# EQUIVARIANT $K$-THEORY OF GKM BUNDLES

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Abstract. Given a fiber bundle of GKM spaces, $\pi\colon M \to B$, we analyze the structure of the equivariant $K$-ring of $M$ as a module over the equivariant $K$-ring of $B$ by translating the fiber bundle, $\pi$, into a fiber bundle of GKM graphs and constructing, by combinatorial techniques, a basis of this module consisting of $K$-classes which are invariant under the natural holonomy action on the $K$-ring of $M$ of the fundamental group of the GKM graph of $B$. We also discuss the implications of this result for fiber bundles $\pi\colon M \to B$ where $M$ and $B$ are generalized partial flag varieties and show how our GKM description of the equivariant $K$-ring of a homogeneous GKM space is related to the Kostant-Kumar description of this ring.

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## Contents



## 1. Introduction

Let $T$ be an $n$-torus and $M$ a compact $T$ manifold. The action of $T$ on $M$ is said to be GKM if $M^T$ is finite and if, in addition, for every codimension one subtorus, $T'$, of $T$ the connected components of $M^{T'}$ are of dimension at most 2. An implication of this assumption is that these fixed point components are either isolated points or diffeomorphic copies of $S^2$ with its standard $S^1$ action, and a convenient way of encoding this fixed point data is by means of the *GKM graph*, $\Gamma$, of $M$. By definition, the $S^2$'s above are the *edges* of this graph, the points on the fixed point set, $M^T$, are the *vertices* of this graph and two vertices are joined by

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an edge if they are the fixed points for the $S^1$ action on the $S^2$ representing that edge. Moreover, to keep track of which $T'$'s correspond to which edges, one defines a labeling function $\alpha$ on the set of oriented edges of $\Gamma$ with values in the weight lattice of $T$. This function (the "axial function" of $\Gamma$) assigns to each oriented edge the weight of the isotropy action of $T$ on the tangent space to the north pole of the $S^2$ corresponding to this edge. (The orientation of this $S^2$ enables one to distinguish the north pole from the south pole.)

The concept of GKM space is due to Goresky-Kottwitz-MacPherson, who showed that the equivariant cohomology ring, $H_T(M)$, can be computed from $(\Gamma, \alpha)$ (see [GKM]). Then Allen Knutson and Ioanid Rosu (see [R]) proved the much harder result that this is also true for the equivariant $K$-theory ring, $K_T(M)$. (We will give a graph theoretic description of this ring in section 2 below.)

Suppose now that $M$ and $B$ are GKM manifolds and $\pi\colon M \to B$ a $T$-equivariant fiber bundle. Then the ring $H_T(M)$ becomes a module over the ring $H_T(B)$ and in [GSZ1] we analyzed this module structure from a combinatorial perspective by showing that the fiber bundle $\pi$ of manifolds gives rise to a fiber bundle, $\pi\colon \Gamma_M \to \Gamma_B$ of GKM graphs, and showing that the salient module structure is encoded in this graph fiber bundle. In this article we will prove analogous results in $K$-theory. More explicitly in the first part of this article (from section 2 to 4) we will review the basic facts about GKM graphs, the notion of "fiber bundle" for these graphs and the definition of the $K$-theory ring of a graph-axial function pair, and we will also discuss an important class of examples: the graphs associated with GKM spaces of the form $M = G/P$, where $G$ is a complex reductive Lie group and $P$ a parabolic subgroup of $G$. The main results of this paper are discussed in section 5 and 6. In section 5 we prove that for a fiber bundle of GKM graphs $(\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ a set of elements $c_1 \ldots, c_k$ of $K_\alpha(\Gamma)$ is a free set of generators of $K_\alpha(\Gamma)$ as a module over $K_{\alpha_B}(\Gamma_B)$, providing their restrictions to $K_{\alpha_p}(\Gamma_p)$ are a basis for $K_{\alpha_p}(\Gamma_p)$, where $\Gamma_p$ is the graph theoretical fiber of $(\Gamma, \alpha)$ over a vertex $p$ of $\Gamma_B$. Then in section 6 we describe an important class of such generators. One property of a GKM fiber bundle is a holonomy action of the fundamental group of $\Gamma_B$ on the fiber and we show how a collection of holonomy invariant generating classes $c'_1, \ldots, c'_k$ of $K_{\alpha_p}(\Gamma_p)$ extend canonically to a free set of generators $c_1, \ldots, c_k$ of $K_\alpha(\Gamma)$. In section 7 and 8 we describe how these results apply to concrete examples: special cases of the $G/P$ examples mentioned above. Finally in section 9 we relate our GKM description of the equivariant $K$-ring, $K_T(G/P)$, to a concise and elegant alternative description of this ring by Bertram Kostant and Shrawan Kumar in [KK], and analyze from their perspective the fiber bundle $G/B \to G/P$.

To conclude these introductory remarks we would like to thank Tudor Ratiu for his support and encouragement and Tara Holm for helpful suggestions about the relations between GKM and Kostant-Kumar.

## 2. $K$-theory of integral GKM graphs

Let $\Gamma = (V, E)$ be a $d$-valent graph, where $V$ is the set of vertices, and $E$ the set of oriented edges; for every edge $e \in E$ from $p$ to $q$, we denote by $\overline{e}$ the edge from $q$ to $p$. Let $i\colon E \to V$ (resp. $t\colon E \to V$) be the map which assigns to each oriented edge $e$ its initial (resp. terminal) vertex (so $i(e) = t(\overline{e})$ and $t(e) = i(\overline{e})$); for every $p \in V$ let $E_p$ be the set of edges whose initial vertex is $p$.

Let $T$ be an $n$-dimensional torus; we define a "$T$-action on $\Gamma$" by the following recipe.

**Definition 2.1.** Let $e = (p, q)$ be an oriented edge in $E$. Then a *connection along* $e$ is a bijection $\nabla_e \colon E_p \to E_q$ such that $\nabla_e(e) = \overline{e}$. A connection on the graph $\Gamma$ is a family $\nabla = \{\nabla_e\}_{e \in E}$ satisfying $\nabla_{\overline{e}} = \nabla_e^{-1}$ for every $e \in \Gamma$.

Let $\mathfrak{t}^*$ be the dual of the Lie algebra of $T$ and $\mathbb{Z}_T^*$ its weight lattice.

**Definition 2.2.** Let $\nabla$ be a connection on $\Gamma$. A $\nabla$-*compatible integral axial function* on $\Gamma$ is a map $\alpha \colon E \to \mathbb{Z}_T^*$ satisfying the following conditions:

(1) $\alpha(\overline{e}) = -\alpha(e)$;
(2) for every $p \in V$ the vectors $\{\alpha(e) \mid e \in E_p\}$ are pairwise linearly independent;
(3) for every edge $e = (p, q)$ and every $e' \in E_p$ we have
$$\alpha(\nabla_e(e')) - \alpha(e') = m(e, e')\alpha(e) \ ,$$
where $m(e, e')$ is an integer which depends on $e$ and $e'$.

An *integral axial function* on $\Gamma$ is a map $\alpha \colon E \to \mathbb{Z}_T^*$ which is $\nabla$-compatible, for some connection $\nabla$ on $\Gamma$.

**Definition 2.3.** An *integral GKM graph* is a pair $(\Gamma, \alpha)$ consisting of a regular graph $\Gamma$ and an integral axial function $\alpha \colon E \to \mathbb{Z}_T^*$

**Remark 2.4.** The graphs we described in the introduction are examples of such graphs. In particular condition 2 in definition 2.2 is a consequence of the fact that, for every codimension one subgroup of $T$, its fixed point components are of dimension at most two, and condition 3 a consequence of the fact that this subgroup acts trivially on the tangent bundles of these component.

Observe that an integral GKM graph is a particular case of an abstract GKM graph, as defined in [GSZ1]; here we require $\alpha$ to take values in $\mathbb{Z}_T^*$ rather than in $\mathfrak{t}^*$, and in definition 2.2 (3) we require $\alpha(\nabla_e(e')) - \alpha(e')$ to be an integer multiple of $\alpha(e)$, for every $e = (p, q) \in E$ and $e' \in E_p$. The necessity of these integrality properties will be clear from the definition of $T$-*action on* $\Gamma$. Let $R(T)$ be the representation ring of $T$; notice that $R(T)$ can be identified with the character ring of $T$, i.e. with the ring of finite sums
$$\sum_k m_k e^{2\pi\sqrt{-1}\alpha_k} \ , \tag{2.1}$$
where the $m_k$'s are integers and $\alpha_k \in \mathbb{Z}_T^*$. So giving an axial function $\alpha \colon E \to \mathbb{Z}_T^*$ is equivalent to giving a map which assigns to each edge $e \in E$ a one dimensional representation $\rho_e$, whose character $\chi_e \colon T \to S^1$ is given by
$$\chi_e(e^{2\pi\sqrt{-1}\xi}) = e^{2\pi\sqrt{-1}\alpha(e)(\xi)} \ .$$
For every $e \in E$, let $T_e = \ker(\chi_e)$, and consider the restriction map
$$r_e \colon R(T) \to R(T_e) \ .$$
Then for every vertex $p \in V$, we also obtain a $d$-dimensional representation
$$\nu_p \simeq \bigoplus_{e \in E_p} \rho_e$$

which, by definition 2.2 (3) satisfies

$$r_e(\nu_{i(e)}) \simeq r_e(\nu_{t(e)}) \; . \tag{2.2}$$

So an integral axial function $\alpha\colon E \to \mathbb{Z}_T^*$ defines a one-dimensional representation $\rho_e$ for every edge $e \in E$ and for every $p \in V$ a $d$-dimensional representation $\nu_p$ satisfying (2.2); this is what we refer to as a *$T$-action on* $\Gamma$.

**Remark 2.5.** Henceforth in this article all GKM graphs will, unless otherwise specified, be *integral* GKM graphs.

We will now define the $K$-ring $K_\alpha(\Gamma)$ of $(\Gamma, \alpha)$. As we remarked in the introduction, Knutson and Rosu have proved that if $(\Gamma, \alpha)$ is the GKM graph associated to a GKM manifold $M$, then

$$K_\alpha(\Gamma) \simeq K_T(M) \; ,$$

where $K_T(M)$ is the equivariant $K$-theory ring of $M$ (cf. [R]).

Let $\mathrm{Maps}(V, R(T))$ be the ring of maps which assign to each vertex $p \in V$ a representation of $T$. Following the argument in [R], we define a subring of $\mathrm{Maps}(V, R(T))$, called the ring of *$K$-classes of* $(\Gamma, \alpha)$.

**Definition 2.6.** Let $f$ be an element of $\mathrm{Maps}(V, R(T))$. Then $f$ is a $K$-class of $(\Gamma, \alpha)$ if for every edge $e = (p, q) \in E$

$$r_e(f(p)) = r_e(f(q)) \; . \tag{2.3}$$

Observe that using the identification of $R(T)$ with the ring of finite sums (2.1), condition (2.3) is equivalent to saying that for every $e = (p, q) \in E$

$$f(p) - f(q) = \beta(1 - e^{2\pi\sqrt{-1}\alpha(e)}), \tag{2.4}$$

for some $\beta$ in $R(T)$.

If $f$ and $g$ are two $K$-classes, then also $f + g$ and $fg$ are; so the set of $K$-classes is a subring of $\mathrm{Maps}(V, R(T))$.

**Definition 2.7.** The *$K$-ring* of $(\Gamma, \alpha)$, denoted by $K_\alpha(\Gamma)$, is the subring of $\mathrm{Maps}(V, R(T))$ consisting of all the $K$-classes.

## 3. GKM fiber bundles

Let $(\Gamma_1, \alpha_1)$ and $(\Gamma_2, \alpha_2)$ be GKM graphs, where $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$, $\alpha_1\colon E_1 \to \mathbb{Z}_{T_1}^* \subset \mathfrak{t}_1^*$ and $\alpha_2\colon E_2 \to \mathbb{Z}_{T_2}^* \subset \mathfrak{t}_2^*$.

**Definition 3.1.** An *isomorphism of GKM graphs* $(\Gamma_1, \alpha_1)$ and $(\Gamma_2, \alpha_2)$ is a pair $(\Phi, \Psi)$, where $\Phi\colon \Gamma_1 \to \Gamma_2$ is an isomorphism of graphs, and $\Psi\colon \mathfrak{t}_1^* \to \mathfrak{t}_2^*$ is an isomorphism of linear spaces such that $\Psi(\mathbb{Z}_{T_1}^*) = \mathbb{Z}_{T_2}^*$, and for every edge $(p, q)$ of $\Gamma_1$ we have $\alpha_2(\Phi(p), \Phi(q)) = \Psi(\alpha_1(p, q))$.

By definition, if $(\Phi, \Psi)$ is an isomorphism of GKM graphs, the diagram

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\alpha_1"'] & E_2 \arrow[d, "\alpha_2"] \\ \mathbb{Z}_{T_1}^* \arrow[r, "\Psi|_{\mathbb{Z}_{T_1}^*}"] & \mathbb{Z}_{T_2}^* \end{tikzcd} \tag{3.1}$$

commutes. We can extend the map $\Psi$ to be a ring homomorphism from $R(T_1)$ to $R(T_2)$ using the identification (2.1) and defining $\Psi(e^{2\pi\sqrt{-1}\alpha}) = e^{2\pi\sqrt{-1}\Psi(\alpha)}$, for every $\alpha \in \mathbb{Z}^*_{T_1}$.

Given $f \in \mathrm{Maps}(V_2, R(T_2))$, let $\Upsilon^*(f) \in \mathrm{Maps}(V_1, R(T_1))$ be the map defined by $\Upsilon^*(f)(p) = \Psi^{-1}(f(\Phi(p)))$. From the commutativity of the diagram (3.1) it's easy to see that if $f \in K_{\alpha_2}(\Gamma_2)$ then $\Upsilon^*(f) \in K_{\alpha_1}(\Gamma_1)$, and $\Upsilon^*$ defines an isomorphism between the two $K$-rings.

We are now ready to define the main combinatorial objects of this paper, GKM fiber bundles. Let $(\Gamma, \alpha)$ and $(\Gamma_B, \alpha_B)$ be two GKM graphs, with $\alpha$ and $\alpha_B$ having images in the same weight lattice $\mathbb{Z}^*_T$. Let $\pi\colon \Gamma = (V, E) \to \Gamma_B = (V_B, E_B)$ be a surjective morphism of graphs. By that we mean that $\pi$ maps the vertices of $\Gamma$ onto the vertices of $\Gamma_B$, such that, for every edge $e = (p, q)$ of $\Gamma$, either $\pi(p) = \pi(q)$ (in which case $e$ is called *vertical*), or $(\pi(p), \pi(q))$ is an edge of $\Gamma_B$ (in which case $e$ is called *horizontal*). Such a morphism of graphs induces a map $(d\pi)_p\colon H_p \to E_{\pi(p)}$ from the set of horizontal edges at $p \in V$ to the set of all edges starting at $\pi(p) \in V_B$. The first condition we impose for $\pi$ to be a GKM fiber bundle is the following:

**1:** For all vertices $p \in V$, $(d\pi)_p\colon H_p \to E_{\pi(p)}$ is a bijection compatible with the axial functions:

$$\alpha_B((d\pi)_p(e)) = \alpha(e),$$

for all $e = (p, q) \in H_p$.

The second condition has to do with the connection on $\Gamma$ and $\Gamma_B$.

**2:** The connection along edges of $\Gamma$ moves horizontal edges to horizontal edges, and vertical edges to vertical edges. Moreover, the restriction of the connection of $\Gamma$ to horizontal edges is compatible with the connection on $\Gamma_B$.

For every vertex $p \in \Gamma_B$, let $V_p = \pi^{-1}(p) \subset V$ and $\Gamma_p$ the induced subgraph of $\Gamma$ with vertex set $V_p$. If the map $\pi$ satisfies condition **1**, then, for every edge $e = (p, q)$ of $\Gamma_B$, it induces a bijection $\Phi_{p,q}\colon V_p \to V_q$ by $\Phi_{p,q}(p') = q'$ if and only if $(p, q) = (d\pi)_p(p', q')$.

**3:** For every edge $(p, q) \in \Gamma_B$, $\Phi_{p,q}\colon \Gamma_p \to \Gamma_q$ is an isomorphism of graphs compatible with the connection $\nabla$ on $\Gamma$ in the following sense: for every lift $e' = (p_1, q_1)$ of $e = (p, q)$ at $p_1$ and every edge $e'' = (p_1, p_2)$ of $\Gamma_p$ the connection along the horizontal edge $(p_1, q_1)$ moves the vertical edge $(p_1, p_2)$ to the vertical edge $(q_1, q_2)$, where $q_i = \Phi_{p,q}(p_i)$, $i = 1, 2$.

We can endow $\Gamma_p$ with a GKM structure, which is just the restriction of the GKM structure of $(\Gamma, \alpha)$ to $\Gamma_p$. The axial function on $\Gamma_p$ is the restriction of $\alpha\colon E \to \mathbb{Z}^*_T$ to the edges of $\Gamma_p$; we refer to it as $\alpha_p$, and it takes values in $\mathfrak{v}^*_p$, the subspace of $\mathfrak{t}^*$ generated by values of axial functions $\alpha(e)$, for edges $e$ of $\Gamma_p$. The next condition we impose on $\pi$ is the following:

**4:** For every edge $(p, q)$ of $\Gamma_B$, there exists an isomorphism of GKM graphs

$$\Upsilon_{p,q} = (\Phi_{p,q}, \Psi_{p,q})\colon (\Gamma_p, \alpha_p) \to (\Gamma_q, \alpha_q)\ .$$

By property (3) of definition 2.2 and the fact that $\alpha_B(e) = \alpha(e')$ we have

$$\alpha(\Phi_{p,q}(p_1), \Phi_{p,q}(p_2)) - \alpha(p_1, p_2) = m'(p_1, p_2)\alpha_B(e)\ , \tag{3.2}$$

where $m'(p_1, p_2)$ is an integer; so by the commutativity of (3.1) we have

$$\Psi_{p,q}(\alpha(p_1, p_2)) - \alpha(p_1, p_2) = m'(p_1, p_2)\alpha_B(e)\ . \tag{3.3}$$

Condition (3.3) defines a map $m'\colon E_p \to \mathbb{Z}$, where $E_p$ is the set of edges of $\Gamma_p$. The next condition is a strengthening of this.

**5:** There exists a function $m\colon \mathfrak{v}_p^* \cap \mathbb{Z}_T^* \to \mathbb{Z}$ such that

$$\Psi_{p,q}(x) = x + m(x)\alpha_B(p,q) \ . \tag{3.4}$$

Observe that if $\Gamma_B$ is connected, then all the fibers of $\pi$ are isomorphic as GKM graphs. More precisely, for any two vertices $p, q \in V_B$ let $\gamma$ be a path in $\Gamma_B$ from $p$ to $q$, i.e. $\gamma\colon p = p_0 \to p_1 \to \cdots \to p_m = q$. Then the map

$$\Upsilon_\gamma = \Upsilon_{p_{m-1},p_m} \circ \cdots \circ \Upsilon_{p_0,p_1} \colon (\Gamma_p, \alpha_p) \to (\Gamma_q, \alpha_q) \ .$$

defines an isomorphism between the GKM graphs $(\Gamma_p, \alpha_p)$ and $(\Gamma_q, \alpha_q)$. As observed before, this isomorphism restricts to an isomorphism between the two $K$-rings, $\Upsilon_\gamma^*\colon K_{\alpha_q}(\Gamma_q) \to K_{\alpha_p}(\Gamma_p)$, which is not an isomorphism of $R(T)$-modules, unless the linear isomorphism

$$\Psi_\gamma = \Psi_{p_0,p_1} \circ \cdots \circ \Psi_{p_{m-1},p_m} \colon \mathfrak{v}_q^* \to \mathfrak{v}_p^*$$

is the identity.

Let $\Omega(p)$ be the set of all loops in $\Gamma_B$ that start and end at $p$, i.e. the set of paths $\gamma\colon p_0 \to p_1 \to \cdots \to p_{m-1} \to p_m$ such that $p_0 = p_m = p$. Then every such $\gamma$ determines a GKM isomorphism $\Upsilon_\gamma$ of the fiber $(\Gamma_p, \alpha_p)$. The *holonomy group* of the fiber $(\Gamma_p, \alpha_p)$ is the subgroup of the GKM isomorphisms of the fiber, $\mathrm{Aut}(\Gamma_p, \alpha_p)$, given by

$$\mathrm{Hol}_\pi(\Gamma_p) = \{\Upsilon_\gamma \mid \gamma \in \Omega(p)\} \leqslant \mathrm{Aut}(\Gamma_p, \alpha_p) \ .$$

## 4. Flag manifolds as GKM fiber bundles

In this section we will discuss some important examples of GKM fiber bundles coming from generalized partial flag varieties.

Let $G$ be a complex semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and $\mathfrak{t} \subset \mathfrak{h}$ a compact real form; let $T$ be the compact torus whose Lie algebra is $\mathfrak{t}$. Let $\Delta \subset \mathbb{Z}_T^* \subset \mathfrak{t}^*$ be the set of roots and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

the Cartan decomposition of $\mathfrak{g}$. Let $\Delta_0 = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta$ be a choice of simple roots and $\Delta^+$ the corresponding positive roots. The set of positive roots determines a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ given by

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \ .$$

Let $\Sigma \subset \Delta_0$ be a subset of simple roots, and $B \leqslant P(\Sigma) \leqslant G$, where $B$ is the Borel subgroup whose Lie algebra is $\mathfrak{b}$ and $P(\Sigma)$ the parabolic subgroup of $G$ corresponding to $\Sigma$. Then $M = G/P(\Sigma)$ is a (generalized) partial flag manifold. In particular, if $\Sigma = \emptyset$, then $P(\emptyset) = B$, and we refer to $M = G/B$ as the (generalized) complete flag manifold.

The torus $T$ with Lie algebra $\mathfrak{t}$ acts on $M = G/P(\Sigma)$ by left multiplication on $G$; this action determines a GKM structure on $M$ with GKM graph $(\Gamma, \alpha)$. In fact, let $W$ be the Weyl group of $\mathfrak{g}$ and $W(\Sigma)$ the subgroup of $W$ generated by reflections across the simple roots in $\Sigma$, and $\langle\Sigma\rangle$ the positive roots which can be written as a linear combination of the roots in $\Sigma$. Then the vertices of $\Gamma$, corresponding to the $T$-fixed points, are in bijection with the right cosets

$$W/W(\Sigma) = \{vW(\Sigma) \mid v \in W\} = \{[v] \mid v \in W\} ,$$

where $[v] = vW(\Sigma)$ is the right $W(\Sigma)$-coset containing $v$. Two vertices $[v]$ and $[w]$ are joined by an edge if and only if there exists $\beta \in \Delta^+ \setminus \langle\Sigma\rangle$ such that $[v] = [ws_\beta]$; moreover the axial function $\alpha$ on the edge $e = ([v], [vs_\beta])$ is given by $\alpha([v], [vs_\beta]) = v\beta$; it's easy to see that this label is well defined. Moreover, given an edge $e' = ([v], [vs_{\beta'}])$ starting at $[v]$, a natural choice of a connection along $e$ is given by $\nabla_e e' = ([vs_\beta], [vs_\beta s_{\beta'}])$. So

$$\alpha(\nabla_e e') - \alpha(e') = vs_\beta\beta' - v\beta' = s_{v\beta}v\beta' - v\beta' = m(e, e')v\beta ,$$

where $m(e, e')$ is a Cartan integer; so this connection satisfies property (3) of definition 2.2.

Consider the natural $T$-equivariant projection $\pi\colon G/B \to G/P(\Sigma)$; this map induces a projection map on the corresponding GKM graphs $\pi\colon (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ given by $\pi(w) = [w]$ for every $w \in W$, where $[w] = wW(\Sigma)$. As we proved in [GSZ1, section 4.3], $\pi$ is a GKM fiber bundle. Moreover, let $\Gamma_0$ be the GKM graph of the fiber containing the identity element of $W$; then

$$\mathrm{Hol}_\pi(\Gamma_0) \simeq W(\Sigma).$$

The key ingredient in the proof of this is the identification of the weights of the $T$ action on the tangent space to the identity coset $p_0$ of $G/P$ with the complement of $\langle\Sigma\rangle$ in $\Delta^+$. Namely for any such root $\alpha$ let $w \in W$ be the Weyl group element $w\colon \mathfrak{t} \to \mathfrak{t}$ associated with the reflection in the hyperplane $\alpha = 0$. Then for the edge $e = (p_0, wp_0)$ of the GKM graph of $G/P$ the GKM isomorphism of $\Gamma_{p_0}$ onto $\Gamma_p$ is the isomorphism $\Phi_{p_0,p}$ associated with the left action of $w$ on $G/B$ and the $T$ automorphism (3.4) is just the action of $w$ on $\mathfrak{t}$ given by compositions of reflections in the hyperplane $\alpha_B(p, q) = 0$ in $\mathfrak{t}$, for some horizontal edge $(p, q)$. (These results are also a byproduct of the identification of the GKM model of $K_T(G/P)$ with the Kostant-Kumar model which we will describe in section 9).

## 5. $K$-theory of GKM fiber bundles

Given a GKM fiber bundle $\pi\colon (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$, we will describe the $K$-ring of $(\Gamma, \alpha)$ in terms of the $K$-ring of $(\Gamma_B, \alpha_B)$. In the proof of the main theorem we will need the following technical lemma.

**Lemma 5.1.** Let $\alpha$ and $\beta$ be linearly independent weights in $\mathfrak{t}^*$, and $P$ an element of $R(T)$ . If $1 - e^{2\pi\sqrt{-1}\alpha}$ divides $(1 - e^{2\pi\sqrt{-1}\beta})P$ then $1 - e^{2\pi\sqrt{-1}\alpha}$ divides $P$.

*Proof.* Let $\alpha = m\alpha_1$, for some $m \in \mathbb{Z}$, where $\alpha_1$ is a primitive element of the weight lattice in $\mathfrak{t}^*$. We can complete $\alpha_1$ to a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ of the lattice. Let $x_j = e^{2\pi\sqrt{-1}\alpha_j}$ for all $j = 1, \ldots, d$.
Then by hypothesis $(1 - x_1^m)$ divides $(1 - x_1^n Q(x_2, \ldots, x_d))P(x_1, \ldots, x_d)$ for some non-constant polynomial $Q(x_2, \ldots, x_d)$ . Consider an element $\xi \in \mathfrak{t} \otimes \mathbb{C}$ such that $x_1(\xi)^m = 1$; then $(1 - x_1^n Q(x_2, \ldots, x_d))P(x_1, \ldots, x_d)(\xi) = 0$. Since in general $(1 - x_1^n Q(x_2, \ldots, x_d))(\xi) \neq 0$, this implies that $(1 - x_1^m)$ divides $P(x_1, \ldots, x_d)$. $\square$

For every $K$-class $f\colon V_B \to R(T)$, define the pull-back $\pi^*(f)\colon V \to R(T)$ by $\pi^*(f)(q) = f(\pi(q))$. It's easy to check that $\pi^*(f)$ is a $K$-class on $(\Gamma, \alpha)$. So $K_\alpha(\Gamma)$

contains $K_{\alpha_B}(\Gamma_B)$ as a subring, and the map $\pi^*\colon K_{\alpha_B}(\Gamma_B) \to K_\alpha(\Gamma)$ gives $K_\alpha(\Gamma)$ the structure of a $K_{\alpha_B}(\Gamma_B)$-module.

**Definition 5.2.** A $K$-class $h \in K_\alpha(\Gamma)$ is called *basic* if $h \in \pi^*(K_{\alpha_B}(\Gamma_B))$.

We denote the subring of basic $K$-classes by $(K_\alpha(\Gamma))_{bas}$; clearly we have

$$(K_\alpha(\Gamma))_{bas} \simeq K_{\alpha_B}(\Gamma_B) \ .$$

**Theorem 5.3.** Let $\pi : (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ be a GKM fiber bundle, and let $c_1, \ldots, c_m$ be $K$-classes on $\Gamma$ such that for every $p \in V_B$ the restriction of these classes to the fiber $\Gamma_p = \pi^{-1}(p)$ form a basis for the $K$-ring of the fiber. Then, as $K_{\alpha_B}(\Gamma_B)$-modules, $K_\alpha(\Gamma)$ is isomorphic to the free $K_{\alpha_B}(\Gamma_B)$-module on $c_1, \ldots, c_m$.

*Proof.* First of all, observe that any linear combination of $c_1, \ldots, c_m$ with coefficients in $(K_\alpha(\Gamma))_{bas} \simeq K_{\alpha_B}(\Gamma_B)$ is an element of $K_\alpha(\Gamma)$. Now we want to prove that the $c_i$'s are independent over $K_{\alpha_B}(\Gamma_B)$. In order to prove so, let $\sum_{k=1}^m \beta_k c_k = 0$ for some $\beta_1, \ldots, \beta_m \in (K_\alpha(\Gamma))_{bas}$. Let $\Gamma_p = \pi^{-1}(p)$ denote the fiber over $p \in B$, $\iota_p : \Gamma_p \to \Gamma$ the inclusion, and $\iota_p^*\colon K_\alpha(\Gamma) \to K_\alpha(\Gamma_p)$ the restriction to the $K$-theory of the fiber. Then $\sum_{k=1}^m \iota_p^*(\beta_k c_k) = 0$ for all $p \in B$. Since the $\beta_k$'s are basic $K$-classes, $\iota_p^*(\beta_k)$ is just an element of $R(T)$ for all $k$. But by assumption $\{\iota_p^*(c_1), \ldots, \iota_p^*(c_m)\}$ is a basis of $K_\alpha(\Gamma_p)$; so $\iota_p^*(\beta_k) = 0$ for all $k = 1, \ldots, m$, for all $p \in \Gamma_B$, which implies that $\beta_k = 0$ for all $k$. We need to prove that the free $K_{\alpha_B}(\Gamma_B)$-module generated by $c_1, \ldots, c_m$ is $K_\alpha(\Gamma)$.

Let $c \in K_\alpha(\Gamma)$. Since the classes $\iota_p^* c_1, \ldots, \iota_p^* c_m$ are a basis for $K_\alpha(\Gamma_p)$, there exist $\beta_1, \ldots, \beta_m \in \mathrm{Maps}(B, R(T))$ such that

$$c = \sum_{k=1}^m \beta_k c_k \ ;$$

we need to prove that the $\beta_k$'s belong to $(K_\alpha(\Gamma))_{bas}$ for all $k$. In order to prove this, it is sufficient to show that (2.4) is satisfied for every edge $(p, q)$ of $\Gamma_B$. Let $e' = (p', q')$ be the lift of $(p, q)$ at $p' \in \Gamma_p$. Then

$$\begin{aligned} c(q') - c(p') &= \sum_{k=1}^m (\beta_k(q) c_k(q') - \beta_k(p) c_k(p')) \\ &= \sum_{k=1}^m (\beta_k(q) - \beta_k(p)) c_k(p') + \sum_{k=1}^m \beta_k(q)(c_k(q') - c_k(p')) \ . \end{aligned}$$

Since $c, c_1, \ldots, c_m$ belong to $K_\alpha(\Gamma)$, by (2.4) the differences $c(q') - c(p')$, $c_k(q') - c_k(p')$ are multiples of $(1 - e^{2\pi\sqrt{-1}\alpha(e')})$, for all $k = 1, \ldots, m$. Therefore, for all $p' \in \Gamma_p$,

$$\sum_{k=1}^m (\beta_k(q) - \beta_k(p)) c_k(p') = (1 - e^{2\pi\sqrt{-1}\alpha(e')}) \eta(p') \ ,$$

where $\eta(p') \in R(T)$. We will show that $\eta\colon \Gamma_p \to R(T)$ belongs to $K_\alpha(\Gamma_p)$.

If $p'$ and $p''$ are vertices in $\Gamma_p$, joined by an edge $(p', p'')$, then

$$\sum_{k=1}^m (\beta_k(q) - \beta_k(p))(c_k(p'') - c_k(p')) = (1 - e^{2\pi\sqrt{-1}\alpha(e')})(\eta(p'') - \eta(p')) \ .$$

Each $c_k$ is a $K$-class on $\Gamma$, so $c_k(p'') - c_k(p')$ is a multiple of $(1 - e^{2\pi\sqrt{-1}\alpha(p', p'')})$, for all $k = 1, \ldots, m$. Then $(1 - e^{2\pi\sqrt{-1}\alpha(p', p'')})$ divides $(1 - e^{2\pi\sqrt{-1}\alpha(e')})(\eta(p'') - \eta(p'))$.

But $\alpha(e')$ and $\alpha(p',p'')$ are linearly independent vectors. Therefore, by Lemma 5.1, $(1-e^{2\pi\sqrt{-1}\alpha(p',p'')})$ divides $\eta(p'')-\eta(p')$, and so $\eta$ is a $K$-class on $\Gamma_p$.

Since the classes $\iota_p^* c_1,\dots,\iota_p^* c_m$ are a basis for $K_\alpha(\Gamma_p)$ there exist $Q_1,\dots,Q_m\in R(T)$ such that

$$\eta=\sum_{k=1}^m Q_k\iota_p^* c_k\ .$$

Then

$$\sum_{k=1}^m(\beta_k(q)-\beta_k(p)-Q_k(1-e^{2\pi\sqrt{-1}\alpha(e')}))\iota_p^* c_k=0\ .$$

But $\iota_p^* c_1,\dots,\iota_p^* c_m$ are linearly independent over $R(T)$, so

$$\beta_k(q)-\beta_k(p)=Q_k(1-e^{2\pi\sqrt{-1}\alpha(e')})\ .$$

Since $\alpha(e')=\alpha(p',q')=\alpha_B(p,q)$, this implies that $\beta_k\in K_{\alpha_B}(\Gamma_B)$. Therefore every $K$-class on $\Gamma$ can be written as a linear combination of classes $c_1,\dots,c_m$, with coefficients in $K_{\alpha_B}(\Gamma_B)$. □

## 6. Invariant classes

Let $\pi\colon(\Gamma,\alpha)\to(\Gamma_B,\alpha_B)$ be a GKM fiber bundle, and let $(\Gamma_p,\alpha_p)$ be one of its fibers. We say that $f\in K_{\alpha_p}(\Gamma_p)$ is an *invariant class* if $\Upsilon_\gamma^*(f)=f$ for every $\Upsilon_\gamma\in\mathrm{Hol}_\pi(\Gamma_p)$. We denote by $(K_{\alpha_p}(\Gamma_p))^{\mathrm{Hol}}$ the subring of $K_{\alpha_p}(\Gamma_p)$ given by invariant classes.

Given any such class $f\in(K_{\alpha_p}(\Gamma_p))^{\mathrm{Hol}}$, we can extend it to be an element of Maps: $V\to R(T)$ by the following recipe: let $q$ be a vertex of $\Gamma_B$, and $\gamma$ a path in $\Gamma_B$ from $q$ to $p$. Let $\Upsilon_\gamma\colon(\Gamma_q,\alpha_q)\to(\Gamma_p,\alpha_p)$ be the isomorphism of GKM graphs associated to $\gamma$; then, as we observed before, $\Upsilon_\gamma^*(f)$ defines an element of $K_{\alpha_q}(\Gamma_q)$. Notice that the invariance of $f$ implies that $\Upsilon_\gamma^*(f)$ only depends on the end-points of $\gamma$; so we denote $\Upsilon_\gamma^*(f)$ by $f_q$.

**Proposition 6.1.** Let $c\in\mathrm{Maps}(V,R(T))$ be the map defined by $c(q')=f_{\pi(q')}(q')$ for any $q'\in V$. Then $c\in K_\alpha(\Gamma)$.

*Proof.* Since $c$ is a class on each fiber, it is sufficient to check the compatibility condition (2.4) on horizontal edges; let $e=(p,q)$ be an edge of $\Gamma_B$, and let $e'=(p',q')$ be its lift at $p'\in V$. If $c(p')=f_p(p')=\sum_k n_k e^{2\pi\sqrt{-1}\alpha_k}$, where $\alpha_k\in\mathbb{Z}_T^*\cap\mathfrak{v}_p^*$ and $n_k\in\mathbb{Z}$ for all $k$, then

$$c(p')-c(q')=f_p(p')-f_q(q')=f_p(p')-\Psi_e(f_p(p'))=\sum_k n_k(e^{2\pi\sqrt{-1}\alpha_k}-e^{2\pi\sqrt{-1}\Psi_e(\alpha_k)})\ .$$

By definition of GKM fiber bundle, $\Psi_e(\alpha_k)=\alpha_k+c(\alpha_k)\alpha_B(p,q)$ for every $k$, where $c(\alpha_k)$ is an integer and $\alpha_B(p,q)=\alpha(p',q')$. So $e^{2\pi\sqrt{-1}\alpha_k}-e^{2\pi\sqrt{-1}\Psi_e(\alpha_k)}=e^{2\pi\sqrt{-1}\alpha_k}(1-e^{2\pi\sqrt{-1}c(\alpha_k)\alpha_B(p,q)})$, and it's easy to see that $(1-e^{2\pi\sqrt{-1}c(\alpha_k)\alpha_B(p,q)})=\beta_k(1-e^{2\pi\sqrt{-1}\alpha_B(p,q)})$ for some $\beta_k\in R(T)$, for every $k$. □

## 7. Classes on projective spaces

Let $T=(S^1)^n$ be the compact torus of dimension $n$, with Lie algebra $\mathfrak{t}=\mathbb{R}^n$, and let $\{y_1,\dots,y_n\}$ be the basis of $\mathfrak{t}^*\simeq\mathbb{R}^n$ dual to the canonical basis of $\mathbb{R}^n$. Let

$\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{C}^n$. The torus $T$ acts componentwise on $\mathbb{C}^n$ by

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n) \ . \tag{7.1}$$

This action induces a GKM action of $T$ on $\mathbb{C}P^{n-1}$, and the GKM graph is $\Gamma = \mathcal{K}_n$, the complete graph on $n$ vertices labelled by $[n] = \{1, \dots, n\}$. The axial function $\alpha$ on the edge $(i,j)$ is $y_i - y_j$, for every $i \neq j$. Let $\mathbb{S} = \mathbb{Z}[y_1, \dots, y_n]$, $(\mathbb{S})$ the field of fractions of $\mathbb{S}$, $\mathcal{M} = \mathrm{Maps}([n], \mathbb{S})$, and

$$H_\alpha(\Gamma) = \{ f \in \mathcal{M} \mid f(j) - f(k) \in (y_j - y_k)\mathbb{S}, \text{ for all } j \neq k \} \ .$$

Then $H_\alpha(\Gamma)$ is an $\mathbb{S}$-subalgebra of $\mathcal{M}$. Let $\int_\Gamma \colon \mathcal{M} \to (\mathbb{S})$ be the map

$$\int_\Gamma f = \sum_{k=1}^n \frac{f(k)}{\prod_{j \neq k}(y_k - y_j)} \ .$$

**Proposition 7.1.** Let $f \in \mathcal{M}$. Then $f \in H_\alpha(\Gamma)$ if and only if $\int_\Gamma f \in \mathbb{S}$.

*Proof.* We have

$$\int_\Gamma f = \sum_{k=1}^n \frac{f(k)}{\prod_{j \neq k}(y_k - y_j)} = \frac{P}{\prod_{j<k}(y_j - y_k)} \ ,$$

where $P \in \mathbb{S}$. The factors in the denominator are distinct and relatively prime, hence $\int_\Gamma f \in \mathbb{S}$ if and only if all factors in the denominator divide $P$.

The factor $y_j - y_k$ comes from

$$\frac{f(k)}{\prod_{i \neq k}(y_i - y_j)} + \frac{f(j)}{\prod_{i \neq j}(y_i - y_j)} = \frac{f(j) - f(k)}{(y_j - y_k)\prod_{i \neq j,k}(y_j - y_i)} + \\ + \frac{f(k)\left(\prod_{i \neq k,k}(y_k - y_i) - \prod_{i \neq j,k}(y_j - y_i)\right)}{(y_j - y_k)\prod_{i \neq j,k}(y_j - y_i)(y_k - y_i)} \ .$$

But $y_j - y_k$ divides the numerator of the second fraction, hence $y_j - y_k$ divides $P$ if and only if it divides $f(j) - f(k)$. $\square$

The permutation group $S_n$ acts on $\mathbb{S}$ by permuting variables, and that action induces an action on $H_\alpha(\Gamma)$ by

$$(w \cdot f)(j) = w^{-1} \cdot f(w(j)) \ .$$

We say that a class $f \in H_\alpha(\Gamma)$ is $S_n$*-invariant* if $w \cdot f = f$ for every $w \in S_n$, i.e.

$$f(w(j)) = w \cdot f(j) \quad \text{for every } \ w \in S_n \ .$$

The goal of this section is to construct bases of the $\mathbb{S}$-module $H_\alpha(\Gamma)$ consisting of $S_n$-invariant classes, and to give explicit formulas for the coordinates of a given class in those bases.

Let $\phi \colon [n] \to \mathbb{S}$, $\phi(j) = y_j$ for all $1 \leqslant j \leqslant n$. Then $\phi$ is an $S_n$-invariant class in $H_\alpha(\Gamma)$. For $1 \leqslant k \leqslant n$, let $f_k = \phi^{k-1}$. Then $f_1, f_2, \dots, f_n$ are $\mathbb{S}$-linearly independent invariant classes.

For $0 \leqslant j \leqslant n$, let $s_j$ be the $j^{th}$ elementary symmetric polynomial in the variables $y_1, \dots, y_n$. Then $s_0 = 1$, $s_1 = y_1 + \cdots + y_n$, $s_2 = y_1y_2 + y_1y_3 + \cdots + y_{n-1}y_n$, $\dots$, $s_n = y_1y_2 \cdots y_n$. For $1 \leqslant k \leqslant n$, let

$$g_k = f_k - s_1 f_{k-1} + s_2 f_{k-2} - \cdots + (-1)^{k-1} s_{k-1} f_1 \ .$$

Then $g_1, g_2, \dots, g_n$ are invariant classes and the transition matrix from the $f$'s to the $g$'s is triangular with ones on the diagonal, hence it is invertible over $\mathbb{S}$. Therefore the classes $g_1, \dots, g_n$ are also $\mathbb{S}$-linearly independent.

Let $\langle .,. \rangle \colon H_\alpha(\Gamma) \times H_\alpha(\Gamma) \to \mathbb{S}$ be the pairing

$$\langle f, g \rangle = \int_\Gamma fg \ .$$

**Theorem 7.2.** The sets of classes $\{f_1, \dots, f_n\}$ and $\{g_n, \dots, g_1\}$ are dual to each other:

$$\langle f_j, g_{n-k+1} \rangle = \delta_{jk} \ .$$

for all $1 \leqslant j, k \leqslant n$.

*Proof.* We have $\int_\Gamma \phi^k = 0$ for all $0 \leqslant k \leqslant n-2$ and $\int_\Gamma \phi^{n-1} = 1$. Moreover

$$\phi^n - s_1\phi^{n-1} + s_2\phi^{n-2} - \ldots + (-1)^n s_n \phi^0 = 0 \ .$$

Let $1 \leqslant j, k \leqslant n$. Then

$$f_j g_{n-k+1} = \phi^{j-1} \left( \phi^{n-k} - s_1\phi^{n-k-1} + \cdots + (-1)^{n-k} s_{n-k} \phi^0 \right) \ .$$

If $j < k$, then

$$f_j g_{n-k+1} = \text{ a combination of powers of } \phi \text{ at most } n-2 \ ,$$

and then $\langle f_j, g_{n-k+1} \rangle = 0$.

If $j = k$, then

$$f_k g_{n-k+1} = \phi^{n-1} + \text{ a combination of powers of } \phi \text{ at most } n-2 \ ,$$

hence $\langle f_k, g_{n-k+1} \rangle = \int_\Gamma \phi^{n-1} = 1$.

If $j > k$, then

$$\begin{aligned} f_j g_{n-k+1} &= \phi^{j-k-1} \left( \phi^n - s_1\phi^{n-1} + \cdots + (-1)^{n-k} s_{n-k} \phi^k \right) = \\ &= -\phi^{j-k-1} \left( (-1)^{n-k-1} s_{n-k+1} \phi^{k-1} + \ldots + (-1)^n s_n \phi^0 \right) = \\ &= \text{ a combination of powers of } \phi \text{ at most } n-2 \ , \end{aligned}$$

and then $\langle f_j, g_{n-k+1} \rangle = 0$. □

The following is an immediate consequence of this result.

**Corollary 7.3.** The sets $\{f_1, f_2, \dots, f_n\}$ and $\{g_1, g_2, \dots, g_n\}$ are dual bases of the $\mathbb{S}$-module $H_\alpha(\Gamma)$, and both bases consist of invariant classes. Moreover, if $h \in H_\alpha(\Gamma)$ and

$$h = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = b_1 g_1 + b_2 g_2 \cdots + b_n g_n \ ,$$

then

$$a_k = \langle g_{n-k+1}, h \rangle \quad \text{ and } \quad b_k = \langle f_{n-k+1}, h \rangle \ .$$

The entire discussion above extends naturally to $K$-theory. Let $z_j = e^{2\pi\sqrt{-1}y_j}$ for $j = 1, \dots, n$. If $y_1, \dots, y_n$ denote a basis of $\mathbb{Z}_T^*$, we have that

$$R(T) = \mathbb{Z}[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}].$$

Let $\psi \colon \mathbb{S} \to R(T)$ be the injective ring morphism determined by $\psi(y_j) = z_j$, for $j = 1, \dots, n$. Its image is $\psi(\mathbb{S}) = R_+(T) = \mathbb{Z}[z_1, \dots, z_n]$. Let

$$K_\alpha(\Gamma) = \{ g \colon [n] \to R(T) \mid f(j) - f(k) \in (z_j - z_k)R(T), \text{ for all } j \neq k \} \ .$$

Then

$$\Phi\colon H_\alpha(\Gamma) \to K_\alpha(\Gamma) \quad , \quad \Phi(f)(j) = \psi(f(j))$$

is an injective morphism of rings and

$$\Phi(qf) = \psi(q)\Phi(f)$$

for all $f \in H_\alpha(\Gamma)$ and $q \in \mathbb{S}$. The image of $\Phi$ is

$$\mathrm{im}\Phi = \{g \in K_\alpha(\Gamma) \mid \mathrm{im}(g) \subset R_+(T)\} .$$

Let $\nu = \Phi(\phi)$; then $\nu\colon [n] \to R(T)$, $\nu(j) = z_j$.

The symmetric group $S_n$ acts on $R(T)$ by simultaneously permuting the variables $z$ and $z^{-1}$, and that action induces an action on $K_\alpha(\Gamma)$. The $K$-class $\nu$ is invariant, and so are its powers.

**Proposition 7.4.** The invariant classes $\{1, \nu, \ldots, \nu^{n-1}\}$ form a basis of $K_\alpha(\Gamma)$ over $R(T)$.

*Proof.* A Vandermonde determinant argument shows that the classes are independent. If $g \in K_\alpha(\Gamma)$, then there exists an invertible element $u \in R(T)$ such that $ug \in \mathrm{im}\Phi$. Let $ug = \Phi(f)$, with $f \in H_\alpha(\Gamma)$. If

$$f = a_0\phi^0 + \cdots + a_{n-1}\phi^{n-1} ,$$

then

$$g = u^{-1}\psi(a_0)\nu^0 + \cdots + u^{-1}\psi(a_{n-1})\nu^{n-1}$$

and therefore the classes also generate $K_\alpha(\Gamma)$. □

## 8. Invariant bases on flag manifolds

In this section we use the same technique as above to produce a basis of invariant $K$-classes on the variety of complete flags in $\mathbb{C}^{n+1}$.

As in the previous section, let $T = (S^1)^{n+1}$ act componentwise on $\mathbb{C}^{n+1}$ by

$$(t_1, \ldots, t_{n+1}) \cdot (z_1, \ldots, z_{n+1}) = (t_1z_1, \ldots, t_{n+1}z_{n+1}),$$

and let $x_1, \ldots, x_{n+1} \in \mathbb{Z}_T^*$ be the weights. This action induces a GKM action both on the flag manifold $Fl(\mathbb{C}^{n+1})$ and $\mathbb{C}P^n$.

The GKM graph $(\Gamma, \alpha)$ associated to $Fl(\mathbb{C}^{n+1})$ is the permutahedron: its vertices are in bijection with the elements of $S_{n+1}$, the group of permutations on $n+1$ elements, and there exists an edge $e$ between two vertices $\sigma$ and $\sigma'$ if and only if $\sigma$ and $\sigma'$ differ by a transposition, i.e. $\sigma' = \sigma(i,j)$, for some $1 \leqslant i < j \leqslant n+1$; the axial function is given by $\alpha(\sigma, \sigma') = x_{\sigma(i)} - x_{\sigma'(i)}$.

The group $S_{n+1}$ acts on the GKM graph by left multiplication on its vertices, and on $\mathfrak{t}^*$ by

$$\sigma \cdot x_i = x_{\sigma(i)}.$$

Using the identification (2.1), this action determines an action on $R(T)$ by defining

$$\sigma \cdot (e^{2\pi\sqrt{-1}x_i}) = e^{2\pi\sqrt{-1}x_{\sigma(i)}} \tag{8.1}$$

and then extending it to the elements of $R(T)$ in the natural way. Using the fiber bundle construction introduced in this paper we will now produce a basis of the $K$-ring of $(\Gamma, \alpha)$ composed of $S_{n+1}$-*invariant classes*, i.e. elements $f \in K_\alpha(\Gamma)$ satisfying

$$f(u) = u \cdot f(id) \quad \text{for every} \quad u \in S_{n+1}.$$

The main idea in the construction of our invariant basis of $K$-classes for $Fl(\mathbb{C}^{n+1})$ is to use the natural projection $\pi$ of $Fl(\mathbb{C}^{n+1})$ to $\mathbb{C}P^n$ with fiber $Fl(\mathbb{C}^n)$ to construct these classes by an induction argument on $n$.

As we saw in section 7, the GKM graph $(\Gamma_B, \alpha_B)$ associated to $\mathbb{C}P^n$ is $(\mathcal{K}_{n+1}, \alpha_B)$, where $\alpha_B(i,j) = x_i - x_j$ for every $1 \leq i \neq j \leq n+1$. The projection $\pi\colon Fl(\mathbb{C}^{n+1}) \to \mathbb{C}P^n$ can be described in a simple way in terms of the vertices of the GKM graphs; in fact $\pi\colon (\Gamma, \alpha) \to (\Gamma_B, \alpha_B)$ is simply given by $\pi(\sigma) = \sigma(1)$, for every $\sigma \in S_{n+1}$.

Let $z_i$ be $e^{2\pi\sqrt{-1}x_i}$ for every $i = 1, \ldots, n+1$ and $\nu\colon [n+1] \to R(T)$ be invariant the $K$-class given by $\nu(i) = z_i$ for $i = 1, \ldots, n+1$. By Proposition 7.4, the classes $1, \nu, \ldots, \nu^n$ form an invariant basis of $K_{\alpha_B}(\mathcal{K}_{n+1})$, and they lift to $S_{n+1}$-invariant basic classes on $\Gamma$.

Let $\Gamma_{n+1}$ be the fiber over $\mathbb{C} \cdot e_{n+1}$, where $\{e_1, \ldots, e_{n+1}\}$ denotes the canonical basis of $\mathbb{C}^{n+1}$; the holonomy group on this fiber is isomorphic to $S_n$ viewed as the subgroup of $S_{n+1}$ which leaves the element $n+1$ fixed. Once we construct a basis of $K_\alpha(\Gamma_{n+1})$ consisting of holonomy invariant classes, we can extend those to a set of classes of $K_\alpha(\Gamma)$, which, together with the invariant basic classes constructed above, will generate a basis of $K_\alpha(\Gamma)$ as a module over $R(T)$. We will show that if we start with a natural choice for the base of the induction, then the global classes generated by this means are indeed $S_{n+1}$ invariant.

Letting $I = [i_1, \ldots, i_n]$ be a multi-index of non-negative integers, define

$$\mathbf{z}^I = z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$$

and let $C_I = C_T(\mathbf{z}^I)\colon S_{n+1} \to R(T)$ be the element defined by

$$C_I(\sigma) = \sigma \cdot \mathbf{z}^I \quad \text{for every} \quad \sigma \in S_{n+1} ;$$

it's easy to verify that $C_I$ is an invariant class of $K_\alpha(\Gamma)$.

**Theorem 8.1.** Let

$$\mathcal{A}_n = \{I = [i_1, \ldots, i_n] \mid 0 \leqslant i_1 \leqslant n, 0 \leqslant i_2 \leqslant n-1, \ldots, 0 \leqslant i_n \leqslant 1\}.$$

Then the set

$$\{C_I = C_T(\mathbf{z}^I) \mid I \in \mathcal{A}_n\}$$

is an invariant basis of $K_\alpha(\Gamma)$ as an $R(T)$-module.

*Proof.* As mentioned above, the proof is by induction. Let $n = 2$, then the fiber bundle $\pi\colon Fl(\mathbb{C}^3) \to \mathbb{C}P^2$ is a $\mathbb{C}P^1$-bundle. Let $p = \mathbb{C} \cdot e_3 \in \mathbb{C}P^2$ be the one dimensional subspace generated by the third vector in the canonical basis of $\mathbb{C}^3$. Then the fiber over $p$ is a copy of $\mathbb{C}P^1$ and the invariant classes $C_{[0]}$ and $C_{[1]}$ form a basis of the $K$-ring of the fiber. We can extend these classes using transition maps between fibers, thus obtaining $S_3$-invariant $K$-classes $C_{[0,0]}$ and $C_{[0,1]}$ on $Fl(\mathbb{C}^3)$. By Proposition 7.4, the invariant classes $1, \nu, \nu^2$ form a basis of the $K$-ring of the base, and they lift to $S_3$-invariant basic classes $C_{[1,0]}$ and $C_{[2,0]}$. By Theorem 5.3, the $K$-ring of $Fl(\mathbb{C}^3)$ is freely generated over $R(T)$ by the invariant classes $C_{[0,0]}$, $C_{[0,1]}$, $C_{[1,0]}$, $C_{[1,1]}$, $C_{[2,0]}$ and $C_{[2,1]}$.

The general statement follows by repeating inductively the same argument, since at each stage the fiber of $\pi\colon Fl(\mathbb{C}^{m+1}) \to \mathbb{C}P^m$ is a copy of $Fl(\mathbb{C}^m)$. $\square$

**Remark 8.2.** This argument can be adapted to give a basis of invariant $K$-classes for the generalized flag variety of type $C_n$. Let $\alpha_i = x_i - x_{i+1}$ for $i = 1, \ldots, n-1$ and $\alpha_n = 2x_n$ be a choice of simple roots of type $C_n$. The corresponding Weyl group

$W$ is the group of signed permutations of $n$ elements. Let $\Sigma = \{\alpha_2, \dots, \alpha_n\}$, then $G/P(\Sigma)$ is a GKM manifold diffeomorphic to a complex projective space $\mathbb{C}P^{2n-1}$; its GKM graph is a complete graph $\mathcal{K}_{2n}$ whose vertices can be identified with the set $\{\pm 1, \dots, \pm n\}$ and the axial function $\alpha$ is simply given by $\alpha(\pm i, \pm j) = \pm x_i \mp x_j$, for every edge $(\pm i, \pm j)$ of the GKM graph. Observe that the procedure in section 7 can be used here to produce a $W$-invariant basis of $K_\alpha(\mathcal{K}_{2n})$. In fact it is sufficient to let $y_1 = x_1, \dots, y_n = x_n, y_{n+1} = -x_1$ and $y_{2n} = -x_n$; the basis of Proposition 7.4 is $S_{2n}$-invariant, and hence in particular $W$-invariant.

Let $G/B$ be the generic coadjoint orbit of type $C_n$, and $(\Gamma, \alpha)$ its GKM graph. If we consider the natural projection $G/B \to G/P(\Sigma)$, the fiber is diffeomorphic to a generic coadjoint orbit of type $C_{n-1}$. Hence we can repeat the inductive argument used in type $A_n$ to produce a $W$-invariant basis of $K_\alpha(\Gamma)$.

## 9. The Kostant-Kumar description

The manifolds in section 4 are also describable in terms of compact groups. Namely, if we let $G_0$ be the compact form of $G$ and $K$ the maximal compact subgroup of $P$, then $M = G/P = G_0/K$. Moreover, $W = W_{G_0}$ and $W(\Sigma) = W_K$, $W_{G_0}$ and $W_K$ being the Weyl groups of $G_0$ and $K$, so $M^T = W_{G_0}/W_K$. A fundamental theorem in equivariant $K$-theory is the Kostant-Kumar theorem, which asserts that $K_T(M)$ is isomorphic to the tensor product

$$R^{W_K} \otimes_{R^W} R \,, \tag{9.1}$$

where $R$ is the character ring $R(T)$, and $R^{W_K}$ and $R^W$ are the subrings of $W_K$ and $W$-invariant elements in $R$. This description of $K_T(M)$ generalizes to $K$-theory the well-known Borel description of the equivariant cohomology ring $H_T(M)$ as the tensor product

$$\mathbb{S}(\mathfrak{t}^*)^{W_K} \otimes_{\mathbb{S}(\mathfrak{t}^*)^W} \mathbb{S}(\mathfrak{t}^*) \tag{9.2}$$

and in [GHZ] the authors showed how to reconcile this description with the GKM description of $H_T(M)$. Mutatis mutandi, their arguments work as well in $K$-theory and we will give below a brief description of the $K$-theoretic version of their theorem.

Let $\Gamma$ be the GKM graph of $M$. As we pointed out above, $M^T = W/W_K$, so the vertices of $\Gamma$ are the elements of $W/W_K$. Now let $f \otimes g$ be a decomposable element of the tensor product (9.1). Then one gets an $R$-valued function, $k(f \otimes g)$, on $W/W_K$ by setting

$$k(f \otimes g)(wW_K) = wfg \,. \tag{9.3}$$

One can show that this defines a ring morphism, $k$, of the ring (9.1) into the ring $\mathrm{Maps}(M^T, R)$, and in fact that this ring morphism is a bijection of the ring (9.1) onto $K_\alpha(\Gamma)$. (For the proof of the analogous assertions in cohomology see section 2.4 of [GHZ].) Moreover the action of $W$ on $K_T(M)$ becomes, under this isomorphism, the action

$$w(f_1 \otimes f_2) = f_1 \otimes wf_2 \tag{9.4}$$

of $W$ on the ring (9.1), so the ring of $W$-invariant elements in $K_T(M)$ gets identified with the tensor product $R^{W_K} \otimes_{R^W} R^W$, which is just the ring $R^{W_K}$ itself. Finally we note that if $M$ is the generalized flag variety, $G/B = G_0/T$, (9.1) becomes the tensor product

$$R \otimes_{R^W} R \tag{9.5}$$

and the ring of $W$-invariant elements in $K_T(M)$ becomes $R$. Moreover, if $\pi$ is the fibration $G_0/T \to G_0/K$, the fiber $F$ over the identity coset of $G_0/K$ is $K/T$; so

$K_T(F)$ is the tensor product $R \otimes_{R^{W_K}} R$, and the subring of $W_K$-invariant elements in $K_T(F)$ is $R$ which, as we saw above, is also the ring of $W_G$-invariant elements in $K_T(G_0/T)$ which is also (see section 6) the ring of invariant elements associated with the fibration $G_0/T \to G_0/K$. Thus most of the features of our GKM description of the fibration $G_0/T \to G_0/K$ have simple interpretations in terms of this Kostant-Kumar model.

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