{ij} = 4$ or 6. The computation is horrendous.

Chapter 15

A geometric interpretation of braid relations

We want to give a geometric interpretation of our result about braid relations. Using the geometric interpretation of cobordism (chapter 7), we will see that a sequence of operators $C_{i_1} \cdots C_{i_k}$ maps the Euler class of the tangent bundle of the flag variety to the class given by a Bott-Samelson desingularization of a Schubert variety. Then we translate this result into a geometric statement for any complex oriented cohomology theory.

Let E be a complex vector bundle of dimension n over a manifold X and let $i : X \to E$ be the zero section. If $1 \in MU^0(X)$ is the class of the identity, then

$$i^* i_* 1 \in MU^{2n}(X)$$

may be taken to be the Euler class of E [Qu]. $i_*(1)$ is the class of the inclusion (i, X). To compute $i^* i_* 1$, we need to adjust i so that it is transversal to itself. By a homotopy, we may change i to a generic section $\tilde{i}$. By definition,

$$\tilde{i}^* i_* 1 = (p_1 : X \times_E X \to X),$$

where the first map $X \to E$ is $\tilde{i}$ and the second is i. Thus, $\tilde{i}^* i_* 1$ is the class of the inclusion of the zeroes of a generic section into X. Now suppose $X = G/T$ and E is its tangent bundle. It is well known that a generic section may be chosen with zeroes at Weyl group elements. Indeed, we may identify the tangent bundle $T(G/T)$ with $G \times_T Lie(G)/Lie(T)$ and define $\tilde{i}(gT) = Adg(X)$, where X is a regular vector in $Lie(T)$. Then the zeroes of $\tilde{i}$ are precisely at points gT where g normalizes T. Hence, the class of

$$i^* i_* 1 = \mid W \mid (pt \to G/T) \tag{15.1}$$

so the Euler class of the tangent bundle is given up to a constant by the inclusion of a point into G/T.

Fix a complex structure $G_{\mathbf{C}}/B$ on G/T. Then $G/H_i \cong G_{\mathbf{C}}/P_i$, where P_i is the complex parabolic subgroup containing B and H_i. Let x be the cobordism class of the inclusion of a point into $G_{\mathbf{C}}/B$. As before, let

$$p_i : G_{\mathbf{C}}/B \to G_{\mathbf{C}}/P_i$$

be the associated $\mathbf{C}P(1)$ fiber bundle. Then

$$p_i{}^* p_{i*}(x) = p_i{}^*(pt \to G_{\mathbf{C}}/P_i).$$

Since p_i is surjective, it is transverse to any map, so

$$p_i{}^*(pt \to G_{\mathbf{C}}/P_i) = G_{\mathbf{C}}/B \times_{G_{\mathbf{C}}/P_i} pt \to G_{\mathbf{C}}/B,$$

which is equivalent to the inclusion $P_i/B \to G_{\mathbf{C}}/B$. Let

$$p_j : G_{\mathbf{C}}/B \to G_{\mathbf{C}}/P_j$$

be the $\mathbf{C}P(1)$ bundle associated to a different parabolic P_j. Then

$$p_j{}^* p_{j*} p_i{}^* p_{i*}(x) = p_j{}^* p_{j*}(P_i/B \to G_{\mathbf{C}}/B))$$

$$= p_j{}^*(P_i/B \to G_{\mathbf{C}}/P_j) = (G_{\mathbf{C}}/B \times_{G_{\mathbf{C}}/P_j} P_i/B \to G_{\mathbf{C}}/B)$$

This last space

$$G_{\mathbf{C}}/B \times_{G_{\mathbf{C}}/P_j} P_i/B = [(xB, x_iB) : xP_j = x_iP_j]$$

$$= (x_i, x_i{}^{-1}x \in P_i \times_B P_j/B$$

The map to $G_{\mathbf{C}}/B$ is

$$(x_1, x_2) \to x_1x_2.$$

Continuing in a similar vein, for any sequence $s_{i_1}, \cdots, s_{i_k}$ we can construct a space

$$Z_{i_1,\cdots,i_k} = P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_k}/B$$

together with a map

$$\theta : (x_1, \cdots, x_k) \to x_1 \cdots x_k \in G_{\mathbf{C}}/B$$

The identification of the B-action in $Z_{i_1,\cdots,i_k}$ is by

$$(x_1, \cdots, x_ib, x_{i+1}, \cdots) = (x_1, \cdots, x_i, bx_{i+1}, \cdots)$$

Definition 15.2 : Let $s_{i_1} \cdots s_{i_k} = w$ be a minimal decomposition of a Weyl group element w. Let

$$Z_{\bar{w}} = P_{i_1} \times_B \cdots \times_B P_{i_k}/B$$

be the space constructed above and let θ be the associated map. Then the pair $(Z_{\bar{w}}, \theta)$ is called a Bott-Samelson variety. Here $\bar{w}$ refers to a Weyl group element together with a reduced decomposition.

Theorem 15.3 : (Demazure [De]) $\theta : Z_{\bar{w}} \to \bar{X}_w$ is a resolution of singularities of $\bar{X}_w$, so θ maps to $\bar{X}_w$, $Z_{\bar{w}}$ is smooth, and θ is proper and birational.

As Haynes Miller has pointed out to me, $(Z_{\bar{w}}, \theta)$ may be interpreted as a cobordism class. It follows from the above discussion that,

Theorem 15.4 : $(Z_{\bar{w}}, \theta) = C_{i_k} \cdots C_{i_1}(x)$.

Corollary 15.5 : Let h^* be any complex oriented cohomology theory. Let $t : MU^* \to h^*$ be the unique natural transformation commuting with push-forwards such that $t(1) = 1$. Let

$$t(Z_{\bar{w}}, \theta) = z_{\bar{w}} \in h^*(G/T)$$

and let e be the identity of the Weyl group. Then

$$z_{\bar{w}} = C_{i_k} \cdots C_{i_1}(z_{\bar{e}})$$

Proof: The above relation is true in cobordism by construction. Since

$$C_i = {p_i}^* p_{i*},$$

C_i commutes with t, so the above relation is true in h^*.

Remark 15.6 : In ordinary cohomology the operator

$$C_{i_k} \cdots C_{i_1} = C_w$$

does not depend on the choice of a reduced decomposition. Hence the class $z_{\bar{w}}$ does not depend on the choice of a reduced decomposition. This class $z_{\bar{w}}$

is dual to the homology class of the Schubert cell $\bar{X}_{w_0 w}$. Thus, using complex cobordism, we can recover one of the basic facts about the operators C_i, which is that they take classes dual to Schubert cells to classes dual to Schubert cells. This argument is in some sense a generalization of an argument for ordinary cohomology due to Arabia [Ara].

Remark 15.7 : I hope to use Theorem 15.4 in a future paper with Bressler as follows. We had hoped to generalize the cohomological and K-theoretic results of BGG, Demazure, Kostant-Kumar (et al) to the case of an arbitrary complex oriented theory h^*. For this purpose, it would suffice to generalize these results to complex cobordism. In particlar, we needed a generalization of the concept of the cohomology class dual to a Schubert cell. In a generalized homology theory there is no good notion for the generalized homology class of a cell. However, for MU, there is a notion of the cohomology class associated to a resolution of singularities. What 15.4 asserts is that if we replace the notion of the cohomology class dual to a cell with the notion of a cobordism class given by a Bott-Samelson variety, then the operators C_i play the same role as in ordinary cohomology. The cobordism classes $(Z_{\bar{w}}, \theta)$ generate $MU^*(G/T)$, but there are relations. We would like to produce a formula for $(Z_{\bar{w}}, \theta) \cup (Z_{\bar{w}'}, \theta)$ for any pair $\bar{w}, \bar{w}'$. Such a formula would specialize to the classical Schubert calculus for the case of ordinary cohomology and $G = SU(n)$, and would also give the results of BGG, Demazure, and Kostant-Kumar. We would like to prove this formula for the case of a Kač-Moody group also. The problem in deriving any such formula is that one has to choose specific orderings for elements $w \in W$ in order to get a basis for $MU^*(G/T)$, and relating the different bases corresponding to different orderings may prove to be too complicated to make the above project worthwhile. At the least, (15.1) and (15.4) provide an answer to the above question in terms of characteristic classes.

Chapter 16

Some nice algebras and some messy algebras

The operators D_i introduced in chapter 13 generate an algebra. In K-theory this algebra has been thoroughly studied. It is a degenerate Hecke algebra in a sense we will make clear. In ordinary cohomology, this algebra has not been studied so thoroughly. We will show how it is a degenerate form of the group algebra of the Weyl group. An intertwining operator will arise from this analysis which has a very interesting interpretation in terms of a completely integrable system. Finally, we will make some comments on this algebra of operators for an arbitrary complex oriented cohomology theory.

We begin with some remarks about the 0-Hecke algebra. These remarks are not new. They are included only for perspective. For an excellent reference, see Carter [Car].

In K-theory the operators D_i are known as the Demazure operators. Their defining relations are

$$(1) D_i^2 = D_i$$

$$(2) D_i D_j D_i \cdots = D_j D_i D_j \cdots (i \neq j, m_{ij} \text{ terms on each side})$$

The algebra they generate is finite dimensional over any coeffient ring. We will refer to it as H_0, the 0-Hecke algebra. The reason for this name is as follows. Let

$$T_i = (q-1)D_i + qs_i$$

Lusztig [Lu] has shown that the T_i generate the Hecke algebra H_q, with defining relations

$$(1)(T_i - q)(T_i + 1) = 0$$

$$(2) T_i T_j T_i \cdots = T_j T_i T_j \cdots (m_{ij} \text{ terms on each side})$$

Thus, the 0-Hecke algebra is the Hecke algebra H_q with q specialized to 0. When $q = 1$, H_q is the group algebra of the Weyl group.

The operators T_i have a topological interpretation due to Lusztig. Let $\pi_i : G/T \to G/H_i$ be the $\mathbf{C}P(1)$ fiber bundle used to define D_i. Let Ω_i^1 be the line bundle over G/T of holomorphic one-forms along the fiber. Then for $E \in K(G/T)$,

$$E + s_i E = D_i(E) - D_i(E \otimes \Omega_i^1)$$

More generally, replace G with $G \times U(1) = M$ and let $U(1)$ act trivially on E and by scalar multiplication on Ω_i^1. Then if we perform the same operation as before,

$$E + T_i E = D_i(E) - D_i(E \otimes \Omega_i^1)$$

Here q implicit in the above description of T_i is a generator for the characters of $U(1)$.

Suppose the coefficient ring for H_q is the complex numbers. Then the Hecke algebra is semi-simple and moreover, its representation theory is identical with that of the group algebra of the Weyl group. The 0-Hecke algebra, however, is far from being semi-simple. In fact, its left regular representation is upper triangular. Norton has proved that H_0 decomposes into a direct sum of 2^l indecomposable submodules, where l is the rank of G. Moreover, she gives an explicit generator for these submodules. Starkey and Carter have proved several interesting results about the relation between this decomposition and various decompositions of the group algebra of the Weyl group. It would be interesting to try to interpret these results in terms of K-theory.

The case of ordinary cohomology is also interesting. There the operators

$$-D_i = A_i = (1 + s_i)\frac{1}{\alpha_i}$$

Their defining relations are

$$(1) A_i^2 = 0$$

$$(2) A_i A_j A_i \cdots = A_j A_i A_j \cdots$$

The nilpotence condition (1) led Kostant and Kumar to call the algebra R it generates the nil Hecke ring. R is strictly upper triangular with respect to the left regular representation. For $w \in W$, let $A_w = A_1 A_2 \cdots A_k$, where $w = s_1 \cdots s_k$ is a minimal decomposition for w. Then the A_w form a basis for R over any coefficient ring.

Proposition 16.1 : R is indecomposable.

Proof: Let $x = \sum_{w \in W} a_w A_w$ be a nonzero element of R. Choose some w so that w is minimal with respect to the Bruhat order such that $a_w \neq 0$. Choose

$v \in W$ such that $vw = w_0$, the long element of the Weyl group. It follows from (2) above that $A_v A_w = A_{w_0}$. Moreover, $A_v A_y = 0$ for $y \not\leq w$, since the A_i preserve order relations and w_0 is the maximum element of the Weyl group. Hence, $A_v x = a_w A_{w_0}$. Thus, every submodule contains A_{w_0} so R is indecomposable.

We want to show that R is a degeneration of the group algebra of the Weyl group $\mathbf{C}[W]$. This fact was first proved by Gutkin and I noticed it independently later. Let

$$C_i = ts_i + A_i.$$

Bressler and I have proved the following result.

Theorem 16.2 : Let $C_i = f(\alpha_i) - g(\alpha_i)s_i$. The C_i satisfy the braid relations if

$$f(\alpha_i) = \frac{1}{\alpha_i} \text{ or } \frac{1}{e^{\alpha_i} - 1},$$

and

$$g(\alpha_i)g(-\alpha_i) - f(\alpha_i)f(-\alpha_i) = c, c \in \mathbf{C}.$$

In our case, $f(\alpha_i) = 1/\alpha_i$ and $g(\alpha_i) = t - \frac{1}{\alpha_i}$. Thus,

$$g(\alpha_i)g(-\alpha_i) - f(\alpha_i)f(-\alpha_i) = t^2,$$

so the condition is satisfied.

It is easy to check that $C_i^2 = 1$ and that the C_i satisfy no further relations. Thus, we have

Proposition 16.3 : (Gutkin) The algebra R_t generated by the C_i is isomorphic to the group algebra of the Weyl group and R_0 is the nil Hecke ring.

For $t \neq 0$, R_t is generated by the operators

$$E_i = 1/tC_i = s_i + 1/tA_i.$$

Let $u = 1/t$ and call the corresponding representation of $\mathbf{C}[W]$ π_u. Thus, in this nomenclature π_0 is the standard representation of $\mathbf{C}[W]$. Thus, we have a $\mathbf{C}$-parameter family of representations of the Weyl group on $H^*(BT, \mathbf{C})$. Filter $H^*(BT, \mathbf{C})$ by

$$S^j = \oplus_{i \leq j} H^i(BT, \mathbf{C})$$

Since the A_i reduce degree and $H^i = 0$ for $i < 0$, the C_i act on S_j, which is finite dimensional over $\mathbf{C}$.

Proposition 16.4 : The π_u representation on S_j is equivalent to the π_0 representation.

Proof: A finite dimensional representation of the group algebra of a finite group is completely determined by its character. Since the A_i reduce degree, they do not contribute to the character. Hence $trs_i = trE_i$.

Since π_u is equivalent to π_0, it follows that there must be an intertwining operator. We can construct one by averaging. Let

$$\phi_u = \frac{1}{\mid W \mid} \sum_{w \in W} \pi_u(w)\pi_0(w^{-1})$$

Then, clearly $\phi_u(\pi_0(w)x) = \pi_u(w)\phi_0(x)$. To show ϕ_u is an intertwining operator we just need to show it is a vector space isomorphism. Filter S^j so

$$S^j_k = \oplus_{i \leq k} H^i(BT, \mathbf{C}).$$

Then $\pi_u = \pi_0$ on GrS^j, since

$$\pi_u - \pi_0 = uA_j = 0$$

on GrS_j. Hence $\phi_u = 1$ on GrS^j, so ϕ_u is an isomorphism.

Its precise form is interesting. Let $w = s_1 \cdots s_k$ be a reduced decomposition. Let $\beta_1 = \alpha_1$ and let

$$\beta_i = s_1 s_2 \cdots s_{i-1}(\alpha_i)$$

$$\pi_u(w)\pi_0(w)^{-1} = (s_1 + tA_1) \cdots (s_k + tA_k)s_k \cdots s_1$$

But,

$$(s_k + tA_k)s_k = 1 - tA_k, \text{ and } s_i(1 - tA_k)s_i = 1 - tA_{s_i(\alpha_k)},$$

where

$$A_\alpha = (1 + s_\alpha)\frac{1}{\alpha}$$

for any root α. Thus,

$$\pi_u(w)\pi_0(w)^{-1} = (1 - tA_{\beta_1})(1 - tA_{\beta_2}) \cdots (1 - tA_{\beta_k})$$

Proposition 16.5 : The operator on S^j

$$\frac{1}{\mid W \mid} \sum_{w \in W} (1 - tA_{\beta_1}) \cdots (1 - tA_{\beta_k})$$

is an intertwining operator between π_0 and π_u.

The above operator is important in Gutkin's solution of the Bethe Ansatz in the field of completely integrable systems[Gu1]. Gutkin showed that a certain Hamiltonian H related to the Laplacian is completely integrable. He proved complete integrability by constructing an integral operator P such that $P\Delta P^{-1} = H$. This operator P is defined on a space of functions over the Cartan subalgebra $\mathbf{t}$. P has a different form for the different Weyl chambers of $\mathbf{t}$. Weyl chambers C are indexed by Weyl group elements. Let $P_w = P$ in the chamber C_w. Then P_w is adjoint to the above operator in the following sense. Consider compactly supported cohomology of BT, $H_c^*(BT, \mathbf{C}) \cong S(\mathbf{t}^*)$.Here $S(V)$ denotes the symmetric algebra of V. We may identify $S(\mathbf{t}^*)$ with polynomial functions on $\mathbf{t}$ and $S(\mathbf{t})$ with constant coefficient differential operators on $\mathbf{t}$. There is a natural Hermitian pairing between $S(\mathbf{t})$ and $S(\mathbf{t}^*)$ given as follows. If D is a differential operator and p is a polynomial then $< D, p >= D\bar{p}(0)$. Thus, there is an integral operator C_i^* on $S(\mathbf{t})$ adjoint to our operators C_i on functions. More generally, we can define an integral operator

$$Q_w = ((1 - tA_{\beta_1}) \cdots (1 - tA_{\beta_k}))^*,$$

adjoint to a summand of the intertwining operator constructed in Proposition 16.5. Then Gutkin's operator $P_w = Q_w$. It would be interesting to try to understand this fact better. It asserts that our $\mathbf{C}$-parameter family of representations of the Weyl group corresponds to a family of completely integrable systems in some sense.

Finally, we want to comment on the algebra of operators generated by the D_i for an arbitrary complex oriented cohomology theory h^*. Our main theorem about these operators was a negative result, in the sense that it asserted certain relations are not satisfied. It would be interesting to determine the precise structure of these algebras. We have no reason to expect them to be finite dimensional, even over $h^*(pt)$. However, the algebra of operators generated by the C_i on $h^*(G/T)$ is finite dimensional over $h^*(pt)$. Our main result about the algebra generated by the D_i is that $D_i^2 = D_i(1)D_i$ (Proposition 13.5). It is not in general true that $D_i(1) \in h^*(pt)$, as is true in ordinary cohomology and in K-theory. The question of what relations should replace braid relations is still unanswered. By a laborious computation, I have managed to show that for the root system A_2,

$$D_iD_jD_iD_jD_iD_j = D_jD_iD_jD_iD_jD_i$$

when $\chi(-\alpha) = -\chi(\alpha)$. It is likely that further results would come from some geometric argument.

References

[AB] M. F. Atiyah and R. Bott, A Lefschetz fixed point theorem for elliptic complexes II, *Ann. of Math.* **88** (1968) 451-491.

[Ad] J.F. Adams,*Stable homotopy and generalised homology*, University of Chicago Press, 1974.

[AH1] M.F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces,*Proc. Sympos. Pure Math.*;Vol. III,7-38,*Amer. Math. Soc.*,(1961)

[AH2] M. F. Atiyah and F. Hirzebruch, The Riemann-Roch theorem for analytic embeddings, *Topology* **1** (1962) 151-166.

[Ara] A. Arabia, Cohomologie T-equivariante de la variete de drapeaux d'un groupe de Kač-Moody, preprint (1987).

[Art] E. Artin, *Galois Theory*, Notre Dame lecture notes, 1959.

[AS] M. F. Atiyah and G. B. Segal, Equivariant K-theory and completion, *Journal of Differential Geometry* **3** (1969) 1-18

[BB] A. A. Beilinson and J. Bernstein, Localisation des $\mathbf{g}-$modules, *C. R. Acad. Sci. Paris* **292** (1981), 15-18.

[BG] J.C. Becker and D.H. Gottlieb, The transfer map and fiber bundles,*Topology* **14** (1975),1-12.

[BGG] J.N. Bernstein, I.M. Gel'fand, S.I. Gel'fand, Schubert cells and the cohomology of spaces G/P, *Russian Math Surveys* **28** (1973),1-26.

[BE1] P. Bressler and S. Evens, On certain Hecke rings,*Proc. Natl. Acad. Sci. USA* **84** (1987),624-625.

[BE2] P. Bressler and S. Evens, Representations of braid groups and generalized cohomology, submitted for publication, (1987).

[BH] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces,I,*Amer. J. Math.* **80** (1958),458-538.

[BM] G. Brumfiel and I. Madsen, Evaluation of the transfer and the universal surgery classes,*Inventiones math.* **32** (1976),133-169.

[Bor] A. Borel, Sur la cohomologie des expaces fibres principaux et des espaces homogenes de groupes de Lie compacts, *Ann. of Math.* **57** (1953), 115-207.

[Bott] R. Bott, Homogeneous vector bundles, *Ann. of Math.* **66** (1957),203-248.

[Bour] N. Bourbaki, *Groupes et algebres de Lie, Chapitres 4,5, et 6*, Hermann, Paris, 1968.

[BS] A. Borel and J.-P. Serre, Le theoreme de Riemann-Roch, *Bull. Soc. Math. de France* **86** (1958), 97-136.

[Ca] R. W. Carter, Representation theory of the 0-Hecke algebra, *Journal of Algebra* **104** (1986), 89-103.

[CF] P.E. Conner and E. E. Floyd, *The relation of cobordism to K-theories*, Springer-Verlag, 1966.

[Ch] C. Chevalley, Invariants of finite groups generated by reflections, *Amer. J. Math.* **77** (1955), 778-782.

[De] M. Demazure, Desingularisation des varietes de Schubert, *Ann. Ec. Norm. Sup.*(**4**) **7** (1974),53-88.

[Do] A. Dold, The fixed point transfer for fiber-preserving maps, Math. Z. **148** (1976) 215-244.

[Dy] E. Dyer, *Cohomology theories*, W.A. Benjamin, 1969.

[Fel] M. Feshbach, The transfer and compact Lie groups, *Trans. Amer.*

Math. Soc. **251** (1979), 139-169.

[Fe2] M. Feshbach, Some general theorems on the cohomology of classifying spaces of compact Lie groups, *Trans. Amer. Math. Soc.* **264** (1981), 49-58.

[GS] E. Gutkin and P. Slodowy, Cohomologies des varietes de drapeaux infinites, *C. R. Acad. Sci. Paris* **296** (1983), 625-627.

[Gu1] E. Gutkin, Integrable systems with delta potential, *Duke Math. J.* **41** (1982), 1-21.

[Gu2] E. Gutkin, Operator calculi associated with reflection groups, *Duke Math. J.* **55** (1987) 1-17.

[Gu3] E. Gutkin, Representations of braid relations, preprint, 1987.

[Ha] M. Hazewinkel, *Formal groups and applications*, Academic Press, 1978.

[HS] P. J. Hilton and U. Stammbach, *A course in homological algebra*, Springer-Verlag, 1971.

[Hum] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, 1972.

[Huse] Dale Husemoller, *Fiber bundles*, McGraw-Hill, 1966.

[Huss] S. Y. Husseini, *The topology of classical groups and related topics*, Gordon and Breach, 1969.

[Ka] V. Kač, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups, *Invent. math* **80** (1985) 69-79.

[KP] V. Kač and D. Peterson, Cohomology of infinite dimensional groups and their flag varieties, to appear.

[KL1] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. math* **53** (1979), 165-184.

[KL2] D. Kazhdan and G. Lusztig, Equivariant K-theory and representations of Hecke algebras II, *Invent. math* **80** (1985),209-231.

[KL3] D. Kazhdan and G. Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebra, *Invent. math* **87** (1987) 153-215.

[KK1] B. Kostant and S. Kumar, The nil Hecke ring and cohomology of G/P for a Kač-Moody group G, *Advances in Math* **62** (1986),187-237.

[KK2] B. Kostant and S. Kumar, T-equivariant K-theory of generalized flag varieties, *Proc. Natl. Acad. Sci. USA* **84** (1987), 4351-4354.

[Ko1] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. of Math.* **74** (1961), 329-387.

[Ko2] B. Kostant, Lie algebra cohomology and generalized Schubert cells, *Ann. of Math.* **77** (1963), 72-144.

[La] P. S. Landweber, Elliptic cohomology and modular forms, preprint, 1986.

[LMS] L. G. Lewis, J. P. May, M. Steinberger, *Equivariant stable homotopy theory*, Springer-Verlag, 1986.

[Lu] G. Lusztig, Equivariant K-theory and representations of Hecke algebras, *Proc. Am. Math. Soc.* **94** (1985),337-342.

[May] J. P. May, "Equivariant constructions of nonequivariant spectra", *Algebraic topology and algebraic K-theory*, ed. William Browder, Princeton University Press, 1987.

[MP] S. A. Mitchell and S. B. Priddy, A double coset formula for Levi subgroups and splitting BGL_n, preprint (1987).

[Qu] D. G. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, *Advances in Math.* **7** (1971), 29-56.

[Se] J.-P. Serre, Representations lineaires et espaces homogenes Kahlerians des groupes de Lie compacts, Expose 100, *Seminaire Bourbaki*, 6^e *annee*, 1953/54, Inst. Henri Poincare, Paris, 1954.

[St] R. Steinberg, On a theorem of Pittie, *Topology* **14** (1975) 173-177.

[Vo] D. A. Vogan, *Representations of real reductive Lie groups*, Birkhauser, 1981.

[Wi] E. Witten, Elliptic genera and quantum field theory, pre-print (1986).

BIOGRAPHICAL NOTE

Sam Evens was born in Chicago, Illinois on May 17, 1961. After a brief tenure in Berkeley, he moved to Evanston, Illinois to make sure he would not learn how to eat alfalfa sprouts on pastrami sandwiches. He graduated with a B.A. in mathematics from Haverford College in 1984. Since then he has been a graduate student at MIT working under the supervision of Professor Bertram Kostant. He hopes to soon move to a state with a name he can spell.