STABILITY FUNCTIONS

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Abstract. In this article we discuss the role of *stability functions* in geometric invariant theory and apply stability function techniques to problems in toric geometry. In particular we show how one can use these techniques to recover results of [BGU] and [STZ] on asymptotic properties of sections of holomorphic line bundles over toric varieties.

1. Introduction

Suppose (M,ω) is a pre-quantizable Kähler manifold, and $(\mathbb{L},\langle\cdot,\cdot\rangle)$ is a pre-quantization of (M,ω). Let ∇ be the metric connection on $\mathbb{L}$. Then the quantizability assumption is equivalent to the condition that the curvature form equals the negative of the Kähler form,

$$curv(\nabla) = -\omega. \tag{1.1}$$

For any positive integer k we will denote the k^{th} tensor power of $\mathbb{L}$ by $\mathbb{L}^k$. The Hermitian structure on $\mathbb{L}$ induces a Hermitian structure on $\mathbb{L}^k$. Denote by $\Gamma_{hol}(\mathbb{L}^k)$ the space of holomorphic sections of $\mathbb{L}^k$. (If M is compact, $\Gamma_{hol}(\mathbb{L}^k)$ is a finite dimensional space, whose dimension is given by the Riemann-Roch theorem,

$$\dim \Gamma_{hol}(\mathbb{L}^k) = k^d \mathrm{Vol}(M) + k^{d-1}\int_M c_1(M)\wedge\omega^{d-1} + \cdots . \tag{1.2}$$

for k sufficiently large.) We equip this space with the L^2 norm induced by the Hermitian structure,

$$\langle s_1, s_2\rangle = \int_M \langle s_1(x), s_2(x)\rangle \frac{\omega^d}{d!}. \tag{1.3}$$

(In semi-classical analysis $\Gamma_{hol}(\mathbb{L}^k)$ is the Hilbert space of quantum states associated to M, and k plays the role of the inverse of Planck's constant.)

If one has a holomorphic action of a compact Lie group on M which lifts to $\mathbb{L}$ one gets from the data above a Hermitian line bundle on the geometric invariant theory (GIT) quotient of M. One of the purposes of this paper is to compare the L^2 norms of holomorphic sections of $\mathbb{L}$ with the L^2 norms of holomorphic sections of this quotient line bundle equipped with the quotient metric. More explicitly,

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suppose G is a connected compact Lie group, $\mathfrak{g}$ its Lie algebra, and τ a holomorphic Hamiltonian action of G on M with a proper moment map Φ. Moreover, assume that there exists a lifting, $\tau^{\#}$, of τ to $\mathbb{L}$, which preserves the Hermitian inner product $\langle\cdot,\cdot\rangle$. If the G-action on $\Phi^{-1}(0)$ is free, the quotient space

$$M_{red} = \Phi^{-1}(0)/G$$

is a compact Kähler manifold. Moreover, the Hermitian line bundle $(\mathbb{L}, \langle\cdot,\cdot\rangle)$ on M naturally descends to a Hermitian line bundle $(\mathbb{L}_{red}, \langle\cdot,\cdot\rangle_{red})$ on M_{red}, and the curvature form of $\mathbb{L}_{red}$ is the reduced Kähler form $-\omega_{red}$, thus $\mathbb{L}_{red}$ is a pre-quantum line bundle over M_{red} (c.f. §2 for more details). From these line bundle identifications one gets a natural map

$$\Gamma_{hol}(\mathbb{L}^k)^G \to \Gamma_{hol}(\mathbb{L}^k_{red}) \tag{1.4}$$

and one can prove

Theorem 1.1 (Quantization commutes with reduction for Kähler manifolds)**.** *Suppose that for some $k_0 > 0$ the set $\Gamma_{hol}(\mathbb{L}^{k_0})^G$ contains a nonzero element. Then the map (1.4) is bijective for every k.*

The proof of this theorem in [GuS82] implicitly involves the notion of *stability function* and one of the goals of this article will be to make the role of this function in geometric invariant theory more explicit. To define this function let $G_{\mathbb{C}}$ be the complexification of G (See §2) and let M_{st} be the $G_{\mathbb{C}}$ flow-out of $\Phi^{-1}(0)$. Modulo the assumptions in the theorem above M_{st} is a Zariski open subset of M, and if G acts freely on $\Phi^{-1}(0)$ then $G_{\mathbb{C}}$ acts freely on M_{st} and

$$M_{red} = \Phi^{-1}(0)/G = M_{st}/G_{\mathbb{C}}.$$

Let π be the projection of M_{st} onto M_{red}. The stability function associated to this data is a real-valued C^∞ map $\psi : M_{st} \to \mathbb{R}$ with the defining property

$$\langle \pi^* s, \pi^* s\rangle = e^{\psi}\pi^*\langle s, s\rangle_{red} \tag{1.5}$$

for one or, equivalently, all sections s of our line bundle $\mathbb{L}_{red}$. This function can also be viewed as a relative potential function, relating the Kähler form ω on M_{st} to the Kähler form on M_{red}, i.e. ψ satisfies

$$\omega - \pi^*\omega_{red} = \sqrt{-1}\partial\bar{\partial}\psi. \tag{1.6}$$

We will show that this function is proper, non-positive, and takes its maximum value 0 precisely on $\Phi^{-1}(0)$. We will also show that this property suffices to determine it in general by showing that on the gradient trajectory of any component of Φ it satisfies a simple ODE. Unfortunately this fact turns out to be hard to implement in practice except in special cases. Most results in this paper will concern only the case where the upstairs space M is the complex d-space $\mathbb{C}^d$ with the flat metric, e.g. the toric varieties and the quiver varieties.

Another basic property of ψ is that, for any point $p \in \Phi^{-1}(0)$, p is the only critical point of the restriction of ψ to the "orbit" $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ (Here $\exp(\sqrt{-1}\mathfrak{g})$ is the "imaginary" part of $G_{\mathbb{C}}$). Let dx be the (Riemannian) volume form on this orbit, which is induced by the restriction to $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ of the Kähler-Riemannian metric on M_{st}. By applying the method of steepest descent, one gets an asymptotic expansion

$$\int_{\exp(\sqrt{-1}\mathfrak{g})\cdot p} e^{\lambda\psi} dx \sim \left(\frac{\lambda}{\pi}\right)^{-m/2} \left(1 + \sum_{i=1}^{\infty} c_i \lambda^{-i}\right) \tag{1.7}$$

for λ large, where m is the dimension of G, and c_i are constants depending on p. (Throughout this paper we will fix the notations $d = \dim_{\mathbb{C}} M, m = \dim_{\mathbb{R}} G$ and $n = d - m = \dim_{\mathbb{C}} M_{red}$.)

The asymptotic formula (1.7) has many applications. First by integrating (1.7) over the G-orbit through p, we get

$$\int_{G_{\mathbb{C}}\cdot p} e^{\lambda\psi} \frac{\omega^m}{m!} \sim \left(\frac{\lambda}{\pi}\right)^{-m/2} V(p) \left(1 + O(\frac{1}{\lambda})\right) \tag{1.8}$$

as $\lambda \to \infty$, where $V(p)$ is the Riemannian volume of the G-orbit through p. Thus for any holomorphic section $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$,

$$\left(\frac{k}{\pi}\right)^{m/2} \|\pi^* s_k\|^2 = \|V^{1/2} s_k\|^2_{red} + O(\frac{1}{k}). \tag{1.9}$$

This can be viewed as a "$\frac{1}{2}$-form correction" which makes the identification of $\Gamma_{hol}(\mathbb{L}^k_{red})$ with $\Gamma_{hol}(\mathbb{L}^k)^G$ an isometry modulo $O(\frac{1}{k})$. (Compare with [HaK], [Li] for similar results on $\frac{1}{2}$-form corrections).

A second application of (1.7) concerns the measures associated with holomorphic sections of $\mathbb{L}^k_{red}$: Let μ and μ_{red} be the symplectic volume forms on M and M_{red} respectively. Given a sequence of "quantum states"

$$s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$$

one can, by (1.7), relate the asymptotics of the measures

$$\langle s_k, s_k \rangle \mu_{red} \tag{1.10}$$

defined by these quantum states for appropriately chosen sequences of s_k's to the asymptotics of the corresponding measures

$$\langle \pi^* s_k, \pi^* s_k \rangle \mu \tag{1.11}$$

on M. In the special case where M is $\mathbb{C}^d$ with flat metric and M_{red} a toric variety with the quotient metric the asymptotics of (1.11) can be computed explicitly by

Mellin transform techniques (see [GuW] and [Wan]) and from this computation together with the identity (1.7) one gets an alternative proof of the asymptotic properties of (1.10) for toric varieties described in [BGU].

One can also regard the function

$$\langle s_k, s_k \rangle : M_{red} \to \mathbb{R} \tag{1.12}$$

as a random variable and study the asymptotic properties of its probability distribution, i.e., the measure

$$\langle s_k, s_k \rangle_* \mu_{red}, \tag{1.13}$$

on the real line. These properties, however, can be read off from the asymptotic behavior of the *moments* of this measure, which are, by definition just the integrals

$$m_{red}(l, s_k, \mu_{red}) = \int_{M_{red}} \langle s_k, s_k \rangle^l d\mu_{red}, \qquad l = 1, 2, \cdots \tag{1.14}$$

and by (1.7) the asymptotics of these integrals can be related to the asymptotics of the corresponding integrals on M viz

$$m(l, \pi^* s_k, \mu) = \int_M \langle \pi^* s_k, \pi^* s_k \rangle^l \mu \ . \tag{1.15}$$

In the toric case Shiffman, Tate and Zelditch showed in [STZ] that if s_k lies in the weight space $\Gamma_{hol}(\mathbb{L}^k)^{\alpha_k}$, where $\alpha_k = k\alpha + O(\frac{1}{k})$, and $\nu = (\Phi_P^* \omega_{FS})^n / n!$ is the pull-back of the Fubini-Study volume form on the projective space via the monomial embedding Φ_P, then, if s_k has L^2 norm 1,

$$\left(\frac{k}{\pi}\right)^{-n(l-1)/2} m_{red}(l, s_k, \nu) \sim \frac{c^l}{l^{n/2}} \tag{1.16}$$

as k tends to infinity, c being a positive constant. From this they derived a "universal distribution law" for such measures. We will give below a similar asymptotic result for the moments associated with another volume form, $V\mu_{red}$, which can be derived from (1.7) and an analogous, but somewhat simpler version of (1.16) for the moments (1.15) upstairs on $\mathbb{C}^d$.

Related to these results is another application of (1.7): Let

$$\pi_N : L^2(\mathbb{L}^N, \mu) \to \Gamma_{hol}(\mathbb{L}^N)$$

be the orthogonal projection of the space of L^2-sections of $\mathbb{L}^N$ onto the space of holomorphic sections of $\mathbb{L}^N$ and for $f \in C^\infty(M)$ let

$$M_f : L^2(\mathbb{L}^N, \mu) \to L^2(\mathbb{L}^N, \mu)$$

be the "multiplication by f" operator. If M is compact (which will be the case below with M replaced by M_{red}) then by contracting this operator to $\Gamma_{hol}(\mathbb{L}^N)$

and taking its trace one gets a measure

$$\mu_N(f) = \mathrm{Tr}(\pi_N M_f \pi_N) \tag{1.17}$$

which one can also write (somewhat less intrinsically) as the "density of states"

$$\mu_N = \sum \langle s_{N,i}, s_{N,i} \rangle \mu, \tag{1.18}$$

the $s_{N,i}$'s being an orthonormal basis of $\Gamma_{hol}(\mathbb{L}^N)$ inside $L^2(\mathbb{L}^N, \mu)$. By a theorem of Boutet de Monvel-Guillemin, $\mu_N(f)$ has an asymptotic expansion,

$$\mu_N(f) \sim \sum_{i=d-1}^{-\infty} a_i(f) N^i \tag{1.19}$$

as $N \to \infty$. One of the main results of this paper is a G-invariant version of Boutet de Monvel-Guillemin's result. More precisely, if we let π_N^G be the orthogonal projection

$$\pi_N^G : L^2(\mathbb{L}^N, \mu) \to \Gamma_{hol}(\mathbb{L}^N)^G,$$

then for any G-invariant function f on M we have the asymptotic expansion

$$\mu_N^G(f) = \mathrm{Tr}(\pi_N^G M_f \pi_N^G) \sim \sum_{i=n-1}^{\infty} a_i^G(f) N^i \tag{1.20}$$

as $N \to \infty$. Moreover, the identity (1.7) enables one to read off this upstairs G-invariant expansion from the downstairs expansion and vice versa. Notice that for this G-invariant expansion, we don't have to require the upstairs manifold to be compact. For example, for toric varieties, the upstairs space, $\mathbb{C}^d$, is not compact, so the space of holomorphic sections is infinite dimensional, and Boutet-Guillemin's result doesn't apply; however the G-invariant version of the upstairs asymptotics can, in this case, be computed directly by Mellin transform techniques ([GuW], [Wan]) together with an Euler-Maclaurin formula for convex lattice polytopes ([GuS06]) and hence one gets from (1.7) an alternative proof of the asymptotic expansion of μ_N for toric varieties obtained in [BGU].

As a last application of the techniques of this paper we discuss "Bohr-Sommerfeld" issues in the context of GIT theory. Let ∇_{red} be the Kählerian connection on $\mathbb{L}_{red}$ with defining property,

$$curv(\nabla_{red}) = -\omega_{red}.$$

A Lagrangian submanifold $\Lambda_{red} \subset M_{red}$ is said to be *Bohr-Sommerfeld* if the connection $\iota^*_{\Lambda_{red}} \nabla_{red}$ is trivial. In this case there exists a covariant constant non-vanishing section, s_{BS}, of $\iota^*_{\Lambda_{red}} \mathbb{L}_{red}$. Viewing s_{BS} as a "delta section" of $\mathbb{L}_{red}$ and projecting it onto $\Gamma_{hol}(\mathbb{L}_{red})$, one gets a holomorphic section $s_{\Lambda_{red}}$ of $\mathbb{L}_{red}$, and one would like to know

(1) Is this section nonzero?

(2) What, in fact, is this section?
(3) What about the sections $s^{(k)}_{\Lambda_{red}}$ of $\mathbb{L}^k_{red}$? Do they have interesting asymptotic properties as $k \to \infty$? Do they, for instance, "concentrate" asymptotically on Λ_{red}?

We will show that the "downstairs" versions of these questions on M_{red} can be translated into "upstairs" versions of these questions on M where they often become more accessible.

Finally a few words on the organization of this paper. It is divided into three parts, with part 1, §2-§5, focusing on the general theory of stability functions, part 2, §6-§8, on stability functions on toric varieties, and part 3, §9-§12, on stability functions on some non-toric varieties.

More precisely, in §2 we go over some basic facts about Kähler reduction and geometric invariant theory and define the reduced line bundle $\mathbb{L}_{red}$. In §3 we prove a number of basic properties of the stability function, and in §4 we derive the asymptotic expansion (1.7), and use it to show that the map between $\Gamma_{hol}(\mathbb{L}^k_{red})$ and $\Gamma_{hol}(\mathbb{L}^k)^G$ can be made into an asymptotic isometry by means of $\frac{1}{2}$-forms. Then in §5 we deduce from (1.7) the results about density of states, probability distributions and Bohr-Sommerfeld sections which we described above. In §6 we review the Delzant description of toric varieties as GIT quotients of $\mathbb{C}^d$ and discuss some of its implications. In §7 we derive an explicit formula, in terms of moment polytope data, for the stability function involved in this description and in §8 specialize the results of §5 to toric varieties and discuss their relation to the results of [BGU] and [STZ] alluded to above.

Finally in the last sections of this paper we make a tentative first step toward generalizing the results of §6-§8 to the non-abelian analogues of toric varieties: *spherical varieties*. The simplest examples of spherical varieties are the coadjoint orbits of $\mathcal{U}(n)$ viewed as $\mathcal{U}(n-1)$-manifolds. Following Shaun Martin, we will show how these varieties can be obtained by symplectic reduction from a linear action of a compact Lie group on $\mathbb{C}^N$, and, as above for toric varieties, compute their stability functions. In §11, following Joel Kamnitzer [Kam] we show how Martin's description of $\mathcal{U}(n)$-coadjoint orbits extends to quiver varieties, and for some special classes of quiver varieties (e.g. the polygon spaces of Kapovich-Millson [KaM]) show that there are stability function formulae similar to those for coadjoint orbits. (For spherical varieties in general the question still seems to be open as to whether they have "nice" description as GIT quotients analogue to the Delzant description of toric varieties.)

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2. Kähler reduction vs geometric invariant theory

2.1. **Kähler reduction.** Suppose (M,ω) is a symplectic manifold, G a connected compact Lie group acting in a Hamiltonian fashion on M, and $\Phi : M \to \mathfrak{g}^*$ a moment map, i.e., Φ is equivariant with respect to the given G-action on M and the coadjoint G-action on $\mathfrak{g}^*$, with the defining property

$$d\langle\Phi, v\rangle = \iota_{v_M}\omega, \qquad v\in\mathfrak{g}, \tag{2.1}$$

where v_M is the vector field on M generated by the one-parameter subgroup $\{\exp(-tv) \mid t\in\mathbb{R}\}$ of G. Furthermore we assume that Φ is proper, 0 is a regular value and that G acts freely on the zero level set $\Phi^{-1}(0)$. Then by the Marsden-Weinstein theorem, the quotient space

$$M_{red} = \Phi^{-1}(0)/G$$

is a connected compact symplectic manifold with symplectic form ω_{red} satisfying

$$\iota^*\omega = \pi_0^*\omega_{red}, \tag{2.2}$$

where $\iota : \Phi^{-1}(0) \hookrightarrow M$ is the inclusion map, and $\pi_0 : \Phi^{-1}(0) \to M_{red}$ the quotient map. Moreover, if ω is integral, so is ω_{red}; and if (M,ω) is Kähler with holomorphic G-action, then M_{red} is a compact Kähler manifold and ω_{red} is a Kähler form.

2.2. **GIT quotients.** The Kähler quotient M_{red} also has the following GIT description:

Let $G_{\mathbb{C}}$ be the complexification of G, i.e., $G_{\mathbb{C}}$ is the unique connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\oplus\sqrt{-1}\mathfrak{g}$ which contains G as its maximal compact subgroup. We will assume that the action of G on M extends canonically to a holomorphic action of $G_{\mathbb{C}}$ on M (This will automatically be the case if M is compact). The infinitesimal action of $G_{\mathbb{C}}$ on M is given by

$$w_M = Jv_M \tag{2.3}$$

for $v\in\mathfrak{g}, w=\sqrt{-1}v$, where J is the automorphism of TM defining the complex structure.

The set of stable points, M_{st}, of M (with respect to this $G_{\mathbb{C}}$ action) is defined to be the $G_{\mathbb{C}}$-flow out of $\Phi^{-1}(0)$:

$$M_{st} = G_{\mathbb{C}}\cdot\Phi^{-1}(0). \tag{2.4}$$

This is an open subset of M on which $G_{\mathbb{C}}$ acts freely, and each $G_{\mathbb{C}}$-orbit in M_{st} intersects $\Phi^{-1}(0)$ in precisely one G-orbit, c.f. [GuS82]. Moreover, for any G-invariant holomorphic section s_k of $\mathbb{L}^k$, M_{st} contains all p with $s_k(p)\neq 0$. (For a proof, see the arguments at the end of §3.2). In addition, if M is compact $M - M_{st}$ is just the common zero sets of these s_k's. Since M_{st} is a principal $G_{\mathbb{C}}$ bundle over M_{red}, the $G_{\mathbb{C}}$ action on M_{st} is proper. The quotient space $M_{st}/G_{\mathbb{C}}$

has the structure of a complex manifold. Moreover, since each $G_{\mathbb{C}}$-orbit in M_{st} intersects $\Phi^{-1}(0)$ in precisely one G-orbit, this GIT quotient space coincides with the symplectic quotient:

$$M_{red} = M_{st}/G_{\mathbb{C}}.$$

In other words, M_{red} is a Kähler manifold with ω_{red} its Kähler form, and the projection map $\pi : M_{st} \to M_{red}$ is holomorphic.

2.3. **Reduction at the quantum level.** Suppose $(\mathbb{L}, \langle\cdot,\cdot\rangle)$ is a pre-quantum line bundle over M. There is a unique holomorphic connection ∇ on $\mathbb{L}$, (called the metric connection), which is compatible with the Hermitian inner product on $\mathbb{L}$, i.e., satisfies the compatibility condition for every locally nonvanishing holomorphic section $s : U \to \mathbb{L}$,

$$\frac{\nabla s}{s} = \partial \log \langle s, s\rangle \in \Omega^{1,0}(U). \tag{2.5}$$

The pre-quantization condition amounts to requiring that the curvature form of the connection ∇ is $-\omega$, i.e.,

$$curv(\nabla) := -\sqrt{-1}\bar{\partial}\partial \log \langle s, s\rangle = -\omega. \tag{2.6}$$

To define reduction on the quantum level, we assume that the G action on M can be lifted to an action $\tau^{\#}$ of G on $\mathbb{L}$ by holomorphic line bundle automorphisms. By averaging, we may assume that $\tau^{\#}$ preserves the metric $\langle\cdot,\cdot\rangle$, and thus preserves the connection ∇ and the curvature form ω. By Kostant's formula ([Kos]), the infinitesimal action of $\mathfrak{g}$ on sections of $\mathbb{L}$ is

$$L_v s = \nabla_{v_M} s - \sqrt{-1}\langle\Phi, v\rangle s \tag{2.7}$$

for all smooth sections $s \in \Gamma(\mathbb{L})$ and all $v \in \mathfrak{g}$. Since G acts freely on $\Phi^{-1}(0)$, the lifted action $\tau^{\#}$ is free on $\iota^*\mathbb{L}$. The quotient

$$\mathbb{L}_{red} = \iota^*\mathbb{L}/G$$

is now a holomorphic line bundle over M_{red}.

On the other hand, by [GuS82], the lifted action $\tau^{\#}$ can be extended canonically to an action $\tau^{\#}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ on $\mathbb{L}$. Denote by $\mathbb{L}_{st}$ the restriction of $\mathbb{L}$ to the open set M_{st}, then $G_{\mathbb{C}}$ acts freely on $\mathbb{L}_{st}$, and we get the GIT description of the quotient line bundle,

$$\mathbb{L}_{red} = \mathbb{L}_{st}/G_{\mathbb{C}}.$$

On $\mathbb{L}_{red}$ there is a naturally defined Hermitian structure, $\langle\cdot,\cdot\rangle_{red}$, defined by

$$\pi_0^*\langle s, s\rangle_{red} = \iota^*\langle \pi^* s, \pi^* s\rangle \tag{2.8}$$

for all $s \in \Gamma(\mathbb{L}_{red})$. Moreover, the induced curvature form of $\mathbb{L}_{red}$ is the reduced Kähler form ω_{red}. In other words, the quotient line bundle $(\mathbb{L}_{red}, \langle\cdot,\cdot\rangle_{red})$ is a pre-quantum line bundle over the quotient space (M_{red}, ω_{red}).

3. The stability function

3.1. **Definition of the stability function.** The stability function $\psi : M_{st} \to \mathbb{R}$ is defined to satisfy

$$\langle \pi^* s, \pi^* s \rangle = e^{\psi} \pi^* \langle s, s \rangle_{red} \ . \tag{3.1}$$

More precisely, suppose U is an open subset in M_{st} and $s : U \to \mathbb{L}_{red}$ a non-vanishing section, then ψ restricted to $\pi^{-1}(U)$ is defined to be

$$\psi = \log \langle \pi^* s, \pi^* s \rangle - \pi^* \log \langle s, s \rangle_{red} \ . \tag{3.2}$$

Obviously this definition is independent of the choice of s.

It is easy to see from the definition that ψ is a G-invariant function on M_{st} which vanishes on $\Phi^{-1}(0)$, and by (2.6),

$$\omega = \pi^* \omega_{red} + \sqrt{-1}\ \bar{\partial}\partial\psi. \tag{3.3}$$

Thus ψ can be thought of as a potential function for the restriction of ω to M_{st} *relative* to ω_{red}.

Remark 3.1. From the definition it is also easy to see that the stability function depends on the metric on the line bundle. One such dependence that is crucial in the whole paper is the following: If $\mathbb{L}$ is the trivial line bundle over $\mathbb{C}$ with the Bargmann metric and ψ the stability function for some Kähler quotient of $\mathbb{C}$ associated with $\mathbb{L}$, then $\mathbb{L}^N$ is still the trivial line bundle over $\mathbb{C}$ but with a slightly different metric, i.e. the N^{th} tensor of the Bargmann metric, and the corresponding stability function becomes $N\psi$.

Remark 3.2. (Reduction by stages) Let $G = G_1 \times G_2$ be a product of compact Lie groups G_1 and G_2. Then by reduction in stages M_{red} can be identified with $(M_{red}^{(1)})^{(2)}$, where $M_{red}^{(1)}$ is the reduction of M with respect to G_1 and $(M_{red}^{(1)})^{(2)}$ the reduction of $M_{red}^{(1)}$ with respect to G_2. Let M_{st}^G and $M_{st}^{G_1}$ be the set of stable points in M with respect to the G-action and G_1-action respectively, and $(M_{red}^{(1)})_{st}^{G_2}$ the set of stable points in $M_{st}^{(1)}$ with respect to the G_2-action. Denote by π_1 the projection of M_{st} onto $M_{red}^{(1)}$. We claim that $M_{st}^G \subset M_{st}^{G_1}$ and $\pi_1^{-1}((M_{red}^{(1)})_{st}^{G_2}) = M_{st}^G$. The first of these assertions is obvious and the second assertion follows from the identification

$$\begin{aligned}
\pi_1^{-1}((M_{red}^{(1)})_{st}^{G_2}) &= \pi_1^{-1}((G_2)_{\mathbb{C}} \bar{\Phi}_2^{-1}(0)) \\
&= (G_1)_{\mathbb{C}} (\pi_1^{-1}((G_2)_{\mathbb{C}} \bar{\Phi}_2^{-1}(0)) \cap \Phi_1^{-1}(0)) \\
&= (G_1)_{\mathbb{C}} (G_2)_{\mathbb{C}} (\pi_1^{-1}(\bar{\Phi}_2^{-1}(0)) \cap \Phi_1^{-1}(0)) \\
&= G_{\mathbb{C}} (\Phi_2^{-1}(0) \cap \Phi_1^{-1}(0)) \\
&= G_{\mathbb{C}} \Phi^{-1}(0).
\end{aligned}$$

Thus $\psi = \psi_1 + \pi_1^* \psi_2^1$, where ψ is the stability function associated with reduction of M by G, ψ_1 the stability function associated with reduction of M by G_1, and ψ_2^1 the stability function associated with the reduction of $M_{red}^{(1)}$ by G_2.

Remark 3.3. (Action on product manifolds) As in the previous remark let $G = G_1 \times G_2$. Let M_i, $i = 1, 2$, be Kählerian G_i manifolds and $\mathbb{L}_i$ pre-quantum line bundles over M_i, satisfying the assumptions in the previous sections. Denote by ψ_i the stability function on M_i associated to $\mathbb{L}_i$. Letting G be the product $G_1 \times G_2$ the stability function on the G-manifold $M_1 \times M_2$ associated with the product line bundle $\mathrm{pr}_1^* \mathbb{L}_1 \otimes \mathrm{pr}_2^* \mathbb{L}_2$ is $\mathrm{pr}_1^* \psi_1 + \mathrm{pr}_2^* \psi_2$.

3.2. **Two useful lemmas.** Recall that by (2.3), the vector field w_M for the "imaginary vector" $w = \sqrt{-1} v \in \sqrt{-1}\mathfrak{g}$ is $w_M = J v_M$.

Lemma 3.4 ([GuS82]). *Suppose $w = \sqrt{-1} v \in \sqrt{-1}\mathfrak{g}$, then w_M is the gradient vector field of $\langle \Phi, v \rangle$ with respect to the Kähler metric g.*

Proof.

$$d\langle \Phi, v \rangle = \iota_{v_M} \omega = \omega(-J w_M, \cdot) = \omega(\cdot, J w_M) = g(w_M, \cdot).$$

□

Lemma 3.5. *Suppose $w = \sqrt{-1} v \in \sqrt{-1}\mathfrak{g}$, then for any nonvanishing G-invariant holomorphic section $\tilde{s} \in \Gamma_{hol}(\mathbb{L})^G$,*

$$L_{w_M} \log \langle \tilde{s}, \tilde{s} \rangle = -2 \langle \Phi, v \rangle. \tag{3.4}$$

Proof. Since

$$J(v_M + \sqrt{-1} w_M) = w_M - \sqrt{-1} v_M = -\sqrt{-1}(v_M + \sqrt{-1} w_M),$$

$v_M + \sqrt{-1} w_M$ is a complex vector field of type (0,1). Since $\tilde{s}$ is holomorphic, the covariant derivative

$$\nabla_{v_M} \tilde{s} = -\sqrt{-1} \nabla_{w_M} \tilde{s}. \tag{3.5}$$

Since $\tilde{s}$ is G-invariant, by Kostant's identity (2.7),

$$0 = L_v \tilde{s} = \nabla_{v_M} \tilde{s} - \sqrt{-1} \langle \Phi, v \rangle \tilde{s}. \tag{3.6}$$

Thus

$$\nabla_{w_M} \tilde{s} = -\langle \Phi, v \rangle \tilde{s}.$$

By metric compatibility, we have for any G-invariant holomorphic section $\tilde{s}$

$$L_{w_M} \log \langle \tilde{s}, \tilde{s} \rangle = -2 \langle \Phi, v \rangle.$$

□

In particular suppose M is compact, let $\tilde{s}$ be a G-invariant holomorphic section of $\mathbb{L}$ and p a point where $\tilde{s}(p) \neq 0$. The function

$$\langle \tilde{s}, \tilde{s} \rangle : \overline{G_{\mathbb{C}} \cdot p} \to \mathbb{R}$$

takes its maximum at some point q and since $\overline{G_{\mathbb{C}} \cdot p}$ is $G_{\mathbb{C}}$-invariant and

$$\langle \tilde{s}, \tilde{s} \rangle(q) \geq \langle \tilde{s}, \tilde{s} \rangle(p) > 0$$

it follows from (3.4) that $\Phi(q) = 0$, i.e. $q \in M_{st}$. But M_{st} is open and $G_{\mathbb{C}}$-invariant. Hence $p \in M_{st}$. Thus we've proved that if $p \in M - M_{st}$, then $s(p) = 0$ for all $s \in \Gamma_{hol}(\mathbb{L})^G$.

3.3. **Analytic properties of the stability function.**

Proposition 3.6. *Suppose* $w = \sqrt{-1}v \in \sqrt{-1}\mathfrak{g}$, *then* $L_{w_M}\psi = -2\langle \Phi, v \rangle$.

Proof. Suppose s is any holomorphic section of the reduced bundle $\mathbb{L}_{red}$. Since $\pi^* \log\langle s, s \rangle_{red}$ is $G_{\mathbb{C}}$-invariant, we have from (3.2),

$$L_{w_M}\psi = L_{w_M} \log\langle \pi^* s, \pi^* s \rangle.$$

Now apply lemma 3.5 to the G-invariant section $\pi^* s$. □

The main result of this section is

Theorem 3.7. *ψ is a proper function which takes its maximum value* 0 *on* $\Phi^{-1}(0)$. *Moreover, for any* $p \in \Phi^{-1}(0)$, *the restriction of* ψ *to the orbit* $\exp \sqrt{-1}\mathfrak{g} \cdot p$ *has only one critical point, namely p itself, and this critical point is a global maximum.*

Proof. As before we take $w = \sqrt{-1}v \in \sqrt{-1}\mathfrak{g}$. Since $G_{\mathbb{C}}$ acts freely on M_{st}, we have a diffeomorphism

$$\kappa : \Phi^{-1}(0) \times \sqrt{-1}\mathfrak{g} \to M_{st}, \; (p, w) \mapsto \tau_{\mathbb{C}}(\exp w)p. \tag{3.7}$$

We define two functions

$$\psi_0(p, w, t) = (\kappa^*\psi)(p, tw) \tag{3.8}$$

and

$$\phi_0(p, w, t) = \langle \kappa^*\Phi(p, tw), v \rangle. \tag{3.9}$$

Then proposition 3.6 leads to the following differential equation

$$\frac{d}{dt}\psi_0 = -2\phi_0, \tag{3.10}$$

with initial conditions

$$\psi_0(p, w, 0) = 0 \tag{3.11}$$

and

$$\phi_0(p, w, 0) = 0. \tag{3.12}$$

Since w_M is the gradient vector field of $\langle \Phi, v\rangle$, and $t \mapsto \kappa(p, tw)$ is an integral curve of w_M, we see that ϕ_0 is a strictly increasing function of t. Thus ψ_0 is strictly increasing for $t < 0$, strictly decreasing for $t > 0$, and takes its maximal value 0 at $t = 0$. This shows that p is the only critical point in the orbit $\sqrt{-1}\mathfrak{g} \cdot p$.

The fact ψ is proper also follows from the differential equation (3.10), since for any $t_0 > 0$ we have

$$\psi_0(p, w, t) \le C_0 - 2(t - t_0)C_1, \qquad t > t_0$$

where

$$C_0 = \max_{|w|=1} \psi_0(p, w, t_0) < 0$$

and

$$C_1 = \min_{|w|=1} \phi_0(p, w, t_0) > 0.$$

□

Remark 3.8. The proof above also gives us an alternate way to compute the stability function, namely we "only" need to solve the differential equation (3.10) along each orbit $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ with initial condition (3.11). (Of course a much more complicated step is to write down explicitly the decomposition of M_{st} as a product $\Phi^{-1}(0) \times \sqrt{-1}\mathfrak{g}$.)

Corollary 3.9. *For any $s \in \Gamma_{hol}(\mathbb{L}_{red})$, the norm $\langle \pi^* s, \pi^* s\rangle(p)$ is bounded on M_{st}, and tends to* 0 *as p goes to the boundary of M_{st}.*

3.4. **Quantization commutes with reduction.** As we have mentioned in the introduction, the properties of the stability function described above were implicitly involved in the proof of the "quantization commutes with reduction" theorem in [GuS82]. We end this section by briefly describing this proof. Assume M compact. Then using elliptic operator techniques one can prove that there exists a non-vanishing $G_{\mathbb{C}}$-invariant holomorphic section $\tilde{s}$ of $\mathbb{L}^k$ for k large. But M_{st} contains all points p with $\tilde{s}(p) \neq 0$. So the complement of M_{st} is contained in a codimension one complex subvariety of M. By the corollary above, we see that for any holomorphic section s of $\mathbb{L}_{red}$, $\pi^* s$ can be extended to a holomorphic section of $\mathbb{L}$ by setting $\pi^* s = 0$ on $M - M_{st}$. This gives the required bijection. For details, c.f. [GuS82].

4. Asymptotic properties of the stability function

4.1. **The basic asymptotics.** From the previous section we have seen that the stability function takes its global maximum 0 exactly at $\Phi^{-1}(0)$. Thus as λ tends to infinity, $e^{\lambda\psi}$ tends to 0 exponentially fast off $\Phi^{-1}(0)$. So in principle, only a very small neighborhood of $\Phi^{-1}(0)$ will contribute to the asymptotics of the integral

$$\int_{M_{st}} f e^{\lambda\psi} \frac{\omega^d}{d!}$$

for f a bounded function in $C^\infty(M_{st})^G$ and for λ large. In this section we will derive an asymptotic expansion in λ for this integral, beginning with (1.7).

The proof of (1.7) is based on the following method of steepest descent: Let X be an m-dimensional Riemannian manifold with volume form dx, $\psi : X \to \mathbb{R}$ a real-valued smooth function which has a unique maximum $\psi(p) = 0$ at a point p, and is bounded away from zero outside a compact set. Suppose moreover that p is a nondegenerate critical point of ψ. If $f \in C^\infty(X)$ satisfies $fe^{\lambda\psi}$ in $\mathcal{L}^1(X, dx)$ for $\lambda \geq 1$, then

$$\int_X f(x) e^{\lambda\psi(x)} dx \sim \sum_{k=0}^{\infty} c_k \lambda^{-\frac{m}{2}-k}, \quad \text{as } \lambda \to \infty \tag{4.1}$$

where the c_k's are constants. (If X is compact or of finite volume we can weaken this assumption to "f bounded", however in the example below X will be neither compact nor have finite volume.) Moreover,

$$c_0 = (2\pi)^{m/2} \tau_p f(p), \tag{4.2}$$

where

$$\tau_p^{-1} = \frac{(\det d^2\psi_p(e_i, e_j))^{1/2}}{|dx_p(e_1, \cdots, e_n)|} \tag{4.3}$$

for any basis $e_1, \cdots, e_m$ of T_pM.

From this general result we obtain:

Theorem 4.1. *Let dx be the Riemannian volume form on $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ induced by the restriction to $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ of the Kähler-Riemannian metric on M_{st}, where p is any point in $\Phi^{-1}(0)$. Let f be a smooth function on M such that the function $fe^{\lambda\psi}$ is integrable on this orbit $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$. Then for λ large,*

$$\int_{\exp\sqrt{-1}\mathfrak{g}\cdot p} f(x) e^{\lambda\psi(x)} dx \sim \left(\frac{\lambda}{\pi}\right)^{-m/2} \left(f(p) + \sum_{i=1}^{\infty} c_i \lambda^{-i}\right), \tag{4.4}$$

where c_i are constants depending on f, ψ and p.

Proof. We need to compute the Hessian of ψ restricted to $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$ at the point p. By proposition 3.6,

$$d(d\psi(w_M)) = d(L_{w_M}\psi) = -2d\langle\Phi, v\rangle = -2\omega(v_M, \cdot),$$

so

$$d^2\psi_p(w_M, w'_M) = -2\omega_p(v_M, w'_M) = -2g_p(v_M, v'_M) = -2g_p(w_M, w'_M).$$

This implies $\tau_p = 2^{-m/2}$. $\square$

4.2. **Asymptotics on submanifolds of M_{st}.** From (4.4) we obtain asymptotic formulas similar to (4.4) for submanifolds of M_{st} which are foliated by the sets $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$. For example, by the Cartan decomposition

$$G_{\mathbb{C}} = G \times \exp(\sqrt{-1}\mathfrak{g})$$

one gets a splitting

$$G_{\mathbb{C}} \cdot p = G \times \exp(\sqrt{-1}\mathfrak{g}) \cdot p.$$

Moreover, this is an orthogonal splitting on $\Phi^{-1}(0)$. Thus if we write

$$\frac{\omega^m}{m!}(p) = g(x) d\nu \wedge dx,$$

where $d\nu$ is the Riemannian volume form on the G-orbit $G \cdot p$, defined by the Kähler-Riemannian metric, we see that $g(x)$ is G-invariant and $g(p) = 1$ on $\Phi^{-1}(0)$. Thus if we apply theorem 4.1 we get

Corollary 4.2. *As $\lambda \to \infty$,*

$$\int_{G_{\mathbb{C}} \cdot p} f(x) e^{\lambda\psi} \frac{\omega^m}{m!} \sim V(p) \left(\frac{\lambda}{\pi}\right)^{-m/2} \left(f(p) + \sum_{i=1}^{\infty} c_i(p) \lambda^{-i}\right) , \tag{4.5}$$

where $V(p) = Vol(G \cdot p)$ is the Riemannian volume of the G orbit through p.

Similarly the diffeomorphism (3.7) gives a splitting of M_{st} into the imaginary orbits $\exp(\sqrt{-1}\mathfrak{g}) \cdot p$, and by the same argument one gets

Corollary 4.3. *As $\lambda \to \infty$,*

$$\int_{M_{st}} e^{\lambda\psi} \frac{\omega^d}{d!} \sim Vol(\Phi^{-1}(0)) \left(\frac{\lambda}{\pi}\right)^{-m/2} \left(1 + \sum_{i=1}^{\infty} C_i \lambda^{-i}\right) . \tag{4.6}$$

4.3. **The Half form correction.** Now we apply corollary 4.2 to prove (1.9). Since $M_{red} = M_{st}/G_{\mathbb{C}}$, we have a decomposition of the volume form

$$\frac{\omega^d}{d!} = \pi^* \frac{\omega_{red}^n}{n!} \wedge d\mu_\pi, \tag{4.7}$$

where $d\mu_\pi$ is the induced volume form on $G_{\mathbb{C}} \cdot p$,

$$d\mu_\pi(x) = h(x) \frac{\omega^m}{m!},$$

with $h(p) = 1$ on $\Phi^{-1}(0)$. Now suppose $s_k \in \Gamma_{hol}(\mathbb{L}_{red}^k)$. Since the stability function of $\mathbb{L}_{red}^k$ is $k\psi$, (1.5) becomes

$$\langle \pi^* s_k, \pi^* s_k \rangle = e^{k\psi} \pi^* \langle s_k, s_k \rangle_{red}.$$

By (4.5),

$$\|\pi^* s_k\|^2 = \int_{M_{red}} \left(\int_{G_{\mathbb{C}}\cdot p} e^{k\psi} d\mu_\pi \right) \langle s_k, s_k\rangle_{red} \frac{\omega_{red}^n}{n!}$$
$$= \left(\frac{k}{\pi}\right)^{-m/2} \left(1 + O(k^{-1})\right) \int_{M_{red}} V(\pi_0^{-1}(q))\langle s_k, s_k\rangle_{red} \frac{\omega_{red}^n}{n!}.$$

In other words,

$$\left(\frac{k}{\pi}\right)^{m/2} \|\pi^* s_k\|^2 = \|V^{1/2} s_k\|_{red}^2 + O(\frac{1}{k}), \tag{4.8}$$

where V is the volume function $V(q) = V(\pi_0^{-1}(q))$.

The presence of the factor V can be viewed as a "$\frac{1}{2}$-form correction" in the Kostant-Souriau version of geometric quantization. Namely, let $\mathbb{K} = \bigwedge^d (T^{1,0}M)^*$ and $\mathbb{K}_{red} = \bigwedge^n (T^{1,0}M_{red})^*$ be the canonical line bundles on M and M_{red} and let $\ll, \gg$ and $\ll, \gg_{red}$ be the Hermitian inner products on these bundles, then

$$\pi_0^* \mathbb{K}_{red} = \iota^* \mathbb{K}$$

and

$$\pi_0^*(V \ll, \gg_{red}) = \iota^* \ll, \gg .$$

So if $\mathbb{K}^{\frac{1}{2}}$ and $\mathbb{K}_{red}^{\frac{1}{2}}$ are "$\frac{1}{2}$-form" bundles on M and M_{red} (i.e., the square roots of $\mathbb{K}$ and $\mathbb{K}_{red}$), then one has a map

$$\Gamma_{hol}(\mathbb{L}^k \otimes \mathbb{K}^{\frac{1}{2}})^G \to \Gamma_{hol}(\mathbb{L}_{red}^k \otimes \mathbb{K}_{red}^{\frac{1}{2}})$$

which is an isometry modulo an error term of order $O(k^{-1})$. (See [HaK] and [Li] for more details.)

5. Applications to spectral problems on Kähler quotients

5.1. **Maximum points of quantum states.** Suppose M is a Kähler manifold with quantum line bundle $\mathbb{L}$, and $\tilde{s} \in \Gamma_{hol}(\mathbb{L})$ is a quantum state. The "invariance of polarization" conjecture of Kostant-Souriau is closely connected with the question: where does the function $\langle \tilde{s}, \tilde{s}\rangle$ take its maximum? If C is the set where $\langle \tilde{s}, \tilde{s}\rangle$ takes its maximum, what can one say about C? What is the asymptotic behavior of the function $\langle \tilde{s}, \tilde{s}\rangle^k$ in a neighborhood of C?

To address these questions we begin by recalling the following results:

Proposition 5.1. *If C above is a submanifold of M, then*
(a) *C is an isotropic submanifold of M;*
(b) *$\iota_C^* \tilde{s}$ is a non-vanishing covariant constant section of $\iota_C^* \mathbb{L}$;*
(c) *Moreover if M is a Kähler G-manifold and $\tilde{s}$ is in $\Gamma_{hol}(\mathbb{L})^G$ then C is contained in the zero level set of Φ.*

Proof. (a) Let $\alpha = \sqrt{-1}\bar{\partial}\log\langle\tilde{s}, \tilde{s}\rangle$. Then $\omega = d\alpha$ and $\alpha_p = 0$ for every $p \in C$, so $\iota_C^*\omega = 0$.
(b) By (2.5), $\nabla s = 0$ on C.
(c) By (2.7),

$$\nabla_{v_M} s = \sqrt{-1}\langle\phi, v\rangle s = 0$$

along C, therefore since s is non-zero on C, $\langle\Phi, v\rangle = 0$ on C. □

We will call a submanifold C of M for which the line bundle $\iota_C^*\mathbb{L}$ admits a nonzero covariant constant section a *Bohr-Sommerfeld* set. Notice that if s_0 is a section of $\iota_C^*\mathbb{L}$ which is non-vanishing, then

$$\frac{\nabla s_0}{s_0} = \alpha_0 \Leftrightarrow d\alpha_0 = \iota_C^*\omega,$$

so if s is covariant constant then C has to be isotropic. The most interesting Bohr-Sommerfeld sets are those which are maximally isotropic, i.e., Lagrangian, and the term "Bohr-Sommerfeld" is usually reserved for these Lagrangian submanifolds .

A basic problem in Bohr-Sommerfeld theory is obtaining converse results to the proposition above. Given a Bohr-Sommerfeld set, C, does there exist a holomorphic section, s, of $\mathbb{L}$ taking its maximum on C, i.e., for which the measure

$$\langle s^k, s^k\rangle \mu_{Liouville} \tag{5.1}$$

becomes more and more concentrated on C as $k \to \infty$. As we pointed out in the introduction this problem is often intractable, however if we are in the setting of GIT theory with M replaced by M_{red}, then the downstairs version of this question can be translated into the upstairs version of this question which is often easier. In §5.2 we will discuss the behavior of measures of type (5.1) in general and then in §5.5 discuss this Bohr-Sommerfeld problem.

5.2. **Asymptotics of the measures (1.10).** We will now apply stability theory to the measure (1.10) on M_{red}. For f an integrable function on M_{red}, consider the asymptotic behavior of the integral

$$\int_{M_{red}} f\langle s_k, s_k\rangle \mu_{red}, \tag{5.2}$$

with $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$ and $k \to \infty$. It is natural to compare (5.2) with the upstairs integral

$$\int_{M_{st}} \pi^* f\langle \pi^* s_k, \pi^* s_k\rangle \mu. \tag{5.3}$$

However, since M_{st} is noncompact, the integral above may not converge in general. To eliminate the possible convergence issues, we multiply the integrand by a cutoff function, i.e., a compactly supported function χ which is identically 1 on a neighborhood of $\Phi^{-1}(0)$. In other words, we consider the integral

$$\int_{M_{st}} \chi\pi^* f\langle \pi^* s_k, \pi^* s_k\rangle \mu. \tag{5.4}$$

Obviously different choices of the cutoff function will not affect the asymptotic behavior of (5.4).

Using the decomposition (4.7) we get

$$\begin{aligned}\int_{M_{st}} \chi\pi^* f\langle \pi^* s_k, \pi^* s_k\rangle \frac{\omega^d}{d!} &= \int_{M_{red}} \left(\int_{G_{\mathbb{C}}\cdot p} e^{k\psi}\chi d\mu_\pi\right) f\langle s_k, s_k\rangle_{red} \frac{\omega_{red}^n}{n!} \\ &\sim \int_{M_{red}} V f\langle s_k, s_k\rangle d\mu_{red},\end{aligned}$$

where $V(q) := V(\pi^{-1}(q))$ is the volume function. We conclude

Proposition 5.2. *As $k \to \infty$ we have*

$$\int_{M_{red}} f\langle s_k, s_k\rangle \mu_{red} \sim (\frac{k}{\pi})^{-m/2} \int_M \chi\tilde{f}\tilde{V}^{-1}\langle \pi^* s_k, \pi^* s_k\rangle \mu,$$

where $\tilde{f} = \pi^* f, \tilde{V} = \pi^* V$ and χ is any cutoff function near $\Phi^{-1}(0)$.

Similarly if we apply the same arguments to the density of states

$$\mu_N = \sum_i \langle s_{N,i}, s_{N,i}\rangle \mu_{red}, \tag{5.5}$$

where $\{s_{N,i}\}$ is an orthonormal basis of $\mathbb{L}^N_{red}$, we get

Proposition 5.3. *As $N \to \infty$,*

$$\int_{M_{red}} f\mu_N \sim (\frac{N}{\pi})^{-m/2} \int_{M_{st}} \chi\tilde{f}\tilde{V}^{-1}\mu_N^G, \tag{5.6}$$

where $\mu_N^G = \sum_i \langle \pi^* s_{N,i}, \pi^* s_{N,i}\rangle \mu$ is the upstairs G-invariant measure (1.20).

5.3. **Asymptotics of the moments.** We next describe the role of "upstairs" versus "downstairs" in describing the asymptotic behavior of the distribution function

$$\sigma_k([t, \infty)) = \mathrm{Vol}\{z \mid \langle s_k, s_k\rangle(z) \geq t\}, \tag{5.7}$$

for $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$, i.e., of the push-forward measure, $\langle s_k, s_k\rangle_*\mu$, on the real line $\mathbb{R}$. The moments (1.14) completely determine this measure, and by theorem 4.1 the moments (1.14) on M_{red} are closely related to the corresponding moments (1.15) on M. In fact, by corollary 4.2 and the decomposition (4.7),

$$\int_{M_{st}} \langle \pi^* s_k, \pi^* s_k \rangle^l \mu = \int_{M_{st}} (\pi^* \langle s_k, s_k \rangle)^l e^{lk\psi} \pi^* \frac{\omega_{red}^n}{n!} \wedge h(x) \frac{\omega^m}{m!}$$

$$\sim \left(\frac{lk}{\pi}\right)^{-m/2} \int_{M_{red}} \langle s_k, s_k \rangle^l V \mu_{red}.$$

We conclude

Proposition 5.4. *For any integer l, the l^{th} moments (1.15) satisfy*

$$m(l, \pi^* s_k, \mu) \sim \left(\frac{lk}{\pi}\right)^{-m/2} m_{red}(l, s_k, V\mu_{red}). \tag{5.8}$$

as $k \to \infty$.

5.4. **Asymptotic expansion of the G-invariant density of states.** For the measure (1.17), Boutet-Guillemin showed that it admits an asymptotic expansion (1.19) in inverse power of N as $N \to \infty$ if the manifold is compact (See the appendix for a proof of this result). By applying stability theory above, we get from the Boutet-Guillemin's expansion for the downstairs manifold a similar asymptotic expansion upstairs for the G-invariant density of states without assuming M to be compact. Namely, since M_{red} is compact, Boutet-Guillemin's theorem gives one an asymptotic expansion

$$\mu_N^{red}(f) = \mathrm{Tr}(\pi_N^{red} M_f \pi_N^{red}) \sim \sum_{i=n-1}^{-\infty} a_i^{red} N^i,$$

and for $\pi_N^G : L^2(\mathbb{L}^N, \mu) \to \Gamma_{hol}(\mathbb{L}^N)^G$ the orthogonal projection onto G-invariant holomorphic sections, we will deduce from this:

Theorem 5.5. *For any compactly supported G-invariant function f on M,*

$$\mathrm{Tr}(\pi_N^G M_f \pi_N^G) \sim \sum_{i=n-1}^{-\infty} a_i^G(f) N^i, \tag{5.9}$$

as $N \to \infty$, and the coefficients a_i^G can be computed explicitly from a_i^{red}. In particular, the leading coefficient $a_{n-1}^G(f) = a_{n-1}^{red}(f_0 V)$, where $f_0(p) = f(\pi_0^{-1}(p))$.

Proof. Let $\{s_{N,j}\}$ be an orthonormal basis of $\Gamma_{hol}(\mathbb{L}_{red}^N)$ with respect to the volume form $V\mu_{red}$, then $\{\pi^* s_{N,j}\}$ is an orthogonal basis of $\Gamma_{hol}(\mathbb{L}^N)^G$, and

$$\mathrm{Tr}(\pi_N^G M_f \pi_N^G) = \int_M \sum_j \frac{\langle \pi^* s_{N,j}, \pi^* s_{N,j} \rangle}{\|\pi^* s_{N,j}\|^2} f \mu,$$

where, by the same argument as in the proof of (4.8), we have

$$\|\pi^* s_{N,j}\|^2 \sim \left(\frac{N}{\pi}\right)^{-m/2}\left(1+\sum_i C_i N^{-i}\right),$$

which implies

$$\frac{1}{\|\pi^* s_{N,j}\|^2} \sim \left(\frac{N}{\pi}\right)^{m/2}\left(1+\sum_i \tilde{C}_i N^{-i}\right).$$

Moreover, it is easy to see that

$$\int_{M_{red}} \sum_j \langle s_{N,j}, s_{N,j}\rangle V f_0 \mu_{red} = \mu_N^{red}(f_0 V).$$

Now the theorem follows from straightforward computations

$$\begin{aligned}
\mathrm{Tr}(\pi_N^G M_f \pi_N^G) &\sim \left(\frac{N}{\pi}\right)^{\frac{m}{2}}(1+\sum_i \tilde{C}_i N^{-i})\int_{M_{st}} \sum_j \langle \pi^* s_{N,j}, \pi^* s_{N,j}\rangle f\mu \\
&\sim \left(\frac{N}{\pi}\right)^{\frac{m}{2}}(1+\sum_i \tilde{C}_i N^{-i})\int_{M_{red}} \left(\int_{G\cdot p} \sum_{i=-\frac{m}{2}}^{-\infty} N^i c_i(f,p)\right)\sum_j \langle s_{N,j}, s_{N,j}\rangle(p) \\
&= \left(\frac{N}{\pi}\right)^{\frac{m}{2}}(1+\sum_i \tilde{C}_i N^{-i})\int_{M_{red}} \sum_{i=-\frac{m}{2}}^{-\infty} \left(N^i \int_{G\cdot p} c_i(f,p)d\nu\right)\sum_j \langle s_{N,j}, s_{N,j}\rangle(p) \\
&= \left(\frac{N}{\pi}\right)^{\frac{m}{2}}(1+\sum_i \tilde{C}_i N^{-i}) \sum_{i=-\frac{m}{2}}^{-\infty} N^i \mu_N^{red}(c_i V) \\
&\sim \sum_{i=n-1}^{-\infty} a_i^G(f) N^i,
\end{aligned}$$

where we used the fact that since f is G-invariant, so is $c_i(f,p)$. This proves (5.9). Moreover, since $c_{-m/2}(f,p) = f(p)/\pi^{m/2}$, we see that

$$a_{n-1}^G(f) = a_{n-1}^{red}(f_0 V),$$

completing the proof. □

5.5. **Bohr-Sommerfeld Lagrangians.** We assume we are in the same setting as before, and denote by ∇_{red} the metric connection on $\mathbb{L}_{red}$. Suppose Λ_{red} is a Bohr-Sommerfeld Lagrangian submanifold of M_{red}, and s_{BS} is a covariant constant section, i.e.,

$$s_{BS} : \Lambda_{red} \to \iota^*_{\Lambda_{red}} \mathbb{L}_{red}, \quad (\iota^*_{\Lambda_{red}} \nabla_{red}) s_{BS} = 0, \tag{5.10}$$

where $\iota_{\Lambda_{red}} : \Lambda_{red} \to M_{red}$ is the inclusion map. Let $\Lambda = \pi_0^{-1}(\Lambda_{red})$, then $\Lambda \subset \Phi^{-1}(0)$ is a G-invariant Lagrangian submanifold of M. Since

$$\pi_0^* \nabla_{red} s_{BS} = \iota_\Lambda^* \nabla \pi_0^* s_{BS}, \tag{5.11}$$

we see that $\pi_0^* s_{BS}$ is a covariant constant section on Λ. In other words, Λ is a Bohr-Sommerfeld Lagrangian submanifold of M. Conversely, if Λ is a G-invariant Bohr-Sommerfeld Lagrangian submanifold of M, then $\Lambda_{red} = \pi_0(\Lambda)$ is a Bohr-Sommerfeld Lagrangian submanifold of M_{red}.

Fixing a volume form μ_Λ on Λ, the pair (Λ_{red}, s_{BS}) defines a functional l on the space of holomorphic sections by

$$l : \Gamma_{hol}(\mathbb{L}_{red}) \to \mathbb{C}, \quad s \mapsto \int_{\Lambda_{red}} \langle \iota^*_{\Lambda_{red}} s, s_{BS} \rangle \mu_{\Lambda_{red}}.$$

This in turn defines a global holomorphic section $s_{\Lambda_{red}} \in \Gamma_{hol}(\mathbb{L}_{red})$ by duality. In other words, $s_{\Lambda_{red}}$ is the holomorphic section on M_{red} with the defining property

$$\int_{M_{red}} \langle s, s_{\Lambda_{red}} \rangle \mu_{red} = \int_{\Lambda_{red}} \langle \iota^*_{\Lambda_{red}} s, s_{BS} \rangle \mu_{\Lambda_{red}} \tag{5.12}$$

for all $s \in \Gamma_{hol}(\mathbb{L}_{red})$. A fundamental problem in Bohr-Sommerfeld theory is to know whether the section $s_{\Lambda_{red}}$ vanishes identically; and if not, to what extent $s_{\Lambda_{red}}$ is "concentrated" on the set Λ_{red}. One can also ask this question for the analogous section of $\mathbb{L}^k_{red}$.

We apply the upstairs-vs-downstairs philosophy to these problems. For the upstairs Bohr-Sommerfeld Lagrangian pair $(\Lambda, \tilde{s}_{BS})$, $\tilde{s}_{BS} = \pi_0^* s_{BS}$, as above one can associate with it a functional $\tilde{l}$ on $\Gamma_{hol}(\mathbb{L})^G$, which by duality defines a global G-invariant section $\tilde{s}_\Lambda \in \Gamma_{hol}(\mathbb{L})^G$. Obviously $l \neq 0$ if and only if $\tilde{l}$ is nonzero on $\Gamma_{hol}(\mathbb{L})^G$. However, since $\tilde{s}_{BS}$ is a G-invariant section,

$$\langle \tilde{s}, \tilde{s}_{BS} \rangle = \langle \tilde{s}^G, \tilde{s}_{BS} \rangle,$$

where $\tilde{s}^G$ is the orthogonal projection of $\tilde{s} \in \Gamma_{hol}(\mathbb{L})$ onto $\Gamma_{hol}(\mathbb{L})^G$. Thus $\tilde{l}$ is nonzero on $\Gamma_{hol}(\mathbb{L})^G$ if and only if it is nonzero on $\Gamma_{hol}(\mathbb{L})$. Thus we proved

Proposition 5.6. $s_{\Lambda_{red}} \neq 0$ *if and only if* $\tilde{s}_\Lambda \neq 0$.

A natural question to ask is whether $\pi^* s_{\Lambda_{red}}$ coincides with $\tilde{s}_\Lambda$ on M_{st}, or alternatively, whether $\pi_0^* s_{\Lambda_{red}} = \iota^* \tilde{s}_\Lambda$ on $\Phi^{-1}(0)$. In view of the $\frac{1}{2}$-form correction, we will modify the definition of the downstairs section $s_{\Lambda_{red}}$ to be

$$\int_{M_{red}} \langle s, s_{\Lambda_{red}} \rangle V \mu_{red} = \int_{\Lambda_{red}} \langle \iota^*_{\Lambda_{red}} s, s_{BS} \rangle V \mu_{\Lambda_{red}} \ , \tag{5.13}$$

for $s, s_{\Lambda_{red}} \in \Gamma_{hol}(\mathbb{L}_{red})$. The upstairs version of this is

$$\int_{M_{st}} \langle \tilde{s}, \tilde{s}_\Lambda \rangle \mu = \int_\Lambda \langle \iota^*_\Lambda \tilde{s}, \pi_0^* s_{BS} \rangle \mu_\Lambda \tag{5.14}$$

for $\tilde{s} = \pi^* s$. Since $\Lambda = \pi_0^{-1}(\Lambda_{red})$, the right hand sides of (5.13) and (5.14) coincide. Thus

$$\int_{M_{st}} \langle \pi^* s, \tilde{s}_\Lambda \rangle \mu = \int_{M_{red}} \langle s, s_{\Lambda_{red}} \rangle V \mu_{red} \tag{5.15}$$

for all $s \in \Gamma_{hol}(\mathbb{L}_{red})$.

Now we assume $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$, s^k_{BS} being the k^{th} tensor power of s_{BS}, and let $s^{(k)}_{\Lambda_{red}}$ and $\tilde{s}^{(k)}_\Lambda$ be the corresponding holomorphic sections. Then equation (5.15) now reads

$$\int_{M_{st}} \langle \pi^* s_k, \tilde{s}^{(k)}_\Lambda \rangle \mu = \int_{M_{red}} \langle s_k, s^{(k)}_{\Lambda_{red}} \rangle V \mu_{red} \tag{5.16}$$

for all $s_k \in \Gamma_{hol}(\mathbb{L}^k_{red})$ (However, the sections $\tilde{s}^{(k)}_\Lambda$ and $s^{(k)}_{\Lambda_{red}}$ are no longer the k^{th} tensor powers of $\tilde{s}_\Lambda$ and $s_{\Lambda_{red}}$ above). Notice that we can choose the two sections in (3.1) to be different nonvanishing sections and still get the same stability function ψ. Thus applying stability theory, one has

$$\int_{M_{st}} \langle \pi^* s_k, \pi^* s^{(k)}_{\Lambda_{red}} \rangle \mu \sim (\frac{k}{\pi})^{m/2} \int_{M_{red}} \langle s_k, s^{(k)}_{\Lambda_{red}} \rangle V \mu_{red}$$

for all s_k as $k \to \infty$. This together with (5.16) implies that asymptotically

$$\pi^* s^{(k)}_{\Lambda_{red}} \sim (\frac{k}{\pi})^{m/2} \tilde{s}^{(k)}_\Lambda, \ k \to \infty.$$

6. Toric varieties

6.1. **The Delzant construction.** Let $\mathbb{L} = \mathbb{C}^d \times \mathbb{C}$ be the trivial line bundle over $\mathbb{C}^d$ equipped with the Hermitian inner product

$$\langle 1, 1 \rangle = e^{-|z|^2}, \tag{6.1}$$

where $1 : \mathbb{C}^d \to \mathbb{L}$, $z \mapsto (z, 1)$ is the standard trivialization of $\mathbb{L}$. The line bundle $\mathbb{L}$ is the pre-quantum line bundle for $\mathbb{C}^d$, since

$$curv(\nabla) = -\sqrt{-1}\bar{\partial}\partial \log \langle 1, 1 \rangle = \sqrt{-1} \sum d\bar{z} \wedge dz = -\omega.$$

Let $K = (S^1)^d$ be the d-torus, which acts on $\mathbb{C}^d$ by the diagonal action,

$$\tau(e^{it_1}, \cdots, e^{it_d}) \cdot (z_1, \cdots, z_d) = (e^{it_1} z_1, \cdots, e^{it_d} z_d).$$

This is a Hamiltonian action with moment map

$$\phi(z) = \sum_{i=1}^d |z_i|^2 e_i^*, \tag{6.2}$$

where $e_1^*, \cdots, e_d^*$ is the standard basis of $\mathfrak{k}^* = \mathbb{R}^d$.

Now suppose $G \subset K$ is an m-dimensional sub-torus of K, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, and $\mathbb{Z}_G^* \subset \mathfrak{g}^*$ the weight lattice. Then the restriction of the K-action to G is still Hamiltonian, with moment map

$$\Phi(z) = L \circ \phi(z) = \sum_{i=1}^{d} |z_i|^2 \alpha_i, \tag{6.3}$$

where $\alpha_i = L(e_i^*) \in \mathbb{Z}_G^*$, and $L : \mathfrak{k}^* \to \mathfrak{g}^*$ is the transpose of the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{k}$.

We assume that the moment map Φ is proper, or alternatively, that the α_i's are polarized: there exists $v \in \mathfrak{g}$ such that $\alpha_i(v) > 0$ for all $1 \le i \le d$. Let $\alpha \in \mathbb{Z}_G^*$ be fixed, with the property that G acts freely on $\Phi^{-1}(\alpha)$. Then the symplectic quotient at level α,

$$M_\alpha = \Phi^{-1}(\alpha)/G,$$

is a symplectic toric manifold; and by Delzant's theorem, all toric manifolds arise this way.

The Hamiltonian action of K on $\mathbb{C}^d$ induces a Hamiltonian action of K on M_α, with moment map Φ_α defined by

$$\phi \circ \iota_\alpha = \Phi_\alpha \circ \pi_\alpha, \tag{6.4}$$

where $\iota_\alpha : \Phi^{-1}(\alpha) \hookrightarrow \mathbb{C}^d$ is the inclusion map, and $\pi_\alpha : \Phi^{-1}(\alpha) \to M_\alpha$ the projection map. The moment polytope of this Hamiltonian action on M_α is

$$\Delta_\alpha = L^{-1}(\alpha) \cap \mathbb{R}_+^d = \{t \in \mathbb{R}^d \mid t_i \ge 0, \ \sum t_i \alpha_i = \alpha\}. \tag{6.5}$$

If we replace $\mathbb{L}$ by $\mathbb{L}^k$, i.e. the trivial line bundle over $\mathbb{C}^d$ with Hermitian inner product $\langle 1, 1 \rangle_k = e^{-k|z|^2}$, then everything proceeds as above, and the moment polytope is changed to $k\Delta_\alpha = \Delta_{k\alpha}$.

6.2. **Line bundles over toric varieties.** As we showed in §2, M_α also admits the following GIT description,

$$M_\alpha = \mathbb{C}_{st}^d(\alpha)/G_{\mathbb{C}},$$

where $G_{\mathbb{C}} \simeq (\mathbb{C}^*)^n$ is the complexification of G, and $\mathbb{C}_{st}^d(\alpha)$ is the $G_{\mathbb{C}}$ flow-out of $\Phi^{-1}(\alpha)$. This flow-out is easily seen to be identical with the set

$$\mathbb{C}_{st}^d(\alpha) = \{z \in \mathbb{C}^d \mid I_z \in I_{\Delta_\alpha}\}, \tag{6.6}$$

where

$$I_z = \{i \mid z_i \ne 0\}$$

and

$$I_{\Delta_\alpha} = \{I_t \mid t \in \Delta_\alpha\}.$$

Now let G acts on the line bundle $\mathbb{L}$ by acting on the trivial section, 1, of $\mathbb{L}$, by weight α. (In Kostant's formula (2.7) this has the effect of shifting the moment map Φ by α, so that the new moment map becomes $\Phi - \alpha$ and the α level set of

Φ becomes the zero level set of $\Phi - \alpha$). This action extends to an action of $G_{\mathbb{C}}$ on $\mathbb{L}$ which acts on the trivial section 1 by the complexification, $\alpha_{\mathbb{C}}$, of the weight α and we can form the quotient line bundle,

$$\mathbb{L}_\alpha = \iota_\alpha^* \mathbb{L}/G = \mathbb{L}_{st}(\alpha)/G_{\mathbb{C}},$$

where $\mathbb{L}_{st}(\alpha)$ is the restriction of $\mathbb{L}$ to $\mathbb{C}^d_{st}(\alpha)$.

The holomorphic sections of $\mathbb{L}_\alpha^k$ are closely related to monomials in $\mathbb{C}^d$. In fact, since $\mathbb{L}$ is the trivial line bundle, the monomials

$$z^m = z_1^{m_1} \cdots z_d^{m_d}$$

are holomorphic sections of $\mathbb{L}$, and by Kostant's formula, z^m is a G-invariant section of $\mathbb{L}$ (with respect to the moment map Φ_α) if and only if

$$\tau^{\#}(\exp v)^* z^m = e^{i\alpha(v)} z^m$$

for all $v \in \mathfrak{g}$; in other words, if and only if m is an integer point in Δ_α. So we obtain

$$\Gamma_{hol}(\mathbb{L})^G = \text{span}\{z^m \mid m \in \Delta_\alpha \cap \mathbb{Z}^d\}. \tag{6.7}$$

In view of (6.6), $\mathbb{C}^d_{st}(\alpha)$ is Zariski open, so the GIT mapping

$$\gamma : \Gamma_{hol}(\mathbb{L})^G \to \Gamma_{hol}(\mathbb{L}_\alpha)$$

is bijective, although $\mathbb{C}^d$ is noncompact. As a result, the sections

$$s_m = \gamma(z^m), \quad m \in \Delta_\alpha \cap \mathbb{Z}^d. \tag{6.8}$$

give a basis of $\Gamma_{hol}(\mathbb{L}_\alpha)$.

To compute the norm of these sections s_m, we introduce the following notation. Let $j : \Delta_\alpha \hookrightarrow \mathbb{R}^+_d$ be the inclusion map, and t_i the standard i^{th} coordinate functions of $\mathbb{R}^d$. Then the *lattice distance* of $x \in \Delta_\alpha$ to the i^{th} facet of Δ_α is $l_i(x) = j^* t_i(x)$. On $\Phi^{-1}(\alpha)$ one has

$$\langle z^m, z^m \rangle = |z_1^{m_1}|^2 \cdots |z_d^{m_d}|^2 e^{-|z|^2},$$

which implies

$$\langle s_m, s_m \rangle_\alpha = (\Phi_\alpha)^* (l_1^{m_1} \cdots l_d^{m_d} e^{-l}) \ , \tag{6.9}$$

where $l = l_1 + \cdots + l_d$. As a corollary, we see that the stability function on $\mathbb{C}^d_{st}(\alpha)$ is

$$\psi(z) = -|z|^2 + \log |z^m|^2 - \pi^* \Phi_\alpha^* (\sum m_i \log l_i - l). \tag{6.10}$$

Finally by the Duistermaat-Heckman theorem the push-forward of the symplectic measure on M_α by Φ_α is the Lebesgue measure $d\sigma$ on Δ_α, so the L^2 norm of s_m is

$$\langle s_m, s_m \rangle_{L^2} = \int_{\Delta_\alpha} l_1^{m_1} \cdots l_d^{m_d} e^{-l} d\sigma.$$

For toric varieties, the "Bohr-Sommerfeld" issues that we discussed in §5.1 are easily dealt with: Let $\tilde{s}$ be the G-invariant section, $z_1^{m_1} \cdots z_d^{m_d}$, of $\mathbb{L}$, with $(m_1, \cdots, m_d) \in \Delta_\alpha$. Then $\langle \tilde{s}, \tilde{s} \rangle$ take its maximum on the set $\Phi^{-1}(m_1, \cdots, m_d)$, and if $(m_1, \cdots, m_d)$ is in the interior of Δ_α, this set is a Lagrangian torus: an orbit of $\mathbb{T}^d$. Moreover, if s is the section of $\mathbb{L}_\alpha$ corresponding to $\tilde{s}$, $\langle s, s \rangle$ takes its maximum on the projection of this orbit in M_α, which is also a Lagrangian submanifold.

6.3. **Canonical affines.** We end this section by briefly describing a covering by natural coordinate charts on M_α – the canonical affines. (For more details c.f. [DuP]). Let v be a vertex of Δ. Since Δ is a Delzant's polytope, $\#I_v = n$ and $\{\alpha_i \mid i \in I_v\}$ is a lattice basis of $\mathbb{Z}_G^*$. Denote by

$$\Delta_v = \{t \in \Delta \mid I_t \supset I_v\}, \tag{6.11}$$

the open subset in Δ_α obtained by deleting all facets which don't contain v and let

$$Z_v = \Phi_\alpha^{-1}(\Delta_v).$$

Definition 6.1. The *canonical affines* in M_α are the open subsets

$$\mathcal{U}_v = Z_v / G. \tag{6.12}$$

Since $\{\alpha_i \mid i \in I_v\}$ is a lattice basis, for $j \notin I_v$ we have $\alpha_j = \sum c_{j,i}\alpha_i$, where $c_{j,i}$ are integers. Suppose $\alpha = \sum a_i \alpha_i$, then Z_v is defined by the equations

$$|z_i|^2 = a_i - \sum_{j \notin I_v} c_{j,i}|z_j|^2, \qquad i \in I_v \tag{6.13}$$

and the resulting inequalities

$$\sum c_{j,i}|z_j|^2 < a_i. \tag{6.14}$$

So $\mathcal{U}_v$ can be identified with the set (6.13). The set

$$z_i = \left(a_i - \sum_{j \notin I_v} c_{j,i}|z_j|^2 \right)^{1/2}$$

is a cross-section of the G-action on Z_v, and the restriction to this cross-section of the standard symplectic form on $\mathbb{C}^d$ is $\sqrt{-1}\ \sum_{i \notin I_z} dz_i \wedge d\bar{z}_i$. So the reduced symplectic form is

$$\omega_\alpha = \sqrt{-1} \sum_{j \notin I_v} dz_j \wedge d\bar{z}_j, \tag{6.15}$$

in other words, the z_j's with $j \notin I_v$ are *Darboux coordinates* on $\mathcal{U}_v$.

7. Stability functions on toric varieties

7.1. **The general formula.** In this section we compute the stability functions for the toric varieties M_α defined above, with upstairs space $\mathbb{C}^d$ and upstairs metric (6.1). For $z \in M_{st}$ there is a unique $g \in \exp\sqrt{-1}\mathfrak{g}$ such that $g \cdot z \in \Phi^{-1}(\alpha)$, and by definition, if $s(z) = z^m = \pi^* s_m$,

$$\begin{aligned}\psi(z) &= \log\langle s,s\rangle(z) - \log\langle s,s\rangle(g\cdot z)\\ &= -|z|^2 + \log|z^m|^2 + |g\cdot z|^2 - \log|(g\cdot z)^m|^2.\end{aligned} \tag{7.1}$$

Moreover, If the circle group $(e^{i\theta}, \cdots, e^{i\theta})$ is contained in G, or alternatively, if $v = (1, \cdots, 1) \in \mathfrak{g}$, or alternatively if M_α can be obtained by reduction from $\mathbb{CP}^{d-1}$, then

$$|z|^2 = \sum \alpha_i(v)|z_i|^2 = \langle \Phi(z), v\rangle,$$

thus

$$|g\cdot z|^2 = \langle \Phi(g\cdot z), v\rangle = \langle \alpha, v\rangle, \tag{7.2}$$

and (7.1) simplifies to

$$\psi(z) = -|z|^2 + \log|z^m|^2 + \alpha(v) - \log|(g\cdot z)^m|^2. \tag{7.3}$$

Given a weight $\beta \in \mathbb{Z}^*_G$ let $\chi_\beta : G_\mathbb{C} \to \mathbb{C}$ be the character of $G_\mathbb{C}$ associated to β. Restricted to $\exp(\sqrt{-1}\mathfrak{g})$, χ_β is the map

$$\chi_\beta(\exp i\xi) = e^{-\beta(\xi)}. \tag{7.4}$$

Now note that by (7.3),

$$\begin{aligned}\psi(z) &= -|z|^2 + \alpha(v) + \log|z^m|^2 - \log(\prod \chi_{\alpha_i}(g)^{2m_i})|z^m|^2\\ &= -|z|^2 + \alpha(v) - \log\prod \chi_{\alpha_i(g)^{2m_i}}.\end{aligned}$$

But $z^m = \pi^* s_m$ for $s_m \in \Gamma_{hol}(\mathbb{L}_\alpha)$ if and only if m is in Δ_α, i.e. $\sum m_i\alpha_i = \alpha$, so we get finally by (7.4), $\prod \chi_{\alpha_i}(g)^{m_i} = \chi_\alpha(g)$ and

$$\psi(z) = -|z|^2 + \alpha(v) - 2\log\chi_\alpha(g). \tag{7.5}$$

Recall now that the map

$$\Phi^{-1}(\alpha) \times \exp(\sqrt{-1}\mathfrak{g}) \to \mathbb{C}^d_{st}$$

is bijective, so the inverse of this map followed by projection onto $\exp(\sqrt{-1}\mathfrak{g})$ gives us a map

$$\gamma : \mathbb{C}^d_{st} \to \exp(\sqrt{-1}\mathfrak{g}), \tag{7.6}$$

and by the computation above we've proved

Theorem 7.1. *The stability function for M_α, viewed as a GIT quotient of $\mathbb{C}^d$ with the trivial line bundle and the flat metric (6.1), is*

$$\psi(z) = -|z|^2 + \alpha(v) - 2(\log \gamma^* \chi_\alpha)(z). \tag{7.7}$$

For example for $\mathbb{CP}^{n-1}$ itself with $\mathbb{C}^n_{st} = \mathbb{C}^n - \{0\}$ and $\alpha = 1$, $\gamma(z) = |z|$ and hence

$$\psi(z) = -|z|^2 + 1 + \log |z|^2. \tag{7.8}$$

The formula (7.7) is valid modulo the assumption that M_α can be obtained by reduction from $\mathbb{CP}^{d-1}$, i.e. modulo the assumption (7.2). Dropping this assumption we have to replace (7.7) by the slightly more complicated formula

$$\psi(z) = -|z|^2 + |\gamma(z)^{-1}z|^2 - 2(\log \gamma^* \chi_\alpha)(z). \tag{7.9}$$

7.2. **Stability function on canonical affines.** We can make the formula (7.7) more explicit by restricting to the canonical affines, $\mathcal{U}_v$, of §6.3. For any vertex v of Δ it is easy to see that

$$\mathcal{U}_v = \mathbb{C}^d_{\Delta_v}/G_{\mathbb{C}},$$

where

$$\mathbb{C}^d_{\Delta_v} = \{z \in \mathbb{C}^d \mid I_z \supset I_v\} \tag{7.10}$$

is an open subset of $\mathbb{C}^d_{st}$. By relabelling we may assume $I_v = \{1, 2 \cdots, n\}$. Since the relabelling makes $\alpha_1, \cdots, \alpha_n \in \mathfrak{g}^*$ into a lattice basis of $\mathbb{Z}^*_G$, $\alpha_k = \sum c_{k,i}\alpha_i$ for $k > n$, where $c_{k,i}$ are integers. Let $f_1, \cdots, f_n$ be the dual basis of the group lattice, $\mathbb{Z}_G$, then the map

$$\mathbb{C}^n \to G_{\mathbb{C}}, \quad (w_1, \cdots, w_n) \mapsto w_1 f_1 + \cdots + w_n f_n \mod \mathbb{Z}_G \tag{7.11}$$

gives one an isomorphism of $G_{\mathbb{C}}$ with the complex torus $(\mathbb{C}^*)^n$ and in terms of this isomorphism the $G_{\mathbb{C}}$-action on $\mathbb{C}^d_{\Delta_v}$ is given by

$$(w_1, \cdots, w_n) \cdot z = \left(w_1 z_1, \cdots, w_n z_n, (\prod_{i=1}^n w_i^{c_{n+1,i}}) z_{n+1}, \cdots, (\prod_{i=1}^n w_i^{c_{d,i}}) z_d \right).$$

Now suppose $z \in \mathbb{C}^d_{\Delta_v}$. Then the system of equations obtained from (6.13) and (7.1),

$$r_i^2 |z_i|^2 + \sum_{k=n+1}^d c_{k,i} (\prod_{j=1}^n r_j^{c_{k,i}})^2 |z_k|^2 = a_i, \quad 1 \le i \le n,$$

has a unique solution, $g = (r_1(z), \cdots, r_n(z)) \in (\mathbb{R}^+)^n = \exp(\sqrt{-1}\mathfrak{g})$, i.e., the g in (7.1) is $(r_1, \cdots, r_n)$. Via the identification (7.10) the weight $\alpha \in \mathbb{Z}^*_G$ corresponds by (7.11) to the weight $(a_1, \cdots, a_n) \in \mathbb{Z}^n$ and by (7.7) and (7.9)

$$\psi|_{\mathbb{C}^d_{\Delta_v}} = -|z|^2 + \alpha(v) - 2 \sum_i a_i \log r_i(z) \tag{7.12}$$

in the projective case and

$$\psi|_{\mathbb{C}^d_{\Delta_v}} = -|z|^2 + \sum_i r_i^2|z_i|^2 + \sum_{k>n} |(\prod_{i=1}^{n} r_i^{c_{k,i}})z_k|^2 - 2\sum_i a_i \log r_i. \tag{7.13}$$

in general.

7.3. **Example: The stability function on the Hirzebruch surfaces.** As an example, let's compute the stability function for Hirzebruch surfaces. Recall that the Hirzebruch surface H_n is the toric 4-manifold whose moment polytope is the polygon with vertices $(0,0),(0,1),(1,1),(n+1,0)$. By the Delzant construction, we see that H_n is in fact the toric manifold obtained from the $\mathbb{T}^2$-action on $\mathbb{C}^4$,

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot z = (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{i\theta_1 - in\theta_2}z_3, e^{i\theta_2}z_4).$$

By the procedure above, we find the stability function

$$\psi(z) = -|z|^2 - a_1 \log r_1 - a_2 \log r_2 + a_1 + a_2 - nr_1^{2n}r_2^2|z_3|^2,$$

where r_1, r_2 are the solution to the system of equations

$$r_1^2|z_1|^2 + r_1^{2n}r_2^2|z_3|^2 = a_1,$$
$$r_2^2|z_2|^2 - nr_1^{2n}r_2^2|z_3|^2 + r_1^2|z_4|^2 = a_2.$$

8. Applications of stability theory to toric varieties

8.1. **Universal rescaled law on toric varieties.** In this section we suppose $\beta \in \Delta_\alpha$ is rational, and N is large with $N\beta \in \mathbb{Z}^d$. One of the main results in [STZ] is the following universal rescaled law for the probability distribution function (5.7) on toric varieties,

$$\lim_{N\to\infty} (\frac{N}{\pi})^{n/2}\sigma_{N,N\beta}((\frac{N}{\pi})^{n/2}t) = \frac{(\log c/t)^{n/2}}{c\Gamma(n/2+1)}. \tag{8.1}$$

By measure theoretic arguments, they deduce this from moment estimates, (c.f. §4.1 of [STZ])

$$\int_{M_\alpha} x^l d\nu_N \to \frac{c^{l-1}}{l^{n/2}}, \qquad N \to \infty, \tag{8.2}$$

where l is any positive integer, ν_N is the push-forward measure

$$\nu_N = \left(\left|(\frac{N}{\pi})^{-n/4}\phi_{N\beta}\right|^2\right)_* \left((\frac{N}{\pi})^{n/2}\nu\right),$$

with $\phi_{N\beta} = s_{N\beta}/\|s_{N\beta}\|$ and ν the pullback of the Fubini-Study form via a projective embedding. By a simple computation it is easy to see that

$$\int x^l d\nu_N(x) = \left(\frac{N}{\pi}\right)^{-\frac{n(l-1)}{2}} \int_{M_\alpha} |\phi_{N\beta}|^{2l}\nu = \left(\frac{N}{\pi}\right)^{-\frac{n(l-1)}{2}} m_\alpha(l, \phi_{N\beta}, \nu). \tag{8.3}$$

Instead of considering the pullback of the Fubini-Study measure we will consider another natural measure on toric varieties: the quotient measure induced by the upstairs flat metric. The upstairs analogue of (8.1) for toric varieties is rather easy to prove:

Lemma 8.1. *For any l, the l^{th} moments*

$$\left(\frac{N}{\pi}\right)^{-d(l-1)/2} m(l, \frac{z^{N\beta}}{\|z^{N\beta}\|}, d\mu) \to \frac{c^{l-1}}{l^{d/2}} \qquad (N \to \infty). \tag{8.4}$$

Proof. See [GuW], or by direct computation. □

Thus we can apply proposition 5.4 to derive (8.2) from (8.4). By (5.8) and (4.8),

$$m_\alpha(l, \frac{s_{N\beta}}{\|s_{N\beta}\|}, \frac{\omega_\alpha^n}{n!}) \sim l^{-m/2} \left(\frac{N}{\pi}\right)^{m(l-1)/2} m(l, \frac{z^{N\beta}}{\|z^{N\beta}\|}, \frac{\omega^d}{d!}).$$

Thus

$$\left(\frac{N}{\pi}\right)^{-n(l-1)/2} m_\alpha(l, \frac{s_{N\beta}}{\|s_{N\beta}\|}, \frac{\omega_\alpha^n}{n!}) \to \frac{c^{l-1}}{l^{n/2}}$$

as $N \to \infty$ for all l. This together with the measure theoretic arguments alluded to above implies the distribution law (8.1) for the special volume form $V\mu_\alpha$ on M_α.

Remark 8.2. Here we only consider the case when β is an interior point of the Delzant polytope, which corresponds to the case $r = 0$ in [STZ]. However, one can modify the arguments above slightly to show the same result for general r and $N\beta$ replaced by $N\beta + o(1)$.

8.2. **Spectral properties of toric varieties.** As we have seen, the stability theory derived in §1-§5 is particularly useful for toric varieties M_α, since the upstairs space is the complex space, $\mathbb{C}^d$, the Lie group G is abelian, its action on $\mathbb{C}^d$ is linear, and the G-invariant sections of $\mathbb{L}$ are just linear combinations of monomials. As a consequence, the expressions (1.11), (1.15), (1.20) etc. are relatively easy to compute.

For example, consider the density of states

$$\mu_N = \sum \langle s_{N,i}, s_{N,i} \rangle \mu_{red},$$

where $\{s_{N,i}\}$ is an orthonormal basis of $\Gamma_{hol}(\mathbb{L}^N)$, then by proposition 5.3,

$$\int_{M_\alpha} f\mu_N \sim \left(\frac{N}{\pi}\right)^{-m/2} \int_{\mathbb{C}^d} \frac{\pi^* f}{\pi^* V} \chi \sum \langle \pi^* s_{N,i}, \pi^* s_{N,i} \rangle \mu.$$

The right hand side has a very simple asymptotic expansion in terms of Stirling numbers of the first kind (See [GuW], [Wan]) and from this and the results of §5.4 one gets an alternative proof of theorem 1.1 of [BGU]:

Theorem 8.3. *There exists differential operators $P_i(x, D)$ of order $2i$ such that*

$$\mu_N(f) \sim \sum_i N^{d-m-i} \int P_i(x, D) f(x) dx, \quad N \to \infty.$$

In this way the coefficients of the downstairs density of states asymptotics can be computed explicitly by the coefficients of the asymptotic expansion of the invariant upstairs density of states asymptotics – the relation of the leading terms is given in theorem 5.5, and the other coefficients depend on the asymptotics of the Laplace integral (4.1) together with the value of the stability function near $\Phi^{-1}(0)$. Similarly theorem 1.2 of [BGU] can be derived from the results of §5.2 and upstairs analogues of these results in [GuW].

9. Martin's construction

The non-abelian generalizations of toric varieties are "spherical" varieties, and the simplest examples of these are coadjoint orbits and varieties obtained from coadjoint orbits by symplectic cuts. In the remainder of this paper we apply stability theory to the coadjoint orbits of the unitary group $\mathcal{U}(n)$. It is well known that the coadjoint orbits of $\mathcal{U}(n)$ can be identified with the sets of isospectral Hermitian matrices $\mathcal{H}(\lambda) \subset \mathcal{H}(n)$, i.e., Hermitian matrices with fixed eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For $\lambda_1 = \cdots = \lambda_{n-1} > \lambda_n$, $\mathcal{H}(\lambda)$ is $\mathbb{CP}^{n-1}$ which is a toric manifold. Thus the first non-toric case is given by $\lambda_1 = \cdots = \lambda_k > \lambda_{k+1} = \cdots = \lambda_n$, $1 < k < n-1$, in which case $\mathcal{H}(\lambda)$ is the complex Grassmannian $Gr(k, \mathbb{C}^n)$.

9.1. **GIT for Grassmannians.** Suppose $k < n$. It is well known that the complex Grassmannian $Gr(k, \mathbb{C}^n)$ can be realized as the quotient space of $\mathbb{C}^{kn}$ by symplectic reduction or as a GIT quotient as follows:

Let $M = \mathfrak{M}_{k,n}(\mathbb{C}) \simeq \mathbb{C}^{kn}$ be the space of complex $k \times n$ matrices. We equip $\mathbb{C}^{kn}$ with its standard Kähler metric, the standard trivial line bundle $\mathbb{C} \times \mathbb{C}^{kn} \to \mathbb{C}^{kn}$, and the standard Hermitian inner product on this line bundle,

$$\langle 1, 1 \rangle (Z) = e^{-\operatorname{Tr} ZZ^*}. \tag{9.1}$$

Now let $G = \mathcal{U}(k)$ act on $\mathfrak{M}_{k,n}$ by left multiplication. This action preserves the inner product (9.1), and thus preserves the Kähler form $\sqrt{-1}\partial\bar{\partial} \operatorname{Tr} ZZ^*$. It is not hard to see that it is a Hamiltonian action with moment map

$$\Phi : \mathfrak{M}_{k,n} \to \mathcal{H}_k, \qquad Z \mapsto ZZ^*, \tag{9.2}$$

where $\mathcal{H}_k$ is the space of $k \times k$ Hermitian matrices. Here we identify $\mathcal{H}_k$ with $\sqrt{-1}\mathcal{H}_k = Lie(\mathcal{U}(k))$, and identify $\mathcal{H}_k$ with $Lie(\mathcal{U}(k))^* = \mathcal{H}_k^*$ via the Killing form. Notice that the identity matrix I lies in the annihilator of the commutator ideal,

$$[\mathcal{H}_k, \mathcal{H}_k]^0 = \{a \in \mathcal{H}_k^* \mid \langle [h_1, h_2], a \rangle = 0 \text{ for all } h_1, h_2 \in \mathcal{H}_k\},$$

so $\Phi - I$ is also a moment map, and it's clear that the reduced space

$$M_{red} = \Phi^{-1}(I)/G$$

is the Grassmannian $Gr(k, \mathbb{C}^n)$.

On the other hand, the complexification of $\mathcal{U}(k)$ is $GL(k, \mathbb{C})$, and it's not hard to see that the set of stable points, M_{st}, is exactly the set of $k \times n$ matrices $A \in M$ which have rank k, and that the quotient $M_{st}/GL(k, \mathbb{C})$ is again $Gr(k, \mathbb{C}^n)$. This gives us the GIT description of $Gr(k, \mathbb{C}^n)$.

As for the reduced line bundle, $\mathbb{L}_{red}$, on M_{red}, this is obtained from the trivial line bundle on M_{st} by "shifting" the action of $GL(k, \mathbb{C})$ on the trivial line bundle in conformity with the shifting, "$\Phi \Rightarrow \Phi - I$", of the moment map, i.e. by letting $GL(k, \mathbb{C})$ act on this bundle by the character

$$\gamma : GL(k, \mathbb{C}) \to \mathbb{C}^*, \gamma(A) = \det(A).$$

9.2. **Martin's reduction procedure.** For general coadjoint orbit of $\mathcal{U}(n)$, Shaun Martin showed that there is an analogous GIT description. Since he never published this result, we will roughly outline his argument here, focusing for simplicity on the case $\lambda_1 > \cdots > \lambda_n$.

Let

$$M = \mathfrak{M}_{1,2}(\mathbb{C}) \times \mathfrak{M}_{2,3}(\mathbb{C}) \times \cdots \times \mathfrak{M}_{n-1,n}(\mathbb{C}).$$

Then each component of M is a linear symplectic space, and M is just the linear symplectic space $\mathbb{C}^{(n-1)n(n+1)/3}$ with standard Kähler form $\omega = -\sqrt{-1}\partial\bar{\partial}\log\rho$, where ρ is the potential function

$$\rho(Z) = \exp(-\sum_{i=1}^{n-1} \operatorname{Tr} Z_i Z_i^*).$$

Consider the group

$$G = \mathcal{U}(1) \times \mathcal{U}(2) \times \cdots \times \mathcal{U}(n-1)$$

acting on M by the recipe:

$$\tau_{(U_1,\cdots,U_{n-1})}(Z_1, \cdots, Z_{n-1}) = (U_1 Z_1 U_2^*, \cdots, U_{n-2} Z_{n-2} U_{n-1}^*, U_{n-1} Z_{n-1}). \tag{9.3}$$

Lemma 9.1. *The action above is Hamiltonian with moment map*

$$\Phi(Z_1, \cdots, Z_{n-1}) = (Z_1 Z_1^*, Z_2 Z_2^* - Z_1^* Z_1, \cdots, Z_{n-1} Z_{n-1}^* - Z_{n-2}^* Z_{n-2}). \tag{9.4}$$

Proof. Given any $H = (H_1, \cdots, H_{n-1}) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_{n-1}$, denote by $\mathcal{U}_H(t)$ the one parameter subgroup of G generated by H, i.e.,

$$\mathcal{U}_H(t)Z = \big(\exp(\sqrt{-1}tH_1) Z_1 \exp(-\sqrt{-1}tH_2), \cdots, \\ \exp(\sqrt{-1}tH_{n-2}) Z_{n-2} \exp(-\sqrt{-1}tH_{n-1}), \exp(\sqrt{-1}tH_{n-1}) Z_{n-1}\big).$$

Let v_H be the infinitesimal generator of this group, then

$$\iota_{v_H}(\sqrt{-1}\partial \log \rho) = -\sqrt{-1}\sum \mathrm{Tr}((\iota_{v_H} dZ_i)Z_i^*).$$

Since

$$\iota_{v_H} dZ_i = \frac{d}{dt}(\exp(\sqrt{-1}tH_i)Z_i \exp(-\sqrt{-1}tH_{i+1}))\Big|_{t=0} = \sqrt{-1}(H_iZ_i - Z_iH_{i+1}),$$

we see that

$$\iota_{v_H}(\sqrt{-1}\partial \log \rho) = \sum \mathrm{Tr}(H_iZ_iZ_i^* - H_{i+1}Z_i^*Z_i) = \langle \Phi(Z), H\rangle.$$

This shows that (9.4) is a moment map of τ. □

Given $a = (a_1, \cdots, a_n) \in \mathbb{R}^n_+$, let

$$\phi^{-1}(aI) = \Phi^{-1}(a_1I_1, \cdots, a_{n-1}I_{n-1}),$$

and let

$$M_a = \Phi^{-1}(aI)/G$$

be the reduced space at level $(a_1I_1, \cdots, a_{n-1}I_{n-1}) \in [\mathfrak{g}, \mathfrak{g}]^0$. Consider the residual action of $GL(n, \mathbb{C})$ on M,

(9.5) $$\kappa : GL(n, \mathbb{C}) \times M \to M, \quad \kappa_A Z = (Z_1, \cdots, Z_{n-2}, Z_{n-1}A^{-1}).$$

Then the actions κ and τ commute, and by the same argument as above we see that $\kappa|_{\mathcal{U}(n)}$ is a Hamiltonian action with a moment map

(9.6) $$\Psi : M \to \mathcal{H}_n, \quad \Psi(Z) = Z_{n-1}^*Z_{n-1} + a_nI_n.$$

We thus get a Hamiltonian action of $\mathcal{U}(n)$ on the reduced space M_a with moment map $\Psi_a : M_a \to \mathcal{H}_n$, which satisfies $\Psi \circ i = \Psi_a \circ \pi_0$, where, as usual, $i : \Phi^{-1}(aI) \hookrightarrow M$ is the inclusion map and $\pi_0 : \Phi^{-1}(aI) \to M_a$ the projection.

Theorem 9.2 ([Mar]). *Ψ_a is a $\mathcal{U}(n)$-equivariant symplectomorphism of M_a onto $\mathcal{H}(\lambda)$, with $\lambda_i = \sum_{j=i}^n a_j$.*

Proof. First we prove that Ψ_a maps M_a onto the isospectral set $\mathcal{H}(\lambda)$. In view of the relation $\Psi \circ i = \Psi_a \circ \pi_0$, we only need to show Image$(\Psi) = \mathcal{H}(\lambda)$. In fact, if $Z_iZ_i^*$ has eigenvalues $(\mu_1, \cdots, \mu_i)$, then the eigenvalues of $Z_iZ_i^*$ are exactly $(\mu_1, \cdots, \mu_i, 0)$, so it is straightforward to see that $Z_2Z_2^* = Z_1Z_1^* + a_2I_2$ has eigenvalues $a_1 + a_2, a_2$, and in general $Z_iZ_i^*$ has eigenvalues $a_1 + \cdots + a_i, a_2 + \cdots + a_i, \cdots, a_i$. This proves that Ψ_a maps M_a into $\mathcal{H}(\lambda)$, and since G acts transitively on $\mathcal{H}(\lambda)$, this map is onto.

Next note that by dimension-counting $\dim M_a = \dim \mathcal{H}(\lambda)$, so Ψ_a is a finite-to-one covering. Since the adjoint orbits of $\mathcal{U}(n)$ are simply-connected, we conclude that this map is also injective, and thus a diffeomorphism.

Since Ψ_a is a moment map, it is a Poisson mapping between M_a and $\mathcal{H}(n)$, i.e.,

$$\{f \circ \Psi_a, g \circ \Psi_a\}_{M_a} = \{f, g\}_{\mathcal{H}(\lambda)} \circ \Psi_a$$

for any $f, g \in C^\infty(\mathcal{H}(\lambda))$. Thus Ψ_a is a symplectomorphism between M_a and $\mathcal{H}(\lambda)$.

Finally the $\mathcal{U}(n)$-equivariance comes from the fact that

$$\Psi(U \cdot Z) = (U^{-1})^* Z_{n-1}^* Z_{n-1} U^{-1} + a_n I_n = U(Z_{n-1}^* Z_{n-1} + a_n I_n)U^{-1} = U \cdot \Psi(Z).$$

This completes the proof. □

The GIT description of this reduction procedure is now clear:

$$Z = (Z_1, \cdots, Z_{n-1}) \in M_{st}$$

if and only if Z_i is of rank i for all i, and

$$M_a = M_{st}/G_{\mathbb{C}}$$

with $G_{\mathbb{C}}$ the product

$$G_{\mathbb{C}} = GL(1, \mathbb{C}) \times \cdots \times GL(n-1, \mathbb{C}).$$

9.3. **Twisted line bundles over $\mathcal{U}(n)-$coadjoint orbits.** As in the toric case, reduction at level 0 of the moment map (9.2) is not very interesting, since the reduced line bundle is the trivial line bundle. To get the Grassmannian, we shifted the moment map by the identity matrix. Equivalently, we "twisted" the action of $GL(k, \mathbb{C})$ on the trivial line bundle $\mathbb{C} \times \mathbb{C}^{kn}$ by a character of $GL(k, \mathbb{C})$. It is to this shifted moment map/twisted action that we applied the reduction procedure to obtain a reduced line bundle on $Gr(k, \mathbb{C}^n)$.

Similarly, for $\mathcal{U}(n)$-coadjoint orbits we will twist the $G_{\mathbb{C}}$ action on the trivial line bundle over M by characters of $G_{\mathbb{C}}$. Every character of $G_{\mathbb{C}}$ is of the form

$$\gamma = \gamma_1^{m_1} \cdots \gamma_{n-1}^{m_{n-1}}, \tag{9.7}$$

where $\gamma_k(A) = \det(A_k)$ for $A = (A_1, \cdots, A_{n-1})$. Let

$$\pi_k : M \to \mathfrak{M}_{k,n}, \quad (Z_1, \cdots, Z_{n-1}) \to Z_k Z_{k+1} \cdots Z_{n-1}.$$

Then π_k intertwines the action of $G_{\mathbb{C}}$ on M with the standard left action of $\mathcal{U}(k)$ on $\mathfrak{M}_{k,n}$, and intertwines the action κ of $\mathcal{U}(n)$ on M with the standard right action of $\mathcal{U}(n)$ on $\mathfrak{M}_{k,n}$. Let $\mathbb{L}_k$ be the holomorphic line bundle on $\mathfrak{M}_{k,n}$ associated with the character

$$\gamma_k : GL(k, \mathbb{C}) \to \mathbb{C}^*, \quad A \mapsto \det(A). \tag{9.8}$$

Then the bundle $\pi_k^* \mathbb{L}_k$ is the holomorphic line bundle on M associated with γ_k and

$$\mathbb{L} := \bigotimes_{k=1}^{n-1} (\pi_k^* \mathbb{L}_k)^{m_k} \tag{9.9}$$

is the holomorphic line bundle associated with the character γ. In particular if s_k is a $GL(k,\mathbb{C})$-invariant holomorphic section of $\mathbb{L}_k$, then

$$(\pi_1^* s_1)^{m_1} \cdots (\pi_{n-1}^* s_{n-1})^{m_{n-1}} \tag{9.10}$$

is a $G_{\mathbb{C}}$-invariant holomorphic section of $\mathbb{L}$, and all $G_{\mathbb{C}}$-invariant holomorphic sections of $\mathbb{L}$ are linear combinations of these sections. Since the representation of $GL(n,\mathbb{C})$ on the space $\Gamma_{hol}(\mathbb{L}_k)$ is its k-th elementary representation we conclude

Theorem 9.3. *The representation of $GL(n,\mathbb{C})$ on the space $\Gamma_{hol}(\mathbb{L})$ is the irreducible representation with highest weight $\sum_{i=1}^{n-1} m_i\alpha_i$, where $\alpha_1, \cdots, \alpha_{n-1}$ are the simple roots of $GL(n,\mathbb{C})$.*

For the canonical trivializing section of $\mathbb{L}$ its Hermitian inner product with itself is

$$\prod_{i=1}^{n-1} \det(Z_i Z_{i+1} \cdots Z_{n-1} Z_{n-1}^* \cdots Z_i^*)^{-m_i}$$

and hence the potential function for the $\mathbb{L}$-twisted Kähler structure on M is

$$\rho_{\mathbb{L}} = \sum_{i=1}^{n-1} \operatorname{Tr} Z_i Z_i^* - m_i \log\det(Z_i \cdots Z_{n-1} Z_{n-1}^* \cdots Z_i^*) \tag{9.11}$$

and the corresponding $\mathbb{L}$-twisted moment map is

$$\Phi_{\mathbb{L}}(Z_1, \cdots, Z_{n-1}) = (Z_1 Z_1^* - m_1 I_1, \cdots, Z_{n-1} Z_{n-1}^* - m_{n-1} I_{n-1}). \tag{9.12}$$

10. Stability theory for coadjoint orbits

10.1. **The stability function on the Grassmannians $Gr(k,\mathbb{C}^n)$.** To compute this stability function, we first look for the G-invariant sections of the twisted line bundle. For any index set

$$J = \{j_1, \cdots, j_k\} \subset \{1, 2, \cdots, n\}$$

denote by $Z_J = Z_{j_1,\cdots,j_k}$ the $k \times k$ sub-matrix consisting of the $j_1, \cdots, j_k$ columns of Z.

Lemma 10.1. *The functions $s_J(Z) = \det(Z_J)$ are G-invariant sections of the trivial line bundle on $\mathfrak{M}_{k,n}$ for the twisted G-action.*

Proof. Let H be any $n \times n$ Hermitian matrix, and v_H the generator of the one-parameter subgroup generated by H. Then by Kostant's identity (2.7) one only needs to show

$$\iota_{v_H} \partial \log\langle s_J, s_J\rangle = -\sqrt{-1}\operatorname{Tr}\left((ZZ^* - I)H\right).$$

This follows from direct computation:

$$\begin{aligned}\iota_{v_H}\partial\log\langle s_J, s_J\rangle &= \iota_{v_H}\partial(-\operatorname{Tr} ZZ^* + \log\det(Z_J\bar{Z}_J))\\ &= -\operatorname{Tr}((\iota_{v_H}dZ)Z^*) + \iota_{v_H}\partial\operatorname{Tr}\log(Z_JZ_J^*)\\ &= -\operatorname{Tr}((\iota_{v_H}dZ)Z^*) + \operatorname{Tr}((\iota_{v_H}dZ_J)Z_J^*(Z_J^*)^{-1}Z_J^{-1})\\ &= -\sqrt{-1}\operatorname{Tr}(H(ZZ^*-I)),\end{aligned}$$

completing the proof. □

Now we are ready to compute the stability function for the Grassmannians. Without loss of generality, we suppose $\{j_1,\cdots,j_k\} = \{1,\cdots,k\}$. For any rank k matrix $Z \in M_{st}$, let $B \in GL(k,\mathbb{C})$ be a nonsingular matrix with $BZ \in \Phi^{-1}(I)$. Thus the stability function at point Z is

$$\begin{aligned}\psi(Z) &= \log\left(|\det(Z_{1,\cdots,k})|^2 e^{-\operatorname{Tr} ZZ^*}\right) - \log\left(|\det((BZ)_{1,\cdots,k})|^2 e^{-\operatorname{Tr} I}\right)\\ &= k - \operatorname{Tr}(ZZ^*) - \log|\det B|^2\end{aligned}$$

Since $B^*B = (Z^*)^{-1}Z^{-1}$, we conclude

$$\psi(Z) = k - \operatorname{Tr}(ZZ^*) + \log\det(ZZ^*). \tag{10.1}$$

Similarly, if we do reduction at mI instead of I, or alternately, use the moment map $\Phi - mI$, then the invariant sections are given by $s_J(Z) = \det(Z_J)^m$, and the stability function is

$$\psi(Z) = km - \operatorname{Tr}(ZZ^*) + m^2\log\det(ZZ^*).$$

10.2. **The stability functions on $\mathcal{U}(n)$-coadjoint orbits.** These stability functions are computed in more or less the same way as above. By the same arguments as in the proof of lemma 10.1, one can see that

$$s(Z_1,\cdots,Z_{n-1}) = \prod(\det(Z_i)_{1,\cdots,i})^{m_i-m_{i-1}} \tag{10.2}$$

is G-invariant for the moment map $\Phi - (m_1I_1,\cdots,m_{n-1}I_{n-1})$.

Now suppose $(Z_1,\cdots,Z_{n-1}) \in M_{st}$, then there are $B_i \in GL(i,\mathbb{C})$ such that

$$B_1Z_1Z_1^*B_1^* = m_1I_1 \tag{10.3}$$

and

$$B_iZ_iZ_i^*B_i^* = Z_{i-1}^*B_{i-1}^*B_{i-1}Z_{i-1} + m_iI_i, \qquad 2 \le i \le n-1. \tag{10.4}$$

From (10.3) we have

$$\det(B_iB_1^*) = m_1\det(Z_1Z_1^*)^{-1},$$

and from this and (10.4) we conclude

$$\begin{aligned}\det(B_iZ_iZ_i^*B_i^*) &= \det(m_iI_i + B_{i-1}Z_{i-1}Z_{i-1}^*B_{i-1}^*)\\ &= \det((m_i+m_{i-1})I_{i-1} + B_{i-2}Z_{i-2}Z_{i-2}^*B_{i-2}^*)\\ &= m_1 + \cdots + m_i.\end{aligned}$$

So we get for all i,

$$\det(B_iB_i^*) = (m_1 + \cdots + m_i)\det(Z_iZ_i^*)^{-1}.$$

Now it is easy to compute

$$\begin{aligned}\psi(Z) = \log&\left(e^{-\sum \mathrm{Tr}(Z_iZ_i^*)}\prod|\det(Z_i)_{1,\cdots,i}|^{2m_i-2m_{i-1}}\right)\\ &- \log\left(e^{-\sum im_i}\prod|\det(B_iZ_i)_{1,\cdots,i}|^{2m_i-2m_{i-1}}\right)\\ &= \sum im_i - \sum \mathrm{Tr}(Z_iZ_i^*) - \sum(m_i - m_{i-1})\log|\det B_i|^2\\ &= \sum im_i - \sum \mathrm{Tr}(Z_iZ_i^*) + \sum(m_i - m_{i-1})(m_1+\cdots+m_i)\log\det(Z_iZ_i^*).\end{aligned}$$

Remark 10.2. Although we only carry out the computations for generic $\mathcal{U}(n)$-coadjoint orbits, i.e., for the isospectral sets with $\lambda_1 < \cdots < \lambda_n$, the same argument apply to all $\mathcal{U}(n)$-coadjoint orbits. In fact, for the isospectral set with $\lambda_1 < \cdots < \lambda_r$ whose multiplicities are $i_1, \cdots, i_r$, we can take the upstairs space to be

$$\mathfrak{M}_{i_1\times(i_1+i_2)} \times \mathfrak{M}_{(i_1+i_2)\times(i_1+i_2+i_3)} \times \mathfrak{M}_{(n-i_r)\times n}$$

and obtain results for these degenerate coadjoint orbits completely analogous to those above.

11. Stability functions on quiver varieties

It turns out that the results above can be generalized to a much larger class of manifolds: quiver varieties. We will give a brief account of this below.

11.1. **Quiver Varieties.** Let's first recall some notations from quiver algebra theory. A *quiver* Q is an oriented graph (I, E), where $I = \{1, 2, \cdots, n\}$ is the set of vertices, and $E \subset I \times I$ the set of edges. A representation, V, of a quiver assigns a Hermitian vector space V_i to each vertex i of the quiver and a linear map $Z_{ij} \in \mathrm{Hom}(V_i, V_j)$ to each edge $(i, j) \in E$. The dimension vector of the quiver representation V is the vector $l = (l_1, \cdots, l_n)$, where $l_i = \dim V_i$. Thus the space of representations of Q with underlying vector spaces V fixed is the complex space

$$M = \mathrm{Hom}(V) := \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j). \tag{11.1}$$

We equip M with its standard symplectic form and consider the unitary group $U(V) = U(V_1) \times \cdots \times U(V_n)$ acting on M by

$$(u_1, \cdots, u_n) \cdot (Z_{ij}) = (u_j Z_{ij} u_i^{-1}). \tag{11.2}$$

The isomorphism classes of representations of Q of dimension l is in bijection with the $GL(V)$-orbits on $\mathrm{Hom}(V)$. Geometrically this quotient space can have bad singularities, and to avoid this problem, one replaces this quotient by its GIT quotient, or equivalently, the Kähler quotient of $\mathrm{Hom}(V)$ by the $U(V)$-action. These quotients are what one calls *quiver varieties.*

Proposition 11.1. *The action (11.2) is Hamiltonian with moment map $\mu : Hom(V) \to \mathfrak{g}^*$,*

(11.3)

$$\mu(Z_{ij}) = \left(\sum_{(j,1)\in E} Z_{j1} Z_{j1}^* - \sum_{(1,j)\in E} Z_{1j}^* Z_{1j}, \cdots, \sum_{(j,n)\in E} Z_{jn} Z_{jn}^* - \sum_{(n,j)\in E} Z_{nj}^* Z_{nj} \right).$$

The proof involves the same computation as in lemma 9.1, so we will omit it.

Notice that by (11.2) the circle group $\{(e^{i\theta} I_{l_1}, \cdots, e^{i\theta} I_{l_n})\}$ act trivially on M, so we get an induced action of the quotient group $G = U(V)/S^1$. The Lie algebra of G is given by

$$\{(H_1, \cdots, H_n) \mid H_i \text{ Hermitian }, \sum \mathrm{Tr}\, H_i = 0\}$$

and this G-action also has μ as its moment map. Letting $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with

$$l_1\lambda_1 + \cdots + l_n\lambda_n = 0,$$

and supposing that the G-action is free on $\mu^{-1}(\lambda I)$, the *quiver variety* associated to λ is by definition the quotient

$$R_\lambda(l) = \mu^{-1}(\lambda I)/G,$$

where $\lambda I = (\lambda_1 I_{l_1}, \cdots, \lambda_n I_{l_n})$.

We can also modify the definition of quiver varieties to get an effective $U(V)$-action. Namely, we attach to Q another collection of Hermitian vector spaces (the "frame"), $\tilde{V} = (\tilde{V}_1, \cdots, \tilde{V}_n)$, with dimension vector $\tilde{l} = (\tilde{l}_1, \cdots, \tilde{l}_n)$, and redefine the space M to be

$$\mathrm{Hom}(V, \tilde{V}) := \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i\in I} \mathrm{Hom}(V_i, \tilde{V}_i).$$

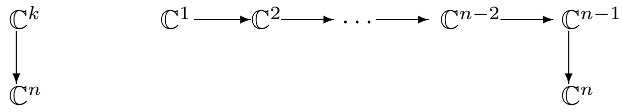
The group $U(V)$ acts on $\mathrm{Hom}(V, \tilde{V})$ by

$$(u_1, \cdots, u_n) \cdot (Z_{ij}, Y_i) = (u_j Z_{ij} u_i^{-1}, Y_i u_i^{-1}).$$

As above the $U(V)$-action is Hamiltonian, and the k^{th} component of its moment map is

$$(\mu(Z_{ij}, Y_i))_k = \sum_{(j,k)\in E} Z_{jk} Z_{jk}^* - \sum_{(k,j)\in E} Z_{kj}^* Z_{kj} - Y_k^* Y_k.$$

Now the center S^1 acts nontrivially on $\mathrm{Hom}(V, \tilde{V})$ providing that the "frames" $\tilde{V}_i$ are not all zero, and we define the *framed quiver variety* $R_\lambda(l, \tilde{l})$ to be the Kähler quotient of $\mathrm{Hom}(V, \tilde{V})$ by the $U(V)$-action above at the level $\lambda = (\lambda_1 I_{l_1}, \cdots, \lambda_n I_{l_n})$. As examples, the Grassmannian and the coadjoint orbit of $\mathcal{U}(n)$ that we considered in the previous section are just the framed quiver varieties whose underlying quivers are depicted below:



11.2. **Stability functions.** As in §10 we equip M with the trivial line bundle and, for actions of $\mathcal{U}(V)$ associated with characters $\prod(\det A_i)^{\lambda_i}$, describe the invariant sections.

Proposition 11.2. *For fixed $\lambda \in \mathbb{Z}^n$, the sections*

$$s(Z_{ij}) = \prod_{(i,j)\in E} \det((Z_{ij})_J)^{\nu_{ij}} \tag{11.4}$$

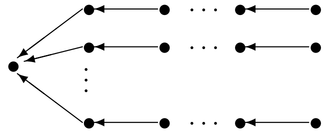
are invariant sections with respect to the moment map $\mu - \lambda I$, where ν_{ij} are integers satisfying

$$\sum_j \nu_{ji} - \sum_j \nu_{ij} = \lambda_i. \tag{11.5}$$

The proof is essentially the same proof as that of Lemma 10.1.

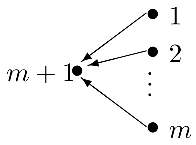
From now on we will require that the quiver,Q, be noncyclic, otherwise there will be infinitely many G-invariant sections. (Moreover, in the cyclic case the quiver variety is not compact.) For a general quiver variety whose underlying quiver is noncyclic, we can, in principle, compute the stability function, using the G-invariant sections above, as we did for toric varieties in §7; but in practice the computation can be quite complicated.

However, in the special case that the quiver is a star quiver, i.e., is of the following shape:



one can write down the stability functions fairly explicitly: on each "arm", we just apply the same technique we used for the coadjoint orbits of $\mathcal{U}(n)$.

As an example, we'll compute the stability function for *polygon space*. This is by definition a quiver variety whose underlying quiver is the oriented graph



and for which the V_i's satisfy $\dim V_i = 1$ for $1 \leq i \leq m$ and $\dim V_{m+1} = 2$. Thus

$$\mathrm{Hom}(V) = \bigoplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) = (\mathbb{C}^2)^m \tag{11.6}$$

and

$$G = (S^1)^m \times U(2)/S^1 \simeq (S^1)^m \times SO(3). \tag{11.7}$$

The moment map for this data is

$$(Z_1, \cdots, Z_m) \mapsto (-|Z_1|^2, \cdots, -|Z_m|^2, Z_1 Z_1^* + \cdots + Z_m Z_m^*), \tag{11.8}$$

where $Z_i = (x_i, y_i) \in \mathbb{C}^2$.

Now consider the quiver variety $\mu^{-1}(\lambda I)/G$, with $\lambda = (\lambda_1, \cdots, \lambda_m, \lambda_{m+1})$ satisfying

$$\lambda_1 + \cdots + \lambda_m + 2\lambda_{m+1} = 0$$

and $\lambda_i < 0$ for $1 \leq i \leq m$. Let's explain why this variety is called "polygon space". The $(S^1)^m$-action on $(\mathbb{C}^2)^m$ is the standard action, so reducing at level $(\lambda_1, \cdots, \lambda_m)$ gives us a product of spheres $S^2_{-\lambda_1} \times \cdots \times S^2_{-\lambda_m}$ of radii $-\lambda_1, \cdots, -\lambda_m$. So we can think of an element of $S^2_{-\lambda_1} \times \cdots \times S^2_{-\lambda_m}$ as a polygon path in $\mathbb{R}^3$ whose i^{th} edge is a vector of length $-\lambda_i$ in $S^2_{-\lambda_i}$. The $SO(3)$-action on this product of spheres is the standard diagonal action, and the moment map sums up the points, i.e. takes as its value the endpoint of the polygon path. However, under the identification (11.7), the Lie algebra of $SO(3)$ gets identified with $\mathcal{H}(2)/\{aI_2\}$. Thus the fact that the last entry of the moment map (11.8) equals $\lambda_{m+1}I_2$ implies that this endpoint is the origin in the Lie algebra of $SO(3)$. In other words, our polygon path is a polygon. So the quiver variety $R_\lambda(1, \cdots, 1, 2)$ is just the space of all polygons in $\mathbb{R}^3$ whose sides are of length $-\lambda_1, \cdots, -\lambda_m$, up to rotation.

Using the invariant section $s(Z) = \prod_{i=1}^m x_i^{-\lambda_i}$ to compute the stability function for this space we have

$$\begin{aligned}\psi(Z) &= -\sum(|x_i|^2 + |y_i|^2) + \sum(-\lambda_i)\log|x_i|^2 + \sum(-\lambda_i) - \sum(-\lambda_i)\log\frac{-\lambda_i|x_i|^2}{|x_i|^2+|y_i|^2} \\ &= 2\lambda_{m+1} - |Z|^2 + \sum \lambda_i \log\frac{-\lambda_i}{|Z_i|^2}.\end{aligned}$$

Finally we point out that everything we said above applies to framed quiver varieties, in which case the $U(V)$-action is free on $\Phi^{-1}(\lambda I)$. The coadjoint orbits of §10 are just special cases of quiver varieties of this type.

APPENDIX

In this appendix we will give a proof of Boutet de Monvel-Guillemin's theorem on the asymptotics of the density of states, (1.20), adapted to the Toeplitz operator setting.

We will begin with a very brief account on the definition of Toeplitz operators. Let W be a compact strictly pseudoconvex domain with smooth boundary ∂W. One defines the space of Hardy functions, H^2, to be the L^2-closure of the space of C^∞ functions on ∂W which can be extended to holomorphic functions on W. The orthogonal projection $\pi : L^2 \to H^2$ is called the Szegö projector, and an operator $T : C^\infty(\partial W) \to C^\infty(\partial W)$ is called a Toeplitz operator if it can be written in the form

$$T = \pi P \pi$$

for some pseudodifferential operator P on ∂W.

Now suppose $(\mathbb{L}, \langle\cdot,\cdot\rangle)$ is a Hermitian line bundle over a compact Kähler manifold X. Let

$$D = \{(x, v) \in \mathbb{L}^* \mid v \in \mathbb{L}^*_x, \|v\| \leq 1\}$$

be the disc bundle in the dual bundle. As observed by Grauert, D is a strictly pseudoconvex domain in $\mathbb{L}$. The manifold we are interested in is its boundary,

$$M = \partial D = \{(x, v) \in \mathbb{L}^* \mid v \in \mathbb{L}^*_x, \|v\| = 1\},$$

the unit circle bundle in the dual bundle. Let Q be the operator

$$Q : H^2 \to H^2,\ Qf(x, v) = \sqrt{-1}\, \frac{\partial}{\partial\theta} f(x, e^{i\theta} v)\Big|_{\theta=0}.$$

This is a first order elliptic operator in the Toeplitz sense and is a Zoll operator (meaning that its spectrum only consists of positive integers). Moreover, the n^{th} eigenspace of Q coincides with $\Gamma_{hol}(\mathbb{L}^n, X)$. For any smooth function $f \in C^\infty(X)$, let M_f be the operator "multiplication by f". We may view $\Gamma_{hol}(\mathbb{L}^n, X)$ as a subspace of H^2, and denote by

$$\pi_n : L^2(\mathbb{L}^n, X) \to \Gamma_{hol}(\mathbb{L}^n, X)$$

the orthogonal projection.

Theorem A. *There is an asymptotic expansion*

$$\mathrm{Tr}(\pi_n M_f \pi_n) \sim \sum_{k=d-1}^{-\infty} a_k(f) n^k, \quad n \to \infty,$$

where $d = \dim X$.

Proof. By functorial properties of Toeplitz operators (c.f. [BoG] §13),

$$\mathrm{Tr}(e^{itQ}M_f) \sim \sum a_k \chi_k(t)$$

where

$$\chi_k(t) = \sum_{n>0} n^k e^{int}.$$

On the other hand,

$$\mathrm{Tr}(e^{itQ}M_f) = \sum e^{int}\,\mathrm{Tr}\,\pi_n M_f \pi_n.$$

By comparing the coefficient of e^{int}, we get the theorem. □

Finally we point out that the coefficients a_k in the asymptotic expansion above are given by the noncommutative residue trace on the algebra of Toeplitz operators, [Gui93]. In fact, for $\Re(z) \gg 0$, theorem A gives

$$\mathrm{Tr}(Q^{-z}\pi_n M_f \pi_n) \sim \sum_{k=d-1}^{-\infty} a_k n^{k-z}.$$

Summing over n,

$$\mathrm{Tr}(Q^{-z}M_f) \sim \sum_k a_k \zeta(z-k),$$

where ζ is the classical zeta function, which implies

$$a_{k-1} = \mathrm{res}_{z=k}(Q^{-z}M_f),$$

the noncommutative residue.

References

[Ati] M. Atiyah, "Convexity and commuting Hamiltonians," *Bull. Lond. Math. Soc.* 14 (1981), 1-15.

[BPU] D. Borthwick, T. Paul and A. Uribe, "Legendrian distributions with applications to relative Poincaré series", *Invent. Math.* 122 (1995) , 359-402.

[BoG] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators,* Annals of Math. Studies 99, Princeton U. Press, Princeton, NJ, 1981.

[BuG] D. Burns and V. Guillemin, "Potential functions and actions of tori on Kähler manifolds", *Com. Anal. Geom.* 12 (2004), 281-303.

[BGL] D. Burns, V. Guillemin and E. Lerman, "Kähler cuts", math.DG/0212062.

[BGU] D. Burns, V. Guillemin and A. Uribe, "The spectral density function of a toric variety", math.SP/0706.3039.

[Can] A. Canas da Silva, "Symplectic Toric Manifolds," in *Symplectic Geometry of Integrable Hamiltonian Systems*, Birkhauser, 2003.

[Cha03] L. Charles, "Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators", *Comm. Partial Differential Equations* 28, (2003), no. 9-10, p. 1527-1566.

[Cha06] L. Charles, "Toeplitz operators and Hamiltonian Torus Action", *Jour. Func. Anal.* 236 (2006), 299-350.

[Col] Y. Colin de Verdiere, "Sur le spectre des oprateurs elliptiques a bicaracteristiques toutes peridiques", *Comment. Math. Helv.* 54 (1979), 508-522.

[DeP] M. Debernardi and R. Paoletti, "Equivariant asymptotics for Bohr-Sommerfeld Lagrangian submanifolds", *Comm. Math. Phys.* 267 (2006), no. 1, 227-263.

[Del] T. Delzant, "Hamiltoniens périodiques et images convexes de l'application moment", *Bull. Soc. Math. France* 116 (1988), 315-339.

[DuP] J. Duistermaat and A. Pelayo, " Reduced phase space and toric variety coordinatizations of Delzant spaces", math.SG/0704.0430.

[Gui93] V. Guillemin, "Residue Traces for Certain Algebras of Fourior Integral Operators", *Jour. Func. Ana.* 115 (1993), 391-417.

[Gui94a] V. Guillemin, "Kähler structures on toric varieties," *Jour. Diff. Geom.* 40 (1994), 285-309.

[Gui94b] V. Guillemin, *Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces,* Progress in Mathematics 122, Birkhauser Boston, Boston, MA, 1994.

[GuO] V. Guillemin and K. Okikiolu, "Spectral Asymptotics of Toeplitz Operators on Zoll Manifolds", *Jour. Func. Anal.* 146 (1997), 496-516.

[GuS82] V. Guillein and S. Sternberg, "Geometric quantization and multiplicities of group representations", *Invent. Math.* 67, 515-538, 1982.

[GuS06] V. Guillemin and S. Sternberg, "Riemann Sums over Polytopes", *Ann. Inst. Fourier* 57 (2007), 2183-2195.

[GuU] V. Guillemin and A. Uribe, "The Laplace operator on the nth tensor power of a line bundle: eigenvalues which are bounded uniformly in n", *Asymptotic Anal.* 1 (1988), 105-113.

[GuW] V. Guillemin and Z. Wang, "The Mellin transform and spectral properties of toric varieties", math.SG/0706.3696, to appear in *Transformation Groups*.

[HaK] B. Hall and W. Kirwin, "Unitarity in 'quantization commutes with reduction' ", *Comm. Math. Phys.* 275 (2007), 401-442.

[HKn] J-C. Hausmann and A. Knutson, "Polygon Spaces and Grassmannians", *L'Enseignement Mathematique*, 43 (1997), 173-198.

[Kam] J. Kamnitzer, "Quiver Varieties from a Symplectic Viewpoint", 2003.

[KaM] M. Kapovich and J. Millson, "The Symplectic Geometry of Polygons in Euclidean Space", *Journal of Diff. Geometry*, Vol. 44 (1996) p. 479-513.

[KeN] G. Kempf and L. Ness, "The length of vectors in representation spaces", in *Algebraic Geometry – Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978*, Lecture notes in Math. 732, 233-243, Springer, Berlin, 1979.

[Kna] A. Knapp, *Lie Groups: Beyond an introduction,* 2nd Edition. Progress in Mathematics 140. Birkhauser 2002.

[KoM] M. Kogan and E. Miller, "Toric degeneration of Schubert varieties and Gelfand-Tsetlin polytopes", Advances in Mathematics 193 (2005), no. 1, 1-17.

[Kos] B. Kostant, "Quantization and unitary representation", in *Lectures in Mordern Analysis and Applications III*, Lecture notes in Math. 170, 87-207, Springer, New York, 1970.

[Li] H. Li, "Singular unitarity in quantization commutes with reduction", math.SG/0706.1471v1.

[Mar] S. Martin, personal communication, 1998.

[MFK] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, Third Edition, Springer-Verlag, Birlin, 1994.

[Nak] H. Nakajima, "Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras", Duke Math. J. 76 (1994), 365-416.

[Pa1] R. Paoletti, "Moment maps and equivariant Szegö kernels", *J. Symplectic Geom.* 2 (2003), no. 1, 133-175.

[Pa2] R. Paoletti, “Scaling limits for equivariant Szegö kernels,” *J. Symplectic Geom.* 6 (2008), no. 1, 9-32.

[Pa3] R. Paoletti, “A note on scaling asymptotics for Bohr-Sommerfeld Lagrangrian submanifolds”, math.SG/0709.3395.

[STZ] B. Shiffman, T. Tate and S. Zelditch, “Distribution laws for integrable eigenfunctions”, Ann. Inst. Fourier (Grenoble), 54 (2004), no. 5, 1497-1546.

[SoZ] J. Song and S. Zelditch, “Bergman metrics and geodesics in the space of Kähler metrics on toric varieties”, math.CV/0707.3082.

[Tho] R. Thomas, “Notes on GIT and symplectic reduction for bundles and varieties”, in *Surveys in Differential Geometry X*, 221-273.

[Wan] Z. Wang, “The Twisted Mellin Transform”, math.CO/0706.2642.

[Won] R. Wong, *Asymptotic Approximations of Integrals,* Academic Press, Boston, 1989.

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