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BRANCHED POLYMERS AND HYPERPLANE ARRANGEMENTS

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ABSTRACT. We generalize the construction of connected branched polymers and the notion of the volume of the space of connected branched polymers studied by Brydges and Imbrie [BI], and Kenyon and Winkler [KW] to any hyperplane arrangement $\mathcal{A}$. The volume of the resulting configuration space of connected branched polymers associated to the hyperplane arrangement $\mathcal{A}$ is expressed through the value of the characteristic polynomial of $\mathcal{A}$ at 0. We give a more general definition of the space of branched polymers, where we do not require connectivity, and introduce the notion of q-volume for it, which is expressed through the value of the characteristic polynomial of $\mathcal{A}$ at $-q$. Finally, we relate the volume of the space of branched polymers to broken circuits and show that the cohomology ring of the space of branched polymers is isomorphic to the Orlik-Solomon algebra.

1. INTRODUCTION

Brydges and Imbrie [BI], and Kenyon and Winkler [KW] study the space of branched polymers of order n in $\mathbb{R}^2$ and $\mathbb{R}^3$. In their definition, a **branched polymer** of order n in $\mathbb{R}^D$ is a connected set of n labeled unit spheres in $\mathbb{R}^D$ with nonoverlapping interiors. To each polymer P corresponds a graph G_P on the vertex set $[n]$ where $(i, j) \in E(G_P)$ if and only if spheres i and j touch in the polymer. The space of branched polymers of order n in $\mathbb{R}^D$ can be parametrized by the D-dimensional angles at which the spheres are attached to each other. If G_P has a cycle, this parametrization becomes ambiguous. However, since polymers containing a cycle of touching spheres have probability zero, this ambiguity is not of concern. Brydges and Imbrie [BI], and Kenyon and Winkler [KW] compute the volume of the space of branched polymers in $\mathbb{R}^2$ and $\mathbb{R}^3$, which are $(n-1)!(2\pi)^{n-1}$ and $n^{n-1}(2\pi)^{n-1}$, respectively. They show that if we relax the requirement that the spheres in the polymer have radii 1, the volume of the space of branched polymers in $\mathbb{R}^2$ remains the same, while the volume of the space of branched polymers in $\mathbb{R}^3$ changes.

Intrigued by the robustness of the space of branched polymers in $\mathbb{R}^2$ under the change of radii, we generalize this notion differently from how it is done

in [BI, KW]. Under our notion of polymers, the volume of the space of polymers is independent of the radii in all cases.

We associate the space of branched polymers to any hyperplane arrangement $\mathcal{A}$. The polymers corresponding to the braid arrangement $\mathcal{B}_n$ coincides with the definition of branched polymers in $\mathbb{R}^2$ given above. We also broaden the notions of branched polymers, by not requiring polymers to be connected. We define the volume of the space of such polymers and show that it is invariant under the change of radii. In the case of the braid arrangement, the volume of the space of connected branched polymers associated to $\mathcal{B}_n$ is $(n-1)!(2\pi)^{n-1} = (-2\pi)^{r(\mathcal{B}_n)}\chi_{\mathcal{B}_n}(0)$, where $r(\mathcal{B}_n)$ is the rank of $\mathcal{B}_n$ and $\chi_{\mathcal{B}_n}(t)$ is the characteristic polynomial of $\mathcal{B}_n$. In our generalized notion of volume, weighted with q, the volume of the space of branched polymers associated to $\mathcal{B}_n$ is $(-2\pi)^{r(\mathcal{B}_n)}\chi_{\mathcal{B}_n}(-q)$. The volume of the space of connected branched polymers associated to $\mathcal{B}_n$ is a specialization of this q-volume at $q = 0$.

A theorem of the same flavor holds for any of our polymers.

Theorem. *The q-volume of the space of branched polymers associated to a central hyperplane arrangement $\mathcal{A}$ is $(-2\pi)^{r(\mathcal{A})}\chi_{\mathcal{A}}(-q)$. Furthermore, the volume of the space of connected branched polymers associated to $\mathcal{A}$ is a specialization of its q-volume at $q = 0$.*

In the case that $\mathcal{A}$ is a graphical arrangement, we recover the G-polymers of [KW], and the Theorem can be rephrased in terms of the chromatic polynomial of the graph G. We also relate the volume of the space of branched polymers to broken circuits and the Orlik-Solomon algebra. Finally, we prove that the cohomology ring of the space of branched polymers is isomorphic to the Orlik-Solomon algebra, and conjecture the same for for the space of connected branched polymers.

The outline of the paper is as follows. In Section 2 we explain how the notion of branched polymers in $\mathbb{R}^2$ from [BI, KW] translates to polymers associated to the braid arrangement $\mathcal{B}_n$. In Section 3 we give a general definition of connected branched polymers, as well as branched polymers (where we do not require connectivity) associated to any arrangement $\mathcal{A}$. We define the notion of q-volume of the space of branched polymers and restate the Theorem about the value of the q-volume of the space of branched polymers. In Section 4 we prove the Theorem. In Section 5 we recover the G-polymers of [KW] from the graphical arrangement $\mathcal{A}_G$. We relate the volumes of branched polymers to broken circuits in Section 7. We conclude in Section 8 by proving that the cohomology ring of the space of branched polymers is the Orlik-Solomon algebra, and we conjecture that the cohomology ring of the space of connected branched polymers is the same.

2. Branched polymers and the braid arrangement

In this section we explain how to think of the space of connected branched polymers and its volume defined in [BI, KW] in terms of the braid arrangement $\mathcal{B}_n$. We also give a more general definition of the space of branched polymers associated to $\mathcal{B}_n$ equipped with the notion of q-volume. For $q = 0$ we recover the volume of the space of connected branched polymers in the sense of [BI, KW]. [1]

Let $V := \mathbb{C}^n/(1,\ldots,1)\mathbb{C} \cong \mathbb{C}^{n-1}$. A point $\mathbf{x} \in V$ has coordinates $(x_1,\ldots,x_n)$ considered modulo simultaneous shifts of all x_i's.

The **braid arrangement** $\mathcal{B}_n$ in V, $n > 1$, consists of the $\binom{n}{2}$ hyperplanes given by the equations

$$h_{ij}(\mathbf{x}) = x_i - x_j = 0, \text{ for } 1 \le i < j \le n, \text{ where } \mathbf{x} \in V.$$

Let

$$H_{ij} = \{\mathbf{x} \in V \mid h_{ij}(\mathbf{x}) = 0\}.$$

Define the **space of branched polymers** associated to the arrangement $\mathcal{B}_n$ and nonnegative scalars R_{ij}, $1 \le i < j \le n$ to be

$$P_{\mathcal{B}_n} = \{\mathbf{x} \in V \mid |h_{ij}(\mathbf{x})| \ge R_{ij} \text{ for all } H_{ij} \in \mathcal{B}_n\}.$$

A connected branched polymer of size n is a connected collection of n labeled disks in $\mathbb{R}^2$ with nonoverlapping interiors (considered up to translation). Think of the collection of n disks in $\mathbb{R}^2 = \mathbb{C}$ as a point $\mathbf{x} \in V$, where x_k is the center of the kth disk. Denote by r_k the radius of the kth disk and let $R_{ij} = r_i + r_j$. The condition that the disks do not overlap can be written as $|x_i - x_j| \ge R_{ij}$. Disks i and j touch exactly if $|x_i - x_j| = R_{ij}$. Thus the space $P_{\mathcal{B}_n}$ consists of points corresponding to branched polymers, which are not necessarily connected.
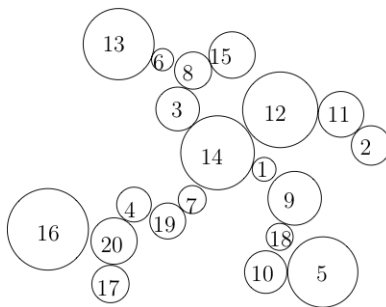

Figure 1. A connected branched polymer.

It is clear from the definition of $P_{\mathcal{B}_n}$ that it decomposes as the disjoint union

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[1]Note that our terminology differs slightly from that of [BI, KW]. The notion of "branched polymer" in [BI, KW] corresponds to our "connected branched polymers."

$$P_{\mathcal{B}_n} = \bigsqcup_{S \subset \mathcal{B}_n} P_{\mathcal{B}_n}^S,$$

where $P_{\mathcal{B}_n}^S$ consists of all points $\mathbf{x} \in P_{\mathcal{B}_n}$ such that $|x_i - x_j| = R_{ij}$ exactly if $H_{ij} \in S$. That is,

$$P_{\mathcal{B}_n}^S = \{\mathbf{x} \in \mathbb{C}^n \mid |h_{ij}(\mathbf{x})| > R_{ij} \text{ for } H_{ij} \in \mathcal{B}_n \backslash S, \ |h_{ij}(\mathbf{x})| = R_{ij} \text{ for } H_{ij} \in S\}.$$

We can consider $\mathcal{B}_n$ as a matroid with ground set the set of hyperplanes in $\mathcal{B}_n$ and independence given by linear independence of the normals to the hyperplanes.

Connected branched polymers, in the sense of [BI, KW], correspond to points in the strata $P_{\mathcal{B}_n}^S$ of $P_{\mathcal{B}_n}$ such that S contains a base of $\mathcal{B}_n$. Otherwise, the configuration of disks would not be connected. Thus, the **space of connected branched polymers** corresponding to $\mathcal{B}_n$ is

$$CP_{\mathcal{B}_n} = \bigsqcup_{S \subset \mathcal{B}_n \ : \ r(S) = r(\mathcal{B}_n)} P_{\mathcal{B}_n}^S,$$

where $r(S)$ denotes the rank of the arrangement S, which is the dimension of the space which the normals to the hyperplanes in S span.

We now define the notion of the volume of the space $CP_{\mathcal{B}_n}$, which coincides with the definition given in [BI, KW].

Let $S = \{h_{i_1 j_1}, \ldots, h_{i_{n-1} j_{n-1}}\}$ be a base of $\mathcal{B}_n$. Let us embed $P_{\mathcal{B}_n}^S$ into $[0, 2\pi]^{n-1}$ by

$$\mathbf{x} \mapsto \varphi(\mathbf{x}) = (\varphi_1, \ldots, \varphi_{n-1}),$$

where $h_{i_k j_k}(\mathbf{x}) = e^{i\varphi_k} R_{i_k j_k}, k \in [n-1]$. Define the volume of $P_{\mathcal{B}_n}^S$ as the volume of its image in $[0, 2\pi]^{n-1}$:

$$\text{vol } \mathrm{P}_{\mathcal{B}_\mathrm{n}}^\mathrm{S} = \text{vol } (\{\varphi(\mathbf{x}) \mid \mathbf{x} \in \mathrm{P}_{\mathcal{B}_\mathrm{n}}^\mathrm{S}\}).$$

If S is a dependent set in $\mathcal{B}_n$, then $P_{\mathcal{B}_n}^S$ has a lower dimension ($\dim P_{\mathcal{B}_n}^S < n-1$), so we let $\text{vol } \mathrm{P}_{\mathcal{B}_\mathrm{n}}^\mathrm{S} = 0$.

Let

$$\text{vol } \mathrm{CP}_{\mathcal{B}_\mathrm{n}} = \sum_{\mathrm{S} \ : \ \mathrm{r}(\mathrm{S}) = \mathrm{r}(\mathcal{B}_\mathrm{n})} \text{vol } \mathrm{P}_{\mathcal{B}_\mathrm{n}}^\mathrm{S} = \sum_{\mathrm{S} \ : \ \mathrm{S} \in \mathcal{B}(\mathcal{B}_\mathrm{n})} \text{vol } \mathrm{P}_{\mathcal{B}_\mathrm{n}}^\mathrm{S},$$

where $\mathcal{B}(\mathcal{M})$ denotes the set of bases of a matroid $\mathcal{M}$.

Recall that the characteristic polynomial of $\mathcal{B}_n$ is $\chi_{\mathcal{B}_n}(t) = (t-1) \cdots (t - (n-1))$ [S, p. 414]. By [KW, Theorem 2], $\text{vol } \mathrm{CP}_{\mathcal{B}_\mathrm{n}} = (\mathrm{n}-1)!(2\pi)^{\mathrm{n}-1}$. Observe that it equals

(1) $$\text{vol } \mathrm{CP}_{\mathcal{B}_\mathrm{n}} = (-2\pi)^{\mathrm{r}(\mathcal{B}_\mathrm{n})} \chi_{\mathcal{B}_\mathrm{n}}(0),$$

where $r(\mathcal{B}_n) = n - 1$ is the rank of $\mathcal{B}_n$. Equation (1) is a special case of Theorem 1 which we state and prove in the following sections.

3. Polymers associated to a hyperplane arrangement

In this section we associate branched polymers to any central hyperplane arrangement. We calculate the volume of the space of connected branched polymers as well as the q-volume of the space of all branched polymers.

Let $\{h_1, \ldots, h_N\} \subset (\mathbb{C}^r)^*$, and assume that $h_1, \ldots, h_N$ span $(\mathbb{C}^r)^*$. Let $\mathcal{A} = \{H_1, \ldots, H_N\}$ be the hyperplane arrangement where the hyperplanes are

$$H_i = \{\mathbf{x} \in \mathbb{C}^r \mid h_i(\mathbf{x}) = 0\}.$$

Note that $r = r(\mathcal{A})$ is the rank of the arrangement.

Define the **space of branched polymers** associated to the arrangement $\mathcal{A}$ and nonnegative scalars R_i, $1 \le i \le N$ to be

$$P_\mathcal{A} = \{\mathbf{x} \in \mathbb{C}^r \mid |h_i(\mathbf{x})| \ge R_i \text{ for all } H_i \in \mathcal{A}\}.$$

This space can be thought of as $\mathbb{C}^r$ with N tubes removed. Namely, let

$$T_i = \{\mathbf{x} \in \mathbb{C}^r \mid |h_i(\mathbf{x})| < R_i\}, \ i \in [N],$$

be the ith tube, which is the set of points in $\mathbb{C}^r$ at distance less than $R_i/\|h_i\|$ from the hyperplane H_i. Clearly then $P_\mathcal{A}$ is the complement to all the tubes in $\mathbb{C}^r$. The space $P_\mathcal{A}$ is related to the well-studied space $C_\mathcal{A} = \mathbb{C}^r \backslash \bigcup_{H \in \mathcal{A}} H$, the complement of the arrangement $\mathcal{A}$ in $\mathbb{C}^r$.

The space of branched polymers $P_\mathcal{A}$ decomposes as the disjoint union

$$(2) \qquad P_\mathcal{A} = \bigsqcup_{S \subset \mathcal{A}} P_\mathcal{A}^S,$$

where

$$P_\mathcal{A}^S = \{\mathbf{x} \in \mathbb{C}^r \mid |h_i(\mathbf{x})| > R_i \text{ for } H_i \in \mathcal{A}\backslash S, \ |h_i(\mathbf{x})| = R_i \text{ for } H_i \in S\}.$$

Consider $\mathcal{A}$ as the matroid with ground set the hyperplanes in $\mathcal{A}$ and independence linear independence of the normals to the hyperplanes.

Define the **space of connected branched polymers** associated to $\mathcal{A}$ as

$$(3) \qquad CP_\mathcal{A} = \bigsqcup_{S \subset \mathcal{A} \ : \ r(S) = r(\mathcal{A})} P_\mathcal{A}^S.$$

We now define the notions of volume vol and q-volume vol_q of $P_\mathcal{A}^S$ for any set $S \subset \mathcal{A}$ with respect to $\mathcal{A}$. Let vol $\mathrm{P}_\mathcal{A}^\mathrm{S} = 0$ for any dependent set S, since $\dim P_\mathcal{A}^S < n$.

Let $S = \{h_{i_1}, \ldots, h_{i_r}\}$ be a base of $\mathcal{A}$. Embed $P_\mathcal{A}^S$ into $[0, 2\pi]^r$ by

$$\mathbf{x} \mapsto \varphi(\mathbf{x}) = (\varphi_1, \ldots, \varphi_r),$$

where $h_{i_k}(\mathbf{x}) = e^{i\varphi_k} R_{i_k}, k \in [r]$. Define the volume of $P_\mathcal{A}^S$ as the volume of its image in $[0, 2\pi]^r$:

$$\text{vol } P_{\mathcal{A}}^{S} = \text{vol } (\{\varphi(\mathbf{x}) \mid \mathbf{x} \in P_{\mathcal{A}}^{S}\}).$$

For any subset $S = \{H_{i_1}, \ldots, H_{i_l}\}$ of $\mathcal{A}$, let

$$\langle S \rangle_{\mathcal{A}} := \{H_i \in \mathcal{A} \mid h_i \text{ is in the span } \langle h_{i_1}, \ldots, h_{i_l} \rangle\}.$$

We consider $\langle S \rangle_{\mathcal{A}}$ as a hyperplane arrangement in $\mathbb{C}^r/(H_{i_1} \cap \ldots \cap H_{i_l}) \cong \mathbb{C}^{r'}$. Its rank is $r(\langle S \rangle_{\mathcal{A}}) = r'$.

Define

$$\text{vol } P_{\mathcal{A}}^{S} := \text{vol } P_{\langle S \rangle_{\mathcal{A}}}^{S} \cdot (2\pi)^{\mathrm{r}(\mathcal{A}) - \mathrm{r}(\langle S \rangle_{\mathcal{A}})},$$

and

$$\text{vol}_q \ P_{\mathcal{A}}^{S} := \text{vol } P_{\mathcal{A}}^{S} \cdot \mathrm{q}^{\mathrm{r}(\mathcal{A}) - \mathrm{r}(\langle S \rangle_{\mathcal{A}})}.$$

Finally, define

$$\text{vol } CP_{\mathcal{A}} = \sum_{S \in \mathcal{B}(\mathcal{A})} \text{vol } P_{\mathcal{A}}^{S}, \ \text{vol } P_{\mathcal{A}} = \sum_{S \in \mathcal{I}(\mathcal{A})} \text{vol } P_{\mathcal{A}}^{S},$$

and

$$\text{vol}_q \ P_{\mathcal{A}} = \sum_{S \in \mathcal{I}(\mathcal{A})} \text{vol}_q \ P_{\mathcal{A}}^{S},$$

where $\mathcal{I}(\mathcal{M})$ denotes the independent sets of the matroid $\mathcal{M}$.

Theorem 1.

(4) $$\text{vol}_q \ P_{\mathcal{A}} = (-2\pi)^{r(\mathcal{A})} \chi_{\mathcal{A}}(-q),$$

where $\chi_{\mathcal{A}}(t)$ is the characteristic polynomial of the central hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^r$ with $\dim(\bigcap_{H \in \mathcal{A}} H) = 0$.

In particular,

(5) $$\text{vol } CP_{\mathcal{A}} = (-2\pi)^{\mathrm{r}(\mathcal{A})} \chi_{\mathcal{A}}(0).$$

We prove Theorem 1 in the next section.

We now point out an alternative way of thinking about the q-volume. Recall that $\langle S \rangle_{\mathcal{A}}$ has the structure of a matroid. Two hyperplanes H_k and H_l are in the same **connected component** of $\langle S \rangle_{\mathcal{A}}$ if they are in a common circuit of the matroid defined by $\langle S \rangle_{\mathcal{A}}$. Let the components of $\langle S \rangle_{\mathcal{A}}$ be $C_1, \ldots, C_k$. Then it is possible to write $S = \bigsqcup_{l=1}^{k} S_l$ such that $C_i = \langle S_i \rangle_{\mathcal{A}}$. We call $S_1, \ldots, S_k$ the **components** of S.

The volume $\text{vol}_q \ P_{\langle S \rangle_{\mathcal{A}}}^{S}$, where $S_1, \ldots, S_k$ are the components of S, can be written as

$$\text{vol}_q \ P_{\langle S \rangle_{\mathcal{A}}}^{S} = \text{vol}_q \ P_{\langle S_1 \rangle_{\mathcal{A}}}^{S_1} \cdots \text{vol}_q \ P_{\langle S_k \rangle_{\mathcal{A}}}^{S_k},$$

where $S_1, \ldots, S_k$ are the components of S.

4. The proof of Theorem 1

In this section we prove Theorem 1. We first state a few lemmas we use in the proof.

Lemma 2. *(Invariance Lemma) The volume $\sum_{S \,:\, |S|=s,\ S \in \mathcal{I}(\mathcal{A})} \text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}}$ is independent of the R_i, $i \in [N]$, where $s \in \{0, 1, \ldots, r(\mathcal{A})\}$.*

Lemma 2 is more general than the Invariance Lemma stated and proved in [KW], but the techniques used in [KW] apply just as well in this case, yielding the proof of Lemma 2.

Lemma 3. *Let $\mathcal{A}' = \mathcal{A}\backslash\{H_1\}$, such that $r(\mathcal{A}) = r(\mathcal{A}')$. Then as $R_1 \to 0$,*

$$\text{vol } \mathrm{P}_{\mathcal{A}'}^{\mathrm{S}} = \text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}},$$

for any independent set $S \subset \mathcal{A}$, with $H_1 \notin S$.

Proof. Note that $r(\langle S \rangle_{\mathcal{A}}) = r(\langle S \rangle_{\mathcal{A}'})$. Let the set of components of $\langle S \rangle_{\mathcal{A}}$ be $\{\langle S_1 \rangle_{\mathcal{A}}, \ldots, \langle S_k \rangle_{\mathcal{A}}\}$, and the set of components of $\langle S \rangle_{\mathcal{A}'}$ be $\{\langle S_{1i} \rangle_{\mathcal{A}'}\}_{i \in I_1} \cup \ldots \cup \{\langle S_{ki} \rangle_{\mathcal{A}'}\}_{i \in I_k}$. Then $\bigcup_{i \in I_l} S_{li} = S_l$, $l \in [k]$, since if two elements x and y in the same component of $\langle S \rangle_{\mathcal{A}'}$, then there is a circuit in $\mathcal{A}$ containing x, y and not containing H_1. Also, if for two elements x and y in different components of $\langle S \rangle_{\mathcal{A}'}$, there is a circuit in $\mathcal{A}$ containing x and y (then this circuit necessarily contains H_1) and if x' and x in a circuit of $\mathcal{A}$ and x and y are in a circuit of $\mathcal{A}$, so are x' and y.

We have

$$\text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}} = \prod_{j=1}^{k} \text{vol } \mathrm{P}_{\langle S_j \rangle_{\mathcal{A}}}^{\mathrm{S_j}} \cdot (2\pi)^{\mathrm{r}(\mathcal{A}) - \mathrm{r}(\langle S \rangle_{\mathcal{A}})}$$

and

$$\text{vol } \mathrm{P}_{\mathcal{A}'}^{\mathrm{S}} = \prod_{j=1}^{k} \prod_{i \in I_j} \text{vol } \mathrm{P}_{\langle S_{ji} \rangle_{\mathcal{A}'}}^{\mathrm{S_{ji}}} \cdot (2\pi)^{\mathrm{r}(\mathcal{A}') - \mathrm{r}(\langle S \rangle_{\mathcal{A}'})}.$$

We claim that for any $j \in [k]$,

$$(6) \qquad \text{vol } \mathrm{P}_{\langle S_j \rangle_{\mathcal{A}}}^{\mathrm{S_j}} = \prod_{i \in I_j} \text{vol } \mathrm{P}_{\langle S_{ji} \rangle_{\mathcal{A}'}}^{\mathrm{S_{ji}}}.$$

Since $r(\mathcal{A}) - r(\langle S \rangle_{\mathcal{A}}) = r(\mathcal{A}') - r(\langle S \rangle_{\mathcal{A}'})$, equation (6) suffices to prove $\text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}} = \text{vol } \mathrm{P}_{\mathcal{A}'}^{\mathrm{S}}$.

Equation (6) follows, since, as $R_1 \to 0$, the presence of the hyperplane H_1 in $\mathcal{A}$ imposes no constraints on the angles φ_j, for $j \in S_{ji}$. Furthermore, if $H_l \in \langle S_{jl} \rangle_{\mathcal{A}'}$ and $H_m \in \langle S_{jm} \rangle_{\mathcal{A}'}$, $l \neq m$, then as $R_1 \to 0$, the span of the angles φ_l and φ_m are independent of each other as all circuits containing both of them also contain H_1. $\qquad\square$

Lemma 4. *Let $\mathcal{A}'' = \mathcal{A}/\{H_1\}$ be the contraction of $\mathcal{A}$ with respect to the hyperplane H_1 such that $r(\mathcal{A}) = r(\mathcal{A}'') + 1$. Then as $R_1 \to 0$,*

$$2\pi \cdot \text{vol } \mathrm{P}_{\mathcal{A}''}^{\mathrm{S}\backslash\{\mathrm{H_1}\}} = \text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}},$$

for any independent set $S \subset \mathcal{A}$ with $H_1 \in S$.

Proof. Note that since S is an independent set in $\mathcal{A}$, so is $S\backslash\{H_1\}$ in $\mathcal{A}''$. Also, $r(\langle S\rangle_{\mathcal{A}}) = r(\langle S\backslash\{H_1\}\rangle_{\mathcal{A}''}) + 1$ and $r(\mathcal{A}) - r(\langle S\rangle_{\mathcal{A}}) = r(\mathcal{A}'') - r(\langle S\backslash\{H_1\}\rangle_{\mathcal{A}''})$. Thus, it suffices to prove that $2\pi\cdot\text{vol } \mathrm{P}^{\mathrm{S}\backslash\{\mathrm{H_1}\}}_{\langle \mathrm{S}\backslash\{\mathrm{H_1}\}\rangle_{\mathcal{A}''}} = \text{vol } \mathrm{P}^{\mathrm{S}}_{\langle \mathrm{S}\rangle_{\mathcal{A}}}$.

Let the set of components of $\langle S\rangle_{\mathcal{A}}$ be $\{\langle S_1\rangle_{\mathcal{A}}, \ldots, \langle S_k\rangle_{\mathcal{A}}\}$ with $H_1 \in S_1$. Note that any circuit in $\langle S\rangle_{\mathcal{A}}$ not involving H_1 automatically carries over to a circuit in $\langle S\rangle_{\mathcal{A}''}$ (easy to see if we assume, without loss of generality, that $h_1(\mathbf{x}) = x_n$). Also, any circuit in $\langle S\rangle_{\mathcal{A}''}$ automatically lifts to a circuit in $\langle S\rangle_{\mathcal{A}}$. Thus, the set of components of $\langle S\backslash\{H_1\}\rangle_{\mathcal{A}''}$ is $\{\langle S_{1j}\rangle_{\mathcal{A}''}\}_{j \in I_1} \cup \{\langle S_2\rangle_{\mathcal{A}''}, \ldots, \langle S_k\rangle_{\mathcal{A}''}\}$, where $\bigcup_{j\in I_1} S_{1j} = S_1$. As $R_1 \to 0$, the volumes $\text{vol } \mathrm{P}^{\mathrm{S_i}}_{\langle \mathrm{S_i}\rangle_{\mathcal{A}''}} = \text{vol } \mathrm{P}^{\mathrm{S_i}}_{\langle \mathrm{S_i}\rangle_{\mathcal{A}}}$, $i \in \{2, \ldots, k\}$. Finally, if $H_l \in \langle S_{1l}\rangle_{\mathcal{A}}$ and $H_m \in \langle S_{1m}\rangle_{\mathcal{A}}$, $l \neq m$, then as $R_1 \to 0$, the span of the angles φ_l and φ_m are independent of each other as all circuits containing both of them also contain H_1. Thus, $2\pi \cdot \prod_{j\in I_1} \text{vol } \mathrm{P}^{\mathrm{S_{1j}}}_{\langle \mathrm{S_{1j}}\rangle_{\mathcal{A}''}} = \text{vol } \mathrm{P}^{\mathrm{S_1}}_{\langle \mathrm{S_1}\rangle_{\mathcal{A}}}$, the 2π accounting for the fact that $H_1 \in S_1$ and as $R_1 \to 0$, φ_1 ranges between 0 and 2π freely. $\square$

Proof of Theorem 1. Equation (5) is a consequence of (4) by the definitions of the spaces $P_{\mathcal{A}}$ and $CP_{\mathcal{A}}$ and their notions of volume. To prove equation (4) we consider two cases depending on whether there is a hyperplane $H \in \mathcal{A}$ such that $r(\mathcal{A}) = r(\mathcal{A}')$, for $\mathcal{A}' = \mathcal{A}\backslash\{H\}$.

Suppose $\mathcal{A}$ is such that if we delete any hyperplane H from it, obtaining the hyperplane arrangement $\mathcal{A}' = \mathcal{A}\backslash\{H\}$, then $r(\mathcal{A}') = r(\mathcal{A}) - 1$. This means that the hyperplanes of $\mathcal{A}$ constitute a base. Then,

$$
\begin{aligned}
\text{vol}_q \ P_{\mathcal{A}} &= \sum_{S \ : \ S\in\mathcal{I}(\mathcal{A})} q^{r(\mathcal{A})-|S|}\text{vol } \mathrm{P}^{\mathrm{S}}_{\mathcal{A}} \\
&= \sum_{i=0}^{r(\mathcal{A})} \binom{r(\mathcal{A})}{i} q^{r(\mathcal{A})-i}(2\pi)^{r(\mathcal{A})} \\
&= (1+q)^{r(\mathcal{A})}(2\pi)^{r(\mathcal{A})} \\
&= (-2\pi)^{r(\mathcal{A})}(-q-1)^{r(\mathcal{A})} \\
&= (-2\pi)^{r(\mathcal{A})}\chi_{\mathcal{A}}(-q).
\end{aligned}
$$

The second equality holds since the independent sets of size i are exactly the i-subset of the base $\mathcal{A}$, of which there are $\binom{r(\mathcal{A})}{i}$ and $\text{vol } \mathrm{P}^{\mathrm{S}}_{\mathcal{A}} = (2\pi)^{\mathrm{r}(\mathcal{A})}$ for any set S, since $\mathcal{A}$ is a base itself. The last equality holds since if $\mathcal{A}$ itself is a base, then $\chi_{\mathcal{A}}(t) = (t-1)^{r(\mathcal{A})}$.

Suppose now that there exists a hyperplane $H \in \mathcal{A}$ such that $r(\mathcal{A}) = r(\mathcal{A}')$ for $\mathcal{A}' = \mathcal{A}\backslash\{H\}$. Assume without loss of generality that $H = H_1$. If $\mathcal{A}'$ and $\mathcal{A}''$ are the deletion and restriction of $\mathcal{A}$ with respect to the hyperplane $h_1(\mathbf{x}) = 0$, then $r(\mathcal{A}) = r(\mathcal{A}') = r(\mathcal{A}'') + 1$. Recall that

(7) $\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$

Let

$$R(\mathcal{A}, q) = (-2\pi)^{r(\mathcal{A})}\chi_{\mathcal{A}}(-q)$$

denote the right hand side of equation (4). By (7)

$$R(\mathcal{A}, q) = R(\mathcal{A}', q) + 2\pi R(\mathcal{A}'', q).$$

In particular,

$$[q^p]R(\mathcal{A}, q) = [q^p]R(\mathcal{A}', q) + 2\pi[q^p]R(\mathcal{A}'', q),$$

for any integer power p of q.

To prove the theorem it suffices to show that the same recurrence holds for

$$\text{(8)} \qquad L(\mathcal{A}, q) = \sum_{S \ : \ S\in\mathcal{I}(\mathcal{A})} q^{r(\mathcal{A})-|S|}\text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}},$$

since the case when we cannot use the recurrence is exactly when removing any hyperplane from $\mathcal{A}$ reduces the rank, and we proved the validity of Theorem 1 for it above.

Recurrence (8) holds, since

$$\sum_{S \ : \ |S|=s, \ S\in\mathcal{I}(\mathcal{A})} \text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}} = \sum_{S' \ : \ |S'|=s, \ S'\in\mathcal{I}(\mathcal{A}')} \text{vol } \mathrm{P}_{\mathcal{A}'}^{\mathrm{S}'}+$$

$$+ 2\pi \sum_{S'' \ : \ |S''|=s-1, \ S''\in\mathcal{I}(\mathcal{A}'')} \text{vol } \mathrm{P}_{\mathcal{A}''}^{\mathrm{S}''},$$

since Lemma 3 and 4 imply

$$\text{(9)} \qquad \sum_{S' \ : \ |S'|=s, \ S'\in\mathcal{I}(\mathcal{A}')} \text{vol } \mathrm{P}_{\mathcal{A}'}^{\mathrm{S}'} = \sum_{S \ : \ H_1\notin S, \ |S|=s, \ S\in\mathcal{I}(\mathcal{A})} \text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}}$$

and

$$\text{(10)} \qquad 2\pi \sum_{S'' \ : \ |S''|=s-1, \ S''\in\mathcal{I}(\mathcal{A}'')} \text{vol } \mathrm{P}_{\mathcal{A}''}^{\mathrm{S}''} = \sum_{S \ : \ H_1\in S, \ |S|=s, \ S\in\mathcal{I}(\mathcal{A})} \text{vol } \mathrm{P}_{\mathcal{A}}^{\mathrm{S}}.$$

$\square$

Example. By Theorem 1 the q-volume of the space of branched polymers associated to the braid arrangement is

$$\text{(11)} \qquad \text{vol}_q \ \mathrm{P}_{\mathcal{B}_n} = (-2\pi)^{r(\mathcal{B}_n)}\chi_{\mathcal{B}_n}(-q)$$
$$= (2\pi)^{n-1}(q+1)\cdots(q+(n-1)).$$

A special case of (11) is equation (1).

5. G-POLYMERS

In this section we discuss G-polymers, as defined in [KW, Section 3], and show how they correspond to graphical arrangements in our setup. We also rewrite the q-volume of the generalized G-polymers in the language of its chromatic polynomial, which is a special case of its Tutte polynomial.

Given a graph $G = ([n], E)$, let R_{ij} be positive scalars for each edge (i, j) of G. A G-**polymer**, following Kenyon and Winkler, is a configuration of points $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that

- $|x_i - x_j| \geq R_{ij}$, for any edge (i, j) of G
- $|x_i - x_j| = R_{ij}$ for all edges (i, j) of some connected subgraph of G.

If G is not connected, there are no G-polymers. The volume of the space of G-polymers is defined by the angles made by the vectors from x_i to x_j, where $(i, j) \in E$ is such that $|x_i - x_j| = R_{ij}$. See [KW] for further details.

Recall that the **graphical arrangement** $\mathcal{A}_G$ corresponding to the graph $G = ([n], E)$ consists of the $|E|$ hyperplanes

$$H_{ij} = \{\mathbf{x} \in \mathbb{C}^n \mid x_i - x_j = 0\} \text{ for } (i, j) \in E, i < j.$$

We consider $\mathcal{A}_G$ as a hyperplane arrangement in $\mathbb{C}^n/(\bigcap_{(i,j)\in E} H_{ij})$. The space connected branched polymers associated to $\mathcal{A}_G$ as defined in (3) coincide with the space of G-polymers.

Let $\chi_G(t)$ be the chromatic polynomial of the graph G, $\widetilde{\chi}_G(t) = \chi_G(t)/t^{k(G)}$, where $k(G)$ is the number of components of G and $r(G) = n - k(G)$. A special case of Theorem 1 is the following proposition.

Proposition 5. ([KW, Theorem 4])

(12) $$\mathrm{vol\,CP}_{\mathcal{A}_\mathrm{G}} = (-2\pi)^{\mathrm{r(G)}}\widetilde{\chi}_\mathrm{G}(0).$$

Proposition 5 follows from Theorem 1 since $\widetilde{\chi}_G(t) = \chi_{\mathcal{A}_G}(t)$ ([S, Theorem 2.7.]) and $r(G) = r(\mathcal{A}_G)$.

Recall also that the chromatic polynomial $\chi_G(t)$ of a graph G is a special case of its Tutte polynomial $T_G(x, y)$: $\chi_G(t) = (-1)^{r(G)}t^{k(G)}T_G(1 - t, 0)$ ([B, Theorem 6, p. 346]). In this light, Proposition 5 coincides with [KW, Theorem 4], which is stated in terms of the Tutte polynomial of G.

The **space of generalized G-polymers** is $P_{\mathcal{A}_G}$ as defined in (2). A special case of Theorem 1 is the following proposition.

Proposition 6.

$$\mathrm{vol}_q\, P_{\mathcal{A}_G} = (-2\pi)^{r(G)}\widetilde{\chi}_G(-q).$$

As Proposition 6 illustrates, the notion of q-volume of the space of generalized G-polymers allows for recovering the entire chromatic polynomial of G.

6. Symmetric polymers

In this section we specialize our results to the type B_n Coxeter arrangement. This arrangement naturally gives rise to polymers symmetric with respect to the origin.

The **type B_n Coxeter arrangement** $\mathcal{A}(B_n)$, $n > 1$, in $\mathbb{C}^n$, consists of the $2\binom{n}{2} + n$ hyperplanes

$$H_{ij}^- = \{\mathbf{x} \in \mathbb{C}^n \mid x_i - x_j = 0\}, H_{ij}^+ = \{\mathbf{x} \in \mathbb{C}^n \mid x_i + x_j = 0\},$$

$$H_k = \{\mathbf{x} \in \mathbb{C}^n \mid x_k = 0\}, \text{ for } 1 \le i < j \le n, k \in [n].$$

We consider $\mathcal{A}(B_n)$ as a hyperplane arrangement in $V = \mathbb{C}^n/(\bigcap_{H \in \mathcal{A}(B_n)} H)$.

The space of branched polymers associated to $\mathcal{A}(B_n)$ and nonnegative scalars R_{ij}^+, R_{ij}^-, R_k, for $1 \le i < j \le n, k \in [n]$ is, by definition,

$$P_{\mathcal{A}(B_n)} = \{\mathbf{x} \in V \mid |x_i + x_j| \ge R_{ij}^+, |x_i - x_j| \ge R_{ij}^-, |x_k| \ge R_k,$$
$$\text{for } 1 \le i < j \le n, k \in [n]\}.$$

To obtain symmetric polymers, place a disk of radius r_k around the points x_k and $-x_k$ in the plane for all $k \in [n]$. Let $R_{ij}^+ = R_{ij}^- = r_i + r_j$ and $R_k = r_k$, for $1 \le i < j \le n, k \in [n]$. Then the condition $|x_i + x_j| \ge R_{ij}^+$ ensures that the disks around x_i and $-x_j$, and $-x_i$ and x_j do not overlap, the condition $|x_i - x_j| \ge R_{ij}^-$ ensures that the disks around x_i and x_j, and $-x_i$ and $-x_j$ do not overlap, and the condition $|x_k| \ge R_k$ ensures that the disks around x_k and $-x_k$ do not overlap.

Since the characteristic polynomial of $\mathcal{A}(B_n)$ is given by

$$\chi_{\mathcal{A}(B_n)}(t) = (t - 1) \cdot (t - 3) \cdots (t - (2n - 1)), \text{ [S, p. 451]}$$

Theorem 1 specializes to the following corollary.

Corollary 7.

(13) $\qquad \mathrm{vol}_q \ P_{\mathcal{A}(B_n)} = (2\pi)^n (q + 1) \cdot (q + 3) \cdots (q + (2n - 1)).$

In particular,

(14) $\qquad\qquad\qquad \mathrm{vol} \ \mathrm{CP}_{\mathcal{A}(B_n)} = (2\pi)^n 1 \cdot 3 \cdots (2n - 1).$

We now outline the geometric picture of connected branched polymers associated to $\mathcal{A}(B_n)$. Since $\mathrm{vol} \ \mathrm{CP}_{\mathcal{A}(B_n)} = \sum_{S \in \mathcal{B}(\mathcal{A}(B_n))} \mathrm{vol} \ \mathrm{P}_{\mathcal{A}(B_n)}^S$ we restrict our attention to the polymers corresponding to the bases of $\mathcal{A}(B_n)$. Biject a subset S of hyperplanes of $\mathcal{A}(B_n)$ with a graph G_S on vertex set $[n]$ by adding an edge labeled by $+$ between vertices i and j if $H_{ij}^+ \in S$, by adding an edge labeled by $-$ between vertices i and j if $H_{ij}^- \in S$, and adding a loop at vertex k if $H_k \in S$. Then the bases of $\mathcal{A}(B_n)$ correspond to graphs G on vertex set $[n]$ with n edges such that each component of G contains exactly one cycle or one loop, and if it contains a cycle then the cycle has an odd number of edges labeled by $+$.
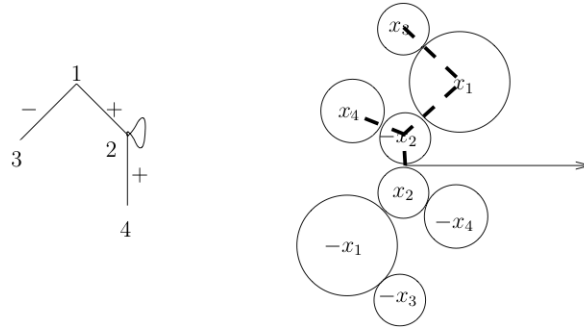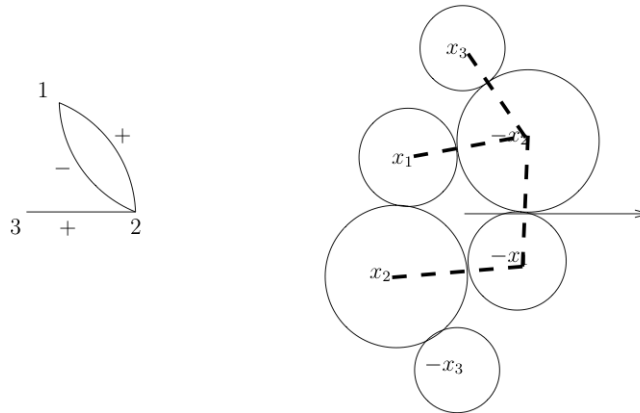
FIGURE 2. The graph representing a base and the corresponding polymer. The half-line drawn on the figure is the positive x-axis. The parametrization can be done by the angles which the dashed line segments make with the x-axis.

FIGURE 3. The graph representing a base and the corresponding polymer. The half-line drawn on the figure is the positive x-axis. The parametrization can be done by the angles which the dashed line segments make with the x-axis.

It is easy to see that if there is more than one loop in G_S, for some base S, then the stratum $P^S_{\mathcal{A}(B_n)}$ is empty. This is true, since a loop based at vertex k in G_S corresponds to two disks of radius $R_k = r_k$ around the points x_k and $-x_k$, touching at the origin. If there was also a loop at vertex $l \neq k$, it would be impossible to also satisfy $|x_k + x_l| \geq r_l + r_k$ and $|x_k - x_l| \geq r_l + r_k$ if $k < l$ or $|x_l - x_k| \geq r_l + r_k$ if $k > l$. The disks corresponding to a component of G_S with a loop at k look like a tree symmetric across the origin and can be parametrized by the angles which the edges of "half" of the tree make with the x-axis and by the angle made by the segment $0 - x_k$ and the x-axis; see Figure 2. The disks corresponding to a component of

G_S with a cycle is symmetric with respect to the origin and contains a cycle of disks symmetrically around the origin. If the cycle is of length $2m$, then the polymer is parametrized by $m + 1$ angles which the edges of the cycle make with the x-axis as well the angles that "half" of the remaining edges make with the x-axis; see Figure 3.

7. VOLUMES OF BRANCHED POLYMERS AND BROKEN CIRCUITS

In this section we relate the volumes of branched polymers associated to $\mathcal{A}$ and broken circuits of $\mathcal{A}$. While the volume vol $CP_{\mathcal{A}}$ is invariant under changing the radii R_i by the Invariance Lemma, the individual volumes of the strata $P_{\mathcal{A}}^S$ change. By picking the radii R_i appropriately, it is possible to construct a stratification, where only strata corresponding to certain special bases S appear in the stratification, and they all have the same volume. The mentioned special bases are the no broken circuit bases as we show in Theorem 8.

A **broken circuit** of $\mathcal{A}$ with respect to a linear ordering $\mathcal{O}$ of the hyperplanes of $\mathcal{A}$ is a set $C - \{u\}$, where C is a circuit and u is the largest element of C in the linear ordering $\mathcal{O}$. While the exact set of broken circuits depends on the linear order we choose, remarkably, the number of sets S of hyperplanes of size k, for an arbitrary fixed k, such that S contains no broken circuit is the same; see [S, Theorem 4.12].

Theorem 8. *Let $\mathcal{A} = \{H_1, \ldots, H_N\}$ be a hyperplane arrangement with $R_1 \ll \cdots \ll R_N$. Then the space $CP_{\mathcal{A}}$ is a disjoint union of $(-1)^{r(\mathcal{A})}\chi_{\mathcal{A}}(0)$ tori $(S^1)^{r(\mathcal{A})}$.*

More precisely, if $\mathcal{O}$ is the order $H_1 < \cdots < H_N$ on the hyperplanes of $\mathcal{A}$ and $BC_{\mathcal{O}}(\mathcal{A})$ are all subsets of $\mathcal{A}$ containing no broken circuits with respect to $\mathcal{O}$, then for $R_1 \ll \cdots \ll R_N$

$$(15) \qquad CP_{\mathcal{A}} = \bigsqcup_{S \in \mathcal{B}(\mathcal{A}) \cap BC_{\mathcal{O}}(\mathcal{A})} P_{\mathcal{A}}^S,$$

where vol $P_{\mathcal{A}}^S = (2\pi)^{r(\mathcal{A})}$ *for* $S \in \mathcal{B}(\mathcal{A}) \cap BC_{\mathcal{O}}(\mathcal{A})$.

Proof. By [S, Theorem 4.12]

$$(16) \qquad \#\{S \mid |S| = i, S \in BC_{\mathcal{O}}(\mathcal{A})\} = (-1)^{r(\mathcal{A})}[q^{r(\mathcal{A})-i}]\chi_{\mathcal{A}}(-q).$$

Thus, $|\mathcal{B}(\mathcal{A}) \cap BC_{\mathcal{O}}(\mathcal{A})| = (-1)^{r(\mathcal{A})}\chi_{\mathcal{A}}(0)$, and it suffices to prove the second part of the theorem's statement.

Recall that by definition

$$CP_{\mathcal{A}} = \bigsqcup_{S \subset \mathcal{A} \,:\, r(S)=r(\mathcal{A})} P_{\mathcal{A}}^S.$$

We first show that if S contains a broken circuit, then $P_{\mathcal{A}}^S$ is empty. Let $\{H_{i_1}, \ldots, H_{i_k}\} \subset S$, $i_1 < \cdots < i_k$, be a broken circuit. Then there

exists $H_i \in \mathcal{A}$ such that $\{H_{i_1}, \ldots, H_{i_k}, H_i\}$, is a circuit and $i_k < i$. Let $h_i = \sum_{j=1}^{k} c_j h_{i_j}$, $c_j \neq 0$. Then for any $\mathbf{x} \in P_{\mathcal{A}}^{S_j}$,

$$\sum_{j=1}^{k} |c_j| R_{i_k} \geq \sum_{j=1}^{k} |c_j| R_{i_j} = \sum_{j=1}^{k} |c_j h_{i_j}(\mathbf{x})| \geq |\sum_{j=1}^{k} c_j h_{i_j}(\mathbf{x})| = |h_i(\mathbf{x})| \geq R_i.$$

However,

$$\sum_{j=1}^{k} |c_j| R_{i_k} < R_i, \text{ since } R_{i_k} \ll R_i.$$

Thus, $P_{\mathcal{A}}^{S}$ is empty.

Next we prove that vol $\mathrm{P}_{\mathcal{A}}^{\mathrm{S}} = (2\pi)^{\mathrm{r}(\mathcal{A})}$ for $S \in \mathcal{B}(\mathcal{A}) \cap BC_{\mathcal{O}}(\mathcal{A})$. Let $S = \{H_{j_1}, \ldots, H_{j_r}\}$ with $j_1 < \cdots < j_r$. Since S contains no broken circuit, it follows that if $i \in [N] \backslash \{j_1, \ldots, j_r\}$, then $i < j_r$. To prove that vol $\mathrm{P}_{\mathcal{A}}^{\mathrm{S}} = (2\pi)^{\mathrm{r}(\mathcal{A})}$, it suffices to show that $|h_{j_l}(\mathbf{x})| = R_{j_l}$ for $l \in [r]$ imply $|h_i(\mathbf{x})| > R_i$ for $i \in [N] \backslash \{j_1, \ldots, j_r\}$. Since S is a base, $S \cup \{H_i\}$ contains a circuit C whose maximal element is $H_{j_k} \neq H_i$, $k \in [r]$, since $S \in BC_{\mathcal{O}}(\mathcal{A})$. Thus,

$$h_{j_k}(\mathbf{x}) = c_i h_i(\mathbf{x}) + \sum_{l \in [N] \backslash \{k\}} c_{j_l} h_{j_l}(\mathbf{x}),$$

where $c_i, c_{j_1}, \ldots, c_{j_r}$ are scalars with $c_i \neq 0$. In particular,

$$|h_{j_k}(\mathbf{x})| \leq |c_i| |h_i(\mathbf{x})| + \sum_{l \in [N] \backslash \{k\}} |c_{j_l}| |h_{j_l}(\mathbf{x})|.$$

Since $|h_{j_l}(\mathbf{x})| = R_{j_l}$ for $l \in [r]$ and $R_{j_k} \gg R_i$ it follows that $|h_i(\mathbf{x})| > R_i$. $\square$

An alternative proof of Theorem 1. By definition,

$$(17) \qquad \mathrm{vol}_q \, P_{\mathcal{A}} = \sum_{i=0}^{r(\mathcal{A})} \sum_{S \in \mathcal{S}_i} (q \cdot 2\pi)^{r(\mathcal{A})-i} \mathrm{vol} \, \mathrm{CP}_{\langle S \rangle_{\mathcal{A}}},$$

where $\mathcal{S}_i$ is a collection of independent sets S of cardinality i such that the arrangements $\langle S \rangle_{\mathcal{A}}$ for $S \in \mathcal{S}_i$ run over all rank i subarrangements of $\mathcal{A}$. By Theorem 8 the right hand side of equation (17) is

$$(18) \quad \sum_{i=0}^{r(\mathcal{A})} \sum_{S \in \mathcal{S}_i} (q \cdot 2\pi)^{r(\mathcal{A})-i} \#\{B \mid B \in \mathcal{B}(\langle S \rangle_{\mathcal{A}}) \cap BC_{\mathcal{O}}(\langle S \rangle_{\mathcal{A}})\} \cdot (2\pi)^i =$$

$$(19) \qquad \sum_{i=0}^{r(\mathcal{A})} q^{r(\mathcal{A})-i} \cdot (2\pi)^{r(\mathcal{A})} \#\{S \mid |S| = i, S \in BC_{\mathcal{O}}(\mathcal{A})\}.$$

Since by [S, Theorem 4.12]

$$(20) \qquad \#\{S \mid |S| = i, S \in BC_{\mathcal{O}}(\mathcal{A})\} = (-1)^{r(\mathcal{A})} [q^{r(\mathcal{A})-i}] \chi_{\mathcal{A}}(-q),$$

it follows from equations (17), (18), (19) and (20) that

$$\text{vol}_q \ P_\mathcal{A} = (-2\pi)^{r(\mathcal{A})} \chi_\mathcal{A}(-q).$$

In particular,

$$\text{vol} \ \text{CP}_\mathcal{A} = (-2\pi)^{\text{r}(\mathcal{A})} \chi_\mathcal{A}(0).$$

$\square$

Note the striking similarity between the formula for the $1/t$-volume of branched polymers $P_\mathcal{A}$,

(21) $$\text{vol}_{1/t} \ P_\mathcal{A} = (-2\pi)^{r(\mathcal{A})} \chi_\mathcal{A}(-1/t)$$

and the generating function

(22) $$\sum_{i \geq 0} \text{rank} \ H^i(C_\mathcal{A}, \mathbb{Z}) t^i = (-t)^{r(\mathcal{A})} \chi_\mathcal{A}(-1/t),$$

where $C_\mathcal{A} = \mathbb{C}^r \backslash \bigcup_{H \in \mathcal{A}} H$, is the complement of the arrangement $\mathcal{A}$ in $\mathbb{C}^r$ and $H^k(C_\mathcal{A}, \mathbb{Z})$ is the kth graded piece of the cohomology ring $H^*(C_\mathcal{A}, \mathbb{Z})$; see [OS]. The right hand sides of (21) and (22) are in fact identical, if we normalize appropriately. This is no surprise, since we related $\text{vol}_{1/t} \ P_\mathcal{A}$ to no broken circuits in Theorem 8, and since the cohomology ring $H^*(C_\mathcal{A}, \mathbb{Z})$ of $C_\mathcal{A}$ is isomorphic to the Orlik-Solomon algebra $A(\mathcal{A})$ associated to the hyperplane arrangement $\mathcal{A}$, and the ith graded piece of $A(\mathcal{A})$ has a basis in terms of no broken circuits.

8. Cohomology rings of the spaces $P_\mathcal{A}$ and $CP_\mathcal{A}$ and the Orlik-Solomon algebra

In this section we prove and conjecture about the cohomology rings of the spaces $P_\mathcal{A}$ and $CP_\mathcal{A}$. The well-known Orlik-Solomon algebra is isomorphic to the cohomology ring $H^*(C_\mathcal{A}, \mathbb{Z})$ of $C_\mathcal{A}$, where $C_\mathcal{A} = \mathbb{C}^r \backslash \bigcup_{H \in \mathcal{A}} H$, is the complement of the arrangement $\mathcal{A}$ in $\mathbb{C}^r$. We prove that the spaces $P_\mathcal{A}$ and $C_\mathcal{A}$ are homotopy equivalent, and we conjecture that the same is true under certain circumstances for the spaces $CP_\mathcal{A}$ and $C_\mathcal{A}$.

Proposition 9. *The map $f : C_\mathcal{A} \to P_\mathcal{A}$ defined by*

$$f : (x_1, \ldots, x_r) \mapsto \max_{i \in [r]}(1, \frac{R_i}{|h_i(\mathbf{x})|}) \cdot (x_1, \ldots, x_r),$$

where $\mathcal{A}$ is a hyperplane arrangements in $\mathbb{C}^r$,

$$C_\mathcal{A} = \mathbb{C}^r \backslash \bigcup_{H \in \mathcal{A}} H,$$

$$P_\mathcal{A} = \{\mathbf{x} \in \mathbb{C}^r \mid |h_i(\mathbf{x})| \geq R_i\}$$

is a deformation retraction.

Proof. It is straightforward to check that f satisfies the following three conditions:

- (i) continuous,
- (ii) $f|_{P_\mathcal{A}} = id$,
- (iii) f is homotopic to the identity. $\qquad\qquad\qquad\qquad\qquad\square$

Corollary 10. *The spaces $P_\mathcal{A}$ and $C_\mathcal{A}$ are homotopy equivalent. In particular, the cohomology rings of the spaces $C_\mathcal{A}$ and $P_\mathcal{A}$, where $\mathcal{A}$ is a hyperplane arrangements in $\mathbb{C}^r$, satisfy*

$$H^*(P_\mathcal{A}, \mathbb{Z}) \cong H^*(C_\mathcal{A}, \mathbb{Z}).$$

In Corollary **??** we proved that for $\mathcal{A} = \{H_1, \dots, H_N\}$ and $R_1 \ll \cdots \ll R_N$ the space $CP_\mathcal{A}$ is a disjoint union of $|\chi_\mathcal{A}(0)|$ tori $(S^1)^{r(\mathcal{A})}$. In particular, $CP_\mathcal{A}$ is disconnected and cannot be homotopy equivalent to $C_\mathcal{A}$. However, if the scalars R_i, $i \in [N]$, satisfy certain generalized triangle inequalities, we conjecture that the spaces $CP_\mathcal{A}$ and $C_\mathcal{A}$ are homotopy equivalent, so in particular, the space $CP_\mathcal{A}$ is connected.

Let $\mathcal{A} = \{H_1, \dots, H_N\}$ be the hyperplane arrangement where the hyperplanes are $H_i = \{\mathbf{x} \in \mathbb{C}^r \mid h_i(\mathbf{x}) = 0\}$. Define the subset $T_\mathcal{A}$ of $\mathbb{R}_{>0}^N$ to consist of all points $\mathbf{R} = (R_1, \dots, R_N) \in \mathbb{R}_{>0}^N$ satisfying the inequalities

$$R_{i_0} \le |c_1| R_{i_1} + \cdots + |c_k| R_{i_k},$$

whenever

$$h_{i_0} = c_1 h_{i_1} + \cdots + c_k h_{i_k}.$$

Example. The set $T_{\mathcal{B}_n}$ consists of points $\mathbf{R} = (R_{ij})_{1 \le i < j \le n}$, where R_{ij} denotes the coordinate corresponding to the hyperplane $H_{ij} = \{\mathbf{x} \in V \mid x_i - x_j = 0\}$, for which the R_{ij}'s satisfy the triangle inequalities $R_{ij} \le R_{ik} + R_{kj}$, for all $1 \le i < k < j \le n$ and $R_{ij} \le R_{ik} + R_{jk}$, for all $1 \le i < j < k \le n$. In particular, $(R_{ij} = r_i + r_j)_{1 \le i < j \le n} \in T_{\mathcal{B}_n}$, where r_i, $i \in [n]$, are nonnegative scalars.

Conjecture 11. *For $(R_1, \dots, R_N) \in T_\mathcal{A}$ the spaces $CP_\mathcal{A}$ and $C_\mathcal{A}$ are homotopy equivalent. In particular, $H^*(CP_\mathcal{A}, \mathbb{Z}) \cong H^*(C_\mathcal{A}, \mathbb{Z})$.*

A proof of Conjecture 11 would yield another explanation, why, while $C_\mathcal{A}$ has $2n$ real dimensions, its top cohomology is in dimension n, since $CP_\mathcal{A}$ has n real dimensions.

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