COHOMOLOGY OF GKM FIBER BUNDLES

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ABSTRACT. The equivariant cohomology ring of a GKM manifold is isomorphic to the cohomology ring of its GKM graph. In this paper we explore the implications of this fact for equivariant fiber bundles for which the total space and the base space are both GKM and derive a graph theoretical version of the Leray-Hirsch theorem. Then we apply this result to the equivariant cohomology theory of flag varieties.

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1. Introduction

Let T be an $n-$dimensional torus, and M a compact, connected $T-$manifold. The equivariant cohomology ring of M, $H_T^*(M;\mathbb{R})$, is an $\mathbb{S}(\mathfrak{t}^*)-$module, where $\mathbb{S}(\mathfrak{t}^*) = H_T^*(\text{point})$ is the symmetric algebra on $\mathfrak{t}^*$, the dual of the Lie algebra of T. If $H_T^*(M)$ is torsion-free, the restriction map

$$i^*\colon H_T^*(M) \to H_T^*(M^T)$$

is injective and hence computing $H_T^*(M)$ reduces to computing the image of $H_T^*(M)$ in $H_T^*(M^T)$. If M^T is finite, then

$$H_T^*(M^T) = \bigoplus_{p\in M^T} \mathbb{S}(\mathfrak{t}^*) \ ,$$

with one copy of $\mathbb{S}(\mathfrak{t}^*)$ for each $p \in M^T$. Determining where $H_T^*(M)$ sits inside this sum is a challenging problem. However, one class of spaces M with $H_T^*(M)$ torsion-free for which this problem has a simple and elegant solution is the one introduced by Goresky-Kottwitz-MacPherson in their seminal paper [GKM]. These are now known as *GKM spaces*: an equivariantly formal space M is a GKM space if M^T is finite and for every codimension one subtorus $T' \subset T$, the connected components of $M^{T'}$ are either points or 2-spheres.

To each GKM space M we attach a graph $\Gamma = \Gamma_M$ by decreeing that the points of M^T are the vertices of Γ and the edges of Γ are these two-spheres. If S is one of the edge two-spheres, then S^T consists of exactly two $T-$fixed points, p and q. If M has an invariant almost complex or symplectic structure, then the isotropy representations on tangent spaces at fixed points are complex representations and their weights are well-defined. These data determine a map

$$\alpha\colon E_\Gamma \to \mathbb{Z}_T^*$$

of oriented edges of Γ into the weight lattice of T. This map assigns to the edge 2-sphere S with North pole p the weight of the isotropy representation of T on the tangent space to S at p. The map α is called the *axial function* of the graph Γ. We use it to define a subring $H_\alpha^*(\Gamma_M)$ of $H_T^*(M^T)$ as follows. Let c be an element of $H_T^*(M^T)$, *i.e.* a function which assigns to each $p \in M^T$ an element $c(p)$ of $H_T^*(\text{point}) = \mathbb{S}(\mathfrak{t}^*)$. Then c is in $H_\alpha^*(\Gamma_M)$ if and only if for each edge e of Γ_M with vertices p and q as end points, $c(p) \in \mathbb{S}(\mathfrak{t}^*)$ and $c(q) \in \mathbb{S}(\mathfrak{t}^*)$ have the same image in $\mathbb{S}(\mathfrak{t}^*)/\alpha_e\mathbb{S}(\mathfrak{t}^*)$. (Without the invariant almost complex or symplectic structure, the isotropy representations are only real representations and the weights are defined only up to sign; however, that does not change the construction of $H_\alpha^*(\Gamma)$.) A consequence of a Chang-Skjelbred result ([CS]) is that $H_\alpha^*(\Gamma_M)$ is the image of i^*, and therefore there is an isomorphism of rings

$$H_T^*(M) \simeq H_\alpha^*(\Gamma_M) \ . \qquad (1)$$

In a companion paper [GSZ] we prove a fiber bundle generalization of this result. Let M and B be $T-$manifolds and $\pi\colon M \to B$ be a $T-$equivariant fiber bundle. If $H_T^*(M)$ is torsion free, then the restriction map

$$i^*\colon H_T^*(M) \to H_T^*(\pi^{-1}(B^T))$$

is injective, and if B^T is finite then $H_T^*(\pi^{-1}(B^T))$ is isomorphic to

$$\bigoplus_{p\in B^T} H_T^*(F_p) \tag{2}$$

with $F_p = \pi^{-1}(p)$. We show in [GSZ] that if B is GKM, then the image of $H_T^*(M)$ in (2) can be computed by a generalized version of (1). Moreover, if the fiber bundle is balanced (as defined in [GSZ]), there is a holonomy action of the groupoid of paths in Γ on the sum (2) and the elements which are invariant under this action form an interesting subring of $H_T^*(M)$.

In this paper we will take the analysis of $H_T^*(M)$ one step further by assuming that M is also GKM. By interpreting this assumption combinatorially one is led to a combinatorial notion which is a central topic of this paper, the notion of a "fiber bundle of a GKM graph (Γ_1, α_1) over a GKM graph (Γ_2, α_2)," and, associated with this, the notion of a "holonomy action" of the groupoid of paths in Γ_2 on the ring $H_{\alpha_1}(\Gamma_1)$. We will explore below the properties of such fiber bundles and apply these results to fiber bundles between generalized flag varieties; *i.e.* fiber bundles of the form

$$\pi\colon G/P_1 \to G/P_2 \tag{3}$$

where G is a semi-simple Lie group and P_1 and P_2 are parabolic subgroups. In particular we will examine in detail the fiber bundle

$$\pi\colon \mathcal{F}l(\mathbb{C}^n) \to \mathcal{G}r_k(\mathbb{C}^n)\ , \tag{4}$$

of complete flags in $\mathbb{C}^n$ over the Grassmannian of $k-$dimensional subspaces of $\mathbb{C}^n$ and the analogue of this fibration for the classical groups of type B_n, C_n, and D_n. For each of these examples we will compute the subring of invariant classes in $H_T^*(M)$ (those elements which are fixed by the holonomy action of the paths in Γ_2) and show how the generators of this ring are related to the usual basis of $H_T^*(M)$, given by equivariant Schubert classes. These results were inspired by and are related to results of Sabatini and Tolman. In [ST] they explore the equivariant cohomology of fiber bundles where the total space and the base space are more general symplectic manifolds with Hamiltonian actions. The theory developed in the present paper can be regarded as a combinatorial version of the geometrical theory of symplectic fibrations of coadjoint orbits, studied in [GLS].

What follows is a brief table of contents for this paper: In Section 2.1 we describe some of the salient features of the fiber bundle (4). In Sections 2.2-2.4 we briefly review the theory of abstract GKM graphs, following [GZ1] and [GZ2]. We then define abstract versions of fibrations and fiber bundles between GKM graphs which incorporate these features, and in Sections 3.1-3.3 we show how to compute the cohomology ring of such graphs. The main ingredient in this computation is a holonomy action of the group of based loops in the base on the cohomology of the fiber graph.

In Section 4 we apply this theory to generalized flag manifolds, which have been extensively studied in the combinatorics literature, but not from the perspective of this paper. Let G be a semisimple Lie group, B a Borel subgroup of G and $P_1 \subset P_2$ parabolic subgroups containing B. Building on results of [GHZ], in Section 4.1 we describe the GKM graph associated with the space P_2/P_1. In Sections 4.3-4.4 we discuss the fibration of GKM graphs associated with the fibration of $T-$manifolds (3) and compute the group of holonomy automorphisms associated with this fibration. In Section 5 we specialize to the case where G is one of the four classical simple Lie group types, A_n, B_n, C_n, or D_n, and, using iterations of fiber bundles, give explicit constructions of bases of invariant classes.

In Section 6 we construct a second explicit basis of $H_T^*(G/B)$ consisting of classes that are $W-$invariant. These invariant classes are obtained from the equivariant Schubert classes by averaging over the action of the Weyl group. In Theorem 6.2 we give explicit combinatorial formulas for the decomposition of twisted Schubert classes, generalizing earlier results of Tymoczko ([T, Theorem 4.9]) from twistings by simple reflections to actions of general Weyl group elements. We then obtain formulas for the transition matrix between the basis of invariant classes consisting of symmetrized Schubert classes and the basis of invariant classes obtained through the iterated fiber bundle construction. In addition we obtain an explicit formula for the decomposition of an invariant class in the basis of equivariant Schubert classes.

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2. GKM Fiber Bundles

2.1. **Motivating example.** Let $T^n = (S^1)^n$ be the compact torus of dimension n, with Lie algebra $\mathfrak{t}_n = \mathbb{R}^n$, and let $\{x_1, \dots, x_n\}$ be the basis of $\mathfrak{t}_n^* \simeq \mathbb{R}^n$ dual to the canonical basis of $\mathbb{R}^n$. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{C}^n$. The torus T^n acts componentwise on $\mathbb{C}^n$ by

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n) \ .$$

This action induces a T^n-action on both $M = \mathcal{F}l(\mathbb{C}^n)$, the manifold of complete flags in $\mathbb{C}^n$, and $B = \mathcal{G}r_k(\mathbb{C}^n)$, the Grassmannian manifold of $k-$dimensional subspaces of $\mathbb{C}^n$. Let $C = \{(t, \dots, t) \mid t \in S^1\}$ be the diagonal circle in T^n and let $T = T^n/C$. Then C acts trivially on the flag manifold and on Grassmannians, and the induced actions of T on $\mathcal{F}l(\mathbb{C}^n)$ and on $\mathcal{G}r_k(\mathbb{C}^n)$ are effective. Let

$$\pi \colon \mathcal{F}l(\mathbb{C}^n) \to \mathcal{G}r_k(\mathbb{C}^n) \ , \tag{5}$$

be the map that sends each complete flag $V_\bullet = (V_1, \dots, V_n)$ to its $k-$dimensional component. Then (M, B, π) is a $T-$equivariant fiber bundle.

Since flag manifolds and Grassmannians are GKM spaces, their $T-$equivariant cohomology rings are determined by fixed point data. These data can be nicely organized using the corresponding GKM graphs, as follows. For a general GKM space M the fixed point set M^T is finite and is the vertex set of the GKM graph Γ. If $T' \subset T$ is a codimension one subtorus of T, then the connected components of the set $M^{T'}$ of $T'-$fixed points are either $T-$fixed points or copies of $\mathbb{C}P^1$ joining two $T-$fixed points. The edges of the graph Γ correspond to these $\mathbb{C}P^1$'s, for all codimension one subtori $T' \subset T$. An edge e corresponding to a connected component of $M^{T'}$ is labeled by an element $\alpha_e \in \mathfrak{t}^*$ such that $\mathfrak{t}' = \ker \alpha_e$. As explained in the introduction, the equivariant cohomology ring $H_T^*(M)$ can be computed from the GKM graph (Γ, α) associated to M, and we will give the details of that construction in Section 3.1.

For the flag manifold $\mathcal{F}l(\mathbb{C}^n)$, the $T-$fixed point set is indexed by S_n, the group of permutations of $[n] = \{1, \dots, n\}$. A permutation $u = u(1) \dots u(n)$ of $[n]$ indexes the fixed flag

$$V_\bullet^u = (V_1^u, \ldots, V_n^u) \ ,$$

given by $V_k^u = \mathbb{C}e_{u(1)} \oplus \cdots \oplus \mathbb{C}e_{u(k)}$, for all $k = 1, \ldots, n$.

The codimension one subtori T' of T for which the fixed point set is not just the set of $T-$fixed points are the subtori $T_{ij} = \{t \in T \mid t_i = t_j\} = \exp(\ker(x_i - x_j))$. For a fixed flag $V_\bullet^u$, the connected component of $\mathcal{F}l(\mathbb{C}^n)^{T_{ij}}$ that passes through $V_\bullet^u$ also contains the fixed flag $V_\bullet^v$, where $v = (i,j)u$ and (i,j) is the transposition that swaps i and j.

The GKM graph Γ of the flag manifold $\mathcal{F}l(\mathbb{C}^n)$ is the Cayley graph (S_n, t) constructed from the group S_n and generating set t, the set of transpositions: the vertices correspond to permutations in S_n and two vertices are joined by an edge if they differ by a transposition. If $u \in S_n$, then $u * (i,j) = (u(i), u(j)) * u$, so two permutations that differ by a transposition on the right (operating on *positions*) also differ by a transposition on the left (operating on *values*). We denote the edge e that joins u and $v = u * (i,j)$ by $u \to v$. If $1 \leqslant i < j \leqslant n$, then the value of the axial function α on this edge is

$$\alpha_e = x_{u(i)} - x_{u(j)} \ .$$

We will refer to Γ as S_n, and it will be clear from the context when S_n is the graph, the vertex set, or the group of permutations. Figure 1(b) shows the Cayley graph (S_3, t). As a general convention throughout this paper, edges that are represented by parallel segments have collinear labels. For example, $\alpha(123, 132) = \alpha(231, 321) = x_2 - x_3$.

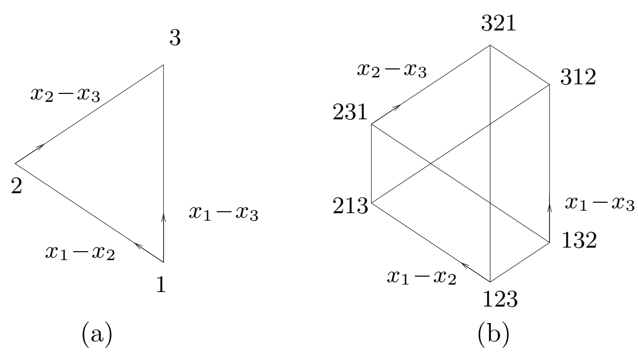

FIGURE 1. The complete graph K_3 (a) and the Cayley graph (S_3, t) (b)

For the Grassmannian $\mathcal{G}r_k(\mathbb{C}^n)$, the $T-$fixed point set is indexed by $k-$element subsets of $[n]$. A subset $I = \{i_1, \ldots, i_k\}$ corresponds to the fixed $k-$dimensional subspace $V_I = \mathbb{C}e_{i_1} \oplus \cdots \oplus \mathbb{C}e_{i_k}$. Two vertices are joined by an edge if the intersection of their corresponding $k-$element subsets is a $(k-1)-$element subset. The resulting graph is the Johnson graph $J(n,k)$. If $I = (I \cap J) \cup \{i\}$ and $J = (I \cap J) \cup \{j\}$, then the value of the axial function on the edge e from I to J is $\alpha_e = x_i - x_j$. In particular, when $k = 1$ we get the complex projective space $\mathbb{C}P^{n-1}$, and the associated graph is the complete graph K_n with n vertices. The complete graph K_3 is shown in Figure 1(a).

The discrete version of (5) is the morphism of graphs $\pi\colon S_n \to J(n,k)$, given by $\pi(u) = \{u(1), \ldots, u(k)\}$. This map is compatible with the axial functions on the two graphs, and for each vertex $A \in J(n,k)$, the fiber $\pi^{-1}(A)$ is a product $S_k \times S_{n-k}$. The axial functions on fibers are not identical, but they are compatible in a natural way.

The GKM fiber bundle $S_4 \to J(4,2)$ is a combinatorial description of the fiber bundle $\mathcal{F}l_4(\mathbb{C}) \to \mathcal{G}r_2(\mathbb{C}^4)$ that sends a complete flag in $\mathcal{F}l_4(\mathbb{C})$ to its two dimensional component. Figure 2 shows the graphical representation of this fiber bundle. The fibers are the squares. (The internal edges of S_4 have been omitted.)

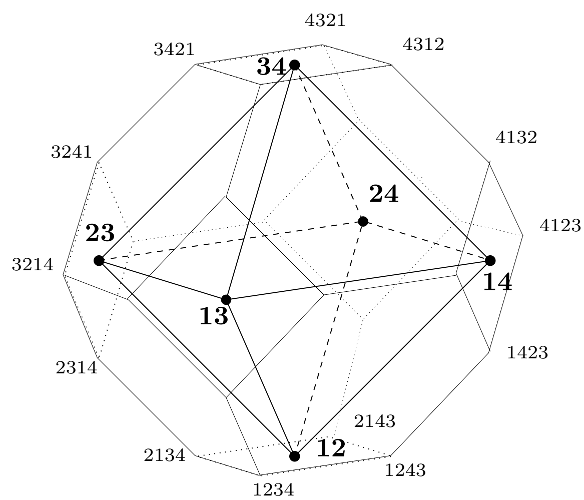

FIGURE 2. The GKM fiber bundle $S_4 \to J(4,2)$

This example motivates one of the main goals of this paper: to define the discrete analog of a fiber bundle between GKM spaces for which the fibers are isomorphic GKM spaces. We then prove a discrete Leray-Hirsch theorem, showing how one can recover the graph cohomology of the total space from the cohomology of the base and invariant classes in the cohomology of the fiber.

Then we will revisit the example $\pi\colon \mathcal{F}l(\mathbb{C}^n) \to \mathcal{G}r_k(\mathbb{C}^n)$ and consider more general fiber bundles $G/B \to G/P$, with $B \subset P \subset G$ a Borel and parabolic subgroup of a complex semisimple Lie group G, and give a combinatorial description/construction of invariant classes for classical groups.

2.2. **Abstract GKM Graphs.** We start by recalling some general definitions (see [GZ1], [GZ2] for more details and motivation). The reader should have in mind the examples of the Cayley graph S_n, the complete graph K_n, and the Johnson graph $J(n,k)$. We will return to these with a summarizing example at the end of Section 2.

Let $\Gamma = (V, E)$ be a regular graph, with V the set of vertices and E the set of oriented edges. We will consider oriented edges, so each unoriented edge e joining vertices p and q will appear twice in E: once as $(p,q) = p \to q$ and a second time as $(q,p) = q \to p$. When e is oriented from p to q, we will call $p = i(e)$ the initial vertex of e, and $q = t(e)$ the terminal vertex of e. For a vertex p, let E_p be the set of oriented edges with initial vertex p.

Definition 2.1. Let $e = (p, q)$ be an edge of Γ, oriented from p to q. A *connection along the edge* e is a bijection $\nabla_e \colon E_p \to E_q$ such that $\nabla_e(p, q) = (q, p)$. A connection on Γ is a family $\nabla = (\nabla_e)_{e\in E}$ of connections along the oriented edges of Γ, such that $\nabla_{(q,p)} = \nabla^{-1}_{(p,q)}$ for every edge $e = (p, q)$ of Γ.

Definition 2.2. Let ∇ be a connection on Γ. A $\nabla-$*compatible axial function* on Γ is a labeling $\alpha \colon E \to \mathfrak{t}^*$ of the oriented edges of Γ by elements of a linear space $\mathfrak{t}^*$, satisfying the following conditions:

(1) $\alpha(q, p) = -\alpha(p, q)$;
(2) For every vertex p, the vectors $\{\alpha(e) \mid e \in E_p\}$ are mutually independent;
(3) For every edge $e = (p, q)$, and for every $e' \in E_p$ we have

$$\alpha(\nabla_e(e')) - \alpha(e') = c\alpha(e) \ ,$$

for some scalar $c \in \mathbb{R}$ that depends on e and e'.

An *axial function* on Γ is a labeling $\alpha \colon E \to \mathfrak{t}^*$ that is a $\nabla-$compatible axial function for some connection ∇ on Γ.

Definition 2.3. A *GKM graph* is a pair (Γ, α) consisting of a regular graph Γ and an axial function $\alpha \colon E \to \mathfrak{t}^*$ on Γ.

Example 2.4 (The complete graph)**.** For the complete graph $\Gamma = K_n$ considered in Section 2.1, the axial function on *oriented* edges is defined as follows. Let $\mathfrak{t}^*$ be an $n-$dimensional linear space and $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{t}^*$. Define $\alpha \colon E \to \mathfrak{t}^*$ by

$$\alpha(i, j) = x_i - x_j \ .$$

If $\nabla_{(i,j)} \colon E_i \to E_j$ sends (i, j) to (j, i) and (i, k) to (j, k) for $k \neq i, j$, then ∇ is a connection compatible with α. The image of α spans the $(n-1)-$dimensional subspace $\mathfrak{t}^*_0$ generated by $\alpha_1 = x_1 - x_2, \ldots, \alpha_{n-1} = x_{n-1} - x_n$.

When $n = 2$, the graph Γ has two vertices, 1 and 2, joined by an edge. The oriented edge from 1 to 2 is labeled $\beta = x_1 - x_2$, and the oriented edge from 2 to 1 is labeled $-\beta = x_2 - x_1$. The second condition in the definition of an axial function is automatically satisfied.

Example 2.5 (The Cayley graph (S_n, t))**.** For the Cayley graph $\Gamma = (S_n, t)$ considered in Section 2.1, the axial function on *oriented* edges is defined as follows. Let $\mathfrak{t}^*$ be an $n-$dimensional linear space and $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{t}^*$. Let $\alpha \colon E \to \mathfrak{t}^*$ be the axial function defined as follows. If $u \to v = u(i, j)$ is an oriented edge, with $1 \leqslant i < j \leqslant n$, define

$$\alpha(u, v) = x_{u(i)} - x_{u(j)} \ .$$

Note that $\alpha(u, v)$ is determined by the *values* changed from u to v. For an edge $e = u \to v = u(i, j)$, define $\nabla_e \colon E_u \to E_v$ by

$$\nabla_e(u, u(a, b)) = (v, v(a, b)) \ . \tag{6}$$

Then ∇ is a connection compatible with α and, as above, the image of α spans the $(n-1)-$dimensional subspace $\mathfrak{t}^*_0$ generated by $\alpha_1 = x_1 - x_2, \ldots, \alpha_{n-1} = x_{n-1} - x_n$.

The examples above show that the image of α may not generate the entire linear space $\mathfrak{t}^*$. Let (Γ, α) be a GKM graph. For a vertex p, let

$$\mathfrak{t}^*_p = \operatorname{span}\{\alpha_e \mid e \in E_p\} \subset \mathfrak{t}^*$$

be the subspace of $\mathfrak{t}^*$ generated by the image of the axial function on edges with initial vertex p. If Γ is connected, then this subspace is the same for all vertices of Γ, and we will denote it by $\mathfrak{t}_0^*$. We can co-restrict the axial function $\alpha\colon E \to \mathfrak{t}^*$ to a function $\alpha_0\colon E \to \mathfrak{t}_0^*$, and the resulting pair (Γ, α_0) is also a GKM graph.

Definition 2.6. An axial function $\alpha\colon E \to \mathfrak{t}^*$ is called *effective* if $\mathfrak{t}_0^* = \mathfrak{t}^*$.

Let (Γ, α) be a GKM graph with $\Gamma = (V, E)$ and axial function $\alpha\colon E \to \mathfrak{t}^*$. Let ∇ be a connection compatible with α. Let $\Gamma_0 = (V_0, E_0)$ be a subgraph of Γ, with $V_0 \subset V$ and $E_0 \subset E$, such that, if $e \in E$ is an edge with $i(e), t(e) \in V_0$, then $e \in E_0$.

Definition 2.7. The connected subgraph Γ_0 is a $\nabla-$*GKM subgraph* if for every edge $e \in E_0$ with $i(e) = p$ and $t(e) = q$, we have $\nabla_e(E_p \cap E_0) = E_q \cap E_0$. The subgraph Γ_0 is a *GKM subgraph* if it is a $\nabla-$GKM subgraph for a connection ∇ compatible with α.

In other words, Γ_0 is a GKM subgraph if, for some connection ∇ compatible with the axial function α, the connection along edges of Γ_0 sends edges of Γ_0 to edges of Γ_0 and edges not in Γ_0 to edges not in Γ_0. Then the connected subgraph Γ_0 is regular, the restriction α_0 of α to E_0 is an axial function on Γ_0, and the connection ∇ induces a connection ∇_0 compatible with α_0. Therefore a GKM subgraph is naturally a GKM graph.

2.2.1. *Isomorphisms of GKM Graphs.* Let (Γ_1, α_1) and (Γ_2, α_2) be two GKM graphs, with $\Gamma_1 = (V_1, E_1)$, $\alpha_1\colon E_1 \to \mathfrak{t}_1^*$ and $\Gamma_2 = (V_2, E_2)$, $\alpha_2\colon E_2 \to \mathfrak{t}_2^*$.

Definition 2.8. An isomorphism of GKM graphs from (Γ_1, α_1) to (Γ_2, α_2) is a pair (Φ, Ψ), where

(1) $\Phi\colon \Gamma_1 \to \Gamma_2$ is an isomorphism of graphs;
(2) $\Psi\colon \mathfrak{t}_1^* \to \mathfrak{t}_2^*$ is an isomorphism of linear spaces;
(3) For every edge (p, q) of Γ_1 we have

$$\alpha_2(\Phi(p), \Phi(q)) = \Psi \circ \alpha_1(p, q)\ .$$

The first condition implies that Φ induces a bijection from E_1 to E_2, and the third condition can be restated as saying that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ \mathfrak{t}_1^* & \xrightarrow{\Psi} & \mathfrak{t}_2^* \end{array}$$

2.3. **Fiber Bundles of Graphs.** We now introduce special types of morphisms between graphs. Later we will add the GKM package (axial function and connection) and define the corresponding types of morphisms between GKM graphs.

2.3.1. *Fibrations.* Let Γ and B be connected graphs and $\pi\colon \Gamma \to B$ be a morphism of graphs. By this we mean that π is a map from the vertices of Γ to the vertices of B such that, if (p, q) is an edge of Γ, then either $\pi(p) = \pi(q)$ or else $(\pi(p), \pi(q))$ is an edge of B.

When (p, q) is an edge of Γ and $\pi(p) = \pi(q)$, we will say that the edge (p, q) is *vertical*; otherwise $(\pi(p), \pi(q))$ is an edge of B and we will say that (p, q) is *horizontal*. For a vertex q of Γ, let $E_q^{\perp}$ be the set of vertical edges with initial vertex q, and let H_q be the set of horizontal edges with initial vertex q. Then $E_q = E_q^{\perp} \cup H_q$ and π canonically induces a map $(d\pi)_q \colon H_q \to (E_B)_{\pi(q)}$ given by

$$(d\pi)_q(q, q') = (\pi(q), \pi(q')) \,. \tag{7}$$

Definition 2.9. The morphism of graphs $\pi \colon \Gamma \to B$ is a *fibration of graphs*[1] if for every vertex q of Γ, the map $(d\pi)_q \colon H_q \to (E_B)_{\pi(q)}$ is bijective.

Fibrations have the *unique lifting of paths* property: Let $\pi \colon \Gamma \to B$ be a fibration, (p_0, p_1) an edge of B, and $q_0 \in \pi^{-1}(p_0)$ a point in the fiber over p_0. Since $(d\pi)_{q_0} \colon H_{q_0} \to (E_B)_{p_0}$ is a bijection, there exists a unique edge (q_0, q_1) such that $(d\pi)_{q_0}(q_0, q_1) = (p_0, p_1)$. We will say that (q_0, q_1) is *the lift* of (p_0, p_1) at q_0. If γ is a path $p_0 \to p_1 \to \cdots \to p_m$ in B and $q_0 \in \pi^{-1}(p_0)$ is a point in the fiber over p_0, then we can lift γ uniquely to a path $\widetilde{\gamma}(q_0) = q_0 \to q_1 \to \cdots \to q_m$ in Γ starting at q_0, by successively lifting the edges of γ.

2.3.2. *Fiber Bundles.* Let $\pi \colon \Gamma \to B$ be a fibration of graphs. For a vertex p of B, let $V_p = \pi^{-1}(p) \subset V$ and let Γ_p be the induced subgraph of Γ with vertex set V_p. For every edge (p, q) of B, define a map $\Phi_{p,q} \colon V_p \to V_q$ as follows. For $p' \in V_p$, define $\Phi_{p,q}(p') = q'$, where (p', q') is the lift of (p, q) at p'. It is easy to see that $\Phi_{p,q}$ is bijective, with inverse $\Phi_{q,p}$. What is not true, in general, is that $\Phi_{p,q}$ is an isomorphism of graphs from Γ_p to Γ_q.

Example 2.10. Let Γ be the regular 3−valent graph consisting of two quadrilaterals (p_1, p_2, p_3, p_4) and (q_1, q_3, q_2, q_4) joined by edges (p_i, q_i) for i=1,2,3,4. (See Figure 3.)

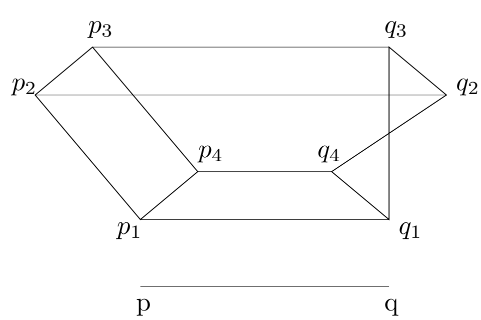

FIGURE 3. Twisted fibration

Let B be a graph with two vertices p and q joined by an edge. Let $\pi \colon \Gamma \to B$ be the morphism of graphs $\pi(p_i) = p$ and $\pi(q_i) = q$ for $i = 1, 2, 3, 4$. Then π is a fibration and $\Phi_{p,q}(p_i) = q_i$ for $i = 1, 2, 3, 4$. However, (p_1, p_2) is an edge in Γ_p, but (q_1, q_2) is not an edge in Γ_q. While the fibers Γ_p and Γ_q are isomorphic as graphs, the map $\Phi_{p,q}$ is not an isomorphism.

We will be interested in fibrations for which $\Phi_{p,q}$ is an isomorphism of graphs from the fiber Γ_p to the fiber Γ_q.

[1]This is what we called *submersion* in [GZ2]. This definition of a fibration of graphs is different from the one introduced in [BV]. We work with undirected graphs, and our morphisms of graphs allow edges to collapse.

Definition 2.11. A fibration $\pi\colon \Gamma \to B$ is a *fiber bundle*[2] if for every edge (p,q) of B, the map $\Phi_{p,q}\colon \Gamma_p \to \Gamma_q$ is a morphism of graphs.

If $\pi\colon \Gamma \to B$ is a fiber bundle then $\Phi_{p,q}$ is bijective, and both $\Phi_{p,q}\colon \Gamma_p \to \Gamma_q$ and $\Phi_{p,q}^{-1} = \Phi_{q,p}\colon \Gamma_q \to \Gamma_p$ are morphisms of graphs. Therefore the maps $\Phi_{p,q}$ are isomorphisms of graphs. The simplest example of a fiber bundle is the projection of a direct product of graphs onto one of its factors, $\pi\colon \Gamma = B \times F \to B$. We will call such fiber bundles *trivial bundles.*

2.4. **GKM Fiber Bundles.** We now add the GKM package to a fibration, and define GKM fibrations. Let (Γ,α) and (B,α_B) be two GKM graphs, with axial functions $\alpha\colon E \to \mathfrak{t}^*$ and $\alpha_B\colon E_B \to \mathfrak{t}^*$ taking values in the same linear space $\mathfrak{t}^*$. Let ∇ and ∇_B be connections on Γ and B, compatible with α and α_B, respectively.

Definition 2.12. A map $\pi\colon (\Gamma,\alpha) \to (B,\alpha_B)$ is a $(\nabla,\nabla_B)-$*GKM fibration* if it satisfies the following conditions:

(1) π is a fibration of graphs;
(2) If e is an edge of B and $\widetilde{e}$ is any lift of e, then $\alpha(\widetilde{e}) = \alpha_B(e)$;
(3) Along every edge e of Γ the connection ∇ sends horizontal edges into horizontal edges and vertical edges into vertical edges;
(4) The restriction of ∇ to horizontal edges is compatible with ∇_B, in the following sense: Let $e=(p,q)$ be an edge of B and $\widetilde{e}=(p',q')$ the lift of e at p'. Let $e' \in E_p$ and $e'' = (\nabla_B)_e(e') \in E_q$. If $\widetilde{e'}$ is the lift of e' at p' and $\widetilde{e''}$ is the lift of e'' at q' then
$$(\nabla)_{\widetilde{e}}(\widetilde{e'}) = \widetilde{e''}\ .$$

A map $\pi\colon (\Gamma,\alpha) \to (B,\alpha_B)$ is a *GKM fibration* if it is a $(\nabla,\nabla_B)-$GKM fibration for some connections ∇ and ∇_B compatible with α and α_B.

If $\pi\colon (\Gamma,\alpha) \to (B,\alpha_B)$ is a GKM fibration, then for each $p \in B$, the fiber (Γ_p,α) is a GKM subgraph of (Γ,α). Let $\mathfrak{v}_p^*$ be the subspace of $\mathfrak{t}^*$ generated by values of axial functions α_e, for edges e of Γ_p. Then the axial function on Γ_p can be co-restricted to α_p, from the oriented edges of Γ_p to $\mathfrak{v}_p^*$, and (Γ_p,α_p) is a GKM graph.

Suppose now that π is both a GKM fibration and a fiber bundle of graphs. Let $e=(p,q)$ be an edge of B. We say that the transition isomorphism $\Phi_{p,q}\colon \Gamma_p \to \Gamma_q$ is *compatible with the connection* on Γ if for every lift $\tilde{e}=(p_1,q_1)$ of e and for every edge $e'=(p_1,p_2)$ of Γ_p, the connection along $\tilde{e}$ moves e' into the edge $e''=(q_1,q_2)=(\Phi_{p,q}(p_1),\Phi_{p,q}(p_2))$ of Γ_q.

Definition 2.13. A GKM fibration $\pi\colon (\Gamma,\alpha) \to (B,\alpha_B)$ is a *GKM fiber bundle* if π is a fiber bundle and for every edge $e=(p,q)$ of B:

(1) The transition isomorphism $\Phi_{p,q}$ is compatible with the connection of Γ.
(2) There exists a linear isomorphism $\Psi_{p,q}\colon \mathfrak{v}_p^* \to \mathfrak{v}_q^*$ such that
$$\Upsilon_{p,q} = (\Phi_{p,q},\Psi_{p,q})\colon (\Gamma_p,\alpha_p) \to (\Gamma_q,\alpha_q)$$
is an isomorphism of GKM graphs.

---

[2]This is what we called *fibration* in [GZ2]

For a GKM fiber bundle $\pi\colon (\Gamma,\alpha)\to(B,\alpha_B)$ we can be more specific about the transition isomorphisms $\Psi_{p,q}$. Let (p,q) be an edge of B, let (p',p'') be an edge of Γ_p, and let (q',q'') be the corresponding edge of Γ_q. The compatibility condition along the edge (p',q') implies that $\alpha_{q',q''}-\alpha_{p',p''}$ is a multiple of $\alpha_{p',q'}=\alpha_{p,q}$, hence there exists a unique constant $c=c(\alpha_{p',p''})$ such that

$$\Psi_{p,q}(\alpha_{p',p''})=\alpha_{p',p''}+c(\alpha_{p',p''})\alpha_{p,q}\ .$$

The linearity of $\Psi_{p,q}$ implies that there exists a unique linear function $c\colon \mathfrak{v}_p^*\to\mathbb{R}$ such that

$$\Psi_{p,q}(x)=x+c(x)\alpha_{p,q}$$

for all $x\in\mathfrak{v}_p^*$.

For a path $\gamma\colon\ p_0\to p_1\to\cdots\to p_{m-1}\to p_m$ in B from p_0 to p_m, let

$$\Upsilon_\gamma=\Upsilon_{p_{m-1},p_m}\circ\cdots\circ\Upsilon_{p_0,p_1}\colon(\Gamma_{p_0},\alpha_{p_0})\to(\Gamma_{p_m},\alpha_{p_m})$$

be the GKM graph isomorphism given by the composition of the transition maps. Let $p\in B$ be a vertex, and let $\Omega(p)$ be the set of all loops in B that start and end at p. If $\gamma\in\Omega(p)$ is a loop based at p, then Υ_γ is an automorphism of the GKM graph (Γ_p,α_p). The *holonomy group* of the fiber Γ_p is the group

$$\mathrm{Hol}_\pi(\Gamma_p)=\{\Upsilon_\gamma\mid\gamma\in\Omega(p)\}\leqslant\mathrm{Aut}(\Gamma_p,\alpha_p)\ .$$

If the base B is connected, then all the fibers are isomorphic as GKM graphs. Let (F,α_F) be a GKM graph isomorphic to all fibers, with $\alpha_F\colon E_F\to\mathfrak{t}_F^*$, and, for each vertex p of B, let $\rho_p=(\varphi_p,\psi_p)\colon(F,\alpha_F)\to(\Gamma_p,\alpha)$ be a fixed isomorphism of GKM graphs. For every edge (p,p') of B, let $\rho_{p,p'}=(\varphi_{p,p'},\psi_{p,p'})\colon(F,\alpha_F)\to(F,\alpha_F)$ be the automorphism of (F,α_F) given by

$$\varphi_{p,p'}=\varphi_{p'}^{-1}\circ\Phi_{p,p'}\circ\varphi_p$$
$$\psi_{p,p'}=\psi_{p'}^{-1}\circ\Psi_{p,p'}\circ\psi_p\ .$$

If γ is any path in B, then the composition of the transition maps along the edges of γ defines an automorphism $\rho_\gamma=(\varphi_\gamma,\psi_\gamma)$ of (F,α_F). Let p be a vertex of B and

$$\mathrm{Hol}(F,p)=\{\rho_\gamma\mid\gamma\in\Omega(p)\}\subset\mathrm{Aut}(F,\alpha_F)\ .$$

Then $\mathrm{Hol}(F,p)$ is a subgroup of $\mathrm{Aut}(F,\alpha_F)$ and if p,p' are vertices of B, then $\mathrm{Hol}(F,p)$ and $\mathrm{Hol}(F,p')$ are conjugated by ρ_γ, where γ is any path in B connecting p and p'.

2.5. **Example.** In this section we return to $\pi\colon\mathcal{F}l(\mathbb{C}^n)\to\mathbb{C}P^{n-1}$ (as a particular case of $\mathcal{F}l(\mathbb{C}^n)\to\mathcal{G}r_k(\mathbb{C}^n)$). We show that the discrete version, $\pi\colon S_n\to K_n$, given by $\pi(u)=u(1)$, is an abstract GKM fiber bundle.

2.5.1. *π is a GKM fibration.* Clearly π is a morphism of graphs, because in K_n all vertices are joined by edges. Moreover, let u and $v=u(i,j)$ (with $1\leqslant i<j\leqslant n$) be adjacent vertices in S_n. If $i\neq 1$, then $\pi(u)=\pi(v)$, hence the edge $u\to v$ is vertical. If $i=1$, then $\pi(v)=u(j)\neq u(1)$, hence the edge $u\to v$ is horizontal.

Let $(d\pi)_u\colon H_u\to E_{\pi(u)}$ be the induced map (7). If $\tilde e$ is the horizontal edge $u\to v=u(1,j)$, then $(d\pi)_u(\tilde e)=e$, the edge of K_n joining $u(1)$ and $u(j)$. Therefore $(d\pi)_u$ is bijective, hence π is a fibration of graphs.

The case $n=3$ is shown in Figure 4. If γ is the cycle $1\to2\to3\to1$ in K_3, then the lift of γ at 123 is the path $\widetilde{\gamma}:123\to213\to312\to132$ in S_3.

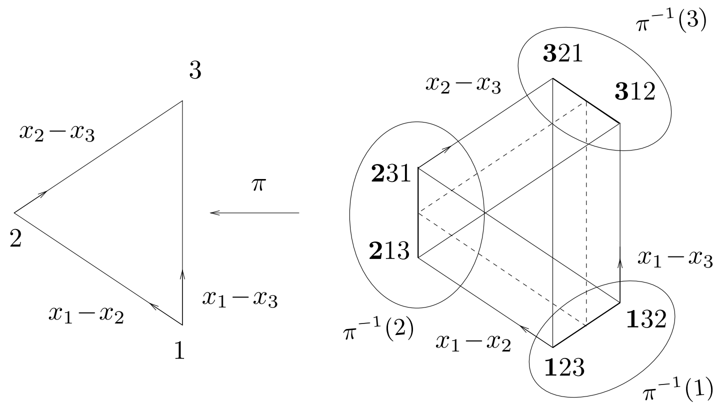

FIGURE 4. Fibration $S_3 \to K_3$

In Section 4.3 we will prove in a more general case that π is a GKM fibration. For now, we notice that it is compatible with the axial functions: if $\tilde{e}$ is the horizontal edge in S_n from u to $v = u(1,j)$, then $\tilde{e}$ is a lift of the edge e in K_n from $u(1)$ to $u(j)$, and both edges have the same label, $x_{u(1)} - x_{u(j)}$.

2.5.2. *Transition isomorphisms.* For each $i \in [n]$, the fiber $\Gamma_i = \pi^{-1}(i)$ consists of all permutations $u \in S_n$ for which $u(1) = i$ and is isomorphic, as a graph, with the Cayley graph (S_{n-1}, t). For $1 \leqslant i \neq j \leqslant n$, the transition map $\Phi_{i,j}\colon \Gamma_i \to \Gamma_j$ is given by $\Phi_{i,j}(u) = (i,j)u$. Let u be a vertex of Γ_i and $u \to u' = u(a,b)$ an edge of Γ_i, hence $2 \leqslant a < b \leqslant n$. If $v = \Phi_{i,j}(u)$ and $v' = \Phi_{i,j}(u')$, then $v' = (i,j)u' = (i,j)u(a,b) = v(a,b)$, hence v and v' are joined by an edge in Γ_j. This shows that $\Phi_{i,j}$'s are morphisms of graphs, and therefore π is a fiber bundle.

The subspace generated by the values of the axial function on the edges of Γ_i is the $(n-1)-$dimensional space

$$\mathfrak{v}_i^* = \mathrm{span}_{\mathbb{R}}\{x_r - x_s \,|\, 1 \leqslant r \neq i \neq s \leqslant n\}$$

and similarly for Γ_j. Let α_i and α_j be the axial functions on Γ_i and Γ_j. Then $\alpha_i(u,u') = x_{u(a)} - x_{u(b)}$, and $\alpha_j(v,v') = x_{(i,j)u(a)} - x_{(i,j)u(b)}$. Let $\Psi_{i,j}$ be the linear automorphism of $\mathfrak{t}^*$ induced by $\Psi_{i,j}(x_r) = x_{(i,j)r}$, for $1 \leqslant r \leqslant n$. Then $\Psi_{i,j}$ induces an isomorphism $\Psi_{i,j}\colon \mathfrak{v}_i^* \to \mathfrak{v}_j^*$ and

$$\alpha_j(\Phi_{i,j}(u), \Phi_{i,j}(u')) = \Psi_{i,j}(\alpha_i(u,u'))\ ,$$

which proves that $(\Phi_{i,j}, \Psi_{i,j})\colon (\Gamma_i, \alpha_i) \to (\Gamma_j, \alpha_j)$ is an isomorphism of GKM graphs. Since the fibers are canonically isomorphic as GKM graphs, the map $\pi\colon S_n \to K_n$ is a GKM fiber bundle.

2.5.3. *Typical fiber.* For $1 \leqslant i \leqslant n$, the fiber (Γ_i, α_i) is isomorphic to S_{n-1}, and we construct an explicit isomorphism $\varphi_i\colon S_{n-1} \to \Gamma_i$. For a permutation $u \in S_{n-1}$, let $\tilde{u} = u(1)u(2)\cdots u(n-1)n \in S_n$. For $1 \leqslant a < b \leqslant n$, let $c_{a,b}$ be the cycle $a \to a+1 \to \cdots \to b \to a$, and let $c_{b,a} = c_{a,b}^{-1}$. Then the map $\varphi_i\colon S_{n-1} \to \Gamma_i$,

$$\varphi_i(u) = c_{i,n}\tilde{u}c_{n,1}$$

is a graph isomorphism between S_{n-1} and Γ_i. The cycle $c_{i,n}$, operating on values, moves the value i to the last position and preserves the relative order of the values on the other positions. The cycle $c_{n,1}$, operating on positions, moves i from the last position to the first and then shifts all the other positions to the right by one.

Let ψ_i be the linear isomorphism induced by $\psi_i(x_k) = x_{c_{i,n}(k)}$ for all $1 \leqslant k \leqslant n$. If $u \in S_{n-1}$ and $v = u(a,b)$, with $1 \leqslant a < b \leqslant n-1$, then

$$\alpha_i(\varphi_i(u), \varphi_i(v)) = \psi_i(\alpha(u,v)) ,$$

hence $(\varphi_i, \psi_i)\colon S_{n-1} \to \Gamma_i$ is an isomorphism of GKM graphs.

2.5.4. *Holonomy action on the fiber.* Let $Hol(\Gamma_n)$ be the holonomy group of the fiber Γ_n. It is generated by compositions of transition isomorphisms along loops in K_n based at n. Each such nontrivial loop can be decomposed into triangles $\gamma_{ij}\colon n \to i \to j \to n$, and for such a triangle we have $(j,n)(i,j)(n,i) = (i,j)$, hence the corresponding element of $Hol(\Gamma_n)$ generated by γ_{ij} is

$$\Upsilon_{\gamma_{ij}} = (\Phi_{\gamma_{ij}}, \Psi_{\gamma_{ij}}) ,$$

with $\Phi_{\gamma_{ij}}(u) = (i,j)u$ and $\Psi_{\gamma_{ij}}(x_r) = x_{(i,j)r}$.

Since every permutation in S_{n-1} can be decomposed into transpositions, it follows that

$$\mathrm{Hol}(\Gamma_n) = \{\Upsilon_w = (\Phi_w, \Psi_w) \mid w \in S_{n-1}\} \simeq S_{n-1} ,$$

where, for a permutation $w \in S_{n-1}$, $\Phi_w\colon \Gamma_n \to \Gamma_n$ is given by $\Phi_w(u) = wu$, and $\Psi_w(x_a) = x_{w(a)}$.

Since the holonomy actions are conjugated, it follows that the holonomy group of all fibers are isomorphic to S_{n-1}.

3. Cohomology of GKM Fiber Bundles

Let $\pi\colon (\Gamma, \alpha) \to (B, \alpha_B)$ be a GKM fiber bundle, with typical fiber (F, α_F). One of the main goals of this paper is to describe how the cohomology ring of the total space (Γ, α) is determined by the cohomology rings of the base (B, α_B) and the fiber (F, α_F) and the holonomy action of the base on the fiber. We start by recalling the construction of the cohomology ring of a GKM graph.

3.1. **Cohomology of GKM graphs.** Let (Γ, α) be a GKM graph, with $\Gamma = (V, E)$ a regular graph and $\alpha\colon E \to \mathfrak{t}^*$ an axial function. Let $\mathbb{S}(\mathfrak{t}^*)$ be the symmetric algebra of $\mathfrak{t}^*$; if $\{x_1, \ldots, x_n\}$ is a basis of $\mathfrak{t}^*$, then $\mathbb{S}(\mathfrak{t}^*) \simeq \mathbb{R}[x_1, \ldots, x_n]$.

Definition 3.1. A cohomology class on (Γ, α) is a map $\omega\colon V \to \mathbb{S}(\mathfrak{t}^*)$ such that for every edge $e = (p,q)$ of Γ, we have

$$\omega(q) \equiv \omega(p) \pmod{\alpha_e} . \tag{8}$$

The compatibility condition (8) means that $\omega(q) - \omega(p) = \alpha_e f$, for some element $f \in \mathbb{S}(\mathfrak{t}^*)$, and is equivalent to $\omega(q) = \omega(p)$ on $\ker(\alpha_e)$. If ω and τ are cohomology classes, then $\omega + \tau$ and $\omega\tau$ are also cohomology classes.

Definition 3.2. The *cohomology ring of* (Γ, α), denoted by $H^*_\alpha(\Gamma)$, is the subring of $\mathrm{Maps}(V, \mathbb{S}(\mathfrak{t}^*))$ consisting of all the cohomology classes.

Moreover, $H^*_\alpha(\Gamma)$ is a graded ring, with the grading induced by $\mathbb{S}(\mathfrak{t}^*)$. We say that $\omega \in H^*_\alpha(\Gamma)$ is a class of degree $2k$ if for every $p \in V$, the polynomial $\omega(p) \in \mathbb{S}^k(\mathfrak{t}^*)$ is homogeneous of degree k. (The fact that the class degree is twice the polynomial degree is a consequence of the convention that elements of $\mathfrak{t}^*$ have degree 2.) If $H^k_\alpha(\Gamma)$ is the space of classes of degree $2k$, then

$$H^*_\alpha(\Gamma) = \bigoplus_{k\geqslant 0} H^{2k}_\alpha(\Gamma) \; .$$

If $\omega \in H^*_\alpha(\Gamma)$ and $h \in \mathbb{S}(\mathfrak{t}^*)$, then $h\omega \in H^*_\alpha(\Gamma)$, hence $H^*_\alpha(\Gamma)$ is an $\mathbb{S}(\mathfrak{t}^*)-$module; it is in fact a graded $\mathbb{S}(\mathfrak{t}^*)-$module.

Remark 3.3. The main motivation behind these constructions is the fact that if M is a GKM manifold and $\Gamma = \Gamma_M$ is its GKM graph, then $H^{odd}_T(M) = 0$ and $H^{2k}_T(M) \simeq H^{2k}_\alpha(\Gamma)$.

Let (Γ_0, α) be a GKM subgraph of (Γ, α). If $f : V \to \mathbb{S}(\mathfrak{t}^*)$ is a cohomology class on Γ, then the restriction of f to V_0 is a cohomology class on Γ_0. Therefore the inclusion $i : (\Gamma_0, \alpha) \to (\Gamma, \alpha)$ induces a ring morphism $i^* : H^*_\alpha(\Gamma) \to H^*_\alpha(\Gamma_0)$.

If $\rho = (\varphi, \psi) : (\Gamma_1, \alpha_1) \to (\Gamma_2, \alpha_2)$ is an isomorphism of GKM graphs, define $\rho^* : \mathrm{Maps}(V_2, \mathbb{S}(\mathfrak{t}^*)) \to \mathrm{Maps}(V_1, \mathbb{S}(\mathfrak{t}^*))$ by

$$(\rho^*(f))(p) = \psi^{-1}(f(\varphi(p))) \; ,$$

for $p \in V_1$, where $\psi^{-1} : \mathbb{S}(\mathfrak{t}^*) \to \mathbb{S}(\mathfrak{t}^*)$ is the algebra isomorphism extending the linear isomorphism $\psi^{-1} : \mathfrak{t}^* \to \mathfrak{t}^*$. Then ρ^* is a ring isomorphism and $(\rho^*)^{-1} = (\rho^{-1})^*$, but, unless $\psi : \mathfrak{t}^* \to \mathfrak{t}^*$ is the identity, ρ^* is not an isomorphism of $\mathbb{S}(\mathfrak{t}^*)-$modules.

3.2. **Cohomology of GKM Fiber Bundles.** Let $\pi : (\Gamma, \alpha) \to (B, \alpha_B)$ be a GKM fiber bundle. For a cohomology class $f : V_B \to \mathbb{S}(\mathfrak{t}^*)$ on the base (B, α_B), define the pull-back $\pi^*(f) : V_\Gamma \to \mathbb{S}(\mathfrak{t}^*)$ by $\pi^*(f)(q) = f(\pi(q))$. Then $\pi^*(f)$ is a cohomology class on (Γ, α), and π defines an injective morphism of rings $\pi^* : H^*_{\alpha_B}(B) \to H^*_\alpha(\Gamma)$. In particular, $H^*_\alpha(\Gamma)$ is an $H^*_{\alpha_B}(B)-$module.

Definition 3.4. A cohomology class $h \in H^*_\alpha(\Gamma)$ is called *basic* if $h \in \pi^*(H^*_{\alpha_B}(B))$.

Let $(H^*_\alpha(\Gamma))_{bas} = \pi^*(H^*_{\alpha_B}(B)) \subseteq H^*_\alpha(\Gamma)$. Then $(H^*_\alpha(\Gamma))_{bas}$ is a subring of $H^*_\alpha(\Gamma)$, and is isomorphic to $H^*_{\alpha_B}(B)$. We will identify $H^*_{\alpha_B}(B)$ and $(H^*_\alpha(\Gamma))_{bas}$ and regard $H^*_{\alpha_B}(B)$ as a subring of $H^*_\alpha(\Gamma)$.

The next theorem is one of the main results of this paper, and shows how the cohomology of the total space Γ is determined by the cohomology of the base B and special sets of cohomology classes with certain properties on fibers.

Theorem 3.5. Let $\pi : (\Gamma, \alpha) \to (B, \alpha_B)$ be a GKM fiber bundle, and let $c_1, \ldots, c_m$ be cohomology classes on Γ such that, for every $p \in B$, the restrictions of these classes to the fiber $\Gamma_p = \pi^{-1}(p)$ form a basis for the cohomology of the fiber. Then, as $H^*_{\alpha_B}(B)-$modules, $H^*_\alpha(\Gamma)$ is isomorphic to the free $H^*_{\alpha_B}(B)-$module on $c_1, \ldots, c_m$.

Proof. A linear combination of $c_1, \ldots, c_m$ with coefficients in $(H^*_\alpha(\Gamma))_{bas} \simeq H^*_{\alpha_B}(B)$ is clearly a cohomology class on Γ. If such a combination is the zero class, then

$$\sum_{k=1}^{m} \beta_k(p) c_k(p') = 0$$

for every $p \in B$ and $p' \in \Gamma_p$. Since the restrictions of $c_1, \ldots, c_m$ to Γ_p are independent, it follows that $\beta_k(p) = 0$ for every $k = 1, \ldots, m$. This is valid for all $p \in B$, hence the classes $\beta_1, \ldots, \beta_m$ are zero. Therefore $c_1, \ldots, c_m$ are independent over $H^*_{\alpha_B}(B)$, and the free $H^*_{\alpha_B}(B)-$module they generate is a submodule of $H^*_\alpha(\Gamma)$.

We prove that this submodule is the entire $H^*_\alpha(\Gamma)$. Let $c \in H^*_\alpha(\Gamma)$ be a cohomology class on Γ. For $p \in B$, the restriction of c to the fiber Γ_p is a cohomology class on Γ_p. Since the restrictions of $c_1, \ldots, c_m$ to Γ_p generate the cohomology of Γ_p, there exist polynomials $\beta_1(p), \ldots, \beta_m(p)$ in $\mathbb{S}(\mathfrak{t}^*)$ such that, for every $p' \in \Gamma_p$,

$$c(p') = \sum_{k=1}^m \beta_k(p) c_k(p') \ .$$

We will show that the maps $\beta_k \colon B \to \mathbb{S}(\mathfrak{t}^*)$ are in fact cohomology classes on B. Let $e = p \to q$ be an edge of B, with weight $\alpha_e = \alpha_{pq} \in \mathfrak{t}^*$. Let $p' \in \Gamma_p$, and $q' \in \Gamma_q$ such that $p' \to q'$ is the lift of $p \to q$. Then $\alpha(p', q') = \alpha(p, q) = \alpha_e$ and

$$\begin{aligned} c(q') - c(p') &= \sum_{k=1}^m (\beta_k(q) c_k(q') - \beta_k(p) c_k(p')) = \\ &= \sum_{k=1}^m (\beta_k(q) - \beta_k(p)) c_k(p') + \sum_{k=1}^m \beta_k(q)(c_k(q') - c_k(p')) \ . \end{aligned}$$

Since $c, c_1, \ldots, c_m$ are classes on Γ, the differences $c(q') - c(p')$, $c_k(q') - c_k(p')$ are multiples of α_e, for all $k = 1, \ldots, m$. Therefore, for all $p' \in \Gamma_p$,

$$\sum_{k=1}^m (\beta_k(q) - \beta_k(p)) c_k(p') = \alpha_e \eta(p') \ ,$$

where $\eta(p') \in \mathbb{S}(\mathfrak{t}^*)$. We will show that $\eta \colon \Gamma_p \to \mathbb{S}(\mathfrak{t}^*)$ is a cohomology class on Γ_p.

If p' and p'' are vertices in Γ_p, joined by an edge (p', p''), then

$$\sum_{k=1}^m (\beta_k(q) - \beta_k(p))(c_k(p'') - c_k(p')) = \alpha_e(\eta(p'') - \eta(p')) \ .$$

Each c_k is a cohomology class on Γ, so $c_k(p'') - c_k(p')$ is a multiple of $\alpha(p', p'')$, for all $k = 1, \ldots, m$. Then $\alpha_e(\eta(p'') - \eta(p'))$ is also a multiple of $\alpha(p', p'')$. But $\alpha_e = \alpha(p', q')$ and $\alpha(p', p'')$ point in different directions as vectors, so, as linear polynomials, they are relatively prime. Therefore $\eta(p'') - \eta(p')$ must be a multiple of $\alpha(p', p'')$. Therefore η is a cohomology class on Γ_p.

The restrictions of $c_1, \ldots, c_m$ form a basis for the cohomology ring of Γ_p, hence there exist polynomials $Q_1, \ldots, Q_m \in \mathbb{S}(\mathfrak{t}^*)$ such that

$$\eta(p') = \sum_{k=1}^m Q_k c_k(p')$$

for all $p' \in \Gamma_p$. Then

$$\sum_{k=1}^m (\beta_k(q) - \beta_k(p) - Q_k \alpha_e) c_k = 0$$

on the fiber Γ_p. Since the classes $c_1, \ldots, c_m$ restrict to linearly independent classes on fibers, it follows that

$$\beta_k(q) - \beta_k(p) = Q_k \alpha_e \ ,$$

hence $\beta_k \in H^*_{\alpha_B}(B)$. Therefore every cohomology class on Γ can be written as a linear combination of classes $c_1, \ldots, c_m$, with coefficients in $H^*_{\alpha_B}(B)$. ∎

3.3. **Invariant Classes.** In this section we describe a method of constructing global classes $c_1, \ldots, c_m$ with the properties required by Theorem 3.5.

Let $\pi\colon (\Gamma, \alpha) \to (B, \alpha_B)$ be a GKM fiber bundle, with typical fiber (F, α_F). Let p be a fixed vertex of B and let $\rho_p = (\varphi_p, \psi_p)\colon (F, \alpha_F) \to (\Gamma_p, \alpha_p)$ be a GKM isomorphism from F to the fiber above p. For a loop $\gamma \in \Omega(p)$, let $\rho_\gamma = (\varphi_\gamma, \psi_\gamma)$ be the GKM automorphism of (F, α_F) determined by γ. Let $K = \mathrm{Hol}(F, p)$ be the holonomy subgroup of $\mathrm{Aut}(F, \alpha_F)$ generated by all automorphisms ρ_γ, and let $f \in (H^*_{\alpha_F}(F))^K$ be a cohomology class on the fiber, invariant under all the automorphisms in K. Then $f_p = (\rho_p^{-1})^*(f) \in H^*_\alpha(\Gamma_p)$ is a class on the fiber over p, invariant under all the automorphisms in $\mathrm{Hol}_\pi(\Gamma_p) \subset \mathrm{Aut}(\Gamma_p, \alpha)$. For any vertex $q \in \Gamma_p$ we have $f_p(q) \in \mathbb{S}(\mathfrak{v}_p^*) \subset \mathbb{S}(\mathfrak{t}^*)$, where $\mathfrak{v}_p^*$ is the subspace of $\mathfrak{t}^*$ generated by the values of α on the edges of Γ_p.

We will extend the class f_p from the fiber Γ_p to the total space Γ. Let p' be a vertex of B, and γ a path in B from p' to p. Let $\Upsilon_\gamma^*\colon H^*_\alpha(\Gamma_p) \to H^*_\alpha(\Gamma_{p'})$ be the ring isomorphism induced by the GKM graph isomorphism $\Upsilon_\gamma\colon (\Gamma_{p'}, \alpha) \to (\Gamma_p, \alpha)$. Since f_p is $\mathrm{Hol}_\pi(\Gamma_p)-$invariant, it follows that if γ_1 and γ_2 are two paths in B from p' to p, then $\Upsilon_{\gamma_1}^*(f_p) = \Upsilon_{\gamma_2}^*(f_p)$. We define $f_{p'} = \Upsilon_\gamma^*(f_p) \in H^*_\alpha(\Gamma_{p'})$, where γ is any path in B from p' to p. Then $f_{p'}(q') \in \mathbb{S}(\mathfrak{t}^*_{p'}) \subset \mathbb{S}(\mathfrak{t}^*)$ for every $q' \in \Gamma_{p'}$.

Proposition 3.6. Let $c = c_{f,p}\colon V_\Gamma \to \mathbb{S}(\mathfrak{t}^*)$ be defined by $c|_{\Gamma_q} = f_q$ for all $q \in B$. Then $c \in H^*_\alpha(\Gamma)$.

Proof. Since the restrictions of c to fibers are classes on fibers, it suffices to show that c satisfies the compatibility conditions along horizontal edges.

Let (q_1, q_2) be a horizontal edge of Γ and let $e = (p_1, p_2)$ be the corresponding edge of B. Then

$$c(q_2) - c(q_1) = f_{p_2}(q_2) - f_{p_1}(q_1) = \Psi_e(f_{p_1}(q_1)) - f_{p_1}(q_1)$$

is a multiple of $\alpha_e = \alpha(q_1, q_2)$, because $\Psi_e(x) = x + c(x)\alpha_e$ on $\mathfrak{v}_{p_1}^*$. □

Note that c depends not only on the class f on the typical fiber F, but also on the point p where we start realizing f on Γ. The choice of p is limited by the fact that f has to be invariant under the subgroup $\mathrm{Hol}(F, p)$ determined by p.

Remark 3.7. Suppose that the $\mathbb{S}(\mathfrak{t}^*)-$module $H^*_{\alpha_F}(F)$ has a basis $\{f_1, \ldots, f_m\}$, consisting of $\mathrm{Hol}(F, p)-$invariant classes, for some $p \in B$. Let $c_j = c_{f_j, p}$, for $j = 1, \ldots, m$. Then the classes $c_1, \ldots, c_m$ have the property that their restrictions to each fiber form a basis for the cohomology of the fiber.

4. Flag Manifolds as GKM Fiber Bundles

Let G be a connected semisimple complex Lie group, let P be a parabolic subgroup of G, and let $M = G/P$ be the corresponding flag manifold. Let T be a maximal compact torus of G, acting on M by left multiplication on G. Then M is a GKM space and the equivariant cohomology ring $H^*_T(M)$ can be computed from the associated GKM graph.

The goal of this section is to briefly review flag manifolds and their GKM graphs. In the last subsection we will describe the discrete analog of the natural fiber bundle $G/P_1 \to G/P_2$, with $T \subset P_1 \subset P_2 \subset G$.

4.1. **Flag Manifolds.** In this subsection we review facts about semisimple Lie algebras and flag manifolds. Details and proofs can be found, for example, in [FH] or [Hu].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and $\mathfrak{t} \subset \mathfrak{h}$ a compact real form. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

be the Cartan decomposition of $\mathfrak{g}$, where $\Delta \subset \mathfrak{t}^*$ is the set of roots. Let Δ^+ be a choice of positive roots and $\Delta_0 = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta$ be the corresponding simple roots. The choice of Δ^+ is equivalent to a choice of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$,

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \,.$$

If G is a connected Lie group with Lie algebra $\mathfrak{g}$ and B is the Borel subgroup with Lie algebra $\mathfrak{b}$, then $M = G/B$ is the manifold of (generalized) complete flags corresponding to G.

For a subset $\Sigma \subset \Delta_0$ of simple roots, let $\langle \Sigma \rangle \subset \Delta^+$ be the set of positive roots that can be written as linear combinations of roots in Σ. Then

$$\mathfrak{p}(\Sigma) = \mathfrak{b} \oplus \bigoplus_{\alpha \in \langle \Sigma \rangle} \mathfrak{g}_{-\alpha} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Sigma \rangle} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \oplus \bigoplus_{\alpha \in \Delta^+ \setminus \langle \Sigma \rangle} \mathfrak{g}_\alpha$$

is a Lie subalgebra of $\mathfrak{g}$, and the corresponding Lie subgroup $P(\Sigma) \leqslant G$ is a parabolic subgroup of G. Up to conjugacy, every parabolic subgroup of G is of this form. The Borel subgroup B corresponds to $\Sigma = \emptyset$, and the whole group G to $\Sigma = \Delta_0$. The homogeneous space $M = G/P(\Sigma)$ is the manifold of (generalized, partial) flags corresponding to G and Σ.

The examples considered in Section 2.1 correspond to $G = SL(n, \mathbb{C})$.

4.2. **GKM Graphs of Flag Manifolds.** In this subsection we outline the construction of the GKM graph (Γ, α) for quotients of parabolic subgroups; more details are available in [GHZ].

4.2.1. *Weyl groups.* For flag manifolds, the construction of the GKM graph involves Weyl groups and their actions on roots, and we start with a few useful results. Let W be the Weyl group of $\mathfrak{g}$, generated by reflections $s_\alpha \colon \mathfrak{t}^* \to \mathfrak{t}^*$ for $\alpha \in \Delta_0$. As a general convention, we will use Greek letters α, β for roots and axial functions (whose values are, in this case, roots, and it will be clear from the context whether α is a root or an axial function), and Roman letters u, v, w, for elements of the Weyl group W. Then $w\beta$ is the element of $\mathfrak{t}^*$ obtained by applying $w \in W$ to $\beta \in \mathfrak{t}^*$, and ws_β is the element of the Weyl group obtained by multiplying $w \in W$ with the reflection $s_\beta \in W$ corresponding to the root β. Then $ws_\beta = s_{w\beta}w$, hence two elements of W that differ by a reflection to the left also differ by a reflection to the right.

For a subset $\Sigma \subset \Delta_0$, let $W(\Sigma)$ be the subgroup of W generated by reflections s_α corresponding to roots $\alpha \in \Sigma$. Then, for a root $\alpha \in \Delta$, the reflection $s_\alpha \in W$ is in $W(\Sigma)$ if and only if $\alpha \in \langle \Sigma \rangle$ ([Hu, 1.14]). For subsets $\Sigma_1 \subset \Sigma_2 \subset \Delta_0$, let $W_1 = W(\Sigma_1)$ and $W_2 = W(\Sigma_2)$; then $W_1 \leqslant W_2 \leqslant W$.

Lemma 4.1. The set $\langle \Sigma_2 \rangle \setminus \langle \Sigma_1 \rangle$ is W_1-invariant.

Proof. If $\beta \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$, then the positive root β is a linear combination of simple roots in Σ_2, with all coefficients non-negative. Since β is not in $\langle\Sigma_1\rangle$, there exists at least one simple root, say α_i, that is not in Σ_1 and appears in β with a strictly positive coefficient. If $\alpha \in \Sigma_1$, then $s_\alpha\beta = \beta - n_{\beta,\alpha}\alpha$, with $n_{\beta,\alpha} \in \mathbb{Z}$. Then $s_\alpha\beta$ and β have the same coefficients in front of the simple roots not in Σ_1. In particular, α_i appears in $s_\alpha\beta$ with a strictly positive coefficient, which proves that $s_\alpha\beta$ is a positive root. The simple roots appearing in α and β are all in Σ_2, hence $s_\alpha\beta \in \langle\Sigma_2\rangle$, and as α_i is not in Σ_1, it follows that $s_\alpha\beta \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$. Since W_1 is generated by the reflections s_α with $\alpha \in \Sigma_1$, we conclude that $\langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$ is W_1-invariant. □

Let $w \in W_2$ and let $w = s_{\beta_1} \cdots s_{\beta_m}$ be a decomposition of w into simple reflections, with $\beta_i \in \Sigma_2$ for all $i = 1, ..., m$. If $\alpha \in \langle\Sigma_1\rangle$ and $\beta \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$ then

$$s_\beta s_\alpha = s_\alpha s_{s_\alpha\beta} \ ,$$

and $s_\alpha\beta \in \langle\Sigma_2\rangle\setminus\langle\Sigma_1\rangle$. We can therefore push all the reflections coming from roots in $\langle\Sigma_1\rangle$ to the left, and get $w = us_{\beta_1'} \dots s_{\beta_k'}$ with $u \in W_1$ and $\beta_1', \dots, \beta_k' \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$. We can also push all the reflections coming from roots in $\langle\Sigma_1\rangle$ to the right, and get $w = s_{\beta_1''} \dots s_{\beta_k''}u$ with $u \in W_1$ and $\beta_1'', \dots, \beta_k'' \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$.

4.2.2. *Quotients of parabolic subgroups.* Let $\Sigma_1 \subset \Sigma_2 \subset \Delta_0$ be subsets of simple roots and $B \leqslant P(\Sigma_1) := P_1 \leqslant P(\Sigma_2) := P_2 \leqslant G$ the corresponding parabolic subgroups. The compact torus T with Lie algebra $\mathfrak{t}$ acts on $M = P_2/P_1$ by left multiplication on P_2, and the space $M = P_2/P_1$ is a GKM space, isomorphic to G'/P' for a Levi subgroup G' of P_1. All flag manifolds are of this type, corresponding to $\Sigma_2 = \Delta_0$.

We describe now the GKM graph (Γ, α) associated to $M = P_2/P_1$. The fixed point set M^T is identified with the set of right cosets

$$W_2/W_1 = \{vW_1 \mid v \in W_2\ \} = \{[v] \mid v \in W_2\ \}\ ,$$

where $[v] = vW_1$ is the right W_1-coset containing $v \in W_2$. Vertices $[w], [v]$ are joined by an edge if and only if $[v] = [ws_\beta]$ for some $\beta \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$. If $[w\sigma_\beta] = [w]$, then $\sigma_\beta \in W_1$, which is impossible if $\beta \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$, because the only reflections in W_1 are those associated to roots in Σ_1. Therefore the endpoints of an edge are distinct and the graph has no loops. For $w \in W_2$ and $\beta \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$, the edge $e = ([w] \to [ws_\beta] = [s_{w\beta}w])$ is labeled by $\alpha_e = \alpha([w], [ws_\beta]) = w\beta$.

We show that the label α_e is independent of the representative $w \in W_2$: if $[w'] = [w]$ and $[ws_\beta] = [w's_\gamma]$ with $\beta, \gamma \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$, then there exist $w_1, w_2 \in W_1$ such that $w' = ww_1$ and $w's_\gamma = ws_\beta w_2$. Then $s_\beta s_{w_1\gamma} = w_2 w_1^{-1} \in W_1$, which implies $w_1\gamma = \pm\beta$. Since $\langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle$ is W_1-invariant, it follows that $w_1\gamma = \beta$ and therefore $w'\gamma = ww_1\gamma = w\beta$.

The connection along the edge $e = ([w], [ws_\beta])$ sends the edge $e' = ([w], [ws_{\beta'}])$ to the edge $e'' = ([ws_\beta], [ws_\beta s_{\beta'}])$.

Then $(\Gamma(W_2/W_1), \alpha)$ is the GKM graph of the GKM space $M = P_2/P_1$. We will refer to it simply as W_2/W_1, and it will be clear from the context when we mean the GKM graph, when just the graph, and when just the vertices.

Example 4.2. We describe the particular cases when $P_2 = G$ or $P_1 = B$, or both.

For $M = G/B$ we have $\Sigma_1 = \emptyset$, $\Sigma_2 = \Delta_0$, $W_1 = \{1\}$ and $W_2 = W$, hence $W_2/W_1 = W$. Vertices $w, v \in W$ of the corresponding GKM graph $\Gamma(W)$ are joined by an edge if and only if $w^{-1}v = s_\beta$ for some $\beta \in \Delta^+$ (or, equivalently, if $v = ws_\beta = s_{w\beta}w$), and the edge $w \to ws_\beta = s_{w\beta}w$ is labeled by $w\beta$.

For $M = P(\Sigma)/B$, we have $\Sigma_1 = \emptyset$, $\Sigma_2 = \Sigma \subset \Delta_0$, $W_2 = W(\Sigma)$, and $W_1 = \{1\}$. The GKM graph $\Gamma(W(\Sigma))$ is the induced subgraph of $\Gamma(W)$ with vertex set $W(\Sigma)$: vertices $w, v \in W(\Sigma)$ are joined by an edge in $\Gamma(W(\Sigma))$ if and only if they are joined by an edge in $\Gamma(W)$. That happens if $v = ws_\beta = s_{w\beta}w$ for some $\beta \in \langle\Sigma\rangle$. The edge $w \to ws_\beta = s_{w\beta}w$ is labeled by $w\beta$.

For $M = G/P(\Sigma)$, we have $\Sigma_2 = \Delta_0$ and $\Sigma_1 = \Sigma \subset \Delta_0$. The GKM graph is a graph with vertex set $W/W(\Sigma)$. Vertices $[w], [v] \in W/W(\Sigma)$ are joined by an edge if and only if $w^{-1}v = s_\beta$ for some $\beta \in \Delta^+ \setminus \langle\Sigma\rangle$; equivalently, if $v = ws_\beta = s_{w\beta}w$. The edge $w \to ws_\beta = s_{w\beta}w$ is labeled by $w\beta$.

4.3. **GKM Fiber Bundles of Flag Manifolds.** Let $\Sigma_1 \subsetneq \Sigma_2 \subset \Delta_0$ be, as above, subsets of simple roots, and let $W_1 = W(\Sigma_1)$ and $W_2 = W(\Sigma_2)$ be the corresponding subgroups of W. For an element $w \in W$, let wW_1 be its class in W/W_1, and wW_2 its class in W/W_2. One has a natural map $\pi\colon W/W_1 \to W/W_2$, given by $\pi(wW_1) = wW_2$, from the vertices of $\Gamma(W/W_1)$ to the vertices of $\Gamma(W/W_2)$. If $\Sigma_2 = \Delta_0$, then the base W/W_2 is just a point and the map π is trivial. For the rest of this section we will assume that $\Sigma_2 \subsetneq \Delta_0$. The goal of this section is to show that π is a GKM fiber bundle between the corresponding GKM graphs.

Theorem 4.3. The projection $\pi\colon W/W_1 \to W/W_2$ is a GKM fiber bundle with typical fiber W_2/W_1.

Proof. Let wW_1 be a vertex of W/W_1 and let $e = (wW_1, ws_\beta W_1)$ be an edge of W/W_1, with $\beta \in \Delta^+ \setminus \langle\Sigma_1\rangle$. This edge is vertical if and only if $s_\beta \in W_2$, and this happens exactly when $\beta \in \langle\Sigma_2\rangle$. Therefore the vertical edges at wW_1 are

$$E^\perp_{wW_1} = \{(wW_1, ws_\beta W_1) \mid \beta \in \langle\Sigma_2\rangle \setminus \langle\Sigma_1\rangle\ \}\ ,$$

and the horizontal edges are

$$H_{wW_1} = \{(wW_1, ws_\beta W_1) \mid \beta \in \Delta^+ \setminus \langle\Sigma_2\rangle\ \}\ .$$

If $(wW_1, ws_\beta W_1)$ is a horizontal edge, then $(wW_2, ws_\beta W_2)$ is an edge of W/W_2, hence π is a morphism of graphs, and $(d\pi)_{wW_1}\colon H_{wW_1} \to E_{wW_2}$, is defined by

$$(d\pi)_w(wW_1, ws_\beta W_1) = (wW_2, ws_\beta W_2)\ .$$

It is clear that $(d\pi)_{wW_1}$ is a bijection, hence π is a fibration of graphs.

Next we show that π is a GKM fibration. Let $e = (wW_2, ws_\beta W_2)$ be an edge of W/W_2, with $\beta \in \Delta^+ \setminus \langle\Sigma_2\rangle$. If vW_1 is a vertex of W/W_1 in the fiber above wW_2, then $v = wu$, for some $u \in W_2$. Let $\beta' = u^{-1}\beta$. By Lemma 4.1 applied to the pair (Δ_0, Σ_2) corresponding to (W, W_2), the set $\Delta^+ \setminus \langle\Sigma_2\rangle$ is W_2-invariant, hence $\beta' \in \Delta^+ \setminus \langle\Sigma_2\rangle$. Therefore $\tilde{e} = (vW_1, vs_{\beta'}W_1)$ is an edge of W/W_1. Since

$$\pi(vs_{\beta'}W_1) = vs_{\beta'}W_2 = wus_{u^{-1}\beta}W_2 = ws_\beta uW_2 = ws_\beta W_2\ ,$$

it follows that $\tilde{e}$ is the lift of e at vW_1. Moreover, if α_1 and α_2 are the axial functions on W/W_1 and W/W_2, respectively, then

$$\alpha_1(vW_1, vs_{\beta'}W_1) = v\beta' = wuu^{-1}\beta = w\beta = \alpha_2(wW_2, ws_\beta W_2)\ ,$$

hence the axial functions are compatible with π.

Let $e = (vW_1, vs_\beta W_1)$ and $e' = (vW_1, vs_{\beta'}W_1)$ be edges of W/W_1. The connection ∇_1 along e moves e' to $e'' = (vs_\beta W_1, vs_\beta s_{\beta'}W_1)$. If $\beta' \in \Delta^+ \setminus \langle\Sigma_2\rangle$, then both

e' and e'' are horizontal, and if $\beta' \in \langle \Sigma_2 \rangle \setminus \langle \Sigma_1 \rangle$, then both are vertical. Hence the connection along any edge of W/W_1 moves horizontal edges to horizontal edges and vertical edges to vertical edges. Moreover, if both e and e' are horizontal (and hence so is e''), then the connection ∇_2 along the projection of e moves the projection of e' to the projection of e'', which shows that the restriction of ∇_1 to horizontal edges is compatible with ∇_2, and we have shown that π is a GKM fibration.

Finally, we prove that π is a GKM fiber bundle. Let $p = wW_2$ and $q = ws_\beta W_2$ be two adjacent vertices of W/W_2, with $\beta \in \Delta^+ \setminus \langle \Sigma_2 \rangle$. A straightforward computation shows that the transition map $\Phi_{p,q} \colon \pi^{-1}(p) \to \pi^{-1}(q)$ is given by

$$\Phi_{p,q}(vW_1) = s_{w\beta} vW_1 \ ,$$

and hence, if $e' = (vW_1, vs_{\beta'}W_1)$ is an edge of $\pi^{-1}(p)$, then

$$e'' = (\Phi_{p,q}(vW_1), \Phi_{p,q}(vs_{\beta'}W_1)) = (s_{w\beta}vW_1, s_{w\beta}vs_{\beta'}W_1)$$

is an edge of $\pi^{-1}(q)$. Therefore $\Phi_{p,q}$ is a morphism of graphs, hence an isomorphism, with inverse $\Phi_{p,q}^{-1} = \Phi_{q,p}$. In addition, the connection ∇_1 along the lift of $e = (p, q)$ at vW_1 moves e' to e''. Moreover

$$\alpha_1(e'') = s_{w\beta} v\beta' = s_{w\beta}(\alpha_1(e')) \ ,$$

hence, if $\Psi_{p,q} \colon \mathfrak{t}^* \to \mathfrak{t}^*$ is given by $\Psi_{p,q}(x) = s_{w\beta}(x)$, then its induced restriction and co-restriction $\Psi_{p,q} \colon \mathfrak{v}_p^* \to \mathfrak{v}_q^*$ is compatible with $\Phi_{p,q}$. This proves that

$$(\Phi_{p,q}, \Psi_{p,q}) \colon (W/W_1)_p \to (W/W_1)_q$$

is an isomorphism of GKM graphs, hence the fibers are canonically isomorphic, through an isomorphism compatible with the connection of Γ_1. We conclude that π is a GKM fiber bundle.

All that remains is to show that the fibers are isomorphic, as GKM graphs, to W_2/W_1. Let p be a vertex of W/W_2 and $w \in W$ a representative for p. Let $\varphi_w \colon W_2/W_1 \to \pi^{-1}(p)$, $\varphi_w(vW_1) = wvW_1$ and ψ_w the restriction and co-restriction of $\psi_w \colon \mathfrak{t}^* \to \mathfrak{t}^*$, $\psi_w(x) = wx$. Note that φ_w and ψ_w depend not just on the class p, but on the *particular* representative w. If $e = (vW_1, vs_\beta W_1)$ is an edge of W_2/W_1, with $\beta \in \langle \Sigma_2 \rangle \setminus \langle \Sigma_1 \rangle$, then $e' = (\varphi_w(vW_1), \varphi_w(vs_\beta W_1) = (wvW_1, wvs_\beta W_1)$ is an edge of the fiber, and

$$\alpha_1(e') = wv\beta = \psi_v(\alpha(e)) \ .$$

It is not hard to see that $(\varphi_w, \psi_w) \colon W_2/W_1 \to \pi^{-1}(p)$ is in fact an isomorphism of GKM graphs, and this concludes the proof of the theorem. □

The example considered in Section 2.5 is the particular case of a root system of type A_{n-1}, with $\Sigma_1 = \emptyset$ and $\Sigma_2 = \Delta_0 \setminus \{\alpha_1\}$. The fiber bundle $\mathcal{F}l_4(\mathbb{C}) \to \mathcal{G}r_2(\mathbb{C}^4)$ shown in Figure 2 corresponds to the root system A_3, with $\Delta_0 = \{\alpha_1, \alpha_2, \alpha_3\}$, $\Sigma_1 = \emptyset$ and $\Sigma_2 = \{\alpha_1, \alpha_3\}$.

4.4. **Holonomy Subgroup.** In this section we determine the holonomy subgroup of $\mathrm{Aut}(W_2/W_1, \alpha)$ determined by loops in the base W/W_2.

Let $w \in W_2$, let $\Phi_w \colon W_2/W_1 \to W_2/W_1$, $\Phi_w(uW_1) = wuW_1$, and $\Psi_w \colon \mathfrak{t}^* \to \mathfrak{t}^*$, $\Psi_w(\beta) = w\beta$. Then $\Upsilon_w = (\Phi_w, \Psi_w) \colon W_2/W_1 \to W_2/W_1$ is a GKM automorphism. Moreover, the map $\Upsilon \colon W_2 \to \mathrm{Aut}(W_2/W_1, \alpha)$, $\Upsilon(w) = \Upsilon_w$ is a morphism of groups with kernel included in W_1. When W_1 is a normal subgroup of W_2, the kernel *is* W_1, and then the image $\Upsilon(W_2)$ is isomorphic with the quotient group W_2/W_1.

Proposition 4.4. The holonomy subgroup of $\mathrm{Aut}(W_2/W_1, \alpha)$ is $\Upsilon(W_2)$.

Proof. For $v_0 \in W$ let $\pi^{-1}(v_0W_2) \subset W/W_1$ be the fiber through v_0W_2, identified with W_2/W_1 by $(\varphi_{v_0}, \psi_{v_0})\colon W_2/W_1 \to \pi^{-1}(v_0W_2)$.

Let $\gamma \in \Omega(v_0W_2)$ be a loop in W/W_2 based at v_0W_2, given by

$$v_0W_2 \to v_1W_2 \to \cdots \to v_{m-1}W_2 \to v_mW_2 = v_0W_2 \ ,$$

where $v_k = v_{k-1}s_{\beta_k}$, with $\beta_k \in \Delta^+ \setminus \langle\Sigma_2\rangle$ for $k = 1, \ldots, m$, and let $w = v_0^{-1}v_m$. Then $w = s_{\beta_1}\cdots s_{\beta_m}$, and since γ is a loop, we have $w \in W_2$.

Let $\varphi_\gamma\colon W_2/W_1 \to W_2/W_1$ be the map

$$\varphi_\gamma = \varphi_{v_0}^{-1} \circ \Phi_\gamma \circ \varphi_{v_0} = \varphi_{v_0}^{-1} \circ \Phi_{v_{m-1}W_2, v_mW_2} \circ \cdots \circ \Phi_{v_0W_2, v_1W_2} \circ \varphi_{v_0} \ .$$

Then

$$\Phi_{v_0W_2, v_1W_2} \circ \varphi_{v_0}(uW_1) = s_{v_0\beta_1}v_0uW_1 = v_0s_{\beta_1}uW_1 = \varphi_{v_1}(uW_1) \ .$$

Continuing with the other edges of γ, we get

$$\varphi_\gamma(uW_1) = \varphi_{v_0}^{-1}\varphi_{v_m}(uW_1) = \Phi_w(uW_1) \ ,$$

hence $\varphi_\gamma = \Phi_w$. Similarly, $\psi_\gamma = \psi_w$, and hence $\rho_\gamma = \Upsilon_w$. We conclude that

$$\mathrm{Hol}(W_2/W_1, v_0W_2) \subset \Upsilon(W_2) \ .$$

We now show that for every $v \in W_2$, there exists a loop γ in W/W_2, starting and ending at v_0W_2, and such that $\rho_\gamma = \Upsilon(v)$.

Let $\alpha_i \in \Sigma_2 \subsetneq \Delta_0$. The Weyl group W acts transitively on Δ, hence there exists $w \in W$ such that $w\alpha_i \in \Delta^+ \setminus \langle\Sigma_2\rangle$. Let $w = uv$ be a decomposition of w such that $u \in W_2$ and $v = s_{\beta_1}\cdots s_{\beta_m}$ with $\beta_1, \ldots, \beta_m \in \Delta^+\setminus\langle\Sigma_2\rangle$. Then $u^{-1}w\alpha_i \in \Delta^+\setminus\langle\Sigma_2\rangle$, because $\Delta^+ \setminus \langle\Sigma_2\rangle$ is W_2-invariant. Consider the path γ in W/W_2 that starts with

$$v_0W_2 \to v_0s_{\beta_m}W_2 \to \cdots \to v_0s_{\beta_m}\cdots s_{\beta_1}W_2 = v_0v^{-1}W_2 \ ,$$

continues with

$$v_0v^{-1}W_2 \to v_0v^{-1}s_{u^{-1}w\alpha_i}W_2 \to v_0v^{-1}s_{u^{-1}w\alpha_i}s_{\beta_1}W_2 \to v_0v^{-1}s_{u^{-1}w\alpha_i}s_{\beta_1}s_{\beta_2}W_2 \ ,$$

and ends with

$$v_0v^{-1}s_{u^{-1}w\alpha_i}s_{\beta_1}s_{\beta_2}W_2 \to \cdots \to v_0v^{-1}s_{u^{-1}w\alpha_i}s_{\beta_1}s_{\beta_2}\cdots s_{\beta_m}W_2 = v_0v^{-1}s_{u^{-1}w\alpha_i}vW_2 \ .$$

This path is a loop because $v_0v^{-1}s_{u^{-1}w\alpha_i}v = v_0s_{\alpha_i}$ and $\alpha_i \in \Sigma_2$, and

$$\rho_\gamma = \Upsilon_{v_0v_0^{-1}s_i} = \Upsilon(s_i) \ .$$

Since W_2 is generated by $s_i = s_{\alpha_i}$ for $\alpha_i \in \Sigma_2$, we conclude that

$$\mathrm{Hol}(W_2/W_1, v_0W_2) = \Upsilon(W_2) \ ,$$

and the holonomy group of the typical fiber does not depend on the base point. □

4.5. **Bases of Invariant Classes.** We use the GKM graph of $M = G/B$ to describe equivariant cohomology classes in $H_T^*(M)$. The ring $H_\alpha^*(W)$ consists of the maps $f\colon W \to \mathbb{S}(\mathfrak{t}^*)$ such that

$$f(ws_\beta) - f(w) \in (w\beta)\mathbb{S}(\mathfrak{t}^*)$$

for every $w \in W$ and $\beta \in \Delta^+$.

The Weyl group action on $\mathfrak{t}^*$ induces an action of W on $H_\alpha^*(W)$, given by

$$w \cdot f = f^w\colon W \to \mathbb{S}(\mathfrak{t}^*) \ , \quad f^w(v) = w^{-1}f(wv) \ .$$

Let K be a compact real form of G containing T. Then (see, for example, [GS, Section 4.7]) the subring of $W-$invariant classes is

$$H^*_\alpha(W)^W \simeq H^*_T(M)^W \simeq H^*_K(M) = H^*_K(G/B) = H^*_T(K/T) \simeq \mathbb{S}(\mathfrak{t}^*) \ ,$$

An explicit ring isomorphism from $\mathbb{S}(\mathfrak{t}^*)$ to $H^*_\alpha(W)^W$ is given by

$$c_T \colon \mathbb{S}(\mathfrak{t}^*) \to H^*_\alpha(W)^W, c_T(q)(v) = v \cdot q \ , \tag{9}$$

for all $q \in \mathbb{S}(\mathfrak{t}^*)$ and $v \in W$. The inverse is $c_T^{-1} \colon H^*_\alpha(W)^W \to \mathbb{S}(\mathfrak{t}^*)$, $c_T^{-1}(f) = f(1)$.

We will show in Section 6.3 that the $\mathbb{S}(\mathfrak{t}^*)-$module $H^*_\alpha(W)$ has bases consisting of $W-$invariant classes. The isomorphism c_T establishes an explicit correspondence between such bases and $\mathbb{S}(\mathfrak{t}^*)^W-$module bases of $\mathbb{S}(\mathfrak{t}^*)$.

Theorem 4.5. Let $q_1, \dots, q_N$ be elements of $\mathbb{S}(\mathfrak{t}^*)$ and $f_i = c_T(q_i)$, $i = 1, \dots, N$ the corresponding $W-$invariant classes. Then $\{f_1, \dots, f_N\}$ is a basis of $H^*_\alpha(W)$ over $\mathbb{S}(\mathfrak{t}^*)$ if and only if $\{q_1, \dots, q_N\}$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ over $\mathbb{S}(\mathfrak{t}^*)^W$.

Proof. Assume first that $\{f_1, \dots, f_N\}$ is a basis of $H^*_\alpha(W)$ over $\mathbb{S}(\mathfrak{t}^*)$.

Suppose that $a_1, \dots, a_N$ are elements of $\mathbb{S}(\mathfrak{t}^*)^W$ such that

$$a_1 q_1 + \cdots + a_N q_N = 0 \ .$$

Then for every $v \in W$ we have

$$v \cdot (a_1 q_1 + \cdots + a_N q_N) = 0 \Longrightarrow a_1 f_1(v) + \cdots + a_N f_N(v) = 0 \ ,$$

and since this is valid for every $v \in W$, we conclude that

$$a_1 f_1 + \cdots + a_N f_N = 0 \ .$$

But the classes $f_1, \dots, f_N$ are independent, hence $a_1 = \cdots = a_N = 0$. Therefore $q_1, \dots, q_N$ are linearly independent over $\mathbb{S}(\mathfrak{t}^*)^W$.

Let $q \in \mathbb{S}(\mathfrak{t}^*)$. Then $c_T(q) \in H^*_\alpha(W)$, hence there exist $a_1, \dots, a_N$ in $\mathbb{S}(\mathfrak{t}^*)$ such that

$$c_T(q) = a_1 f_1 + \cdots + a_N f_N \ .$$

Then for every $v \in W$ we have

$$\begin{aligned} c_T(q)(v^{-1}) = a_1 f_1(v^{-1}) + \cdots + a_N f_N(v^{-1}) &\Longrightarrow \\ v^{-1} \cdot q = a_1 v^{-1} \cdot q_1 + \cdots + a_N v^{-1} \cdot q_N &\Longrightarrow \\ q = (v \cdot a_1)\, q_1 + \cdots + (v \cdot a_N)\, q_N \ . & \end{aligned}$$

Averaging over W we get

$$q = b_1 q_1 + \cdots + b_N q_N \ ,$$

where for each $k = 1, \dots, N$,

$$b_k = \frac{1}{|W|} \sum_{v \in W} v \cdot a_k$$

is an element of $\mathbb{S}(\mathfrak{t}^*)^W$. This proves that $q_1, \dots, q_N$ also generate $\mathbb{S}(\mathfrak{t}^*)$ over $\mathbb{S}(\mathfrak{t}^*)^W$, and therefore $\{q_1, \dots, q_N\}$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ over $\mathbb{S}(\mathfrak{t}^*)^W$.

Conversely, assume now that $\{q_1, \dots, q_N\}$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ over $\mathbb{S}(\mathfrak{t}^*)^W$.

Let $\{\sigma_1, \dots, \sigma_N\}$ be a basis of $H^*_\alpha(W)$ consisting of $W-$invariant classes. There must be exactly N such classes, because by the first part $\{r_i = \sigma_i(1) \mid i = 1, .., N\}$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ over $\mathbb{S}(\mathfrak{t}^*)^W$, and all bases of a free module over a commutative ring have the same number of elements.

Let $A \in GL_N(\mathbb{S}(\mathfrak{t}^*)^W) \subset GL_N(\mathbb{S}(\mathfrak{t}^*))$ be the change-of-basis matrix from the basis $\{r_1, \ldots, r_N\}$ to the basis $\{q_1, \ldots, q_N\}$:

$$q_i = a_{1i}r_1 + \cdots + a_{Ni}r_N$$

for all $i = 1 \ldots, N$. Since the entries of A are $W-$invariant, for $v \in W$ we have

$$f_i(v) = v \cdot q_i = a_{1i}v \cdot r_1 + \cdots + a_{Ni}v \cdot r_N = a_{1i}\sigma_1(v) + \cdots + a_{Ni}\sigma_N(v)$$

and therefore

$$f_i = a_{1i}\sigma_1 + \cdots + a_{Ni}\sigma_N$$

for all $i = 1 \ldots, N$. Since $\{\sigma_1, \ldots, \sigma_N\}$ is a basis and A is invertible, it follows that $\{f_1, \ldots, f_N\}$ is also a basis, and that concludes the proof. □

5. Fibrations of Classical Groups

In this section we consider the GKM bundle $W \to W/W_S$ when $S = \Delta_0 \setminus \{\alpha_1\}$, where Δ_0 is the set of simple roots for a classical root system and α_1 is one of the endpoint roots in the Dynkin diagram. By recursively applying Theorem 3.5, we construct a basis of $H^*_\alpha(W)$ consisting of $W-$invariant classes.

5.1. **Type A.** The set of simple roots of A_n (for $n \geqslant 2$) is $\Delta_0 = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i = x_i - x_{i+1}$, for $i = 1, \ldots, n$. The set of positive roots is

$$\Delta^+ = \{x_i - x_j \mid 1 \leqslant i < j \leqslant n+1\}$$

and $x_i - x_j = \alpha_i + \ldots + \alpha_{j-1}$. If $S = \{\alpha_2, \ldots, \alpha_n\}$, then

$$\langle S \rangle = \{x_i - x_j \mid 2 \leqslant i < j \leqslant n+1\} ,$$

is the set of positive roots for a root system of type A_{n-1}, and

$$\Delta^+ \setminus \langle S \rangle = \{\beta_j \mid \beta_j = x_1 - x_j, 2 \leqslant j \leqslant n+1\} = \{\alpha_1 + \cdots + \alpha_j \mid 1 \leqslant j \leqslant n\} .$$

Let

$$\omega_1 = [id] \quad \text{and} \quad \omega_j = [s_{\beta_j}] \text{ , for } 2 \leqslant j \leqslant n+1 .$$

Then $W/W_S = \{\omega_1, \ldots, \omega_{n+1}\}$, and the graph structure of W/W_S is that of a complete graph with $n+1$ vertices. If $\tau \colon W/W_S \to \mathfrak{t}^*$ is given by $\tau(\omega_i) = x_i$ for all $i = 1, \ldots, n+1$, then the axial function α on W/W_S is given by

$$\alpha(\omega_i, \omega_j) = \tau(\omega_i) - \tau(\omega_j) = x_i - x_j$$

and $\tau \in H^1_\alpha(W/W_S)$ is a class of degree 1. Using a Vandermonde determinant argument, one can show that the classes $\{1, \tau, \ldots, \tau^n\}$ are linearly independent over $\mathbb{S}(\mathfrak{t}^*)$, and in fact form a basis of the free $\mathbb{S}(\mathfrak{t}^*)-$module $H^*_\alpha(W/W_S)$.

The Weyl group W is isomorphic to the symmetric group S_{n+1}, acting on roots by

$$w \cdot (x_i - x_j) = x_{w(i)} - x_{w(j)} .$$

The simple reflection s_i acts as the transposition $(i, i+1)$, and, more generally, the reflection associated to the root $x_i - x_j$ acts as the transposition (i, j). The subgroup W_S is the subgroup of $W = S_{n+1}$ consisting of the permutations that fix the element 1. With the identification $W/W_S \simeq K_{n+1}$, the projection $\pi \colon W \to W/W_S$ is the map $\pi \colon S_{n+1} \to K_{n+1}$, $\pi(w) = w(1)$.

Remark 5.1. This is essentially the example discussed in Section 2.5, and corresponds to the fiber bundle of complete flags over a projective space. The group G is $SL_{n+1}(\mathbb{C})$, the Borel subgroup B is the subgroup of upper triangular matrices, and the parabolic subgroup P is the subgroup of G consisting of block-diagonal matrices, with one block of size 1×1 and a second block of size $n \times n$. Then $G/B \simeq \mathcal{F}l(\mathbb{C}^{n+1})$ and $G/P \simeq \mathbb{C}P^n$. The projection $\pi\colon \mathcal{F}l(\mathbb{C}^{n+1}) \to \mathbb{C}P^n$ sends the flag

$$V_\bullet \colon V_1 \subset V_2 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}$$

to $\pi(V_\bullet) = V_1$. For an element $L \in \mathbb{C}P^n$, hence a one-dimensional subspace of $\mathbb{C}^{n+1}$, the fiber $\pi^{-1}(L)$ is diffeomorphic to $\mathcal{F}l(\mathbb{C}^{n+1}/L) \simeq \mathcal{F}l(\mathbb{C}^n)$.

For a multi-index $I = [i_1, \ldots, i_n]$ of non-negative integers, we define

$$\mathbf{x}^I = {x_1}^{i_1} {x_2}^{i_2} \cdots x_n^{i_n}$$

and let $c_I = c_T(\mathbf{x}^I)$ be the corresponding $W-$invariant class $c_I \colon S_{n+1} \to \mathbb{S}(\mathfrak{t}^*)$,

$$c_I(u) = u \cdot \mathbf{x}^I = x_{u(1)}^{i_1} \cdots x_{u(n)}^{i_n} \ ;$$

then $\mathbf{x}^I = c_I(id)$, where id is the identity element of the Weyl group $W = S_{n+1}$. We will construct a basis of the $\mathbb{S}(\mathfrak{t}^*)-$module $H^*_\alpha(W)$ consisting of classes of the type c_I for specific indices I.

Consider the GKM fiber bundle $\pi\colon S_3 \to K_3$, $\pi(u) = u(1)$. The fiber $\pi^{-1}(3)$ is canonically isomorphic to S_2, and since $S_2 \simeq K_2$, the cohomology of S_2 is a free $\mathbb{S}(\mathfrak{t}^*)-$module with a basis given by the invariant classes $c_{[0]}$ and $c_{[1]}$. The invariant class $c_{[0]}$ on this fiber is extended, using transition maps between fibers to the constant class $c_{[0,0]} \equiv 1$ on the total space. The invariant class $c_{[1]}$ extends to the class $c_{[0,1]}$; the shift in index is due to the fact that the axial functions on fibers are different. The cohomology of the base K_3 is generated, over $\mathbb{S}(\mathfrak{t}^*)$, by 1, τ, and τ^2, and these classes lift to basic classes $c_{[0,0]}$, $c_{[1,0]}$, and $c_{[2,0]}$ on S_3. Theorem 3.5 implies that the cohomology of S_3 is a free $\mathbb{S}(\mathfrak{t}^*)-$module, with a basis given by

$$\{c_I \mid I = [i_1, i_2], 0 \leqslant i_1 \leqslant 2, 0 \leqslant i_2 \leqslant 1\} \ .$$

Their values on $W(A_2) = S_3$ are given in Table 1.

		$c_{[0,0]}$	$c_{[0,1]}$	$c_{[1,0]}$	$c_{[1,1]}$	$c_{[2,0]}$	$c_{[2,1]}$
id	123	1	x_2	x_1	x_1x_2	x_1^2	$x_1^2x_2$
s_1	213	1	x_1	x_2	x_2x_1	x_2^2	$x_2^2x_1$
s_2	132	1	x_3	x_1	x_1x_3	x_1^2	$x_1^2x_3$
s_1s_2	231	1	x_3	x_2	x_2x_3	x_2^2	$x_2^2x_3$
s_2s_1	312	1	x_1	x_3	x_3x_1	x_3^2	$x_3^2x_1$
$s_1s_2s_1$	321	1	x_2	x_3	x_3x_2	x_3^2	$x_3^2x_2$

TABLE 1. Invariant classes on $W(A_2)$

Repeating the procedure further, we get the following result.

Theorem 5.2. Let

$$\mathcal{A}_n = \{I = [i_1, \ldots, i_n] \mid 0 \leqslant i_1 \leqslant n, 0 \leqslant i_2 \leqslant n-1, \ldots, 0 \leqslant i_n \leqslant 1\} \ .$$

Then

$$\{c_I = c_T(\mathbf{x}^I) \mid I \in \mathcal{A}_n\}$$

is an $\mathbb{S}(\mathfrak{t}^*)-$module basis of $H^*_\alpha(A_n)$, consisting of invariant classes.

By Theorem 4.5 it follows that, in type A_n, $\{\mathbf{x}^I \mid I \in \mathcal{A}_n\}$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ as an $\mathbb{S}(\mathfrak{t}^*)^W-$module. Observe that the top degree class is generated by the top degree Schubert polynomial.

5.2. **Type B.** The set of simple roots of B_n (for $n \geqslant 2$) is $\Delta_0 = \{\alpha_1, \dots, \alpha_n\}$, where $\alpha_i = x_i - x_{i+1}$, for $i = 1, \dots, n-1$ and $\alpha_n = x_n$. The set of positive roots is

$$\Delta^+ = \{x_i \mid 1 \leqslant i \leqslant n\} \cup \{x_i \pm x_j \mid 1 \leqslant i < j \leqslant n\} .$$

If $S = \{\alpha_2, \dots, \alpha_n\}$, then

$$\langle S \rangle = \{x_i \mid 2 \leqslant i \leqslant n\} \cup \{x_i \pm x_j \mid 2 \leqslant i < j \leqslant n\}$$

is the set of positive roots for a root system of type B_{n-1}, and

$$\Delta^+ \setminus \langle S \rangle = \{\beta_1 = x_1\} \cup \{\beta_j^\pm = x_1 \mp x_j \mid 2 \leqslant j \leqslant n\} .$$

Let

$$\begin{aligned} \omega_1^+ =& [id] \quad , \quad \omega_1^- = [s_{\beta_1}] \\ \omega_j^+ =& [s_{\beta_j^+}] = [s_{x_1 - x_j}] \text{ for } 2 \leqslant j \leqslant n \\ \omega_j^- =& [s_{\beta_j^-}] = [s_{x_1 + x_j}] \text{ for } 2 \leqslant j \leqslant n . \end{aligned}$$

Then $W/W_S = \{\omega_1^+, \omega_1^-, \dots, \omega_n^+, \omega_n^-\}$, and the graph structure of W/W_S is that of a complete graph with $2n$ vertices. If τ is the map $\tau \colon W/W_S \to \mathfrak{t}^*$, $\tau(\omega_j^\epsilon) = \epsilon x_j$, with $1 \leqslant j \leqslant n$ and $\epsilon \in \{+,-\}$, then the axial function α

$$\begin{aligned} \alpha(\omega_i^{\epsilon_i}, \omega_j^{\epsilon_j}) =& \tau(\omega_i^{\epsilon_i}) - \tau(\omega_j^{\epsilon_j}), \text{ for } 1 \leqslant i \neq j \leqslant n \\ \alpha(\omega_i^{\epsilon_i}, \omega_i^{-\epsilon_i}) =& \frac{1}{2}(\tau(\omega_i^{\epsilon_i}) - \tau(\omega_i^{-\epsilon_i})) \qquad \text{for } 1 \leqslant i \leqslant n . \end{aligned}$$

Note that although W/W_S and K_{2n} are isomorphic as graphs, they are not isomorphic as GKM graphs. One way to see that is to notice that

$$\alpha(\omega_1^+, \omega_1^-) + \alpha(\omega_1^-, \omega_2^-) + \alpha(\omega_2^-, \omega_1^+) = -x_1 \neq 0 .$$

Nevertheless, as in the K_{2n} case, the set of classes $\{1, \tau, \dots, \tau^{2n-1}\}$ is a basis for the free $\mathbb{S}(\mathfrak{t}^*)-$module $H^*_\alpha(W/W_S)$.

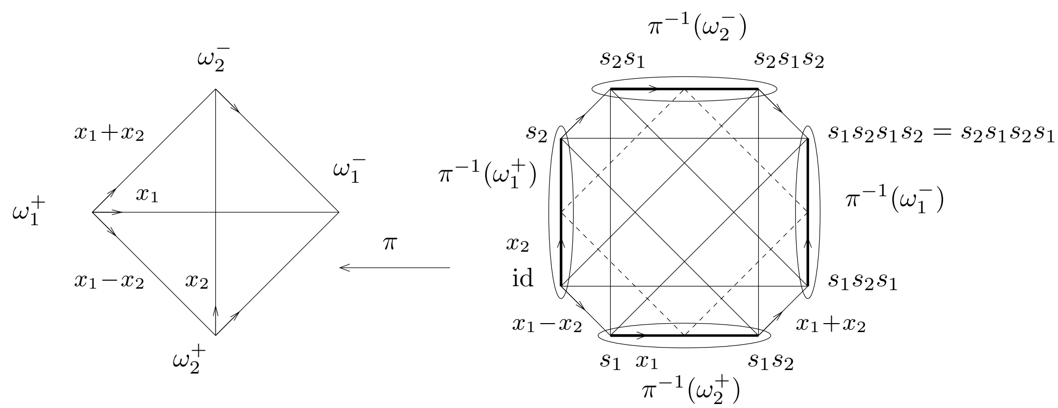

FIGURE 5. Fibration of B_2

An alternative description of the Weyl group W is that of the group of signed permutations (u,ϵ), with $u \in S_n$ and $\epsilon = (\epsilon_1,\ldots,\epsilon_n)$, $\epsilon_j = \pm 1$. The element (u,ϵ) is represented as $(\epsilon_1 u(1),\ldots,\epsilon_n u(n))$ or by underlying the negative entries.

Then s_{x_i} is just a change of the sign of x_i, $s_{x_i - x_j}$ corresponds to the transposition (i,j), with no sign changes, and $s_{x_i+x_j}$ corresponds to the transposition (i,j) with both signs changed. In particular, id is the identity permutation with no sign changes, s_{β_1} is the identity permutation with the sign of 1 changed, $s_{\beta_j^+}$ is the transposition $(1,j)$ with no sign changes, and $s_{\beta_j^-}$ is the transposition $(1,j)$ with sign changes for 1 and j. In general, if $u \in S_n$ and $\epsilon = (\epsilon_1,\ldots,\epsilon_n) \in \mathbb{Z}_2^n$, then the element $w = (u,\epsilon) \in W$ acts by $(u,\epsilon)\cdot x_k = \epsilon_k x_{u(k)}$. Then W/W_S can be identified with $\{\pm 1, \pm 2, \ldots, \pm n\}$ by $\omega_j^\epsilon \to \epsilon j$, and the projection $\pi \colon W \to W/W_S$ is $\pi((u,\epsilon)) = \epsilon_1 u(1)$.

For $I = [i_1,\ldots,i_n]$, let $c_I = c_T(\mathbf{x}^I) \colon W \to \mathbb{S}(\mathfrak{t}^*)$ be given by

$$c_I((u,\epsilon)) = (\epsilon_1 x_{u(1)})^{i_1}\cdots(\epsilon_n x_{u(n)})^{i_n}\ .$$

Then $c_I \in (H_\alpha^*(W))^W$ is an invariant class, and we will construct a basis of the free $\mathbb{S}(\mathfrak{t}^*)-$module $H_\alpha^*(W)$ consisting of classes of the type c_I, for specific indices I.

When $n=2$, the fiber over 2 is $\pi^{-1}(2) = \{(2,1),(2,-1)\}$ and is identified with $W_S = S_2 = \{1,-1\}$. A basis for $H_\alpha^*(W_S)$ is given by the invariant classes $\{c_{[0]}, c_{[1]}\}$, where $c_{[0]} \equiv 1$ and $c_{[1]}(1) = x_1$, $c_{[1]}(-1) = -x_1$. These classes are extended to the invariant classes $c_{[0,0]}$ and $c_{[0,1]}$ on W.

The classes 1, τ, τ^2, and τ^3 on the base lift to the basic classes $c_{[0,0]}$, $c_{[1,0]}$, $c_{[2,0]}$, and $c_{[3,0]}$ on W. Then a basis for the free $\mathbb{S}(\mathfrak{t}^*)-$module $H_\alpha^*(W)$ is

$$\{c_I \mid I = [i_1,i_2],\ 0 \leqslant i_1 \leqslant 3, 0 \leqslant i_2 \leqslant 1\}\ .$$

The values of these classes on the elements of $W(B_2)$ are shown in Table 2.

		$c_{[0,0]}$	$c_{[0,1]}$	$c_{[1,0]}$	$c_{[1,1]}$	$c_{[2,0]}$	$c_{[2,1]}$	$c_{[3,0]}$	$c_{[3,1]}$
id	12	1	x_2	x_1	x_1x_2	x_1^2	$x_1^2x_2$	x_1^3	$x_1^3x_2$
s_1	21	1	x_1	x_2	x_1x_2	x_2^2	$x_2^2x_1$	x_2^3	$x_2^3x_1$
s_2	$1\underline{2}$	1	$-x_2$	x_1	$-x_1x_2$	x_1^2	$-x_1^2x_2$	x_1^3	$-x_1^3x_2$
s_1s_2	$2\underline{1}$	1	$-x_1$	x_2	$-x_1x_2$	x_2^2	$-x_2^2x_1$	x_2^3	$-x_2^3x_1$
s_2s_1	$\underline{2}1$	1	x_1	$-x_2$	$-x_1x_2$	x_2^2	$x_2^2x_1$	$-x_2^3$	$-x_2^3x_1$
$s_1s_2s_1$	$\underline{1}2$	1	x_2	$-x_1$	$-x_1x_2$	$-x_1^2$	$x_1^2x_2$	$-x_1^3$	$-x_1^3x_2$
$s_2s_1s_2$	$\underline{21}$	1	$-x_1$	$-x_2$	x_1x_2	x_2^2	$-x_2^2x_1$	$-x_2^3$	$x_2^3x_1$
$s_1s_2s_1s_2$	$\underline{12}$	1	$-x_2$	$-x_1$	x_1x_2	x_1^2	$-x_1^2x_2$	$-x_1^3$	$x_1^3x_2$

TABLE 2. Invariant classes on $W(B_2)$

Repeating the procedure further, we get the following result.

Theorem 5.3. Let

$$\mathcal{B}_n = \{I = [i_1,\ldots,i_n] \mid 0 \leqslant i_1 \leqslant 2n-1, 0 \leqslant i_2 \leqslant 2n-3, \ldots, 0 \leqslant i_n \leqslant 1\}$$

Then

$$\{c_I \mid I \in \mathcal{B}_n\}$$

is an $\mathbb{S}(\mathfrak{t}^*)-$module basis of $H_\alpha^*(W(B_n))$ consisting of $W-$invariant classes.

By Theorem 4.5 it follows that, in type B_n, $\{\mathbf{x}^I \mid I \in \mathcal{B}_n\}$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ as an $\mathbb{S}(\mathfrak{t}^*)^W-$module.

5.3. **Type C.** The set of simple roots of C_n (for $n \geqslant 2$) is $\Delta_0 = \{\alpha_1, \dots, \alpha_n\}$, where $\alpha_i = x_i - x_{i+1}$, for $i = 1, \dots, n-1$ and $\alpha_n = 2x_n$. The set of positive roots is

$$\Delta^+ = \{2x_i \mid 1 \leqslant i \leqslant n\} \cup \{x_i \pm x_j \mid 1 \leqslant i < j \leqslant n\} .$$

If $S = \{\alpha_2, \dots, \alpha_n\}$, then

$$\langle S \rangle = \{2x_i \mid 2 \leqslant i \leqslant n\} \cup \{x_i \pm x_j \mid 2 \leqslant i < j \leqslant n\}$$

is the set of positive roots for a root system of type C_{n-1}, and

$$\Delta^+ \setminus \langle S \rangle = \{\beta_1 = 2x_1\} \cup \{\beta_j^{\pm} = x_1 \mp x_j \mid 2 \leqslant j \leqslant n\} .$$

Let

$$\begin{aligned} \omega_1^+ =& [id] \quad , \quad \omega_1^- = [s_{\beta_1}] \\ \omega_j^+ =& [s_{\beta_j^+}] = [s_{x_1 - x_j}] \text{ for } 2 \leqslant j \leqslant n \\ \omega_j^- =& [s_{\beta_j^-}] = [s_{x_1 + x_j}] \text{ for } 2 \leqslant j \leqslant n . \end{aligned}$$

This is essentially the same as the type B case, and $W(C_n) \simeq W(B_n)$ is the group of signed permutations of n letters. Then $W/W_S = \{\omega_1^+, \omega_1^-, \dots, \omega_n^+, \omega_n^-\}$, and the graph structure of W/W_S is that of a complete graph with $2n$ vertices. The axial function on W/W_S is given by

$$\alpha(\omega_i^{\epsilon_i}, \omega_j^{\epsilon_j}) = \tau(\omega_i^{\epsilon_i}) - \tau(\omega_j^{\epsilon_j}) ,$$

hence W/W_S is isomorphic, as a GKM graph, with a projection of the complete graph K_{2n}. Then $H_\alpha^*(W(C_n)) \simeq H_\alpha^*(W(B_n))$, with $\mathcal{B}(C_n) = \mathcal{B}(B_n)$ as a basis consisting of invariant classes.

5.4. **Type D.** The set of simple roots of D_n (for $n \geqslant 3$) is $\Delta_0 = \{\alpha_1, \dots, \alpha_n\}$, where $\alpha_i = x_i - x_{i+1}$, for $i = 1, \dots, n-1$ and $\alpha_n = x_{n-1} + x_n$. The set of positive roots is

$$\Delta^+ = \{x_i - x_j \mid 1 \leqslant i < j \leqslant n\} \cup \{x_i + x_j \mid 1 \leqslant i < j \leqslant n\} .$$

If $S = \{\alpha_2, \dots, \alpha_n\}$, then

$$\langle S \rangle = \{x_i - x_j \mid 2 \leqslant i < j \leqslant n\} \cup \{x_i + x_j \mid 2 \leqslant i < j \leqslant n\} .$$

If $n \geqslant 4$, then $\langle S \rangle$ is the set of positive roots for a root system of type D_{n-1} and if $n = 3$, then $\langle S \rangle$ is the set of positive roots of $A_1 \times A_1$. In both cases

$$\Delta^+ \setminus \langle S \rangle = \{\beta_i^+ = x_1 - x_i \mid 2 \leqslant i \leqslant n\} \cup \{\beta_i^- = x_1 + x_i \mid 2 \leqslant i \leqslant n\} .$$

Let

$$\begin{aligned} &\omega_1^+ = [id], && \omega_1^- = [s_{\beta_j^-} s_{\beta_j^+}] = [s_{\beta_j^+} s_{\beta_j^-}], && \text{for all } 2 \leqslant j \leqslant n \\ &\omega_i^+ = [s_{\beta_i^+}], && \omega_i^- = [s_{\beta_i^-}], && \text{for all } 2 \leqslant i \leqslant n . \end{aligned}$$

Then $W/W_S = \{\omega_1^+, \omega_1^-, \dots, \omega_n^+, \omega_n^-\}$ and the graph structure of W/W_S is that of the complete $n-$partite graph K_2^n, with partition classes $\{\omega_i^+, \omega_i^-\}$ for $1 \leqslant i \leqslant n$. If $\tau \colon W/W_S \to \mathfrak{t}^*$ is given by $\tau(\omega_i^\epsilon) = \epsilon x_i$, where $\epsilon \in \{+, -\}$, then the axial function α on W/W_S is

$$\alpha(\omega_i^{\epsilon_i}, \omega_j^{\epsilon_j}) = \tau(\omega_i^{\epsilon_i}) - \tau(\omega_j^{\epsilon_j}) = \epsilon_i x_i - \epsilon_j x_j .$$

Then $H_\alpha^*(W/W_S)$ is a free $\mathbb{S}(\mathfrak{t}^*)-$module, and a Vandermonde determinant argument shows that a basis is given by $1, \tau, \dots, \tau^{2n-2}$, and $\eta = x_1 \cdots x_n \tau^{-1}$.

An alternative description of the Weyl group W is that of the group of signed permutations (u,ϵ) with an even number of sign changes. Then $s_{x_i-x_j}$ corresponds to the transposition (i,j), with no sign changes, and $s_{x_i+x_j}$ corresponds to the transposition (i,j) with both signs changed. In particular, id is the identity permutation with no sign changes, $s_{\beta_j^+}$ is the transposition $(1,j)$ with no sign changes, $s_{\beta_j^-}$ is the transposition $(1,j)$ with sign changes for 1 and j, and $s_{\beta_j^+}s_{\beta_j^-}$ is the identity permutation with the sign changes for 1 and j. In general, if $u\in S_n$ and $\epsilon=(\epsilon_1,\ldots,\epsilon_n)\in\mathbb{Z}_2^n$ with $\epsilon_1\cdots\epsilon_n=1$, then the element $w=(u,\epsilon)\in W$ acts by $(u,\epsilon)\cdot x_k=\epsilon_k x_{u(k)}$. Then W/W_S can be identified with $\{\pm1,\pm2,\ldots,\pm n\}$ by $\omega_i^\epsilon\to\epsilon i$, and the projection $\pi\colon W\to W/W_S$ is $\pi((u,\epsilon))=\epsilon_1 u(1)$.

For $I=[i_1,\ldots,i_n]$, let $c_I=c_T(\mathbf{x}^I)\colon W\to\mathbb{S}(\mathfrak{t}^*)$ be given by

$$c_I((u,\epsilon))=(\epsilon_1 x_{u(1)})^{i_1}\cdots(\epsilon_n x_{u(n)})^{i_n}\ .$$

Then $c_I\in(H_\alpha^*(W))^W$ is an invariant class, and we will construct a basis of the free $\mathbb{S}(\mathfrak{t}^*)-$module $H_\alpha^*(W)$ consisting of classes of the type c_I, for specific indices I.

When $n=3$, the fiber $\pi^{-1}(3)$ of the GKM fiber bundle $\pi\colon D_3\to K_2^3$ is

$$\pi^{-1}(3)=\{(3,1,2),(3,2,1),(3,-2,-1),(3,-1,-2)\}$$

and is identified with $W_S=S_2\times S_2=\{(1,2),(2,1),(-2,-1),(-1,-2)\}$. Then $H_\alpha^*(W_S)$ is generated by the W_S-invariant classes $\{c_I\mid I\in\mathcal{D}_2\}$, where

$$\mathcal{D}_2=\{[0,0],[1,0],[2,0],[0,1]\}\ .$$

The classes 1, τ, τ^2, τ^3, τ^4, η on K_2^3 lift to the basic classes $c_{[0,0,0]}$, $c_{[1,0,0]}$, $c_{[2,0,0]}$, $c_{[3,0,0]}$, $c_{[4,0,0]}$, and $c_{[0,1,1]}$. Then a basis for the free $\mathbb{S}(\mathfrak{t}^*)-$module $H_\alpha^*(W)$ is

$$\{c_I\mid I=[i_1,i_2,i_3]\in\mathcal{D}_3\}\ ,$$

where $\mathcal{D}_3$ is the set of triples $[i_1,i_2,i_3]\in\mathbb{Z}^3_{\geqslant0}$, such that $i_1i_2i_3=0$ and either $i_1\leqslant4, i_2\leqslant2, i_3\leqslant1$ or $[i_1,i_2,i_3]=[0,1,2]$ or $[0,3,1]$.

Repeating this process further, we get the following general result.

Theorem 5.4. Let $\mathcal{D}_n$ be a set of multi-indices defined inductively by

(1) $\mathcal{D}_2=\{[0,0],[1,0],[2,0],[0,1]\}$;
(2) $[i_1,\ldots,i_n]\in\mathcal{D}_n$ if
- $0\leqslant i_1\leqslant 2n-2$ and $[i_2,\ldots,i_n]\in\mathcal{D}_{n-1}$, or
- $i_1=0$ and $[i_2-1,\ldots,i_n-1]\in\mathcal{D}_{n-1}$.

Then

$$\{c_I\mid I\in\mathcal{D}_n\}\ .$$

is an $\mathbb{S}(\mathfrak{t}^*)-$module basis of $H_\alpha^*(D_n)$ consisting of $W-$invariant classes.

By Theorem 4.5 it follows that, in type D_n, $\{\mathbf{x}^I\mid I\in\mathcal{D}_n\}$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ as a free $\mathbb{S}(\mathfrak{t}^*)^W-$module.

6. Symmetrization of Schubert Classes

In Section 5 we constructed invariant classes for classical groups by iterating the GKM fiber bundle construction. In this section we present a different method of constructing invariant classes.

6.1. **Symmetrization of Classes.** Recall that the ring $H^*_\alpha(W)$ consists of the maps $f\colon W \to \mathbb{S}(\mathfrak{t}^*)$ such that

$$f(ws_\beta) - f(w) \in (w\beta)\mathbb{S}(\mathfrak{t}^*)$$

for every $w \in W$ and $\beta \in \Delta^+$, and the holonomy action of the Weyl group W is

$$w \cdot f = f^w \colon W \to \mathbb{S}(\mathfrak{t}^*) \ , \quad f^w(v) = w^{-1} f(wv) \ .$$

For every $u \in W$, there exists a unique class $\tau_u \in H^*_\alpha(W)$, called the equivariant Schubert class of u, that satisfies the following conditions:

(1) τ_u is homogeneous of degree $2\ell(u)$, where $\ell(u)$ is the length of u;
(2) τ_u is supported on $\{v \mid u \preccurlyeq v\}$, where $\preccurlyeq$ is the strong Bruhat order, and
(3) τ_u is normalized by the condition

$$\tau_u(u) = \prod\{\beta \mid \beta \in \Delta^+, u^{-1}\beta \in \Delta^-\}$$

The set $\{\tau_u \mid u \in W\}$ of equivariant Schubert classes is a basis of the $\mathbb{S}(\mathfrak{t}^*)-$module $H^*_\alpha(W)$; however, these classes are not invariant under the action of W on $H^*_\alpha(W)$.

For $f \in H^*_\alpha(W)$ we define the $W-$invariant class $f^{sym}\colon W \to \mathbb{S}(\mathfrak{t}^*)$ by

$$f^{sym} = \frac{1}{|W|} \sum_{w \in W} f^w \ ,$$

where the permuted class $f^w \colon W \to \mathbb{S}(\mathfrak{t}^*)$ is given by $f^w(u) = w^{-1} \cdot f(wu)$, $u \in W$.

For every $w \in W$, the permuted classes $\{\tau_u^w \mid u \in W\}$ form a basis of the $\mathbb{S}(\mathfrak{t}^*)-$module $H^*_\alpha(W)$. The main result of this section is that the *symmetrized* classes also form a basis of the $H^*_\alpha(W)$, and these classes *are* $W-$invariant.

6.2. **NilCoxeter Rings.** We start by recalling a few things about nilCoxeter rings. More details are available, for example, in [Ku].

These rings are defined for general Coxeter groups, but we will only need them for Weyl groups, for which we will use the notation introduced in Section 4.

Let W be a Weyl group, with simple positive roots $\{\alpha_1, \dots, \alpha_n\}$ and let $s_i = s_{\alpha_i}$ be the reflection generated by the simple root α_i, for $1 \leqslant i \leqslant n$. The nilCoxeter ring $\mathcal{H}$ is the ring with generators $\{u_i \mid i = 1, \dots, n\}$ satisfying $u_i^2 = 0$ for all $i = 1, \dots, n$ and the same commutation relations as $\{s_i \mid i = 1, \dots, n\}$.

If $w = s_{i_1} \cdots s_{i_r}$ is a reduced decomposition of $w \in W$ (hence $\ell(w) = r$), we define

$$u_w = u_{i_1} \cdots u_{i_r} \ .$$

The definition does not depend on the reduced decomposition, and

$$u_w u_v = \begin{cases} u_{wv}, & \text{if } \ell(wv) = \ell(w) + \ell(v) \\ 0, & \text{otherwise.} \end{cases}$$

For every $i = 1, \dots, n$, let $h_i(x) = 1 + xu_i$, where x is a variable that commutes with all the generators $u_1, \dots, u_n$. Then $h_i(x)$ is invertible and $h_i(x)^{-1} = h_i(-x)$.

If $w = s_{i_1} \cdots s_{i_r}$ is a reduced decomposition of $w \in W$, define $H_w \in \mathcal{H} \otimes \mathbb{S}(\mathfrak{t}^*)$ by

$$\begin{aligned} H_w =& h_{i_1}(\alpha_{i_1}) h_{i_2}(s_{i_1}\alpha_{i_2}) \cdots h_{i_r}(s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r}) = \\ =& (1 + \alpha_{i_1} u_{i_1})(1 + s_{i_1}\alpha_{i_2} u_{i_2}) \cdots (1 + s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r} u_{i_r}) \end{aligned} \tag{10}$$

The definition of H_w does not depend on the reduced decomposition of w.

In [Bi, Theorem 3], Billey showed that

$$H_w = \sum_{v\in W} \tau_v(w) u_v \tag{11}$$

and used this formula to prove an explicit positive formula for $\tau_v(w)$, as a sum of products of positive roots (see also [AJS, Appendix D]). In particular,

$$\tau_v(w) \in \mathbb{Z}_{\geqslant 0}^{\ell(v)}[\alpha_1, \ldots, \alpha_n]$$

is a homogeneous polynomial of degree $\ell(v)$ in the simple positive roots $\alpha_1, \ldots, \alpha_n$, with nonnegative integer coefficients. Moreover, H_w is invertible, and

$$H_w^{-1} = h_{i_r}(-s_{i_1}\cdots s_{i_{r-1}}\alpha_{i_r})\cdots h_{i_1}(-\alpha_{i_1}) = \sum_{v\in W} (-1)^{\ell(v^{-1})}\tau_{v^{-1}}(w) u_v\ . \tag{12}$$

Lemma 6.1. If $w, v \in W$, then

$$H_{wv} = H_w \cdot wH_v\ . \tag{13}$$

Proof. If $\ell(v) = 0$, then $v = 1$, $H_v = 1$, and the formula is clearly true.

The proof is made in four steps.

Step 1: $v = s_i$ and $\ell(ws_i) = \ell(w) + 1$. Let $w = s_{i_1}\cdots s_{i_r}$ be a reduced decomposition of w; then $ws_i = s_{i_1}\cdots s_{i_r}s_i$ is a reduced decomposition for ws_i, hence

$$\begin{aligned} H_{ws_i} =& h_{i_1}(\alpha_{i_1})h_{i_2}(s_{i_1}\alpha_{i_2})\cdots h_{i_r}(s_{i_1}\cdots s_{i_{r-1}}\alpha_{i_r})h_{i_r}(s_{i_1}\cdots s_{i_r}\alpha_i) = \\ =& H_w \cdot h_i(w\alpha_i) = H_w \cdot wh_i(\alpha_i) = H_w \cdot wH_{s_i}\ . \end{aligned}$$

Step 2: $\ell(wv) = \ell(w) + \ell(v)$. If $v = s_{i_1}\cdots s_{i_r}$ is a reduced decomposition for v, then $ws_{i_1}\cdots s_{i_k}$ is a reduced decomposition for every $k = 1, \ldots, r$, and hence Step 1 applies in all those cases. Hence

$$\begin{aligned} H_{wv} =& H_{ws_{i_1}\cdots s_{i_{r-1}}s_{i_r}} = H_{ws_{i_1}\cdots s_{i_{r-1}}} \cdot ws_{i_1}\cdots s_{i_{r-1}}h_{i_r}(\alpha_{i_r}) = \\ =& H_{ws_{i_1}\cdots s_{i_{r-1}}} \cdot wh_{s_{i_r}}(s_{i_1}\cdots s_{i_{r-1}}\alpha_{i_r}) = \\ =& H_w \cdot wh_{i_1}(\alpha_{i_1})\cdots wh_{i_r}(s_{i_1}\cdots s_{i_{r-1}}\alpha_{i_r}) = H_w \cdot wH_v\ . \end{aligned}$$

Step 3: $v = s_i$ and $\ell(ws_i) = \ell(w) - 1$. Let $w = s_{i_1}\cdots s_{i_r}$ be a reduced decomposition of w; then by the Exchange Condition, there exists an index k such that $ws_i = w_1w_2$, where $w_1 = s_{i_1}\cdots s_{i_{k-1}}$ and $w_2 = s_{i_{k+1}}\cdots s_{i_r}$ are reduced decompositions. Let $j = i_k$. Then $w = w_1s_jw_2$, $s_jw_2 = w_2s_i$, and $\ell(w_2s_i) = \ell(w_2) + 1$.

Then, by the result of Step 2, we have $H_w = H_{w_1s_jw_2} = H_{w_1} \cdot w_1H_{s_jw_2}$, hence

$$\begin{aligned} H_w \cdot wH_{s_i} =& H_{w_1} \cdot w_1H_{s_jw_2} \cdot w_1s_jw_wH_{s_i} = H_{w_1} \cdot w_1(H_{w_2s_i} \cdot w_2s_iH_{s_i}) = \\ =& H_{w_1} \cdot w_1(H_{w_2} \cdot w_2H_{s_i} \cdot w_2s_iH_{s_i}) = H_{w_1} \cdot w_1(H_{w_2} \cdot w_2(H_{s_i} \cdot s_iH_{s_i}))\ . \end{aligned}$$

But $H_{s_i} \cdot s_iH_{s_i} = (1 + \alpha_iu_i)(1 - \alpha_iu_i) = 1$, hence

$$H_w \cdot wH_{s_i} = H_{w_1} \cdot w_1H_{w_2} = H_{w_1w_2} = H_{ws_i}\ .$$

At this point we have proved that the formula is true for all w and $v = s_i$.

Step 4: For the general case we follow the same argument as for Step 2, using Step 1 or 3 to move over a simple reflection in the reduced decomposition of v. □

We use Lemma 6.1 to obtain the transition matrices between a basis of permuted Schubert classes and the original basis of Schubert classes.

Theorem 6.2. Let $a, b, w \in W$. Then

$$\tau_a = \sum_{b \leqslant_L a} \tau_{ab^{-1}}(w^{-1})\tau_b^w \ , \tag{14}$$

$$\tau_a^w = \sum_{b \leqslant_L a} (-1)^{\ell(ba^{-1})}\tau_{ba^{-1}}(w^{-1})\tau_b \ . \tag{15}$$

where $\leqslant_L$ is the left weak order, defined by $v \leqslant_L u \Longleftrightarrow \ell(uv^{-1}) = \ell(u) - \ell(v)$.

Proof. Let $v \in W$. By equation (13) we have

$$H_v = H_{w^{-1}} \cdot w^{-1} H_{wv} \tag{16}$$

which, using equation (11) and identifying the corresponding coefficients, yields

$$\tau_a(v) = \sum_{\substack{tb=a \\ \ell(t)+\ell(b)=\ell(a)}} \tau_t(w^{-1}) \cdot w^{-1}\tau_b(wv) = \sum_{b \leqslant_L a} \tau_{ab^{-1}}(w^{-1})\tau_b^w(v) \, .$$

Since this is true for all $v \in W$, we get (14).

From (16) we get

$$w^{-1}H_{wv} = H_{w^{-1}}^{-1} H_v \ ,$$

which, using (11)-(12) and identifying the corresponding coefficients, yields

$$\tau_a^w(v) = \sum_{\substack{tb=a \\ \ell(t)+\ell(b)=\ell(a)}} (-1)^{\ell(t^{-1})}\tau_{t^{-1}}(w^{-1})\tau_b(v) = \sum_{b \leqslant_L a} (-1)^{\ell(ba^{-1})}\tau_{ba^{-1}}(w^{-1})\tau_b(v) \ .$$

Since this is true for all $v \in W$, we get (15). □

If $w \in W$ then $\mathcal{B}^w = \{\tau_u^w \mid u \in W\}$ is a basis of $H_\alpha^*(W)$ as an $\mathbb{S}(\mathfrak{t}^*)-$module. By (14) the transition matrix a^w between $\mathcal{B}^w$ and the basis $\mathcal{B} = \{\tau_u \mid u \in W\}$ is the lower triangular (with respect to the weak left order) matrix

$$a_{u,v}^w = \begin{cases} (-1)^{\ell(vu^{-1})}\tau_{vu^{-1}}(w^{-1}) \ , & \text{if } v \leqslant_L u \\ 0 \ , & \text{otherwise.} \end{cases}$$

Since $\tau_{vu^{-1}}(w^{-1}) \in \mathbb{Z}_{\geqslant 0}[\alpha_1, \ldots, \alpha_n]$ is homogeneous of degree $\ell(vu^{-1})$, we have

$$a_{u,v}^w \in \mathbb{Z}_{\geqslant 0}^{\ell(u)-\ell(v)}[-\alpha_1, \ldots, -\alpha_n] \ .$$

Hence the nonzero entries of a^w are homogeneous polynomials in the negative simple roots, with non-negative integer coefficients, and the diagonal entries are 1. By (15), the inverse of a^w is the lower triangular matrix b^w with entries

$$b_{u,v}^w = \begin{cases} \tau_{uv^{-1}}(w^{-1}) \ , & \text{if } v \leqslant_L u \\ 0 \ , & \text{otherwise.} \end{cases}$$

The nonzero entries of b^w are homogeneous polynomials in the positive simple roots, with non-negative integer coefficients, and, again, the diagonal entries are 1.

6.3. **Symmetrized Schubert Classes.** The next result gives the decompositions of symmetrized Schubert classes in terms of Schubert classes and proves that the set $\mathcal{B}^{sym} = \{\tau_u^{sym} \mid u \in W\}$ of symmetrized classes is a basis of $H_\alpha^*(W)$.

Theorem 6.3. For every $u \in W$, let τ_u^{sym} be the symmetrization of τ_u. If

$$\tau_u^{sym} = \sum_{v \in W} a_{u,v}\tau_v \ , \tag{17}$$

is the decomposition of τ_u^{sym} in the Schubert basis, then

(1) The matrix $(a_{u,v})_{u,v}$ is lower triangular with respect to the left weak order:

$$a_{u,v} \neq 0 \Longrightarrow v \leqslant_L u \; .$$

(2) The entries on the diagonal are all 1:

$$a_{u,u} = 1 \quad \text{ for all } u \in W \; .$$

(3) The set $\mathcal{B}^{sym} = \{\tau_u^{sym} \mid u \in W\}$ is a basis of the $\mathbb{S}(\mathfrak{t}^*)-$module $H^*_\alpha(W)$.

Proof. If $u \in W$ then

$$\tau_u^{sym} = \frac{1}{|W|} \sum_{w\in W} \tau_u^w = \frac{1}{|W|} \sum_{w\in W} \sum_{v \leqslant_L u} a_{u,v}^w \tau_v = \sum_{v\leqslant_L u} \left(\frac{1}{|W|} \sum_{w \in W} a_{u,v}^w \right) \tau_v \; .$$

Therefore

$$a_{u,v} = \frac{1}{|W|} \sum_{w\in W} a_{u,v}^w \; ,$$

hence $(a_{u,v})_{u,v}$ is lower triangular with respect to the left weak order, with entries on the diagonal equal to 1. Such a matrix is invertible, and since $\mathcal{B}$ is a basis of $H^*_\alpha(W)$, it follows that $\mathcal{B}^{sym}$ is also a basis. □

Remark 6.4. For $v \leqslant_L u$ we have

$$|W| a_{u,v} \in \mathbb{Z}_{\geqslant 0}^{\ell(u)-\ell(v)}[-\alpha_1, \ldots, -\alpha_n] \; ,$$

because for all $w \in W$, $a_{u,v}^w$ is a homogeneous polynomial of degree $\ell(u) - \ell(v)$ in the negative simple roots, with non-negative integer coefficients.

6.4. **Decomposition of Invariant Classes.** Theorem 6.3 gives the decomposition of a symmetrized Schubert class τ_u^{sym} in the Schubert basis $\{\tau_w\}_w$. In this section we show how a general invariant class $c_f = c_T(f) \in H^*_\alpha(W)^W$, defined by (9), decomposes in the Schubert basis.

For $i = 1, \ldots, n$ let $\partial_i \colon \mathbb{S}(\mathfrak{t}^*) \to \mathbb{S}(\mathfrak{t}^*)$ be the divided difference operator

$$\partial_i E = \frac{E - s_i \cdot E}{\alpha_i} \; .$$

If $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced decomposition for $w \in W$, let $\epsilon(w) = (-1)^{\ell(w)}$ and

$$\partial_w = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_m} \; ;$$

the notation is justified by the fact that the result of the composition depends only on w and not on the reduced decomposition of w.

Proposition 6.5. If $f \in \mathbb{S}(\mathfrak{t}^*)$, then

$$c_f = \sum_{w\in W} \left(\epsilon(w) \partial_w f \right) \tau_w \; .$$

Proof. We have to show that for every $v \in W$ we have

$$v \cdot f = \sum_{w \in W} \left(\epsilon(w) \partial_w f \right) \tau_w(v) \; ,$$

and we prove this by induction on the length $\ell(v)$ of v.

When $\ell(v) = 0$ we have $v = 1$ and the only Schubert class τ_w that has a nonzero value at $v = 1$ is the one corresponding to $w = 1$, with $\tau_1(1) = 1$. Then $\partial_w f = f$ and the formula is obviously true.

Now suppose the formula is true for all v such that $\ell(v) \leqslant k$ and let $u \in W$ such that $\ell(u) = k+1$. Then u can be written as $u = s_i v$ for some $i = 1, \dots, n$ and some $v \in W$ such that $\ell(v) = \ell(u) - 1 = k$. Then

$$\sum_{w\in W} (\epsilon(w)\partial_w f)\tau_w(u) = \sum_{w\in W} (\epsilon(w)\partial_w f)\tau_w(s_i v)\,.$$

But

$$\tau_w(s_i v) = s_i \tau_w(v) + \begin{cases} \alpha_i s_i \tau_{s_i w}(v), & \text{if } s_i w \prec w \\ 0 & \text{otherwise} \end{cases}$$

This follows from $\tau_w(s_i v) = s_i \cdot \tau_w^{s_i}(v)$ and our formula for $\tau_w^{s_i}$ or from [Kn]. Hence

$$\sum_{w\in W} (\epsilon(w)\partial_w f)\,\tau_w(s_i v) = \sum_{w\in W} (\epsilon(w)\partial_w f)\, s_i \tau_w(v) + \sum_{s_i w \prec w} (\epsilon(w)\partial_w f)\,\alpha_i s_i \tau_{s_i w}(v)\ .$$

However, since

$$\partial_i \partial_{s_i w} = \begin{cases} \partial_w & \text{if } s_i w \prec w \\ 0 & \text{otherwise}\,, \end{cases}$$

we can rewrite the last sum and using $\epsilon(w) = -\,\epsilon(s_i w)$ we get

$$\begin{aligned}
\sum_{w\in W} (\epsilon(w)\partial_w f)\tau_w(s_i v) &= \sum_{w\in W} (\epsilon(w)\partial_w f) s_i \tau_w(v) - \sum_{w\in W} (\epsilon(s_i w)\partial_i \partial_{s_i w} f)\alpha_i s_i \tau_{s_i w}(v) = \\
&= \sum_{w\in W} (\epsilon(w)\partial_w f) s_i \tau_w(v) - \sum_{w\in W} (\epsilon(w)\partial_i \partial_w f)\alpha_i s_i \tau_w(v) = \\
&= \sum_{w\in W} (\epsilon(w)\partial_w f) s_i \tau_w(v) - \sum_{w\in W} \epsilon(w)\frac{\partial_w f - s_i \partial_w f}{\alpha_i}\,\alpha_i s_i \tau_w(v) = \\
&= \sum_{w\in W} \epsilon(w) s_i(\partial_w f) s_i \tau_w(v) = s_i \sum_{w\in W} (\epsilon(w)\partial_w f)\tau_w(v)\ .
\end{aligned}$$

From the induction hypothesis the last sum is $v \cdot f$ and therefore

$$\sum_{w\in W} (\epsilon(w)\partial_w f)\tau_w(u) = \sum_{w\in W} (\epsilon(w)\partial_w f)\tau_w(s_i v) = s_i \cdot (v \cdot f) = (s_i v)\cdot f = u \cdot f\ .$$

The induction is complete and that concludes the proof. □

Remark 6.6. Comparing Proposition 6.5 with [Hi, p. 65], we see that

$$c_T \colon \mathbb{S}(\mathfrak{t}^*) \to H_K^*(M) = H_\alpha^*(W)^W$$

is an equivariant version of the characteristic homomorphism $c\colon \mathbb{S}(\mathfrak{t}^*) \to H^*(M)$.

6.5. **Decomposition of symmetrized Schubert classes.** For $w \in W$, the symmetrized Schubert class τ_w^{sym} is an invariant class and $\tau_w^{sym} = c_T(f_w)$, where

$$f_w = \tau_w^{sym}(1) = \frac{1}{|W|}\sum_{v\in W} v^{-1}\cdot \tau_w(v) \in \mathbb{S}(\mathfrak{t}^*)\ .$$

In this subsection we prove a simple formula for f_w and we use it to revisit the decomposition of τ_w^{sym} in terms of the equivariant classes τ_u's.

Theorem 6.7. Let w_0 be the longest element of W and $\Lambda_0 = \tau_{w_0}(w_0) = \prod_{\alpha \succ 0} \alpha$ the product of all positive roots. If $w \in W$, then

$$f_w = \frac{\varepsilon(w)}{|W|}\partial_{w^{-1}w_0}(\Lambda_0) \tag{18}$$

Proof. We prove this result by descending induction on $\ell(w)$. For $w = w_0$ we have

$$f_{w_0} = \frac{1}{|W|}\sum_{v\in W} v^{-1}\cdot \tau_{w_0}(v) = \frac{1}{|W|} w_0^{-1}\cdot \Lambda_0 = \frac{\varepsilon(w_0)}{|W|}\partial_{w_0^{-1}w_0}(\Lambda_0)\ ,$$

because the action of $w_0^{-1} = w_0$ changes all positive roots to negative roots.

Suppose that (18) is true for u and let $w = us_i \prec w$ for a simple reflection s_i. Then $w^{-1}w_0 = s_i u^{-1} w_0$ and $\ell(w) = \ell(u) - 1$. This implies

$$\ell(s_i u^{-1} w_0) = \ell(w_0) - \ell(s_i u{-}1) = \ell(w_0) - \ell(us_i) = \ell(w_0) - \ell(u) + 1 = \ell(s_i) + \ell(u^{-1}w_0)$$

and therefore $\partial_{w^{-1}w_0} = \partial_i \partial_{u^{-1}w_0}$. Hence the right hand side of (18) becomes

$$\frac{\varepsilon(w)}{|W|}\partial_{w^{-1}w_0}(\Lambda_0) = -\frac{\varepsilon(u)}{|W|}\partial_i\partial_{u^{-1}w_0}(\Lambda_0) = -\partial_i(f_u) = -\frac{1}{\alpha_i}(f_u - s_i\cdot f_u)\ .$$

But

$$-\frac{1}{\alpha_i}(f_u - s_i\cdot f_u) = \frac{-1}{|W|\alpha_i}\left(\sum_{v\in W} v^{-1}\cdot\tau_u(v) - \sum_{v\in W} s_i v^{-1}\cdot \tau_u(v)\right)$$

and, after a change of variables in the second sum and using [Kn, Prop. 2],

$$\frac{\varepsilon(w)}{|W|}\partial_{w^{-1}w_0}(\Lambda_0) = \frac{-1}{|W|\alpha_i}\sum_{v\in W} v^{-1}\cdot(\tau_u(v) - \tau_u(vs_i)) =$$

$$= \frac{1}{|W|}\sum_{v\in W} v^{-1}\cdot\left(\frac{\tau_u(v)-\tau_u(vs_i)}{v\cdot(-\alpha_i)}\right) = \frac{1}{|W|}\sum_{v\in W} v^{-1}\cdot\tau_w(v) = f_w\ ,$$

completing the proof. □

Remark 6.8. Combining Theorem 6.7 and Proposition 6.5 we get

$$\tau_w^{sym} = \frac{1}{|W|}\sum_{v\leqslant_L w}(\varepsilon(v)\varepsilon(w)\partial_{vw^{-1}w_0}(\Lambda_0))\tau_v\ ,$$

hence the entries of the transition matrix in Theorem 6.3 are given, for $v \leqslant_L u$, by

$$a_{u,v} = \frac{1}{|W|}\varepsilon(u)\varepsilon(v)\partial_{vu^{-1}w_0}(\Lambda_0) = \frac{\varepsilon(vu^{-1})}{|W|}\partial_{vu^{-1}w_0}(\Lambda_0)\ . \tag{19}$$

Remark 6.9. Since $\{\tau_w^{sym}\}_w$ is a basis of $H_\alpha^*(W)$ over $\mathbb{S}(\mathfrak{t}^*)$, Theorem 4.5 implies that $\{f_w\}_w$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ over $\mathbb{S}(\mathfrak{t}^*)^W$. Therefore, if Λ_0 is the product of positive roots, then $\{\partial_w\Lambda_0\}_w$ is a basis of $\mathbb{S}(\mathfrak{t}^*)$ over $\mathbb{S}(\mathfrak{t}^*)^W$.

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