Topics in Double Field Theory

by

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Abstract

The existence of momentum and winding modes of closed string on a torus leads to a natural idea that the field theoretical approach of string theory should involve winding type coordinates as well as the usual space-time coordinates. Recently developed double field theory is motivated from this idea and it implements T-duality manifestly by doubling the coordinates. In this thesis we will mainly focus on the double field theory formulation of different string theories in its low energy limit: bosonic, heterotic, type II and its massive extensions, and $\mathcal{N} = 1$ supergravity theory.

In chapter 2 of the thesis we study the equivalence of different formulations of double field theory. There are three different formulations of double field theory: background field $\mathcal{E}$ formulation, generalized metric $\mathcal{H}$ formulation, and frame field $E_A{}^M$ formulation. Starting from the frame field formalism and choosing an appropriate gauge, the equivalence of the three formulations of bosonic theory are explicitly verified. In chapter 3 we construct the double field theory formulation of heterotic strings. The global symmetry enlarges to $O(D, D+n)$ for heterotic strings and the enlarged generalized metric features this symmetry. The structural form of bosonic theory can directly be applied to the heterotic theory with the enlarged generalized metric. In chapter 4 we develop a unified framework of double field theory for type II theories. The Ramond-Ramond potentials fit into spinor representations of the duality group $O(D, D)$ and the theory displays $\mathrm{Spin}^+(D, D)$ symmetry with its self-duality relation. For a specific form of RR 1-form the theory reduces to the massive deformation of type IIA theory due to Romans. In chapter 5 we formulate the $\mathcal{N} = 1$ supersymmetric extension of double field theory including the coupling to n abelian vector multiplets. This theory features a local $O(1, 9+n) \times O(1, 9)$ tangent space symmetry under which the fermions transform.

Thesis Supervisor: Barton Zwiebach
Title: Professor

Dedication

To My Loving Parents and Fiancée

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Contents

Chapter 1

Introduction

String theory is one of the most exciting fields in theoretical physics [1]. In this theory all elementary particles are treated as one dimensional objects, strings, rather than points as in quantum field theory [2]. String theory is best suited for physics of very short distances since it does not display the short-distance divergences of quantum field theory. This fact makes it possible for string theory to be a consistent theory of quantum gravity, something that quantum field theory failed to achieve. Most of all, string theory is an excellent candidate for a unified theory of all forces in nature and it is believed by many physicists to be the strongest candidate available. Hence, understanding string theory would improve our knowledge about the principles of nature that go beyond the Standard Model of particle physics.

The term duality is often used in physics quite generally. It is used when seemingly different physical systems are in fact equivalent [2]. One of the most famous examples is particle-wave duality in quantum mechanics. Another example is a duality in the Ising model that the physics of a spin system at temperature T is identical to that at inverse temperature $1/T$. In string theory the term duality is used for a class of symmetries in physics that link different string theories. There are a few different types of dualities in string theory: T-duality, S-duality, U-duality, etc. These dualities relate different types of string theory (e.g. type I and $SO(32)$ heterotic theory under S-duality) or theories in different backgrounds.

T-duality [3] is an old but still very intriguing duality in string theory. The 'T' in T-duality stands for 'toroidal' as it comes from the toroidal compactification of closed string theory. The most well known example of this duality is 'one-circle inversion' in bosonic string theory, which relates toroidal backgrounds T^d with radius R to backgrounds with radius α'/R in one of the compact directions. The physics of closed string theory for a compactification with two different radii R and α'/R are in fact indistinguishable. This implies that a compactification with extremely large radius is equivalent to a compactification with extremely small radius in closed string theory, which is quite striking. The idea of T-duality can be extended to 10-dimensional superstring theories. It is another famous example that T-duality relates type IIA and IIB theories when compactified on a circle. The one-circle inversion maps the type IIA theory on the background $\mathbb{R}^{8,1} \times S^1$ of radius R to the type IIB theory on the same background but with radius α'/R.

T-duality arises from the existence of momentum and winding modes of closed string in toroidal compactification. If we consider the physics of a particle with one of the spatial dimensions compactified, then the momentum of the particle gets quantized. In closed string theory additional winding states arise from the compactification of a spatial dimension as strings can wrap around the compactified dimension. Hence, T-duality is the symmetry of closed string that mixes momentum and winding modes in such a way that the physics stays invariant. In the example given above, 'one circle inversion', two different backgrounds define an equivalent theory with the exchange of momentum and winding excitations. In general, T-duality acts linearly on momentum and winding modes of closed string via the non-compact duality group $O(d,d;\mathbb{Z})$. Compared to this linear action of T-duality on momentum and winding modes, the transformation of backgrounds takes a complicated nonlinear form known as 'Buscher rules'.

It is interesting that T-duality leads to deviations from our usual intuition about geometry. As winding excitations are dual modes of momentum excitations under T-duality, it is a quite natural idea that the geometrical understanding of string theory should involve not only the usual space-time coordinates x^a but also winding-type coordinates $\tilde{x}_a$. Such a scenario is already realized in closed string field theory and consequently a space-time action can be determined in a way implementing this idea, at least perturbatively. This motivation is the starting point of the recently developed 'double field theory', where the fields depend both on space-time coordinates x^a and winding coordinates $\tilde{x}_a$. In double field theory T-duality is implemented as an explicit $O(d,d;\mathbb{Z})$ symmetry acting linearly on the torus coordinates x^a and $\tilde{x}_a$. In terms of action this can be written as

$$S = \int dx^a d\tilde{x}_a dx^\mu \mathcal{L}(x^a, \tilde{x}_a, x^\mu) \ , \tag{1.1}$$

where x^a are compact coordinates, $\tilde{x}_a$ are their dual coordinates, and x^μ are non-compact coordinates. The action is explicitly invariant under the $O(d,d;\mathbb{Z})$ symmetry. In fact if the coordinates are non-compact, then the symmetry of double field theory enlarges to $O(d,d;\mathbb{R})$.

The work of Tseytlin [4] is an early paper which takes this idea seriously, where a first-quantized approach is used with non-covariant actions for left and right-moving string coordinates on the torus. A few years later, Siegel [5] introduced a duality-covariant geometrical formalism using a frame-field with a local $GL(D) \times GL(D)$ symmetry. Recently Hull and Zwiebach [6] constructed double field theory from the closed string field theory, up to a cubic order in fields. Afterwards, background independent action of double field theory was introduced by Hohm, Hull, and Zwiebach [7] and the same authors also developed the generalized metric formulation of the theory [8]. There are many papers following these original works, including [9–15].

In the rest of the introduction chapter I will sketch briefly the background material needed to understand some basic features of double field theory, including T-duality. Then I review the work of Hull and Zwiebach [6, 16] and Hohm, Hull, and Zwiebach [7, 8]. Most of my works are based on these papers and it is fruitful to summarize important results of them before introducing the main results of [10–15]. Finally, I will summarize [10–15] and then finish with some concluding remarks.

1.1 Background material

Below we start with the world-sheet action to clarify the idea of T-duality. It should be emphasized that the T-duality is not an actual symmetry of the world-sheet theory, but rather an equivalence of conformal field theories. It is a duality as the $O(d,d;\mathbb{Z})$ duality transformation changes the background structure rather than the physical fields in the world-sheet theory. However, as a part of the gauge group in string theory, T-duality is the symmetry of string theory under which the physics is invariant. The space-time theory should not be confused with the world-sheet theory.

The world-sheet action from the first-quantized string theory is given by

$$S = -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int_{-\infty}^{\infty} d\tau \left(\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}\right) , \tag{1.2}$$

where

$$\eta^{\alpha\beta} = \mathrm{diag}(-1,1), \quad \epsilon^{01} = -1, \quad \partial_\alpha = (\partial_\tau, \partial_\sigma), \tag{1.3}$$

$$X^i = (X^a, X^\mu), \quad X^a \sim X^a + 2\pi, \quad i = 0, \cdots, D-1 . \tag{1.4}$$

The X^a are coordinates for compact dimensions and D is the total number of dimensions. The closed string background fields G and B are

$$G_{ij} = \begin{pmatrix} \hat{G}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} , \quad B_{ij} = \begin{pmatrix} \hat{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix} , \tag{1.5}$$

where these backgrounds are constant $D \times D$ matrices. The choice of flat world-sheet metric $\eta_{\alpha\beta}$ is possible since the theory has no Weyl anomaly. We define the matrix $E = G + B$ to arrange the data of G and B in a single matrix.

As the action is given, the usual procedures lead us to the canonical momentum P_i associated with X^i:

$$2\pi P_i = G_{ij}\dot{X}^j + B_{ij}X'^j , \tag{1.6}$$

and the Hamiltonian is

$$H = \frac{1}{4\pi}\int_0^{2\pi} d\sigma \begin{pmatrix} X' & 2\pi P \end{pmatrix} \mathcal{H}(E) \begin{pmatrix} X' \\ 2\pi P \end{pmatrix} , \tag{1.7}$$

where the dot is used for ∂_τ and the prime is used for ∂_σ. The $2D \times 2D$ symmetric matrix $\mathcal{H}(E)$ is called the 'generalized metric' and takes the form

$$\mathcal{H}(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} . \tag{1.8}$$

This object transforms linearly under T-duality transformation, or $O(d,d;\mathbb{Z})$ transformation, and it is an element of the $O(d,d;\mathbb{R})$ group.

Along with the Hamiltonian and its spectrum, there is a constraint in closed string theory which is called 'level-matching condition'. This constraint matches the levels of the right and the left moving excitations in any physical state. In bosonic string theory it can be written in terms of Virasoro operators as

$$L_0 - \bar{L}_0 \;=\; 0 \,. \tag{1.9}$$

To further analyze the spectrum of the Hamiltonian and the level-matching condition, it is convenient to take a mode expansion of string coordinate X^i and the canonical momentum P_i in terms of momenta, winding, and oscillators. The explicit steps are not presented here and we refer [3] for details. After integration and normal ordering the Hamiltonian and the level-matching condition reads, respectively,

$$H = \frac{1}{2} Z^t \mathcal{H}(E) Z + N + \bar{N} \,, \tag{1.10}$$

$$N - \bar{N} = \frac{1}{2} Z^t \eta Z \,, \tag{1.11}$$

where N and $\bar{N}$ are number operators counting the excitations. The $2D$ column vector Z is

$$Z = \begin{pmatrix} w^i \\ p_i \end{pmatrix} , \tag{1.12}$$

which consists of integer winding and momentum quantum numbers. The metric η is an off-diagonal $2D \times 2D$ matrix of the form

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{1.13}$$

Consider a linear transformation of the vector Z with some $2D \times 2D$ invertible matrix h

$$Z = h^t Z' , \qquad Z' = (h^{-1})^t Z . \tag{1.14}$$

For this transformation to define the equivalent theory, the Hamiltonian should remain the same and the level-matching condition should hold under the transformation. Under this transformation the number operators N and $\bar{N}$ are invariant. Then the level-matching condition leads to

$$Z^t \eta Z = Z'^t h \eta h^t Z' = Z'^t \eta Z' , \tag{1.15}$$

and this requires the matrix h to be an element of $O(d, d; \mathbb{Z})$ group, which is defined by

$$h \eta h^t = \eta , \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d, d; \mathbb{Z}) , \tag{1.16}$$

where the element of the matrix h are constrained to be integers. If the entries are taken to be real numbers, the group becomes $O(d, d; \mathbb{R})$. The level-matching condition requires the vector of winding and momentum numbers to transform under $O(d, d; \mathbb{Z})$. It is worth noting that the generalized metric (1.8) satisfies an identity

$$\eta \mathcal{H} \eta = \mathcal{H}^{-1} , \tag{1.17}$$

since $\mathcal{H}$ is a symmetric element of $O(d, d; \mathbb{R})$.

The invariance of the Hamiltonian under the transformation h requires $h \in O(d,d;\mathbb{Z})$. There are well known Buscher rules that apprear often in string theory. According to these rules, under $h \in O(d,d;\mathbb{Z})$ transformation a background field E transforms to E' as

$$E \longrightarrow E' = h(E) = (aE+b)(cE+d)^{-1} \, . \tag{1.18}$$

The generalized metric $\mathcal{H}$ we introduced above is written in terms of background fields G and B, which are symmetric and antisymmetric part of E. Hence, the transformation of a background field E also leads to the transformation of the generalized metric. Extracting the transformation of the generalized metric from the transformation of background fields is not straightforward, but the result is quite simple,

$$\mathcal{H}(E) \longrightarrow \mathcal{H}(E') = h\mathcal{H}(E)h^t \, . \tag{1.19}$$

Then it is straightforward that the Hamiltonian is invariant under the $O(d,d;\mathbb{Z})$ transformation.

Thus far we briefly skimmed how T-duality, or an $O(d,d;\mathbb{Z})$ group duality, emerges from the closed string world-sheet action. We also introduced many useful objects that are used later in the thesis, for example, a background field E and generalized metric $\mathcal{H}$. In double field theory these fields depend on both spacetime coordinates x^i and winding coordinates $\tilde{x}_i$. The $O(d,d;\mathbb{Z})$ transformation acts linearly on these coordinates as it acts linearly on momentum and winding quantum numbers. In matrix form this transformation is written as

$$X \longrightarrow X' = hX \, , \quad X \equiv \begin{pmatrix} \tilde{x} \\ x \end{pmatrix} \, , \tag{1.20}$$

where X is a $2D$ column vector as Z.

The $O(d,d;\mathbb{Z})$ group has three types of generators: $GL(d;\mathbb{Z})$, b-shifts, and factorized dualities (or one circle inversions). Each of the generators has the matrix form, respectively,

$$h_r = \begin{pmatrix} r & 0 \\ 0 & (r^t)^{-1} \end{pmatrix} , \qquad h_\theta = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} , \qquad h_i = \begin{pmatrix} 1 - e_i & e_i \\ e_i & 1 - e_i \end{pmatrix} , \tag{1.21}$$

where θ is an antisymmetric matrix and e_i is the matrix with zeros except on the (i,i) entry. The first two generators have determinant 1 while the determinant of one circle inversion is -1. Finally, if the inversion acts on all directions in compact dimensions, then it generates

$$h_J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{1.22}$$

From (1.20) this duality transformation exchanges all winding coordinates with all spacetime coordinates and hence exchanges the conjugate winding numbers and momentum. This duality generates a $\mathbb{Z}_2$ transformation and will be often used later.

Before proceeding it would be helpful to introduce some useful conventions and notations. Firstly, the double field theory notation covers both the non-compactified and compactified cases. If the double field theory is formulated in fully non-compact $\mathbb{R}^{2D}$ spacetime, the symmetry of the theory is $O(D,D)$ (or $O(D,D;\mathbb{R})$ to be exact). For spacetime $R^{n-1,1} \times T^d$, where $R^{n-1,1}$ is n-dimensional Minkowski space and T^d is a torus, the $O(D,D)$ symmetry breaks to $O(n-1,1) \times O(d,d;\mathbb{Z})$. For notational convenience I will refer to the symmetry of either case as $O(D,D)$ henceforth.

Secondly, indices are put on matrices and vectors introduced above. The generalized metric we defined in (1.8) is identified as

$$\mathcal{H} \leftrightarrow \mathcal{H}^{MN} , \quad \mathcal{H}^{-1} \leftrightarrow \mathcal{H}_{MN}. \tag{1.23}$$

Then the identity (1.17) can be written as

$$\mathcal{H}^{MN} \eta_{MP} \eta_{NQ} = \mathcal{H}_{PQ} . \tag{1.24}$$

The $O(D,D)$ group element h is identified as $h^M{}_N$ and the transformation of coordinates (1.20) under this transformation can be written as

$$X'^M = h^M{}_N X^N \,. \tag{1.25}$$

The indices $M, N, \cdots$ runs from 1 to $2D$ and can be though of as an $O(D, D)$ group indices. Thus if an action is only written in terms of contraction of objects with $O(D, D)$ indices, then the action is manifestly $O(D, D)$ invariant. This is the strength of generalized metric formulation of double field theory. If the double field theory is written with respect to the background fields E then $O(D, D)$ invariance is less manifest.

As a final remark, I would like to make a comment on constraints that double field theory has. The level-matching condition (1.9) is a well known constraint in string theory that a physical state is required to satisfy. For our particular truncation of fields where $N = \bar{N} = 1$ this condition leads to a constraint

$$\partial^M \partial_M A \;=\; \eta^{MN} \partial_M \partial_N A \;=\; 0\,, \qquad \eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.26}$$

for all fields and gauge parameters A. This constraint, which is a direct consequence of level-matching condition, is referred to as 'weak constraint' throughout the thesis. There is a more restrictive constraint so called 'strong constraint' used in many places. This constraint includes the weak constraint (1.26) and an additional condition

$$\partial^M A \, \partial_M B \;=\; 0\,, \tag{1.27}$$

for all fields and gauge parameters A and B. It can be proved that all products of fields and parameters are annihilated by $\partial^M \partial_M$ under the strong constraint. Geometrically, this constraint has a deeper implication that locally all fields depend only on half of the coordinates , e.g., only on the x^i or the $\tilde{x}_i$. The strong constraint can be interpreted as a stronger form of the level-matching condition. After solving the strong constraint by choosing some T-duality frame, double field theory can be related to generalized geometry, which involves doubling the tangent space by replacing the tangent bundle T with $T \oplus T^*$.

The weak constraint should be satisfied in all double field theory since this constraint has its root in the level-matching condition. However, the strong constraint does not have a direct interpretation in string theory as the weak constraint. Thus, the relaxation of this constraint is one of the goals of constructing 'true' double field theory. As these two constraints appear quite often, it would be helpful to keep track of what constraints are imposed on each paper.

1.2 Double Field Theory

In this section let us start by briefly introducing Hull and Zwiebach [6]. This paper initiated recent active research on double field theory along with the relevant work of generalized geometry. Hull and Zwiebach computed an $O(D, D)$ duality invariant action to cubic order in fluctuation, directly from closed string field theory. The field contents of the action are gravity field h_{ij}, antisymmetric tensor field b_{ij}, and dilaton d, which is a truncation of string theory to a massless subsector with $N = \bar{N} = 1$. These fields depend both on spacetime coordinates x^i and winding coordinates $\tilde{x}_i$. The action is gauge invariant under (an incomplete version of) double diffeomorphisms, which nonlinearly embed usual diffeomorphisms and antisymmetric tensor gauge transformations. When there is no $\tilde{x}$ dependence of fields, the action reduces to the linearised version of the standard Einstein-Kalb-Ramond-dilaton action

$$S = \int dx \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right] . \tag{1.28}$$

The weak constraint is imposed for the gauge invariance of the action to cubic order and the closure of the gauge algebra.

In Hull and Zwiebach [16], they imposed the strong constraint. Then they derived the full gauge transformation of background field e, which is $h+b$ in its leading order. If fields are restricted to the null space, i.e. the space where the strong constraint is satisfied, then the gauge algebra defines C-bracket which takes the form

$$([\xi_1,\xi_2]_C)^M = \xi_1^N\partial_N\xi_2^M - \xi_2^N\partial_N\xi_1^M - \frac{1}{2}\xi_1^N\partial^M\xi_{2N} + \frac{1}{2}\xi_2^N\partial^M\xi_{1N} \,, \tag{1.29}$$

where

$$\xi^M = \begin{pmatrix}\tilde{\xi}_i\\ \xi^i\end{pmatrix} \,, \qquad X^M = \begin{pmatrix}\tilde{x}_i\\ x^i\end{pmatrix} \,, \qquad \partial_M = \begin{pmatrix}\tilde{\partial}^i\\ \partial_i\end{pmatrix} \,. \tag{1.30}$$

In (1.29) the indices are $O(D,D)$ group indices and the gauge parameters ξ transform as vectors under T-duality. This ensures that the C-bracket is $O(D,D)$ covariant. When there is no $\tilde{x}_i$ dependence, this C-bracket reduces to the Courant bracket, which is a central construction in generalized geometry defined on smooth sections of $T\oplus T^*$. Neither the C-bracket nor the Courant bracket satisfies the Jacobi identity.

The previous two papers have a limitation that the theory has an explicit dependence on background fields $E = G + B$. The background independent action for double field theory was first introduced by Hohm, Hull, and Zwiebach [7]. The field content includes g_{ij}, the antisymmetric tensor b_{ij}, and a dilaton field d. The action is neatly written in terms of $\mathcal{E}_{ij} = g_{ij} + b_{ij}$ and a dilaton d as

$$S = \int dx d\tilde{x}\ \mathcal{L}(\mathcal{E}, d) \,, \tag{1.31}$$

where

$$\begin{aligned}\mathcal{L}(\mathcal{E},d) =& e^{-2d}\Big[-\frac{1}{4}g^{ik}g^{jl}\mathcal{D}^p\mathcal{E}_{kl}\mathcal{D}_p\mathcal{E}_{ij} + \frac{1}{4}g^{kl}(\mathcal{D}^j\mathcal{E}_{ik}\mathcal{D}^i\mathcal{E}_{jl} + \bar{\mathcal{D}}^j\mathcal{E}_{ki}\bar{\mathcal{D}}^i\mathcal{E}_{lj})\\ &+ (\mathcal{D}^i d\ \bar{\mathcal{D}}^j\mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\ \mathcal{D}^j\mathcal{E}_{ji}) + 4\mathcal{D}^i d\ \mathcal{D}_i d\Big] \,.\end{aligned} \tag{1.32}$$

The calligraphic derivatives are defined by

$$\mathcal{D}_i \equiv \partial_i - \mathcal{E}_{ik}\tilde{\partial}^k \,, \quad \bar{\mathcal{D}}_i \equiv \partial_i + \mathcal{E}_{ki}\tilde{\partial}^k. \tag{1.33}$$

These two derivatives originate from right/left factorization of closed string theory. In terms of constant background fields these are analogue of independent derivatives with respect to right- and left-moving coordinates $\tilde{x}_i - E_{ij}x^j$ and $\tilde{x}_i + E^t_{ij}x^j$, respectively. The action is invariant under the gauge transformation

$$\begin{aligned} \delta\mathcal{E}_{ij} &= \mathcal{D}_i\tilde{\xi}_j - \bar{\mathcal{D}}_j\tilde{\xi}_i + \xi^M\partial_M\mathcal{E}_{ij} + \mathcal{D}_i\xi^k\mathcal{E}_{kj} + \bar{\mathcal{D}}_j\xi^k\mathcal{E}_{ik} , \\ \delta d &= -\frac{1}{2}\partial_M\xi^M + \xi^M\partial_M d , \end{aligned} \tag{1.34}$$

where ξ^i is the gauge parameter of standard diffeomorphism and $\tilde{\xi}_i$ is the parameter of Kalb-Ramond gauge transformation. The gauge parameters form an $O(D,D)$ vector ξ^M

$$\xi^M \longrightarrow \xi'^M = h^M{}_N\xi^N , \qquad \xi^M = \begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix} , \qquad h \in O(D,D) . \tag{1.35}$$

Spacetime coordinates and winding coordinates form an $O(D,D)$ vector X^M and the derivatives ∂_M with respect to doubled coordinates transform as a contravariant vector

$$X^M \longrightarrow h^M{}_N X^N , \; \partial_M \longrightarrow h_M{}^N\partial_N , \qquad X^M = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} , \; \partial_M = \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix} , \tag{1.36}$$

where $h_M{}^N$ is the inverse of $h^M{}_N$. Using matrix notation,

$$X \longrightarrow hX , \; \partial \longrightarrow (h^{-1})^t\partial . \tag{1.37}$$

These $O(D,D)$ indices are lowered and raised with the $O(D,D)$ invariant metric η_{MN} and its inverse η^{MN}, where the matrix forms are the same for both. Under an $O(D,D)$ transformation $\mathcal{E}$ field transform as

$$\mathcal{E}(X) \rightarrow (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1} , \qquad h \in O(D,D) , \tag{1.38}$$

where a, b, c, and d are block matrices of h defined in (1.16). The dilaton stays invariant

$$d(X) \rightarrow d(X) , \tag{1.39}$$

which is an $O(D,D)$ scalar.

The gauge transformation of dilaton in the second equation of (1.34) is invariant under an $O(D,D)$ transformation with the postulation of (1.39). It is straightforward from the contractions of $O(D,D)$ indices since T-duality or an $O(D,D)$ is a global transformation. However, the gauge transformation of $\mathcal{E}$ field and the action is not written in a manifestly $O(D,D)$ covariant way. It needs complicated computations to see the $O(D,D)$ covariance of the action and the $O(D,D)$ covariance of the gauge transformation of $\mathcal{E}$ fields. To see the $O(D,D)$ covariance of the theory it is useful to introduce

$$M(X) \equiv d^t - \mathcal{E}(X)c^t\,, \qquad \bar{M}(X) \equiv d^t + \mathcal{E}^t(X)c^t\,, \tag{1.40}$$

where matrices c and d are from (1.16). The field $\mathcal{E}_{ij}$ does not transform as a tensor under an $O(D,D)$ transformation as the Buscher rules have the fractional linear form (1.38). However, the calligraphic derivatives defined in (1.33) and the variation of the $\mathcal{E}$ field transform as $O(D,D)$ tensors

$$\mathcal{D}_i = M_i{}^k \mathcal{D}'_k\,, \qquad \bar{\mathcal{D}}_i = \bar{M}_i{}^k \bar{\mathcal{D}}'_k\,, \qquad \delta\mathcal{E}_{ij} = M_i{}^k \bar{M}_j{}^l \delta\mathcal{E}'_{kl}\,, \tag{1.41}$$

and the inverse metric transforms in two equivalent forms under the $O(D,D)$ transformation

$$g^{ij} = (M^{-1})_k{}^i (M^{-1})_l{}^j (g')^{kl}\,, \qquad g^{ij} = (\bar{M}^{-1})_k{}^i (\bar{M}^{-1})_l{}^j (g')^{kl}\,. \tag{1.42}$$

From $\mathcal{D}\mathcal{E} = \mathcal{D}\delta\mathcal{E}$ and $\bar{\mathcal{D}}\mathcal{E} = \bar{\mathcal{D}}\delta\mathcal{E}$, all the objects in the action (1.31) indeed transform as tensors under the $O(D,D)$ transformation. Since there are two types of tensors for each index $i, j, \cdots$ (one transforms with M and the other transforms with $\bar{M}$), the only remaining step is to check the contractions between indices are made with the same type. This leads to the $O(D,D)$ invariance of the action (1.31).

To show that the gauge transformation of $\mathcal{E}$ field is $O(D,D)$ covariant, [7] made the redefinition of gauge parameters such that the new parameters transform as $O(D,D)$ tensors. Also some $O(D,D)$ covariant derivatives are defined to verify that the gauge transformation is $O(D,D)$ covariant. These steps can be neatly organized using the frame field formalism developed by Siegel [5] and it will be presented in chapter 2. Remarkably, the action (1.31) reproduces the cubic action in [6] when expanded around constant backgrounds. Also the action reduces to the standard Einstein-Kalb-Ramond-dilaton action (1.28) nonperturbatively when there is no $\tilde{x}_i$ dependence. It is worth pointing out that the action is gauge invariant only with the strong constraint thus the theory is restricted to the null space.

We investigated in [9] this background independent action of double field theory further. The equation of motion of the action with respect to dilaton d is given in [7] and it is simply $\mathcal{R}(\mathcal{E}, d) = 0$. The field equation for $\mathcal{E}$ is $\mathcal{K}_{pq} = 0$ where $\mathcal{K}_{pq}$ is a Ricci-like tensor which takes a complicated form. $\mathcal{K}_{pq}$ is $O(D,D)$ covariant as the first index transforms with M and the second index transforms with $\bar{M}$ under an $O(D,D)$ transformation

$$\mathcal{K}_{pq} = M_p{}^k \bar{M}_q{}^l \mathcal{K}'_{kl}\,. \tag{1.43}$$

In the limit of no $\tilde{x}$ dependence this tensor reduces to

$$\mathcal{K}_{pq}\Big|_{\tilde{\partial}=0} = \left[- R_{pq} + \frac{1}{4} H_p{}^{rs} H_{qrs} - 2\nabla_p \nabla_q \phi \right] + \left[\frac{1}{2} \nabla^s H_{spq} - H_{spq} \nabla^s \phi \right] , \tag{1.44}$$

where terms in the first square bracket are symmetric and terms in the second square bracket are antisymmetric in p and q. In this limit of no $\tilde{x}$ dependence these field equations are the linear combination of field equations of Einstein-Kalb-Ramond-dilaton action (1.28). Also in the paper a generalized version of Bianchi identity is obtained from the gauge invariance of the action, which again reduces to the standard Bianchi identity in general relativity in the same limit. The action, gauge transformation of fields, and the field equations are all $O(D,D)$ covariant in double field theory. A detailed examination of double field theory in [9] gives an impression that the theory can be formulated more geometrically as double field theory reduces to the standard general relativity (with additional Kalb-Ramond field and dilaton) in a certain limit. This is indeed the case in the work of Siegel [5], which formulated a Riemann-like curvature tensor, a Ricci-like tensor, and a Ricci-like scalar in terms of frame fields and spin connections. Siegel [5] will be covered in the next chapter in detail.

In (1.35) and (1.36) gauge parameters and coordinates neatly form $O(D,D)$ vectors in double field theory. Therefore, it is natural to expect that the action can be written in terms of more natural objects in terms of an $O(D,D)$ symmetry. As mentioned above, we already have an $O(D,D)$ covariant object which include the gravity field g_{ij} and the antisymmetric tensor field b_{ij}: generalized metric $\mathcal{H}^{MN}$. Hohm, Hull, and Zwiebach [8] developed the generalized metric formulation of double field theory, which has an advantage that the $O(D,D)$ symmetry is manifest.

The double field theory action written in terms of $\mathcal{H}^{MN}$ and d is

$$S = \int dx d\tilde{x}\, \mathcal{L}(\mathcal{H}, d)\,, \tag{1.45}$$

where

$$\begin{aligned}\mathcal{L}(\mathcal{H}, d) &= e^{-2d}\Big[\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK} \\ &\quad - 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d\Big]\,.\end{aligned} \tag{1.46}$$

In the paper it is verified that the action (1.45) is equivalent to that of [7]. The action with respect to $\mathcal{E}$ and d has an advantage that the theory can be easily shown to reduce to the cubic action in [6], which is derived from closed string field theory and not constructed from the symmetries imposed by hand. However, proving the $O(D,D)$ invariance and the gauge invariance of the action need elaborate and lengthy calculations. This is because the background field $\mathcal{E}$ is not an $O(D,D)$ covariant object and the gauge transformation of this field is rather complicated. In this sense the action (1.45) is written in such a way that the $O(D,D)$ symmetry is manifest since all the fields are $O(D,D)$ covariant objects. These objects form an $O(D,D)$ scalar $\mathcal{R}(\mathcal{H}, d)$ by contracting all $O(D,D)$ indices. Furthermore, the gauge transformation of generalized metric $\mathcal{H}^{MN}$ is quite simple in terms of 'generalized Lie derivative'. For a tensor $A_M{}^N$ the generalized Lie derivative is defined to be

$$\widehat{\mathcal{L}}_\xi A_M{}^N \equiv \xi^P \partial_P A_M{}^N + (\partial_M \xi^P - \partial^P \xi_M)\, A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P\,, \qquad (1.47)$$

and the gauge transformation of $\mathcal{H}^{MN}$ is

$$\delta_\xi \mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi \mathcal{H}^{MN}\,. \qquad (1.48)$$

The indices of these objects are lowered and raised with the $O(D,D)$ invariant metric η_{MN} and its inverse η^{MN} as before. The commutator of the generalized Lie derivatives define the gauge algebra to close according to the C-bracket

$$\left[\widehat{\mathcal{L}}_{\xi_1}, \widehat{\mathcal{L}}_{\xi_2}\right] = -\widehat{\mathcal{L}}_{[\xi_1,\xi_2]_C}\,, \qquad (1.49)$$

where the C-bracket is defined in (1.29). This is in accordance with the results of [16] as they show that the complete gauge algebra on fields $\mathcal{E}$ and d (to be precise, the fluctuation of $\mathcal{E}$ field) of double field theory closes with the C-bracket. The strong constraint is required for the gauge algebra to close as in [16].

The $O(D,D)$ covariance of this formulation of double field theory is quite straightforward as mentioned above. The gauge invariance of the action is not as direct as the $O(D,D)$ invariance of the action but this formulation has computational strengths compared to the background independent action introduced in [7]. To prove the gauge invariance of the action it is convenient to define the curvature scalar $\mathcal{R}(\mathcal{H},d)$ in terms of $\mathcal{H}$ and d

$$\mathcal{R}(\mathcal{H},d) = e^{2d}\mathcal{L}(\mathcal{H},d) + e^{2d}\partial_M \left(e^{-2d}[\partial_N \mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_N d]\right)\,. \qquad (1.50)$$

Then the action (1.45) is equivalent to, up to a boundary term,

$$S' = \int dx d\tilde{x}\, e^{-2d}\mathcal{R}(\mathcal{H},d)\,. \qquad (1.51)$$

The dilaton transforms as density under the gauge transformation, which can be seen from (1.34). Hence it is sufficient to show that the Ricci-like scalar $\mathcal{R}(\mathcal{H}, d)$ transforms as a scalar under the generalized Lie derivative

$$\delta_\xi \mathcal{R}(\mathcal{H}, d) \;=\; \widehat{\mathcal{L}}_\xi \mathcal{R}(\mathcal{H}, d) \;=\; \xi^M \partial_M \mathcal{R}(\mathcal{H}, d) \,. \tag{1.52}$$

This indeed shows that the action (1.51) is gauge invariant.

I would like to examine the moduli space for toroidal backgrounds as a final remark related to this paper. We first construct the parameterization of background fields

$$h_\mathcal{E} = \begin{pmatrix} e & b(e^t)^{-1} \\ 0 & (e^t)^{-1} \end{pmatrix} \,, \qquad g = ee^t \,, \tag{1.53}$$

where e is not the fluctuation of the constant background field E but a vielbein for the metric g. This parameterization $h_\mathcal{E}$ is an element of $O(d, d; \mathbb{R})$. Any specific background $\mathcal{E}$ can be created by the action of $h_\mathcal{E}$ on the identity background $\mathcal{I}$

$$h_\mathcal{E}(\mathcal{I}) = (e \cdot \mathcal{I} + b(e^t)^{-1})(0 \cdot \mathcal{I}(X) + (e^t)^{-1})^{-1} = \mathcal{E} \,. \tag{1.54}$$

The generalized metric $\mathcal{H}$ corresponding to the background $\mathcal{E}$ can also be constructed from $h_\mathcal{E}$ as

$$\mathcal{H}(\mathcal{E}) = h_\mathcal{E} h_\mathcal{E}^t = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} \,. \tag{1.55}$$

Thus $h_\mathcal{E}$ parameterizes both the background fields $\mathcal{E}$ and the corresponding generalized metric $\mathcal{H}(\mathcal{E})$. Since $\mathcal{H}$ is also an element of $O(d, d; \mathbb{R})$, the map $i : h_\mathcal{E} \to \mathcal{H} = h_\mathcal{E} h_\mathcal{E}^t$ defines a map from $O(d, d; \mathbb{R})$ to $O(d, d; \mathbb{R})$. Then the image of this map i is the moduli space of $\mathcal{H}$ and, hence, the moduli space of backgrounds $\mathcal{E}$.

Let $h'_\mathcal{E}$ and $h_\mathcal{E}$ be two different $O(d, d; \mathbb{R})$ elements that map to the same $\mathcal{H}$. One can always write $h'_\mathcal{E} = h_\mathcal{E} \cdot h$ with $h \in O(d, d; \mathbb{R})$ and therefore

$$\mathcal{H} = h_\mathcal{E} h h^t h_\mathcal{E}^t = h_\mathcal{E} h_\mathcal{E}^t \implies hh^t = I \implies h \in O(2d; \mathbb{R}) \,. \tag{1.56}$$

Then the group of elements h is the maximal compact subgroup $O(d; \mathbb{R}) \times O(d; \mathbb{R})$ of

$O(d,d;\mathbb{R})$. Thus the moduli space for toroidal compactification is $O(d,d;\mathbb{R})/O(d;\mathbb{R}) \times O(d;\mathbb{R})$, i.e. the inequivalent backgrounds take the values of this coset space. This analysis of the moduli space implies that the $O(D,D)$ covariant double field theory has a local $O(d;\mathbb{R}) \times O(d;\mathbb{R})$ symmetry. For notational convenience I will use a notation $O(D) \times O(D)$ for this symmetry.

Siegel is the first to analyze the theory of massless fields (gravity g, Kalb-Ramond field b, and dilaton d) starting from a particular local symmetry with the strong constraint. The work of Siegel [5] implements a frame field $E_A{}^M$ with a local $GL(D) \times GL(D)$ symmetry. This is a larger group than the actual local symmetry $O(D) \times O(D)$ of double field theory. This choice of local symmetry allows an additional gauge-fixing of a local $GL(D) \times GL(D)$ symmetry to $O(D) \times O(D)$ group, which identifies the Siegel's formalism to the generalized metric formulation. By making some specific gauge choice the work of Siegel also reduces to the background independent double field theory of [7].

In this paper [5] Siegel takes a starting point from the $GL(D) \times GL(D)$ covariance of the gauge algebra. The torsion is modified due to this constraint and is different from the standard torsion of general relativity, which is defined via the commutator of covariant derivatives. Due to this modification the standard Riemann tensor, which is also defined via the commutator of covariant derivatives, is no longer $GL(D) \times GL(D)$ covariant. Siegel found a $GL(D) \times GL(D)$ covariant Riemann-like tensor by adding terms compensating these non-covariant terms. The $GL(D) \times GL(D)$ covariant Ricci-like tensor and Ricci-like scalar are also determined in a similar way. Siegel also extends his argument to supersymmetric fields.

In the standard theory of gravity, the torsion constraint (defined via the commutator of covariant derivatives[1]) fully fixes spin connections in terms of vielbein. Then the theory is formulated in the second order formalism, where the spin connections are not independent fields. However, in Siegel's formalism the spin connections are not fully determined by the 'modified' torsion constraint: Some particular components of

[1]The tangent space symmetry covariance of the gauge algebra also yields the same torsion in the usual theory of general relativity.

the spin connections are fixed and the rest parts remain undetermined. Interestingly, only determined components of spin connections appear in the action and the field equations. The Ricci-like tensor and the Ricci-like scalar are written soley in terms of frame fields ${E_A}^M$ but the Riemann-like tensor stays undetermined. As we will see later the Ricci-like scalar is equivalent to $\mathcal{R}(\mathcal{E}, d)$ and $\mathcal{R}(\mathcal{H}, d)$ when the local $GL(D) \times GL(D)$ symmetry is appropriately fixed.

Summarizing the materials covered in this section, there are three different formulations of double field theory: background field $\mathcal{E}$ formulation, generalized metric $\mathcal{H}$ formulation, and frame field ${E_A}^M$ formulation. The dilaton field d plays the same role in each of the formulations. The action in $\mathcal{E}$ formulation is the most easily identified as the cubic action in [6] when treated perturbatively. The cubic action is derived from the closed string field theory and the physical relevance is the most clear in this formulation. On the other hand the generalized metric formulation treats the theory in a more $O(D, D)$ covariant way. The $O(D, D)$ invariant metric η_{MN} plays a crucial role in this formulation and the gauge transformation of fields is written in terms of generalized Lie derivatives. Lastly, Siegel's formalism or frame field ${E_A}^M$ formulation assumes that the tangent symmetry of the theory be $GL(D) \times GL(D)$ group, which is larger than the actual local symmetry of double field theory $O(D) \times O(D)$. This formulation treats the double field theory more geometrically. All these three formulations impose the strong constraint and the gauge algebra with a parameter ξ^M defines C-bracket.

1.3 Summary of Results

In this section I will summarize the results of [10–15]. The results of [9] are briefly presented in the previous section and will not be presented in detail in the thesis. Let us first start from the brief introduction of [10]. The main results of the paper are that the equivalence of three different formulations ($\mathcal{E}$ formulation, $\mathcal{H}$ formulation, and ${E_A}^M$ formulation) are proved explicitly and the previous results of [7, 8] are understood in a more geometrical framework.

A frame field $E_A{}^M$ takes the form

$$E_A{}^M = \begin{pmatrix} E_{ai} & E_a{}^i \\ E_{\bar{a}i} & E_{\bar{a}}{}^i \end{pmatrix} , \tag{1.57}$$

where a flat index A corresponds to a local $GL(D) \times GL(D)$ symmetry. Here I use the splitting $M = ({}_i\,, {}^i\,)$ of the $O(D,D)$ index and $A = (a, \bar{a})$ is the $GL(D) \times GL(D)$ index. As explained in the previous section Siegel constructed a Ricci-like scalar and a Ricci-like tensor in terms of a frame field $E_A{}^M$ and dilaton field d. He added terms 'by hand' to a Riemann-like tensor and Ricci-like scalar to make them local $GL(D) \times GL(D)$ covariant.

The relation to formulation with $\mathcal{E}$ is easily identified with the gauge choice

$$E_A{}^M = \begin{pmatrix} E_{ai} & E_a{}^i \\ E_{\bar{a}i} & E_{\bar{a}}{}^i \end{pmatrix} = \begin{pmatrix} -\mathcal{E}_{ai} & \delta_a{}^i \\ \mathcal{E}_{i\bar{a}} & \delta_{\bar{a}}{}^i \end{pmatrix} . \tag{1.58}$$

In this gauge the spacetime indices are identified with either of $GL(D)$ indices. Among the elements of a frame field $E_A{}^M$, only E_{ai} and $E_{\bar{a}i}$ carry the degrees of freedom, which are encoded in $\mathcal{E}$. It is straightforward from this gauge-fixing that the action and the field equations are equivalent to those constructed in [7,9]. Siegel derived the Bianchi identities

$$\nabla_a \mathcal{R} + \nabla^{\bar{b}} \mathcal{R}_{a\bar{b}} = 0 , \qquad \nabla_{\bar{a}} \mathcal{R} - \nabla^{b} \mathcal{R}_{b\bar{a}} = 0 , \tag{1.59}$$

from the gauge invariance of the action. The Bianchi identities in [9] are equivalent to those of Siegel with the choice of gauge (1.58).

The relation to the generalized metric formulation is rather intricate. Since the local symmetry of $\mathcal{H}$ formulation is $O(D) \times O(D)$ group, we have to choose an appropriate gauge, one of which is

$$\mathcal{G}_{AB} = \begin{pmatrix} -\delta_{ab} & 0 \\ 0 & \delta_{\bar{a}\bar{b}} \end{pmatrix} . \tag{1.60}$$

With this choice of the tangent space metric $\mathcal{G}_{AB}$ the generalized metric $\mathcal{H}^{MN}$ is simplified in terms of frame fields as

$$\mathcal{H}^{MN} = \delta^{AB} E_A{}^M E_B{}^N \, . \tag{1.61}$$

Then the action in Siegel [5] written in terms of frame fields $E_A{}^M$ can be interpreted in terms of $\mathcal{H}^{MN}$. This results in the action (1.45) and proves, up to a boundary term,

$$\int dx d\tilde{x}\, e^{-2d} \mathcal{R}_{\text{Siegel}} = \int dx d\tilde{x}\, \mathcal{L}(\mathcal{H}, d) = \int dx d\tilde{x}\, \mathcal{L}(\mathcal{E}, d) \, . \tag{1.62}$$

The frame field formalism developed by Siegel displays a geometrical understanding of double field theory by adopting the local $GL(D) \times GL(D)$ symmetry, which is analogous to the local Lorentz symmetry of Riemannian geometry in Einstein's theory.

I would like to make a comment on the different formulations of double field theory. The $\mathcal{E}$ field formulation has computational complexity since the $O(D, D)$ symmetry is less manifest in the formulation. It makes the $\mathcal{E}$ formulation less popular but this formulation is quite useful for some applications. In the thesis generalized metric formulation and frame field formulation are mainly used for most of the results. Each of the two formulations has its own strength and either formulation is used depending on a topic.

Generalized metric formulation is manifestly $O(D, D)$ invariant and the gauge transformation of an $O(D, D)$ tensor is simply given by the generalized Lie derivatives in the formulation. This enables us to identify an element of $O(D, D)$ group in a more intuitive way and makes computations easy in many cases. In [12, 13] we constructed $O(D, D)$ spinor fields and the generalized metric formulation is turned out to be useful, especially for constructing the action in a concrete manner.

On the other hand, the frame field formalism is useful when the geometrical structure becomes important. For example, when constructing supergravity theories, the spinor fields and Clifford algebra are generally defined on tangent space. In such theories the spacetime dependence of fields is usually absorbed in a vielbein. In this case a frame field should be introduced since the generalized metric formulation does not capture this type of geometrical aspects very well.

Now let us move on to the double field theory formulation of heterotic strings [11]. The low-energy effective action of the heterotic string including n abelian gauge fields $A_i{}^\alpha$ is given by

$$S = \int dx\sqrt{g}e^{-2\phi}\Big[R + 4(\partial\phi)^2 - \frac{1}{12}\hat{H}^{ijk}\hat{H}_{ijk} - \frac{1}{4}F^{ij\alpha}F_{ij\alpha}\Big], \tag{1.63}$$

where $F_{ij}^\alpha = \partial_i A_j{}^\alpha - \partial_j A_i{}^\alpha$ with an internal symmetry index $\alpha = 1, \cdots, n$ and $\hat{H}_{ijk} = \partial_{[i}b_{jk]} - \partial_{[i}A_j{}^\alpha A_{k]\alpha}$, where square parenthesis on indices means antisymmetrization. In heterotic string theory the T-duality group enlarges to $O(D, D+n)$ and the double field theory construction of heterotic strings should implement this enhanced duality. It turns out that the bosonic action (1.45) can be applied to the heterotic theory with an enlarged generalized metric of $(2D+n)\times(2D+n)$ size

$$\begin{aligned}\mathcal{H}^{MN} &= \begin{pmatrix} \mathcal{H}_{ij} & \mathcal{H}_i{}^j & \mathcal{H}_i{}^\beta \\ \mathcal{H}^i{}_j & \mathcal{H}^{ij} & \mathcal{H}^{i\beta} \\ \mathcal{H}^\alpha{}_j & \mathcal{H}^{\alpha j} & \mathcal{H}^{\alpha\beta} \end{pmatrix} \\ &= \begin{pmatrix} g_{ij} + c_{ki}g^{kl}c_{lj} + A_i{}^\gamma A_{j\gamma} & -g^{jk}c_{ki} & c_{ki}g^{kl}A_l{}^\beta + A_i{}^\beta \\ -g^{ik}c_{kj} & g^{ij} & -g^{ik}A_k{}^\beta \\ c_{kj}g^{kl}A_l{}^\alpha + A_j{}^\alpha & -g^{jk}A_k{}^\alpha & \delta^{\alpha\beta} + A_k{}^\alpha g^{kl}A_l{}^\beta \end{pmatrix},\end{aligned} \tag{1.64}$$

where $c_{ij} = b_{ij} + \frac{1}{2}A_i{}^\alpha A_{j\alpha}$. The gauge transformation of g, b, and an abelian vector field A can be derived from the gauge transformation $\delta_\xi\mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi\mathcal{H}^{MN}$ of generalized metric and this indeed yields the correct gauge transformations in the limit of no $\tilde{x}$ dependence. The coordinates, the derivatives, and the gauge parameters should also be enlarged to

$$X^M = \left(\tilde{x}_i, x^i, y^\alpha\right), \qquad \partial_M = \left(\tilde{\partial}^i, \partial_i, \partial_\alpha\right), \qquad \xi^M = \left(\tilde{\xi}_i, \xi^i, \Lambda^\alpha\right). \tag{1.65}$$

The gauge transformation closes with an enlarged parameter ξ^M, where the gauge algebra is determined by C-bracket as before

$$[\delta_{\xi_1}, \delta_{\xi_2}] \;=\; -\delta_{[\xi_1,\xi_2]_C} \;. \tag{1.66}$$

The $O(D,D)$ covariant constraints, either in its weak form or strong form, hold with the enlarged coordinates and take the same form as in (1.26) and (1.27). This comes from the modification of level-matching condition for the heterotic string theory. The level-matching condition in terms of coordinate-conjugate quantities is

$$2p_i w^i + q^\alpha q_\alpha = 0 \;, \tag{1.67}$$

where q_α is the internal quantum number conjugate to coordinates y^α and is a vector in the root lattice of $E_8 \times E_8$ or $SO(32)$. This condition amounts to the weak constraint (1.26) and it is totally analogous to that of the bosonic theory.

In this paper we also thoroughly investigated the non-abelian extension of $O(D, D+n)$ covariant double field theory. However, in the aspects of string theory there is no such valid interpretations of the constraint (1.26) for non-abelian theory. Also the non-abelian theory requires the number of internal coordinates to be the number of dimensions of the gauge group, which is generally very large. For example, in heterotic string theory $n = 16$ and there are 16 abelian gauge fields in the theory. If we apply this to the non-abelian extension, the number of internal coordinates is $n = 496$. Since there is little justification about this non-abelian $O(D, D+n)$ covariant double field theory in terms of string theory, I will not introduce those results in this section. However, the theory is still interesting in terms of classical supergravity theory and it will be covered in the corresponding chapter.

So far, the double field theory formulation of bosonic and heterotic theories involve fields that transform as bosons under $O(D,D)$. Interestingly, we need an introduction of spinor fields to construct the double field theory [12, 13] of the Ramond-Ramond (RR henceforth) sector of 10-dimensional type II theory. The RR potentials are bosonic fields but the corresponding fields in double field theory transform as spinors under $O(D,D)$. This $O(D,D)$ spinor field χ encodes all the RR potentials, either even or odd forms depending on the chirality of the theory. In the limit of no $\tilde{x}$ dependence the theory reduces to type II theory written in the democratic formulation. The double field theory construction of type II theory provides a unified description of the low energy limit of type IIA and type IIB superstring theory.

The understanding of the results in [12, 13] requires a lot of detailed preliminary work. The construction of an $O(D, D)$ spinor field accompanies the definition of $\mathrm{Pin}(D, D)$ or $\mathrm{Spin}(D, D)$ Clifford algebra and group homomorphism in a specific basis. These steps are rather technical but necessary to understand how the RR-sector of type II string theory beautifully fits into the $O(D, D)$ covariant double field theory. I will not present details but only significant results in this section but they will be thoroughly examined in a relevant chapter.

The action we constructed in bosonic string theory is still valid as an NS-NS sector action and the full bosonic action of type II theory is

$$\begin{aligned} S &= \int dx d\tilde{x} \left(e^{-2d} \mathcal{R}(\mathcal{H}, d) + \frac{1}{4} (\not{\partial}\chi)^\dagger \, \mathbb{S} \, \not{\partial}\chi \right), \\ \mathcal{H} &= \rho(\mathbb{S}), \quad \mathbb{S} \in \mathrm{Spin}^-(D, D), \quad \mathbb{S}^\dagger = \mathbb{S}. \end{aligned} \tag{1.68}$$

It should be emphasized here that the $\mathrm{Spin}^-(D, D)$ element $\mathbb{S}$ is viewed as the dynamical field, rather than a spin representative of the generalized metric $\mathcal{H}$. The generalized metric is uniquely determined by the group homomorphism $\rho : \mathrm{Pin}(D, D) \to O(D, D)$. $\mathcal{H}$ takes a symmetric form from the hermitian property of $\mathbb{S}$.

This action is invariant under the $\mathrm{Spin}(D, D)$ duality transformations. In fact, the action is originally $\mathrm{Pin}(D, D)$ duality invariant, but the Weyl condition on spinor fields reduce the duality group to $\mathrm{Spin}(D, D)$. Imposing the self-duality constraint

$$\not{\partial}\chi = -C^{-1} \mathbb{S} \, \not{\partial}\chi, \tag{1.69}$$

the duality group finally reduces to $\mathrm{Spin}^+(D, D)$. The self-duality relation is needed for the democratic formulation of the standard action of type II theory. The action is also gauge invariant under both the usual abelian gauge symmetries of the RR potentials parameterized by λ and the generalized diffeomorphism symmetry parameterized by ξ^M.

There is the massive extension of type IIA supergravity theory by Romans [28]. The deformation of type II theory introduces a 9-form RR potential, which carries no propagating degrees of freedom due to its field equation. Such a massive extension does not exist for type IIB theory. The work in [14] constructed the massive type IIA theory and its dual type II theory in the language of the double field theory of type II strings. In the paper a specific form of RR 1-form is assumed

$$C^{(1)}(x,\tilde{x}) \;=\; C_i(x)dx^i + m\tilde{x}_1 dx^1 \;, \tag{1.70}$$

and all other RR potentials are assumed to only depend on the spacetime coordinates x. With these RR potentials the double field theory precisely reduces to the massive type IIA theory.

The interesting point of this paper is that the strong constraint is relaxed, but not quite to the weak constraint. In fact the RR 1-from (1.70) explicitly violates the strong constraint but the theory (at least of RR sector) is still consistent without the constraint. For example the gauge invariance of the RR action does not require the strong constraint. This relaxation of constraints is indeed necessary to construct the true double field theory. When the strong constraint is imposed, the results of double field theory are closely related to those of generalized geometry. However, fields can have nontrivial dependence on both x and $\tilde{x}$ coordinates without imposing the strong constraint. In this sense relaxing the strong constraint is very important to understand the full double field theory beyond the scope of generalized geometry.

Double field theory has an explicit $O(D,D)$ symmetry, which is manifest in the generalized metric formulation. This enables the original theory in a specific frame to be readily interpreted in terms of its dual theory. In this paper [14] we take an one-circle inversion along the 10th direction (which can be chosen to be any other direction from 2 to 9). This choice of duality transformation exchanges type IIA and type IIB theory. The resulting type IIB theory, which is the dual of massive type IIA theory, is just the standard type IIB theory with some peculiar field redefinition. It is again confirmed that the dual theory of the massive type IIA is not a massive type IIB theory. This is consistent with the fact that there is no massive type IIB theory.

The final paper I would like to discuss in this section is [15]. In this paper the $\mathcal{N}=1$ supersymmetric extension of double field theory is constructed in the frame field formalism. The field contents of the theory are the usual NS-NS fields plus a gravitino Ψ_a and a dilatino ρ. In generalized geometry the supersymmetric type II theory was already formulated by [27] and $\mathcal{N}=1$ theory can be constructed from the straightforward truncation from $\mathcal{N}=2$ to $\mathcal{N}=1$. Therefore, the results of [15] are largely contained in [27] but they are written in terms of double field theory, not generalized geometry. The action is greatly simplified as

$$S_{\mathcal{N}=1} = \int d^{10}x\, d^{10}\tilde{x}\, e^{-2d}\left(\mathcal{R}(E,d) - \bar{\Psi}^a\gamma^{\bar{b}}\nabla_{\bar{b}}\Psi_a + \bar{\rho}\gamma^{\bar{a}}\nabla_{\bar{a}}\rho + 2\bar{\Psi}^a\nabla_a\rho\right)\,, \quad (1.71)$$

where $E_A{}^M$ is a frame field. The standard minimal $\mathcal{N}=1$ supergravity action is generally quite complicated and hence it needs laborious computations to check the supersymmetry invariance of the action. However, the action and the supersymmetric transformation of fields take simple forms in double field theory and the supersymmetry invariance of the action is rather straightforward.

The introduced fermions Ψ_a and ρ transform under the two copies of local Lorentz group $O(1,9)_L \times O(1,9)_R$. As mentioned before, the frame formalism of double field theory can be used to incorporate the tangent space symmetry. The tangent space symmetry group here is $O(1,9)_L \times O(1,9)_R$, which is different from the local $GL(D) \times GL(D)$ symmetry in Siegel's formalism. Under the generalized diffeomorphisms and T-duality, the fermions transform as scalars. Consequently, the supersymmetry algebra of double field theory shows that the supersymmetry transformations close into generalized diffeomorphisms plus the two copies of local Lorentz transformation.

In the paper [15] we also construct the double field theory extension of 10-dimensional $\mathcal{N}=1$ supergravity coupled to an arbitrary number of vector multiplets.

These are new and novel results. As already seen in double field theory formulation of heterotic strings, the coupling of gauge vectors ${A_i}^{\alpha}$ can be neatly described by enlarging the generalized metric (or the frame field). Then the T-duality group also enlarges to $O(10+n, 10)$. In frame formalism we can also enlarge the tangent space symmetry to $O(1, 9+n) \times O(1, 9)$ group. This leads to a remarkable result that the action (5.2) written with respect to the enlarged fields reproduces exactly $\mathcal{N}=1$ supergravity coupled to abelian vector multiplets. The gauginos are naturally embedded as components of the enlarged gravitino Ψ_a.

1.4 Conclusions and Remarks

Double field theory is a field theoretical approach to implement T-duality manifestly by doubling the coordinates: the usual space-time coordinates x^i are supplemented by winding coordinates $\tilde{x}_i$. T-duality is an intriguing feature of string theory but the supergravity limits of string theory do not capture manifestly this property. By doubling the coordinates double field theory captures certain features of string theory, especially those related to T-duality. In addition, the low energy effective theory of string theory is repackaged into simple and nice structures.

In double field theory coordinates, derivatives, and gauge parameters form $O(D, D)$ vectors such that T-duality is naturally incorporated. The information of gravity field g_{ij} and the antisymmetric tensor b_{ij} are encoded in an $O(D, D)$ covariant object: background independent $\mathcal{E}_{ij}$, generalized metric $\mathcal{H}^{MN}$, or frame field ${E_A}^M$, depending on the formulation we use. The background independent formulation ($\mathcal{E}$ formulation), the generalized metric formulation, and the Siegel's frame formalism are shown to be equivalent in [10] under the strong constraint. The gauge algebra closes according to the C-bracket, which locally reduces to the Courant bracket in generalized geometry.

In this thesis we mainly constructed the double field theory formulation of different string theories in its low energy limit: heterotic, type II, massive type IIA, and $\mathcal{N}=1$ supersymmetric theory. In particular, we focused on the massless subsector of string theory. In heterotic theory T-duality group enlarges to $O(D, D+n)$ and the enlarged generalized metric along with enlarged coordinates, derivatives, and gauge parameters features this symmetry. For type II theory, we developed a unified framework of double field theory which features $\mathrm{Spin}^+(D, D)$ symmetry with its self-duality relation. The massive extension of type IIA theory can be constructed in the double field theory formulation with a specific form of RR 1-form, which requires some relaxation of the strong constraint. The $\mathcal{N} = 1$ supersymmetric extension of double field theory in frame formalism features a local $O(1,9)_L \times O(1,9)_R$ symmetry under which the gravitino Ψ_a and the dilatino ρ transform. When the theory is extended to include n vector multiplets, the gauginos are naturally encoded as additional components of the gravitino with a local $O(1, 9+n) \times O(1,9)$ symmetry.

I would like to make a few comments about the double field theory formulation of type II theories. When we take a T-duality transformation in type II theory, there are two types: spacelike T-duality and timelike T-duality. The spacelike T-duality links type IIA and type IIB theories as we expect from the T-duality transformations. In particular, starting from type IIA theory, odd number of spacelike T-duality inversions leads to type IIB theory and even number of spacelike T-duality inversions keeps the theory to be the same. However, when odd number of timelike T-duality inversions is applied, type IIB* is obtained from type IIA theory and type IIA* from type IIB theory, as proposed by Hull [63].

It is worth mentioning that the minimal $\mathcal{N} = 1$ supersymmetric extension of double field theory does not have direct interpretation in terms of string theory. It is rather a truncation of type II theory, where the generalized geometry formulation is constructed in [27]. The extension of the minimal $\mathcal{N} = 1$ supersymmetric theory to include $n = 16$ vector multiplets can be thought of as the double field theory formulation of either type I theory or heterotic theory, as they are related by the field redefinition of the dilaton or S-duality. When the number of vector multiplets is different from 16, the theory also does not have a direct interpretation in string theory.

I hope that our contributions inspire ones who are interested in double field theory. The topics discussed in this thesis are basic extensions of double field theory and there are a lot of topics remaining to be investigated. However, there are also some obstructions that double field theory has: the relaxation of the strong constraint, the issue of consistent truncation, other global issues, etc. For double field theory to be used for interesting applications, these obstructions must be overcome. I believe that future research in double field theory will provide the answers for these obstructions. Furthermore I wish that the active research in this research area improves the general understanding of string theory, which is the goal of double theory, especially the aspects of string theory related to T-duality and possibly larger duality group.

Although I only introduced papers that are crucial to understand the results in [10–15], there are other interesting works in this field. The work of [25] introduced the manifestly $O(D, D)$ covariant formulation of double field theory compatible with projections. Recently they used their own formulation to construct the double field theory formulation of RR sector of type II theories and $\mathcal{N} = 1$ supersymmetric theories [26]. In [18] the perturbative expansion of the low-energy gravity action of closed string theory around a flat background is discussed in double field theory formulation. In this expansion the left-right factorization is exhibited at the level of Lagrangian to all orders. In the previous section we briefly discussed the existence of Riemann-like tensor in Siegel's formalism of double field theory. The work of [19] gives a quite detailed discussion about this object and the related higher-order action. According to this work, it is challenging to construct the α' corrected action in double field theory.

Still there are many attractive topics in double field theory that need to be investigated. We could relax the strong constraint for the RR-sector in the formulation of massive type IIA theory but it is not possible for general double field theory. The strong constraint is still needed in most cases, especially for the gauge symmetry of NS-NS sector and the closure of gauge algebra. The strong constraint is so strong that locally all fields depend only on half of the coordinates. However, the full relaxation of the strong constraint to the weak constraint is presumabley not possible for double field theory to be a consistent truncation of string theory. If possible, the relaxation of the strong constraint in double field theory would be a huge improvement. Relevant works [23] implement the compactification of 'Scherk-Schwarz' type in double field theory. In Scherk-Schwarz reduction, the dependence on internal coordinates enters through the group element of symmetry transformations but the dependence does not appear in the compactified theory. In their works the internal coordinates include both the usual space coordinates and the winding type coordinates. The use of the strong constraint is relaxed there. Very recent work [21] investigates large, or finite, gauge transformations in double field theory. This study of 'doubled manifold' may help our understanding of the geometrical role of the strong constraint in double field theory.

Double field theory can be extended to implement dualities other than T-duality. The motivation of double field theory originates from the toroidal compactification of string theory but it is completely natural to extend the double field theory to other theories with a duality group different from $O(D, D)$. For example double field theory can be formulated to describe the low energy limit of M-theory by implementing U-duality group. The M-theory extension of double field theory is studied mostly in generalized geometry context. [24] gives a unified description of bosonic eleven-dimensional supergravity, which is the low energy limit of M-theory, restricted to a d-dimensional manifold for $d \leq 7$. The theory is constructed using generalized geometry and the tangent space features $E_{d(d)} \times \mathbb{R}^+$ symmetry. Berman et al. [74] also studies the M theory extension of double field theory in terms of generalized geometry, for different duality groups that appear on the reduction of 11-dimensional supergravity.

Non-geometric compactification is also a very interesting application of double field theory. There are configurations of non-geometric compactification that are T-dual to the configurations of well-known geometric compactification. As T-duality transformations are naturally incorporated in double field theory, it can be a useful tool to study non-geometric compactification. Recent work [20] used double field theory to study non-geometric Q and R fluxes in 4 dimensional (gauged) supergravity theory. They started from well-known NSNS H-flux, which can be obtained from the geometric compactification of 10 dimensional (ungauged) supergravity theory. They also suggested some relaxation of the strong constraint to explain more configurations of non-geometric compactification in terms of double field theory.

I believe that there are many unknown but promising research topics in double field theory that I am missing in this thesis. I would like to leave those questions and topics to readers.

Chapter 2

Frame-like Geometry of Double Field Theory

A bulk of this chapter appeared in"Frame-like geometry of double field theory" with Olaf Hohm [10] and is reprinted with the permission of *Journal of Physics A*.

Summary : We relate two formulations of the recently constructed double field theory to a frame-like geometrical formalism developed by Siegel. A self-contained presentation of this formalism is given, including a discussion of the constraints and its solutions, and of the resulting Riemann tensor, Ricci tensor and curvature scalar. This curvature scalar can be used to define an action, and it is shown that this action is equivalent to that of double field theory.

2.1 Introduction

We briefly reviewed the results of [7] and [8] in the introduction chapter. These two papers introduce two different formulations of double field theory: $\mathcal{E}$ field formulation and generalized metric $\mathcal{H}$ formulation. It was mentioned in the introduction chapter that the frame formalism introduced by Siegel [5] is equivalent to these two formulations in a certain gauge and, therefore, the three different formulations are equivalent. The equivalence of $\mathcal{E}$ formulation and $\mathcal{H}$ formulation is already shown in [8]. The work [10] fills the remaining gap of this proof.

The understanding of Siegel's formalism requires quite a lot of preliminary knowledge. This will be provided in the next two sections and these two sections are mainly a review of [5]. In section 4 and 5 the main results of the paper [10] are presented, where we relate explicitly the frame formalism to the formulations in terms of $\mathcal{E}$ and $\mathcal{H}$. Specifically, in section 4 we show the equivalence of the scalar curvature and the corresponding scalar found in [7], and relate in particular '$O(D,D)$ covariant derivatives' introduced there to the $GL(D) \times GL(D)$ connections. In section 5 we give an independent proof of the equivalence of the curvature scalars in the formulation with $\mathcal{H}^{MN}$ given in [8].

2.2 Geometrical frame formalism

In this section we first review a few properties of gauge transformations parametrized by ξ^M and the associated C-bracket. Next we introduce frame fields which are subject to the tangent space symmetry $GL(D) \times GL(D)$ together with connections for this symmetry. Finally, a covariant curvature tensor is discussed. One important note here is that we use a different notation $E_A{}^M$ from that of [10] for a frame field to keep consistent and unified notations throughout the thesis.

2.2.1 Generalized Lie derivatives, Courant bracket and frame fields

The generalized Lie derivative is defined for tensors with an arbitrary number of upper and lower $O(D,D)$ indices by the straightforward extension of

$$\widehat{\mathcal{L}}_\xi A_M{}^N \equiv \xi^P \partial_P A_M{}^N + (\partial_M \xi^P - \partial^P \xi_M)\, A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P\,, \tag{2.1}$$

which is given in the equation (1.47) in the introduction chapter. With this definition the gauge transformation of generalized metric is simply $\delta_\xi \mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi \mathcal{H}^{MN}$. In general we will refer to $O(D,D)$ tensors that transform according to the generalized Lie derivative under gauge transformations parameterized by ξ^M as 'generalized tensors' or as transforming covariantly under ξ^M.

An important consistency property of this formalism is that the $O(D,D)$ invariant metric that is used in (2.1) to raise and lower indices has vanishing generalized Lie derivative,

$$\widehat{\mathcal{L}}_\xi \eta^{MN} \;=\; \xi^P\partial_P\,\eta^{MN} - \partial^N\xi^M - \partial^M\xi^N + \partial^N\xi^M + \partial^M\xi^N \;=\; 0\,. \qquad (2.2)$$

Accordingly, in this formalism it is consistent to have a *constant* tensor with two upper or two lower 'curved' or 'world' indices.

The closure of the gauge transformations spanned by ξ^M or, equivalently, the algebra of generalized Lie derivatives can be straightforwardly determined in this formulation by (1.49). It is governed by the 'C-bracket'

$$\big[\xi_1,\xi_2\big]_{\rm C}^M \;\equiv\; \xi_1^N\partial_N\xi_2^M - \frac{1}{2}\,\xi_1^P\partial^M\xi_{2P} - (1\leftrightarrow 2)\,. \qquad (2.3)$$

This bracket is the $O(D,D)$ covariant double field theory extension of the Courant bracket of generalized geometry [29–31], as has been shown in [16]. An important property that will be used later is that the C-bracket of two generalized vectors is again a generalized vector. Let X^M and Y^M be transforming as $\delta_\xi X^M = \widehat{\mathcal{L}}_\xi X^M$ and $\delta_\xi Y^M = \widehat{\mathcal{L}}_\xi Y^M$, respectively, then we find

$$\delta_\xi\big[X,Y\big]_{\rm C}^M \;=\; \widehat{\mathcal{L}}_\xi\big[X,Y\big]_{\rm C}^M\,. \qquad (2.4)$$

This establishes the covariance of the C-bracket.

In general, partial derivatives of generalized tensors are not generalized tensors. An exception is a generalized scalar S which according to (2.1) simply transforms as

$$\delta_\xi S \;=\; \widehat{\mathcal{L}}_\xi S \;=\; \xi^P\partial_P S\,. \qquad (2.5)$$

Therefore, its partial derivative transforms as

$$\delta_\xi(\partial_M S) = \partial_M(\xi^P\partial_P S) = \xi^P\partial_P(\partial_M S) + (\partial_M\xi^P - \partial^P\xi_M)\partial_P S \equiv \widehat{\mathcal{L}}_\xi(\partial_M S) , \quad (2.6)$$

where in the second equality we were allowed to add the third term because it is zero by the strong constraint. Thus, $\partial_M S$ transforms covariantly, i.e., as a generalized covariant tensor. This covariant transformation behavior does not hold for partial derivatives of higher tensors, not even for antisymmetrized combinations like $\partial_{[M}V_{N]}$ — in contrast to conventional diffeomorphisms.

In the following we will introduce a frame field which allows to convert arbitrary tensors from 'world'-tensors into 'tangent space'-tensors and thereby into scalars under ξ^M. Specifically, following Siegel [5] we introduce a frame field $E_A{}^M$, which is a generalized vector and has a flat index A corresponding to a local $GL(D)\times GL(D)$ symmetry, i.e.,

$$E_A{}^M = \begin{pmatrix} E_{ai} & E_a{}^i \\ E_{\bar{a}i} & E_{\bar{a}}{}^i \end{pmatrix} . \quad (2.7)$$

We assume this vielbein to be invertible and denote the inverse by $E_M{}^A$. In (2.7) we used the splitting $M = ({}_i\,,{}^i)$ of the $O(D,D)$ index and $A=(a,\bar{a})$ is the $GL(D)\times GL(D)$ index. Given the $O(D,D)$ invariant metric η_{MN} we can build an X-dependent 'tangent space' metric of signature (D,D),

$$\mathcal{G}_{AB} = E_A{}^M E_B{}^N \eta_{MN} , \quad (2.8)$$

with inverse $\mathcal{G}^{AB} = \eta^{MN}E_M{}^A E_N{}^B$, which will be used to raise and lower flat indices. The raising and lowering of world indices with η and of flat indices with $\mathcal{G}$ is consistent with inverting the frame field (2.7) in that

$$E_M{}^A = \eta_{MN}\mathcal{G}^{AB}E_B{}^N \qquad \Rightarrow \qquad E_M{}^A E_A{}^N = \delta_M{}^N , \quad (2.9)$$

as follows from the definition (2.8). In order for $E_A{}^M$ to describe only the physical degrees of freedom it turns out to be necessary to impose the $GL(D)\times GL(D)$

covariant constraint

$$\mathcal{G}_{a\bar{b}} \;=\; 0 \quad \Leftrightarrow \quad E_{(a}{}^{i}\, E_{\bar{b})i} \;=\; 0\;, \tag{2.10}$$

which is related to the left-right factorization of closed string theory [5].[1]

Using the frame field one can introduce a 'flattened' derivative e_A, defined by

$$E_A \;\equiv\; E_A{}^M\,\partial_M\;. \tag{2.11}$$

We note that the strong constraint takes the following form in terms of flat indices,

$$E^A X\, E_A Y \;=\; \mathcal{G}^{AB} E_A{}^M E_B{}^N \partial_M X\, \partial_N Y \;=\; \eta^{MN}\partial_M X\, \partial_N Y \;=\; 0\;, \tag{2.12}$$

for arbitrary functions X and Y. Due to the covariance of the partial derivative of a generalized scalar discussed above, the action of E_A on an arbitrary tensor with only flat indices, $E_A X_{BC\ldots}$, is covariant under ξ^M transformations. Of course, it will not be covariant under the local frame rotations, and so covariant derivatives have to be introduced. Thereby, the problem of defining derivative operations that are covariant under generalized diffeomorphisms parameterized by ξ^M has been converted to the problem of introducing appropriate covariant derivatives and connections for the $GL(D) \times GL(D)$ tangent space symmetry, to which we turn now.

2.2.2 $GL(D) \times GL(D)$ connections and constraints

We define the infinitesimal local $GL(D) \times GL(D)$ transformations to be

$$\delta_\Lambda V_A \;=\; \Lambda_A{}^B V_B\;, \qquad \delta_\Lambda V^A \;=\; -\Lambda_B{}^A V^B\;, \tag{2.13}$$

and analogously for tensors with an arbitrary number of upper and lower indices. Since we are dealing with $GL(D) \times GL(D)$, the non-vanishing parameters are $\Lambda_a{}^b$

[1] An alternative motivation of this constraint starting from generalized geometry and the generalized metric $\mathcal{H}$ has been given in [8], c.f. the discussion after eq. (3.101) below.

and $\Lambda_{\bar{a}}{}^{\bar{b}}$. Covariant derivatives with flattened indices are given by

$$\nabla_A V_B \;=\; E_A V_B + \omega_{AB}{}^C V_C\,, \qquad \nabla_A V^B \;=\; E_A V^B - \omega_{AC}{}^B V^C\,, \tag{2.14}$$

where we have introduced connections $\omega_{AB}{}^C$. Again, since we are dealing with gauge group $GL(D)\times GL(D)$ the only non-vanishing components of the connections are

$$\omega_{AB}{}^C\;: \qquad \omega_{Ab}{}^c\,, \quad \omega_{A\bar{b}}{}^{\bar{c}}\,. \tag{2.15}$$

Moreover, the constraint (2.10) implies that the same holds for connections with all indices lowered. We will frequently make use of the fact that components like $\omega_{ab}{}^{\bar{c}}$ and $\omega_{ab\bar{c}}$ vanish. We require that the connections transform under ξ^M as scalars and therefore, as discussed above, the covariant derivatives (5.31) transform as scalars, too. They transform also covariantly under $GL(D)\times GL(D)$ if we require that the $\omega_{AB}{}^C$ transform as connections, i.e.,

$$\delta\omega_{Aa}{}^b \;=\; -\nabla_A\Lambda_a{}^b + \Lambda_A{}^B\omega_{Ba}{}^b\,, \qquad \nabla_A\Lambda_a{}^b \;=\; E_A\Lambda_a{}^b + \omega_{Aa}{}^c\Lambda_c{}^b - \omega_{Ac}{}^b\Lambda_a{}^c\,, \tag{2.16}$$

and analogously for barred indices. We note that the additional term in $\delta\omega_{Aa}{}^b$ as compared to the familiar transformation rule for a Yang-Mills gauge potential is due to the conversion of the 1-form index into a flat one.

Next we have to impose covariant constraints that allow us to solve for (part of) the connections in terms of the physical fields. There are three covariant constraints in total:

1. The torsion constraints

$$\mathcal{T}_{AB}{}^C \;=\; 0\,, \tag{2.17}$$

where the torsion $\mathcal{T}_{AB}{}^C$ is defined by[2]

$$\mathcal{T}_{AB}{}^C = \Omega_{AB}{}^C + 2\left(\omega_{[AB]}{}^C + \frac{1}{2}\omega^C{}_{[AB]}\right) , \tag{2.18}$$

and $\Omega_{AB}{}^C$ is the generalized coefficient of anholonomy, which is given by

$$\Omega_{AB}{}^C = 2\left(f_{[AB]}{}^C + \frac{1}{2}f^C{}_{[AB]}\right) , \qquad f_{ABC} \equiv (E_A E_B{}^M)E_{CM} . \tag{2.19}$$

2. The metricity condition that the metric $\mathcal{G}_{AB}$ is covariantly constant [5],

$$\nabla_A \mathcal{G}_{BC} = 0 \qquad \Leftrightarrow \qquad E_A \mathcal{G}_{BC} + 2\omega_{A(BC)} = 0 . \tag{2.20}$$

3. The partial integration constraint

$$\int e^{-2d}\, V \nabla_A V^A = -\int e^{-2d}\, V^A \nabla_A V = -\int e^{-2d}\, V^A E_A V \tag{2.21}$$

for arbitrary V and V^A. This constraint enables the integration by parts in an action using the covariant derivatives [5].

The consistency of this and the previous constraints will be confirmed in the next subsection by providing the explicit solutions.

The torsion tensor defined here is different from that of Rimannian geometry. In ordinary Riemannian geometry the torsion constraint of the Levi-Civita connection implies that in the Lie bracket of two vector fields the partial derivatives can be replaced by covariant derivatives. In the double field theory context the Lie bracket is replaced by the C-bracket in that only the latter transforms covariantly under generalized diffeomorphisms. Since we are dealing here with flattened derivatives it is thus natural to define a torsion tensor in such a way that it vanishes if and only if in the C-bracket with flattened parameter $\xi_{12}^A = \left[\xi_1, \xi_2\right]_C^M e_M{}^A$ the partial derivatives

[2] In this paper we employ the convention that symmetrization and anti-symmetrization involves the combinatorial factor, e.g., $X_{[ab]} = \frac{1}{2}(X_{ab} - X_{ba})$. In some formulas this leads to numerical factors that are different from those in [5].

are replaced by $GL(D) \times GL(D)$ covariant derivatives, i.e.,

$$\xi_{12}^A = \xi_1^B \nabla_B \xi_2^A - \frac{1}{2}\xi_{1B}\nabla^A \xi_2^B - (1 \leftrightarrow 2) + \xi_1^B \xi_2^C \mathcal{T}_{BC}{}^A , \tag{2.22}$$

and the torsion tensor is defined from this equation. We note here that the torsion tensor defined like this does not coincide with the usual definition via the commutator of covariant derivatives.

2.2.3 Solving the constraints

We solve now the above constraints and show their mutual consistency. Each constraint given in the previous subsection leads to the following conditions for spin connections, respectively:

1. The torsion constraint (2.17) leads to the conditions

$$\omega_{[ABC]} = -\frac{1}{3}\Omega_{[ABC]} = -f_{[ABC]} , \qquad \omega_{a\bar{b}\bar{c}} = -\Omega_{a\bar{b}\bar{c}} , \qquad \omega_{\bar{a}bc} = -\Omega_{\bar{a}bc} . \tag{2.23}$$

2. The metricity condition (2.20) can be trivially solved,

$$\omega_{A(BC)} = -\frac{1}{2}E_A \mathcal{G}_{BC} = -f_{A(BC)} , \tag{2.24}$$

   and determines the part symmetric in the 'group indices' of ω_{ABC} completely.

3. Finally, we solve the constraint (5.23) and obtain

$$\omega_{BA}{}^B = -\tilde{\Omega}_A \equiv -e^{2d}\partial_M\big(E_A{}^M e^{-2d}\big) = -\partial_M E_A{}^M + 2E_A d , \tag{2.25}$$

   where we introduced $\tilde{\Omega}_A$ for notational convenience.

We conclude this section by summarizing which connections are determined by the above constraints (2.17), (2.20) and (5.23). First, the 'off-diagonal' components $\omega_{\bar{a}bc}$ and $\omega_{a\bar{b}\bar{c}}$ are completely determined according to the last two conditions in (5.21). For the 'diagonal' components ω_{abc} and $\omega_{\bar{a}\bar{b}\bar{c}}$ the parts symmetric in the last two indices are fully determined by (2.24). Therefore, it is sufficient for the remaining components to focus on the part antisymmetric in the last two indices, whose irreducible parts, say for ω_{abc}, are given by the following tensor product

$$\omega_{a[bc]} \ : \qquad \square \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \ = \ \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \,, \tag{2.26}$$

where the Young tableaux refer to the left $GL(D)$ group. In here, the completely antisymmetric part $\omega_{[abc]}$ is determined by the first condition in (5.21). For the 'mixed-Young tableaux' representation on the right-hand side of (2.26) the trace parts are determined by (5.24) in terms of the dilaton, leaving precisely the trace-free part of this $(2,1)$ representation as the undetermined connections. Its dimension is given by $\frac{1}{3}D(D+2)(D-2)$ and therefore, taking the right $GL(D)$ into account, the number of undetermined components is twice this value. That not all components are determined by the above constraints limits the extent to which invariant curvatures can be constructed out of the physical fields, which will be discussed in the next subsection.

2.2.4 Covariant cuvature tensor

Let us now turn to the construction of invariant curvature tensors for the $GL(D) \times GL(D)$ connections. In general, given covariant derivatives one can define curvatures through their commutator, say, acting on V_C,

$$\big[\nabla_A, \nabla_B\big] V_C \ = \ T_{AB}{}^D \nabla_D V_C + R_{ABC}{}^D V_D \,. \tag{2.27}$$

This leads to the standard expressions

$$T_{AB}{}^C \ = \ \Omega_{AB}{}^C + 2\omega_{[AB]}{}^C \,, \tag{2.28}$$

$$\begin{aligned} R_{ABC}{}^D \ = \ & E_A \omega_{BC}{}^D - E_B \omega_{AC}{}^D + \omega_{AC}{}^E \omega_{BE}{}^D - \omega_{BC}{}^E \omega_{AE}{}^D \\ & -\Omega_{AB}{}^E \omega_{EC}{}^D \,. \end{aligned} \tag{2.29}$$

We note that the torsion tensor $T_{AB}{}^C$ defined like this does not coincide with the torsion $\mathcal{T}_{AB}{}^C$ defined earlier in (2.18). Given the modification of the ξ^M gauge transformations as compared to the standard diffeomorphisms it was, however, only consistent to set $\mathcal{T}_{AB}{}^C = 0$. We conclude that the *conventional* torsion is necessarily non-zero when imposing (2.17), after which the commutator of covariant derivatives reads

$$\big[\nabla_A, \nabla_B\big]V_C \;=\; -\omega^D{}_{[AB]}\nabla_D V_C + R_{ABC}{}^D V_D \,. \tag{2.30}$$

An immediate consequence is that $R_{ABC}{}^D$ as defined in (2.29) cannot be fully covariant with respect to $GL(D)\times GL(D)$, because the left-hand side of (2.30) is manifestly covariant but the right-hand side contains a bare gauge field.

At this stage a comment is in order regarding the non-covariance of the curvature tensor R, because formally it coincides with a conventional field strength (with flattened indices) that would be covariant with respect to (frame-)transformations of an arbitrary gauge group. The subtlety here is that the generalized coefficients of anholonomy $\Omega_{AB}{}^C$ defined in (5.15) rather than the conventional ones appear in the last term of (2.29). Actually, eq. (2.27) does not determine whether (2.28) should contain the generalized coefficients of anholonomy or the conventional ones. The choice made here is covariant under ξ^M gauge transformations, at the cost of violating the $GL(D)\times GL(D)$ covariance.

This violation of the covariance is fixed in Siegel [5] by hand. The $GL(D)\times GL(D)$ covariant curvature tensor takes a form

$$\mathcal{R}'_{ABCD} \;\equiv\; \frac{1}{2}\Big[R_{ABCD} + R_{CDAB}\Big] + \frac{1}{4}\Big[\omega_{ECD}\,\omega^E{}_{BA} + \omega_{EAB}\,\omega^E{}_{DC}\Big]\,. \tag{2.31}$$

Since the proof of covariance requires the use of the metricity condition, $\mathcal{R}'$ transforms only covariantly after imposing this constraint. This can, however, be relaxed by adding further terms that are zero upon imposing the constraints. Specifically, defining

$$\mathcal{R}_{ABCD} \;=\; \mathcal{R}'_{ABCD} - \frac{1}{4}\omega_{ECD}\,\nabla^E\mathcal{G}_{AB} - \frac{1}{4}\omega_{EAB}\,\nabla^E\mathcal{G}_{CD}\,, \tag{2.32}$$

we obtain a tensor that is fully covariant independently of the constraints. In the remainder of this chapter we will assume that all constraints are satisfied, for which $\mathcal{R} = \mathcal{R}'$, unless stated differently. Since these are further analysis of Riemann-like tensor in Siegel [5], we omitted a lot of detailed steps, which can be found in the paper [10].

In the rest of this section we examine the symmetry properties and identities of $\mathcal{R}_{ABCD}$. We start with the original curvature $R_{ABC}{}^{D}$, which has the following symmetries

$$R_{ABCD} \;=\; -R_{BACD} \;=\; -R_{ABDC}\,. \tag{2.33}$$

Moreover, since the gauge group is $GL(D) \times GL(D)$ the 'off-diagonal' components in the group indices of $R_{ABC}{}^{D}$, i.e., in the last two indices, are zero,

$$R_{ABc\bar{d}} \;=\; R_{AB\bar{c}d} \;=\; 0\,, \tag{2.34}$$

corresponding to the fact that the only non-zero connections are (2.15).

Next, we turn to the symmetry properties of $\mathcal{R}$. In general, the correction terms proportional to the connections in (2.31) have no specific symmetry. If we focus on off-diagonal $GL(D) \times GL(D)$ components, however, these extra terms vanish, see (2.15), and so the antisymmetry properties of R elevate to $\mathcal{R}$. For instance,

$$\mathcal{R}_{ab\bar{c}\bar{d}} \;=\; \frac{1}{2}\Big[R_{ab\bar{c}\bar{d}} + R_{\bar{c}\bar{d}ab}\Big] \;=\; \frac{1}{2}R_{\bar{c}\bar{d}ab} \;=\; -\frac{1}{2}R_{\bar{c}\bar{d}ba} \;=\; -\mathcal{R}_{ba\bar{c}\bar{d}}\,. \tag{2.35}$$

The same conclusion applies to all other components that have precisely three unbarred or three barred indices.

We close this section with a brief discussion of a curvature scalar that will be used in the next section to define an action. The scalar that is obtained by tracing $\mathcal{R}$ turns out to be zero by virtue of the constraints. Specifically, prior to imposing any constraints, one can prove that[3]

$$\mathcal{R}_{AB}{}^{AB} \;=\; 2\nabla_A \tilde{T}^A + \tilde{T}_A{}^2 + \nabla_A \nabla_B \mathcal{G}^{AB} - \frac{1}{6}\mathcal{T}_{[ABC]}{}^2 - \frac{3}{8}\nabla_{(A}\mathcal{G}_{BC)}{}^2 . \tag{2.36}$$

Each term vanishes separately after imposing the constraints, and therefore

$$0 \;=\; \mathcal{R}_{AB}{}^{AB} \;=\; \mathcal{R}_{ab}{}^{ab} + \mathcal{R}_{\bar{a}\bar{b}}{}^{\bar{a}\bar{b}} . \tag{2.37}$$

Thus, there is a unique way to define a (non-vanishing) scalar,

$$\mathcal{R} \;:=\; -\frac{1}{2}\mathcal{R}_{ab}{}^{ab} \;=\; \frac{1}{2}\mathcal{R}_{\bar{a}\bar{b}}{}^{\bar{a}\bar{b}} , \tag{2.38}$$

which by construction is a scalar under ξ^M transformations and $GL(D) \times GL(D)$.

An expression for $\mathcal{R}$ that makes the invariance under $O(D,D)$ and frame transformations manifest is the following,

$$\begin{aligned} \mathcal{R} \;=\; & -(\nabla^a \nabla_a d - \nabla^{\bar{a}} \nabla_{\bar{a}} d) - \frac{1}{2}\Big(\nabla^a (E_a{}^M \nabla^{\bar{b}} E_{\bar{b}M}) - \nabla^{\bar{a}}(E_{\bar{a}}{}^M \nabla^b E_{bM})\Big) \\ & -\frac{1}{4}\Big(E_a{}^M \nabla^b E^{\bar{c}}{}_M \, E^{aN} \nabla_b E_{\bar{c}N} - E_{\bar{a}}{}^M \nabla^{\bar{b}} E^c{}_M \, E^{\bar{a}N} \nabla_{\bar{b}} E_{cN}\Big) \\ & +\frac{1}{2}\Big(E_{\bar{c}}{}^M \nabla_a E_{bM} \, E^{\bar{c}N} \nabla^b E^a{}_N - E_c{}^M \nabla_{\bar{a}} E_{\bar{b}M} \, E^{cN} \nabla^{\bar{b}} E^{\bar{a}}{}_N\Big) \\ & -\Big(\nabla^a d \, (E_a{}^M \nabla^{\bar{b}} E_{\bar{b}M}) - \nabla^{\bar{a}} d \, (E_{\bar{a}}{}^M \nabla^b E_{bM})\Big) - (\nabla^a d \, \nabla_a d - \nabla^{\bar{a}} d \, \nabla_{\bar{a}} d) . \end{aligned} \tag{2.39}$$

It is not manifest either from the definition (2.38) or the explicit form (2.39) that the scalar curvature depends only on the connection components that have been determined by the constraints. A somewhat lengthy calculation shows, however, that $\mathcal{R}$ can be written as[4]

$$\mathcal{R} \;=\; E_a \tilde{\Omega}^a + \frac{1}{2}\tilde{\Omega}_a{}^2 + \frac{1}{2}E_a E_b \mathcal{G}^{ab} - \frac{1}{4}\Omega_{ab\bar{c}}{}^2 - \frac{1}{12}\Omega_{[abc]}{}^2 + \frac{1}{8}E^a \mathcal{G}^{bc} \, E_b \mathcal{G}_{ac} . \tag{2.40}$$

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[3] We note that this expression differs from that in sec. VIII of [5] because of different conventions regarding symmetrization. Moreover, it differs by an overall factor and a relative factor in the last term.

[4] Again, this expression differs from that given in sec. VIII of [5] because of different conventions regarding antisymmetrization, but it also corrects a typo in the fourth term.

This proves that $\mathcal{R}$ is a well-defined function of the physical fields.

2.3 General action principle

In this section we briefly introduce an Einstein-Hilbert like action principle based on the invariant curvature scalar discussed above, and derive Bianchi identities from its gauge invariance.

2.3.1 Gauge invariant action

Having the scalar $\mathcal{R}$ at our disposal we can define the following action principle

$$S \;=\; \int dx d\tilde{x}\, e^{-2d}\mathcal{R}\;, \tag{2.1}$$

which, by virtue of e^{-2d} transforming as a density, is manifestly invariant under all symmetries.

There are a number of conclusions that can be derived from this invariance. First, the variation with respect to d has to be a $GL(D)\times GL(D)$ invariant scalar and therefore it must be proportional to $\mathcal{R}$ defined in (2.38) [5], which conclusion agrees with the results of [7,8], as we will show below. Second, the general variation with respect to $E_A{}^M$ is non-trivial only in its off-diagonal component, in the following sense. Introducing a variation with both indices flat,

$$\Delta E_{AB} \;:=\; E_B{}^M \delta E_{AM}\;, \tag{2.2}$$

we infer that the $GL(D)\times GL(D)$ transformations (2.13) read

$$\Delta E_{AB} \;=\; E_B{}^M \Lambda_A{}^C E_{CM} \;=\; \Lambda_A{}^C \mathcal{G}_{BC} \;=\; \Lambda_{AB}\,. \tag{2.3}$$

By the constraint (2.10) this implies

$$\Delta E_{ab} \;=\; \Lambda_{ab}\;, \qquad \Delta E_{\bar{a}\bar{b}} \;=\; \Lambda_{\bar{a}\bar{b}}\;, \qquad \Delta E_{a\bar{b}} \;=\; -\Delta E_{\bar{b}a} \;=\; 0\,. \tag{2.4}$$

Consequently, the local $GL(D)\times GL(D)$ symmetry of the action implies the 'Bianchi identity' that the diagonal parts of the field equations obtained by variation with respect to ΔE_{ab} and $\Delta E_{\bar{a}\bar{b}}$ vanish identically. Thus, the only non-trivial part of the field equation is obtained by variation with respect to, say, $\Delta E_{a\bar{b}}$. In total, the variation of (2.1) can be written as

$$\delta S \;=\; \int dx d\tilde{x}\, e^{-2d}\left(-2\delta d\,\mathcal{R}+\Delta E_{a\bar{b}}\mathcal{R}^{a\bar{b}}\right)\,, \tag{2.5}$$

giving rise to the field equations

$$\mathcal{R} \;=\; 0\,, \qquad \mathcal{R}_{a\bar{b}} \;=\; 0\,. \tag{2.6}$$

Next we discuss some general properties of these tensors. As indicated by the suggestive notation it is natural to assume that the 'Ricci tensor' $\mathcal{R}_{a\bar{b}}$ derived from (2.1) indeed follows from contracting the covariant curvature tensor introduced above. There are two candidates, $\mathcal{R}_{\bar{c}a\bar{b}}{}^{\bar{c}}$ and $\mathcal{R}_{c\bar{b}a}{}^{c}$. The explicit expression for the first is

$$\begin{aligned}
\mathcal{R}_{a\bar{b}} \;&=\; 2\mathcal{R}_{\bar{c}a\bar{b}}{}^{\bar{c}} \;=\; R_{\bar{c}a\bar{b}}{}^{\bar{c}} \\
&=\; E_{\bar{c}}\omega_{a\bar{b}}{}^{\bar{c}}-E_{a}\omega_{\bar{c}\bar{b}}{}^{\bar{c}}+\omega_{\bar{c}\bar{b}}{}^{\bar{d}}\omega_{a\bar{d}}{}^{\bar{c}}-\omega_{a\bar{b}}{}^{\bar{d}}\omega_{\bar{c}\bar{d}}{}^{\bar{c}}-\Omega_{\bar{c}a}{}^{E}\omega_{E\bar{b}}{}^{\bar{c}} \\
&=\; E_{\bar{c}}\omega_{a\bar{b}}{}^{\bar{c}}-E_{a}\omega_{\bar{c}\bar{b}}{}^{\bar{c}}+\omega_{d\bar{b}}{}^{\bar{c}}\omega_{\bar{c}a}{}^{d}-\omega_{a\bar{b}}{}^{\bar{d}}\omega_{\bar{c}\bar{d}}{}^{\bar{c}}\,,
\end{aligned} \tag{2.7}$$

where the torsion constraint (2.17) has been used in the first line. The second expression is given by

$$\begin{aligned}
\mathcal{R}_{\bar{b}a} \;=\; R_{c\bar{b}a}{}^{c} \;&=\; E_{c}\omega_{\bar{b}a}{}^{c}-E_{\bar{b}}\omega_{ca}{}^{c}+\omega_{ca}{}^{d}\omega_{\bar{b}d}{}^{c}-\omega_{\bar{b}a}{}^{d}\omega_{cd}{}^{c}-\Omega_{c\bar{b}}{}^{E}\omega_{Ea}{}^{c} \\
&=\; E_{c}\omega_{\bar{b}a}{}^{c}-E_{\bar{b}}\omega_{ca}{}^{c}+\omega_{\bar{d}a}{}^{c}\omega_{c\bar{b}}{}^{\bar{d}}-\omega_{\bar{b}a}{}^{d}\omega_{cd}{}^{c}\,,
\end{aligned} \tag{2.8}$$

and we will confirm that this is equivalent to (2.7). Writing out all connection components explicitly, the Ricci tensor can thus be written as

$$\mathcal{R}_{a\bar{b}} \;=\; \mathcal{R}_{\bar{b}a} \;=\; E_{\bar{b}}\tilde{\Omega}_{a}-E_{c}\Omega_{\bar{b}a}{}^{c}+\Omega_{c\bar{b}}{}^{\bar{d}}\,\Omega_{\bar{d}a}{}^{c}-\Omega_{\bar{b}a}{}^{c}\,\tilde{\Omega}_{c}\,. \tag{2.9}$$

Below we will prove that the curvature scalar, upon gauge fixing, reduces to the one of double field theory given in [7], and that the corresponding field equations for $\mathcal{E}_{ij}$ as determined in [9] give rise to the tensors in (2.7) or (2.8), thus showing their equivalence. This proves that the tensors defined by the general variation (5.28) are indeed the curvature scalar and Ricci tensor.

2.3.2 Covariant gauge variation and Bianchi identity

In this subsection we derive a Bianchi identity from the invariance of (2.1) under ξ^M gauge transformations. To this end it is convenient to first rewrite the gauge transformations in terms of the $GL(D) \times GL(D)$ covariant derivatives. For this we use the following form of the gauge transformation in terms of the C-bracket (c.f. eqs. (3.29) and (3.30) in [8])

$$\delta_\xi E_A{}^M \;=\; \big[\xi, E_A\big]_{\mathrm{C}}^M + \frac{1}{2}\partial^M\big(E_A{}^N\xi_N\big)\,, \tag{2.10}$$

and the fact that in the C bracket we can replace curved by flat indices if we use the $GL(D) \times GL(D)$ covariant derivatives, i.e.,

$$\begin{aligned}\big[\xi, E_A\big]_{\mathrm{C}}^B \;&=\; \xi^C\hat{\nabla}_C E_A{}^B - E_A{}^C\nabla_C\xi^B - \frac{1}{2}\xi_C\hat{\nabla}^B E_A{}^C + \frac{1}{2}E_{AC}\nabla^B\xi^C \\ &=\; -\xi^C\omega_{CA}{}^B - \nabla_A\xi^B + \frac{1}{2}\xi^C\omega^B{}_{AC} + \frac{1}{2}\mathcal{G}_{AC}\nabla^B\xi^C\,.\end{aligned} \tag{2.11}$$

Here we have to stress that the covariant derivatives in the first line do not act on the index A, which we indicated by the notation $\hat{\nabla}$, because A is in (2.10) and (2.11) only a 'spectator' index. Consequently, using $E_A{}^B \equiv E_A{}^M E_M{}^B = \delta_A{}^B$ and $E_{AC} \equiv E_{AM}E_C{}^M = \mathcal{G}_{AC}$, we have $\hat{\nabla}_C E_A{}^B = -\omega_{CA}{}^B$, from which the second equality follows. Using (2.11) in (2.10) we obtain

$$\begin{aligned}\delta_\xi E_A{}^M \;&=\; E_B{}^M\big[\xi, E_A\big]_{\mathrm{C}}^B + \frac{1}{2}\partial^M\xi_A \\ &=\; -\xi^C\omega_{CA}{}^B\,E_B{}^M - E_B{}^M\nabla_A\xi^B + \frac{1}{2}E_B{}^M\omega^B{}_{AC}\xi^C + \frac{1}{2}E_B{}^M\nabla^B\xi_A + \frac{1}{2}\partial^M\xi_A\,.\end{aligned} \tag{2.12}$$

The third and last term combine into a covariant derivative, which in turn combines with the fourth term. Moreover, the first term can be viewed as a field-dependent $GL(D) \times GL(D)$ transformation with parameter $\Lambda_A{}^B = -\xi^C \omega_{CA}{}^B$ and can thus be discarded. Therefore, the final form reads

$$\delta_\xi E_A{}^M \;=\; -E_B{}^M \left(\nabla_A \xi^B - \nabla^B \xi_A\right) \,, \tag{2.13}$$

or, in terms of the variation (5.27),

$$\Delta E_{AB} \;=\; \nabla_B \xi_A - \nabla_A \xi_B \,. \tag{2.14}$$

For the dilaton one finds

$$\begin{aligned}
\delta_\xi d \;&=\; \xi^M \partial_M d - \frac{1}{2}\partial_M \xi^M \;=\; \xi^A E_A d - \frac{1}{2}\partial_M \left(\xi^A E_A{}^M\right) \\
&=\; -\frac{1}{2} E_A \xi^A + \frac{1}{2}\xi^A \left(-\partial_M E_A{}^M + 2E_A d\right) \;=\; -\frac{1}{2}\left(E_A \xi^A - \omega_{BA}{}^B \xi^A\right) \\
&=\; -\frac{1}{2}\nabla_A \xi^A \,,
\end{aligned} \tag{2.15}$$

where we used (5.24) in the second line.

We can now read off the Bianchi identity following from the gauge invariance of (2.1). Using (2.14) and (2.15) in (5.28) we infer

$$\begin{aligned}
0 \;=\; \delta_\xi S \;&=\; \int dx d\tilde{x}\, e^{-2d} \Big(\left(\nabla_a \xi^a + \nabla_{\bar{a}} \xi^{\bar{a}}\right) \mathcal{R} + \left(\nabla_{\bar{b}} \xi_a - \nabla_a \xi_{\bar{b}}\right) \mathcal{R}^{a\bar{b}} \Big) \\
&=\; -\int dx d\tilde{x}\, e^{-2d} \Big(\xi^a \left(\nabla_a \mathcal{R} + \nabla^{\bar{b}} \mathcal{R}_{a\bar{b}}\right) + \xi^{\bar{a}} \left(\nabla_{\bar{a}} \mathcal{R} - \nabla^b \mathcal{R}_{b\bar{a}}\right) \Big) \,,
\end{aligned} \tag{2.16}$$

which implies the Bianchi identities [5]

$$\nabla_a \mathcal{R} + \nabla^{\bar{b}} \mathcal{R}_{a\bar{b}} \;=\; 0 \,, \qquad \nabla_{\bar{a}} \mathcal{R} - \nabla^b \mathcal{R}_{b\bar{a}} \;=\; 0 \,. \tag{2.17}$$

These are equivalent to similar Bianchi identities derived from the double field theory, as we will show in the next section, and reduce to the usual Bianchi identities for R_{ij} and H_{ijk} when $\tilde{\partial} = 0$ [9].

2.4 Relation to formulation with $\mathcal{E}_{ij}$

Here we start the detailed 're-derivation' of the original double field theory formulations introduced in [7] from Siegel's geometrical formalism. We identify the 'non-symmetric' metric $\mathcal{E}_{ij}$ as components of $e_A{}^M$ after a particular gauge fixing. This allows us to study the non-linear realization of the $O(D,D)$ symmetry and to find a rather direct relation between the action (1.31) and the geometrical Einstein-Hilbert like action.

2.4.1 Gauge choice

One way to identify $\mathcal{E}_{ij}$ in the frame-like formalism is to gauge-fix the local $GL(D) \times GL(D)$ symmetry by setting the components $E_a{}^i$ and $E_{\bar{a}}{}^i$ in (2.7) equal to the unit matrix (assuming certain invertibility properties). Taking the constraint (2.10) into account, the remaining components are then parametrized by a general $D \times D$ matrix which we identify with $\mathcal{E}_{ij}$,

$$E_A{}^M = \begin{pmatrix} E_{ai} & E_a{}^i \\ E_{\bar{a}i} & E_{\bar{a}}{}^i \end{pmatrix} = \begin{pmatrix} -\mathcal{E}_{ai} & \delta_a{}^i \\ \mathcal{E}_{i\bar{a}} & \delta_{\bar{a}}{}^i \end{pmatrix} . \tag{2.1}$$

In this gauge, the 'space-time' indices $i, j, \ldots$ can be identified with the frame indices of either $GL(D)$ factor via the trivial vielbeins $\delta_a{}^i$ or $\delta_{\bar{a}}{}^i$. The calligraphic derivatives then coincide with the 'flattened' partial derivatives (2.11),

$$E_a = E_a{}^M \partial_M = \partial_a - \mathcal{E}_{ai}\tilde{\partial}^i \equiv \mathcal{D}_a , \qquad E_{\bar{a}} = E_{\bar{a}}{}^M \partial_M = \partial_{\bar{a}} + \mathcal{E}_{i\bar{a}}\tilde{\partial}^i \equiv \bar{\mathcal{D}}_{\bar{a}} . \tag{2.2}$$

Moreover, the metric $g_{ij} = \mathcal{E}_{(ij)}$ can be identified with either of the two 'tangent space' metrics

$$g_{ab} = -\frac{1}{2} E_a{}^M E_b{}^N \eta_{MN} , \qquad g_{\bar{a}\bar{b}} = \frac{1}{2} E_{\bar{a}}{}^M E_{\bar{b}}{}^N \eta_{MN} , \tag{2.3}$$

as one may verify directly from (2.1). From this it follows that (2.8) is given by

$$\mathcal{G}_{AB} = \begin{pmatrix} -2g_{ab} & 0 \\ 0 & 2g_{\bar{a}\bar{b}} \end{pmatrix} . \tag{2.4}$$

The relative factors of ± 2 appearing here lead, after the gauge fixing (2.1) and the corresponding identification of indices, to an ambiguity regarding the contraction of indices. We will follow the convention that contractions are done with respect to the tangent space metric $\mathcal{G}_{AB}$ when the indices are letters from the beginning of the latin alphabet (i.e., either $a, b \ldots$ or $\bar{a}, \bar{b}, \ldots$), and that contractions are only done with respect to g_{ij} if the indices are letters from the middle of the latin alphabet $(i, j, \ldots)$.

For the comparison with the action (1.31) it is instructive to re-interpret derivatives like $\mathcal{D}_i\mathcal{E}_{jk}$ in a more covariant way. Specifically, in analogy to the modified variation (5.27), we can write this as

$$\mathcal{D}_a\mathcal{E}_{b\bar{c}} = E_b{}^M E_a E_{\bar{c}M} = -E_{\bar{c}}{}^M E_a E_{bM} . \tag{2.5}$$

This follows from the gauge-fixed forms (2.1) and (2.2), and is manifestly $O(D, D)$ invariant. Remarkably, it can also be made manifestly $GL(D) \times GL(D)$ invariant by observing that in

$$E_b{}^M \nabla_a E_{\bar{c}M} = E_b{}^M \big(E_a E_{\bar{c}M} + \omega_{a\bar{c}}{}^{\bar{d}} E_{\bar{d}M} \big) \tag{2.6}$$

the connection term is zero by the constraint (2.10). The same conclusion applies to the barred derivative $E_{\bar{a}} = \bar{\mathcal{D}}_{\bar{a}}$, and so we find in total the following identifications

$$\begin{aligned} \mathcal{D}_a\mathcal{E}_{b\bar{c}} &\equiv E_b{}^M \nabla_a E_{\bar{c}M} = -E_{\bar{c}}{}^M \nabla_a E_{bM} , \\ \bar{\mathcal{D}}_{\bar{a}}\mathcal{E}_{b\bar{c}} &\equiv E_b{}^M \nabla_{\bar{a}} E_{\bar{c}M} = -E_{\bar{c}}{}^M \nabla_{\bar{a}} E_{bM} , \end{aligned} \tag{2.7}$$

which are manifestly covariant with respect to $O(D, D)$ and tangent space transformations.

In the following we will examine how the $O(D, D)$ duality symmetry is realized after this gauge fixing. Acting with a general $O(D, D)$ transformation on (2.1) violates the gauge condition and thus requires a compensating $GL(D) \times GL(D)$ transformation. In order to determine the transformation that restores the form of the vielbein (2.1), we consider a finite $O(D,D)$ and $GL(D)$ transformation,

$$E_a{}^{M\prime}(X') \;=\; h^M{}_N\,(M^{-1}(X))_a{}^b\,E_b{}^N(X)\,. \tag{2.8}$$

Here we denoted the $GL(D)$ matrix by M^{-1} for later convenience, and h is the $O(D,D)$ matrix, whose components read

$$h^M{}_N \;=\; \begin{pmatrix} h_i{}^j & h_{ij} \\ h^{ij} & h^i{}_j \end{pmatrix} \;=\; \begin{pmatrix} a_i{}^j & b_{ij} \\ c^{ij} & d^i{}_j \end{pmatrix}\,. \tag{2.9}$$

Applied to the gauge-fixed component we find

$$E_a{}^{i\prime} \;=\; (M^{-1})_a{}^b\left(h^i{}_j E_b{}^j + h^{ij}E_{bj}\right) \;=\; \left(M^{-1}(d^t - \mathcal{E}c^t)\right)_a{}^i \;=\; \delta_a{}^i\,, \tag{2.10}$$

where we used matrix notation and suppressed the X-dependence. The last equation expresses the condition that the gauge fixing condition be preserved. Analogously, one finds for the other component

$$E_{\bar{a}}{}^i \;=\; (\bar{M}^{-1})_{\bar{a}}{}^{\bar{b}}\left(h^i{}_j E_{\bar{b}}{}^j + h^{ij}E_{\bar{b}j}\right) \;=\; \left(\bar{M}^{-1}(d^t + \mathcal{E}^t c^t)\right)_{\bar{a}}{}^i \;=\; \delta_{\bar{a}}{}^i\,, \tag{2.11}$$

where we denoted the matrix corresponding to the second $GL(D)$ factor by $\bar{M}^{-1}$. The two conditions (2.10) and (2.11) thus determine the compensating $GL(D) \times GL(D)$ transformations uniquely in terms of c and d,

$$M(X) \;=\; d^t - \mathcal{E}(X)c^t\,, \qquad \bar{M}(X) \;=\; d^t + \mathcal{E}^t(X)c^t\,, \tag{2.12}$$

which are both X-dependent through their dependence on $\mathcal{E}_{ij}$. Finally, using this form of the compensating gauge transformations it is straightforward to verify that $\mathcal{E}_{ij}$ transforms under $O(D,D)$ in the required non-linear representation according to the Buscher rules, which we repeat here for the reader's convenience

$$\mathcal{E}'(X') \;=\; (a\mathcal{E}(X)+b)(c\mathcal{E}(X)+d)^{-1}\,, \quad d'(X') \;=\; d(X)\,, \quad X'=hX\,. \tag{2.13}$$

With the above analysis of the non-linear realization of $O(D,D)$ we have in fact recovered the formalism that has been used in [7] (extending the background-dependent formalism in [6, 32]) in order to prove the $O(D,D)$ invariance of the action (1.31). More precisely, in this formalism every index is thought of either as an unbarred or barred index and to transform, accordingly, either under M or $\bar{M}$ in (2.12). For instance, we have just verified that the calligraphic derivatives (2.2) transform with M or $\bar{M}$, respectively. Moreover, due to the manifestly $O(D,D)$ and $GL(D)\times GL(D)$ covariant rewriting of the calligraphic derivatives of $\mathcal{E}$ in (2.7), it follows that after gauge fixing

$$\mathcal{D}_a\mathcal{E}_{b\bar{c}} \;=\; M_a{}^d\, M_b{}^e\, \bar{M}_{\bar{c}}{}^{\bar{f}}\, \mathcal{D}'_d\mathcal{E}'_{e\bar{f}}\,, \qquad \bar{\mathcal{D}}_{\bar{a}}\mathcal{E}_{b\bar{c}} \;=\; \bar{M}_{\bar{a}}{}^{\bar{d}}\, M_b{}^e\, \bar{M}_{\bar{c}}{}^{\bar{f}}\, \bar{\mathcal{D}}'_{\bar{d}}\mathcal{E}'_{e\bar{f}}\,. \tag{2.14}$$

Thus, we can think of the first index on $\mathcal{E}$ (under $\mathcal{D}$ or $\bar{\mathcal{D}}$) as unbarred and the second index as barred. From the definition (2.3) we conclude that the indices on g can be thought of either as both barred or both unbarred, because g can be viewed as a tensor either of the left $GL(D)$ or the right $GL(D)$ such that it transforms after gauge fixing as

$$g_{\bar{a}\bar{b}} \;=\; \bar{M}_{\bar{a}}{}^{\bar{c}}\, \bar{M}_{\bar{b}}{}^{\bar{d}}\, g'_{\bar{c}\bar{d}}\,, \qquad g_{ab} \;=\; M_a{}^c\, M_b{}^d\, g'_{cd}\,, \tag{2.15}$$

and similarly for the inverse. The $O(D,D)$ invariance of the action is then a consequence of the fact, which one may easily confirm by inspection of the action (1.31), that only like-wise indices are contracted [7].

2.4.2 $O(D,D)$ covariant derivatives and gauge variation

In the previous subsection we have seen that in the formulation using $\mathcal{E}_{ij}$ the $O(D,D)$ transformations are governed by the matrices M and $\bar{M}$ in (2.12). Since these matrices are X-dependent, it follows that derivatives of objects that transform 'covariantly' with M and $\bar{M}$ according to their index structure are in general not covariant in the same sense. This led ref. [7] to introduce '$O(D,D)$ covariant derivatives' — despite $O(D,D)$ being a *global* symmetry with constant parameters. There are two types of covariant derivatives, $\nabla_i(\Gamma)$ and $\bar{\nabla}_i(\Gamma)$, i.e., unbarred and barred, and various connections Γ depending on the index structure of the object on which the derivative acts. Here we indicate the dependence on the connections explicitly, in order to distinguish these 'covariant' derivatives from the $GL(D) \times GL(D)$ covariant derivatives introduced before.

Since we have here realized the global non-linear $O(D,D)$ transformations according to M and $\bar{M}$ through compensating $GL(D) \times GL(D)$ transformations, it is natural to assume that, after gauge fixing, the $GL(D) \times GL(D)$ covariant derivatives are related to the '$O(D,D)$ covariant derivatives' of [7]. This indeed turns out to be the case, and so we are able to give a more conventional interpretation of these covariant derivatives.

As a first test of this relation we reproduce a manifestly $O(D,D)$ covariant form of the ξ^M gauge transformations that has been found in [7]. Specifically, introducing the following change of basis for the gauge parameters (which is suggested by the gauge structure in string field theory [6]),

$$\eta_i \;=\; -\tilde{\xi}_i + \mathcal{E}_{ij}\xi^j \,, \qquad \bar{\eta}_i \;=\; \tilde{\xi}_i + \xi^j\mathcal{E}_{ji} \,, \tag{2.16}$$

the gauge transformations of $\mathcal{E}_{ij}$ take the remarkable form

$$\delta\mathcal{E}_{ij} \;=\; \nabla_i(\Gamma)\bar{\eta}_j + \bar{\nabla}_j(\Gamma)\eta_i \,. \tag{2.17}$$

The corresponding result using the $GL(D) \times GL(D)$ connections follows almost immediately. First, the flattened gauge parameters

$$\eta_a \;:=\; -\xi_a \;\equiv\; -E_a{}^M\xi_M \,, \qquad \bar{\eta}_{\bar{a}} \;:=\; \xi_{\bar{a}} \;\equiv\; E_{\bar{a}}{}^M\xi_M \,, \tag{2.18}$$

coincide with (2.16) upon using (2.1). Moreover, after the gauge fixing (2.1), any variation of $\mathcal{E}$ coincides with the Δ variation in (5.27),

$$\delta\mathcal{E}_{a\bar{b}} \;=\; \Delta E_{\bar{b}a} \;=\; E_a{}^M\,\delta E_{\bar{b}M} \;=\; E_a{}^i\,\delta E_{\bar{b}i} + E_{ai}\,\delta E_{\bar{b}}{}^i\,. \tag{2.19}$$

This follows because the last term is zero by the gauge fixing condition. More precisely, for the ξ^M gauge variation this term will vanish by a compensating frame rotation that restores the chosen gauge. The advantage of using the Δ variation is that this compensating transformation need not to be determined explicitly. Applying now (2.14) one finds in the basis (2.18)

$$\delta\mathcal{E}_{a\bar{b}} \;=\; \nabla_a\bar{\eta}_{\bar{b}} + \nabla_{\bar{b}}\eta_a\,, \tag{2.20}$$

which agrees with (2.17), using that after gauge fixing the indices $i, j, \ldots$ can be identified with the flat indices.

We note in passing that the original form of the gauge transformations of $\mathcal{E}_{ij}$ field also follows easily by use of the Δ variation as in (2.19),

$$\begin{aligned}
\delta\mathcal{E}_{a\bar{b}} \;&=\; \Delta E_{\bar{b}a} \;=\; E_a{}^M\delta E_{\bar{b}M} \;=\; E_a{}^M\left(\xi^N\partial_N E_{\bar{b}M} + (\partial_M\xi^N - \partial^N\xi_M)E_{\bar{b}N}\right) \\
&=\; \xi^N\partial_N\mathcal{E}_{a\bar{b}} + \mathcal{D}_a\xi^N E_{\bar{b}N} - \bar{\mathcal{D}}_{\bar{b}}\xi_M E_a{}^M \\
&=\; \xi^N\partial_N\mathcal{E}_{a\bar{b}} + \mathcal{D}_a\tilde{\xi}_j\,E_{\bar{b}}{}^j + \mathcal{D}_a\xi^j\,E_{\bar{b}j} - \bar{\mathcal{D}}_{\bar{b}}\tilde{\xi}_j\,E_a{}^j - \bar{\mathcal{D}}_{\bar{b}}\xi^j\,E_{aj} \\
&=\; \xi^N\partial_N\mathcal{E}_{a\bar{b}} + \mathcal{D}_a\tilde{\xi}_{\bar{b}} + \mathcal{D}_a\xi^j\mathcal{E}_{j\bar{b}} - \bar{\mathcal{D}}_{\bar{b}}\tilde{\xi}_a + \bar{\mathcal{D}}_{\bar{b}}\xi^j\mathcal{E}_{aj}\,.
\end{aligned} \tag{2.21}$$

Here we used (2.2) in the second line and the gauge fixed form (2.1) in the last line, where we again identified indices. Thus we have derived the gauge transformation of $\mathcal{E}_{ij}$ from the fundamental gauge transformation of the vielbein, as in [8], but without invoking the compensating frame rotation explicitly.

The previous results show that the '$O(D,D)$ covariant derivatives' coincide with the $GL(D)\times GL(D)$ covariant derivatives after gauge fixing, at least when acting on η and $\bar{\eta}$ as in (4.207). The complete set of connections Γ is not fixed by $O(D,D)$ covariance and therefore have been given in [7] only provisionally. Here we display for completeness their relation after gauge fixing,

$$
\begin{aligned}
\omega_{i\bar{j}}{}^{\bar{k}} &= -\frac{1}{2}g^{kl}\Big(\mathcal{D}_i\mathcal{E}_{lj} + \bar{\mathcal{D}}_j\mathcal{E}_{il} - \bar{\mathcal{D}}_l\mathcal{E}_{ij}\Big) \;=\; -\Gamma^{\bar{k}}_{i\bar{j}}\,, \\
\omega_{\bar{i}j}{}^{k} &= -\frac{1}{2}g^{kl}\Big(\bar{\mathcal{D}}_i\mathcal{E}_{jl} + \mathcal{D}_j\mathcal{E}_{li} - \mathcal{D}_l\mathcal{E}_{ji}\Big) \;=\; -\Gamma^{k}_{\bar{i}j}\,, \\
\omega_{ji}{}^{j} &= \frac{1}{2}(\bar{\mathcal{D}}^j - \mathcal{D}^j)\mathcal{E}_{ij} + 2\mathcal{D}_i d \;=\; -\Gamma^{j}_{ji} + \frac{1}{2}\bar{\mathcal{D}}^j\mathcal{E}_{ij} + 2\mathcal{D}_i d\,, \\
\omega_{\bar{j}\bar{i}}{}^{\bar{j}} &= \frac{1}{2}(\mathcal{D}^j - \bar{\mathcal{D}}^j)\mathcal{E}_{ji} + 2\bar{\mathcal{D}}_i d \;=\; -\Gamma^{\bar{j}}_{\bar{j}\bar{i}} + \frac{1}{2}\mathcal{D}^j\mathcal{E}_{ji} + 2\bar{\mathcal{D}}_i d\,.
\end{aligned}
\tag{2.22}
$$

We see that they are equivalent in the 'off-diagonal' parts but differ in the trace parts. In fact, it has already been noted, c.f. the discussion around eq. (4.13) in [7], that modifying the definition as suggested by (2.22) would have the advantage of simplifying the gauge transformation of d in that

$$
\delta d \;=\; -\frac{1}{4}\nabla_i\eta^i - \frac{1}{4}\bar{\nabla}_i\bar{\eta}^i\,. \tag{2.23}
$$

Here we see that this is a direct consequence of (2.15), where we recall that according to our index conventions g rather than $\mathcal{G}$ is used to raise indices in (2.23), and that there is a relative sign in the definition (2.18) of η_i. In [7], however, there was no justification from symmetry arguments for this modification, but here we see it emerging naturally from Siegel's frame formalism.

Given the precise correspondence between the $O(D,D)$ and $GL(D)\times GL(D)$ connections, we have verified that the curvature scalar and Ricci tensor of Siegel's formalism agree with the corresponding expressions obtained in [7] and [9]. More precisely, the scalar curvature constructed from Siegel's frame formalism is $\frac{1}{4}$ times $\mathcal{R}(\mathcal{E},d)$ as given in [7]. Taking this factor as well as the relative factors of $\pm\frac{1}{2}$ in (2.3) into account, the Bianchi identities (2.17) reduce to

$$
\nabla^i\mathcal{R}_{i\bar{j}} + \frac{1}{2}\mathcal{D}_j\mathcal{R}(\mathcal{E},d) \;=\; 0\,, \qquad \bar{\nabla}^j\mathcal{R}_{i\bar{j}} + \frac{1}{2}\mathcal{D}_i\mathcal{R}(\mathcal{E},d) \;=\; 0\,, \tag{2.24}
$$

which agree with [9].

Starting from the expression (2.39) for the scalar curvature we can actually immediately compare with the double field theory action (1.31) in terms of $\mathcal{E}_{ij}$. Using that the covariant derivatives allow for partial integration in presence of the dilaton density, we infer that the first line in (2.39) contributes only total derivatives under an integral, and thus the resulting Lagrangian is equivalent to

$$\begin{aligned} \mathcal{L}' \;=\; e^{-2d}\Big(& -\frac{1}{2}E^{aM}\nabla^{b}E^{\bar{c}}{}_{M}\,E_{a}{}^{N}\nabla_{b}E_{\bar{c}N}+\frac{1}{2}E_{\bar{c}}{}^{M}\nabla_{a}E_{bM}\,E^{\bar{c}N}\nabla^{b}E^{a}{}_{N} \\ & -\frac{1}{2}E_{c}{}^{M}\bar{\nabla}_{\bar{a}}E_{\bar{b}M}\,E^{cN}\bar{\nabla}^{\bar{b}}E^{\bar{a}}{}_{N}-\nabla^{a}d\,E_{a}{}^{M}\bar{\nabla}^{\bar{b}}E_{\bar{b}M} \\ & +\bar{\nabla}^{\bar{a}}d\,E_{\bar{a}}{}^{M}\nabla^{b}E_{bM}-2\nabla^{a}d\,\nabla_{a}d\Big)\;. \end{aligned} \tag{2.25}$$

Taking into account the relation (2.3) between g and the tangent space metric, and using that the latter is covariantly constant, it then immediately follows by virtue of the identifications (2.7) that (2.25) agrees with the action (1.31) up to the global factor of 4.

2.5 Relation to formulation with $\mathcal{H}^{MN}$

In this section we introduce the formulation in terms of the generalized metric $\mathcal{H}^{MN}$ from the point of view of the frame formalism and discuss Christoffel-type connections that are introduced via a vielbein postulate.

2.5.1 Gauge choice and generalized coset formulation

We next identify the generalized metric and the corresponding formulation in the geometrical frame formalism. In general, one can define $\mathcal{H}^{MN}$ in terms of the frame field through [8]

$$\mathcal{H}^{MN} \;=\; 2\mathcal{G}^{\bar{a}\bar{b}}\,E_{\bar{a}}{}^{M}E_{\bar{b}}{}^{N}-\eta^{MN} \;=\; -2\mathcal{G}^{ab}\,E_{a}{}^{M}E_{b}{}^{N}+\eta^{MN}\,, \tag{2.1}$$

where the second equation is a consequence of the definition (2.8) and the constraint (2.10). The generalized metric is a constrained field in that

$$\mathcal{H}^{MK}\mathcal{H}_{KN} \;=\; \delta^{M}{}_{N}\,, \tag{2.2}$$

where the indices are lowered, as usual, with η_{MN}. In the standard parametrization

$$\mathcal{H}^{MN} \;=\; \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix}\,, \tag{2.3}$$

this can be checked by a direct computation. Here, it can be verified with either one of the definitions in (3.101). We note, however, that if we use for the first $\mathcal{H}$ in (2.2), say, the first expression in (3.101) and for the second $\mathcal{H}$ the second expression, then the constraint (2.10) is required in order to verify this.

For later use we note that (3.101) implies for the flattened components of the generalized metric

$$\mathcal{H}^{AB} \;=\; \mathcal{H}^{MN}\,E_{M}{}^{A}\,E_{N}{}^{B} \;=\; \begin{pmatrix} -\mathcal{G}^{ab} & 0 \\ 0 & \mathcal{G}^{\bar{a}\bar{b}} \end{pmatrix}\,, \tag{2.4}$$

where again (2.10) has been used.

In the following, we find it convenient to fix the $GL(D)\times GL(D)$ symmetry by setting the tangent space metric (2.8) to

$$\mathcal{G}_{AB} \;=\; \begin{pmatrix} -\delta_{ab} & 0 \\ 0 & \delta_{\bar{a}\bar{b}} \end{pmatrix}\,. \tag{2.5}$$

This implies $g_{ab} = 2\delta_{ab}$ and $g_{\bar{a}\bar{b}} = 2\delta_{\bar{a}\bar{b}}$ from the definition (2.3) and also $\mathcal{H}_{AB} = \delta_{AB}$ from (2.4). This leaves a residual local $O(D)\times O(D)$ symmetry. Therefore, the resulting formulation can be viewed as a generalized coset model based on $O(D,D)/(O(D)\times O(D))$ [8]. In fact, from (2.5) we conclude with (2.8) that $E_{A}{}^{M}$ is an $O(D,D)$ element (up to a similarity transformation) in that it transforms the $O(D,D)$ metric

η into the $O(D, D)$ metric, but written in the form (2.5). Thus, e can be viewed as a group-valued coset representative with a local $O(D) \times O(D)$ action from the left. Moreover, (2.4) implies

$$\mathcal{H}^{MN} \;=\; \delta^{AB}\, E_A{}^M\, E_B{}^N\,, \tag{2.6}$$

and so $\mathcal{H}$ can be viewed as the $O(D) \times O(D)$ invariant combination $E^t E$. For completeness we record that the form of the coset representative that leads to the standard parametrization (2.3) for $\mathcal{H}^{MN}$ according to (3.102) is given by

$$E_A{}^M \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} e_{ai} + b_{ij}e_a{}^j & e_a{}^i \\ -e_{\bar{a}i} + b_{ij}e_{\bar{a}}{}^j & e_{\bar{a}}{}^i \end{pmatrix}\,, \tag{2.7}$$

where $e_i{}^a$ is the conventional vielbein for the metric g_{ij}, i.e., $g_{ij} = e_i{}^a e_{ja}$, with inverse $e_a{}^i$. We recall that an explicit parametrization like this requires a further gauge fixing of the local $O(D) \times O(D)$ symmetry.

2.5.2 Scalar curvature

Next, we prove that the Ricci scalar (2.40) reduces upon the gauge fixing (2.5) to the function $\mathcal{R}(\mathcal{H}, d)$ given in [8], and thus that the actions in (2.1) and (1.45) are equivalent. The scalar curvature (2.40) reduces to

$$\mathcal{R} \;=\; E_a\tilde{\Omega}^a + \frac{1}{2}\tilde{\Omega}_a^2 - \frac{1}{4}\Omega_{ab\bar{c}}{}^2 - \frac{1}{12}\Omega_{[abc]}{}^2\,. \tag{2.8}$$

We first evaluate the dilaton-dependent terms, which originate only from the first two terms. Using (5.24) we find

$$E_a\tilde{\Omega}^a + \frac{1}{2}\tilde{\Omega}_a^2\Big|_d = -2E^{aN}\partial_N\left(E_a{}^M\partial_M d\right) - 2\partial_M E_a{}^M\, E^{aN}\partial_N d + 2E_a{}^M\partial_M d\, E^{aN}\partial_N d \tag{2.9}$$

With the expression for $\mathcal{H}^{MN}$ from (3.101) this reduces to

$$E_a\tilde{\Omega}^a + \frac{1}{2}\tilde{\Omega}_a^2\Big|_d \;=\; \mathcal{H}^{MN}\partial_M\partial_N d + \partial_N\mathcal{H}^{MN}\,\partial_M d - \mathcal{H}^{MN}\partial_M d\,\partial_N d\,, \tag{2.10}$$

where we used the strong constraint.

We turn next to the pure $E_A{}^M$-dependent terms which are more involved. The first two terms in (2.8) yield

$$E_a\tilde{\Omega}^a + \frac{1}{2}\tilde{\Omega}_a^2\Big|_e = -\frac{1}{4}\partial_M\partial_N\mathcal{H}^{MN} - \frac{1}{2}\partial_N E^{aM}\partial_M E_a{}^N . \tag{2.11}$$

In order to compute the third term in (2.8) we obtain

$$\begin{aligned}\Omega_{ab\bar{c}}{}^2 = &-\frac{1}{4}\mathcal{H}^{KL}(\mathcal{H}_{MN} - \eta_{MN})\partial_K E_b{}^M\partial_L E^{bN} \\ &+ (\mathcal{H}^{NK} - \eta^{NK})(\mathcal{H}_{ML} + \eta_{ML})\partial_K E_b{}^M\partial^L E^b{}_N .\end{aligned} \tag{2.12}$$

For the final term in (2.8) we compute

$$\Omega_{[abc]}{}^2 = \frac{3}{4}\big[\mathcal{H}_{KL}(\mathcal{H}_{MN}-\eta_{MN})-2(\mathcal{H}_{ML}-\eta_{ML})(\mathcal{H}_{NK}-\eta_{NK})\big]\partial^K E_b{}^M\,\partial^L E^{bN} . \tag{2.13}$$

In total, the third and the fourth term of $\mathcal{R}$ in (2.8) combine as follows:

$$\begin{aligned}-\frac{1}{4}\Omega_{ab\bar{c}}{}^2 - \frac{1}{12}\Omega_{[abc]}{}^2 = \;& \frac{1}{8}\mathcal{H}^{KL}(\mathcal{H}_{MN} + \eta_{MN})\partial_K E_b{}^M\partial_L E^{bN} \\ &+\frac{1}{4}(\mathcal{H}_{MN} + \eta_{MN})E_b E_a{}^M\, E^a E^{bN} \\ &-\frac{1}{8}(\mathcal{H}^{NK} - \eta^{NK})(\mathcal{H}_{ML} + 3\eta_{ML})\partial_K E_b{}^M\partial^L E^b{}_N .\end{aligned} \tag{2.14}$$

Adding (2.11) and (2.15) one obtains after some work

$$\begin{aligned}\mathcal{R}\Big|_{\mathcal{H}} = \;& -\frac{1}{4}\partial_M\partial_N\mathcal{H}^{MN} + \frac{1}{32}\mathcal{H}^{KL}\partial_K\mathcal{H}_{MN}\partial_L\mathcal{H}^{MN} \\ &-\frac{1}{8}\mathcal{H}^{ML}\partial_K\mathcal{H}_{MN}\partial_L\mathcal{H}^{NK} .\end{aligned}$$

In combination with (2.10) we obtain in total

$$\begin{aligned}\mathcal{R} = \;& \mathcal{H}^{MN}\partial_M\partial_N d + \partial_N\mathcal{H}^{MN}\,\partial_M d - \mathcal{H}^{MN}\partial_M d\,\partial_N d \\ &-\frac{1}{4}\partial_M\partial_N\mathcal{H}^{MN} + \frac{1}{32}\mathcal{H}^{KL}\partial_K\mathcal{H}^{MN}\partial_L\mathcal{H}_{MN} - \frac{1}{8}\mathcal{H}^{ML}\partial_L\mathcal{H}^{NK}\partial_K\mathcal{H}_{MN} .\end{aligned} \tag{2.15}$$

This coincides with the curvature scalar $\mathcal{R}(\mathcal{H}, d)$ constructed in [8], up to the same irrelevant overall factor of 4 encountered above, and thus we have established independently the equivalence of the two action principles. Many detailed computations are omitted in this section of the thesis and can be found in the main text and the appendix of [10].

2.6 Conclusions and Discussions

In this chapter we have shown that the duality-covariant formalism developed by Siegel already some time ago in [5] provides a geometrical framework of double field theory in terms of frame fields, connections and curvatures for the gauge group $GL(D) \times GL(D)$. For the convenience of the reader we summarize here the main differences to ordinary Riemannian geometry.

First of all, a central object is the $O(D, D)$ invariant metric η which is a constant 'world tensor' with two upper or two lower indices. In Riemannian geometry such an object would not be well-defined, but here the constancy of η has a gauge invariant meaning due to the modified form of the gauge transformations, governed by the 'generalized Lie derivatives' (2.1). In contrast to the 'world' metric η^{MN}, the 'tangent space metric' $\mathcal{G}_{AB}$ is space-time dependent, and thus we have the opposite of the usual situation. It is instructive to compare this with a reformulation of conventional Riemannian geometry that resembles the formalism presented here in that there is an enlarged group of frame transformations, the general linear group $GL(D)$ rather than the Lorentz group, and a space-time dependent tangent space metric g_{ab} that enters together with the vielbein $e_a{}^m$ as an independent field (see sec. IX.A.2 in [33]). Imposing a metricity condition and the usual torsion constraint,

$$\nabla_a g_{bc} \;=\; 0\,, \qquad T_{ab}{}^c \;=\; -2e_a{}^m e_b{}^n \nabla_{[m} e_{n]}{}^c \;=\; 0\,, \tag{2.1}$$

allows one to solve for the connections ω_{abc} in terms of derivatives of $e_a{}^m$ and g_{ab}. The local $GL(D)$ symmetry can then be fixed by setting either $e_a{}^m = \delta_a{}^m$, in which case g_{ab} can be identified with the usual metric and the ω_{abc} reduce to the Christoffel symbols Γ_{abc}, or one can set $g_{ab} = \delta_{ab}$, in which case $e_a{}^m$ carries the physical degrees of freedom and ω_{abc} reduces to the usual spin connection. This formalism differs, however, from the present frame formalism, at least in the form discussed in this paper, in several respects. For instance, here it is not the tangent space metric $\mathcal{G}_{AB}$ that is introduced as an independent object but rather the constant $O(D,D)$ invariant metric η^{MN}, while $\mathcal{G}_{AB}$ is defined in terms of η_{MN} by use of the frame fields. Moreover, the torsion constraint is modified as compared to (2.1).

Perhaps the most important difference to Riemannian geometry is the novel gauge symmetry parametrized by ξ^M, whose algebra is governed by the C-bracket rather than the Lie bracket of the usual diffeomorphisms. This has a number of consequences. Most importantly, due to the modified torsion constraint, the Riemann-like tensor defined through the commutator of covariant derivatives is generally not covariant under frame rotations. Following [33] this can be repaired 'by hand', but is should be stressed that the resulting curvature tensor, which is fully covariant, is not in all components independent on the undetermined connections. The resulting Ricci-like tensor and scalar curvature are, however, fully expressible in terms of the physical fields, and are equivalent to the field equations and Lagrangian of double field theory, respectively.

Chapter 3

Double Field Theory Formulation of Heterotic Strings

A bulk of this chapter appeared in"Double Field Theory Formulation of Heterotic Strings" with Olaf Hohm [11] and is reprinted with the permission of *JHEP*.

Summary : We extend the recently constructed double field theory formulation of the low-energy theory of the closed bosonic string to the heterotic string. The action can be written in terms of a generalized metric that is a covariant tensor under $O(D, D+n)$, where n denotes the number of gauge vectors, and n additional coordinates are introduced together with a covariant constraint that locally removes these new coordinates. For the abelian subsector, the action takes the same structural form as for the bosonic string, but based on the enlarged generalized metric, thereby featuring a global $O(D, D+n)$ symmetry. After turning on non-abelian gauge couplings, this global symmetry is broken, but the action can still be written in a fully $O(D, D+n)$ covariant fashion, in analogy to similar constructions in gauged supergravities.

3.1 Introduction and Overview

In this chapter we extend the double field theory formulation of [8] to the low-energy action of the heterotic string [34], which features extra non-abelian gauge fields. In its low-energy limit, this theory is described by an effective two-derivative action which extends the standard Einstein-Kalb-Ramond-dilaton action by n non-abelian gauge fields $A_i{}^\alpha$, $\alpha = 1, \dots, n$, [35],

$$S \;=\; \int dx\sqrt{g}e^{-2\phi}\Big[R + 4(\partial\phi)^2 - \frac{1}{12}\hat{H}^{ijk}\hat{H}_{ijk} - \frac{1}{4}F^{ij\alpha}F_{ij\alpha}\Big]\,, \tag{3.1}$$

where

$$F_{ij}{}^\alpha \;=\; \partial_i A_j{}^\alpha - \partial_j A_i{}^\alpha + g_0\big[A_i, A_j\big]^\alpha \tag{3.2}$$

is the non-abelian field strength of the gauge vectors, and the field strength of the b-field gets modified by a Chern-Simons 3-form,

$$\hat{H}_{ijk} \;=\; 3\left(\partial_{[i}b_{jk]} - \kappa_{\alpha\beta}A_{[i}{}^\alpha\Big(\partial_j A_{k]}{}^\beta + \tfrac{1}{3}g_0\big[A_j, A_{k]}\big]^\beta\Big)\right)\,. \tag{3.3}$$

Here g_0 denotes the gauge coupling constant and $\kappa_{\alpha\beta}$ is the invariant Cartan-Killing form. With the gauge field transforming as

$$\delta_\Lambda A_i{}^\alpha \;=\; \partial_i\Lambda^\alpha + g_0\big[A_i, \Lambda\big]^\alpha\,, \tag{3.4}$$

the b-field transforms under Λ^α as

$$\delta_\Lambda b_{ij} \;=\; \frac{1}{2}\big(\partial_i A_j{}^\alpha - \partial_j A_i{}^\alpha\big)\,\Lambda_\alpha\,, \tag{3.5}$$

such that (3.3) is invariant. At the level of the classical supergravity action the gauge group is arbitrary, but in heterotic string theory it is either $SO(32)$ or $E_8 \times E_8$.

For the abelian subsector the double field theory extension of the heterotic string is straightforward. To this end, the coordinates are further extended by n extra coordinates y^α and, correspondingly, the generalized metric $\mathcal{H}^{MN}$ is enlarged to a $(2D+n)\times(2D+n)$ matrix that naturally incorporates the additional fields $A_i{}^\alpha$ in precise analogy to the coset structure appearing in dimensional reductions. This suggests an enhancement of the global symmetry to $O(D, D+n)$. Indeed, if we formally keep the action

$$S = \int dx d\tilde{x}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d) \,, \tag{3.6}$$

and the form of the gauge transformations $\delta_\xi \mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi \mathcal{H}^{MN}$, but with respect to the enlarged $\mathcal{H}^{MN}$, we obtain precisely the (abelian subsector of the) required action (3.1) and the correct gauge transformations in the limit that the new coordinates are set to zero. In this construction, the number n of new coordinates is not constrained, but the case relevant for heterotic string theory is $n = 16$, where the y^α can be thought of as the coördinates of the internal torus corresponding to the Cartan subalgebra of $SO(32)$ or $E_8 \times E_8$.

The double field formulation of heterotic strings can be extended to the non-abelian gauge fields. In this case the group $O(D, D+n)$ is broken. More precisely, the reduction of the low-energy effective action (i.e., of heterotic supergravity) on a torus T^D gives rise to a theory with a global $O(D, D+n)$ symmetry only in the abelian limit $g_0 \to 0$ [36]. Remarkably, however, we find that the action can be extended to incorporate the non-abelian gauge couplings in a way that formally preserves $O(D, D+n)$, where n equals the dimension of the full gauge group. More precisely, we write the extended action in terms of a tensor $f^M{}_{NK}$, which encodes the structure constants of the gauge group, and the generalized metric $\mathcal{H}^{MN}$. The consistency of this construction requires a number of $O(D, D+n)$-covariant constraints on $f^M{}_{NK}$, which reads

$$f^M{}_{NK}\, \partial_M = 0 \,. \tag{3.7}$$

Moreover, the gauge variations parametrized by ξ^M get deformed by $f^M{}_{NK}$ in that, say, a 'vector' V^M transforms as

$$\delta_\xi V^M = \widehat{\mathcal{L}}_\xi V^M - \xi^K f^M{}_{KL} V^L \,, \tag{3.8}$$

Thus, the ξ^M gauge transformations represent a curious mix between diffeomorphism-like symmetries (which simultaneously treat each index as upper and lower index) and the adjoint rotations with respect to some Lie group. The invariance of the action under these deformed gauge transformations then requires new couplings to be added to (3.6), whose Lagrangian reads (without the e^{-2d} prefactor)

$$\begin{aligned}\mathcal{L}_f &= -\frac{1}{2}f^M{}_{NK}\mathcal{H}^{NP}\mathcal{H}^{KQ}\partial_P\mathcal{H}_{QM} \\ &\quad -\frac{1}{12}f^M{}_{KP}f^N{}_{LQ}\mathcal{H}_{MN}\mathcal{H}^{KL}\mathcal{H}^{PQ} - \frac{1}{4}f^M{}_{NK}f^N{}_{ML}\mathcal{H}^{KL} - \frac{1}{6}f^{MNK}f_{MNK}\,.\end{aligned} \tag{3.9}$$

Despite the $O(D, D+n)$ covariant form of the action, any non-vanishing choice for the $f^M{}_{NK}$ will actually break the symmetry to the subgroup that leaves this tensor invariant, because $f^M{}_{NK}$ is not a dynamical field and therefore does not transform under the T-duality group. For instance, if we choose $f^M{}_{NK}$ to be non-vanishing only for the components $f^\alpha{}_{\beta\gamma}$ that are the structure constants of a semi-simple Lie group G, the remaining symmetry will be $O(D, D) \times G$, where G is the rigid subgroup of the gauge group. In this case, the new couplings (3.9) precisely constitute the non-abelian gauge couplings required by (3.1), while the gauge variations (3.8) evaluated for $\mathcal{H}^{MN}$ reduce to the non-abelian Yang-Mills transformations.

It should be stressed that the abelian and non-abelian cases are conceptually quite different. The abelian case is closely related to the original construction in [6]. Specifically, if we choose $n = 16$, the $O(D, D)$ covariant constraint in its strong form can be interpreted as a stronger form of the level-matching condition. Moreover, the winding coordinates $\tilde{x}_i$ and the y^α have a direct interpretation in the full string theory. In contrast, the non-abelian case requires the new constraint (3.7), which has no obvious interpretation in string theory, and formally we introduce as many new coordinates as the dimension of the gauge group, i.e., $n = 496$ for the case relevant to heterotic string theory. However, the number n is a free parameter at the level of the double field theory constructions discussed here, and therefore we will not introduce different notations for n in the two cases.

The original construction of double field theory is closely related to a frame-like geometrical formalism developed by Siegel in important independent work [5]. The precise relation to the formulation in terms of a generalized metric is by now well-understood both at the level of the symmetry transformations [8] and the action [10]. Siegel's formalism as presented in [5] is already adapted to include the abelian subsector of the heterotic theory. Using the recent results of [10], it is straightforward to verify the equivalence of this formalism with the generalized metric formulation in the abelian limit.

This chapter is organized as follows. In sec. 2 we extend the double field theory construction to the heterotic string for the abelian subsector. In sec. 3 we discuss the non-abelian extension of this formulation. The physical implication of $O(D,D)$ covariant constraints in heterotic string context is investigated in sec. 4. We show that Siegel's frame formalism naturally encodes the double field theory formulation of heterotic strings, either abelian or non-abelian extensions, in sec. 5.

3.2 Double field theory with abelian gauge fields

In this section we introduce the double field theory formulation for the abelian subsector of the low-energy theory of the heterotic string. We first define the enlarged generalized metric and then show that the action (3.6) and the gauge transformations of $\mathcal{H}^{MN}$ reduce to the required form when the dependence on the new coordinates is dropped.

3.2.1 Conventions and generalized metric

The coordinates are grouped according to

$$X^M \;=\; \left(\tilde{x}_i,\, x^i,\, y^\alpha\right)\;, \tag{3.10}$$

which transforms as a fundamental $O(D,D+n)$ vector,

$$X'^M \;=\; h^M{}_N X^N\;, \qquad h \,\in\, O(D,D+n)\;. \tag{3.11}$$

Here, $O(D, D+n)$ is the group leaving the metric of signature $(D, D+n)$ invariant,

$$\eta^{MN} = h^M{}_P h^N{}_Q \eta^{PQ} , \tag{3.12}$$

where

$$\eta_{MN} = \begin{pmatrix} \eta^{ij} & \eta^i{}_j & \eta^i{}_\beta \\ \eta_i{}^j & \eta_{ij} & \eta_{i\beta} \\ \eta_\alpha{}^j & \eta_{\alpha j} & \eta_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix} . \tag{3.13}$$

Here, we introduced κ to denote the matrix corresponding to the Cartan-Killing metric of the gauge group. In the present abelian case, this is simply given by the unit matrix, $\kappa_{\alpha\beta} = \delta_{\alpha\beta}$, but we kept the notation more general for the later extension to the non-abelian case.

According to these index conventions, the derivatives and gauge parameters are

$$\partial_M = \left(\tilde{\partial}^i, \partial_i, \partial_\alpha\right) , \qquad \xi^M = \left(\tilde{\xi}_i, \xi^i, \Lambda^\alpha\right) , \tag{3.14}$$

which combines the gauge parameters of diffeomorphism, Kalb-Ramond and abelian gauge transformations into an $O(D, D+n)$ vector. The strong constraint reads explicitly

$$\partial_M \partial^M A = 2\tilde{\partial}^i \partial_i A + \partial_\alpha \partial^\alpha A = 0 , \tag{3.15}$$

$$\partial_M A \, \partial^M B = \tilde{\partial}^i A \, \partial_i B + \partial_i A \, \tilde{\partial}^i B + \partial_\alpha A \, \partial^\alpha B = 0 , \tag{3.16}$$

for arbitrary fields and gauge parameters A and B. As for the bosonic theory, this constraint is a stronger version of the level-matching condition and it implies that locally there is always an $O(D, D+n)$ transformation that rotates into a frame in which the fields depend only on the x^i.

Next, we introduce the extended form of the generalized metric $\mathcal{H}^{MN}$ and require that it transforms covariantly under $O(D, D+n)$,

$$\mathcal{H}'^{MN}(X') = h^M{}_P h^N{}_Q \mathcal{H}^{PQ}(X) , \qquad d'(X') = d(X) . \tag{3.17}$$

In analogy to the structure encountered in dimensionally reduced theories [36], we make the ansatz

$$\begin{aligned}\mathcal{H}_{MN} &= \begin{pmatrix} \mathcal{H}^{ij} & \mathcal{H}^{i}{}_{j} & \mathcal{H}^{i}{}_{\beta} \\ \mathcal{H}_{i}{}^{j} & \mathcal{H}_{ij} & \mathcal{H}_{i\beta} \\ \mathcal{H}_{\alpha}{}^{j} & \mathcal{H}_{\alpha j} & \mathcal{H}_{\alpha\beta} \end{pmatrix} \\ &= \begin{pmatrix} g^{ij} & -g^{ik}c_{kj} & -g^{ik}A_{k\beta} \\ -g^{jk}c_{ki} & g_{ij}+c_{ki}g^{kl}c_{lj}+A_i{}^{\gamma}A_{j\gamma} & c_{ki}g^{kl}A_{l\beta}+A_{i\beta} \\ -g^{jk}A_{k\alpha} & c_{kj}g^{kl}A_{l\alpha}+A_{j\alpha} & \kappa_{\alpha\beta}+A_{k\alpha}g^{kl}A_{l\beta} \end{pmatrix},\end{aligned} \tag{3.18}$$

where gauge group indices $\alpha, \beta, \ldots$ are raised and lowered with $\kappa_{\alpha\beta}$, and

$$c_{ij} \;=\; b_{ij} + \frac{1}{2}A_i{}^{\alpha}A_{j\alpha}\,. \tag{3.19}$$

The generalized metric defined like this is still symmetric, $\mathcal{H}_{MN} = \mathcal{H}_{NM}$. Raising all indices with η^{MN}, we obtain

$$\begin{aligned}\mathcal{H}^{MN} &= \begin{pmatrix} \mathcal{H}_{ij} & \mathcal{H}_{i}{}^{j} & \mathcal{H}_{i}{}^{\beta} \\ \mathcal{H}^{i}{}_{j} & \mathcal{H}^{ij} & \mathcal{H}^{i\beta} \\ \mathcal{H}^{\alpha}{}_{j} & \mathcal{H}^{\alpha j} & \mathcal{H}^{\alpha\beta} \end{pmatrix} \\ &= \begin{pmatrix} g_{ij}+c_{ki}g^{kl}c_{lj}+A_i{}^{\gamma}A_{j\gamma} & -g^{jk}c_{ki} & c_{ki}g^{kl}A_l{}^{\beta}+A_i{}^{\beta} \\ -g^{ik}c_{kj} & g^{ij} & -g^{ik}A_k{}^{\beta} \\ c_{kj}g^{kl}A_l{}^{\alpha}+A_j{}^{\alpha} & -g^{jk}A_k{}^{\alpha} & \kappa^{\alpha\beta}+A_k{}^{\alpha}g^{kl}A_l{}^{\beta} \end{pmatrix}.\end{aligned} \tag{3.20}$$

This is the inverse of (3.18), and so the generalized metric satisfies the constraint $\mathcal{H}^{MK}\mathcal{H}_{KN} = \delta^{M}{}_{N}$. This implies that, viewed as a matrix, it is an element of $O(D, D+n)$ in that it satisfies

$$\mathcal{H}^{-1} \;=\; \eta\,\mathcal{H}\,\eta\,. \tag{3.21}$$

The $O(D, D+n)$ action (3.17) defines the generalized Buscher rules for the abelian subsector of heterotic string theory.

3.2.2 Gauge symmetries

We turn now to the gauge transformations of the component fields that follow from the extended form of the generalized metric (3.20) and the generalized Lie derivatives with respect to the extended parameter (3.14). Specifically, we verify that for $\tilde{\partial}^i = \partial_\alpha = 0$ the gauge transformations of the component fields take the required form.

For the gauge variation of $\mathcal{H}^{ij}$ we find

$$\begin{aligned} \delta_\xi \mathcal{H}^{ij} &= \delta_\xi g^{ij} = \xi^k \partial_k \mathcal{H}^{ij} - \partial^P \xi^i \, \mathcal{H}_P{}^j - \partial^P \xi^j \, \mathcal{H}^i{}_P \\ &= \xi^k \partial_k g^{ij} - \partial_k \xi^i \, g^{kj} - \partial_k \xi^j \, g^{ik} = \mathcal{L}_\xi g^{ij} \, , \end{aligned} \tag{3.22}$$

i.e., the metric g_{ij} transforms as expected with the Lie derivative under diffeomorphisms parametrized by ξ^i and is inert under the other gauge symmetries. For the component $\mathcal{H}^{i\beta}$ we infer

$$\begin{aligned} \delta_\xi \mathcal{H}^{i\beta} &= \delta_\xi \big(- g^{ik} A_k{}^\beta \big) = \xi^k \partial_k \mathcal{H}^{i\beta} - \partial^P \xi^i \, \mathcal{H}_P{}^\beta - \partial^P \xi^\beta \, \mathcal{H}^i{}_P \\ &= \xi^k \partial_k \mathcal{H}^{i\beta} - \partial_k \xi^i \, \mathcal{H}^{k\beta} - \partial_k \xi^\beta \, \mathcal{H}^{ik} \\ &= \xi^k \partial_k \big(- g^{il} A_l{}^\beta \big) - \partial_k \xi^i \, \big(- g^{kl} A_l{}^\beta \big) - \partial_k \Lambda^\beta g^{ik} \\ &= \mathcal{L}_\xi \big(- g^{ik} A_k{}^\beta \big) - g^{ik} \partial_k \Lambda^\beta \, . \end{aligned} \tag{3.23}$$

Together with the form of $\delta_\xi g^{ij}$ determined above, this implies for the gauge vectors

$$\delta_\xi A_k{}^\beta = \mathcal{L}_\xi A_k{}^\beta + \partial_k \Lambda^\beta \, , \tag{3.24}$$

which represents the expected diffeomorphism and abelian gauge transformation. Finally, for the component $\mathcal{H}^i{}_j$ we derive

$$\begin{aligned} \delta_\xi \mathcal{H}^i{}_j &= \delta_\xi \big(- g^{ik} c_{kj} \big) = \xi^k \partial_k \mathcal{H}^i{}_j - \partial^P \xi^i \, \mathcal{H}_{Pj} + \big(\partial_j \xi^P - \partial^P \xi_j \big) \mathcal{H}^i{}_P \\ &= \xi^k \partial_k \mathcal{H}^i{}_j - \partial_k \xi^i \, \mathcal{H}^k{}_j + \partial_j \xi^k \, \mathcal{H}^i{}_k + \partial_j \tilde{\xi}_k \, \mathcal{H}^{ik} + \partial_j \xi^\beta \, \mathcal{H}^i{}_\beta - \partial_k \tilde{\xi}_j \, \mathcal{H}^{ik} \\ &= \mathcal{L}_\xi \mathcal{H}^i{}_j + \big(\partial_j \tilde{\xi}_k - \partial_k \tilde{\xi}_j \big) \mathcal{H}^{ik} + \partial_j \xi^\beta \, \mathcal{H}^i{}_\beta \\ &= \mathcal{L}_\xi \big(- g^{ik} c_{kj} \big) + \big(\partial_j \tilde{\xi}_k - \partial_k \tilde{\xi}_j \big) g^{ik} + \partial_j \Lambda^\beta \, \big(- g^{ik} A_{k\beta} \big) \, . \end{aligned} \tag{3.25}$$

Using again the known form of the gauge transformation $\delta_\xi g^{ij}$, this implies for the tensor defined in (3.19)

$$\delta_\xi c_{ij} \;=\; \mathcal{L}_\xi c_{ij} + \big(\partial_i\tilde{\xi}_j - \partial_j\tilde{\xi}_i\big) + A_{i\beta}\partial_j\Lambda^\beta\,. \tag{3.26}$$

In order to derive the gauge transformation of b_{ij}, we project this onto the symmetric and antisymmetric part,

$$\begin{aligned}
\delta_\xi c_{(ij)} \;&=\; \delta_\xi\big(\tfrac{1}{2}A_{i\beta}A_j{}^\beta\big) \;=\; \mathcal{L}_\xi\big(\tfrac{1}{2}A_{i\beta}A_j{}^\beta\big) + \tfrac{1}{2}\big(A_{i\beta}\partial_j\Lambda^\beta + A_{j\beta}\partial_i\Lambda^\beta\big)\,, &(3.27)\\
\delta_\xi c_{[ij]} \;&=\; \delta_\xi b_{ij} \;=\; \mathcal{L}_\xi b_{ij} + \big(\partial_i\tilde{\xi}_j - \partial_j\tilde{\xi}_i\big) + \tfrac{1}{2}\big(A_{i\beta}\partial_j\Lambda^\beta - A_{j\beta}\partial_i\Lambda^\beta\big)\,. &(3.28)
\end{aligned}$$

The first equation is consistent with the gauge transformation of the gauge field as obtained above, while the second equation yields the gauge transformation of b_{ij}.

To summarize, the gauge transformations in the limit $\tilde{\partial}^i = \partial_\alpha = 0$ read

$$\begin{aligned}
\delta g_{ij} \;&=\; \mathcal{L}_\xi g_{ij}\,, &(3.29)\\
\delta A_i{}^\alpha \;&=\; \mathcal{L}_\xi A_i{}^\alpha + \partial_i\Lambda^\alpha\,, &(3.30)\\
\delta b_{ij} \;&=\; \mathcal{L}_\xi b_{ij} + \big(\partial_i\tilde{\xi}_j - \partial_j\tilde{\xi}_i\big) + \frac{1}{2}\big(A_{i\alpha}\partial_j\Lambda^\alpha - A_{j\alpha}\partial_i\Lambda^\alpha\big)\,. &(3.31)
\end{aligned}$$

For metric and gauge vector, these give the expected result, but for b_{ij} a parameter redefinition is required in order to obtain (3.5). If we redefine the one-form parameter $\tilde{\xi}_i$ according to

$$\tilde{\xi}_i' \;:=\; \tilde{\xi}_i - \frac{1}{2}A_i{}^\alpha\Lambda_\alpha\,, \tag{3.32}$$

the gauge variation of b_{ij} becomes

$$\delta b_{ij} \;=\; \partial_i\tilde{\xi}_j' - \partial_j\tilde{\xi}_i' + \frac{1}{2}F_{ij}{}^\alpha\Lambda_\alpha\,, \tag{3.33}$$

with the abelian field strength $F_{ij}{}^\alpha$, in accordance with (3.5).

We close this section with a brief discussion of the closure of the gauge transformations. The gauge algebra of bosonic double field theory immediately generalizes to the present case:

$$\left[\delta_{\xi_1}\,,\delta_{\xi_2}\right] \;=\; -\delta_{[\xi_1,\xi_2]_{\mathrm{C}}}\,, \tag{3.34}$$

where

$$[\xi_1,\xi_2]_{\mathrm{C}}^M \;\equiv\; \xi_1^N\partial_N\xi_2^M - \frac{1}{2}\xi_1^P\partial^M\xi_{2P} - (1\leftrightarrow 2)\,. \tag{3.35}$$

Let us see how this generalizes after adding the n additional components for ξ^M. Setting now also $\partial_\alpha = 0$, we obtain for the various components of (3.35)

$$\left([\xi_1,\xi_2]_{\mathrm{C}}\right)^i \;=\; \xi_1^j\partial_j\xi_2^i - \xi_2^j\partial_j\xi_1^i \;\equiv\; [\xi_1,\xi_2]^i\,, \tag{3.36}$$

which is unmodified and given by the usual Lie bracket,

$$\begin{aligned}\left([\xi_1,\xi_2]_{\mathrm{C}}\right)_i &= \mathcal{L}_{\xi_1}\tilde{\xi}_{2i} - \mathcal{L}_{\xi_2}\tilde{\xi}_{1i} - \frac{1}{2}\partial_i\big(\tilde{\xi}_{2j}\xi_1^j\big) + \frac{1}{2}\partial_i\big(\tilde{\xi}_{1j}\xi_2^j\big)\\ &\quad - \frac{1}{2}\big(\Lambda_{1\alpha}\partial_i\Lambda_2{}^\alpha - \Lambda_{2\alpha}\partial_i\Lambda_1{}^\alpha\big)\,,\end{aligned} \tag{3.37}$$

which receives a new contribution involving Λ, and finally

$$\left([\xi_1,\xi_2]_{\mathrm{C}}\right)^\alpha \;=\; \xi_1^j\partial_j\Lambda_2{}^\alpha - \xi_2^j\partial_j\Lambda_1{}^\alpha\,, \tag{3.38}$$

which is the (antisymmetrized) Lie derivative of Λ. The Courant bracket is defined as a structure on the direct sum of tangent and cotangent bundle over the space-time base manifold M, $(T\oplus T^*)M$, whose sections are formal sums $\xi+\tilde{\xi}$ of vectors and one-forms. Thus, for the given generalization it is natural to consider a bundle that is further extended to $T\oplus T^*\oplus V$, where we identify the sections of V with the Λ^α. The sections of the total bundle are then written as $\xi+\tilde{\xi}+\Lambda$, and in this language, the results (3.36), (3.37) and (3.38) can be summarized by

$$\begin{aligned}\left[\xi_1+\tilde{\xi}_1+\Lambda_1\,,\,\xi_2+\tilde{\xi}_2+\Lambda_2\right] \;=\;& [\xi_1,\xi_2]+\mathcal{L}_{\xi_1}\tilde{\xi}_2 - \mathcal{L}_{\xi_2}\tilde{\xi}_1\\ &-\frac{1}{2}d\big(i_{\xi_1}\tilde{\xi}_2 - i_{\xi_2}\tilde{\xi}_1\big) - \frac{1}{2}\big(\langle\Lambda_1,d\Lambda_2\rangle - \langle\Lambda_2,d\Lambda_1\rangle\big)\\ &+\mathcal{L}_{\xi_1}\Lambda_2 - \mathcal{L}_{\xi_2}\Lambda_1\,,\end{aligned} \tag{3.39}$$

where $\langle\Lambda_1,\Lambda_2\rangle = \kappa_{\alpha\beta}\Lambda_1^\alpha\Lambda_2^\beta$ denotes the inner product, and i is the canonical product between vectors and one-forms. Here, the term on the right-hand side in the first line represents the vector part, the terms in the second line represent the one-form part, and finally the terms in the last line represent the V-valued part. For $\Lambda = 0$ this reduces to the Courant bracket.

The bracket (3.39) implies in particular that the abelian gauge transformations parametrized by Λ^α close into the gauge transformations of the 2-form. This can also be confirmed directly from (3.30) and (3.31),

$$\big[\delta_{\Lambda_1},\delta_{\Lambda_2}\big]b_{ij} \;=\; \delta_{\tilde{\xi}}b_{ij}\;, \qquad \tilde{\xi}_i \;=\; \frac{1}{2}\big(\Lambda_{1\alpha}\partial_i\Lambda_2{}^\alpha - \Lambda_{2\alpha}\partial_i\Lambda_1{}^\alpha\big)\;. \tag{3.40}$$

We stress, however, that this result depends on a choice of basis for the gauge parameters. In fact, after the parameter redefinition (3.32), the 2-form varies into the gauge invariant field strength according to (3.33) and thus the commutator trivializes.

3.2.3 The action

Let us now turn to the action (3.6) applied to the extended form (3.20) of the generalized metric. We show that for $\tilde{\partial}^i = \partial_\alpha = 0$ it reduces to the (abelian) low-energy action (3.1) of the heterotic string.

The relevant terms in the action, setting $\tilde{\partial}^i = \partial_\alpha = 0$, are given by

$$\begin{aligned} S \;=\; \int dx\, e^{-2d}\,\Big(\,&\frac{1}{8}\mathcal{H}^{ij}\partial_i\mathcal{H}^{KL}\,\partial_j\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{Mi}\partial_i\mathcal{H}^{Kj}\,\partial_j\mathcal{H}_{MK} \\ &- 2\,\partial_i d\,\partial_j\mathcal{H}^{ij} + 4\mathcal{H}^{ij}\,\partial_i d\,\partial_j d\,\Big)\,. \end{aligned} \tag{3.41}$$

The last two terms are unchanged as compared to the original case without gauge vectors since the component $\mathcal{H}^{ij} = g^{ij}$ is unmodified. Thus, we only need to examine the first two terms. The first term reads

$$
\begin{aligned}
&\frac{1}{8}\mathcal{H}^{ij}\partial_i\mathcal{H}^{KL}\,\partial_j\mathcal{H}_{KL} \\
= \;& \frac{1}{4}\partial_i\mathcal{H}^{kl}\,\partial^i\mathcal{H}_{kl}+\frac{1}{4}\partial_i\mathcal{H}_k{}^l\,\partial^i\mathcal{H}^k{}_l+\frac{1}{2}\partial_i\mathcal{H}^{\alpha l}\,\partial^i\mathcal{H}_{\alpha l}+\frac{1}{8}\partial_i\mathcal{H}^{\alpha\beta}\,\partial^i\mathcal{H}_{\alpha\beta} \\
= \;& \frac{1}{4}\partial_i g^{lp}\,\partial^i\big(g_{lp}+c_{kp}g^{kq}c_{ql}+A_l{}^\alpha A_{p\alpha}\big)+\frac{1}{4}\partial_i\big(g^{lp}c_{pk}\big)\,\partial^i\big(g^{kq}c_{ql}\big) \\
& -\frac{1}{2}\partial_i\big(g^{lp}A_p{}^\alpha\big)\,\partial^i\big(c_{ql}g^{qk}A_{k\alpha}+A_{l\alpha}\big)+\frac{1}{8}\partial_i\big(A_p{}^\alpha g^{pl}A_l{}^\beta\big)\,\partial^i\big(A_{k\alpha}g^{kq}A_{q\beta}\big)\,.
\end{aligned}
\tag{3.42}
$$

After some work, this can be simplified to

$$
\frac{1}{8}\mathcal{H}^{ij}\,\partial_i\mathcal{H}^{KL}\,\partial_j\mathcal{H}_{KL} \;=\; \frac{1}{4}g^{ij}\,\partial_i g^{kl}\partial_j g_{kl}-\frac{1}{2}g^{ij}g^{kl}\partial_i A_{k\alpha}\,\partial_j A_l{}^\alpha-\frac{1}{4}\tilde{H}_{ijk}\tilde{H}^{ijk}\,, \tag{3.43}
$$

where $\tilde{H}_{ijk}=\partial_i b_{jk}-\partial_i A_{[j}{}^\alpha\,A_{k]\alpha}$.

Next we consider the second term in (3.41), which yields

$$
\begin{aligned}
-\frac{1}{2}\mathcal{H}^{Mi}\partial_i\mathcal{H}^{Kj}\,\partial_j\mathcal{H}_{MK} = & -\frac{1}{2}\mathcal{H}^{mi}\big(\partial_i\mathcal{H}^{kj}\,\partial_j\mathcal{H}_{mk}+\partial_i\mathcal{H}_k{}^j\,\partial_j\mathcal{H}_m{}^k+\partial_i\mathcal{H}^{\alpha j}\,\partial_j\mathcal{H}_{m\alpha}\big) \\
& -\frac{1}{2}\mathcal{H}_m{}^i\big(\partial_i\mathcal{H}^{kj}\,\partial_j\mathcal{H}^m{}_k+\partial_i\mathcal{H}_k{}^j\,\partial_j\mathcal{H}^{mk}+\partial_i\mathcal{H}^{\alpha j}\,\partial_j\mathcal{H}^m{}_\alpha\big) \\
& -\frac{1}{2}\mathcal{H}^{\beta i}\big(\partial_i\mathcal{H}^{kj}\,\partial_j\mathcal{H}_{\beta k}+\partial_i\mathcal{H}_k{}^j\,\partial_j\mathcal{H}_\beta{}^k+\partial_i\mathcal{H}^{\alpha j}\,\partial_j\mathcal{H}_{\beta\alpha}\big)\,.
\end{aligned}
\tag{3.44}
$$

To simplify the evaluation of these terms, it is convenient to work out the following structures separately,

$$
-\frac{1}{2}\mathcal{H}^{Mi}\partial_i\mathcal{H}^{Kj}\,\partial_j\mathcal{H}_{MK}\big|_{(\partial g)^2} \;=\; -\frac{1}{2}g^{ij}\partial_j g^{kl}\,\partial_l g_{ik}\,, \tag{3.45}
$$

$$
-\frac{1}{2}\mathcal{H}^{Mi}\partial_i\mathcal{H}^{Kj}\,\partial_j\mathcal{H}_{MK}\big|_{(\partial g)^1} \;=\; 0\,, \tag{3.46}
$$

$$
-\frac{1}{2}\mathcal{H}^{Mi}\partial_i\mathcal{H}^{Kj}\,\partial_j\mathcal{H}_{MK}\big|_{(\partial g)^0} \;=\; \frac{1}{2}g^{ik}g^{jl}\partial_i A_l{}^\alpha\,\partial_j A_{k\alpha}-\frac{1}{2}\tilde{H}_{ijk}\tilde{H}^{jki}\,. \tag{3.47}
$$

Combining these three structures, we obtain

$$
-\frac{1}{2}\mathcal{H}^{Mi}\partial_i\mathcal{H}^{Kj}\,\partial_j\mathcal{H}_{MK} \;=\; -\frac{1}{2}g^{ij}\partial_j g^{kl}\,\partial_l g_{ik}+\frac{1}{2}g^{ik}g^{jl}\partial_i A_l{}^\alpha\,\partial_j A_{k\alpha}-\frac{1}{4}\tilde{H}_{ijk}(\tilde{H}^{jki}+\tilde{H}^{kij})\,. \tag{3.48}
$$

Finally, using (4.209) and (3.48), the reduced action (3.41) can be written as

$$
\begin{aligned}
S \;=\; \int dx\, e^{-2d}\,\Big(&\frac{1}{4}g^{ij}\partial_i g^{kl}\partial_j g_{kl}-\frac{1}{2}g^{ij}\partial_j g^{kl}\,\partial_l g_{ik}-2\,\partial_i d\,\partial_j g^{ij}+4g^{ij}\,\partial_i d\,\partial_j d\\
&-\frac{1}{12}\hat{H}^2-\frac{1}{4}F_{ij\alpha}F^{ij\alpha}\Big)\,.
\end{aligned}
\tag{3.49}
$$

Up to boundary terms, the terms in the first line are equivalent to the Einstein-Hilbert term coupled to the dilaton, compare eq. (3.18) in [7]. Thus, the reduced action coincides precisely with (3.1).

3.3 Non-abelian generalization

In this section we generalize the previous results to non-abelian gauge groups. This will be achieved by introducing a 'duality-covariant' form of the structure constants of the gauge group. While this object is not an invariant tensor under $O(D, D+n)$ and so the T-duality group is no longer a proper symmetry, remarkably the action and gauge transformations can still be written in an $O(D, D+n)$ invariant fashion.

3.3.1 Duality-covariant structure constants

We encode the structure constant in an object $f^M{}_{NK}$ that formally can be regarded as a tensor under $O(D, D+n)$, even though it is ultimately fixed to be constant and thus not to transform according to its index structure. To be specific, let us fix an n-dimensional semi-simple Lie group G whose Lie algebra has the structure constants $f^\alpha{}_{\beta\gamma}$. Then we can define

$$
f^M{}_{NK} \;=\; \begin{cases} f^\alpha{}_{\beta\gamma} & \text{if } (M,N,K)=(\alpha,\beta,\gamma) \\ 0 & \text{else} \end{cases}\;. \tag{3.50}
$$

This is *not* an invariant tensor under $O(D, D+n)$, rather it will break this symmetry to $O(D, D) \times G$. The advantage of this formulation is, however, that the explicit form of the prototypical example (4.241) is not required for the general analysis: it is sufficient to impose duality-covariant constraints, which in general may have different solutions.

Let us now turn to the constraints. First, we require that η^{MN} is an invariant tensor under the adjoint action with $f^M{}_{NK}$,

$$f^{(M}{}_{PK}\,\eta^{N)K} \;=\; 0\,. \tag{3.51}$$

This is satisfied for (4.241) with η^{MN} defined by (3.13), and we recall that the component $\eta_{\alpha\beta}$ is identified with the invariant Cartan-Killing form of G. Together with the antisymmetry of $f^M{}_{NK}$ in its lower indices, the constraint (3.51) implies that f with all indices raised or lowered with η is totally antisymmetric,

$$f_{MNK} \;=\; f_{[MNK]}\,, \qquad f^{MNK} \;=\; f^{[MNK]}\,. \tag{3.52}$$

Next, we require that $f^M{}_{NK}$ satisfies the Jacobi identity

$$f^M{}_{N[K}\,f^N{}_{LP]} \;=\; 0\,, \tag{3.53}$$

which is satisfied for (4.241) by virtue of the Jacobi identity for $f^\alpha{}_{\beta\gamma}$.

Apart from these algebraic constraints, we have to impose one new condition in addition to the strong constraint : we require the differential constraint

$$f^M{}_{NK}\,\partial_M \;=\; 0\,, \tag{3.54}$$

when acting on fields or parameters. By (3.52) this implies that all derivatives act trivially that are contracted with any index of $f^M{}_{NK}$. For the choice (4.241) this implies $\partial_\alpha = 0$, as we will prove below.

To summarize, we impose the $O(D,D+n)$ covariant constraints (3.51), (3.53) and (3.54). Any $f^M{}_{NK}$ satisfying these conditions will lead to a consistent, that is, gauge invariant deformation of the abelian theory discussed above. A particular solution of these constraints is given by (4.241) with $\partial_\alpha = 0$ where, as we shall see below, the theory reduces to the non-abelian low-energy action of the heterotic string. We stress, however, that any solution obtained from this one by an $O(D, D+n)$ transformation also satisfies the constraints. We will return to this point in sec. 4.

We close this section by introducing the modified or deformed gauge transformations. Each $O(D, D+n)$ index will give rise to a adjoint rotation with the structure constants $f^M{}_{NK}$. In (3.8) we displayed this transformation for a tensor with an upper index,

$$\delta_\xi V^M \;=\; \widehat{\mathcal{L}}_\xi V^M - \xi^N f^M{}_{NK} V^K \,, \tag{3.55}$$

and the transformation for a tensor with a lower index is given by

$$\delta_\xi V_M \;=\; \widehat{\mathcal{L}}_\xi V_M + \xi^K f^N{}_{KM} V_N \,. \tag{3.56}$$

This extends in a straightforward way to tensors with an arbitrary number of upper and lower indices, such that the generalized metric transforms as

$$\delta_\xi \mathcal{H}^{MN} \;=\; \widehat{\mathcal{L}}_\xi \mathcal{H}^{MN} - 2\,\xi^P f^{(M}{}_{PK}\,\mathcal{H}^{N)K} \,. \tag{3.57}$$

By virtue of the constraints (3.51), the $O(D, D+n)$ invariant metric η is invariant under these transformations, $\delta_\xi \eta^{MN} = 0$, which is a generalization of the analogous property in the abelian case. Moreover, the constraint (3.54) has two immediate consequences for these deformed gauge transformations. First, the partial derivative of a scalar transforms covariantly,

$$\delta_\xi(\partial_M S) \;=\; \widehat{\mathcal{L}}_\xi(\partial_M S) \;=\; \widehat{\mathcal{L}}_\xi(\partial_M S) + \xi^L f^K{}_{LM}\partial_K S \,. \tag{3.58}$$

Second, any gauge transformation with a parameter that is a gradient acts trivially,

$$\xi^M \;=\; \partial^M \chi \qquad \Rightarrow \qquad \delta_\xi \mathcal{H}^{MN} \;=\; 0 \,, \tag{3.59}$$

i.e., as for the abelian case there is a 'gauge symmetry for gauge symmetries'.

3.3.2 The non-abelian gauge transformations

Let us now verify that the deformed gauge transformations (3.57) indeed lead to the required non-abelian gauge transformations if we choose (4.241) and set $\tilde{\partial}^i = \partial_\alpha = 0$. The Yang-Mills gauge field transforms as[1]

$$\delta_\Lambda A_i{}^\alpha \;=\; \partial_i \Lambda^\alpha + f^\alpha{}_{\beta\gamma} A_i{}^\beta \Lambda^\gamma \,. \tag{3.60}$$

The b-field transforms according to (3.5) and thus its transformation rule is not modified as compared to the abelian case.

We apply (3.57) to particular components of $\mathcal{H}^{MN}$, where we focus on the new terms proportional to $f^M{}_{NK}$, which we denote by δ'. The variation of $\mathcal{H}^{ij}$ does not receive any modification since by (4.241) the f-dependent term in (3.57) is zero for external indices i, j. Thus, the metric g_{ij} is still inert under Λ transformations, as expected. For components with external index α, however, we find, e.g.,

$$\delta'_\xi \mathcal{H}^{i\alpha} \;=\; -g^{ik}\delta A_k{}^\alpha \;=\; -\Lambda^\beta f^\alpha{}_{\beta\gamma}\mathcal{H}^{i\gamma} \quad\Rightarrow\quad \delta'_\Lambda A_k{}^\alpha \;=\; f^\alpha{}_{\beta\gamma} A_k{}^\beta \Lambda^\gamma \,, \tag{3.61}$$

which amounts to the required transformation rule (3.60). Next, from $\mathcal{H}^i{}_j = -g^{ik}c_{kj}$ we infer that δc_{ij} does not get corrected. In (3.19) the symmetric combination quadratic in A is invariant under the non-abelian part of (3.60), as one may easily confirm, and therefore we conclude that also δb_{ij} does not get modified as compared to the abelian case, in agreement with (3.5). Thus, (3.57) yields precisely the required gauge transformations.

In the remainder of this subsection, we discuss the closure of the deformed gauge transformations. It is sufficient (and simplifies the analysis) to compute the closure on a vector V^M whose gauge variation is given in (3.55). The commutator of two

[1] In order to simplify the notation, we assume from now on that the gauge coupling constant g_0 has been absorbed into the structure constants $f^\alpha{}_{\beta\gamma}$, such that it does not appear explicitly in the formulas below.

such gauge transformations is then given by

$$
\begin{aligned}
[\delta_{\xi_1}, \delta_{\xi_2}] V^M &= \delta_{\xi_1}\big(\xi_2^N \partial_N V^M + (\partial^M \xi_{2N} - \partial_N \xi_2^M) V^N - \xi_2^K f^M{}_{KN} V^N\big) - (1 \leftrightarrow 2) \\
&= \big[\widehat{\mathcal{L}}_{\xi_1}, \widehat{\mathcal{L}}_{\xi_2}\big] V^M - \xi_2^N \partial_N\big(\xi_1^K f^M{}_{KP} V^P\big) \\
&\quad - \big(\partial^M \xi_{2N} - \partial_N \xi_2^M\big) \xi_1^K f^N{}_{KP} V^P - \xi_2^K f^M{}_{KN}\big(\xi_1^P \partial_P V^N \\
&\quad + (\partial^N \xi_{1P} - \partial_P \xi_1^N) V^P - \xi_1^P f^N{}_{PQ} V^Q\big) - (1 \leftrightarrow 2) \, .
\end{aligned} \tag{3.62}
$$

Using the constraints (3.54) and (3.53) it is now relatively straightforward to check that this can be rewritten as

$$
[\delta_{\xi_1}, \delta_{\xi_2}] V^M = \widehat{\mathcal{L}}_{\xi_{12}} V^M - \xi_{12}^N f^M{}_{NK} V^K \, , \tag{3.63}
$$

where

$$
\xi_{12}^M = \xi_2^N \partial_N \xi_1^M - \frac{1}{2} \xi_{2N} \partial^M \xi_1^N - (1 \leftrightarrow 2) - f^M{}_{NK} \xi_2^N \xi_1^K \, . \tag{3.64}
$$

Thus, we have verified the closure of the gauge algebra and thereby arrived at a generalization of the C-bracket that is deformed by the structure constants $f^M{}_{NK}$,

$$
\big[X, Y\big]_f^M = \big[X, Y\big]_{\mathrm{C}}^M - f^M{}_{NK} X^N Y^K \, . \tag{3.65}
$$

The C-bracket does not satisfy the Jacobi identities, but the resulting non-trivial Jacobiator gives rise to a trivial gauge transformation that leaves the fields invariant. The deformed bracket (3.65) has a similar property, which we investigate now. First, we evaluate the Jacobiator,

$$
J_f(X, Y, Z) = \big[\big[X, Y\big]_f, Z\big]_f + \big[\big[Y, Z\big]_f, X\big]_f + \big[\big[Z, X\big]_f, Y\big]_f \, . \tag{3.66}
$$

We compute from (3.65)

$$
\begin{aligned}
\big[[X,Y]_{f},Z\big]_{f}^{M} &= \big[[X,Y]_{\mathrm{C}},Z\big]_{\mathrm{C}}^{M} + f^{M}{}_{NK}f^{N}{}_{PQ}X^{P}Y^{Q}Z^{K} \\
&\quad + f^{M}{}_{NK}\big(Z^{P}\partial_{P}(X^{N}Y^{K}) - (X^{P}\partial_{P}Y^{N} - Y^{P}\partial_{P}X^{N})Z^{K}\big) \qquad (3.67) \\
&\quad + \frac{1}{2}f^{N}{}_{KL}\big(X^{K}Y^{L}\partial^{M}Z_{N} - Z_{N}\partial^{M}(X^{K}Y^{L})\big)\,,
\end{aligned}
$$

where we used the constraint (3.54). Using the Jacobi identity (3.53) we obtain after a brief computation

$$
J_{f}(X,Y,Z)^{M} \;=\; J_{\mathrm{C}}(X,Y,Z)^{M} - \frac{1}{2}\partial^{M}\big(f_{NKL}X^{N}Y^{K}Z^{L}\big)\,. \qquad (3.68)
$$

Here, J_{C} is the Jacobiator of the C-bracket, which has been proved in [16] to be a gradient. Thus, we infer from (3.68)

$$
J_{f}(X,Y,Z)^{M} \;=\; \partial^{M}\Big(\chi_{\mathrm{C}}(X,Y,Z) - \frac{1}{2}f_{NKL}X^{N}Y^{K}Z^{L}\Big)\,, \qquad (3.69)
$$

where χ_{C} is given in eq. (8.29) of [16]. We have seen in (3.59) that a gauge parameter that takes the form of a pure gradient gives rise to a trivial gauge transformation on the fields. Thus, in precise analogy to [16], the non-vanishing Jacobiator is consistent with the fact that the infinitesimal gauge transformations δ_{ξ} automatically satisfy the Jacobi identity.

We finally note that, in analogy to the discussion at the end of sec. 2.2, the modified form of the gauge algebra is consistent with the closure property

$$
\big[\delta_{\Lambda_1},\delta_{\Lambda_2}\big]b_{ij} \;=\; \big(\delta_{\tilde{\xi}} + \delta_{\Lambda}\big)b_{ij}\,, \qquad \Lambda^{\alpha} \;=\; f^{\alpha}{}_{\beta\gamma}\Lambda_{1}^{\beta}\Lambda_{2}^{\gamma}\,, \qquad (3.70)
$$

where $\tilde{\xi}_{i}$ is given by (3.40). In the mathematical terminology of sec. 2.2, the closure property (3.64) or (3.70) amounts to a further generalization of the Courant bracket, involving the structure of a non-abelian Lie algebra, in that the term $[\Lambda_1,\Lambda_2]$ has to be added in the last line of (3.39).

3.3.3 The non-abelian action

Next, we construct a deformation of the double field theory action parametrized by the $f^M{}_{NK}$ in such a way that it is gauge invariant under (3.57) and leads to the required low-energy action. For this we will start from the action written in Einstein-Hilbert like form [8],

$$S = \int dx\, d\tilde{x}\, e^{-2d}\,\mathcal{R}(\mathcal{H},d)\,, \tag{3.71}$$

where $\mathcal{R}(\mathcal{H},d)$ is given by

$$\begin{aligned}\mathcal{R} \;\equiv\; & 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} \\ & - 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d \\ & + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}\,.\end{aligned} \tag{3.72}$$

It is defined such that it is a scalar under generalized Lie derivatives,

$$\delta_\xi\mathcal{R} \;=\; \xi^P\partial_P\mathcal{R}\,, \tag{3.73}$$

which, together with the gauge variation of the dilaton

$$\delta_\xi(e^{-2d}) = \partial_M(e^{-2d}\xi^M)\,, \tag{3.74}$$

implies gauge invariance of the action. Here we modify the form of $\mathcal{R}$ such that (3.73) be preserved under the deformed gauge transformations (3.57).

The result for the deformed scalar curvature is given by

$$\begin{aligned}\mathcal{R}_f \;=\; & \mathcal{R} - \frac{1}{2}f^M{}_{NK}\,\mathcal{H}^{NP}\mathcal{H}^{KQ}\partial_P\mathcal{H}_{QM} \\ & - \frac{1}{12}f^M{}_{KP}f^N{}_{LQ}\mathcal{H}_{MN}\mathcal{H}^{KL}\mathcal{H}^{PQ} - \frac{1}{4}f^M{}_{NK}f^N{}_{ML}\mathcal{H}^{KL} - \frac{1}{6}f^{MNK}f_{MNK}\,,\end{aligned} \tag{3.75}$$

and reduces for the abelian case $f=0$ to the previous expression. Remarkably, the structure in the second line is precisely analogous to the scalar potential appearing for

Kaluza-Klein reduction on group manifolds [42] and, for instance, in $\mathcal{N}=4$ gauged supergravity in $D=4$ [43].[2] We next verify that this action evaluated for (4.241) and $\tilde{\partial}^i=\partial_\alpha=0$ gives rise to the required non-abelian form of the low-energy action of the heterotic string.

The non-abelian field strength with structure constants $f^\alpha{}_{\beta\gamma}$ is given by

$$F_{ij}{}^\alpha \;=\; \partial_i A_j{}^\alpha-\partial_j A_i{}^\alpha+f^\alpha{}_{\beta\gamma}A_i{}^\beta A_j{}^\gamma\,, \tag{3.76}$$

while the field strength of the b-field is modified by the Chern-Simons 3-form and thus reads explicitly

$$\hat{H}_{ijk} \;=\; 3\left(\partial_{[i}b_{jk]}-\kappa_{\alpha\beta}A_{[i}{}^\alpha\Big(\partial_j A_{k]}{}^\beta+\tfrac{1}{3}f^\beta{}_{\gamma\delta}A_j{}^\gamma A_{k]}{}^\delta\Big)\right)\,. \tag{3.77}$$

We recall that here we do not indicate the gauge coupling constant explicitly, but rather absorb it into the structures constants. Using (3.76) and (3.77), the f-dependent non-abelian couplings in the low-energy Lagrangian in (3.1) are found to be

$$\begin{aligned}\mathcal{L}_f \;=\; & -f_{\alpha\beta\gamma}\,g^{ik}\,g^{jl}\,\partial_i A_j{}^\alpha A_k{}^\beta A_l{}^\gamma-\frac{1}{4}f^\alpha{}_{\beta\gamma}f_{\alpha\delta\epsilon}\,g^{ik}\,g^{jl}\,A_i{}^\beta A_j{}^\gamma A_k{}^\delta A_l{}^\epsilon \\ & +\frac{1}{2}f_{\alpha\beta\gamma}\,g^{ik}\,g^{jl}\,g^{pq}\,\partial_i b_{jp}\,A_k{}^\alpha\,A_l{}^\beta\,A_q{}^\gamma-\frac{1}{2}f_{\alpha\beta\gamma}\,g^{ik}\,g^{jl}\,g^{pq}\,A_{i\delta}\,\partial_j A_p{}^\delta\,A_k{}^\alpha\,A_l{}^\beta\,A_q{}^\gamma \\ & -\frac{1}{12}f_{\alpha\beta\gamma}f_{\delta\epsilon\zeta}\,g^{ik}\,g^{jl}\,g^{pq}\,A_i{}^\alpha A_j{}^\beta A_p{}^\gamma A_k{}^\delta A_l{}^\epsilon A_q{}^\zeta\,,\end{aligned} \tag{3.78}$$

where the first line originates from the Yang-Mills terms and the second and third line from the non-abelian parts of the Chern-Simons 3-form.

Evaluating the new terms in (3.75) yields precisely these terms for (4.241) and $\tilde{\partial}^i=\partial_\alpha=0$. We omit detailed steps for this verification, which can be found in [11].

[2] In fact, the scalar potential in $\mathcal{N}=4$ gauged supergravity for so-called electric gaugings is, up to an overall prefactor, precisely given by the second line of (3.75), see eq. (2.2) in [44].

3.3.4 Proof of gauge invariance

We turn now to the proof that the deformed action defined by (3.75) is invariant under the deformed gauge transformations (3.57). The unmodified $\mathcal{R}$ transforms as a scalar under the unmodified gauge transformations. We have to prove that its variation under the modified part of the gauge transformation, which is proportional to f, cancels against the variation of the new terms involving f.

Since all $O(D, D+n)$ indices are properly contracted it is sufficient to focus on the subset of variations that are non-covariant and which we will denote by Δ_ξ. Specifically, in $\mathcal{R}$ the new non-covariant contributions originate from partial derivatives only. For instance, for the following structure the f-dependent terms in the gauge variation, denoted by δ'_ξ, read

$$\delta'_\xi(\partial_M\mathcal{H}^{KL}) \;=\; \xi^P f^Q{}_{PM}\partial_Q\mathcal{H}^{KL} - 2\xi^P f^{(K}{}_{PQ}\,\partial_M\mathcal{H}^{L)Q} - 2\partial_M\xi^P f^{(K}{}_{PQ}\mathcal{H}^{L)Q}\,, \quad (3.79)$$

where the first term has been added by hand, which is allowed since it is zero by the constraint (3.54). The first two terms represent the covariant contributions, while the last term is non-covariant. We thus find

$$\Delta_\xi(\partial_M\mathcal{H}^{KL}) \;=\; -2\partial_M\xi^P f^{(K}{}_{PQ}\mathcal{H}^{L)Q}\,. \quad (3.80)$$

Since we saw that η^{MN} can be viewed as an invariant tensor under the modified gauge transformations (3.57), we can derive from this result, by lowering indices with η, the following form

$$\Delta_\xi(\partial_M\mathcal{H}_{KL}) \;=\; 2\partial_M\xi^P f^Q{}_{P(K}\mathcal{H}_{L)Q}\,. \quad (3.81)$$

Moreover, from (3.58) we infer

$$\Delta_\xi(\partial_M d) \;=\; 0\,. \quad (3.82)$$

Using this and (3.80), it is straightforward to see that all dilaton-dependent terms in (3.72) are separately invariant under the deformed part of the gauge transformations.

For instance

$$\Delta_\xi\big(4\partial_M\mathcal{H}^{MN}\partial_N d\big) \;=\; -8\partial_M\xi^P f^{(M}{}_{PQ}\,\mathcal{H}^{N)Q}\,\partial_N d \;=\; 0 \tag{3.83}$$

easily follows with (3.54). All other d-dependent terms can also be seen to be gauge invariant by virtue of (3.54). Similarly, the term involving a second derivative of $\mathcal{H}$ is gauge invariant,

$$\delta'_\xi\left(\partial_M\partial_N\mathcal{H}^{MN}\right) \;=\; -2\partial_M\big(\xi^P f^{(M}{}_{PQ}\,\partial_N\mathcal{H}^{N)Q} + \partial_N\xi^P f^{(M}{}_{PQ}\mathcal{H}^{N)Q}\big) \;=\; 0\;, \tag{3.84}$$

where (3.79) has been used. Thus, we have to focus only on the terms in the last line of (3.72), whose variation with a little work can be brought to the form

$$\Delta_\xi\mathcal{R} \;=\; -\frac{1}{2}\partial_N\xi^L f^M{}_{LK}\mathcal{H}^{NP}\mathcal{H}^{QK}\partial_P\mathcal{H}_{MQ} - \partial^M\xi_L f^L{}_{NK}\mathcal{H}^{NP}\mathcal{H}^{KQ}\partial_P\mathcal{H}_{QM}\;. \tag{3.85}$$

These terms have to be cancelled by the variations of the new terms in $\mathcal{R}_f$.

There are various contributions to the gauge transformations of the f-dependent terms in (3.75). First, the partial derivative of $\mathcal{H}$ in the first line transforms non-covariantly already under the unmodified part of the gauge transformations, but it can be easily checked, using eq. (4.36) from [8], that this contribution is zero by (3.54). Next, we have to keep in mind that $f^M{}_{NP}$ is constant and thus does not transform with a generalized Lie derivative with respect to ξ^M. The resulting non-covariant terms can be accounted for by assigning a fictitious non-covariant variation to f (with the opposite sign),

$$\Delta_\xi f^M{}_{NK} \;=\; -\widehat{\mathcal{L}}_\xi f^M{}_{NK} \;=\; -\partial^M\xi_P\, f^P{}_{NK} - \partial_N\xi^P\, f^M{}_{PK} - \partial_K\xi^P\, f^M{}_{NP}\;, \tag{3.86}$$

where the constancy of f and (3.54) has been used in the final step. Using this, the variation of the f-dependent term in the first line of (3.75) can be seen to precisely cancel (3.85), which in turn fixes the coefficient of this term in $\mathcal{R}_f$ uniquely.

Next, using (3.81), the term in the first line of (3.75) gives a variation proportional to f^2,

$$\begin{aligned} -\frac{1}{2}\Delta_\xi\big(f^M{}_{NK}\mathcal{H}^{NP}\mathcal{H}^{KQ}\partial_P\mathcal{H}_{QM}\big) \;=\; & -\frac{1}{2}f^M{}_{NK}f^L{}_{RQ}\partial_P\xi^R\mathcal{H}_{ML}\mathcal{H}^{NP}\mathcal{H}^{KQ} \\ & -\frac{1}{2}f^M{}_{NK}f^K{}_{RM}\partial_P\xi^R\mathcal{H}^{NP}\;. \end{aligned} \tag{3.87}$$

Thus, we get two contributions: one cubic in $\mathcal{H}$ and one linear in $\mathcal{H}$. The cubic term is cancelled by the variation of the first term in the second line of (3.75) according to (3.86), which in turn fixes the coefficient of this term. The term linear in $\mathcal{H}$ is cancelled by the variation (3.86) of the second term in the second line of (3.75), which finally fixes the coefficient of this term. The last term in (3.75) is constant and thus trivially gauge invariant. In total, we have proved that the modified scalar curvature $\mathcal{R}_f$ transforms as in (3.73), i.e., as a scalar, under the deformed gauge transformations (3.57), and thus that the Einstein-Hilbert like action (3.71) is gauge invariant.

3.4 The covariant constraints and their solutions

In this section we discuss the $O(D, D+n)$ covariant differential constraints and (3.7) and their solutions. Before that, we explain the relation of the $O(D, D+n)$ covariant constraints to the level-matching condition in string theory.

3.4.1 Relation to level-matching condition

In the abelian case, for which (3.7) trivializes, the remaining constraint has a rather direct relation to the level-matching condition of closed string theory. In the original double field theory construction for the bosonic string, the level-matching requires for the massless sector [6]

$$L_0 - \bar{L}_0 \;=\; -p_i w^i \;=\; 0\;, \tag{3.88}$$

where p_i and w^i are the momenta and winding modes on the torus, respectively. Upon Fourier transformation, this implies that in string field theory all fields and parameters need to be annihilated by the differential operator $\tilde{\partial}^i\partial_i$. Here, we require the stronger form that also all products of fields and parameters are annihilated. Similarly, the extended form (3.15) and (3.16) of the constraint is the stronger version of the level-matching condition in heterotic string theory, which will be discussed next.

We start by recalling the (bosonic part of) the world-sheet action for heterotic string theory, which is given by [45]

$$S = \frac{1}{2\pi}\int d\tau d\sigma\Big[G_{ij}\partial_a X^i\partial^a X^j + \varepsilon^{ab}B_{ij}\partial_a X^i\partial_b X^j + \partial_a X_\alpha\partial^a X^\alpha + \varepsilon^{ab}A_{i\alpha}\partial_a X^i\partial_b X^\alpha\Big]\,. \tag{3.89}$$

Here, $X^i \sim X^i + 2\pi k^i$, $k^i \in \mathbb{Z}$, denotes the periodic coordinates of the torus, and we have not displayed the non-compact coordinates. The X^α are 16 internal left-moving coordinates, i.e., satisfying the constraint $(\partial_\tau - \partial_\sigma)X^\alpha = 0$. In this subsection, the indices a, b label the world-sheet coordinates τ, σ, and G, B and A are the backgrounds. We split the world-sheet scalars into left- and right-moving parts, $X^i = X^i_L + X^i_R$, whose zero-modes are

$$\begin{aligned} X^i_L(\tau+\sigma) &= \tfrac{1}{2}x^i_0 + \tfrac{1}{2}p^i_L(\tau+\sigma)\,, \\ X^i_R(\tau-\sigma) &= \tfrac{1}{2}x^i_0 + \tfrac{1}{2}p^i_R(\tau-\sigma)\,, \\ X^\alpha(\tau+\sigma) &= x^\alpha_0 + p^\alpha_L(\tau+\sigma)\,. \end{aligned} \tag{3.90}$$

Following the canonical quantization of [45] (see also the discussion around eqs. (11.6.17) in [2]), the left- and right-moving momenta can in turn be written as

$$\begin{aligned} p_{Li} &= \tfrac{1}{2}p_i + (G_{ij} - B_{ij})\,w^j - \tfrac{1}{2}A_{i\alpha}\left(q^\alpha + \tfrac{1}{2}A_j{}^\alpha w^j\right)\,, \\ p_{Ri} &= \tfrac{1}{2}p_i - (G_{ij} + B_{ij})\,w^j - \tfrac{1}{2}A_{i\alpha}\left(q^\alpha + \tfrac{1}{2}A_j{}^\alpha w^j\right)\,, \\ p^\alpha_L &= q^\alpha + A_i{}^\alpha w^i\,, \end{aligned} \tag{3.91}$$

where the momentum and winding quantum numbers p_i and w^i, respectively, are integers as a consequence of the periodicity of the X^i, while the q^α take values in the root lattice of $E_8 \times E_8$ or $SO(32)$.

Let us now turn to the level-matching condition, where for definiteness we work in the Green-Schwarz formalism. We truncate to the massless subsector of the heterotic string spectrum with 16 abelian gauge fields, i.e., taking values in the Cartan subalgebra. In other words, we restrict to the massless spectrum with $N = 0$ and $\bar{N} = 1$ and thereby truncate out the 480 remaining gauge fields, which appear for $N = 0$ and $\bar{N} = 0$, were N and $\bar{N}$ are the number operators. The level-matching condition for this subsector is given by

$$L_0 - \bar{L}_0 + a_L - a_R \;=\; L_0 - \bar{L}_0 + 1 \;=\; (p_R^i)^2 - (p_L^i)^2 - (p_L^\alpha)^2 \;=\; 0\,, \tag{3.92}$$

where the normal ordering constants are $a_L = 1$ and $a_R = 0$. Inserting (3.91) into (3.92), we obtain

$$2p_i w^i + q^\alpha q_\alpha \;=\; 0\,. \tag{3.93}$$

If we interpret the q_α, like p_i and w^i, as the Fourier numbers corresponding to a torus, this condition translates in coordinate space precisely into the differential constraint (3.15). More precisely, the q^α are vectors in the root lattice of $E_8 \times E_8$ or $SO(32)$ rather than T^{16}, but these are topologically equivalent, and so we conclude that, in precise analogy to the case of bosonic string theory originally analyzed in [6], the level-matching condition amounts to the differential constraint (3.15) (and, correspondingly, (3.16) represents the stronger form of this constraint). We stress that the non-abelian case to be discussed in the next subsection is conceptually very different because it requires formally the introduction of 496 extra coordinates together with the novel constraint (3.7), which have no direct interpretation in the full string theory.

3.4.2 Solutions of the constraints

Next, we turn to the discussion of the solutions of the strong constraint. As in the bosonic string, we will show that all solutions of this constraint are locally related via an $O(D, D+n)$ rotation to solutions for which fields and parameters depend only on the x^i. To see this, consider the Fourier expansion of all fields and parameters, denoted generically by A, which take the form

$$A(x, \tilde{x}, y) \;=\; A\, e^{i(p_i x^i + w^i \tilde{x}_i + q_\alpha y^\alpha)}\,, \tag{3.94}$$

where we indicated for simplicity only a single Fourier mode. The quantum numbers combine into a vector of $O(D, D+n)$,

$$P_M \;=\; \big(\, w^i \,,\; p_i \,,\; q_\alpha \,\big)\;. \tag{3.95}$$

The strong constraint now implies that

$$\eta^{MN}\, P_M^{\mathfrak{a}}\, P_N^{\mathfrak{b}} \;=\; 0\;, \tag{3.96}$$

for all $\mathfrak{a}, \mathfrak{b}$ (which label the Fourier modes of all fields and parameters). Thus, all momenta are null and mutually orthogonal. In other words, they lie in a totally null or isotropic subspace of $\mathbb{R}^{2D+n}$. The canonical example of such a subspace is given by a space with $w^i = q_\alpha = 0$, corresponding to a situation where all fields and parameters depend only on the x^i. Since the flat metric on $\mathbb{R}^{2D+n}$ has signature $(D, D+n)$, the maximal dimension of any isotropic subspace is D. It is a rather general result, related to Witt's theorem (see the discussion and references in [7]), that all isotropic subspaces of the same dimension are related by isometries of the full space, i.e., here they are related by $O(D, D+n)$ transformations. In particular, one can always find an $O(D, D+n)$ transformation to a T-duality frame where $w^i = q_\alpha = 0$ and therefore one can always rotate into a frame where fields and parameters depend only on x^i, as we wanted to show.

Next, we discuss the general non-abelian theory. In this case, the global $O(D, D+n)$ symmetry is broken by a choice of non-vanishing structure constants $f^M{}_{NK}$ and, therefore, we have no longer all T-duality transformations to our disposal in order to rotate into a frame in which the fields depend only on x^i. This is, however, compensated by the additional constraint (3.54) which eliminates further coordinates for non-vanishing structure constants.

To illustrate this point, suppose that we choose $f^M{}_{NK}$ as in (4.241), i.e., the only non-vanishing components $f^\alpha{}_{\beta\gamma}$ are given by the structure constants of a semi-simple Lie group G. We can view G as the subgroup of $SO(n)$ that leaves the tensor $f^\alpha{}_{\beta\gamma}$

invariant,[3] and so the global symmetry group is then broken to $O(D,D)\times G$, where we view G as the global subgroup of the gauge group. The constraint (3.54) can now be multiplied with the structure constants, which implies

$$0 \;=\; f^{\gamma}{}_{\delta\alpha}\, f^{\delta}{}_{\gamma\beta}\,\partial^{\beta} \;=\; -2\,\kappa_{\alpha\beta}\,\partial^{\beta}\;, \tag{3.97}$$

where $\kappa_{\alpha\beta}$ is the Cartan-Killing form. As $\kappa_{\alpha\beta}$ is invertible for a semi-simple Lie algebra, we conclude $\partial_{\alpha}=0$, i.e., the constraint implies that all fields are independent of y^{α}. The unbroken $O(D,D)$ transformations can then be used as above in order to rotate into a T-duality frame in which the fields are independent of $\tilde{x}$. In total, the constraints are still sufficient in order to guarantee that the dependence on the 'unphysical' coordinates $\tilde{x}$ and y is either eliminated directly or removable by a surviving T-duality transformation.

Let us now turn to a more general situation where $f^{M}{}_{NK}$ is of the form (4.241), but with the gauge group G having some $U(1)$ factors. Suppose, the gauge group is of the form

$$G \;=\; U(1)^{p}\times G_0\;, \tag{3.98}$$

where G_0 is semi-simple and embedded into $O(n-p)$. If we split the indices accordingly, $\alpha=(\underline{\alpha},\bar{\alpha})$, with $\underline{\alpha}=1,\dots,p$ and $\bar{\alpha}=1,\dots,n-p$, the non-vanishing components of $f^{M}{}_{NK}$ are given by the structure constants $f^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}}$ of G_0. The constraint (3.54) implies in this case only $\partial_{\bar{\alpha}}=0$, i.e., that the fields are independent of the $n-p$ coordinates $y^{\bar{\alpha}}$. The unbroken T-duality group is, however, given by $O(D,D+p)$ and thus larger than in the previous example. Therefore, as in the above discussion of the abelian case, these transformations can be used in order to rotate into a T-duality frame in which the fields are both independent of $\tilde{x}_i$ but also of the remaining p coordinates $y^{\underline{\alpha}}$. Thus, the constraints and residual T-duality transformations are again sufficient in order to remove the dependence on $\tilde{x}$ and y.

We finally note that by virtue of the $O(D,D+n)$ covariance of the constraints any

[3] Any compact n-dimensional Lie group G can be canonically embedded into $SO(n)$. If we denote the generators of $\mathfrak{so}(n)$ by $K^{\alpha\beta}=-K^{\beta\alpha}$, the generators t^{α} of G are embedded as $t^{\alpha}=\frac{1}{2}f^{\alpha}{}_{\beta\gamma}K^{\beta\gamma}$.

$f^M{}_{NK}$ obtained from (4.241) by a duality transformation also solves the constraints. Presumably, these have to be regarded as physically equivalent to (4.241) and thereby to the conventional low-energy action of heterotic string theory. It remains to be investigated, however, whether there are different solutions to the constraints. This is particularly interesting in the context of (generalized) Kaluza-Klein compactifications, where the fields are independent of some of the x^i and for which the differential constraints may allow for more general solutions. We leave this to future work.

3.5 Frame formulation

Here, we reformulate the above results in a frame-like language in order to make contact with the formalism developed by Siegel [5], as has been done in [10] for the double field theory extension of the bosonic string. We first discuss the abelian case, which is straightforward, and then turn to the non-abelian case which requires an extension of the formalism.

3.5.1 Frame fields and coset formulation

The basic field in the formalism of Siegel is a vielbein or frame field $e_A{}^M$ that is a vector under gauge transformations parameterized by ξ^M and which is subject to local tangent space transformations indicated by the flat index A. In the present case, the tangent space group is $GL(D) \times GL(D+n)$ and the index splits as $A = (a, \bar{a})$. Using the frame field and η_{MN}, one can define a tangent-space metric of signature $(D, D+n)$,

$$\mathcal{G}_{AB} \;=\; e_A{}^M \, e_B{}^N \, \eta_{MN} \;, \tag{3.99}$$

and the frame field is constrained to satisfy

$$\mathcal{G}_{a\bar{b}} \;=\; 0 \;. \tag{3.100}$$

Starting from this frame field and the local tangent space symmetry, one may introduce connections for this gauge symmetry, impose covariant constraints and construct invariant generalizations of the Ricci tensor and scalar curvature. Rather than repeating this construction here, we will just mention in the following the new aspects in the case of the heterotic string theory and refer to [5] and [10] for more details.

The generalized metric can be defined as follows

$$\mathcal{H}^{MN} \;=\; 2\mathcal{G}^{\bar{a}\bar{b}}\, e_{\bar{a}}{}^{M} e_{\bar{b}}{}^{N} - \eta^{MN} \;=\; -2\mathcal{G}^{ab}\, e_{a}{}^{M} e_{b}{}^{N} + \eta^{MN}\,, \tag{3.101}$$

where the equivalence of the two definitions is a consequence of the constraint (3.100). Next, it is convenient to gauge-fix the tangent space symmetry by setting $\mathcal{G}_{AB}$ equal to η_{MN} (up to a similarity transformation, c.f. the discussion after eq. (5.22) in [8]), such that (3.99) and (3.101) imply [8]

$$\mathcal{H}^{MN} \;=\; \delta^{AB}\, e_{A}{}^{M}\, e_{B}{}^{N}\,. \tag{3.102}$$

This leaves a local $O(D) \times O(D+n)$ symmetry unbroken, and in this gauge we can think of the frame field $e_A{}^M$ as a $O(D, D+n)$-valued coset representative that is subject to local $O(D) \times O(D+n)$ transformations. Thus, this formulation can be viewed as a generalized coset space construction based on $O(D, D+n)/(O(D) \times O(D+n))$, in analogy to the structure appearing in dimensional reduction of heterotic supergravity [36]. Fixing the local symmetry further, one may give explicit parametrizations of the frame field $e_A{}^M$ in terms of the physical fields that give rise to the form (3.20) of $\mathcal{H}^{MN}$ according to (3.102), see, e.g., eq. (4.12) in [36].

We turn now to the definition of the scalar curvature $\mathcal{R}$ that can be used to define an invariant action as in (3.71). It can be written in terms of 'generalized coefficients of anholonomy' $\Omega_{AB}{}^{C}$ that are defined via the C-bracket (3.35),

$$\big[e_A, e_B\big]_C^M \;=\; \Omega_{AB}{}^{C}\, e_C{}^{M}\,. \tag{3.103}$$

Defining[4]

$$h_{ABC} \;=\; (e_A e_B{}^M) e_{CM} \,, \tag{3.104}$$

where $e_A = e_A{}^M \partial_M$, one obtains explicitly

$$\Omega_{ABC} \;=\; 2h_{[AB]C} + h_{C[AB]} \;=\; h_{ABC} + h_{BCA} + h_{CAB} \;=\; 3h_{[ABC]} \,. \tag{3.105}$$

Here we used that the gauge condition implies that $\mathcal{G}_{AB}$ is constant and therefore $h_{ABC} = -h_{ACB}$ from the definition (3.104). Finally, defining

$$\tilde{\Omega}_A \;=\; \partial_M e_A{}^M - 2e_A d \,, \tag{3.106}$$

the scalar curvature is given by

$$\mathcal{R} \;=\; e_a \tilde{\Omega}^a + \frac{1}{2}\tilde{\Omega}_a{}^2 + \frac{1}{2} e_a e_b \mathcal{G}^{ab} - \frac{1}{4}\Omega_{ab\bar{c}}{}^2 - \frac{1}{12}\Omega_{[abc]}{}^2 + \frac{1}{8} e^a \mathcal{G}^{bc} \, e_b \mathcal{G}_{ac} \,. \tag{3.107}$$

In [10] it has been verified that starting from this expression for $\mathcal{R}$ and using the definition of $\mathcal{H}^{MN}$ in terms of the frame fields, this reduces precisely to the form given above in (3.72), up to an overall factor of 4. This proof immediately generalizes to the abelian case of the heterotic string, as all expressions, including the definition (3.101) of $\mathcal{H}^{MN}$, are formally the same.

3.5.2 Non-abelian extension

Let us now turn to the non-abelian generalization, which has also been mentioned in [41]. A natural starting point is the deformed bracket (3.65) of gauge transformations. We further generalize the coefficients of anholonomy by defining

$$\big[e_A, e_B\big]_f^M \;=\; \widehat{\Omega}_{AB}{}^C e_C{}^M \,. \tag{3.108}$$

[4] We note that we changed notation as compared to [5,10], where this quantity has been denotes by f, in order to distinguish it from the structure constants.

By (3.65) and (3.103) this implies

$$\widehat{\Omega}_{AB}{}^{C} \;=\; \Omega_{AB}{}^{C} - f^{C}{}_{AB}\,, \qquad f^{C}{}_{AB} \;=\; f^{M}{}_{NK}\, e_{M}{}^{C}\, e_{A}{}^{N}\, e_{B}{}^{K}\,, \tag{3.109}$$

where we introduced structure constants with flattened indices. The f-bracket of two vectors that transform covariantly under the deformed gauge transformations transforms covariantly in the same sense, i.e.,

$$\delta_\xi\big[X,Y\big]_f^M \;=\; \widehat{\mathcal{L}}_\xi\big[X,Y\big]_f^M - \xi^N f^M{}_{NK}\big[X,Y\big]_f^K\,. \tag{3.110}$$

To see this, we recall from [10] that the C-bracket is invariant under the generalized Lie derivative. Thus, it remains to be shown that the non-covariant part of the variation of the C-bracket due to the deformed gauge variation cancels against the variation of the new term in the f-bracket. As in the proof of gauge invariance of the action above, we denote the non-covariant part of the variation by Δ_ξ and compute

$$\begin{aligned}\Delta_\xi\big[X,Y\big]_{\rm C}^M \;&=\; -\xi^P f^N{}_{PK}X^K\partial_N Y^M + \frac{1}{2}\xi^P f^N{}_{PK}X^K\partial^M Y_N \\ &\quad - X^N\partial_N\big(\xi^P f^M{}_{PK}Y^K\big) + \frac{1}{2}X^N\partial^M\big(\xi^P f_{NP}{}^K Y_K\big) - \{X\leftrightarrow Y\}\,.\end{aligned} \tag{3.111}$$

Using the constraint (3.54), it is straightforward to verify that this can be rewritten as

$$\Delta_\xi\big[X,Y\big]_{\rm C}^M \;=\; -\xi^N f^M{}_{NK}\big[X,Y\big]_{\rm C}^K - \big(\widehat{\mathcal{L}}_\xi f^M{}_{NK}\big)X^N Y^K\,. \tag{3.112}$$

The second term here is precisely cancelled by the non-covariant variation of the f-dependent term in the f-bracket, which finally proves the covariance relation (3.110).

Next, we discuss the extension of the scalar curvature (3.107). Given the covariance of the f-bracket, it follows from (3.108) that $\widehat{\Omega}$ is a scalar under ξ^M transformations, while its frame transformations are as in the abelian case. Therefore, if we replace in (3.107) Ω by $\widehat{\Omega}$, the resulting expression will also be a scalar. The Ricci scalar is modified as

$$\mathcal{R}_f \;:=\; e_a\tilde{\Omega}^a + \frac{1}{2}\tilde{\Omega}_a{}^2 + \frac{1}{2}e_a e_b \mathcal{G}^{ab} - \frac{1}{4}\widehat{\Omega}_{ab\bar{c}}{}^2 - \frac{1}{12}\widehat{\Omega}_{[abc]}{}^2 + \frac{1}{8}e^a\mathcal{G}^{bc}\,e_b\mathcal{G}_{ac}\,. \qquad (3.113)$$

Inserting here the definition (3.109), we infer

$$\mathcal{R}_f \;=\; \mathcal{R} - \frac{1}{4}\big(-2\Omega_{ab\bar{c}}f^{ab\bar{c}} + f_{ab\bar{c}}f^{ab\bar{c}}\big) - \frac{1}{12}\big(-2\Omega_{[abc]}f^{abc} + f_{abc}f^{abc}\big)\,. \qquad (3.114)$$

With lengthy computations one can verify that this expression indeed agrees with the definition (3.75) above, using

$$e_a{}^M e^{aN} \;=\; \frac{1}{2}\big(\eta^{MN} - \mathcal{H}^{MN}\big)\,, \qquad (3.115)$$

$$e_{\bar{a}}{}^M e^{\bar{a}N} \;=\; \frac{1}{2}\big(\eta^{MN} + \mathcal{H}^{MN}\big)\,. \qquad (3.116)$$

We omit the detailed computations here, which can be found in [11] for interested readers.

3.6 Conclusions

In this chapter we have extended the double field theory formulation of [8] to the low-energy action of the heterotic string, which features extra non-abelian gauge fields. These extra gauge fields neatly assemble with the massless fields of closed bosonic string theory into an enlarged generalized metric that transforms covariantly under the enhanced T-duality group $O(D, D+n)$ and thereby represent a further 'unification'. For the abelian subsector, the action takes the same structural form as for the bosonic string, but based on the enlarged generalized metric. In the non-abelian case, the T-duality group is broken to a subgroup, but interestingly the action can still be written in a covariant fashion, with new couplings which are precisely analogous to those encountered in lower-dimensional gauged supergravities. These new couplings are parametrized by a tensor $f^M{}_{NK}$, and any such tensor satisfying a number of covariant constraints defines a consistent deformation of the abelian theory.

This means that rather than having a proper global $O(D, D+n)$ symmetry, there is an action of this group on the 'space of consistent deformations' of the abelian theory. Whether this space consists of a single $O(D, D+n)$ orbit or whether there are more general solutions to the constraints that are inequivalent to (4.241) (and thereby to the conventional Yang-Mills-type theory) remains to be seen.

Chapter 4

Double Field Theory of Type II Strings and its Massive Extension

A bulk of this chapter appeared in"Double Field Theory of Type II Strings" with Olaf Hohm and Barton Zwiebach [13], and "Massive Type II in Double Field Theory" with Olaf Hohm [14]. They are reprinted with the permission of *JHEP*.

Summary : We use double field theory to give a unified description of the low energy limits of type IIA and type IIB superstrings. The Ramond-Ramond potentials fit into spinor representations of the duality group $O(D, D)$ and field-strengths are obtained by acting with the Dirac operator on the potentials. The action, supplemented by a $\text{Spin}^+(D, D)$ covariant self-duality condition on field strengths, reduces to the IIA and IIB theories in different frames. As usual, the NS-NS gravitational variables are described through the generalized metric. Our work suggests that the fundamental gravitational variable is a hermitian element of the group $\text{Spin}(D, D)$ whose natural projection to $O(D, D)$ gives the generalized metric.

For the special case that only the RR one-form of type IIA depend simultaneously on the 10-dimensional space-time coordinates and linearly on the dual winding coordinates, we obtain the massive deformation of type IIA supergravity due to Romans. For T-dual configurations we obtain a massive but non-covariant formulation of type IIB, in which the 10-dimensional diffeomorphism symmetry is deformed by the mass parameter.

4.1 Introduction

T-duality transformations along circles of compactified type II superstrings show that type IIA and type IIB superstrings are, in fact, the same theory for toroidal backgrounds of odd dimension (see [2] and references therein). This naturally leads to the question of whether there exists a formulation of type II theories that makes this feature manifest. In this chapter we will address this question in terms of double field theory formulation of type II strings developed in [12, 13].

In the papers we construct the double field theory of the RR massless sector of superstring theory. The NS-NS massless sector is described by the same theory that describes the massless sector of the bosonic string. The RR sector requires some new ingredients. The first one is that the RR gauge fields fit naturally into the spinor representation of $O(D, D)$. In the case of interest, the physical dimension is $D = 10$ and we have a spinor of $O(10, 10)$. Calling χ the spinor that encodes the RR forms we have the duality transformations

$$\text{Duality transformations:} \quad \chi \;\to\; S\chi\,, \qquad S \in \text{Spin}(D, D)\,. \tag{4.1}$$

We show that this implies

$$\not{\partial}\chi \;\to\; S\not{\partial}\chi\,, \qquad S \in \text{Spin}(D, D)\,, \tag{4.2}$$

where $\not{\partial}$ is a Dirac operator, which will be defined in the next section. Since $\not{\partial}$ is first order in derivatives, $\not{\partial}\chi$ is naturally interpreted as the field strength associated to the RR potentials, to which it indeed reduces for $\tilde{\partial}^i = 0$.

Following the insights of [61] it is natural to consider the spin group representative of $\mathcal{H}$ to discuss the coupling of the RR fields to the NS-NS fields. The generalized metric $\mathcal{H}$ is a symmetric matrix that is also an $O(D, D)$ element. Since the determinant of $\mathcal{H}$ is plus one, we actually have $\mathcal{H} \in SO(D, D)$. The group $SO(D, D)$ has two disconnected components: the subgroup $SO^+(D, D)$ that contains the identity and a coset denoted by $SO^-(D, D)$. One can check that in Lorentzian signature $\mathcal{H}$

is actually in $SO^{-}(D,D)$. The associated spin representatives are in $\text{Spin}^{-}(D,D)$; they are elements S and $-S$, such that $\rho(\pm S) = \mathcal{H}$.

It turns out to be impossible to choose a spin representative in a single-valued and continuous way over the space of possible $\mathcal{H}$. We note that this phenomenon occurs whenever a *timelike* T-duality is employed, and therefore does not arise in Euclidean signature where $\mathcal{H} \in SO^{+}(D,D)$ and a lift to $\text{Spin}^{+}(D,D)$ can be chosen continuously. In light of this topological subtlety we suggest that instead of viewing $\mathcal{H}$ as the fundamental gravitational field, from which a spin representative needs to be constructed, we view the spin element itself as the dynamical field, denoted by $\mathbb{S} \in \text{Spin}^{-}(D,D)$. The generalized metric can then be defined uniquely by the homomorphism: $\mathcal{H} = \rho(\mathbb{S})$. The condition that $\mathcal{H}$ is symmetric requires that $\mathbb{S}$ be hermitian, $\mathbb{S} = \mathbb{S}^{\dagger}$. Under the duality transformation (4.1) we declare that

$$\text{Duality transformations:} \qquad \mathbb{S} \to \mathbb{S}' = (S^{-1})^{\dagger}\, \mathbb{S}\, S^{-1} \qquad S \in \text{Spin}(D,D)\,. \tag{4.3}$$

This transformation is consistent with that of the generalized metric, namely, $\rho(S)$ is an $SO(D,D)$ transformation that takes $\mathcal{H} = \rho(\mathbb{S})$ to $\mathcal{H}' = \rho(\mathbb{S}')$.

We can now discuss the double field theory action for type II theories, whose independent fields are $\mathbb{S}$, χ and d. It is the sum of the action (1.45) for the NS-NS sector and a new action for the RR sector:

$$\begin{aligned} S &= \int dx d\tilde{x}\, \Big(e^{-2d}\, \mathcal{R}(\mathcal{H}, d) + \frac{1}{4} (\partial\!\!\!/ \chi)^{\dagger}\, \mathbb{S}\, \partial\!\!\!/ \chi \Big)\,, \\ \mathcal{H} &= \rho(\mathbb{S}), \quad \mathbb{S} \in \text{Spin}^{-}(D,D)\,, \quad \mathbb{S}^{\dagger} = \mathbb{S}\,. \end{aligned} \tag{4.4}$$

The duality invariance of the RR action is manifest on account of (4.2) and (4.3). The definition of the theory also requires the field strength $\partial\!\!\!/ \chi$ to satisfy a self-duality constraint that can be written in a manifestly duality covariant way,

$$\partial\!\!\!/ \chi = -C^{-1} \mathbb{S}\, \partial\!\!\!/ \chi\,. \tag{4.5}$$

Here the charge conjugation matrix C satisfies $C^{-1} \Gamma^{M} C = (\Gamma^{M})^{\dagger}$. While the action is invariant under $\mathrm{Spin}(D,D)$, the self-duality constraint breaks the duality symmetry down to $\mathrm{Spin}^+(D,D)$.

The RR potentials have the usual abelian gauge symmetries in which the form fields are shifted by exact forms. This symmetry also takes a manifestly duality covariant form,

$$\delta_\lambda \chi \;=\; \not{\partial}\lambda\,, \tag{4.6}$$

and leaves (4.4) invariant because $\not{\partial}^2 = 0$. More nontrivially, the invariance of the theory under the gauge symmetries parameterized by ξ^M requires that χ transform as

$$\delta_\xi \chi \;=\; \widehat{\mathcal{L}}_\xi \chi \;\equiv\; \xi^M \partial_M \chi + \frac{1}{2}\partial_M \xi_N \Gamma^M \Gamma^N \chi\,. \tag{4.7}$$

In here we defined the generalized Lie derivative $\widehat{\mathcal{L}}_\xi$ acting on a spinor.

We find that the natural form of the gauge transformation of $\mathbb{S}$ is

$$\delta_\xi \mathbb{S} \;=\; \xi^M \partial_M \mathbb{S} + \frac{1}{2} C\big[\Gamma^{PQ}, C^{-1}\mathbb{S}\big]\partial_P \xi_Q\,. \tag{4.8}$$

and the action (4.4) is gauge invariant under the gauge transformations.

Let us now discuss the evaluation of the action in different T-duality frames. Suppose we have chosen a chirality of χ and a parametrization of $\mathbb{S}$ such that the theory reduces for $\tilde{\partial}^i = 0$ to type IIA. All other solutions of the strong constraint can be obtained from this one by an $O(D,D)$ transformation. Unlike in bosonic double field theory, T-duality generally relates different type II theories to each other. If, for instance, the theory reduces in one frame to type IIA, we will see that it reduces in any other frame obtained by an *odd* number of spacelike T-duality inversions to type IIB, and vice versa. If, on the other hand, the frames are related by an even number of spacelike T-duality inversions, the theory reduces in both frames to the same theory, either IIA or IIB.

A timelike T-duality transformation gives quite a different result. If we start from a T-duality frame in which the double field theory reduces to type IIA (IIB), we indeed find that the same theory reduces to IIB* (IIA*), which differs by an overall sign in the RR kinetic terms from IIB (IIA), in any frame obtained by a timelike T-duality transformation. In summary, the manifestly T-duality invariant double field theory defined by (4.4) and (4.5) unifies these four different type II theories in that each of them arises in particular T-duality frames.

There is the massive extension of type IIA supergravity due to Romans [28], which can be motivated as follows. If one introduces for each RR p-form the dual $8-p$ form, type IIA contains all odd forms with $p = 1, \ldots, 7$. We can also introduce a 9-form potential, but imposing the standard field equations sets its field strength $F^{(10)} = dC^{(9)}$ to a constant and so a 9-form carries no propagating degrees of freedom. We can think of massive type IIA as obtained by choosing this integration constant to be non-zero and equal to the mass parameter m. In the resulting theory, m enters as a cosmological constant and deforms the gauge transformations corresponding to the NS-NS b-field such that the RR 1-form transforms with a Stückelberg shift symmetry. It does not admit a maximally symmetric vacuum, but its most symmetric solution is the D8 brane solution that features 9-dimensional Poincaré invariance [71].

As we have already seen above, the $O(10,10)$ spinor representation of RR potentials is isomorphic to the set of all even or odd forms, depending on the chirality of the spinor, and so for type IIA the theory contains already a 9-form potential. However, the duality relations

$$*\widehat{F}^{(10)} \;=\; -\widehat{F}^{(0)} \;\equiv\; 0 \tag{4.9}$$

imply that its field strength is zero, because there is no non-trivial $\widehat{F}^{(0)}$ due to the absence of '(-1)–form' potentials, and therefore the 9-form is on-shell determined to be pure gauge. Formally, one may introduce a (-1)–form potential $C^{(-1)}$ and then set $m = F^{(0)} = dC^{(-1)}$, as has been done in [72], but so far it has been unclear how to find a mathematically satisfactory interpretation of such objects. In this note we will show that a non-trivial 0-form field strength (and thus a mass parameter) is naturally included in the type II double field theory by assuming that the RR 1-form depends linearly on the winding coordinates,

$$C^{(1)}(x,\tilde{x}) \;=\; C_i(x)dx^i + m\tilde{x}_1 dx^1\;, \tag{4.10}$$

where C_i and all other fields depend only on the 10-dimensional coordinates. We will see that the second term in (4.10) effectively acts as a (-1)–form and that the double field theory reduces precisely to massive type IIA.

It should be stressed that the consistency of the ansatz (4.10) is non-trivial in terms of $O(D,D)$ covariant constraints of double field theory. These constraints are necessary for gauge invariance of the action and closure of the gauge algebra. In its *weak* form, which requires $\partial^M\partial_M = 2\tilde{\partial}^i\partial_i$ to annihilate all fields and parameters, it is a direct consequence of the level matching condition of closed string theory, and it allows for field configurations such as (4.10) that depend locally both on x and $\tilde{x}$. The double field theory constructions completed so far, however, impose the stronger form. In this form the constraint implies that locally all fields depend only on half of the coordinates, and so (4.10) violates the strong constraint. Remarkably, as we will show here, the gauge transformations can be reformulated on the RR fields so that the strong constraint can be relaxed. It cannot be relaxed to the weak constraint as formulated above, but it is sufficient for the ansatz (4.10) to be consistent. In particular, this formulation guarantees that in the action and gauge transformations the linear $\tilde{x}$ dependence drops out, such that the resulting theory has a conventional 10-dimensional interpretation.

This chapter is organized as follows. In sec. 2 we review the properties of the spinor representation of $O(D,D)$ and of its double covering group. Due to the aforementioned topological subtleties, we find it necessary to delve in some detail into the mathematical issues. In sec. 3 we discuss the field that is interpreted as the spinor representative of the generalized metric. The duality covariant form of the action and duality relations is introduced in sec. 4, while their evaluation in particular T-duality frames is done in sec. 5 and 6. The massive deformation of type IIA double field theory is discussed in sec. 7.

4.2 O(D,D) spinor representation

In this section we review properties of the T-duality group $O(D,D)$ and its spinor representation or, more precisely, the properties its two-fold covering group $\mathrm{Pin}(D,D)$ and its representations. Convenient references for this section are [61], [31], and [22].

4.2.1 O(D,D), Clifford algebras, and Pin(D,D)

In order to fix our conventions, we start by recalling some basic properties of $O(D,D)$. This group is defined to be the group leaving the metric of signature $(\mathbf{1}_D,-\mathbf{1}_D)$ invariant. We choose a basis where the metric takes the form

$$\eta \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{4.1}$$

and we denote it by η^{MN} or η_{MN} which, viewed as matrices, are equal. The indices M,N run over the $2D$ values $1,2,\ldots,2D$. The preservation of η implies that group elements $h\in O(D,D)$, viewed as matrices, satisfy

$$\eta^{MN} \;=\; h^M{}_P\, h^N{}_Q\, \eta^{PQ} \quad\Leftrightarrow\quad \eta \;=\; h\,\eta\, h^T\,. \tag{4.2}$$

This implies that $\det(h) = \pm 1$. The subgroup of $O(D,D)$ whose elements have determinant plus one is denoted by $SO(D,D)$. While the group $O(D,D)$ has four connected components, $SO(D,D)$ has two connected components. In $SO(D,D)$ the component connected to the identity is the subgroup denoted as $SO^+(D,D)$. It can be shown that in the basis where the metric takes the diagonal form $\mathrm{diag}(\mathbf{1}_D,-\mathbf{1}_D)$, the two $D\times D$ block-diagonal matrices of any $SO^+(D,D)$ element have positive determinant. The other component of $SO(D,D)$ is denoted by $SO^-(D,D)$. It is not a subgroup of $SO(D,D)$ but rather a coset of $SO^+(D,D)$.

The Lie algebra of $O(D,D)$ is spanned by generators $T^{MN} = -T^{NM}$ satisfying

$$\left[T^{MN},T^{KL}\right] \;=\; \eta^{MK}\,T^{LN} - \eta^{NK}\,T^{LM} - \eta^{ML}\,T^{KN} + \eta^{NL}\,T^{KM}\,. \tag{4.3}$$

Any group element connected to the identity can be written as an exponential of Lie algebra generators,

$$h^M{}_N \;=\; \left[\exp\left(\tfrac{1}{2}\Lambda_{PQ}T^{PQ}\right)\right]^M{}_N\,, \tag{4.4}$$

where

$$(T^{MN})^K{}_L \;=\; 2\eta^{K[M}\delta^{N]}{}_L\,, \tag{4.5}$$

is the fundamental representation of the Lie algebra (4.3). We use the anti-symmetrization convention $X_{[MN]} \equiv \frac{1}{2}(X_{MN} - X_{NM})$.

We turn now to the spinor representation of $O(D,D)$ and to the groups $\mathrm{Spin}(D,D)$ and $\mathrm{Pin}(D,D)$, whose properties will be instrumental below. The (reducible) spinor representation of $O(D,D)$ has dimension 2^D and can be chosen to be real or Majorana. Imposing an additional Weyl condition will yield two spinor representations of opposite chirality, both of dimension 2^{D-1}. These can be identified with even and odd forms and thus with the RR fields in type II.

To begin with, we introduce the Clifford algebra $C(D,D)$ associated to the quadratic form $\eta\,(\,\cdot,\,\cdot)$ on $\mathbb{R}^{2D}$. With basis vectors Γ_M, $M = 1,\ldots,2D$, we have

$$\eta_{MN} = \eta\,(\Gamma_M,\Gamma_N) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{4.6}$$

The main relation of the Clifford algebra states that for any $V \in \mathbb{R}^{2D}$

$$V \cdot V = \eta(V,V)\,\mathbf{1}\,, \tag{4.7}$$

where $\mathbf{1}$ is the unit element and the dot indicates the product in the algebra. This algebra is generated by the unit and basis vectors Γ_M. Writing $V = V^M\Gamma_M$, substitution in (4.7) gives

$$\left\{\Gamma_M,\Gamma_N\right\} \;\equiv\; \Gamma_M\cdot\Gamma_N + \Gamma_N\cdot\Gamma_M =\; 2\,\eta_{MN}\,. \tag{4.8}$$

Using the quadratic form η_{MN} and its inverse η^{MN} to raise and lower indices, we can write arbitrary vectors as $V = V^M\Gamma_M = V_M\Gamma^M$, which then allows to write (4.8) with all indices raised.

An explicit representation of the Clifford algebra (and below of the Pin group) can be conveniently constructed using fermionic oscillators ψ^i and ψ_i, $i = 1, \ldots, D$, satisfying

$$\{\psi_i, \psi^j\} = \delta_i{}^j \,, \qquad \{\psi_i, \psi_j\} = 0 \,, \qquad \{\psi^i, \psi^j\} = 0 \,, \tag{4.9}$$

where

$$(\psi_i)^\dagger = \psi^i \,. \tag{4.10}$$

Defining

$$\Gamma_i = \sqrt{2}\psi_i \,, \qquad \Gamma^i = \sqrt{2}\psi^i \,, \tag{4.11}$$

the oscillators realize the algebra (4.8). Spinor states can be defined introducing a Clifford vacuum $|0\rangle$ annihilated by the ψ_i for all i:

$$\psi_i|0\rangle = 0 \,, \quad \forall\, i \,. \tag{4.12}$$

From this, we derive a convenient identity that will be useful below,

$$\psi_j\psi^{i_1}\cdots\psi^{i_p}|0\rangle = p\,\delta_j{}^{[i_1}\psi^{i_2}\cdots\psi^{i_p]}|0\rangle \,. \tag{4.13}$$

A spinor χ in the 2^D-dimensional space can then be identified with a general state

$$|\chi\rangle = \sum_{p=0}^{D}\frac{1}{p!}C_{i_1\ldots i_p}\,\psi^{i_1}\ldots\psi^{i_p}|0\rangle \,, \tag{4.14}$$

where the coefficients are completely antisymmetric tensors. Thus, there is a natural identification of the spinor representation with the p-forms on $\mathbb{R}^D$. We define $\langle 0|$ to be the the 'dagger' of the state $|0\rangle$ and declare:

$$\langle 0|0\rangle = 1 \,. \tag{4.15}$$

For more general states,

$$(\psi^{i_1}\ldots\psi^{i_p}|0\rangle)^\dagger = \langle 0|\psi_{i_p}\ldots\psi_{i_i}\,. \tag{4.16}$$

We work on a real vector space, so the $\dagger$ operation does not affect the numbers multiplying the vectors. In the notation where dagger takes $|a\rangle$ to $\langle a|$ and vice versa, we can quickly show that $\langle a|b\rangle = \langle b|a\rangle$. We see from these definitions that in the spinor representation $(\Gamma^i)^\dagger$ is indeed equal to Γ_i. Since all matrix elements are real, the dagger operation is just transposition.

Let us now turn to the definition of the groups $\mathrm{Spin}(D, D)$ and $\mathrm{Pin}(D, D)$, which act on the spinor states. These groups are, respectively, double covers of the groups $SO(D, D)$ and $O(D, D)$. To describe these groups we need to introduce an anti-involution $\star$ of the Clifford algebra $C(D, D)$, which is defined by

$$(V_1\cdot V_2\ldots\cdot V_k)^\star \equiv (-1)^k\, V_k\cdot\ldots V_2\cdot V_1\,. \tag{4.17}$$

Note that for any vector V in $\mathbb{R}^{2D}$, $V^\star = -V$. For arbitrary elements S, T of the Clifford algebra one has $(S + T)^\star = S^\star + T^\star$ and $(S\cdot T)^\star = T^\star\cdot S^\star$. The group $\mathrm{Pin}(D, D)$ is now defined as follows:

$$\mathrm{Pin}(D,D) \;:=\; \left\{S\in C(D,D)\,|\,S\cdot S^\star = \pm 1\,,\; V\in\mathbb{R}^{2D} \;\Rightarrow\; S\cdot V\cdot S^{-1}\in\mathbb{R}^{2D}\right\}\,. \tag{4.18}$$

The first condition implies for all group elements that $S^\star$ is, up to a sign, the inverse of S. The second condition indicates that acting by conjugation with S on any vector $V\in\mathbb{R}^{2D}$ results in a vector in $\mathbb{R}^{2D}$. One readily checks that $S\in\mathrm{Pin}(D, D)$ implies $S^\star\in\mathrm{Pin}(D, D)$. In what follows we will omit the dot indicating Clifford multiplication whenever no confusion can arise. We finally note that the Lie algebras of $O(D, D)$ and $\mathrm{Pin}(D, D)$ are isomorphic, and in spinor representation the generators are given by

$$T^{MN} \;=\; \frac{1}{2}\Gamma^{MN} \;\equiv\; \frac{1}{4}[\Gamma^M,\Gamma^N]\,, \tag{4.19}$$

which satisfy (4.3).

Next, we define a group homomorphism

$$\rho : \ \mathrm{Pin}(D,D) \rightarrow O(D,D)\,, \tag{4.20}$$

with kernel $\{\mathbf{1}, -\mathbf{1}\}$, that encodes the two-fold covering of $O(D,D)$. It is defined via its action on a vector $V \in \mathbb{R}^{2D}$ according to

$$\rho(S)V \ = \ SVS^{-1}\,. \tag{4.21}$$

The map ρ can be written in a basis using $V = V^M\Gamma_M$ for the original vector and $V' = V'^M\Gamma_M$, with $V'^M = h^M{}_N V^N$, for the rotated vector, where $h^M{}_N$ is an $O(D,D)$ element. With this, the map in (4.21) becomes

$$\rho(S)V \ = \ V' \ = \ SVS^{-1} \quad \rightarrow \quad h^M{}_N V^N \Gamma_M \ = \ S\,V^M \Gamma_M\,S^{-1}\,. \tag{4.22}$$

Relabeling and canceling out the vector components we find

$$S\,\Gamma_M\,S^{-1} \ = \ \Gamma_N\,h^N{}_M\,. \tag{4.23}$$

Here $\rho(S) = h$, and h — with matrix representative $h^M{}_N$ — is the $O(D,D)$ element associated with S. We rewrite the above equation by raising the indices. Using the invariance property $\eta_{MN}(h^{-1})^N{}_K = \eta_{KN}h^N{}_M$, we find

$$S\,\Gamma^M\,S^{-1} \ = \ (h^{-1})^M{}_N\Gamma^N\,. \tag{4.24}$$

Rewritten as $h^M{}_N\,S\,\Gamma^N\,S^{-1} = \Gamma^M$, this is the familiar statement that gamma matrices are invariant under the combined action of $\mathrm{Pin}(D,D)$ on the spinor and vector indices.

Let us now turn to the definition of the subgroup $\mathrm{Spin}(D,D)$ of $\mathrm{Pin}(D,D)$. It is obtained if in (4.18) we have $S \in C(D,D)^{\text{even}}$, which is the Clifford subalgebra spanned by elements with an even number of products of basis vectors. In this case the homomorphism ρ above restricts to a homomorphism

$$\rho: \ \mathrm{Spin}(D,D) \to SO(D,D)\,, \tag{4.25}$$

with kernel $\{\mathbf{1}, -\mathbf{1}\}$. If, in addition to restricting to $C(D,D)^{\mathrm{even}}$, the normalization condition is changed to $SS^\star = \mathbf{1}$, the resulting group is $\mathrm{Spin}^+(D,D)$ and ρ would map to $SO^+(D,D)$.

Let us consider a set of useful elements S of $\mathrm{Pin}(D,D)$. We write the elements using the oscillators ψ_i and ψ^i,[1]

$$\begin{aligned} S_b \ &\equiv e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}\,, \\ S_r \ &\equiv \frac{1}{\sqrt{\det r}}\, e^{\psi^i R_i{}^j \psi_j}\,, \quad (r = (r_i{}^j) = e^{R_i{}^j} \in GL^+(D))\,, \\ S_i \ &\equiv \psi^i + \psi_i\,, \quad (i = 1, \ldots, D)\,, \end{aligned} \tag{4.26}$$

where $GL^+(D)$ is the group of $D \times D$ matrices with strictly positive determinant. It is instructive and straightforward to verify that the first condition in (4.18) holds. Noting that $(e^x)^\star = e^{x^\star}$ we have

$$(S_b)^\star = (S_b)^{-1}\,, \qquad (S_r)^\star = (S_r)^{-1}\,, \qquad S_i^\star = -S_i = -S_i^{-1}\,. \tag{4.27}$$

We note that $S_b \in \mathrm{Spin}^+(D,D)$, $S_r \in \mathrm{Spin}^+(D,D)$, and $S_i \in \mathrm{Pin}(D,D)$, while *even* powers of the S_i are in $\mathrm{Spin}(D,D)$.

Using the definition (4.21) we can calculate the $O(D,D)$ elements associated with these $\mathrm{Spin}(D,D)$ elements. For this we expand (4.23) to find

$$\begin{aligned} S\,\Gamma_i\,S^{-1} \ &= \ \Gamma_k\,h^k{}_i + \Gamma^k\,h_{ki}\,, \\ S\,\Gamma^i\,S^{-1} \ &= \ \Gamma_k\,h^{ki} + \Gamma^k\,h_k{}^i\,, \end{aligned} \tag{4.28}$$

[1] Here we are closely following [61] with a slightly different notation.

and we build the h matrix as follows

$$h^M{}_N = \begin{pmatrix} h_i{}^k & h_{ik} \\ h^{ik} & h^i{}_k \end{pmatrix} . \tag{4.29}$$

Applying the above to (4.26) one finds the $O(D, D)$ matrices associated to the Pin elements:

$$\begin{aligned}
h_b \equiv \rho(S_b) &= \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} , \quad b^T = -b , & (4.30)\\
h_r \equiv \rho(S_r) &= \begin{pmatrix} r & 0 \\ 0 & (r^{-1})^T \end{pmatrix} , \quad r \in GL^+(D) , & (4.31)\\
h_i \equiv \rho(S_i) &= -\begin{pmatrix} 1 - e_i & -e_i \\ -e_i & 1 - e_i \end{pmatrix} , \quad (e_i)_{jk} \equiv \delta_{ij}\delta_{ik} , \quad (i = 1, \ldots, D) . & (4.32)
\end{aligned}$$

The group elements h_b, h_r and even powers of the h_i generate the component $SO^+(D, D)$ connected to the identity.

4.2.2 Conjugation in Pin(D, D)

We turn next to the definition of the charge conjugation matrix. The charge conjugation matrix C can be viewed as an element of Pin(D, D) in general and as an element of Spin(D, D) for even D. It is defined in terms of the oscillators by

$$C \equiv \begin{cases} C_+ \equiv (\psi^1 + \psi_1)(\psi^2 + \psi_2) \cdots (\psi^D + \psi_D) , & \text{if } D \text{ odd} , \\ C_- \equiv (\psi^1 - \psi_1)(\psi^2 - \psi_2) \cdots (\psi^D - \psi_D) , & \text{if } D \text{ even} . \end{cases} \tag{4.33}$$

Noticing that with i not summed $(\psi^i \pm \psi_i)(\psi^i \pm \psi_i) = \pm\{\psi^i , \psi_i\} = \pm 1$, simple calculations show that

$$C_+(C_+)^\star = (-1)^D , \quad C_-(C_-)^\star = 1 . \tag{4.34}$$

It is useful to note that the charge conjugation matrix is proportional to its inverse,

$$C^{-1} \;=\; (-1)^{D(D-1)/2}\, C\,. \tag{4.35}$$

Since C and C^{-1} just differ by a sign, all expressions of the form $C\ldots C^{-1}$ can be rewritten as $C^{-1}\ldots C$. It is straightforward to show that

$$\begin{aligned} C_+\psi_i(C_+)^{-1} &= -(-1)^D\psi^i\,, \qquad & C_+\psi^i(C_+)^{-1} &= -(-1)^D\psi_i\,,\\ C_-\psi_i(C_-)^{-1} &= \;\;(-1)^D\psi^i\,, \qquad & C_-\psi^i(C_-)^{-1} &= \;\;(-1)^D\psi_i\,. \end{aligned} \tag{4.36}$$

It then follows from (4.241) that in *all* dimensions

$$C\,\psi_i\,C^{-1} \;=\; \psi^i\,, \qquad C\,\psi^i\,C^{-1} \;=\; \psi_i\,. \tag{4.37}$$

As $\psi^i = (\psi_i)^\dagger$, these relations can be written in terms of gamma matrices as follows

$$C\,\Gamma^M\,C^{-1} = (\Gamma^M)^\dagger\,, \quad \text{or} \quad C\,\Gamma_M\,C^{-1} = (\Gamma_M)^\dagger\,. \tag{4.38}$$

Introducing the $O(D,D)$ element

$$J^{\bullet}{}_{\bullet} \;=\; J \;\equiv\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.39}$$

we can use (4.23) to write the second equation in (4.38) as

$$C\,\Gamma_M\,C^{-1} = \Gamma_N\,(\rho(C))^N{}_M \;=\; (\Gamma_M)^\dagger = \Gamma_N J^N{}_M\,. \tag{4.40}$$

We thus learn that

$$\rho(C) \;=\; J\,. \tag{4.41}$$

Since C and C^{-1} just differ by a sign, $\rho(C^{-1}) = J$ and equation (4.38) also implies that

$$C^{-1}\,\Gamma^M\,C \;=\; (\Gamma^M)^\dagger\,. \tag{4.42}$$

More generally we define the action of dagger by stating that $\mathbf{1}^\dagger = \mathbf{1}$, and that on vectors V dagger is realized by C conjugation:

$$V^\dagger \equiv CVC^{-1} = J\,V\,. \tag{4.43}$$

On general elements of the Clifford algebra we define dagger using

$$(V_1 \cdot V_2 \cdot \ldots \cdot V_n)^\dagger \equiv V_n^\dagger \cdot \ldots \cdot V_2^\dagger \cdot V_1^\dagger\,, \tag{4.44}$$

so that for general elements $(S_1 \cdot S_2)^\dagger = S_2^\dagger \cdot S_1^\dagger$. A short calculation gives

$$C^\dagger \;=\; C^{-1}\,. \tag{4.45}$$

It is straightforward to verify that $S \in \mathrm{Pin}(D,D)$ implies $S^\dagger \in \mathrm{Pin}(D,D)$. It is then natural to ask how the homomorphism ρ behaves under the dagger conjugation.

To answer this and related questions it is convenient to describe the dagger operation in $C(D,D)$ in terms of C conjugation and the anti-involution τ defined by

$$\tau(V_1 \cdot V_2 \cdot \ldots \cdot V_n) = V_n \cdot \ldots \cdot V_2 \cdot V_1\,, \tag{4.46}$$

which satisfies $\tau(S_1 S_2) = \tau(S_2)\tau(S_1)$. Indeed, it is clear that

$$S^\dagger = C\,\tau(S)\,C^{-1}\,. \tag{4.47}$$

Then taking ρ of this equation gives

$$\rho(S^\dagger) = \rho(S)^T\,. \tag{4.48}$$

For elements S of $\mathrm{Spin}(D,D)$, $\tau(S) = S^\star$, thus (4.47) becomes

$$S^\dagger = C\,S^\star\,C^{-1}\,, \quad S \in \mathrm{Spin}(D,D)\,. \tag{4.49}$$

Using that $S^\star = \pm S^{-1}$ for $S \in \mathrm{Spin}^\pm(D,D)$, this implies

$$\begin{aligned} S^\dagger &= C\,S^{-1}C^{-1} \quad \text{for} \quad S \in \mathrm{Spin}^+(D,D)\,, \\ S^\dagger &= -C\,S^{-1}C^{-1} \quad \text{for} \quad S \in \mathrm{Spin}^-(D,D)\,. \end{aligned} \tag{4.50}$$

4.2.3 Chiral spinors

We close this section with a brief discussion of the chirality conditions to be imposed on the spinors. To this end it is convenient to introduce a 'fermion number operator' N_F, defined by

$$N_F \;=\; \sum_k \psi^k \psi_k\,. \tag{4.51}$$

It acts on a spinor state that is of degree p in the oscillators as follows

$$\begin{aligned} N_F|\chi\rangle_p \;&\equiv\; N_F\Big(\frac{1}{p!}C_{i_1\ldots i_p}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\Big) \\ &=\; \sum_k p\frac{1}{p!}C_{i_1\ldots i_p}\psi^k\delta_k{}^{[i_1}\psi^{i_2}\cdots\psi^{i_p]}|0\rangle \;=\; p|\chi\rangle_p\,, \end{aligned} \tag{4.52}$$

where (4.13) has been used. Thus, acting with $(-1)^{N_F}$ on a general spinor state (4.14), one obtains

$$(-1)^{N_F}\chi \;=\; \sum_{p=0}^{D}(-1)^p\,C_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,. \tag{4.53}$$

We conclude that the eigenstates of $(-1)^{N_F}$ consist of a χ that is a linear combination of only even forms, with eigenvalue $+1$, or of a χ that is a linear combination of only odd forms, with eigenvalue -1. Given an arbitrary spinor χ, one can project onto the two respective chiralities,

$$\chi_\pm \;\equiv\; \frac{1}{2}\big(1\pm(-1)^{N_F}\big)\chi \qquad \Rightarrow \qquad (-1)^{N_F}\chi_\pm \;=\; \pm\chi_\pm\,. \tag{4.54}$$

Then χ_+ has positive chirality, consisting only of even forms, and χ_- has negative chirality, consisting only of odd forms. The operator $(-1)^{N_F}$ is the analogue of the γ^5 matrix in four dimensions.

Finally, we note that the chirality is preserved under an arbitrary $\text{Spin}(D, D)$ transformation. In fact, since the group elements of $\text{Spin}(D, D)$ contain only an even number of fermionic oscillators, they map even forms into even forms and odd forms into odd forms. In contrast, a general $\text{Pin}(D, D)$ transformation can act with an odd number of oscillators and thereby map spinors of positive chirality to spinors of negative chirality and vice versa. Thus, when fixing the chirality, as for the action to be introduced below, we break the symmetry from $\text{Pin}(D, D)$ to $\text{Spin}(D, D)$.

4.3 Spin representative of the generalized metric

In this section we discuss the spin representative $S_{\mathcal{H}}$ of the generalized metric $\mathcal{H}_{MN}$. We determine its transformation behavior under gauge symmetries and T-duality. More fundamentally, we will adopt the point of view that $S_{\mathcal{H}}$ is just a particular parametrization of the fundamental field $\mathbb{S}$.

4.3.1 The generalized metric in $\text{Spin}(D, D)$

We take the fundamental field to be $\mathbb{S}$, satisfying

$$\mathbb{S} \; = \; \mathbb{S}^\dagger \,, \qquad \mathbb{S} \in \text{Spin}^-(D, D) \,. \tag{4.55}$$

The generalized metric $\mathcal{H}_{MN}$ will then be *defined* as

$$\mathcal{H} \; \equiv \; \rho(\mathbb{S}) \qquad \Rightarrow \qquad \mathcal{H}^T \; = \; \rho(\mathbb{S}^\dagger) \; = \; \mathcal{H} \,, \quad \mathcal{H} \in SO^-(D, D) \,. \tag{4.56}$$

Moreover, we constrain $\mathcal{H}$ and thereby $\mathbb{S}$ by requiring that the upper-left $D \times D$ block matrix encoding g^{-1} has Lorentzian signature. An immediate consequence of (4.55) follows with (4.50)

$$\mathbb{S} \; = \; \mathbb{S}^\dagger \; = \; -C \, \mathbb{S}^{-1} \, C^{-1} \,. \tag{4.57}$$

Equivalently, recalling that $C = \pm C^{-1}$,

$$\mathbb{S} C \mathbb{S} \;=\; -C\,. \tag{4.58}$$

It is also possible to adopt the opposite point of view, i.e., to take the group element $\mathcal{H}$ as given and then determine a corresponding spin group representative $S_{\mathcal{H}}$ as a derived object. However, as we will discuss in more detail below, this cannot be done in a consistent way globally over the space of $\mathcal{H}$. In the following we first determine a spin representative $S_{\mathcal{H}}$ locally from $\mathcal{H}$, but we stress that this should be viewed as just a particular parameterization of $\mathbb{S}$ — in the same sense that the explicit form of $\mathcal{H}_{MN}$ in terms of g and b is just a particular parametrization of $\mathcal{H}$.

We start by writing the $O(D,D)$ matrix $\mathcal{H}_{MN}$ as a product of simple group elements,[2]

$$\mathcal{H} \;=\; \begin{pmatrix} g^{-1} & -g^{-1}b \\ bg^{-1} & g - bg^{-1}b \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \;\equiv\; h_b^T\, h_{g^{-1}}\, h_b\,. \tag{4.59}$$

The matrices defined in the last equation are analogous to the matrices defined in (4.30) and (4.31). More precisely, this is true for h_b while for h_g (or $h_{g^{-1}} \;=\; h_g^{-1}$) eq. (4.31) is only valid if g has euclidean signature, because then $g \in GL^+(D)$. Here, however, we assume that g has Lorentzian signature $(- + \cdots +)$. Accordingly, $\mathcal{H}$ is indeed an element of $SO^-(D,D)$.

In order to find the corresponding spinor representative for h_g and thereby for $\mathcal{H}$, it is convenient to introduce vielbeins in the usual way,

$$g_{ij} \;=\; e_i{}^{\alpha}\, e_j{}^{\beta}\, k_{\alpha\beta}\,, \qquad k_{\alpha\beta} \;=\; \text{diag}(-1, 1, \ldots, 1)\,, \tag{4.60}$$

where $\alpha, \beta, \ldots = 1, \ldots, D$ are flat Lorentz indices with invariant metric $k_{\alpha\beta}$. In matrix notation, we also write

$$g \;=\; e\, k\, e^T\,. \tag{4.61}$$

[2]We note that our conventions differ slightly from those in [8] in that what we denote by $\mathcal{H}$ has been denoted $\mathcal{H}^{-1}$ there. All other conventions, however, are the same.

We can choose e to have positive determinant, and thus its spin representative can be chosen to be S_e as defined in (4.26). The spin representative of $\mathrm{diag}(k,k)$ can be taken to be

$$S_k \;=\; \psi^1\psi_1 - \psi_1\psi^1\,, \tag{4.62}$$

where the label one denotes the timelike direction. We note that

$$S_k = S_k^\dagger = S_k^{-1} = -S_k^\star\,. \tag{4.63}$$

Since $S_k S_k^\star = -1$, we confirm that $S_k \in \mathrm{Spin}^-(D,D)$.

Thus, we can choose the spinor representative of g to be

$$S_g \;\equiv\; S_e\, S_k\, S_e^\dagger \;=\; \frac{1}{\det(e)}\, e^{\psi^i E_i{}^j\psi_j}(\psi^1\psi_1 - \psi_1\psi^1)e^{\psi^i (E^T)_i{}^j\psi_j}\,, \tag{4.64}$$

where $e_i{}^\alpha = \exp(E)_i{}^\alpha$, and we used $(E^T)_i{}^j = E_j{}^i$. From its definition it follows that

$$S_g^\dagger \;=\; S_g\,. \tag{4.65}$$

Similarly,

$$S_g^{-1} \;\equiv\; (S_e^{-1})^\dagger\, S_k\, S_e^{-1} \;=\; \det e\, e^{-\psi^i (E^T)_i{}^j\psi_j}(\psi^1\psi_1 - \psi_1\psi^1)e^{-\psi^i E_i{}^j\psi_j}\,. \tag{4.66}$$

We note that S_g is an element of $\mathrm{Spin}^-(D,D)$ because it is the product of $S_k \in \mathrm{Spin}^-(D,D)$ times elements of $\mathrm{Spin}^+(D,D)$. From this and (4.49) we also infer that

$$S_g^\dagger \;=\; S_g \;=\; C\,S_g^\star\,C^{-1} \;=\; -C\,S_g^{-1}\,C^{-1}\,. \tag{4.67}$$

We can finally define the element $S_{\mathcal{H}}$ of $\mathrm{Spin}(D,D)$ as follows

$$S_{\mathcal{H}} \;\equiv\; S_b^\dagger\, S_g^{-1}\, S_b \;=\; e^{\frac{1}{2}b_{ij}\psi_i\psi_j}\, S_g^{-1}\, e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}\,. \tag{4.68}$$

Using (4.65) we infer that

$$S_{\mathcal{H}}^{\dagger} \;=\; S_{\mathcal{H}}\,. \tag{4.69}$$

By construction, the image of $S_{\mathcal{H}}$ under the group homomorphism ρ is precisely $\mathcal{H}$:

$$\rho(S_{\mathcal{H}}) = \rho(S_b)^T\rho(S_g^{-1})\rho(S_b) \;=\; h_b^T h_g^{-1} h_b = \mathcal{H}\,. \tag{4.70}$$

Since $S_b, S_b^{\dagger} \in \mathrm{Spin}^+(D,D)$ and $S_g^{-1} \in \mathrm{Spin}^-(D,D)$, we have $S_{\mathcal{H}} \in \mathrm{Spin}^-(D,D)$. As a result, $S_{\mathcal{H}}$ satisfies the identities (4.57) and (4.58) and therefore gives a consistent parametrization of $\mathbb{S}$.

The flat Minkowski background $g = k$ with zero b-field gives a generalized metric that we denote as $\mathcal{H}_0 \equiv \mathrm{diag}(k, k)$. Since $S_g = S_k$ and $S_b = \mathbf{1}$, we have

$$S_{\mathcal{H}_0} = S_k^{-1} = S_k = \psi^1\psi_1 - \psi_1\psi^1\,. \tag{4.71}$$

4.3.2 Duality transformations

We discuss now the transformation behavior of $\mathbb{S}$ under some arbitrary element $S \in \mathrm{Pin}(D,D)$. Since we view $\mathbb{S}$ as an elementary field we can postulate such a transformation. The transformation of $\mathbb{S}$, however, must be consistent with the transformation of the associated $\mathcal{H} = \rho(\mathbb{S})$. Writing also $\mathcal{H}' = \rho(\mathbb{S}')$, we want to postulate a transformation for which

$$\mathbb{S} \;\xrightarrow{S}\; \mathbb{S}' \;\text{ implies }\; \mathcal{H} \;\xrightarrow{\rho(S)}\; \mathcal{H}'\,. \tag{4.72}$$

In words, the $O(D,D)$ transformation $\rho(S)$ associated with $S \in \mathrm{Pin}(D,D)$ relates the corresponding generalized metrics. The generalized metric appears explicitly in the NS-NS action.

Recall that under an $O(D,D)$ transformation h the generalized metric transforms as

$$\mathcal{H}'_{MN} \;=\; \mathcal{H}_{PQ}(h^{-1})^P{}_M(h^{-1})^Q{}_N\,. \tag{4.73}$$

In matrix notation, we will write $\mathcal{H}$ transformations as follows:

$$\mathcal{H}' = h \circ \mathcal{H} \;\equiv\; (h^{-1})^T \mathcal{H}\, h^{-1}\,. \tag{4.74}$$

For an element $S \in \mathrm{Pin}(D,D)$ we postulate the following $\mathbb{S}$ transformation:

$$\mathbb{S}'(X') \;=\; (S^{-1})^\dagger\, \mathbb{S}(X)\, S^{-1}\,. \tag{4.75}$$

Here $X' = hX$, where $h = \rho(S)$. The compatibility with (4.74) is verified by taking ρ on both sides. Suppressing the coordinate arguments, we indeed find

$$\begin{aligned} \mathcal{H}' \;&=\; \rho(\mathbb{S}') \;=\; \rho\big((S^{-1})^\dagger\, \mathbb{S}\, S^{-1}\big) \;=\; \rho\big((S^{-1})^\dagger\big)\, \rho(\mathbb{S})\, \rho(S^{-1}) \\ &=\; (\rho(S)^{-1})^T \mathcal{H}\, \rho(S)^{-1} \;=\; (h^{-1})^T \mathcal{H}\, h^{-1} \;=\; h \circ \mathcal{H}\,. \end{aligned} \tag{4.76}$$

We infer that $\mathcal{H}'$ satisfies (4.74).

Independently of the postulated transformation rule (4.75), we can ask how $S_{\mathcal{H}}$, defined in (4.68) in terms of $\mathcal{H}$, transforms under a duality transformation generated by an element $S \in \mathrm{Pin}(D,D)$. This transformation is simply given by

$$S: \;\; S_{\mathcal{H}} \;\rightarrow\; S_{\mathcal{H}'}\,, \;\;\text{where}\;\; \mathcal{H}' = \rho(S) \circ \mathcal{H}\,. \tag{4.77}$$

It is of interest to compare

$$(S^{-1})^\dagger\, S_{\mathcal{H}}\, S^{-1} \;\;\longleftrightarrow\;\; S_{\mathcal{H}'}\,. \tag{4.78}$$

Under ρ they both map to $\mathcal{H}'$, thus the two can be equal or can differ by a sign. Perhaps surprisingly, there is a sign factor that depends nontrivially on $\rho(S)$ and on $\mathcal{H}$. We will write

$$(S^{-1})^\dagger\, S_{\mathcal{H}}\, S^{-1} \;=\; \sigma_{\rho(S)}(\mathcal{H})\;\, S_{\rho(S)\circ\mathcal{H}}\,. \tag{4.79}$$

In the remainder of this section we determine this sign factor.

There is a large set of $O(D,D)$ transformations h for which the sign in (4.79) is plus.

$$(S_h^{-1})^\dagger S_{\mathcal{H}} S_h^{-1} \;=\; +S_{h\circ\mathcal{H}}\,, \qquad \text{when} \quad h \in GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}\,. \tag{4.80}$$

The group $GL(D) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$ is that generated by successive applications of $GL(D)$ transformations and b-shifts, transformations h_b of the form indicated in (4.30), which define the abelian subgroup $\mathbb{R}^{\frac{1}{2}D(D-1)}$.

It is the T-dualities that produce sign changes. We therefore consider the sign factor in

$$(S_i^{-1})^\dagger\, S_{\mathcal{H}}\, S_i^{-1} \;=\; \sigma_i(\mathcal{H})\, S_{h_i\circ\mathcal{H}}\,. \tag{4.81}$$

As we can see, the sign factor depends on the particular $\mathcal{H}$ appearing on the left-hand side above. Our final result is:

$$\sigma_i(\mathcal{H}) \;=\; \mathrm{sgn}(g_{ii})\,. \tag{4.82}$$

It follows from this equation that for a general background $\mathcal{H}$ whose metric has Lorentzian signature the duality transformation J about all of the spacetime coordinates gives the sign factor:

$$\sigma_J(\mathcal{H}) = -1\,. \tag{4.83}$$

There seems to be some tension between the defined duality transformation of $\mathbb{S}$ in (4.75), which has no signs, and the duality transformation (4.79) of its particular parametrization $S_{\mathcal{H}}$, which shows some signs. The sign-free transformation of $\mathbb{S}$ implies that the double field theory action is fully invariant under all duality transformations, including those, like timelike T-dualities, that give a sign in (4.79). Once we choose a parametrization by setting $\mathbb{S} = S_{\mathcal{H}}$, the sign factors in (4.79) have two consequences. First, it follows that the Spin(D, D) invariance of the action cannot be fully realized through transformations of the conventional fields g and b. More precisely, it can only be realized for $SO(D, D)$ transformations that do not involve a timelike T-duality. This means that if we take timelike T-dualities seriously, we inevitably have to view $\mathbb{S}$ as the fundamental field. Second, when comparing the double field theory evaluated in one T-duality frame (as $\tilde{\partial}^i = 0$) to the same theory evaluated in another T-duality frame obtained by a timelike T-duality transformation (as $\partial_i = 0$), the conventional effective RR action changes sign. This sign change corresponds precisely to the transition from type II to type II* theories expected for timelike T-dualities. Correspondingly, the freedom in the choice of parametrization for $\mathbb{S}$, namely $\pm S_{\mathcal{H}}$, has no physical significance in that it merely fixes for which coordinates (x or $\tilde{x}$) we obtain the type II and for which we obtain the type II* theory. Similarly, the actual sign of the RR term in the double field theory action (4.4) has no physical significance. Therefore, we find a consistent picture, though certain invariances of the action cannot be fully realized on the conventional gravitational fields.

4.3.3 Gauge transformations

In this section we determine the gauge transformation of the spinor representative $\mathbb{S}$ in such a way that it us consistent with the known gauge variation of the generalized metric $\mathcal{H}_{MN}$. This variation can be rewritten as:

$$\delta_\xi \mathcal{H}^M{}_P \;=\; \xi^L\partial_L\mathcal{H}^M{}_P + \left(\partial^M\xi_K - \partial_K\xi^M\right)\mathcal{H}^K{}_P + \left(\partial_P\xi^K - \partial^K\xi_P\right)\mathcal{H}^M{}_K\,, \qquad (4.84)$$

where we used that the metric η_{MN} that lowers indices is gauge invariant. We have positioned the indices of the generalized metric as in $\mathcal{H}^\bullet{}_\bullet$ to emphasize its role as an $O(D,D)$ group element. We also recall that $\mathcal{H}^M{}_K\mathcal{H}^K{}_N = \delta^M{}_N$. The matrix $\mathcal{H}$ used so far represents $\mathcal{H}_{\bullet\bullet}$.

It turns out to be convenient to write the gauge variation in terms of the spin variable $\mathcal{K}$ defined by

$$\mathcal{K} \;\equiv\; C^{-1}\mathbb{S}\,. \qquad (4.85)$$

This combination will be used to prove the gauge invariance of the action in section 4.4.2. While $\mathbb{S}$ is a spin representative of $\mathcal{H}_{\bullet\bullet}$, we now check that $\mathcal{K}$ is the spin representative of $\mathcal{H}^{\bullet}{}_{\bullet}$. Indeed recalling that $\rho(C^{-1}) = J$ with J defined in (4.39), we have

$$\rho(\mathcal{K}) = \rho(C^{-1})\rho(\mathbb{S}) = J\mathcal{H}_{\bullet\bullet} = \mathcal{H}^{\bullet}{}_{\bullet}\,, \tag{4.86}$$

since J is identical to the matrix η^{-1} that raises indices. We write this conclusion as

$$S_{\mathcal{H}^{\bullet}{}_{\bullet}} = \pm\,\mathcal{K}\,. \tag{4.87}$$

The gauge transformation of $\mathcal{K}$ compatible with that of $\mathcal{H}^{\bullet}{}_{\bullet}$ takes the form

$$\delta_{\xi}\mathcal{K} \;=\; \xi^{M}\partial_{M}\mathcal{K} + \frac{1}{2}\big[\Gamma^{PQ},\,\mathcal{K}\big]\partial_{P}\xi_{Q}\,, \tag{4.88}$$

where $\Gamma^{PQ} \equiv \frac{1}{2}[\Gamma^{P}, \Gamma^{Q}]$. The proof of this form is based on a postulation that the gauge transformation $\mathcal{K}$ is consistent with that of the generalized metric. Detailed steps are provided in [13] for interested readers.

4.4 Action, duality relations, and gauge symmetries

In this section we introduce the $O(D, D)$ covariant double field theory formulation of the RR action and the duality relations. We prove T-duality invariance and gauge invariance, and we determine the $O(D, D)$ covariant form of the field equations.

4.4.1 Action, duality relations, and $O(D, D)$ invariance

The dynamical field we will use to write an action is a spinor of $\text{Pin}(D, D)$ written as in (4.14):

$$\chi \;\equiv\; |\chi\rangle \;=\; \sum_{p=0}^{D}\frac{1}{p!}C_{i_1\ldots i_p}\,\psi^{i_1}\ldots\psi^{i_p}|0\rangle\,. \tag{4.89}$$

Here the component forms $C_{i_1\ldots i_p}(x, \tilde{x})$ are the dynamical fields and, as is usual in double field theory, they are real functions of the full collection of $2D$ coordinates x

and $\tilde{x}$. We will assume χ to have a definite chirality. Thus, as discussed in sec. 4.2.3, it consists either of only odd forms or even forms. The bra associated with this ket is called $\chi^\dagger$ and is defined by

$$\chi^\dagger \;\equiv\; \langle\chi| \;=\; \sum_{p=0}^{D} \frac{1}{p!} C_{i_1\ldots i_p} \langle 0|\psi_{i_p}\ldots\psi_{i_1} \,. \tag{4.90}$$

We conventionally define the conjugate spinor using the C matrix defined in section 4.2.2:

$$\bar{\chi} \;\equiv\; \chi^\dagger C \,. \tag{4.91}$$

We will make use of a Dirac operator on spinors that behaves just as an exterior derivative on the associated forms:

$$\partial\!\!\!/ \equiv \frac{1}{\sqrt{2}}\Gamma^M\partial_M = \; \psi^i\partial_i + \psi_i\tilde{\partial}^i \,, \tag{4.92}$$

where we used (4.11). The $\partial\!\!\!/$ operator behaves like the exterior derivative d in that its repeated action gives zero:

$$\partial\!\!\!/^2 \;=\; \frac{1}{2}\Gamma^M\Gamma^N\partial_M\partial_N \;=\; \frac{1}{4}\{\Gamma^M,\Gamma^N\}\partial_M\partial_N \;=\; \frac{1}{2}\eta^{MN}\partial_M\partial_N \;=\; 0\,, \tag{4.93}$$

by the strong constraint. The $\partial\!\!\!/$ operator will be used to define field strengths in a Pin(D,D) covariant way. It is clear that acting on forms that do not depend on $\tilde{x}$, the only term that survives, $\psi^i\partial_i$, both differentiates with respect to x and increases the degree of the form by one. More details will be given in section 4.5.

We turn now to a discussion of the double field theory action. We claim that the RR action is $S = \int dx d\tilde{x}\mathcal{L}$, where the Lagrangian density $\mathcal{L}$ is simply given bys

$$\mathcal{L} \;=\; \frac{1}{4}(\partial\!\!\!/\chi)^\dagger\, \mathbb{S}\, \partial\!\!\!/\chi\,. \tag{4.94}$$

The above Lagrangian is manifestly real: $\mathcal{L}^\dagger = \mathcal{L}$ because the spinor χ is Grassmann even and $\mathbb{S}$ is Hermitian. The Lagrangian can be written using conjugate spinors and the kinetic operator $\mathcal{K} = C^{-1}\mathbb{S}$. The above Lagrangian is equivalent to

$$\mathcal{L} \;=\; \frac{1}{8}\partial_M\bar{\chi}\Gamma^M\mathcal{K}\Gamma^N\partial_N\chi\,. \tag{4.95}$$

The properties of bar conjugation allow us to recognize that

$$\overline{\partial\!\!\!/\chi} = \frac{1}{\sqrt{2}}(\Gamma^M\partial_M\chi)^\dagger C = \frac{1}{\sqrt{2}}\partial_M\chi^\dagger(\Gamma^M)^\dagger C = \frac{1}{\sqrt{2}}\partial_M\bar{\chi}C^{-1}(\Gamma^M)^\dagger C = \frac{1}{\sqrt{2}}\partial_M\bar{\chi}\Gamma^M\,, \tag{4.96}$$

and therefore we can write the action more compactly as

$$\mathcal{L} \;=\; \frac{1}{4}\overline{\partial\!\!\!/\chi}\,\mathcal{K}\,\partial\!\!\!/\chi\,. \tag{4.97}$$

Our first task now is to establish the global Spin(D,D) invariance of this Lagrangian (the $dxd\tilde{x}$ measure is $O(D,D)$ invariant). This is the maximal invariance group that is consistent with the fixed chirality of χ. Under the action of a Spin(D,D) element S, whose associated $O(D,D)$ element is $h=\rho(S)$, the spinor field χ transforms as follows:

$$\chi \to \chi' \;=\; S\chi\,. \tag{4.98}$$

Implicit in here is that the coordinates the fields depend on are also transformed: primed fields depend on primed coordinates $X'^M \;=\; h^M{}_N X^N$. Note also that the daggered state transforms as

$$\chi^\dagger \;\to\; \chi^\dagger S^\dagger\,. \tag{4.99}$$

Then the gauge transformation of $\partial\!\!\!/\chi$ can be readily computed as

$$\partial\!\!\!/\chi \;\to\; \frac{1}{\sqrt{2}}S\,h^M{}_P\Gamma^P(h^{-1})^N{}_M\partial_N\chi = \frac{1}{\sqrt{2}}S\,\Gamma^N\partial_N\chi = S\,\partial\!\!\!/\chi\,, \tag{4.100}$$

We have thus leaned that $\partial\!\!\!/\chi$ transforms just like χ. In other words, the Dirac operator $\partial\!\!\!/$ is Spin(D,D) invariant. Recalling the transformation of $\mathbb{S}$ in (4.75) :

$\mathbb{S} \to (S^{-1})^\dagger \mathbb{S}\, S^{-1}$, the invariance of the Lagrangian (4.94) is essentially manifest:

$$\mathcal{L} \;=\; \frac{1}{4}(\partial\!\!\!/\chi)^\dagger\,\mathbb{S}\,\partial\!\!\!/\chi \;\;\to\;\; \frac{1}{4}(\partial\!\!\!/\chi)^\dagger S^\dagger\,(S^{-1})^\dagger\,\mathbb{S}\,S^{-1}\,S\partial\!\!\!/\chi \;=\; \mathcal{L}\,. \tag{4.101}$$

The action must be supplemented by duality constraints among the field strengths. We can write $\mathrm{Spin}^+(D,D)$ covariant versions of the duality relations that relate all RR field strengths:[3]

$$\partial\!\!\!/\chi \;=\; -\mathcal{K}\,\partial\!\!\!/\chi\,. \tag{4.102}$$

According to (4.100), the left-hand side transforms covariantly with $S \in \mathrm{Spin}(D,D)$. The right-hand side transforms in the same way, since

$$-\mathcal{K}\,\partial\!\!\!/\chi \;\;\to\;\; -C^{-1}(S^{-1})^\dagger\,\mathbb{S}\,S^{-1}S\partial\!\!\!/\chi \;=\; -S\,C^{-1}\,\mathbb{S}\,\partial\!\!\!/\chi \;=\; -S\,\mathcal{K}\,\partial\!\!\!/\chi\,, \tag{4.103}$$

where we used that (4.50) implies $C^{-1}(S^{-1})^\dagger = SC^{-1}$ for $S \in \mathrm{Spin}^+(D,D)$. Thus, the duality relations are actually only invariant under $\mathrm{Spin}^+(D,D)$. This is to be expected since already for conventional duality relations the presence of an epsilon tensor breaks the symmetry to the group $GL^+(D)$ of parity-preserving transformations.

The relations (4.102) require a consistency condition. Acting on both sides of (4.102) with $\mathcal{K}$, we see that consistency requires $\mathcal{K}^2 = 1$, which in turn implies

$$\mathcal{K}^2 \;=\; C^{-1}\,\mathbb{S}\,C^{-1}\,\mathbb{S} \;=\; C(\mathbb{S}\,C\,\mathbb{S}) \;=\; C(-C) \;=\; -(-1)^{\frac{1}{2}D(D-1)} \;=\; 1\,, \tag{4.104}$$

where we used (4.58) and (4.35). Thus, the duality relations are self-consistent in dimensions for which $\frac{1}{2}D(D-1)$ is odd. For $D \leq 10$, these are

$$D \;=\; \{\,10\,,7\,,6\,,3\,,2\,\}\,. \tag{4.105}$$

We note that the even dimensions above are precisely those for which conventional self-duality relations can be imposed consistently. Indeed, the middle degree forms

[3]For the special case of type IIA, a similar $O(D,D)$-covariant form of the duality relations has also been proposed in the second reference of [59].

corresponding to the self-dual field strengths are then odd, and for them $\star^2 = 1$ in Lorentzian signature. As we will show in sec. 4.5.1 the component form of (4.102) contains one self-duality relation in even dimensions, so this result is to be expected. In the following we will focus on $D = 10$, but we note that $D = 2, 6$ can be seen as type II toy models. The possible significance of theories with odd D will not be discussed here.

We close by giving the equations of motion of χ, which are readily derived from (4.95),

$$\not{\partial}(\mathcal{K}\not{\partial}\chi) \;=\; 0\,. \tag{4.106}$$

As it should be, the equation of motion is the integrability condition for the duality relations: acting with a $\not{\partial}$ on both sides of (4.102), and using $\not{\partial}^2 = 0$, we recover the field equation.

4.4.2 Gauge invariance

In this subsection we give the gauge transformation of the RR fields. The p-form gauge transformations are manifestly invariances of the Lagrangian and of the duality constraints. For the gauge transformations parameterized by ξ^M the transformation of χ is nontrivial and so are the checks of gauge invariance of the Lagrangian and the duality constraints.

Gauge transformations

We start by introducing the double field theory version of the abelian gauge symmetries of the p-form gauge fields. These are parameterized by a spacetime dependent spinor λ:

$$\delta_\lambda\chi = \not{\partial}\lambda\,. \tag{4.107}$$

Since λ encodes a set of forms and $\not{\partial}$ acts as an exterior derivative, the above transformations are the familiar ones. It follows that

$$\delta_\lambda\not{\partial}\chi = \not{\partial}\not{\partial}\lambda = 0\,, \tag{4.108}$$

and this implies the gauge invariance of the Lagrangian density (4.94) and of the duality constraint (4.102).

For the gauge parameter ξ^M that encodes the diffeomorphism and Kalb-Ramond gauge symmetries, we postulate the gauge transformation

$$\begin{aligned} \delta_\xi \chi \;&=\; \widehat{\mathcal{L}}_\xi \chi \;\equiv\; \xi^M \partial_M \chi \;+\; \frac{1}{\sqrt{2}} \partial\!\!\!/ \xi^M \Gamma_M \chi \\ &=\; \xi^M \partial_M \chi \;+\; \frac{1}{2} \partial_N \xi_M \Gamma^N \Gamma^M \chi \,. \end{aligned} \tag{4.109}$$

In the second form it is simple to verify that a gauge parameter of the form $\xi_M = \partial_M \Theta$ is trivial in that it generates no gauge transformations:

$$\delta_{\partial\Theta} \chi \;=\; \partial^M \Theta \, \partial_M \chi \;+\; \frac{1}{2} \partial_N \partial_M \Theta \, \Gamma^N \Gamma^M \chi \;=\; \partial_N \partial_M \Theta \, \eta^{MN} \chi \;=\; 0 \,. \tag{4.110}$$

A short calculation gives the gauge transformation of the conjugate spinor $\bar{\chi}$:

$$\delta_\xi \bar{\chi} \;=\; \xi^M \partial_M \bar{\chi} \;+\; \frac{1}{2} \partial_N \xi_M \, \bar{\chi} \Gamma^M \Gamma^N \,. \tag{4.111}$$

Let us now turn to the gauge algebra. We have shown that the gauge transformations parametrized by λ and ξ^M close as follows

$$\left[\delta_\lambda \,, \delta_\xi\right] = \delta_{\widehat{\mathcal{L}}_\xi \lambda} \,, \tag{4.112}$$

where the right-hand side is the double field theory version of p-form gauge transformation with parameter $\widehat{\mathcal{L}}_\xi \lambda$. We have also verified that, as expected, $[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{[\xi_1,\xi_2]_{\rm C}}$, where $[\,\cdot\,,\,\cdot]_{\rm C}$ is the C-bracket discussed in [8].

Gauge invariance of the action and the duality constraints

The action is manifestly invariant under p-form gauge transformations. Here we check the invariance under δ_ξ. We use the Lagrangian in (4.95):

$$\mathcal{L} \;=\; \frac{1}{4} \overline{\partial\!\!\!/ \chi} \, \mathcal{K} \, \partial\!\!\!/ \chi \,. \tag{4.113}$$

As usual, when we vary the Lagrangian, which has the index structure of a scalar, we obtain a transport term and a 'non-covariant' term

$$\delta_\xi \mathcal{L} \;=\; \xi^M \partial_M \mathcal{L} + \Delta_\xi \mathcal{L}\,. \tag{4.114}$$

Since Δ_ξ acts as a derivation and commutes with bar-conjugation,

$$\Delta_\xi \mathcal{L} = \frac{1}{4}\Big(\overline{(\Delta_\xi \partial\!\!\!/ \chi)}\;\mathcal{K}\,\partial\!\!\!/\chi \;+\; \overline{\partial\!\!\!/\chi}\,(\Delta_\xi\mathcal{K})\,\partial\!\!\!/\chi \;+\; \overline{\partial\!\!\!/\chi}\,\mathcal{K}\,\Delta_\xi\partial\!\!\!/\chi\Big)\,. \tag{4.115}$$

For the action to be gauge invariant, $\Delta_\xi\mathcal{L}$ must be such that $\delta_\xi\mathcal{L}$ in (4.114) is a total derivative. Indeed, we find

$$\Delta_\xi\mathcal{L} = \frac{1}{4}\,\partial_M\xi^M\;\overline{\partial\!\!\!/\chi}\,\mathcal{K}\,\partial\!\!\!/\chi \;=\; \partial_M\xi^M\,\mathcal{L}\,. \tag{4.116}$$

Back in (4.114) we get $\delta_\xi\mathcal{L} = \xi^M\partial_M\mathcal{L} + (\partial_M\xi^M)\mathcal{L} = \partial_M(\xi^M\mathcal{L})$, which confirms the gauge invariance of the action.

Finally, we have to prove gauge covariance of the duality constraints $\partial\!\!\!/\chi = -\mathcal{K}\partial\!\!\!/\chi$. We now take the gauge variation δ_ξ of both sides of the duality constraint. The transport terms on both sides are identical, using the duality constraint. So only the non-covariant terms matter, and we can evaluate Δ_ξ on both sides of the constraint, finding

$$\Delta_\xi\partial\!\!\!/\chi \;=\; -(\Delta_\xi\mathcal{K})\partial\!\!\!/\chi \;-\; \mathcal{K}\Delta_\xi\partial\!\!\!/\chi\,. \tag{4.117}$$

Our task is to verify that this holds, using the duality constraint. Bringing all terms to one side we must check that

$$\Delta_\xi\partial\!\!\!/\chi \;+\; (\Delta_\xi\mathcal{K})\partial\!\!\!/\chi \;+\; \mathcal{K}\Delta_\xi\partial\!\!\!/\chi \;=\; 0\,. \tag{4.118}$$

Using our earlier results we find that the left-hand side is equal to

$$\frac{1}{2}\,\partial_P\xi_Q\,\Big(\Gamma^P\Gamma^Q + [\Gamma^{PQ},\mathcal{K}\,] + \;\mathcal{K}\,\Gamma^P\Gamma^Q\Big)\partial\!\!\!/\chi\,. \tag{4.119}$$

Expanding the commutator and using the duality constraint we find that the above becomes

$$\frac{1}{2}\partial_P\xi_Q\Big((\Gamma^P\Gamma^Q-\Gamma^{PQ})+\ \mathcal{K}(\Gamma^P\Gamma^Q-\Gamma^{PQ})\Big)\not{\partial}\chi\ =\ \frac{1}{2}\partial_P\xi_Q\,\eta^{PQ}(1+\ \mathcal{K})\not{\partial}\chi\ =\ 0\,. \tag{4.120}$$

This concludes our proof.

4.4.3 General variation of $\mathbb{S}$ and gravitational equations of motion

In this section we determine the general variation of the action under a variation of $\mathbb{S}$ in order to determine the contribution of the new action to the field equations. This is non-trivial since $\mathbb{S}$ is a *constrained* field in that it takes values in Spin(D, D). The corresponding problem for the constrained variable given by the generalized metric $\mathcal{H}$ has been discussed in [8], and the method employed there can be elevated to $\mathbb{S}$, as we discuss next.

In [8], sect. 4, it was shown that a general variation of the constrained variable $\mathcal{H}$ can be parametrized in terms of a symmetric but otherwise unconstrained matrix $\mathcal{M}^{MN}$ as follows

$$\begin{aligned}\delta\mathcal{H}^{MN} &= \frac{1}{4}\left[(\delta^M{}_P+\mathcal{H}^M{}_P)(\delta^N{}_Q-\mathcal{H}^N{}_Q)+(\delta^M{}_P-\mathcal{H}^M{}_P)(\delta^N{}_Q+\mathcal{H}^N{}_Q)\right]\mathcal{M}^{PQ}\\ &= \frac{1}{2}\Big[\ \mathcal{M}^{MN}-\mathcal{H}^M{}_P\,\mathcal{M}^{PQ}\,\mathcal{H}^N{}_Q\ \Big]\,.\end{aligned} \tag{4.121}$$

We now form the Lie-algebra element

$$(\delta\mathcal{H}^M{}_P)\mathcal{H}^P{}_N\ =\ \frac{1}{2}\mathcal{M}_{PR}\mathcal{H}^R{}_Q\big(T^{PQ}\big)^M{}_N\,, \tag{4.122}$$

where we made repeated use of the symmetry properties of $\mathcal{H}$ and $\mathcal{M}$ and used (4.5).

In the spin representation this equation yields

$$(\delta\mathcal{K})\,\mathcal{K}^{-1} \;=\; \frac{1}{4}\mathcal{M}_{PR}\mathcal{H}^{R}{}_{Q}\Gamma^{PQ} \;=\; \frac{1}{4}\mathcal{M}_{MN}\mathcal{H}^{M}{}_{P}\,\Gamma^{NP}\,, \tag{4.123}$$

after some index relabeling. Our final result for the variation is therefore

$$\delta\mathcal{K} \;=\; \frac{1}{4}\mathcal{M}_{MN}\mathcal{H}^{M}{}_{P}\,\Gamma^{NP}\,\mathcal{K}\,. \tag{4.124}$$

This, with $\mathcal{H}^{\bullet}{}_{\bullet} = \rho(\mathcal{K})$, is the general variation of $\mathcal{K}$ consistent with its group property $\mathcal{K} \in \mathrm{Spin}(D,D)$. It is consistent with the variation of generalized metric, and thus the variation of the NS-NS action is unmodified as compared to the discussion in [8].

Next, we apply (4.124) in order to compute the variation of the RR action

$$\delta\mathcal{L} \;=\; \frac{1}{4}\,\overline{\partial\!\!\!/\chi}\,\delta\mathcal{K}\,\partial\!\!\!/\chi \;=\; \frac{1}{16}\mathcal{M}_{MN}\,\mathcal{H}^{M}{}_{P}\,\overline{\partial\!\!\!/\chi}\,\Gamma^{NP}\mathcal{K}\,\partial\!\!\!/\chi\,. \tag{4.125}$$

Since $\mathcal{M}$ is an arbitrary symmetric matrix, we read off that the contribution to the field equations is given by the symmetric 'stress-tensor'

$$\mathcal{E}^{MN} \;=\; \frac{1}{16}\mathcal{H}^{(M}{}_{P}\,\overline{\partial\!\!\!/\chi}\,\Gamma^{N)P}\,\mathcal{K}\,\partial\!\!\!/\chi\,. \tag{4.126}$$

It is possible to verify that, as required, the above symmetric tensor is real $(\mathcal{E}^{MN})^{\dagger} = \mathcal{E}^{MN}$. It is also important to note that $\mathcal{E}^{MN}$ transforms covariantly under duality:

$$\mathcal{E}'^{MN}(X') \;=\; h^{M}{}_{P}h^{N}{}_{Q}\,\mathcal{E}^{PQ}(X)\,. \tag{4.127}$$

Taking the variation of the NS-NS action into account, which leads to the tensor $\mathcal{R}_{MN}$ defined in eq. (4.58) of [8], this leads to the $O(D,D)$ covariant form of the type II field equations,

$$\mathcal{R}_{MN} + \mathcal{E}_{MN} \;=\; 0\,, \tag{4.128}$$

supplemented by the duality constraint (4.102). In fact, the duality constraint allows us to simplify $\mathcal{E}^{MN}$ considerably:

$$\mathcal{E}^{MN} \;=\; -\frac{1}{16}\,\mathcal{H}^{(M}{}_{P}\,\overline{\partial\!\!\!/\chi}\,\Gamma^{N)P}\,\partial\!\!\!/\chi\,. \tag{4.129}$$

One may try to verify again the reality of this stress-tensor. A short calculation shows that it is only real whenever $CC = -1$. This is precisely the constraint for consistent duality constraints, as discussed at the end of section 4.4.1. Since we work with real numbers throughout, a non-real stress-tensor can only be equal to zero.

4.5 Action and duality relations in the standard frame

In this section we examine the form of the action and duality relations when choosing the 'standard' duality frame $\tilde{\partial}^i = 0$, and we show that they reduce to the conventional democratic formulation of type II theories. For this we have to assume that we are in a region with a well-defined metric, so that we can choose the parametrization $\mathbb{S} = S_{\mathcal{H}}$. The physical significance of this particular parametrization will be discussed in the next section. Note that the review of democratic formulation of type II theories is omitted in the thesis and can be found in [13].

4.5.1 Action and duality relations in $\tilde{\partial} = 0$ frame

In this section we evaluate the action and duality relations in the standard frame $\tilde{\partial} = 0$. We begin by introducing some relations which will turn out to be useful for this analysis. In order to determine the action of $S_g = S_e S_k S_e^\dagger$ on general states, we compute the action of the respective factors. For S_e, we introduce $e = \exp(E)$ and we have

$$S_e\,\psi^i\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\,e^{\psi^j E_j{}^k \psi_k}\,\psi^i\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\left(\exp E\right)_j{}^i\,\psi^j\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\,e_j{}^i\psi^j\,|0\rangle\,. \tag{4.130}$$

For $S_e^\dagger$ we find an expression with unusual index position

$$S_e^\dagger\, \psi^i\,|0\rangle \;=\; \frac{1}{\sqrt{\det e}}\, e_i{}^j\, \psi^j\,|0\rangle\,. \tag{4.131}$$

The action of S_k can be easily computed,

$$S_k\, \psi^p\,|0\rangle \;=\; \left(\psi^1\psi_1 - \psi_1\psi^1\right)\psi^p\,|0\rangle \;=\; -k_{pq}\,\psi^q|0\rangle\,, \tag{4.132}$$

using the flat Lorentz metric $k = \mathrm{diag}(-1, 1, \ldots, 1)$ defined in (4.60). Using (4.130), (4.131) and (4.132), the action of S_g is then given by

$$S_g\, \psi^i\,|0\rangle \;=\; S_e S_k S_e^\dagger\, \psi^i\,|0\rangle \;=\; -\frac{1}{\sqrt{|\det g|}}\, g_{iq}\, \psi^q\,|0\rangle\,, \tag{4.133}$$

where we used the definition of the metric in (4.60) and wrote $\det e = \sqrt{|\det g|}$. Similarly, for S_g^{-1} one finds

$$S_g^{-1}\, \psi^i\,|0\rangle \;=\; -\sqrt{|\det g|}\; g^{ij}\, \psi^j\,|0\rangle\;, \tag{4.134}$$

where g^{ij} is, as usual, the inverse of the metric g_{ij}.

All of the above relations straightforwardly extend to the case where S_g acts on multiple fermionic oscillators, for which eqs. (4.133) and (4.134) are generalized to

$$\begin{aligned} S_g^{-1}\, \psi^{i_1}\cdots\psi^{i_p}|0\rangle \;&=\; -\sqrt{|\det g|}\, g^{i_1 j_1}\cdots g^{i_p j_p}\; \psi^{j_1}\cdots\psi^{j_p}\,|0\rangle\,, \\ S_g\, \psi^{i_1}\cdots\psi^{i_p}|0\rangle \;&=\; -\frac{1}{\sqrt{|\det g|}}\, g_{i_1 j_1}\cdots g_{i_p j_p}\, \psi^{j_1}\cdots\psi^{j_p}\,|0\rangle\,. \end{aligned} \tag{4.135}$$

With these ingredients we are now ready to evaluate the action.

The action

We start by writing the action in the duality frame $\tilde\partial = 0$. For this choice, the field strength

$$|F\rangle \equiv \not{\partial}|\chi\rangle\,, \tag{4.136}$$

reduces to

$$
\begin{aligned}
|F\rangle\Big|_{\bar\partial=0} &= \sum_{p=0}^{D}\frac{1}{p!}\partial_j C_{i_1\ldots i_p}\,\psi^j\psi^{i_1}\cdots\psi^{i_p}|0\rangle \;=\; \sum_{p=1}^{D}\frac{1}{(p-1)!}\partial_{[i_1}C_{i_2\ldots i_p]}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle \\
&= \sum_{p=1}^{D}\frac{1}{p!}F_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,, \qquad (4.137)
\end{aligned}
$$

where we performed an index shift and relabeled the indices. Thus, the components are given by the conventional field strengths

$$F_{i_1\ldots i_p} \;=\; p\,\partial_{[i_1}C_{i_2\ldots i_p]}\,. \qquad (4.138)$$

It is sometimes useful to avoid explicit indices and combinatorial factors by using the language of differential forms. In general, we identify a spinor state $|G_p\rangle$ with a p-form $G^{(p)}$ as follows

$$|G_p\rangle = \frac{1}{p!}G_{i_1\cdots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle \;\longleftrightarrow\; G^{(p)} = \frac{1}{p!}G_{i_1\cdots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}\,. \qquad (4.139)$$

Whenever we speak of a p-form $G^{(p)}$ and its components $G_{i_1\ldots i_p}$, we will assume a normalization that includes the $p!$ coefficient shown above. It is now straightforward to translate (4.138) to form language:

$$F^{(p)} = dC^{(p-1)}\,. \qquad (4.140)$$

We now collect all field strengths of different degrees into a single form $F = \sum_p F^{(p)}$ and do the same for the potentials $C = \sum_p C^{(p)}$. We then have that (4.140), or for that matter (4.138), for all relevant p is summarized by

$$F = dC\,. \qquad (4.141)$$

In order to evaluate the action we need to choose a parameterization for $\mathbb{S}$, which we take to be $S_{\mathcal{H}}$,

$$\mathbb{S} \;=\; S_{\mathcal{H}} \;=\; e^{\frac{1}{2}b_{ij}\psi_i\psi_j}\, S_g^{-1}\, e^{-\frac{1}{2}b_{ij}\psi^i\psi^j} \,. \tag{4.142}$$

The b-dependent terms in $S_{\mathcal{H}}$ suggest the definition of modified field strengths, related to the original field strengths $|F\rangle = \not{\partial}|\chi\rangle$ by the addition of Chern-Simons like terms:

$$|\widehat{F}\rangle \;\equiv\; e^{-\frac{1}{2}b_{ij}\psi^i\psi^j}|F\rangle \;=\; \sum_{p=1}^{D}\frac{1}{p!}\widehat{F}_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle \,. \tag{4.143}$$

This relation is summarized in form language by

$$\widehat{F} \;=\; e^{-b^{(2)}}\wedge F \;=\; e^{-b^{(2)}}\wedge dC\,, \quad \text{with} \quad b^{(2)} \equiv \frac{1}{2}b_{ij}\,dx^i\wedge dx^j\,. \tag{4.144}$$

The bra corresponding to $|\widehat{F}\rangle$ is given by

$$\langle\widehat{F}| \;=\; \sum_{p=1}^{D}\frac{1}{p!}\langle 0|\psi_{i_p}\cdots\psi_{i_1}\,\widehat{F}_{i_1\ldots i_p} \,. \tag{4.145}$$

Next, we can evaluate the Lagrangian (4.94) using (4.142), (4.143) and (4.145), which yields

$$\mathcal{L} \;=\; \frac{1}{4}\langle\widehat{F}|\,S_g\,|\widehat{F}\rangle \;=\; \frac{1}{4}\sum_{p,q=1}^{D}\frac{1}{q!p!}\widehat{F}_{i_1\ldots i_p}\widehat{F}_{j_1\ldots j_q}\langle 0|\psi_{i_p}\cdots\psi_{i_1}\,S_g^{-1}\,\psi^{j_1}\cdots\psi^{j_q}|0\rangle \,. \tag{4.146}$$

Using now (4.135) for the action of S_g^{-1} and the normalization

$$\langle 0|\psi_{i_p}\cdots\psi_{i_1}\psi^{m_1}\ldots\psi^{m_q}|0\rangle \;=\; \delta_{pq}\,p!\,\delta_{i_1}{}^{[m_1}\cdots\delta_{i_p}{}^{m_p]}\,, \tag{4.147}$$

following from $\langle 0|0\rangle = 1$, the action reduces to

$$\mathcal{L} \;=\; -\frac{1}{4}\sqrt{g}\sum_{p=1}^{D}\frac{1}{p!}\,g^{i_1j_1}\cdots g^{i_pj_p}\,\widehat{F}_{i_1\ldots i_p}\widehat{F}_{j_1\ldots j_p}\,, \tag{4.148}$$

where we used the short-hand notation $\sqrt{g} = \sqrt{|\det g|}$. This can also be written as

$$\mathcal{L} \;=\; -\frac{1}{4}\sqrt{g}\sum_{p=1}^{D} |\widehat{F}^{(p)}|^2 \quad , \tag{4.149}$$

where we define for any p-form $\omega^{(p)}$:

$$|\omega^{(p)}|^2 \equiv \frac{1}{p!}\, g^{i_1 j_1}\cdots g^{i_p j_p}\, \omega_{i_1\ldots i_p}\omega_{j_1\ldots j_p}\;. \tag{4.150}$$

The result in (4.149) is the required sum of kinetic terms for all p-form gauge fields (of odd or even degree, depending on the chirality of χ), which appear in the democratic formulation. This action needs to be supplemented by the duality relations, ensuring that we propagate only the physical degrees of freedom of type II. We consider these next.

Self-duality relations in terms of field strengths

Here we show that for $\tilde{\partial} = 0$ the self-duality conditions $\not{\partial}\chi = -\mathcal{K}\not{\partial}\chi$, c.f. eq. (4.102), reduce to

$$\widehat{F}^{(p)} \;=\; (-1)^{\frac{(D-p)(D-p-1)}{2}} * \widehat{F}^{(D-p)}\;. \tag{4.151}$$

These are conventional duality relations for p-form field strengths. In here we use the following definition of the Hodge-dual form:

$$(*A)_{i_1\cdots i_p} \equiv \frac{1}{(D-p)!}\, g_{i_1 j_1}\cdots g_{i_p j_p}\varepsilon^{k_{p+1}\cdots k_D\, j_1\cdots j_p} A_{k_{p+1}\cdots k_D}\;. \tag{4.152}$$

Our conventions for the epsilon symbols are as follows:

$$\begin{aligned} \epsilon^{1\,2\ldots D} &= +1\,, \qquad & \varepsilon^{i_1\ldots i_D} &= \frac{1}{\sqrt{g}}\,\epsilon^{i_1\ldots i_D} \\ \epsilon_{1\,2\ldots D} &= -1\,, \qquad & \varepsilon_{i_1\ldots i_D} &= \sqrt{g}\,\epsilon_{i_1\ldots i_D}\;, \end{aligned} \tag{4.153}$$

i.e., ϵ is a tensor density, while ε is a (pseudo-)tensor. As usual, lowering the indices on $\varepsilon^{i_1\ldots i_D}$ with g_{ij} yields $\varepsilon_{i_1\ldots i_D}$, and ε and ϵ coincide on flat space. We note the familiar relation for the square of the Hodge star on forms of degree p in a D-dimensional spacetime with signature s:

$$* * \omega^{(p)} = (-1)^{p(D-p)} s\, \omega^{(p)} . \tag{4.154}$$

We can ask when is (4.151) consistent with repeated application of the Hodge star operation. A calculation gives the condition

$$s\,(-1)^{\frac{1}{2}D(D-1)} = 1 . \tag{4.155}$$

Not surprisingly, in Lorenzian signature this agrees with the result in (4.104). Finally, for $D = 10$, the duality constraints (4.151) take the form

$$\widehat{F}^{(p)} = (-1)^{\frac{1}{2}p(p+1)} * \widehat{F}^{(D-p)} . \tag{4.156}$$

We can now begin our calculation. Let us first introduce the short-hand notation

$$\mathbf{B} \;=\; \tfrac{1}{2} b_{ij} \psi^i \psi^j , \quad \mathbf{B}^\dagger \;=\; -\tfrac{1}{2} b_{ij}\, \psi_i \psi_j , \tag{4.157}$$

which allows us to write $S_{\mathcal{H}}$ in (4.68) as follows

$$S_{\mathcal{H}} \;=\; e^{-\mathbf{B}^\dagger} S_g^{-1}\, e^{-\mathbf{B}} . \tag{4.158}$$

The self-duality conditions $\not{\partial}\chi = -\mathcal{K}\not{\partial}\chi$ can now be written as

$$e^{-\mathbf{B}} |\not{\partial}\chi\rangle \;=\; -e^{-\mathbf{B}}\, C^{-1}\, e^{-\mathbf{B}^\dagger} S_g^{-1}\, e^{-\mathbf{B}} |\not{\partial}\chi\rangle\ , \tag{4.159}$$

where we multiplied the factor $e^{-\mathbf{B}}$ from the left to form the modified field strengths $|\widehat{F}\rangle$ defined in (4.143):

$$|\widehat{F}\rangle \;=\; -e^{-\mathbf{B}}\, C^{-1}\, e^{-\mathbf{B}^\dagger}\, S_g^{-1}\, |\widehat{F}\rangle . \tag{4.160}$$

Using (4.37) we readily verify that

$$Ce^{-\mathbf{B}}C^{-1} = e^{-C\mathbf{B}C^{-1}} = e^{-\frac{1}{2}b_{ij}\psi_i\psi_j} = e^{\mathbf{B}^\dagger}\,, \tag{4.161}$$

and, as a result,

$$|\widehat{F}\rangle \;=\; -C^{-1}\,S_g^{-1}|\widehat{F}\rangle \;=\; S_g\,C^{-1}|\widehat{F}\rangle \;=\; -\,S_g\,C|\widehat{F}\rangle\,, \tag{4.162}$$

using $S_g = -C^{-1}S_g^{-1}C$ and $C^{-1} = -C$. This is the simplest possible form of the duality constraints.

We can now examine (4.162) in terms of component fields, as defined in (4.143). We find

$$\sum_{p=1}^{D}\frac{1}{p!}\widehat{F}_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle \;=\; -\sum_{p=1}^{D}\frac{1}{p!}\,\widehat{F}_{i_1\ldots i_p}\,S_g\,\psi_{i_1}\cdots\psi_{i_p}\,C\,|0\rangle\;, \tag{4.163}$$

where we used (4.37). Next, we show that the charge conjugation matrix in (4.163) effectively acts like an epsilon symbol. In fact, by multiple application of the oscillator algebra one can verify that

$$\psi_{i_1}\cdots\psi_{i_p}\,C\,|0\rangle \;=\; \frac{1}{(D-p)!}(-1)^{\frac{p(p-1)}{2}}\epsilon^{i_1 i_2\cdots i_p\,j_{p+1}\cdots j_D}\,\psi^{j_{p+1}}\cdots\psi^{j_D}|0\rangle\;. \tag{4.164}$$

Back in (4.163) we have

$$\sum_{p=1}^{D}\frac{1}{p!}\widehat{F}_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle \;=\; \sum_{p=1}^{D}(-1)^{\frac{(D-p)(D-p-1)}{2}}\frac{1}{p!}\,(*\widehat{F})_{i_1\cdots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;. \tag{4.165}$$

In obtaining this result we made use of (4.135), the definition (4.152) and some simple manipulations. Thus, we have shown that the duality constraint implies the claimed duality relations (4.151).

4.5.2 Conventional gauge symmetries

Let us now verify that the gauge transformations parameterized by ξ^M and λ reduce to the conventional gauge symmetries of type II theories in the frame $\tilde{\partial}^i = 0$. We start with the p-form gauge symmetries (4.107) whose parameter we write in components as

$$|\lambda\rangle \;=\; \sum_{p=0}^{D} \frac{1}{p!}\lambda_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,. \tag{4.166}$$

For $\tilde{\partial} = 0$ this implies

$$\delta_\lambda|\chi\rangle \;=\; \not{\partial}|\lambda\rangle \;=\; \psi^j\partial_j|\lambda\rangle \;=\; \sum_{p=1}^{D}\frac{1}{(p-1)!}\partial_{[i_1}\lambda_{i_2\ldots i_p]}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,, \tag{4.167}$$

from which we read off

$$\delta_\lambda C_{i_1\ldots i_p} \;=\; p\,\partial_{[i_1}\lambda_{i_2\ldots i_p]}\,. \tag{4.168}$$

These are the conventional p-form gauge transformations. In form language they read

$$\delta_\lambda C \;=\; d\lambda\,. \tag{4.169}$$

Let us now discuss the gauge transformations parameterized by $\xi^M = (\tilde{\xi}_i, \xi^i)$. We first claim that the C forms transform as p-forms under diffeomorphisms parameterized by ξ^i. To see this, we compute

$$\delta_\xi|\chi\rangle \;=\; \left(\xi^j\partial_j + \partial_j\xi^k\,\psi^j\psi_k\right)\sum_{p=0}^{D}\frac{1}{p!}\,C_{i_1\ldots i_p}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,. \tag{4.170}$$

The transport term just gives rise to the transport term of the component fields. The second term can be evaluated using (4.13), which then implies for the components

$$\delta_\xi C_{i_1\ldots i_p} \;=\; \xi^j\partial_j C_{i_1\ldots i_p} + p\,\partial_{[i_1}\xi^j\,C_{|j|i_2\ldots i_p]} \;\equiv\; \mathcal{L}_\xi C_{i_1\ldots i_p}\,. \tag{4.171}$$

This is the usual diffeomorphism symmetry which infinitesimally acts via the Lie derivative.

We now consider the $\tilde{\xi}_i$ parameters, which are parameters for the b-field gauge transformations. It turns out that the C forms transform non-trivially under this symmetry. In order to see this we compute for $\tilde{\partial} = 0$

$$\begin{aligned}
\delta_{\tilde{\xi}}|\chi\rangle &= \partial_k\tilde{\xi}_l\,\psi^k\psi^l|\chi\rangle = \sum_{p=0}^{D}\frac{1}{p!}\partial_{[i_1}\tilde{\xi}_{i_2}\,C_{i_3\ldots i_{p+2}]}\psi^{i_1}\cdots\psi^{i_{p+2}}|0\rangle \\
&= \sum_{p=2}^{D}\frac{1}{(p-2)!}\partial_{[i_1}\tilde{\xi}_{i_2}\,C_{i_3\ldots i_p]}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,,
\end{aligned} \tag{4.172}$$

where we performed an index shift $p \to p+2$ in the last equation. We thus read off

$$\delta_{\tilde{\xi}}C_{i_1\ldots i_p} = p(p-1)\partial_{[i_1}\tilde{\xi}_{i_2}\,C_{i_3\ldots i_p]}\,. \tag{4.173}$$

In the language of forms the above equation reads

$$\delta_{\tilde{\xi}}C = d\tilde{\xi}\wedge C\,. \tag{4.174}$$

Note that this implies that

$$\delta_{\tilde{\xi}}C^{(0)} = \delta_{\tilde{\xi}}C^{(1)} = 0\,, \quad \delta_{\tilde{\xi}}C^{(2)} = d\tilde{\xi}\cdot C^{(0)}\,, \quad \ldots\,, \quad \delta_{\tilde{\xi}}C^{(p)} = d\tilde{\xi}\wedge C^{(p-2)}\,. \tag{4.175}$$

4.6 IIA versus IIB

In the previous section we have seen that for fields with no $\tilde{x}$ dependence or, equivalently, setting $\tilde{\partial}^i = 0$, the proposed double field theory reduces to the type IIA or type IIB theory in the democratic formulation, depending on the chosen chirality of χ. It is equally consistent with the strong constraint, however, to keep the $\tilde{x}$ dependence of fields while dropping the x dependence by setting $\partial_i = 0$. We will see that if the theory reduces to type IIA when setting $\tilde{\partial}^i = 0$, the same theory reduces to type IIA* when setting $\partial_i = 0$, and vice versa. Similarly, for the opposite chirality of χ, in one frame the theory reduces to type IIB and in the other frame to type IIB*.

More generally, we can consider intermediate frames that originate from the $\tilde{x}_i = 0$

frame by an arbitrary $O(D,D)$ transformation. Specifically, with the subgroup $O(n-1,1)\times O(d,d)\subset O(D,D)$ acting on coordinates $(x^\mu, x^a, \tilde{x}_a)$, with $\mu=0,\ldots,n-1$ and $a=1,\ldots,d$, we can consider the $O(d,d)$ transformation that maps the $\tilde{x}_a=0$ frame to the $x^a=0$ frame. Here we find that the resulting theory is equivalent to the original one if d is even or to the theory with opposite chirality if d is odd. In other words, for d odd, if we start with a chirality such that the theory reduces to IIA for $\tilde{x}_a=0$, the same theory reduces to type IIB for $x^a=0$, and vice versa.

The two T-duality frames $\tilde{\partial}^i=0$ and $\partial_i=0$ are mapped into each other by the $O(D,D)$ transformation J that exchanges x and $\tilde{x}$,

$$J^M{}_N \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{4.176}$$

The action evaluated in one duality frame is equivalent to the action evaluated in the other duality frame, but written in terms of field variables that are redefined according to the $O(D,D)$ transformation (4.176). To make this more explicit, we introduce

$$\tilde{\mathcal{H}} \;\equiv\; J\mathcal{H}J \;=\; \mathcal{H}^{-1}\,. \tag{4.177}$$

In components, we obtain

$$\tilde{\mathcal{H}} \;=\; \begin{pmatrix} g_{ij}-b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix}\,. \tag{4.178}$$

If we view $\tilde{\mathcal{H}}$ as the generalized metric associated with a new metric g' and a new antisymmetric field b', we would write

$$\tilde{\mathcal{H}} \;=\; \begin{pmatrix} g'^{ij} & -g'^{ik}b'_{kj} \\ b'_{ik}g'^{kj} & g'_{ij}-b'_{ik}g'^{kl}b'_{lj} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{ij} & -\tilde{g}_{ik}\tilde{b}^{kj} \\ \tilde{b}^{ik}\tilde{g}_{kj} & \tilde{g}^{ij}-\tilde{b}^{ik}\tilde{g}_{kl}\tilde{b}^{lj} \end{pmatrix}\,, \tag{4.179}$$

where in the second step we defined the tilde fields by

$$\tilde{g}^{ij}\equiv g'_{ij} \quad\to\quad \tilde{g}_{ij}=g'^{\,ij}\,, \quad\text{ and }\quad \tilde{b}^{ij}\equiv b'_{ij}\,. \tag{4.180}$$

The duality transformations of the metric imply that they satisfy [7]:

$$\tilde{g}_{ij} \;=\; \mathcal{E}_{ki}\, g^{kl}\, \mathcal{E}_{lj}\;, \qquad g^{ij} \;=\; \tilde{\mathcal{E}}^{ik}\, \tilde{g}_{kl}\, \tilde{\mathcal{E}}^{jl}\;, \tag{4.181}$$

where $\mathcal{E}_{ij} = g_{ij} + b_{ij}$ and $\tilde{\mathcal{E}}^{ij} = \left(\mathcal{E}^{-1}\right)^{ij} = \tilde{g}^{ij} + \tilde{b}^{ij}$.

We note that the field redefinitions (4.180) interchange upper with lower indices in order to work consistently with the *lower* indices of the dual coordinates $\tilde{x}_i$. In particular, the diffeomorphisms in the dual coordinates are generated by $\tilde{\xi}_i$ in that the gauge transformations (see (2.37) and (2.38) of [7]) reduce for $\partial_i = 0$ to

$$\delta_{\tilde{\xi}}\tilde{\mathcal{E}}^{ij} \;=\; \tilde{\xi}_k\tilde{\partial}^k\tilde{\mathcal{E}}^{ij} + \tilde{\partial}^i\tilde{\xi}_k\,\tilde{\mathcal{E}}^{kj} + \tilde{\partial}^j\tilde{\xi}_k\,\tilde{\mathcal{E}}^{ik}\;. \tag{4.182}$$

Viewing $\tilde{\mathcal{E}}^{ij}$ with upper indices as a *covariant* rather than a contravariant tensor, this is the conventional transformation of such a tensor under infinitesimal diffeomorphisms.

The double field theory action $S_{\text{NS-NS}}$ for the NS-NS fields is, of course, the same as the double field theory action S_{DFT} for the low energy bosonic string. We thus write

$$S_{\text{NS-NS}}\Big|_{\tilde{\partial}=0} \;=\; S_{\text{DFT}}\Big|_{\tilde{\partial}=0} \;=\; S\Big[g, b, d, \partial\Big]\;, \tag{4.183}$$

with S a function of the four arguments written above. In the dual frame $\partial = 0$ we have

$$S_{\text{NS-NS}}\Big|_{\partial=0} \;=\; S_{\text{DFT}}\Big|_{\partial=0} \;=\; S\Big[\tilde{g}, \tilde{b}, \tilde{d}, \tilde{\partial}\Big]\;. \tag{4.184}$$

The replacements in the arguments of S are, explicitly,

$$g_{ij} \to \tilde{g}^{ij}\,, \quad g^{ij} \to \tilde{g}_{ij}\,, \quad b_{ij} \to \tilde{b}^{ij}\,, \quad \partial_i \to \tilde{\partial}^i\,. \tag{4.185}$$

Let us now see how this generalizes in presence of the RR fields. Before we give a general discussion in the next section, it will be instructive to first examine more explicitly, along the lines reviewed above, what happens in the frame $\partial_i = 0$ with

$\tilde{\partial}^i \neq 0$. Let us first evaluate the field strength $|F\rangle$ in this frame,

$$|F\rangle\Big|_{\partial_i=0} = \sum_{p=0}^{D-1} \frac{1}{p!}\tilde{\partial}^j C_{ji_1\ldots i_p}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,. \tag{4.186}$$

We introduce a dual potential $\tilde{C}$ according to

$$C_{i_1\ldots i_p} = \alpha_p\,\epsilon_{i_1\ldots i_p j_1\ldots j_{D-p}}\,\tilde{C}^{j_1\ldots j_{D-p}}\,, \tag{4.187}$$

where the numerical coefficients have values $\alpha_p = (-1)^{\frac{1}{2}p(p-1)+1}/(D-p)!$ whose derivation is omitted in this chapter. Then (4.186) reads

$$\begin{aligned} |F\rangle\Big|_{\partial_i=0} &= \sum_{p=0}^{D-1} \frac{\alpha_{p+1}}{p!}\epsilon_{ji_1\ldots i_p j_1\ldots j_{D-p-1}}\tilde{\partial}^j\tilde{C}^{j_1\ldots j_{D-p-1}}\psi^{i_1}\cdots\psi^{i_p}|0\rangle \\ &\equiv \sum_{p=0}^{D-1} \frac{\alpha_{p+1}(-1)^p}{p!(D-p)}\,\epsilon_{i_1\ldots i_p j_1\ldots j_{D-p}}\tilde{F}^{j_1\ldots j_{D-p}}\,\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,, \end{aligned} \tag{4.188}$$

where we introduced in analogy to (4.138)

$$\tilde{F}^{j_1\ldots j_p} = p\,\tilde{\partial}^{[j_1}\tilde{C}^{j_2\ldots j_p]}\,. \tag{4.189}$$

We should stress that (4.187) does not involve any metric and so this is *not* the Hodge dual. Consequently, $\tilde{C}$ is not a covariant tensor in the usual sense. However, what we actually have to verify is that, as in (4.182), this is a tensor in the T-dual sense that it transforms under $\tilde{\xi}_i$ rather than ξ^i with a Lie derivative. To see this, we examine the gauge transformation (4.109)

$$\delta_{\tilde{\xi}}|\chi\rangle = \tilde{\xi}_j\tilde{\partial}^j|\chi\rangle + \tilde{\partial}^j\tilde{\xi}_k\,\psi_j\psi^k|\chi\rangle\,. \tag{4.190}$$

The transport term gives manifestly rise to the correct structure, so we focus on the second term, denoted by $\bar{\delta}_{\tilde{\xi}}$, which yields

$$\bar{\delta}_{\tilde{\xi}}|\chi\rangle \;=\; \sum_{p=0}^{D}\frac{\alpha_p(p+1)}{p!}\tilde{\partial}^j\tilde{\xi}_{[j}\,\epsilon_{i_1\dots i_p]k_1\dots k_{D-p}}\tilde{C}^{k_1\dots k_{D-p}}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;. \tag{4.191}$$

To simplify this, we use that a fully antisymmetric tensor with $D+1$ indices in D dimensions vanishes identically,

$$\begin{aligned} 0 \;&=\; (D+1)\tilde{\partial}^j\tilde{\xi}_{[j}\,\epsilon_{i_1\dots i_p k_1\dots k_{D-p}]} \\ &=\; (p+1)\tilde{\partial}^j\tilde{\xi}_{[j}\,\epsilon_{i_1\dots i_p]k_1\dots k_{D-p}} - (D-p)\tilde{\partial}^j\tilde{\xi}_{[k_1}\,\epsilon_{|i_1\dots i_p j|k_2\dots k_{D-p}]}\;. \end{aligned} \tag{4.192}$$

Using this in (4.191), one obtains

$$\bar{\delta}_{\tilde{\xi}}|\chi\rangle \;=\; \sum_{p=0}^{D}\frac{\alpha_p(D-p)}{p!}\epsilon_{i_1\dots i_p k_1\dots k_{D-p}}\tilde{\partial}^{k_1}\tilde{\xi}_j\,\tilde{C}^{jk_2\dots k_{D-p}}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;, \tag{4.193}$$

where we relabeled $k_1 \leftrightarrow j$. In total, we read off

$$\delta_{\tilde{\xi}}\tilde{C}^{i_1\dots i_{D-p}} \;=\; \tilde{\xi}_j\tilde{\partial}^j\tilde{C}^{i_1\dots i_{D-p}} + (D-p)\tilde{\partial}^{[i_1}\tilde{\xi}_k\,\tilde{C}^{|k|i_2\dots i_{D-p}]} \;\equiv\; \mathcal{L}_{\tilde{\xi}}\tilde{C}^{i_1\dots i_{D-p}}\;. \tag{4.194}$$

This is the dual Lie derivative with respect to $\tilde{\xi}_i$ of a dual p-form, where we note that upper indices are now covariant indices and so the signs in (4.194) are the conventional ones, c.f. (4.171) and (4.182).

So far we have seen explicitly that the field strengths in the dual frame $\partial_i = 0$, $\tilde{\partial}^i \neq 0$, take the conventional form when written in terms of the right 'T-dual' variables $\tilde{C}^{i_1\cdots i_p}$. We will now prove more generally that the action and duality relations in the frame $\partial_i = 0$ yield the T-dual type II theory written in terms of the T-dual variables $\tilde{g}$ and $\tilde{b}$ for the NS-NS fields and $\tilde{C}$ for the RR fields. Since the $O(D,D)$ transformation inverts all space-time dimensions, it contains a timelike T-duality and thus maps, say, IIA and IIA* into each other.

To proceed, we describe the field redefinition (4.187) by introducing the following tilde variable of the $O(D,D)$ spinor,

$$\tilde{\chi} \;=\; S_J\,\chi\,, \qquad S_J \;=\; C\,. \tag{4.195}$$

This corresponds to the action of the spinor representative of the $O(D,D)$ transformation $J = J^{-1}$ that exchanges x^i and $\tilde{x}_i$, which for convenience we have chosen to be C, but we stress that this field redefinition does not affect the coordinate arguments.

In terms of the tilde variables (4.195) we have, using (4.24),

$$\begin{aligned} \partial\!\!\!/\chi \;&=\; \frac{1}{\sqrt{2}}\Gamma^M\partial_M(S_J^{-1}\tilde{\chi}) \;=\; \frac{1}{\sqrt{2}}\Gamma^M S_J^{-1}\partial_M\tilde{\chi} \\ &=\; \frac{1}{\sqrt{2}}J^M{}_N S_J^{-1}\Gamma^N\partial_M\tilde{\chi} \;=\; S_J^{-1}\frac{1}{\sqrt{2}}\Gamma^N(J^M{}_N\partial_M)\tilde{\chi} \;=\; S_J^{-1}\hat{\partial\!\!\!/}\tilde{\chi}\,, \end{aligned} \tag{4.196}$$

where we introduced a redefined derivative and Dirac operator,

$$\hat{\partial\!\!\!/} \;\equiv\; \frac{1}{\sqrt{2}}\Gamma^N\hat{\partial}_N\,, \qquad \hat{\partial}_N \;\equiv\; J^M{}_N\partial_M\,. \tag{4.197}$$

Recalling that the matrix $J^M{}_N$ has only the non-vanishing matrix elements J^{ij} and J_{ij} that are equal to Kronecker deltas we find that

$$\hat{\partial\!\!\!/} \;=\; \psi^i\tilde{\partial}^i + \psi_i\partial_i\,. \tag{4.198}$$

As expected, the ∂_i and $\tilde{\partial}^i$ derivatives have been exchanged. For the Lagrangian we now find

$$\mathcal{L} \;=\; \frac{1}{4}(\partial\!\!\!/\chi)^\dagger\, S_{\mathcal{H}}\,\partial\!\!\!/\chi \;=\; \frac{1}{4}(\hat{\partial\!\!\!/}\tilde{\chi})^\dagger(S_J^{-1})^\dagger\, S_{\mathcal{H}}\, S_J^{-1}\hat{\partial\!\!\!/}\tilde{\chi} \;=\; -\frac{1}{4}(\hat{\partial\!\!\!/}\tilde{\chi})^\dagger\, S_{\tilde{\mathcal{H}}}\,\hat{\partial\!\!\!/}\tilde{\chi}\,, \tag{4.199}$$

where we used the sign factor in (4.83). We see that in tilde-variables the RR action takes the same form as in the original variables, up to a sign. It can also be checked that the duality constraints in the dual frame take the form

$$\hat{\partial\!\!\!/}\tilde{\chi} \;=\; C^{-1}S_{\tilde{\mathcal{H}}}\,\hat{\partial\!\!\!/}\tilde{\chi}\,, \tag{4.200}$$

which differs from the constraints in the original frame by a sign factor.

It follows now that setting $\partial_i = 0$ in the evaluation of the Lagrangian as written in the first form in (4.199) is equivalent to setting $\hat{\not{\partial}} = \psi^i \tilde{\partial}^i$ in the evaluation of the Lagrangian as written in the last form in (4.199). But this latter evaluation is identical to our original computation in sec. 5, with ∂_i derivatives replaced by $\tilde{\partial}^i$ derivatives and $C_{i_1 \dots i_p}$ replaced by $\tilde{C}^{i_1 \dots i_p}$. Of course, this time we get an extra minus sign.

Due to this sign change in the RR action we conclude that if the theory reduces for $\tilde{\partial}^i = 0$ to IIA, the same theory reduces for $\partial_i = 0$ to IIA*, but written in terms of the T-dual variables. We thus have, for instance,

$$S_{\mathrm{DFT_{II}}}\Big|_{\tilde{\partial}=0} = S_{\mathrm{IIA}}\Big[g, b, d, C, \partial\Big]\,, \qquad S_{\mathrm{DFT_{II}}}\Big|_{\partial=0} = S_{\mathrm{IIA}^\star}\Big[\tilde{g}, \tilde{b}, \tilde{d}, \tilde{C}, \tilde{\partial}\Big]\,, \qquad (4.201)$$

where we indicated by $S_{\mathrm{DFT_{II}}}$ the full double field theory action of type II, while S_{IIA} and $S_{\mathrm{IIA}^\star}$ are the low-energy actions of IIA and IIA*, respectively. Moreover, the corresponding duality constraints differ by a sign. This is the expected sign given that the stress-tensor from the RR sector in the dual frame must have sign opposite to the one in the original frame.

Similarly, if the chosen chirality is such that the theory reduces in the $\tilde{\partial}^i = 0$ frame to type IIB, the same theory reduces in the $\partial_i = 0$ frame to type IIB*. We finally note that had we chosen the equally valid parametrization $\mathbb{S} = -S_{\mathcal{H}}$, we would have obtained either IIA* or IIB* in the frame $\tilde{\partial}^i = 0$ and the conventional IIA or IIB theories in the opposite frame.

We close this section with a brief discussion of intermediate frames, which we illustrate with the simplest case of one T-duality inversion. Thus, we split the indices as $x^i = (x^1, x^a)$ and assume that the non-trivial derivatives are $(\tilde{\partial}^1, \partial_a)$, where '1' denotes the special direction. As above, we consider a field redefinition that takes the form of the T-duality inversion,

$$
\begin{aligned}
\chi' = S_1\chi &= (\psi^1+\psi_1)\sum_p \frac{1}{p!}\left(C_{a_1\ldots a_p}\psi^{a_1}\cdots\psi^{a_p} + pC_{1a_1\ldots a_{p-1}}\psi^1\psi^{a_1}\cdots\psi^{a_{p-1}}\right)|0\rangle \\
&= \sum_p \frac{1}{p!}\left(C_{a_1\ldots a_p}\psi^1\psi^{a_1}\ldots\psi^{a_p} + pC_{1\,a_1\ldots a_{p-1}}\psi^{a_1}\cdots\psi^{a_{p-1}}\right)|0\rangle \\
&\equiv \sum_p \frac{1}{p!}C'_{i_1\ldots i_p}\psi^{i_1}\cdots\psi^{i_p}|0\rangle\,. \qquad (4.202)
\end{aligned}
$$

This implies that the redefined $C^{(p)}$ are given in terms of the original ones by

$$
C'_{i_1\ldots i_p} = \begin{cases} C_{a_2\ldots a_p} & \text{if } i_1=1,\ i_2=a_2,\ \ldots\ ,i_p=a_p \\ C_{1a_1\ldots a_p} & \text{if } i_1=a_1,\ \ldots\ ,i_p=a_p\,. \end{cases} \qquad (4.203)
$$

Put differently, the new p-forms are obtained from the original ones by adding or deleting the special index. It follows that this redefinition interchanges even and odd forms and thus changes the chirality of χ. The field strength then reads

$$
\partial\!\!\!/\chi = (\psi^a\partial_a+\psi_1\tilde{\partial}^1)(\psi^1+\psi_1)\chi' = (\psi^1+\psi_1)(\psi^a(-\partial_a)+\psi^1\tilde{\partial}^1)\chi' = S_1\,\psi^i\partial'_i\chi'\,, \qquad (4.204)
$$

where we recognized the transformed (primed) derivatives $\partial'_i = (\tilde{\partial}^1, -\partial_a)$, recalling that the transformation h_i in (4.32) changes the overall sign of the coordinates x^a.

In precise analogy to (4.199), we can now conclude that the action in the frame with $\tilde{\partial}^1$, $\partial_a \neq 0$ takes the same form as in the frame $\tilde{\partial}^i = 0$, just with all field variables replaced by primed variables. Since the primed variables have the opposite chirality, it follows that if the theory reduced for $\tilde{\partial}^i = 0$ to, say, type IIA, in the new frame it reduces to type IIB if g_{11} is positive and to type IIB* if g_{11} is negative. More generally, if we evaluate the theory in any frame that results from the $\tilde{\partial}^i = 0$ frame by an $O(d,d)$ transformation, we obtain the corresponding T-dual theory.

4.7 Massive type IIA theories

4.7.1 Reformulation of gauge symmetries

The gauge invariance of the type II double field theory requires for ξ^M transformations the strong form of the constraint, but for λ transformations only the weak constraint. Here, we will perform a change of basis for the gauge parameters such that, for the RR sector, the ξ^M transformations are consistent with a weaker form of the constraint.

We start by rewriting the ξ^M gauge transformation of χ as follows

$$\begin{aligned} \delta_\xi \chi &= \xi^M \partial_M \chi + \frac{1}{2}\partial_M \xi_N \Gamma^M \Gamma^N \chi \\ &= \xi^M \partial_M \chi - \frac{1}{2}\Gamma^M \Gamma^N \xi_N \partial_M \chi + \frac{1}{2}\Gamma^M \partial_M \left(\xi_N \Gamma^N \chi\right) . \end{aligned} \tag{4.205}$$

The last term is of the form of a field-dependent λ gauge transformation $\not{\partial}\lambda$ and can therefore be ignored. We then use the Clifford algebra in the second term,

$$\delta_\xi \chi \;=\; \xi^M \partial_M \chi - \frac{1}{2}(2\eta^{MN} - \Gamma^N \Gamma^M)\xi_N \partial_M \chi \;=\; \frac{1}{2}\Gamma^N \Gamma^M \xi_N \partial_M \chi \,. \tag{4.206}$$

Using the 'slash' notation (4.92), we finally get

$$\delta_\xi \chi \;=\; \not{\xi}\,\not{\partial}\chi \,, \tag{4.207}$$

which is the form of the ξ^M gauge transformations we will use from now on.

We will show next that, starting from (4.207), gauge invariance of the RR action and closure of the gauge algebra uses only the constraint

$$\eta^{MN}\partial_M \partial_N A \;=\; \partial^M \partial_M A \;=\; 0\,, \qquad A \;=\; \left\{\chi, \lambda, \xi^M\right\} . \tag{4.208}$$

In particular, we do not need to use the strong form of the constraint, $\partial^M A \partial_M B = 0$. This observation does not imply, however, that the RR sector is 'weakly constrained' in the sense that fields but not their products need to satisfy the constraint. In fact, (4.207) is not a consistent transformation rule assuming that χ and ξ are weakly constrained. Before discussing this in more detail, we investigate some consequences of the form (4.207) of the gauge transformations.

The original gauge transformations have the property that a gauge parameter of the form $\xi^M = \partial^M\Theta$ is 'trivial' in that it generates no gauge transformation. After the above redefinition, this statement is modified. We compute

$$\delta_{\partial\Theta}\chi \;=\; \frac{1}{2}\Gamma^N\Gamma^M\partial_N\Theta\,\partial_M\chi \;=\; \frac{1}{2}\Gamma^N\partial_N\big(\Theta\,\Gamma^M\partial_M\chi\big)\,, \tag{4.209}$$

assuming only the weaker form (4.208) of the constraint. Thus, the gauge variation (4.209) takes the form of a field-dependent λ gauge transformation,

$$\delta_{\partial\Theta}\chi \;=\; \partial\!\!\!/\lambda\,, \qquad \lambda \;=\; \Theta\,\partial\!\!\!/\chi\,. \tag{4.210}$$

Therefore, the statement that $\xi^M = \partial^M\Theta$ leads to a trivial gauge transformation leaving the fields invariant has to be relaxed to the statement that it leaves the fields invariant *up to a λ gauge transformation*, but it has the advantage that in this weaker form only the constraint (4.208) is required.

We compute next the gauge variation of $\partial\!\!\!/\chi$ under (4.207), which is needed in order to verify gauge invariance,

$$\begin{aligned}\delta_\xi(\partial\!\!\!/\chi) \;&=\; \partial\!\!\!/(\xi\!\!\!/\,\partial\!\!\!/\chi) \;=\; \partial\!\!\!/\xi\!\!\!/\,\partial\!\!\!/\chi+\frac{1}{2\sqrt{2}}\Gamma^M\Gamma^N\Gamma^P\xi_N\partial_M\partial_P\chi\\ &=\; \partial\!\!\!/\xi\!\!\!/\,\partial\!\!\!/\chi+\frac{1}{2\sqrt{2}}(2\eta^{MN}-\Gamma^N\Gamma^M)\Gamma^P\xi_N\partial_M\partial_P\chi\,.\end{aligned} \tag{4.211}$$

The last term contains $\Gamma^{(M}\Gamma^{P)} = \eta^{MP}$ and therefore vanishes by the weak constraint (4.208), while the second term reduces to $\xi^M\partial_M\partial\!\!\!/\chi$. In total we have

$$\delta_\xi(\partial\!\!\!/\chi) \;=\; \xi^M\partial_M\partial\!\!\!/\chi+\partial\!\!\!/\xi\!\!\!/\,\partial\!\!\!/\chi\,. \tag{4.212}$$

This result agrees with the variation under the original form of the gauge transformations determined in [12] (as it should be, because the modification is a λ gauge transformation that leaves $\partial\!\!\!/\chi$ invariant), but in the original derivation the strong constraint was used. As the proof of gauge invariance of the action and the self-duality constraint given in [12] requires only the transformation rule (4.212), we conclude that gauge invariance requires only the weaker constraint (4.208).

Let us verify that also closure of the gauge transformations on χ requires only this weaker constraint. First, for the modified form of the gauge transformations there is no non-vanishing commutator between λ and ξ gauge transformations because $\not{\partial}\chi$ is λ-invariant. Thus, it remains to verify closure of the ξ^M transformations, for which we find

$$\left[\delta_{\xi^1}, \delta_{\xi^2}\right]\chi \;=\; \delta_{\xi^{12}}\chi + \delta_{\lambda}\chi \,. \tag{4.213}$$

Here,

$$\xi_{12}^M \;=\; \left[\xi_2, \xi_1\right]_{\rm C}^M \;=\; \xi_2^N\partial_N\xi_1^M - \frac{1}{2}\xi_{2N}\partial^M\xi_1^N - (1\leftrightarrow 2)\,, \tag{4.214}$$

which is given by the usual 'C-bracket' that characterizes the closure of ξ^M transformations on the NS-NS fields [6, 7], and

$$\lambda \;=\; -\frac{1}{2}(\not{\xi}_2\not{\xi}_1 - \not{\xi}_1\not{\xi}_2)\not{\partial}\chi \,. \tag{4.215}$$

The verification of (4.213) is a straightforward though somewhat tedious exercise in gamma matrix algebra, which we defer to the appendix. The computation makes repeated use of the constraints, but only in its relaxed form (4.208). Thus, on the RR field χ all gauge symmetries close using only this weaker constraint.

We close this section by computing the form of these redefined ξ^M gauge transformations (4.207) for $\tilde{\partial}^i = 0$. For the diffeomorphism parameter ξ^i we find

$$\delta_\xi\chi \;=\; \xi^i\psi_i\psi^j\partial_j\chi \;=\; \sum_p \frac{1}{p!}\xi^i\partial_j C_{i_1\cdots i_p}\psi_i\psi^j\psi^{i_1}\cdots\psi^{i_p}|0\rangle \,. \tag{4.216}$$

Using the oscillator algebra (4.9) to simplify this, we obtain

$$\delta_\xi C_{i_1\cdots i_p} \;=\; (p+1)\xi^j\partial_{[j}C_{i_1\cdots i_p]} \;=\; \xi^j F_{ji_1\cdots i_p} \,. \tag{4.217}$$

For the b-field gauge parameter $\tilde{\xi}_i$ one obtains

$$\delta_{\tilde{\xi}}\chi \;=\; \tilde{\xi}_i\psi^i\psi^j\partial_j\chi \;=\; \sum_p \frac{1}{p!}\,\tilde{\xi}_i\partial_j C_{i_1\ldots i_p}\psi^i\psi^j\psi^{i_1}\cdots\psi^{i_p}|0\rangle\;, \tag{4.218}$$

from which we read off

$$\delta_{\tilde{\xi}}C \;=\; \tilde{\xi}\wedge F\;. \tag{4.219}$$

The diffeomorphism symmetry in the form (4.217) is sometimes referred to as 'improved diffeomorphisms'. They can be introduced for any p-form gauge field by adding to the familiar diffeomorphism symmetry (4.171) a field-dependent gauge transformation with $(p-1)$-form parameter

$$\lambda_{i_1\ldots i_{p-1}} \;=\; -\xi^j C_{ji_1\ldots i_{p-1}}\;. \tag{4.220}$$

Similarly, (4.219) is obtained from the original $\tilde{\xi}$ transformation (4.174) by adding an abelian gauge transformation with parameter $\lambda = -\tilde{\xi}\wedge C$. Thus, the redefinition of the gauge transformations leading to (4.207) is precisely the double field theory analogue of the improved diffeomorphisms in conventional gauge theories. In this form the gauge field appears only under a derivative, which will be instrumental for the generalization we discuss next.

4.7.2 Massive type IIA

In the previous section we have seen that gauge invariance and closure of the gauge algebra requires only the weaker constraint (4.208) for the RR sector. Naively, this would allow for field configurations like

$$\chi(x,\tilde{x}) \;=\; \chi_0(x)+\chi_1(\tilde{x})\;, \tag{4.221}$$

where $\chi_{0,1}$ are arbitrary functions of their arguments, and similarly for the gauge parameters. However, as mentioned above, there is a subtlety, because the gauge variations (4.207) are not consistent assuming only the weak constraint. In fact, $\delta_\xi\chi$

on the left-hand side should satisfy the constraint, but with χ and ξ being weakly constrained their product on the right-hand side in general does not satisfy the constraint. Rather, one should introduce a projector that restricts to the part satisfying the weak constraint [6], while our computation above did not keep track of these projectors. After the insertion of projectors, the gauge invariance of the action and closure of the gauge algebra does not follow from our computation (and is most likely not true). Moreover, the RR fields interact with the NS-NS sector that is still strongly constrained, and so it is presumably inconsistent to have a weakly constrained RR sector. Thus, a complete relaxation of the strong constraint must await a resolution of this problem for the NS-NS sector. However, if we only assume the function χ_1 in (4.221) to depend linearly on $\tilde{x}$, the resulting gauge variations and field equations are independent of $\tilde{x}$, and therefore the constraint is satisfied without insertion of projectors. (In particular, the energy-momentum tensor of the RR fields depends only on $\not{\partial}\chi$ [12] and is thereby independent of $\tilde{x}$.) An ansatz with linear $\tilde{x}$ dependence is therefore consistent, and we will investigate its consequences in what follows.

We will show that the type II double field theory defined by (4.4) and (4.5) leads to massive type IIA if we assume that the RR spinor χ depends on the 10-dimensional space-time coordinates and, in its 1-form part, also linearly on a winding coordinate. We thus write

$$\chi(x,\tilde{x}) \;=\; \Big(\sum_p \frac{1}{p!}C_{i_1\dots i_p}(x)\,\psi^{i_1}\dots\psi^{i_p} + m\tilde{x}_1\psi^1\Big)|0\rangle\;, \tag{4.222}$$

where we assume that χ is of negative chirality such that the sum extends only over odd p. Here we have singled out a particular (winding) coordinate direction, but we stress that this choice is immaterial for the final result: we could have chosen any linear combination of the $\tilde{x}_i$, which would merely amount to a rescaling of the mass parameter m. Let us also note that it would be consistent to allow for a linear $\tilde{x}$ dependence in other p-form parts, both in χ and in its gauge parameter λ. We will comment on this more general case below.

Let us next evaluate the field strength $\not{\partial}\chi$ for (4.222). In contrast to (4.137), the term $\psi_i\tilde{\partial}^i$ in $\not{\partial}$ acts now non-trivially,

$$\begin{aligned}
\not{\partial}\chi &= \sum_p \frac{1}{p!}\partial_j C_{i_1\ldots i_p}\,\psi^j\psi^{i_1}\cdots\psi^{i_p}|0\rangle + \psi_j\tilde{\partial}^j(m\tilde{x}_1)\psi^1|0\rangle \\
&= \sum_p \frac{1}{(p+1)!}(p+1)\partial_{[i_1}C_{i_2\ldots i_{p+1}]}\,\psi^{i_1}\cdots\psi^{i_{p+1}}|0\rangle + m|0\rangle \\
&\equiv \sum_p \frac{1}{(p+1)!}(F_m)_{i_1\ldots i_{p+1}}\psi^{i_1}\cdots\psi^{i_{p+1}}|0\rangle\;,
\end{aligned} \tag{4.223}$$

where we used the oscillator algebra (4.9). We observe that the non-trivial action of $\psi_i\tilde{\partial}^i$ leads to a reduction of the form degree such that the '1-form potential' precisely leads to a non-vanishing 0-form field strength or, in other words, that the $\tilde{x}$ dependent part acts effectively like a '(-1)-form'. The m-deformed field strengths defined in the last line of (4.223) then read

$$F_m^{(0)} = m\;, \qquad F_m^{(p+1)} = F^{(p+1)} = dC^{(p)} \quad \text{for} \quad p\geq 0\;. \tag{4.224}$$

In the action the modified field strengths (4.144) enter, which are now deformed according to (4.224),

$$\widehat{F}_m = e^{-b^{(2)}}\wedge(dC+m)\;. \tag{4.225}$$

This reads explicitly

$$\begin{aligned}
\widehat{F}_m^{(0)} &= m \\
\widehat{F}_m^{(2)} &= F^{(2)} - mb^{(2)} \\
\widehat{F}_m^{(4)} &= F^{(4)} - b^{(2)}\wedge F^{(2)} + \frac{1}{2}mb^{(2)}\wedge b^{(2)}\;, \quad \text{etc.}
\end{aligned} \tag{4.226}$$

These are precisely the m-deformed field strengths appearing in massive type IIA, see, e.g., [71].

We turn now to the gauge symmetries acting on (4.222), starting with the λ-transformations (4.107). In analogy to (4.222) it is natural to allow here also for a linear $\tilde{x}$ dependence in the 0-form part of λ, but such a contribution will be annihilated by $\not{\partial}$ due to $\psi_i|0\rangle = 0$. We note, however, that a linear $\tilde{x}$ dependence in the higher-form components of λ can lead to a *rigid* shift of the RR forms, which is trivially a global symmetry since all RR potentials appear under a derivative. We conclude that the λ gauge transformations are unchanged compared to the massless case (4.169). The ξ^M transformation (4.207) evaluated for the diffeomorphism parameter ξ^i yields no new contribution since

$$\delta_{\xi^i}\chi\Big|_{\partial_i=0} \;=\; \xi^i\psi_i\psi_j\tilde{\partial}^j\chi \;=\; 0\,, \tag{4.227}$$

due to the action of two annihilation operators ψ_i on (4.222). Thus, the diffeomorphism symmetry is given by (4.217), as for $m=0$. Finally, the gauge transformation of the b-field gauge parameter $\tilde{\xi}_i$ receives a non-trivial modification,

$$\delta_{\tilde{\xi}_i}\chi\Big|_{\partial_i=0} \;=\; \tilde{\xi}_i\psi^i\psi_j\tilde{\partial}^j\chi \;=\; m\tilde{\xi}_i\psi^i\psi_1\psi^1|0\rangle \;=\; m\tilde{\xi}_i\psi^i|0\rangle\,. \tag{4.228}$$

Together with the gauge transformation (4.219) for $m=0$ we thus obtain

$$\delta_{\tilde{\xi}}C \;=\; \tilde{\xi}\wedge dC + m\tilde{\xi}\,. \tag{4.229}$$

Therefore, for $m\neq 0$ the RR 1-form $C^{(1)}$ transforms with a Stückelberg shift symmetry under the b-field gauge transformations, which is precisely the expected result for massive type IIA [71]. We note that the modified field strengths $\widehat{F}_m$ are manifestly invariant under the λ gauge transformations. The invariance under $\tilde{\xi}$ transformations can be easily verified with $\delta_{\tilde{\xi}}b^{(2)} = d\tilde{\xi}$,

$$\begin{aligned}\delta_{\tilde{\xi}}\widehat{F}_m \;&=\; \delta_{\tilde{\xi}}\big(e^{-b^{(2)}}\wedge(dC+m)\big) \;=\; -d\tilde{\xi}\wedge\widehat{F}_m + e^{-b^{(2)}}\wedge d\big(\tilde{\xi}\wedge dC + m\tilde{\xi}\big) \\ &=\; -d\tilde{\xi}\wedge\widehat{F}_m + e^{-b^{(2)}}\wedge d\tilde{\xi}\wedge(dC+m) \;=\; -d\tilde{\xi}\wedge\widehat{F}_m + d\tilde{\xi}\wedge\widehat{F}_m \;=\; 0\,.\end{aligned} \tag{4.230}$$

Let us now consider the double field theory action and duality relations (4.4) and (4.5), evaluated for (4.222), and compare with the dynamics of massive type IIA. As in (4.148), the action reduces to the sum of kinetic terms, but here for the modified field strengths (4.225),

$$\mathcal{L}_{\mathrm{RR}} = \frac{1}{4}\sum_{p=0}^{10} \widehat{F}_m^{(p)} \wedge *\widehat{F}_m^{(p)} = \frac{1}{4}\sum_{p\geq 1}^{10} \widehat{F}_m^{(p)} \wedge *\widehat{F}_m^{(p)} + \frac{1}{4}m^2 * 1 \ . \tag{4.231}$$

The action contains now also the 0-form field strength, which contributes a cosmological term proportional to m^2, as made explicit in the second equation. Moreover, we can use the Stückelberg gauge symmetry (4.229) with parameter $\tilde{\xi}$ to set $C^{(1)} = 0$. From the second equation in (4.226) we then infer that the kinetic term for $C^{(1)}$ reduces to a mass term for the b-field. Thus, the b-field becomes massive by 'eating' the RR 1-form.

The self-duality constraint (4.5) reduces to the same duality relations as in (4.156), again with all field strengths being m-deformed,

$$\widehat{F}_m^{(p)} = -(-1)^{\frac{1}{2}p(p+1)} * \widehat{F}_m^{(10-p)} \ . \tag{4.232}$$

This democratic formulation is equivalent to the conventional formulation of massive type IIA. In the following we compare the two formulations in a little more detail.

The RR action of massive type IIA in the standard formulation is given by [28,71]

$$\begin{aligned} S_{\mathrm{RR}} = & \frac{1}{2}\int\Big(\widehat{F}_m^{(2)} \wedge *\widehat{F}_m^{(2)} + \widehat{F}_m^{(4)} \wedge *\widehat{F}_m^{(4)} + m^2 * 1\Big) \\ & + \frac{1}{2}\int\Big(b^{(2)}(dC^{(3)})^2 - (b^{(2)})^2 dC^{(1)} dC^{(3)} + \frac{1}{3}(b^{(2)})^3 (dC^{(1)})^2 \\ & \qquad + \frac{1}{3}m(b^{(2)})^3 dC^{(3)} - \frac{1}{4}m(b^{(2)})^4 dC^{(1)} + \frac{1}{20}m^2 (b^{(2)})^5\Big) \ , \end{aligned} \tag{4.233}$$

where for simplicity we have omitted all wedge products between forms in the topological Chern-Simons terms S_{CS} in the second and third line.[4] We note in passing that this Chern-Simons action simplifies significantly if we formally introduce a (-1)-form $C^{(-1)}$ and then define

$$\widehat{A} = e^{-b^{(2)}} \wedge \big(C + C^{(-1)}\big) \ , \tag{4.234}$$

[4]This action differs from eq. (2.8) of [71] in certain numerical factors, which is due to different conventions regarding differential forms. Moreover, there is a mismatch of a relative factor of $\frac{1}{2}$ between kinetic and Chern-Simons terms, but (4.233) is consistent with [28].

where C still represents the formal sum of all (odd) p-forms with $p \geq 1$. The Chern-Simons action can then simply be written as

$$S_{\rm CS} \;=\; \frac{1}{2}\int b^{(2)} \wedge d\widehat{A}^{(3)} \wedge d\widehat{A}^{(3)}\,. \tag{4.235}$$

More precisely, expanding (4.235) according to (4.234), the resulting action can be written, up to total derivatives, such that $C^{(-1)}$ enters only under an exterior derivative, and then setting $m = dC^{(-1)}$ reproduces precisely the Chern-Simons terms in (4.233). Formally, this drastic simplification can be understood as a consequence of the b-field gauge transformations (4.229), which we rewrite here as

$$\delta_{\tilde{\xi}} C \;=\; \tilde{\xi} \wedge d\big(C + C^{(-1)}\big) \;=\; d\tilde{\xi} \wedge \big(C + C^{(-1)}\big) - d\big(\tilde{\xi} \wedge \big(C + C^{(-1)}\big)\big)\,. \tag{4.236}$$

The last term takes the form of a field-dependent λ gauge transformation and can thus be ignored. The $\widehat{A}$ defined in (4.234) is then $\tilde{\xi}$ gauge invariant,

$$\delta_{\tilde{\xi}} \widehat{A} \;=\; -d\tilde{\xi} \wedge e^{-b^{(2)}} \wedge \big(C + C^{(-1)}\big) + e^{-b^{(2)}} \wedge \big(d\tilde{\xi} \wedge \big(C + C^{(-1)}\big)\big) \;=\; 0\,, \tag{4.237}$$

where we have taken $C^{(-1)}$ to be gauge invariant. From this we infer that (4.235) is the only term invariant under $\tilde{\xi}$ gauge transformations (up to a boundary term). Note that we could have included the (-1)-form potential into the sum of all p-forms, in which case the gauge transformations would be formally as in the massless case.

We have verified the exact equivalence between the equations of motion following from (4.233) and those derived by varying (4.231) and then supplementing them by the duality relations (4.232). For the Einstein equations this is easy to see because the Chern-Simons terms that are present in the conventional formulation do not contribute to the variation of the metric. The energy-momentum tensor then agrees for both formulations owing to the relative factor of $\frac{1}{2}$ between the kinetic terms in (4.231) and (4.233), which compensates for the doubling of fields in the democratic formulation. For the field equations of the p-forms the on-shell equivalence is a consequence of the Bianchi identities

$$d\widehat{F}_m^{(p)} = -H^{(3)} \wedge \widehat{F}_m^{(p-2)}\,, \tag{4.238}$$

following from (4.225). More precisely, the duality relations yield the second-order field equations as integrability conditions of $d^2 = 0$, including the required source terms originating from the Chern-Simons terms in the conventional formulation. Thus, the double field theory leads precisely to massive type IIA.

4.7.3 T-duality and massive type IIB

We discuss now the double field theory evaluated for fields depending on coordinates that result from the 10-dimensional space-time coordinates x^i by a T-duality inversion. The $O(10,10)$ invariance of the constraints implies that fields resulting by an $O(10,10)$ transformation from fields depending only on the x^i (thereby satisfying the constraint) also satisfy the constraint. For instance, we may perform a single T-duality inversion in one direction, which exchanges a 'momentum coordinate' x^i with the corresponding 'winding coordinate' $\tilde{x}_i$. The double field theory evaluated for this field configuration then reduces to the T-dual theory. If it reduces to type IIA in one 'T-duality frame', it reduces to type IIB in the other frame, when expressed in the right T-dual field variables [12]. The mapping of (massless) type IIA into type IIB under T-duality can therefore be discussed without reference to dimensional reduction, while in the usual approach this relation is inferred from the equivalence of type IIA and type IIB upon reduction on a circle [54].

Our task is now to see how this generalizes in the massive case. The usual point of view is as follows [71]. Massive type IIA reduced on a circle leads to a massive $N = 2$ theory in nine dimensions, but there is no corresponding massive deformation of type IIB that could lead to the same nine-dimensional theory upon standard reduction. Rather, to identify the proper T-duality rules one has to perform a Scherk-Schwarz reduction [73] of massless type IIB, which introduces a mass parameter and leads to the same massive $N = 2$ theory in nine dimensions. In contrast, in double field theory the T-dual theory is identified without any dimensional reduction, as we discussed above, and so the puzzle arises what the T-dual to massive type IIA is if there is no massive type IIB in ten dimensions.

In order to address this issue let us analyze the double field theory evaluated for fields in which one space-time coordinate, say x^{10}, is replaced by the corresponding winding coordinate. We split the coordinates as $x^i = (x^\mu, x^{10})$, $\mu = 1, \ldots, 9$, and replace (4.222) by the ansatz

$$\chi(x, \tilde{x}) \;=\; \Big(\sum_p \frac{1}{p!} C_{i_1 \ldots i_p}(x^\mu, \tilde{x}_{10})\, \psi^{i_1} \cdots \psi^{i_p} + m\tilde{x}_1 \psi^1\Big)|0\rangle \,, \tag{4.239}$$

where again the sum extends over all odd p. In the massless case the double field theory reduces to type IIB, which can be made manifest by performing a field redefinition that takes the form of a T-duality inversion in the 10th direction [12].[5] This T-duality transformation acts on the RR spinor via the spin representative $S_{10} = \psi^{10} + \psi_{10}$, i.e., we define

$$\chi' \;=\; S_{10}\,\chi \;=\; \Big(\sum_p \frac{1}{p!} C'_{i_1 \ldots i_p}\, \psi^{i_1} \cdots \psi^{i_p} + m\tilde{x}_1 (\psi^{10} + \psi_{10})\psi^1\Big)|0\rangle \,, \tag{4.240}$$

where in the first term we introduced redefined variables denoted by C'. As S_{10} is linear in the fermionic oscillators the sum extends now over all *even* p. Specifically, one finds (compare eq. (6.41) in [13])

$$C'_{i_1 \ldots i_p} \;=\; \begin{cases} C_{\mu_2 \ldots \mu_p} & \text{if} \;\; i_1 = 10,\; i_2 = \mu_2 \,, \;\ldots\;, i_p = \mu_p \\ C_{10\mu_1 \ldots \mu_p} & \text{if} \;\; i_1 = \mu_1 \,, \;\ldots\;, i_p = \mu_p \,. \end{cases} \tag{4.241}$$

Thus, the dual field variables are obtained by adding or deleting the special index, thereby mapping odd forms into even forms, as required for the transition from type IIA to type IIB. By performing this field redefinition (and renaming the coordinates)

[5] Here we assume that x^{10} is a space-like direction, $g_{10,10} > 0$. For T-dualities along time-like directions the dual theories are the so-called type II* theories [12], which have a reversed sign for the RR kinetic terms [63]. Similarly, the double field theory discussed here contains also a massive type IIA*.

one infers that evaluating the theory for fields depending on x^μ and $\tilde{x}_{10}$ is equivalent to evaluating the theory for fields depending on x^i, but with the opposite chirality for the spinor, i.e., replacing odd forms by even forms. (See sec. 6.2 in [13] for more details.) Now, in the massive case we have to take into account the second term in (4.240), which reduces to $m\tilde{x}_1\psi^{10}\psi^1$. Thus, our task is to evaluate the double field theory for

$$\chi(x,\tilde{x}) \;=\; \Big(\sum_p \frac{1}{p!}C_{i_1\ldots i_p}(x)\,\psi^{i_1}\cdots\psi^{i_p} + m\tilde{x}_1\psi^{10}\psi^1\Big)|0\rangle\;, \tag{4.242}$$

dropping the primes from now on. In other words, we have to evaluate the double field theory for a field configuration in which the 2-form part depends now linearly on $\tilde{x}$,

$$\big(\chi(x,\tilde{x})\big|_{2-\text{form}}\big)_{ij} \;=\; C_{ij}(x) + 2m\tilde{x}_1\,\delta_{[i}{}^{10}\,\delta_{j]}{}^1\;, \tag{4.243}$$

with all other fields still depending only on the 10-dimensional space-time coordinates.

We start by computing the field strength

$$F \;=\; \partial\!\!\!/\chi \;=\; F_{m=0} - \psi_1\tilde{\partial}^1(m\tilde{x}_1)\psi^1\psi^{10}|0\rangle \;=\; F_{m=0} - m\psi^{10}|0\rangle\;. \tag{4.244}$$

Therefore, the field strength of the RR 0-form $C^{(0)}$ gets modified in the 10th component,

$$F^{(1)} \;=\; dC^{(0)} - m dx^{10} \qquad \Leftrightarrow \qquad F_i \;=\; \partial_i C^{(0)} - m\delta_i{}^{10}\;, \tag{4.245}$$

while all other field strengths $F^{(p)}$, $p \neq 1$, remain unchanged. The 'hatted' field strength (4.144) then receives corresponding modifications,

$$\widehat{F} \;=\; e^{-b^{(2)}} \wedge \big(dC - m dx^{10}\big)\;, \tag{4.246}$$

and thus in components

$$\widehat{F}^{(3)} \;=\; F^{(3)} - b^{(2)}\wedge dC^{(0)} + m b^{(2)}\wedge dx^{10}\;, \quad \text{etc.} \tag{4.247}$$

The dynamics is described by the same action (4.148) and duality relations (4.156) as before, but with all field strengths replaced by their m-deformed version (4.246).

This theory breaks manifest 10-dimensional covariance in that the 10th coordinate is treated on a different footing in (4.245). We observe, however, that this theory can be obtained from standard (covariant) type IIB by performing the redefinition

$$C^{(0)} \;\rightarrow\; C^{(0)} - mx^{10}\,, \tag{4.248}$$

as is apparent from (4.245). Thus, the 'deformation' induced by the m-dependent 2-form contribution in (4.242) can be absorbed into a redefinition of the lower RR form $C^{(0)}$, and therefore the obtained theory is nothing but standard type IIB after a somewhat peculiar (non-covariant) redefinition. For this reason we do not introduce a new symbol for the 'deformed' field strengths.

In order to understand the consequences of the non-covariance let us inspect the gauge symmetries. As above, the λ gauge transformations are unchanged compared to the massless case. The gauge transformations (4.207) parametrized by ξ^M applied to (4.242) give

$$\begin{aligned}\delta_\xi\chi \;&=\; (\psi^i\tilde{\xi}_i + \psi_i\xi^i)\partial\!\!\!/\chi \;=\; \delta_\xi\chi\Big|_{m=0} - m(\tilde{\xi}_i\psi^i\psi^{10} + \xi^i\psi_i\psi^{10})|0\rangle \\ &=\; \delta_\xi\chi\Big|_{m=0} - m(\tilde{\xi}_\mu\psi^\mu\psi^{10} + \xi^{10})|0\rangle\,.\end{aligned} \tag{4.249}$$

We read off the m-deformed gauge transformations which are modified on $C_{\mu 10}$,

$$\delta_{\tilde{\xi}}C_{\mu 10} \;=\; 2\tilde{\xi}_{[\mu}F_{10],m=0} - m\tilde{\xi}_\mu \;=\; 2\tilde{\xi}_{[\mu}F_{10]}\,, \tag{4.250}$$

and on $C^{(0)}$

$$\delta_\xi C^{(0)} \;=\; \xi^j\partial_j C^{(0)} - m\xi^{10} \;=\; \xi^j F_j\,, \tag{4.251}$$

where we used (4.245) for both equations in the last step. Thus, the nine-component parameter $\tilde{\xi}_\mu$ acts as a Stückelberg symmetry on the off-diagonal RR 2-form components, while the 10th diffeomorphism parameter ξ^{10} acts as a Stückelberg symmetry on the RR 0-form. The field strength of $C_{\mu 10}$ read off from (4.247),

$$\widehat{F}_{\mu\nu 10} \;=\; 2\partial_{[\mu}C_{\nu]10} + mb_{\mu\nu} + \partial_{10}C_{\mu\nu} - b_{\mu\nu}\partial_{10}C^{(0)} - 2b_{10[\mu}\,\partial_{\nu]}C^{(0)}\;, \tag{4.252}$$

is invariant under the $\tilde{\xi}_\mu$ shift symmetry. Moreover, (4.245) is invariant under ξ^{10}, i.e., the theory is diffeomorphism invariant under $x^{10} \to x^{10} - \xi^{10}(x)$ and (4.251),

$$\delta_{\xi^{10}}F^{(1)} \;=\; -md\xi^{10} + md\xi^{10} \;=\; 0\;. \tag{4.253}$$

Thus, despite the non-covariant formulation that treats the 10th direction on a different footing, the theory is still fully diffeomorphism invariant, as it should be in view of the fact that it results from standard type IIB by the redefinition (4.248). Since this invariance under non-covariant diffeomorphisms is somewhat unconventional, let us also verify this for the component form given in (4.245),

$$\begin{aligned}\delta_\xi F_i \;&=\; \partial_i\big(\xi^j\partial_jC^{(0)} - m\xi^{10}\big) \;=\; \xi^j\partial_j\big(\partial_iC^{(0)}\big) + \partial_i\xi^j\,\partial_jC^{(0)} - m\partial_i\xi^{10}\\ &=\; \xi^j\partial_jF_i + \partial_i\xi^j\big(\partial_jC^{(0)} - m\delta_j{}^{10}\big) \;=\; \xi^j\partial_jF_i + \partial_i\xi^jF_j\;.\end{aligned} \tag{4.254}$$

Thus, the m-deformed field strength transforms under the m-deformed diffeomorphisms (4.251) with the usual Lie derivative of a 1-form field strength. Therefore, the action and duality relations build with this field strength are diffeomorphism invariant.

To summarize, we have identified the 10-dimensional theory that is the T-dual to massive type IIA and that can be seen as a 'massive' formulation of type IIB. It is unconventional in that the 10-dimensional diffeomorphism symmetry is not realized in the usual way, but non-linearly in the 10th direction. This is, however, analogous to the deformation of the gauge transformation of $C^{(1)}$ under the b-field gauge parameter in massive type IIA, and since the diffeomorphisms and b-field gauge symmetries are on the same footing in double field theory this result is not surprising.

Let us now discuss the physical content. We can choose a gauge for the $\tilde{\xi}_\mu$ Stückelberg symmetries by setting $C_{\mu 10} = 0$. From (4.252) we then infer that their kinetic terms give mass terms for the 9-dimensional components of the b-field, rendering these components massive. This is analogous to massive type IIA, but in the latter case the full 10-dimensional b-field becomes massive, carrying 36 massive degrees of freedom, while here only the 9-dimensional components become massive, carrying 28 massive degrees of freedom. It turns out that the 8 missing degrees of freedom are carried instead by the Kaluza-Klein vector field. In order to see this, let us perform a Kaluza-Klein decomposition of the kinetic term involving $C^{(0)}$ (but we stress that we are not performing a reduction in that the fields still depend on all 10 coordinates). The standard Kaluza-Klein decomposition of the (inverse) metric reads

$$g^{ij} = \begin{pmatrix} \gamma^{\mu\nu} & -A^{\mu} \\ -A^{\nu} & \ell^{-1} + A^{\rho}A_{\rho} \end{pmatrix}, \tag{4.255}$$

where $\gamma_{\mu\nu}$ denotes the 9-dimensional metric, A_{μ} is the Kaluza-Klein vector and ℓ the Kaluza-Klein scalar. If we choose a gauge for the ξ^{10} Stückelberg symmetry by setting $C^{(0)} = 0$, we infer with (4.245) that the relevant term in the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}\sqrt{g}\, g^{ij} F_i F_j = -\frac{1}{4}\sqrt{g} g^{10,10} F_{10} F_{10} = -\frac{1}{4} m^2 \sqrt{\gamma}\sqrt{\ell}\left(\ell^{-1} + A^{\mu}A_{\mu}\right) . \tag{4.256}$$

Therefore, the Kaluza-Klein vector receives a mass term and so becomes massive by 'eating' the RR scalar $C^{(0)}$, thus carrying 8 massive degrees of freedom.

We have to point out that the above analysis of the physical content was somewhat naive. In fact, one may wonder why this theory, if obtained from massless type IIB by the mere redefinition (4.248), exhibits a spectrum that is rather different from the usual physical content of type IIB, e.g., with (parts of) the b-field becoming massive and a cosmological term in (4.256). The point is that such a classification of the masses of various fields is only meaningful with respect to a particular background. For instance, type IIB admits a 10-dimensional Minkowski solution, with all field strengths zero in the background, and it is with respect to this background that the b-field is massless. Now, after the redefinition (4.248) the theory of course still admits the same Minkowski vacuum, but now we have to switch on a 'background flux' in order to realize this solution,

$$\langle g_{ij} \rangle \;=\; \eta_{ij} \;, \qquad \langle dC^{(0)} \rangle \;=\; m dx^{10} \;, \tag{4.257}$$

because only then we have $\langle \widehat{F} \rangle = 0$ in the Einstein equations, as follows with (4.245). Around this background, the b-field is still massless.

Thus, there is no conflict of our above analysis of 'massive' type IIB with the usual way type IIB is presented. The presence of massive fields just means that the background space-time we consider is *not* flat space, but rather a background that is appropriate for the comparison to the T-dual massive type IIA. In fact, massive type IIA does not admit a Minkowski (or AdS) vacuum, but instead the D8-brane solution that is invariant under the 9-dimensional Poincaré group corresponding to its world-volume [71]. The T-dual configuration is the D7-brane solution of type IIB, which is only invariant under the 8-dimensional Poincaré group [71], and the above analysis has to be understood with respect to such a background.

Let us close this section by comparing our result with the usual story that relates massive type IIA to the Scherk-Schwarz reduction of massless type IIB [71, 72]. In Scherk-Schwarz reduction one allows some fields to depend non-trivially on the internal coordinates in such a way that this dependence drops out in the effective lower-dimensional theory. For the Scherk-Schwarz reduction of type IIB to nine dimensions relevant for T-duality, the Kaluza-Klein ansatz allows for a linear x^{10} dependence for the RR scalar $C^{(0)}$,

$$C^{(0)}(x^{\mu}, x^{10}) \;=\; c^{(0)}(x^{\mu}) - m x^{10} \;, \tag{4.258}$$

where $c^{(0)}$ denotes the nine-dimensional field. For all other fields the ansatz is as for circle reductions, i.e., the fields are simply assumed to be independent of x^{10}. In the resulting action the dependence on x^{10} drops out, leaving a massive deformation of the usual circle reduction of type IIB.

Instead of this Scherk-Schwarz reduction one may first perform the redefinition (4.248) and then employ a standard reduction, as is apparent by comparing (4.258) with (4.248). We conclude that the Scherk-Schwarz reduction of massless type IIB gives the same 9-dimensional theory as the conventional reduction of the 'massive' formulation of type IIB. Thus, our results are consistent with [71, 72], and the formulation of type IIB that appears naturally in double field theory is already adapted to the Scherk-Schwarz reduction.

4.8 Conclusions

In this chapter we introduced a double field theory formulation for the low-energy limit of type II strings. T-duality relates different type II theories, a feature that does not occur in bosonic or heterotic string theory. In the double field theory built here each of the type II theories can be obtained by choosing different 'slicings' within the doubled coordinates. Consistent slicings are those allowed by the $O(D, D)$ covariant strong constraint $\partial^M \partial_M = 0$ that originates from the $L_0 - \bar{L}_0 = 0$ constraint of closed string theory. If we consider two slicings related by an odd number of T-duality inversions and one yields type IIA, the other must yield type IIB. The double field theory necessarily features the so-called type IIA* and type IIB* theories, which are related to the conventional type II theories via T-dualities along *timelike* directions.

Despite this unification, the actual invariance group of the theory is only $\mathrm{Spin}^+(D, D)$ and therefore does not contain any of the T-duality transformations that relate different type II theories. This means that the $\mathrm{Pin}(D, D)$ transformations that are not in $\mathrm{Spin}^+(D, D)$ must be viewed as dualities rather than invariances. More precisely, while we fix the chirality of the spinor χ from the outset, the opposite chirality is obtained by the field redefinition induced by the appropriate T-duality transformation.

In section 7 of this chapter, we have shown that the type II double field theory defined by (4.4) and (4.5) can be extended by slightly relaxing the strong constraint such that the RR fields may depend simultaneously on all 10-dimensional space-time coordinates and linearly on the winding coordinates. In case that only the RR 1-form carries such a dependence, the double field theory reduces precisely to the massive type IIA theory. We have shown that the T-dual configuration corresponds to the case that the RR 2-form (4.243) of type IIB carries such a dependence. This gives rise to a 'massive' version of type IIB, whose circle reduction to nine dimensions yields the same theory as the Scherk-Schwarz reduction of conventional type IIB. This massive formulation of type IIB is still invariant under 10-dimensional diffeomorphisms, with the 10th diffeomorphism being deformed by the mass parameter.

Chapter 5

$\mathcal{N} = 1$ Supersymmetric Double Field Theory

A bulk of this chapter appeared in "$\mathcal{N} = 1$ Supersymmetric Double Field Theory" with Olaf Hohm [15] and is reprinted with the permission of *JHEP*.

Summary : We construct the $\mathcal{N} = 1$ supersymmetric extension of double field theory for $D = 10$, including the coupling to an arbitrary number n of abelian vector multiplets. This theory features a local $O(1, 9+n) \times O(1, 9)$ tangent space symmetry under which the fermions transform. It is shown that the supersymmetry transformations close into the generalized diffeomorphisms of double field theory.

5.1 Introduction

In this chapter we construct the $\mathcal{N} = 1$ supersymmetric extension of double field theory for $D = 10$. This theory features two copies of the local Lorentz group as tangent space symmetries, under which the fermions naturally transform. Consequently, the formulation of double field theory that is most useful for our present purpose is the frame or vielbein formulation. As usual, we may introduce frame fields $E_M{}^A$, using the splitting $M = ({}_i\,, {}^i)$ of the $O(D, D)$ index and $A = (a, \bar{a})$ is the flat or frame index. In the frame formulation there is an $O(1, 9)_L \times O(1, 9)_R$ 'tangent space' gauge symmetry, with $a, b \ldots = 0, \ldots, 9$ and $\bar{a}, \bar{b} \ldots = 0, \ldots, 9$ denoting $O(1, 9)_L$ and

$O(1,9)_R$ vector indices, respectively. Such a frame formalism has been developed by Siegel prior to the generalized metric formulation [5]. Actually, Siegel's formalism allows also for the larger tangent space group $GL(D) \times GL(D)$, but here we will restrict to the Lorentz subgroups in order to be able to define the corresponding spinor representations. In this formalism one may introduce connections for the local frame symmetry and construct invariant curvatures. This, in turn, allows one to write an Einstein-Hilbert like action based on a generalized curvature scalar $\mathcal{R}$, which provides an equivalent definition of double field theory,

$$S = \int d^{10}x\, d^{10}\tilde{x}\, e^{-2d}\, \mathcal{R}(E,d)\,, \tag{5.1}$$

where we defined $e^{-2d} = \sqrt{g}e^{-2\phi}$. In the frame formulation the theory has a global $O(10,10)$ symmetry, a $O(1,9)_L \times O(1,9)_R$ gauge invariance and a 'generalized diffeomorphism' symmetry.

In this paper we will introduce fermions that, as usual in supergravity, are scalars under (generalized) diffeomorphisms and $O(10,10)$, but which transform under the local tangent space group $O(1,9)_L \times O(1,9)_R$. The fermionic sector of supergravity is thereby rewritten in a way that enlarges the local Lorentz group. Similar attempts have in fact a long history, going back to the work of de Wit and Nicolai in the mid 80's, in which they showed that 11-dimensional supergravity can be reformulated such that it permits an enhanced tangent space symmetry [75]. More recently, a very interesting paper appeared which showed in the context of generalized geometry that type II supergravity can be reformulated such that it permits a doubled Lorentz group [27], as in double field theory, and our results are closely related (see also [26]).

We will introduce a gravitino field Ψ_a that is a spinor under $O(1,9)_R$ and a vector under $O(1,9)_L$, together with a dilatino ρ, that is a spinor under $O(1,9)_R$. The minimally supersymmetric extension of (5.1) can then be written as

$$S_{\mathcal{N}=1} = \int d^{10}x\, d^{10}\tilde{x}\, e^{-2d}\Big(\mathcal{R}(E,d) - \bar{\Psi}^a\gamma^{\bar{b}}\nabla_{\bar{b}}\Psi_a + \bar{\rho}\gamma^{\bar{a}}\nabla_{\bar{a}}\rho + 2\bar{\Psi}^a\nabla_a\rho\Big)\ . \tag{5.2}$$

Here, the $\gamma^{\bar{a}}$ are ten-dimensional gamma matrices, which have to be thought of as gamma matrices of $O(1,9)_R$, so that all suppressed spinor indices in (5.1) are $O(1,9)_R$ spinor indices. Moreover, the covariant derivatives ∇ are with respect to the connections introduced by Siegel [5], and therefore the action is manifestly $O(1,9)_L \times O(1,9)_R$ invariant.

We will show that (5.2), up to field redefinitions, reduces precisely to the standard minimal $\mathcal{N}=1$ action in ten dimensions. In this paper we will not consider higher-order fermi terms. Formally, (5.2) is contained in the results of [27] through the straightforward truncation from $\mathcal{N}=2$ to $\mathcal{N}=1$. The main difference between generalized geometry, which was the starting point in [27], and double field theory is that in the former the coordinates are not doubled but only the tangent space. Consequently, in generalized geometry only the tangent space symmetry is enhanced, while double field theory features also a global $O(D,D)$ symmetry. With the fermions being singlets under $O(D,D)$, this symmetry is somewhat trivially realized on the fermionic sector, and therefore our results for the minimal $\mathcal{N}=1$ theory are to some extent contained in those of generalized geometry given in [27]. In the context of double field theory, however, it remains to verify closure of the supersymmetry transformations into generalized diffeomorphisms and supersymmetric invariance of (5.2), both modulo the $O(D,D)$ invariant constraint. This will be done in sec. 2 of this paper.

As the main new result, we will present in sec. 3 the double field theory extension of $\mathcal{N}=1$ supergravity in $D=10$ coupled to an arbitrary number n of (abelian) vector multiplets. For $n=16$ this is the low-energy effective action of heterotic superstring theory truncated to the Cartan subalgebra of $SO(32)$ or $E_8 \times E_8$. As has been shown in [11], the coupling of gauge vectors $A_i{}^\alpha$ can be neatly described by enlarging the generalized metric to an $O(10+n,10)$ matrix that naturally contains the $A_i{}^\alpha$. In the frame formulation this theory features, in addition, a $O(1,9+n) \times O(1,9)$ tangent space symmetry. The fermionic fields will still be spinors under $O(1,9)$, but Ψ_a is now a vector under $O(1,9+n)$. Remarkably, it turns out that the same action (5.2), but written with respect to these enlarged fields, reproduces precisely the $\mathcal{N}=1$ supergravity coupled to abelian vector multiplets, with the gauginos originating from the additional components of the Ψ_a. Apart from exhibiting a further 'unification' of the massless sector of heterotic superstring theory, this formulation provides a significant technical simplification of the effective action, as should be apparent by comparing (5.81) with (5.2). Moreover, the proof of supersymmetric invariance (up to the higher order fermi terms) is much simpler than in the standard formulation, being essentially reduced to a two-line calculation in (5.36).

5.2 Minimal $\mathcal{N}=1$ Double Field Theory for $D=10$

In this section we introduce the minimal $\mathcal{N}=1$ theory. First, we review the vielbein formalism with local $O(1,9)_L \times O(1,9)_R$ symmetry. Second, we introduce the $\mathcal{N}=1$ double field theory and prove its supersymmetric invariance. In the third subsection we verify that it reduces to conventional $\mathcal{N}=1$ supergravity upon setting the new derivatives to zero.

5.2.1 Vielbein formulation with local $O(1,9)\times O(1,9)$ symmetry

We start by reviewing some generalities on the vielbein formulation of double field theory, which is contained in Siegel's frame formalism [5]. We refer to [10] for a self-contained presentation of this formulation. The fundamental bosonic fields are the frame field $E_A{}^M$ and the dilaton d that depend both on doubled coordinates $X^M=(\tilde{x}_i,x^i)$. The frame field is subject to local $O(1,9)_L\times O(1,9)_R$ transformations acting on the index $A=(a,\bar{a})$ and global $O(10,10)$ transformations acting on the index M, which read infinitesimally

$$\delta E_A{}^M \;=\; k^M{}_N\,\Lambda_A{}^B(X)\,E_B{}^N\;, \qquad k\;\in\;\mathfrak{o}(10,10)\;,\quad \Lambda(X)\;\in\;\mathfrak{o}(1,9)_L\oplus\mathfrak{o}(1,9)_R\;, \tag{5.3}$$

where the parameters take values in the respective Lie algebras. The double field theory is invariant under a 'generalized diffeomorphism' symmetry parameterized by $\xi^M=(\tilde{\xi}_i,\xi^i)$ that combines the b-field 1-form gauge parameter $\tilde{\xi}_i$ with the vectorvalued diffeomorphism parameter ξ^i,

$$\delta_\xi E_A{}^M \;=\; \widehat{\mathcal{L}}_\xi E_A{}^M \;\equiv\; \xi^N\partial_N E_A{}^M + \big(\partial^M\xi_N - \partial_N\xi^M\big)E_A{}^N \;. \tag{5.4}$$

Here, $\partial_M = (\tilde{\partial}^i, \partial_i)$ are the doubled partial derivatives. The right-hand side of (5.4) defines a generalized Lie derivative that can similarly be defined for an $O(D, D)$ tensor with an arbitrary number of upper and lower indices. On the dilaton d these gauge transformations read

$$\delta_\xi d \;=\; \xi^M\partial_M d - \frac{1}{2}\partial_M\xi^M \;. \tag{5.5}$$

The gauge transformations close and leave the action invariant modulo the 'strong constraint'

$$\eta^{MN}\partial_M\partial_N \;=\; 0\;, \qquad \eta^{MN} \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\;, \tag{5.6}$$

when acting on arbitrary fields and parameters and all their products. Here, η_{MN} denotes the $O(10, 10)$ invariant metric, which will be used to raise and lower $O(10, 10)$ indices. This constraint implies that locally all fields depend only on half of the coordinates, for instance only on the x^i.

We have to impose covariant constraints on the frame field in order to describe only the physical degrees of freedom. These constraints are written in terms of the tangent space metric

$$\mathcal{G}_{AB} \;\equiv\; E_A{}^M E_B{}^N \eta_{MN}\;, \tag{5.7}$$

resulting from the $O(10, 10)$ invariant metric η, and which will be used to raise and lower flat indices. We require the $O(1, 9)_L \times O(1, 9)_R$ covariant constraints

$$\mathcal{G}_{a\bar{b}} \;=\; 0\;, \qquad \mathcal{G}_{ab} \;=\; \eta_{ab}\;, \qquad \mathcal{G}_{\bar{a}\bar{b}} \;=\; -\eta_{\bar{a}\bar{b}}\;. \tag{5.8}$$

Note that the relative minus sign entering here is necessary due to the $(10, 10)$ signature of $\mathcal{G}_{AB}$. It is a matter of convention to which metric we assign the minus sign, but once the choice is made the symmetry between unbarred and barred indices is broken. Since flat indices are raised and lowered with $\mathcal{G}_{AB}$, (5.8) leads to some unconventional signs when comparing below to standard expressions for, say, the spin connection. We will comment on this in due course.

A particular solution of these constraints, giving rise to the generalized metric according to

$$\mathcal{H}_{MN} = E_M{}^A E_N{}^B \hat{\eta}_{AB} , \qquad \hat{\eta}_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix} , \tag{5.9}$$

is given by

$$E_A{}^M = \begin{pmatrix} E_{ai} & E_a{}^i \\ E_{\bar{a}i} & E_{\bar{a}}{}^i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_{ia} + b_{ij} e_a{}^j & e_a{}^i \\ -e_{i\bar{a}} + b_{ij} e_{\bar{a}}{}^j & e_{\bar{a}}{}^i \end{pmatrix} , \tag{5.10}$$

where e is the vielbein of the conventional metric, $g = e\,\eta\,e^T$. We stress that when writing (5.10) the tangent space symmetry is gauge-fixed to the diagonal subgroup of $O(1,9)_L \times O(1,9)_R$, as is clear from the fact that e carries in (5.10) both unbarred and barred indices. In order to define the supersymmetric double field theory, however, (5.10) is never used. Rather, we view the (constrained) vielbein $E_A{}^M$ as the fundamental field and so the construction is manifestly invariant under two copies of the local Lorentz group. It is only when comparing to the standard formulation of supergravity that we have to use (5.10) and to partially gauge-fix.[1]

Let us now turn to the definition of connections and covariant derivatives. We first note that the partial derivative of a field S that transforms as a scalar under ξ^M, i.e.,

$$\delta_\xi S = \xi^M \partial_M S , \tag{5.11}$$

transforms covariantly with a generalized Lie derivative [10]. This does not hold for higher tensors, which in turn necessitates the introduction of covariant derivatives.

[1]This differs from the construction in [27] and [25,26], where two independent vielbein fields are introduced, one transforming under $O(1,9)_L$ and one transforming under $O(1,9)_R$. Then, however, the constraint should be imposed that both vielbein fields give rise to the same space-time metric g_{ij}, i.e., that they are equal up to local Lorentz transformations, and it is not obvious to us how this should be done in an $O(D,D)$ covariant way.

Given the frame field ${E_A}^M$, we introduce the 'flattened' partial derivative[2]

$$E_A \equiv \sqrt{2}{E_A}^M \partial_M \,. \tag{5.12}$$

We can then introduce $O(1,9)_L \times O(1,9)_R$ covariant derivatives

$$\nabla_A V_B = E_A V_B + {\omega_{AB}}^C V_C \,, \qquad \nabla_A V^B = E_A V^B - {\omega_{AC}}^B V^C \,, \tag{5.13}$$

where we stress that the only non-trivial connections are ${\omega_{Ab}}^c$ and ${\omega_{A\bar{b}}}^{\bar{c}}$.

Next, we briefly summarize which connection components can be determined in terms of ${E_A}^M$ and d upon imposing covariant constraints. First, in order to be compatible with the constancy of the tangent space metric $\mathcal{G}_{AB}$, the symmetric part $\omega_{A(BC)}$, where indices have been lowered with $\mathcal{G}$, is zero. Thus, ω_{ABC} is antisymmetric in its last two indices. Second, we can impose a generalized torsion constraint, which reads

$$\mathcal{T}_{ABC} \equiv \Omega_{ABC} + 3\omega_{[ABC]} = 0 \,, \tag{5.14}$$

where we introduced the 'generalized coefficients of anholonomy'

$$\Omega_{ABC} = 3f_{[ABC]} \,, \qquad f_{ABC} \equiv (E_A {E_B}^M) E_{CM} \,. \tag{5.15}$$

We note that f_{ABC} is antisymmetric in its last two indices as a consequence of the constancy of $\mathcal{G}_{AB}$. Specializing the constraint (5.14) to $\mathcal{T}_{a\bar{b}\bar{c}} = 0$ and $\mathcal{T}_{\bar{a}bc} = 0$, we derive the following solution for the 'off-diagonal' components

$$\omega_{a\bar{b}\bar{c}} = -\Omega_{a\bar{b}\bar{c}} \,, \qquad \omega_{\bar{a}bc} = -\Omega_{\bar{a}bc} \,. \tag{5.16}$$

For later use let us determine these connection components for the gauge choice (5.10)

[2]Here we introduced a factor of $\sqrt{2}$ for later convenience. With the constraints on the connections to be imposed below, the covariant derivatives ∇_A given here are $\sqrt{2}$ times the covariant derivatives in [10].

of the frame field, setting $\tilde{\partial}^i = 0$. We compute with (5.15)

$$f_{a\bar{b}\bar{c}} = e_a{}^i e_{[\bar{b}}{}^j \partial_i e_{j\bar{c}]} + \frac{1}{2} e_a{}^i e_{\bar{b}}{}^k e_{\bar{c}}{}^j \partial_i b_{jk} , \qquad f_{\bar{b}a\bar{c}} = e_{\bar{b}}{}^i e_{(a}{}^j \partial_i e_{j\bar{c})} + \frac{1}{2} e_{\bar{b}}{}^i e_a{}^k e_{\bar{c}}{}^j \partial_i b_{jk} , \tag{5.17}$$

from which we derive

$$\omega_{a\bar{b}\bar{c}} = -\omega^{\mathrm{L}}_{a\bar{b}\bar{c}}(e) + \frac{1}{2} e_a{}^i e_{\bar{b}}{}^j e_{\bar{c}}{}^k H_{ijk} , \tag{5.18}$$

where ω^{L} denotes the standard Levi-Civita spin connection expressed in terms of the vielbein,

$$\omega^{\mathrm{L}}_{a\bar{b}\bar{c}}(e) = e_{[a}{}^i e_{\bar{b}]}{}^j \partial_i e_{j\bar{c}} - e_{[\bar{b}}{}^i e_{\bar{c}]}{}^j \partial_i e_{ja} + e_{[\bar{c}}{}^i e_{a]}{}^j \partial_i e_{j\bar{b}} . \tag{5.19}$$

Similarly, one finds

$$\omega_{\bar{a}bc} = \omega^{\mathrm{L}}_{\bar{a}bc}(e) + \frac{1}{2} H_{\bar{a}bc} , \tag{5.20}$$

where we flattened the indices of H as in (5.18).[3]

For the 'diagonal' components, having either only unbarred or barred indices, the totally antisymmetric parts are determined by (5.14) as follows

$$\omega_{[abc]} = -\frac{1}{3}\Omega_{[abc]} = -f_{[abc]} , \qquad \omega_{[\bar{a}\bar{b}\bar{c}]} = -\frac{1}{3}\Omega_{[\bar{a}\bar{b}\bar{c}]} = -f_{[\bar{a}\bar{b}\bar{c}]} . \tag{5.21}$$

Again, we may determine these connections for the gauge choice (5.10) and $\tilde{\partial}^i = 0$. One finds,

$$\omega_{[abc]} = \omega^{\mathrm{L}}_{[abc]}(e) + \frac{1}{6} H_{abc} , \qquad \omega_{[\bar{a}\bar{b}\bar{c}]} = -\omega^{\mathrm{L}}_{[\bar{a}\bar{b}\bar{c}]}(e) + \frac{1}{6} H_{\bar{a}\bar{b}\bar{c}} , \tag{5.22}$$

where we flattened the indices on H.

The torsion constraint leaves the mixed Young tableaux representation in ω_{abc} and $\omega_{\bar{a}\bar{b}\bar{c}}$ undetermined, but its trace part can be fixed by imposing a covariant constraint

[3] We note that the relative sign between $\omega_{a\bar{b}\bar{c}}$ and $\omega^{\mathrm{L}}_{a\bar{b}\bar{c}}$ in (5.18) is due to the fact that we lower barred indices with $\mathcal{G}_{\bar{a}\bar{b}} = -\eta_{\bar{a}\bar{b}}$, see eq. (5.8), while in the standard expression (5.19) for the spin connection the index is lowered with $\eta_{\bar{a}\bar{b}}$. Correspondingly, there is no relative sign in (5.20) because here indices are lowered with $\mathcal{G}_{ab} = \eta_{ab}$.

that allows for partial integration in presence of the dilaton density,

$$\int e^{-2d} V \nabla_A V^A \;=\; -\int e^{-2d} V^A \nabla_A V \,, \tag{5.23}$$

for arbitrary V and V^A. This implies

$$\omega_{BA}{}^{B} \;=\; -\tilde{\Omega}_A \;\equiv\; -\sqrt{2} e^{2d} \partial_M \big(E_A{}^M e^{-2d} \big) \,, \tag{5.24}$$

where we introduced $\tilde{\Omega}_A$ for later use. Note that this determines precisely $\omega_{ba}{}^{b}$ and $\omega_{\bar{b}\bar{a}}{}^{\bar{b}}$, because the last two indices cannot be mixed.

Finally, we can introduce an invariant scalar curvature and Ricci tensor. In the frame formalism there is an invariant curvature tensor $\mathcal{R}_{ABCD}$, but it is generally not a function of the determined connections only. For the derived curvature scalar and Ricci tensor, however, it depends only on the determined connections. Without repeating the details of the construction, we give the explicit expressions.

The scalar curvature can be defined as the trace over, say, barred indices as follows

$$\begin{aligned} \mathcal{R} \;\equiv\; -\mathcal{R}_{\bar{a}\bar{b}}{}^{\bar{a}\bar{b}} \;&=\; -2 E_{\bar{a}} \omega_{\bar{b}}{}^{\bar{a}\bar{b}} - \frac{3}{2} \omega_{[\bar{a}\bar{b}\bar{c}]} \, \omega^{[\bar{a}\bar{b}\bar{c}]} + \omega_{\bar{a}}{}^{\bar{c}\bar{a}} \, \omega_{\bar{b}\bar{c}}{}^{\bar{b}} - \frac{1}{2} \omega_{a\bar{b}\bar{c}} \, \omega^{a\bar{b}\bar{c}} \\ &=\; 2 E_{\bar{a}} \tilde{\Omega}^{\bar{a}} + \tilde{\Omega}_{\bar{a}}^2 - \frac{1}{2} \Omega_{\bar{a}\bar{b}c}{}^2 - \frac{1}{6} \Omega_{[\bar{a}\bar{b}\bar{c}]}{}^2 \,, \end{aligned} \tag{5.25}$$

where we have written in the second line the explicit expression in terms of Ω and thereby in terms of the physical fields. The Ricci tensor reads

$$\mathcal{R}_{a\bar{b}} \;=\; E_{\bar{c}} \omega_{a\bar{b}}{}^{\bar{c}} - E_a \omega_{\bar{c}\bar{b}}{}^{\bar{c}} + \omega_{d\bar{b}}{}^{\bar{c}} \, \omega_{\bar{c}a}{}^{d} - \omega_{a\bar{b}}{}^{\bar{d}} \, \omega_{\bar{c}\bar{d}}{}^{\bar{c}} \,. \tag{5.26}$$

These curvature invariants can be obtained by variation of the (bosonic) double field theory action. In order to see this it is convenient to introduce the variation

$$\Delta E_{AB} \;:=\; E_B{}^M \delta E_{AM} \,. \tag{5.27}$$

Under the local $O(1,9)_L \times O(1,9)_R$ this variation reads $\Delta E_{ab} = \Lambda_{ab}$ and $\Delta E_{\bar{a}\bar{b}} = \Lambda_{\bar{a}\bar{b}}$. Thus, only the off-diagonal variation is not pure-gauge and the corresponding general variation of the action (5.1) can be written in terms of the curvatures as

$$\delta S \;=\; -2\int dx d\tilde{x}\, e^{-2d}\left(\delta d\,\mathcal{R} + \Delta E_{a\bar{b}}\,\mathcal{R}^{a\bar{b}}\right)\,, \tag{5.28}$$

which will be used below.

5.2.2 $\mathcal{N}=1$ Double Field Theory

We give now the $\mathcal{N}=1$ supersymmetric extension of double field theory in the frame formulation reviewed above. The fermionic fields are the 'gravitino' ψ_a and the 'dilatino' ρ, and we will later see how they are related to the conventional gravitino and dilatino via a field redefinition. These fields are scalars under $O(10,10)$ and generalized diffeomorphisms and, together with the $\mathcal{N}=1$ supersymmetry parameter ϵ, transform under the local $O(1,9)_L \times O(1,9)_R$ as follows

$$\begin{aligned}
\Psi_a &: \quad \text{vector of } O(1,9)_L\,, \;\text{ spinor of } O(1,9)_R\,,\\
\rho &: \quad \text{spinor of } O(1,9)_R\,,\\
\epsilon &: \quad \text{spinor of } O(1,9)_R\,.
\end{aligned} \tag{5.29}$$

The $\mathcal{N}=1$ supersymmetric extension of (5.1) is given by (5.2),

$$S_{\mathcal{N}=1} \;=\; \int dx d\tilde{x}\, e^{-2d}\left(\mathcal{R}(E,d) - \bar{\Psi}^a\gamma^{\bar{b}}\nabla_{\bar{b}}\Psi_a + \bar{\rho}\gamma^{\bar{a}}\nabla_{\bar{a}}\rho + 2\bar{\Psi}^a\nabla_a\rho\right)\,, \tag{5.30}$$

where all covariant derivatives are with respect to the connections introduced above. We will see below that in here and in the supersymmetry rules all undetermined connections drop out. When acting on $O(1,9)_R$ spinors the covariant derivatives are given by

$$\nabla_a \;=\; E_a - \frac{1}{4}\omega_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\,, \qquad \nabla_{\bar{a}} \;=\; E_{\bar{a}} - \frac{1}{4}\omega_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\,. \tag{5.31}$$

We observe that (5.30) is manifestly $O(1,9)_L \times O(1,9)_R$ invariant, because unbarred and barred indices are properly contracted, and the $\gamma^{\bar{a}}$ are gamma matrices of $O(1,9)_R$, so that all suppressed spinor indices belong to $O(1,9)_R$. More precisely, we define the

$\gamma^{\bar{a}}$ to satisfy

$$\{\gamma^{\bar{a}}, \gamma^{\bar{b}}\} \;=\; -2\mathcal{G}^{\bar{a}\bar{b}} \;=\; 2\eta^{\bar{a}\bar{b}}\,, \tag{5.32}$$

where the signs are such that the $\gamma^{\bar{a}}$ can be chosen to be conventional gamma matrices in ten dimensions. We note that, according to our convention, on $\gamma_{\bar{a}}$ the index is lowered with $\mathcal{G}_{\bar{a}\bar{b}} = -\eta_{\bar{a}\bar{b}}$ so that it differs from the conventional ten-dimensional gamma matrix with a lower index by a sign. Similarly, the minus signs in (5.31) are due to the lowering of indices on $\omega_{A\bar{b}\bar{c}}$ with $\mathcal{G}_{\bar{a}\bar{b}}$. Let us finally stress that the assignment (5.29) of $O(1,9)_L \times O(1,9)_R$ representations is related to the constraint (5.8). We could have chosen the opposite signatures for $\mathcal{G}_{ab}$ and $\mathcal{G}_{\bar{a}\bar{b}}$, but then supersymmetry would require the gravitino to be a vector under $O(1,9)_R$ and a spinor under $O(1,9)_L$.

The action (5.30) is manifestly invariant under generalized diffeomorphisms,

$$\begin{aligned} \delta_\xi E_A{}^M \;&=\; \widehat{\mathcal{L}}_\xi E_A{}^M\,, \qquad & \delta_\xi d \;&=\; \xi^M\partial_M d - \frac{1}{2}\partial_M\xi^M\,, \\ \delta_\xi \Psi_a \;&=\; \xi^M\partial_M\Psi_a\,, \qquad & \delta_\xi\rho \;&=\; \xi^M\partial_M\rho\,, \end{aligned} \tag{5.33}$$

because with the fermions transforming as scalars the (flattened) derivatives in (5.30) transform covariantly. Moreover, the action is invariant under the $\mathcal{N}=1$ supersymmetry transformations

$$\begin{aligned} \Delta_\epsilon E_{a\bar{b}} \;&=\; -\frac{1}{2}\bar{\epsilon}\,\gamma_{\bar{b}}\Psi_a\,, \qquad & \delta_\epsilon d \;&=\; -\frac{1}{4}\bar{\epsilon}\rho\,, \\ \delta_\epsilon\Psi_a \;&=\; \nabla_a\epsilon\,, \qquad & \delta_\epsilon\rho \;&=\; \gamma^{\bar{a}}\nabla_{\bar{a}}\epsilon\,. \end{aligned} \tag{5.34}$$

Here, we have written the transformation of the frame field in terms of the variation (5.27). Due to the $O(1,9)_L \times O(1,9)_R$ gauge freedom, we can assume for the diagonal supersymmetry variations $\Delta_\epsilon E_{ab} = \Delta_\epsilon E_{\bar{a}\bar{b}} = 0$.

Let us now verify that (5.30) is invariant under (5.34), again up to higher-order fermi terms. We start with the variation of the bosonic part, which can be obtained directly by inserting the fermionic supersymmetry rules of (5.34) into (5.28),

$$\delta_\epsilon\mathcal{L}_{\rm B} \;=\; \frac{1}{2}\bar{\epsilon}\rho\mathcal{R} + \bar{\epsilon}\gamma_{\bar{b}}\Psi_a\mathcal{R}^{a\bar{b}}\,, \tag{5.35}$$

where we denoted the bosonic Lagrangian (without the density e^{-2d}) by $\mathcal{L}_{\rm B}$. Denoting the fermionic part similarly by $\mathcal{L}_{\rm F}$, one finds

$$\begin{aligned}\delta_\epsilon\mathcal{L}_{\rm F} &= -2\bar{\Psi}^a\gamma^{\bar{b}}\nabla_{\bar{b}}\nabla_a\epsilon + 2\bar{\rho}\gamma^{\bar{a}}\nabla_{\bar{a}}\big(\gamma^{\bar{b}}\nabla_{\bar{b}}\epsilon\big) + 2\nabla^a\bar{\epsilon}\,\nabla_a\rho + 2\bar{\Psi}^a\nabla_a\big(\gamma^{\bar{b}}\nabla_{\bar{b}}\epsilon\big) \\ &= -2\bar{\Psi}^a\big[\gamma^{\bar{b}}\nabla_{\bar{b}},\nabla_a\big]\epsilon + 2\bar{\rho}\,\Big(\gamma^{\bar{a}}\nabla_{\bar{a}}\gamma^{\bar{b}}\nabla_{\bar{b}} - \nabla^a\nabla_a\Big)\,\epsilon\,.\end{aligned} \tag{5.36}$$

Here we have used that according to (5.23) the covariant derivatives allow us to freely partially integrate in presence of the dilaton density. Moreover, in the second line we have combined the first and last and the second and third term. We can now use the identities [27]

$$\begin{aligned}\Big(\gamma^{\bar{a}}\nabla_{\bar{a}}\gamma^{\bar{b}}\nabla_{\bar{b}} - \nabla^a\nabla_a\Big)\,\epsilon &= -\frac{1}{4}\mathcal{R}\epsilon\,, \\ \Big[\gamma^{\bar{b}}\nabla_{\bar{b}},\,\nabla_a\Big]\,\epsilon &= -\frac{1}{2}\gamma^{\bar{b}}\mathcal{R}_{a\bar{b}}\epsilon\,,\end{aligned} \tag{5.37}$$

which will be proved in the appendix, to see that this cancels precisely the variation (5.35) of the bosonic term, proving supersymmetric invariance.

We turn now to the closure of the supersymmetry transformations. Since these are an invariance of the action (5.30) they must close into the other local symmetries of the theory, which are generalized diffeomorphisms and the doubled local Lorentz transformations $O(1,9)_L \times O(1,9)_R$. It is instructive, however, to investigate this explicitly, and so we verify in the following closure on the bosonic fields. For the dilaton we compute

$$\big[\delta_{\epsilon_1},\delta_{\epsilon_2}\big]d \;=\; \frac{1}{4}\big(\bar{\epsilon}_1\gamma^{\bar{a}}\nabla_{\bar{a}}\epsilon_2 - \bar{\epsilon}_2\gamma^{\bar{a}}\nabla_{\bar{a}}\epsilon_1\big) \;=\; \frac{1}{4}\bar{\epsilon}_1\gamma^{\bar{a}}\big(E_{\bar{a}} - \frac{1}{4}\omega_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\big)\epsilon_2 - (1\leftrightarrow 2)\,. \tag{5.38}$$

Let us work out the first term in here,

$$\frac{1}{4}\bar{\epsilon}_1\gamma^{\bar{a}}E_{\bar{a}}\epsilon_2 - (1\leftrightarrow 2) \;=\; \frac{\sqrt{2}}{4}\bar{\epsilon}_1\gamma^{\bar{a}}E_{\bar{a}}{}^M\partial_M\epsilon_2 - (1\leftrightarrow 2) \;=\; \frac{1}{2\sqrt{2}}E_{\bar{a}}{}^M\partial_M\big(\bar{\epsilon}_1\gamma^{\bar{a}}\epsilon_2\big)\,, \tag{5.39}$$

using $\bar{\epsilon}_1\gamma^{\bar{a}}\epsilon_2 = -\bar{\epsilon}_2\gamma^{\bar{a}}\epsilon_1$. For the second term we compute

$$-\frac{1}{16}\omega_{\bar{a}\bar{b}\bar{c}}\bar{\epsilon}_1\gamma^{\bar{a}}\gamma^{\bar{b}\bar{c}}\epsilon_2 - (1\leftrightarrow 2) \;=\; -\frac{1}{16}\omega_{\bar{a}\bar{b}\bar{c}}\bar{\epsilon}_1\big(\gamma^{\bar{a}\bar{b}\bar{c}} - 2\mathcal{G}^{\bar{a}[\bar{b}}\gamma^{\bar{c}]}\big)\epsilon_2 - (1\leftrightarrow 2)\,. \tag{5.40}$$

The first term in here vanishes due to the antisymmetrization in $(1 \leftrightarrow 2)$ and $\bar{\epsilon}_1\gamma^{\bar{a}\bar{b}\bar{c}}\epsilon_2 = \bar{\epsilon}_2\gamma^{\bar{a}\bar{b}\bar{c}}\epsilon_1$. The second term gives with (5.24)

$$-\frac{1}{4}\omega_{\bar{a}\bar{c}}{}^{\bar{a}}\bar{\epsilon}_1\gamma^{\bar{c}}\epsilon_2 \;=\; \frac{1}{2\sqrt{2}}\big(\partial_M E_{\bar{c}}{}^M - 2E_{\bar{c}}{}^M\partial_M d\big)\bar{\epsilon}_1\gamma^{\bar{c}}\epsilon_2\,. \tag{5.41}$$

The first term in here combines with (5.39) to give $\frac{1}{2\sqrt{2}}\partial_M(E_{\bar{c}}{}^M\bar{\epsilon}_1\gamma^{\bar{c}}\epsilon_2)$. The second term takes the form of a transport term so that we have shown in total

$$\big[\delta_{\epsilon_1},\delta_{\epsilon_2}\big]d \;=\; \xi^M\partial_M d - \frac{1}{2}\partial_M\xi^M\,, \qquad \xi^M \;=\; -\frac{1}{\sqrt{2}}E_{\bar{a}}{}^M\,\bar{\epsilon}_1\gamma^{\bar{a}}\epsilon_2\,. \tag{5.42}$$

Thus, the supersymmetry transformations close into generalized diffeomorphisms, as required.

Next, we verify closure on $E_A{}^M$. We compute

$$\begin{aligned}\big[\delta_{\epsilon_1},\delta_{\epsilon_2}\big]E_{aM} \;&=\; \delta_{\epsilon_1}\big(E_M{}^B E_B{}^N\delta_{\epsilon_2}E_{aN}\big) - (1\leftrightarrow 2)\\ &=\; \delta_{\epsilon_1}\big(E_M{}^{\bar{b}}\Delta_{\epsilon_2}E_{a\bar{b}}\big) - (1\leftrightarrow 2) \;=\; -\frac{1}{2}\delta_{\epsilon_1}\big(E_M{}^{\bar{c}}\bar{\epsilon}_2\gamma_{\bar{c}}\Psi_a\big) - (1\leftrightarrow 2)\,,\end{aligned} \tag{5.43}$$

where we used that we can set $\Delta_\epsilon E_{ab} = 0$ by an appropriate $O(1,9)_L$ transformation, and we relabeled an index in the last equality. In order to disentangle the generalized diffeomorphisms and local $O(1,9)_L \times O(1,9)_R$ transformations we project (5.43) by multiplying with $E_{\bar{b}}{}^M$ and $E_b{}^M$, respectively. For the first we obtain

$$\begin{aligned}E_{\bar{b}}{}^M\big[\delta_{\epsilon_1},\delta_{\epsilon_2}\big]E_{aM} \;&=\; -\frac{1}{2}E_{\bar{b}}{}^M E_M{}^{\bar{c}}\bar{\epsilon}_2\gamma_{\bar{c}}\nabla_a\epsilon_1 - (1\leftrightarrow 2)\\ &=\; -\frac{1}{2}\bar{\epsilon}_2\gamma_{\bar{b}}\big(\sqrt{2}E_a{}^N\partial_N - \frac{1}{4}\omega_{a\bar{c}\bar{d}}\gamma^{\bar{c}\bar{d}}\big)\epsilon_1 - (1\leftrightarrow 2)\,,\end{aligned} \tag{5.44}$$

where we used that only the variation of Ψ_a is non-trivial as a consequence of $\Delta_\epsilon E_{\bar{a}\bar{b}} =$

0. The first term in here reads

$$-\frac{1}{\sqrt{2}}\big(\bar{\epsilon}_2\gamma_{\bar{b}}\partial_N\epsilon_1 - \bar{\epsilon}_1\gamma_{\bar{b}}\partial_N\epsilon_2\big)E_a{}^N \;=\; \frac{1}{\sqrt{2}}\partial_N\big(\bar{\epsilon}_1\gamma_{\bar{b}}\epsilon_2\big)E_a{}^N\,. \tag{5.45}$$

For the second term we use as above that the $\gamma^{(3)}$ structure drops due to the antisymmetrization in $(1 \leftrightarrow 2)$. The remaining structure proportional to $\gamma^{(1)}$ is then automatically antisymmetric in $(1 \leftrightarrow 2)$ and thus reads

$$-\frac{1}{2}\omega_{a\bar{c}\bar{d}}\,\bar{\epsilon}_2\,\delta_{\bar{b}}{}^{[\bar{c}}\gamma^{\bar{d}]}\epsilon_1 \;=\; \frac{1}{2}\omega_{a\bar{b}\bar{c}}\,\bar{\epsilon}_1\,\gamma^{\bar{c}}\epsilon_2\,. \tag{5.46}$$

The spin connection is given by

$$\omega_{a\bar{b}\bar{c}} \;=\; -3f_{[a\bar{b}\bar{c}]} \;=\; \sqrt{2}\left(E_a{}^K E_{\bar{b}}{}^N\partial_K E_{\bar{c}N} - E_{\bar{b}}{}^K\partial_K E_{\bar{c}}{}^N E_{aN} - E_{\bar{c}}{}^K E_{\bar{b}}{}^N\partial_K E_{aN}\right)\,. \tag{5.47}$$

Inserting this into (5.46) and combining with (5.45) we obtain in total

$$E_{\bar{b}}{}^M\big[\delta_{\epsilon_1},\delta_{\epsilon_2}\big]E_{aM} \;=\; E_{\bar{b}}{}^M\big(\xi^N\partial_N E_{aM} + \big(\partial_M\xi^N - \partial^N\xi_M\big)E_{aN}\big)\,, \tag{5.48}$$

where

$$\xi^M \;=\; -\frac{1}{\sqrt{2}}E_{\bar{a}}{}^M\bar{\epsilon}_1\,\gamma^{\bar{a}}\epsilon_2\,, \tag{5.49}$$

is the same parameter as in (5.42).

Next, we turn to the other projection,

$$\begin{aligned} E_b{}^M\big[\delta_{\epsilon_1},\delta_{\epsilon_2}\big]E_{aM} \;&=\; -\frac{1}{2}E_b{}^M\delta_{\epsilon_1}E_M{}^{\bar{c}}\,\bar{\epsilon}_2\gamma_{\bar{c}}\Psi_a - (1\leftrightarrow 2) \\ &=\; \frac{1}{2}\Delta_{\epsilon_1}E_{b\bar{c}}\,\bar{\epsilon}_2\gamma^{\bar{c}}\Psi_a - (1\leftrightarrow 2) \;=\; \frac{1}{2}\big(\bar{\epsilon}_1\gamma_{\bar{c}}\Psi_{[a}\big)\big(\bar{\epsilon}_2\gamma^{\bar{c}}\Psi_{b]}\big)\,. \end{aligned} \tag{5.50}$$

The last term is antisymmetric in a, b and can thus be interpreted as a field-dependent $O(1,9)_L$ gauge transformation. Here we would have expected also a generalized diffeomorphism with parameter (5.49), but for this particular projection such a term can actually be absorbed into an $O(1,9)_L$ gauge transformation. To show this it suffices to note that by definition (5.4)

$$E_b{}^M \delta_\xi E_{aM} \;=\; \xi^N E_b{}^M \partial_N E_{aM} - 2\partial_M \xi_N E_{[a}{}^M E_{b]}{}^N \;, \tag{5.51}$$

is antisymmetric in a, b. Thus, equivalently, (5.50) closes into the required generalized diffeomorphisms and into local $O(1,9)_L$ transformations with parameter

$$\Lambda_{ab} \;=\; \frac{1}{2}\big(\bar{\epsilon}_1 \gamma_{\bar{c}} \Psi_{[a}\big)\big(\bar{\epsilon}_2 \gamma^{\bar{c}} \Psi_{b]}\big) + \xi^N E_{[a}{}^M \partial_N E_{b]M} + 2\partial_M \xi_N E_{[a}{}^M E_{b]}{}^N \;, \tag{5.52}$$

with ξ^M given by (5.49). In total, combining (5.48) and (5.50), we have verified closure,

$$\big[\delta_{\epsilon_1}, \delta_{\epsilon_2}\big] E_{aM} \;=\; \widehat{\mathcal{L}}_\xi E_{aM} + \Lambda_a{}^b E_{bM} \;, \tag{5.53}$$

with parameters given by (5.49) and (5.52). The verification for $E_{\bar{a}M}$ is completely analogous. In particular, the corresponding $O(1,9)_R$ parameter is given by

$$\Lambda_{\bar{a}\bar{b}} \;=\; \frac{1}{2}\big(\bar{\epsilon}_1 \gamma_{[\bar{a}} \Psi^c\big)\big(\bar{\epsilon}_2 \gamma_{\bar{b}]} \Psi_c\big) + \xi^N E_{[\bar{a}}{}^M \partial_N E_{\bar{b}]M} + 2\partial_M \xi_N E_{[\bar{a}}{}^M E_{\bar{b}]}{}^N \;. \tag{5.54}$$

In general, the supersymmetry transformations close according to

$$\big[\delta_{\epsilon_1}, \delta_{\epsilon_2}\big] \;=\; \widehat{\mathcal{L}}_\xi + \delta_\Lambda + \delta_{\bar{\Lambda}} \;, \tag{5.55}$$

with ξ given by (5.49), Λ by (5.52) and $\bar{\Lambda}$ by (5.54). We finally note that even though we have not employed the field equations for the above computation, in general the gauge algebra (5.55) will only hold on-shell. In fact, without auxiliary fields supersymmetry transformations close on the fermions only modulo their field equations. In contrast, for the bosons the field equations do not enter on dimensional grounds, because they are second-order in derivatives.

5.2.3 Reduction to standard $\mathcal{N} = 1$ supergravity

Let us now verify that the action (5.30) and the supersymmetry rules (5.34) reduce to the conventional $\mathcal{N} = 1$ supergravity in $D = 10$ upon setting $\tilde{\partial}^i = 0$. As discussed above, this comparison requires a partial gauge fixing of the local $O(1,9)_L \times O(1,9)_R$ to the diagonal subgroup. We can then write the frame field as in (5.10) in terms of b_{ij} and the conventional vielbein $e_i{}^a$. In the following we will show that the conventional $\mathcal{N} = 1$ theory is related to the action following from (5.30) by a field redefinition.

We start by recalling minimal $\mathcal{N} = 1$, $D = 10$ supergravity in the string frame. The field content is given by

$$(e_i{}^a \; , \; b_{ij} \; , \; \phi \; , \; \psi_i \; , \; \lambda) \; , \tag{5.56}$$

where the fermionic fields are the gravitino ψ_i and the dilatino λ. The action reads[4]

$$\begin{aligned} S \;=\; \int d^{10}x \, e \, e^{-2\phi} \Big[\Big(& R + 4\partial^i\phi\,\partial_i\phi - \frac{1}{12} H^{ijk} H_{ijk} \Big) \\ & - \bar{\psi}_i \gamma^{ijk} D_j \psi_k + 2\bar{\psi}^i (\partial_i\phi)\gamma^j \psi_j - 2\bar{\lambda}\gamma^i D_i \lambda - \bar{\psi}_i (\partial\!\!\!/ \phi)\gamma^i \lambda \\ & + \frac{1}{24} H_{ijk} \left(\bar{\psi}_m \gamma^{mijkn} \psi_n + 6\bar{\psi}^i \gamma^j \psi^k - 2\bar{\psi}_m \gamma^{ijk}\gamma^m \lambda \right) \Big] \; , \end{aligned} \tag{5.57}$$

where $H_{ijk} = 3\partial_{[i} b_{jk]}$ and $e = \det(e_i{}^a)$. Here, we denoted the covariant derivatives with respect to the standard torsion-free Levi-Civita connection by D_i in order to distinguish them from the covariant derivatives ∇ with respect to Siegel's connections. If a non-trivial connection, say $\hat{\omega}$, is used this will be indicated explicitly as $D_i(\hat{\omega})$. We stress that the spin connection defining the Ricci scalar and thus the Einstein-Hilbert term is also the conventional torsion-free connection rather than the super-covariant one. We will not take into account terms higher order in fermions. Up to this order,

[4]This form of the supergravity action is $\frac{1}{2}$ times the one obtained from eq. (10) of [76] by performing the redefinitions $\phi^{-\frac{3}{2}} \to e^{-\phi}$, $\lambda \to \sqrt{2}\lambda$, $F_{ijk} \to \frac{1}{3\sqrt{2}} H_{ijk}$, $B_{ij} \to \frac{1}{\sqrt{2}} b_{ij}$.

the supersymmetry transformations leaving (5.57) invariant read

$$\begin{aligned}
\delta_\epsilon e_i{}^a &= \frac{1}{2}\bar{\epsilon}\,\gamma^a\psi_i - \frac{1}{4}\bar{\epsilon}\,\lambda\, e_i{}^a\,, \\
\delta_\epsilon\phi &= -\bar{\epsilon}\lambda\,, \\
\delta_\epsilon\psi_i &= D_i\epsilon - \frac{1}{8}\gamma_i(\partial\!\!\!/\phi)\epsilon + \frac{1}{96}(\gamma_i{}^{klm} - 9\delta_i{}^k\gamma^{lm})H_{klm}\epsilon\,, \\
\delta_\epsilon\lambda &= -\frac{1}{4}(\partial\!\!\!/\phi)\epsilon + \frac{1}{48}\gamma^{ijk}H_{ijk}\epsilon\,, \\
\delta_\epsilon b_{ij} &= \frac{1}{2}(\bar{\epsilon}\,\gamma_i\psi_j - \bar{\epsilon}\,\gamma_j\psi_i) - \frac{1}{2}\bar{\epsilon}\,\gamma_{ij}\lambda\,.
\end{aligned} \tag{5.58}$$

Next, we perform some field redefinitions that are necessary in order to compare with the double field theory variables [27],

$$\Psi_i \;\equiv\; \psi_i - \frac{1}{2}\gamma_i\lambda\,, \qquad \rho \;\equiv\; \gamma^i\psi_i - \lambda \;=\; \gamma^i\Psi_i + 4\lambda\,. \tag{5.59}$$

Moreover, as usual we introduce the T-duality invariant dilaton $e^{-2d} = e\,e^{-2\phi}$. Written in terms of these variables, the action (5.57) reads

$$\begin{aligned}
S \;=\; \int d^{10}x\, e^{-2d}\Big[\Big(R + 4\partial^i\phi\,\partial_i\phi - \frac{1}{12}H^{ijk}H_{ijk}\Big) - \bar{\Psi}^j\gamma^i D_i\Psi_j + 2\bar{\Psi}^i D_i\rho \\
+ \bar{\rho}\gamma^i D_i\rho + \frac{1}{4}\bar{\Psi}^i H\!\!\!/\,\Psi_i - \frac{1}{4}\bar{\rho}H\!\!\!/\,\rho + \frac{1}{2}H_{ijk}\bar{\Psi}^i\gamma^j\Psi^k + \frac{1}{4}H_{ijk}\bar{\rho}\gamma^{ij}\Psi^k\Big]\,,
\end{aligned} \tag{5.60}$$

where $H\!\!\!/ = \frac{1}{3!}\gamma^{ijk}H_{ijk}$. This is the final form of the action that is suitable for the comparison with double field theory. The supersymmetry variations written in terms of (5.59) are

$$\begin{aligned}
\delta_\epsilon e_i{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\Psi_i\,, \\
\delta_\epsilon b_{ij} &= \bar{\epsilon}\,\gamma_{[i}\Psi_{j]}\,, \\
\delta_\epsilon d &= -\frac{1}{4}\bar{\epsilon}\rho\,, \\
\delta_\epsilon\Psi_i &= D_i(\hat{\omega})\epsilon\,, \\
\delta_\epsilon\rho &= \gamma^i D_i\epsilon - \frac{1}{24}H_{ijk}\gamma^{ijk}\epsilon - (\partial\!\!\!/\phi)\epsilon\,,
\end{aligned} \tag{5.61}$$

where we introduced a redefinition of the Levi-Civita spin connection ω^{L},

$$\hat{\omega}_{abc} \;=\; \omega^{\mathrm{L}}_{abc} - \frac{1}{2}H_{abc}\;, \tag{5.62}$$

because this is the combination that appears naturally in double field theory, see (5.18).

Let us now return to the double field theory action and supersymmetry transformations (5.30) and (5.34). We first observe that the kinetic terms in (5.30) and (5.60) agree, upon converting flat into curved indices. We will show next that the extra terms in the action (5.60) and the supersymmetry rules (5.61) as compared to double field theory are precisely reproduced by the non-trivial connections inside the covariant derivatives in double field theory.

We start with the supersymmetry transformations. First we note that the variation of ψ_i agrees with the double field theory variation (5.34), because (5.62) coincides with (5.18). Next, consider the variation of the dilatino ρ in (5.34), which reads

$$\delta_\epsilon\rho \;=\; \gamma^{\bar{a}}\nabla_{\bar{a}}\epsilon \;=\; \gamma^{\bar{a}}\big(E_{\bar{a}} - \frac{1}{4}\omega_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\big)\epsilon\;. \tag{5.63}$$

We can now work out the connection term in here,

$$\omega_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{a}}\gamma^{\bar{b}\bar{c}} \;=\; \omega_{\bar{a}\bar{b}\bar{c}}\big(\gamma^{\bar{a}\bar{b}\bar{c}} - \mathcal{G}^{\bar{a}\bar{b}}\gamma^{\bar{c}} + \mathcal{G}^{\bar{a}\bar{c}}\gamma^{\bar{b}}\big) \;=\; \omega_{[\bar{a}\bar{b}\bar{c}]}\gamma^{\bar{a}\bar{b}\bar{c}} + 2\omega_{\bar{a}\bar{b}}{}^{\bar{a}}\gamma^{\bar{b}}\;, \tag{5.64}$$

where we used that ω is antisymmetric in its last two indices. Insertion into (5.63) then yields

$$\delta_\epsilon\rho \;=\; \big(\gamma^{\bar{a}}E_{\bar{a}} - \frac{1}{4}\omega_{[\bar{a}\bar{b}\bar{c}]}\gamma^{\bar{a}\bar{b}\bar{c}} - \frac{1}{2}\omega_{\bar{a}\bar{b}}{}^{\bar{a}}\gamma^{\bar{b}}\big)\epsilon\;. \tag{5.65}$$

We see that only the totally antisymmetric and trace parts of the connections enter, which in turn are fully determined by the constraints. This observation, which has first been made in [27], will be used repeatedly below. Inserting now (5.22) and (5.24) for these determined connections we can rewrite (5.63) as

$$\delta_\epsilon\rho \;=\; \gamma^i D_i\epsilon - \frac{1}{24}H_{ijk}\gamma^{ijk}\epsilon - (\partial\!\!\!/\phi)\epsilon\;, \tag{5.66}$$

which agrees with the required supersymmetry variation of ρ in (5.61). Thus, we have shown that the supersymmetry variations of the fermions in double field theory reproduce the transformations required by $\mathcal{N}=1$ supergravity. For the supersymmetry variations of the bosonic fields consistency with double field theory is manifest for the dilaton d, while for the metric and b-field a short computation is required: variation of (5.10) yields

$$\Delta_\epsilon E_{a\bar{b}} \;=\; e_{\bar{b}}{}^i\delta_\epsilon e_{ia} + e_a{}^i\delta_\epsilon e_{i\bar{b}} - \frac{1}{2}e_a{}^i e_{\bar{b}}{}^j\delta_\epsilon b_{ij} \;=\; -\frac{1}{2}\bar{\epsilon}\gamma_{\bar{b}}\Psi_a\,. \tag{5.67}$$

Due to the relative sign in the contraction of barred indices discussed after eq. (5.32) we have to identify $\gamma_i = -e_i{}^{\bar{a}}\gamma_{\bar{a}}$. Projecting (5.67) onto its antisymmetric part we then read off $\delta_\epsilon b_{ij} = \bar{\epsilon}\gamma_{[i}\Psi_{j]}$, in precise agreement with (5.61). In addition, the symmetric projection of (5.67) determines the symmetric part of the supersymmetry variation $e_b{}^i\delta_\epsilon e_{ia}$. Its antisymmetric part is undetermined, as it should be, because this freedom reflects the diagonal local Lorentz group that is left unbroken by the gauge-fixed form (5.10). It is then easy to see that, up to these local Lorentz transformations, (5.67) yields $\delta_\epsilon e_i{}^a$ as in (5.61). In total, the supersymmetry transformations of double field theory reduce precisely to (5.61).

We turn now to the action. Similarly to the discussion of the supersymmetry transformations it is easy to see that all connections are determined and that writing them out in terms of the Levi-Civita connection reproduces the H-dependent terms in (5.60).

Let us start with the covariant derivative $\nabla_{\bar{b}}$ in the first fermionic term in (5.30), which acts on Ψ_a as an $O(1,9)_R$ spinor and as an $O(1,9)_L$ vector, i.e.,

$$\gamma^{\bar{b}}\nabla_{\bar{b}}\Psi_a \;=\; \gamma^{\bar{b}}\big(E_{\bar{b}}\Psi_a - \frac{1}{4}\omega_{\bar{b}\bar{c}\bar{d}}\gamma^{\bar{c}\bar{d}}\Psi_a + \omega_{\bar{b}a}{}^c\Psi_c\big)\,. \tag{5.68}$$

As in (5.66), the first two terms combine into $\gamma^i D_i\Psi_j$ and $\frac{1}{4}\not{H}\Psi_i$, while a d-dependent term drops out as a consequence of $\Psi^j\gamma^i\Psi_j = 0$. The last term gives the contribution

$$-\bar{\Psi}^a\gamma^{\bar{b}}\omega_{\bar{b}a}{}^c\,\Psi_c \;=\; -\bar{\Psi}^a\gamma^{\bar{b}}\Big(\omega^{\rm L}_{\bar{b}a}{}^c + \frac{1}{2}H_{\bar{b}a}{}^c\Big)\Psi_c \;=\; -\bar{\Psi}^a\gamma^{\bar{b}}\omega^{\rm L}_{\bar{b}a}{}^c\,\Psi_c + \frac{1}{2}H_{a\bar{b}c}\bar{\Psi}^a\gamma^{\bar{b}}\Psi^c\,, \tag{5.69}$$

reproducing the term $\frac{1}{2}H_{ijk}\bar{\Psi}^i\gamma^j\Psi^k$ in (5.60).

Next, we consider the kinetic term of ρ which as in (5.66) reduces to

$$\bar{\rho}\gamma^{\bar{a}}\nabla_{\bar{a}}\rho \;=\; \bar{\rho}\gamma^i D_i\rho - \frac{1}{24}\bar{\rho}H_{ijk}\gamma^{ijk}\rho\,. \tag{5.70}$$

Finally, the last structure in (5.30) yields

$$2\bar{\Psi}^a\nabla_a\rho \;=\; 2\bar{\Psi}^a\big(E_a\rho - \frac{1}{4}\omega_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\rho\big) \;=\; 2\bar{\Psi}^i D_i\rho + \frac{1}{4}H_{ijk}\bar{\rho}\gamma^{ij}\Psi^k\,. \tag{5.71}$$

Collecting the term $\frac{1}{4}\bar{\Psi}^i\not{H}\Psi_i$ originating from (5.68) together with (5.69), (5.70) and (5.71) we infer that the double field theory action reproduces (5.60). Summarizing, we have shown that the $\mathcal{N}=1$ supersymmetric double field theory reduces for $\tilde{\partial}^i=0$ to minimal $\mathcal{N}=1$ supergravity in $D=10$.

5.3 Heterotic Supersymmetric Double Field Theory

In this section we extend the above construction to the coupling of an arbitrary number n of abelian vector multiplets. For $n=16$ this completes the construction of [11] by the fermionic or NS-R sector of heterotic superstring theory truncated to the Cartan subalgebra of $E_8\times E_8$ or $SO(32)$. We first review the extension of the frame formalism, in which the tangent space group is extended to $O(1,9+n)\times O(1,9)$. Then we show that the same $\mathcal{N}=1$ double field theory action (5.2), but interpreted with respect to the enlarged frame and spinor fields, reduces to $\mathcal{N}=1$ supergravity coupled to n vector multiplets upon setting the extra derivatives to zero.

5.3.1 $\mathcal{N}=1$ Double Field Theory with local $O(1,9+n)\times O(1,9)$ symmetry

Let us begin by reviewing the double field theory formulation in presence of n abelian gauge vectors $A_i{}^\alpha$ [11]. The generalized metric is extended to an $O(10+n,10)$ group element, naturally encoding these additional fields. Correspondingly, there are $20+n$ coordinates,

$$X^M \;=\; (\tilde{x}_i\,,\, y^\alpha\,,\, x^i)\,, \qquad \partial_M \;=\; (\tilde{\partial}^i\,,\, \partial_\alpha\,,\, \partial_i)\,, \tag{5.72}$$

transforming as an $O(10+n,10)$ vector, with indices that are raised and lowered with

$$\eta_{MN} \;=\; \begin{pmatrix} 0 & 0 & \mathbf{1}_{10} \\ 0 & \mathbf{1}_n & 0 \\ \mathbf{1}_{10} & 0 & 0 \end{pmatrix}\,. \tag{5.73}$$

We still impose the constraint $\eta^{MN}\partial_M\partial_N = 0$, using the $O(10+n,10)$ invariant metric (5.73). It implies that one can always rotate into a frame in which $\tilde{\partial}^i = \partial_\alpha = 0$.

Next, we can introduce an enlarged frame field as in (5.9), but now with indices $a, b, \ldots$ taking $10+n$ values and with the upper-left block of $\hat{\eta}_{AB}$ being

$$\eta_{ab} \;=\; \begin{pmatrix} \eta_{\underline{ab}} & 0 \\ 0 & \delta_{\underline{\alpha\beta}} \end{pmatrix}\,. \tag{5.74}$$

Here and in the following we split flat indices as

$$A \;=\; (a\,,\, \bar{a}) \;=\; (\underline{a}\,,\, \underline{\alpha}\,,\, \bar{a})\,, \qquad \underline{a} = 1,\ldots,10\,, \quad \underline{\alpha} = 1,\ldots,n\,. \tag{5.75}$$

The frame field is constrained by requiring that the tangent space metric $\mathcal{G}_{AB}$ still satisfies (5.8), which reads explicitly

$$\mathcal{G}_{a\bar{b}} \;=\; 0\,, \qquad \mathcal{G}_{\underline{ab}} \;=\; \eta_{\underline{ab}}\,, \qquad \mathcal{G}_{\bar{a}\bar{b}} \;=\; -\eta_{\bar{a}\bar{b}}\,, \qquad \mathcal{G}_{\underline{\alpha\beta}} \;=\; \delta_{\underline{\alpha\beta}}\,. \tag{5.76}$$

We can then choose a gauge and parametrize the frame field as follows

$$E_A{}^M \;=\; \begin{pmatrix} E_{\underline{a}i} & E_{\underline{a}}{}^\beta & E_{\underline{a}}{}^i \\ E_{\underline{\alpha}i} & E_{\underline{\alpha}}{}^\beta & E_{\underline{\alpha}}{}^i \\ E_{\bar{a}i} & E_{\bar{a}}{}^\beta & E_{\bar{a}}{}^i \end{pmatrix} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} e_{i\underline{a}} - e_{\underline{a}}{}^k c_{ki} & -e_{\underline{a}}{}^k A_k{}^\beta & e_{\underline{a}}{}^i \\ \sqrt{2}A_{i\underline{\alpha}} & \sqrt{2}\delta_{\underline{\alpha}}{}^\beta & 0 \\ -e_{i\bar{a}} - e_{\bar{a}}{}^k c_{ki} & -e_{\bar{a}}{}^k A_k{}^\beta & e_{\bar{a}}{}^i \end{pmatrix}\,, \tag{5.77}$$

where we defined $c_{ij} = b_{ij} + \frac{1}{2}A_i{}^{\alpha}A_{j\alpha}$, and we freely raise and lower gauge group indices with the Kronecker delta $\delta_{\alpha\beta}$.

All results of the frame formalism reviewed in sec. 5.2.1 extend directly to the present generalization. In particular, all statements about determined connection components can be readily applied. Moreover, the supersymmetric extension (5.30) is well-defined for these extended fields in that the gamma matrices $\gamma^{\bar{a}}$ and all spinor indices are still to be interpreted with respect to $O(1,9)$. The check of supersymmetric invariance and closure of the supersymmetry transformations immediately generalizes to the present case, as it is never used whether a takes 10 or $10+n$ values. Assuming the parametrization (5.77) and setting $\tilde{\partial}^i = \partial_\alpha = 0$ we compute the following components:

$$
\begin{aligned}
\omega_{\underline{a}\bar{b}\bar{c}} &= -\left(\omega^{\mathrm{L}}_{\underline{a}\bar{b}\bar{c}}(e) - \frac{1}{2}\hat{H}_{\underline{a}\bar{b}\bar{c}}\right) , \qquad \omega_{\bar{a}\underline{bc}} = \omega^{\mathrm{L}}_{\bar{a}\underline{bc}}(e) + \frac{1}{2}\hat{H}_{\bar{a}\underline{bc}} , \\
\omega_{[\bar{a}\bar{b}\bar{c}]} &= -\left(\omega^{\mathrm{L}}_{[\bar{a}\bar{b}\bar{c}]}(e) - \frac{1}{6}\hat{H}_{\bar{a}\bar{b}\bar{c}}\right) , \\
\omega^{\underline{\alpha}}{}_{\bar{b}\bar{c}} &= \frac{1}{\sqrt{2}}F_{\bar{b}\bar{c}}{}^{\underline{\alpha}} , \qquad \omega_{\bar{b}\underline{a}}{}^{\underline{\alpha}} = -\omega_{\bar{b}}{}^{\underline{\alpha}}{}_{\underline{a}} = \frac{1}{\sqrt{2}}F_{\bar{b}\underline{a}}{}^{\underline{\alpha}} ,
\end{aligned}
\tag{5.78}
$$

where

$$
\begin{aligned}
F_{ab}{}^{\alpha} &= e_a{}^i e_b{}^j \left(\partial_i A_j{}^{\alpha} - \partial_j A_i{}^{\alpha}\right) , \\
\hat{H}_{abc} &= 3 e_a{}^i e_b{}^j e_c{}^k \left(\partial_{[i} b_{jk]} - A_{[i}{}^{\alpha}\partial_j A_{k]\alpha}\right) .
\end{aligned}
\tag{5.79}
$$

In particular, we obtain the required Chern-Simons modification of the field strength H.

5.3.2 Reduction to $\mathcal{N}=1$ Supergravity with n vector multiplets

We will now show that the $\mathcal{N}=1$ double field theory action with tangent space symmetry $O(1,9+n)\times O(1,9)$ reproduces standard $\mathcal{N}=1$ supergravity with n abelian vector multiplets upon setting $\tilde{\partial}^i = \partial_\alpha = 0$. Let us first recall $\mathcal{N}=1$

supergravity coupled to n vector multiplets

$$\left(A_i{}^\alpha\,,\ \chi^\alpha\right)\,, \qquad \alpha \;=\; 1,\ldots,n\,. \tag{5.80}$$

The action is given by

$$\begin{aligned} S \;=\; \int d^{10}x\, e\, e^{-2\phi}\Big[&\Big(R + 4\partial^i\phi\,\partial_i\phi - \frac{1}{12}\hat{H}^{ijk}\hat{H}_{ijk} - \frac{1}{4}F_{ij}F^{ij}\Big) \\ &- \bar{\psi}_i\gamma^{ijk}D_j\psi_k - 2\bar{\lambda}\gamma^i D_i\lambda - \frac{1}{2}\bar{\chi}^\alpha \not{D}\chi_\alpha \\ &+ 2\bar{\psi}^i(\partial_i\phi)\gamma^j\psi_j - \bar{\psi}_i(\not{\partial}\phi)\gamma^i\lambda - \frac{1}{4}\bar{\chi}_\alpha\gamma^i\gamma^{jk}F_{jk}{}^\alpha(\psi_i + \frac{1}{6}\gamma_i\lambda) \\ &+ \frac{1}{24}\hat{H}_{ijk}\Big(\bar{\psi}_m\gamma^{mijkn}\psi_n + 6\bar{\psi}^i\gamma^j\psi^k - 2\bar{\psi}_m\gamma^{ijk}\gamma^m\lambda + \frac{1}{2}\bar{\chi}^\alpha\gamma^{ijk}\chi_\alpha\Big)\Big]\,, \end{aligned} \tag{5.81}$$

where $\hat{H}_{ijk}$ is the H-field strength modified by the Chern-Simons 3-form, as in (5.79). This action is invariant under the supersymmetry transformation:

$$\begin{aligned} \delta_\epsilon e_i{}^a \;&=\frac{1}{2}\bar{\epsilon}\,\gamma^a\psi_i - \frac{1}{4}\bar{\epsilon}\,\lambda\, e_i{}^a\,, \\ \delta_\epsilon\phi \;&=\; -\,\bar{\epsilon}\lambda \quad , \quad \delta_\epsilon A_i{}^\alpha \;=\; \frac{1}{2}\bar{\epsilon}\,\gamma_i\chi^\alpha \quad , \quad \delta_\epsilon\chi^\alpha \;=\; -\frac{1}{4}\gamma^{ij}F_{ij}{}^\alpha\epsilon \\ \delta_\epsilon\psi_i \;&=\; D_i\epsilon - \frac{1}{8}\gamma_i(\not{\partial}\phi)\epsilon + \frac{1}{96}(\gamma_i{}^{klm} - 9\delta_i{}^k\gamma^{lm})\hat{H}_{klm}\epsilon\,, \\ \delta_\epsilon\lambda \;&=\; -\frac{1}{4}(\not{\partial}\phi)\epsilon + \frac{1}{48}\gamma^{ijk}\hat{H}_{ijk}\epsilon\,, \\ \delta_\epsilon b_{ij} \;&=\; \frac{1}{2}(\bar{\epsilon}\,\gamma_i\psi_j - \bar{\epsilon}\,\gamma_j\psi_i) - \frac{1}{2}\bar{\epsilon}\,\gamma_{ij}\lambda + \frac{1}{2}\bar{\epsilon}\gamma_{[i}\chi^\alpha A_{j]\alpha}\,. \end{aligned} \tag{5.82}$$

Next, we perform the same field redefinition (5.59) as for the minimal theory. We obtain for the action

$$\begin{aligned} S_F \;=\; \int d^{10}x\, e^{-2d}\Big[&-\bar{\Psi}^j\gamma^i D_i\Psi_j + 2\bar{\Psi}^i D_i\rho + \bar{\rho}\gamma^i D_i\rho - \frac{1}{2}\bar{\chi}^\alpha\gamma^i D_i\chi_\alpha - \frac{1}{4}\bar{\chi}^\alpha\gamma^{jk}F_{jk\alpha}\rho \\ &-\bar{\chi}^\alpha\gamma^k F_{ik\alpha}\Psi^i + \frac{1}{4}\bar{\Psi}^i\hat{\not{H}}\Psi_i - \frac{1}{4}\bar{\rho}\hat{\not{H}}\rho + \frac{1}{2}\hat{H}_{ijk}\bar{\Psi}^i\gamma^j\Psi^k + \frac{1}{4}\hat{H}_{ijk}\bar{\rho}\gamma^{ij}\Psi^k + \frac{1}{8}\bar{\chi}^\alpha\hat{\not{H}}\chi_\alpha\Big]\,, \end{aligned} \tag{5.83}$$

and the supersymmetry transformations are given by

$$
\begin{aligned}
\delta_\epsilon e_i{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\Psi_i\,, \qquad \delta_\epsilon\Psi_i \;=\; D_i(\hat{\omega})\epsilon\,,\\
\delta_\epsilon b_{ij} &= \bar{\epsilon}\,\gamma_{[i}\Psi_{j]} + \frac{1}{2}\bar{\epsilon}\gamma_{[i}\chi A_{j]}\,,\\
\delta_\epsilon d &= -\frac{1}{4}\bar{\epsilon}\rho\,, \qquad \delta_\epsilon\rho \;=\; \gamma^i D_i\epsilon - \frac{1}{24}\hat{H}_{ijk}\gamma^{ijk}\epsilon - (\partial\!\!\!/\phi)\epsilon\\
\delta_\epsilon A_i{}^\alpha &= \frac{1}{2}\bar{\epsilon}\gamma_i\chi^\alpha \quad , \quad \delta_\epsilon\chi^\alpha \;=\; -\frac{1}{4}\gamma^{ij}F_{ij}{}^\alpha\epsilon\,.
\end{aligned}
\tag{5.84}
$$

Let us now verify that the above action and supersymmetry rules are reproduced by supersymmetric double field theory for $\tilde{\partial}^i = \partial_\alpha = 0$. Here, our discussion will be a little briefer than above because it suffices to focus on the new structures involving the gauge vectors and gauginos. It turns out that the comparison requires the identification

$$
\Psi_a \;=\; \big(\Psi_{\underline{a}}\,,\;\Psi_{\underline{\alpha}}\big) \;\equiv\; \big(e_{\underline{a}}{}^i\Psi_i\,,\;\tfrac{1}{\sqrt{2}}\chi_{\underline{\alpha}}\big)\,,
\tag{5.85}
$$

i.e., the gauginos are naturally identified with the additional components of the 'gravitino'. We start with the supersymmetry transformations. The gaugino variation $\delta_\epsilon\chi^\alpha$ can be obtained by considering

$$
\delta_\epsilon\Psi_{\underline{\alpha}} \;=\; \frac{1}{\sqrt{2}}\delta_\epsilon\chi_{\underline{\alpha}} \;=\; \nabla_{\underline{\alpha}}\epsilon \;=\; \left(\sqrt{2}E_{\underline{\alpha}}{}^i\partial_i\epsilon - \frac{1}{4}\omega_{\underline{\alpha}\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\epsilon\right) \;=\; -\frac{1}{4\sqrt{2}}F_{\bar{b}\bar{c}\underline{\alpha}}\,\gamma^{\bar{b}\bar{c}}\epsilon\,,
\tag{5.86}
$$

where we used (5.78) and $E_{\underline{\alpha}}{}^i = 0$ for the gauge choice (5.77). We read off

$$
\delta_\epsilon\chi^{\underline{\alpha}} \;=\; -\frac{1}{4}F_{\bar{b}\bar{c}}{}^{\underline{\alpha}}\,\gamma^{\bar{b}\bar{c}}\epsilon\,.
\tag{5.87}
$$

Comparison with (5.84) shows that we obtained the expected supersymmetry variation. For the supersymmetry variations of the vielbein $e_i{}^a$, the b-field and the gauge vectors we compute as in (5.67) the variation of the gauge-fixed frame field (5.77)

$$
\Delta_\epsilon E_{\underline{a}\bar{b}} \;=\; e_{\bar{b}}{}^i\delta_\epsilon e_{i\underline{a}} + e_{\underline{a}}{}^i\delta_\epsilon e_{i\bar{b}} - \frac{1}{2}e_{\underline{a}}{}^i e_{\bar{b}}{}^j\delta_\epsilon b_{ij} - \frac{1}{2}e_{\underline{a}}{}^i e_{\bar{b}}{}^j A_{[i}{}^{\underline{\alpha}}\delta_\epsilon A_{j]\underline{\alpha}} \;=\; -\frac{1}{2}\bar{\epsilon}\gamma_{\bar{b}}\Psi_{\underline{a}}\,,
\tag{5.88}
$$

and

$$
\Delta_\epsilon E_{\underline{\alpha}\bar{b}} \;=\; \frac{\sqrt{2}}{2} e_{\bar{b}}{}^{i}\delta_\epsilon A_{i\underline{\alpha}} \;=\; -\frac{1}{2\sqrt{2}}\bar{\epsilon}\gamma_{\bar{b}}\chi_{\underline{\alpha}}\,. \tag{5.89}
$$

Combining these two gives the required supersymmetry transformations (5.84).

Let us now turn to the action and show that it produces the required χ-dependent terms. For the first fermionic term in (5.2) we obtain

$$
\begin{aligned}
-\bar{\Psi}^{a}\gamma^{\bar{b}}\nabla_{\bar{b}}\Psi_{a}\Big|_{\chi} \;&=\; -\,\bar{\Psi}^{\underline{\alpha}}\gamma^{\bar{b}}D_{\bar{b}}(\hat{\omega})\Psi_{\underline{\alpha}} - \bar{\Psi}^{\underline{a}}\gamma^{\bar{b}}\omega_{\bar{b}\underline{a}}{}^{\underline{\alpha}}\Psi_{\underline{\alpha}} - \bar{\Psi}^{\underline{\alpha}}\gamma^{\bar{b}}\omega_{\bar{b}\underline{\alpha}}{}^{\underline{a}}\Psi_{\underline{a}} \\
&=\; -\,\frac{1}{2}\bar{\chi}^{\underline{\alpha}}\gamma^{\bar{b}}D_{\bar{b}}\chi_{\underline{\alpha}} + \frac{1}{8}\bar{\chi}^{\underline{\alpha}}\gamma^{\bar{b}\bar{c}\bar{d}}\omega_{[\bar{b}\bar{c}\bar{d}]}\chi_{\underline{\alpha}} - \bar{\chi}_{\underline{\alpha}}\gamma^{\bar{b}}F_{\underline{a}\bar{b}}{}^{\underline{\alpha}}\,\Psi^{\underline{a}} \\
&=\; -\,\frac{1}{2}\bar{\chi}^{\underline{\alpha}}\gamma^{\bar{b}}D_{\bar{b}}\chi_{\underline{\alpha}} + \frac{1}{8}\bar{\chi}^{\underline{\alpha}}\bar{\not{H}}\chi_{\underline{\alpha}} - \bar{\chi}_{\underline{\alpha}}\gamma^{\bar{b}}F_{\underline{a}\bar{b}}{}^{\underline{\alpha}}\,\Psi^{\underline{a}}\,,
\end{aligned} \tag{5.90}
$$

where we used in the first line that the last two terms are equal. The second fermionic term in the action (5.2) does not give any χ-dependent contribution. The third term reads

$$
2\bar{\Psi}^{a}\nabla_{a}\rho\Big|_{\chi} \;=\; 2\bar{\Psi}^{\underline{\alpha}}\nabla_{\underline{\alpha}}\rho \;=\; \frac{2}{\sqrt{2}}\bar{\chi}^{\underline{\alpha}}\Big(-\frac{1}{4}\omega_{\underline{\alpha}\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\Big)\rho \;=\; -\frac{1}{4}\bar{\chi}_{\underline{\alpha}}\gamma^{\bar{b}\bar{c}}F_{\bar{b}\bar{c}}{}^{\underline{\alpha}}\rho\,, \tag{5.91}
$$

reproducing the required coupling in (5.83). Thus, we have shown that all new F-dependent terms due to the coupling of vector multiplets are precisely reproduced by the extended connections of the $O(1,9+n)\times O(1,9)$ tangent space symmetry.

5.4 Conclusions

In this chapter we have constructed the $\mathcal{N}=1$ supersymmetric extension of double field theory for $D=10$. This theory features two copies of the local Lorentz group as tangent space symmetries, under which the fermions naturally transform. Interestingly, the generalization to the coupling of n abelian vector multiplets amounts only to the extension of the T-duality group to $O(10+n,10)$ and, correspondingly, to the extension of the tangent space group to $O(1,9+n)\times O(1,9)$. The 'gravitino' Ψ_a thereby receives n additional components that can be identified with the gauginos. Apart from exhibiting a further 'unification' of the massless sector of heterotic superstring theory, this formulation provides a significant technical simplification of the effective action, as should be apparent by comparing (5.81) with (5.2). Moreover, the proof of supersymmetric invariance (up to the higher order fermi terms) is much simpler than in the standard formulation, being essentially reduced to a two-line calculation in (5.36).

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