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Open superstring field theory I: gauge fixing, ghost structure, and propagator

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Abstract

The WZW form of open superstring field theory has linearized gauge invariances associated with the BRST operator Q and the zero mode η_0 of the picture minus-one fermionic superconformal ghost. We discuss gauge fixing of the free theory in a simple class of gauges using the Faddeev-Popov method. We find that the world-sheet ghost number of ghost and antighost string fields ranges over all integers, except one, and at any fixed ghost number, only a finite number of picture numbers appear. We calculate the propagators in a variety of gauges and determine the field-antifield content and the free master action in the Batalin-Vilkovisky formalism. Unlike the case of bosonic string field theory, the resulting master action is not simply related to the original gauge-invariant action by relaxing the constraint on the ghost and picture numbers.

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1 Introduction and summary

String field theory is an approach to string theory that aims to address non-perturbative questions that are difficult to study in the context of first quantization. Classical solutions that represent changes of the open string background are of particular interest, and considerable progress was made in this subject in the last few years (see, for example, [1, 2, 3, 4, 5]).

A covariant string field theory should satisfy a series of consistency checks. The kinetic term, for example, must define the known spectrum of the theory. The full action, with the inclusion of interaction terms, has nontrivial gauge invariances. It must be possible to gauge fix these symmetries, derive a propagator, and set up a perturbation theory that produces off-shell amplitudes that, on-shell, agree with the amplitudes in the first-quantized theory. The purpose of these checks is not necessarily to construct off-shell amplitudes, but rather to test the consistency and understand better the structure of the theory. Indeed that was the way it turned out for open bosonic string field theory [6]. The Faddeev-Popov quantization of the theory quickly suggested that the full set of required ghost and antighost fields could be obtained by relaxing the ghost number constraint on the classical string field [7, 8, 9].[1] Moreover, the Batalin-Vilkovisky (BV) quantization approach [10, 11] turned out to be surprisingly effective [12]. The full master action for open bosonic string field theory—the main object in this quantization scheme—is simply the classical action evaluated with the unconstrained string field. For the closed bosonic string field theory, the BV master equation was useful in the

[1] In this paper we refer to the string field in the gauge-invariant action before gauge fixing as the "classical" string field, distinguishing it from ghost and antighost fields introduced by gauge fixing.

construction of the full quantum action, since it has a close relation with the constraint that ensures proper covering of the moduli spaces of Riemann surfaces [13]. As is the case for open strings, the closed string field theory master action is simply obtained by relaxing the ghost number constraint on the classical string field.

It is the purpose of this paper to begin a detailed study of gauge fixing of the WZW open superstring field theory [14] using the Faddeev-Popov method and the Batalin-Vilkovisky formalism. This theory describes the Neveu-Schwarz sector of open superstrings using the 'large' Hilbert space of the superconformal ghost sector in terms of ξ, η, and ϕ [15]. As opposed to some alternative formulations [16, 17, 18] no world-sheet insertions of picture-changing operators are required and the string field theory action takes the form

$$S = \frac{1}{2g^2}\Big\langle (e^{-\Phi}Qe^{\Phi})(e^{-\Phi}\eta_0 e^{\Phi}) - \int_0^1 dt(e^{-t\Phi}\partial_t e^{t\Phi})\big\{(e^{-t\Phi}Qe^{t\Phi}),(e^{-t\Phi}\eta_0 e^{t\Phi})\big\}\Big\rangle\,. \tag{1.1}$$

Here $\{A,B\} \equiv AB+BA$, g is the open string coupling constant, η_0 denotes the zero mode of the superconformal ghost field η, and Q denotes the BRST operator. These two operators anticommute and square to zero:

$$\{Q,\eta_0\}=0, \quad Q^2=\eta_0^2=0\,. \tag{1.2}$$

The string field Φ is Grassmann even and has both ghost and picture number zero. Both Q and η_0 have ghost number one. While η_0 carries picture number minus one, Q carries no picture number. Products of string fields are defined using the star product in [6], and the BPZ inner product of string fields A and B is denoted by $\langle\, A\, B\,\rangle$ or by $\langle\, A\,|B\,\rangle$. The action is defined by expanding all exponentials in formal Taylor series, and we evaluate the associated correlators recalling that in the large Hilbert space

$$\langle \xi(z)c\partial c\partial^2 c(w)e^{-2\phi(y)}\rangle \neq 0\,. \tag{1.3}$$

The action can be shown to be invariant under gauge transformations with infinitesimal gauge parameters Λ and Ω:

$$\delta e^{\Phi} = (Q\Lambda)e^{\Phi} + e^{\Phi}(\eta_0\Omega)\,, \tag{1.4}$$

and the equation of motion for the string field is

$$\eta_0(e^{-\Phi}Qe^{\Phi}) = 0\,. \tag{1.5}$$

In this paper we focus on the linearized theory. For notational simplicity we will simply set the open string coupling equal to one: $g=1$. To linearized order the action reduces to S_0 given by

$$S_0 \;=\; \frac{1}{2}\big\langle\,(Q\Phi)\,(\eta_0\Phi)\,\big\rangle\,. \tag{1.6}$$

Using bra and ket notation, the kinetic term can be written as

$$S_0 \;=\; -\frac{1}{2}\,\big\langle \Phi_{(0,0)}\big|\, Q\eta_0\,\big|\Phi_{(0,0)}\big\rangle\,. \tag{1.7}$$

Here we have written $\Phi = \Phi_{(0,0)}$ to emphasize that the classical string field has both ghost number and picture number zero. Unless indicated otherwise, we take $X_{(g,p)}$ to be an object that carries ghost number g and picture number p. To this order the equation of motion (1.5) becomes

$$\eta_0 Q\,\Phi_{(0,0)} \;=\; 0\,, \tag{1.8}$$

and the gauge transformations (1.4) become

$$\delta_0 \Phi_{(0,0)} = Q\Lambda \,+\, \eta_0 \Omega\,. \tag{1.9}$$

Let us use ϵ for gauge parameters and rewrite (1.9) as

$$\delta_0 \Phi_{(0,0)} = Q\epsilon_{(-1,0)} \,+\, \eta_0 \epsilon_{(-1,1)}\,, \tag{1.10}$$

where we have indicated the appropriate ghost and picture numbers in the subscripts. Note that both $\epsilon_{(-1,0)}$ and $\epsilon_{(-1,1)}$ are Grassmann odd, both have ghost number minus one, but differ in picture number. The gauge invariances (1.10) have their own gauge invariances. We can change $\epsilon_{(-1,0)}$ and $\epsilon_{(-1,1)}$ without changing $\delta_0 \Phi_{(0,0)}$. Indeed, with

$$\begin{aligned} \delta_1 \epsilon_{(-1,0)} &= Q\epsilon_{(-2,0)} + \eta_0 \epsilon_{(-2,1)}\,, \\ \delta_1 \epsilon_{(-1,1)} &= Q\epsilon_{(-2,1)} + \eta_0 \epsilon_{(-2,2)}\,, \end{aligned} \tag{1.11}$$

we readily verify that $\delta_1(\delta_0 \Phi_{(0,0)}) = 0$, making use of (1.2). At this stage we have introduced three gauge parameters, all of ghost number minus two, and with pictures zero, one, and two. The above redundant transformations have their own redundancy:

$$\begin{aligned} \delta_2 \epsilon_{(-2,0)} &= Q\epsilon_{(-3,0)} + \eta_0 \epsilon_{(-3,1)}\,, \\ \delta_2 \epsilon_{(-2,1)} &= Q\epsilon_{(-3,1)} + \eta_0 \epsilon_{(-3,2)}\,, \\ \delta_2 \epsilon_{(-2,2)} &= Q\epsilon_{(-3,2)} + \eta_0 \epsilon_{(-3,3)}\,, \end{aligned} \tag{1.12}$$

and this time we verify that $\delta_2(\delta_1 \epsilon_{(-1,0)}) \;=\; \delta_2(\delta_1 \epsilon_{(-1,1)}) = 0$. At step n, in matrix notation, we have

$$\delta_n \begin{pmatrix} \epsilon_{(-n,0)} \\ \epsilon_{(-n,1)} \\ \epsilon_{(-n,2)} \\ \vdots \\ \epsilon_{(-n,n)} \end{pmatrix} \;=\; \begin{pmatrix} Q & \eta_0 & 0 & \cdots & 0 & 0 \\ 0 & Q & \eta_0 & \cdots & 0 & 0 \\ 0 & 0 & Q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q & \eta_0 \end{pmatrix} \begin{pmatrix} \epsilon_{(-(n+1),0)} \\ \epsilon_{(-(n+1),1)} \\ \epsilon_{(-(n+1),2)} \\ \vdots \\ \epsilon_{(-(n+1),n)} \\ \epsilon_{(-(n+1),n+1)} \end{pmatrix}. \tag{1.13}$$

The above describes the full structure of redundant symmetries of the theory at linearized level. It is the starting point for the BRST quantization of the theory, where we select gauge-fixing conditions and add suitable Faddeev-Popov terms to the action. The above gauge parameters turn into ghosts

$$\Phi_{(-n,p)}\,, \quad n \geq 1\,, \quad p = 0, 1, \ldots, n\,. \tag{1.14}$$

It follows from the BRST prescription that all of the above ghost fields are Grassmann even, just like the classical string field $\Phi_{(0,0)}$.[2] Antighosts must be also added. The Faddeev-Popov quantization is carried out using a set of gauge conditions that enable us to confirm that the free gauge-fixed action, after elimination of auxiliary fields, coincides with that of Witten's free theory [16] in Siegel gauge. The gauge-fixing conditions here are of type $(b_0, \xi_0; \alpha)$, meaning that ghosts and antighosts are required to be annihilated by operators made of the zero modes b_0 and ξ_0 with a parameter α. (See (2.90).)

We then turn to the calculation of the propagator of the theory, for which the free gauge-fixed action is sufficient. As usual, we add to the action linear couplings that associate unconstrained sources with the classical field, with each ghost, and with each antighost. The propagator is then the matrix that defines the quadratic couplings of sources in the action, and it is obtained by solving for all fields in terms of sources using the classical equations of motion. We examine this propagator for a few types of gauges. In the $(b_0, \xi_0; \alpha)$ type gauges, the propagator matrix contains the zero mode $X_0 = \{Q, \xi_0\}$ of the picture-changing operator and its powers. The propagator is quite complicated for $\alpha \neq 0$ and simplifies somewhat for $\alpha = 0$, where it takes the form of matrices of triangular type.

A more intriguing class of gauges are of type $(b_0, d_0; \alpha)$. Here d_0 is the zero mode of the operator $d = [Q, b\xi]$. In the language of the twisted $N = 2$ superconformal algebra [19], $d_0 = \widetilde{G}_0^-$ is a counterpart of $b_0 = G_0^-$. Corresponding to the relation $\{Q, b_0\} = L_0$, the anticommutation relation $\{\eta_0, d_0\} = L_0$ holds. In fact, the gauge-fixing conditions $b_0\Phi_{(0,0)} = d_0\Phi_{(0,0)} = 0$ were used in the calculation of a four-point amplitude in [20]. The propagators in this class of gauges are much simpler than in the $(b_0, \xi_0; \alpha)$ type gauges and do not involve picture-changing operators. They further simplify when $\alpha = 1$ (see (3.59), (3.60), and (3.61)). We expect this form of the propagator to be useful in the study of loop amplitudes.

The gauge structure of the free theory is infinitely reducible. In fact, the equations in (1.13) determine the "field/antifield" structure of the theory following the usual Batalin-Vilkovisky procedure [10, 11] (reviewed in [21, 22, 23]). We write the original gauge symmetry of the classical fields ϕ^{α_0} schematically as

$$\delta\phi^{\alpha_0} \;=\; R^{\alpha_0}_{(0)\alpha_1}\, \epsilon^{\alpha_1}\,, \tag{1.15}$$

where sum over repeated indices is implicit and R is possibly field dependent. The symmetry is infinitely reducible if there are gauge invariances of gauge invariances at every stage, namely

$$\begin{aligned} \delta\epsilon^{\alpha_1} &= R^{\alpha_1}_{(1)\alpha_2}\, \epsilon^{\alpha_2} \\ \delta\epsilon^{\alpha_2} &= R^{\alpha_2}_{(2)\alpha_3}\, \epsilon^{\alpha_3} \\ \vdots\; &= \;\vdots \end{aligned} \tag{1.16}$$

[2] The spacetime fields in such string fields can be even or odd depending on the Grassmann parity of the CFT basis states.

with the following on-shell relations

$$R^{\alpha_n}_{(n)\alpha_{n+1}} R^{\alpha_{n+1}}_{(n+1)\alpha_{n+2}} = 0\,, \quad \text{for} \quad n = 0, 1, 2, \ldots\,. \tag{1.17}$$

In this case one introduces fields ϕ^{α_n} with $n \geq 1$ and antifields $\phi^*_{\alpha_n}$ with $n \geq 0$ such that the BV action reads

$$\begin{aligned} S &= S_0(\phi^{\alpha_0}) + \sum_{n=0}^{\infty} \phi^*_{\alpha_n} R^{\alpha_n}_{(n)\alpha_{n+1}} \phi^{\alpha_{n+1}} + \ldots \\ &= S_0 + \phi^*_{\alpha_0} R^{\alpha_0}_{(0)\alpha_1} \phi^{\alpha_1} + \phi^*_{\alpha_1} R^{\alpha_1}_{(1)\alpha_2} \phi^{\alpha_2} + \ldots\,, \end{aligned} \tag{1.18}$$

where the dots represent terms at least cubic in ghosts and antifields that are needed for a complete solution of the master equation. Since all string fields for "fields" in open superstring field theory are Grassmann even, the R's are Grassmann odd, and since the inner product with (1.3) needed to form the action couples states of the same Grassmann parity, the string fields for "antifields" are Grassmann odd.[3] The antifield $\Phi^*_{(g,p)}$associated with the field $\Phi_{(g,p)}$ is $\Phi_{(2-g,-1-p)}$:

$$\Phi^*_{(g,p)} = \Phi_{(2-g,-1-p)}\,. \tag{1.19}$$

This follows from (1.18) where each term in the sum takes the form $\phi^*_{\alpha_n}(\delta\phi^{\alpha_n})$, with the gauge parameter replaced by a ghost field of the same ghost and picture number. This implies that the inner product with (1.3) must be able to couple a field to its antifield. Since this inner product requires a total ghost number violation of two and a total picture number violation of minus one, the claim in (1.19) follows. The full field/antifield structure of the theory is therefore

$$\begin{array}{cccccccccccc} \cdots & \Phi_{(-2,2)} & & & & & & & & & \\ \cdots & \Phi_{(-2,1)} & \Phi_{(-1,1)} & & & & & & & \uparrow p & \\ \cdots & \Phi_{(-2,0)} & \Phi_{(-1,0)} & \Phi_{(0,0)} & - & & & & & \longrightarrow g & \\ & & & & & \Phi_{(2,-1)} & \Phi_{(3,-1)} & \Phi_{(4,-1)} & \cdots & & \\ & & & & & & \Phi_{(3,-2)} & \Phi_{(4,-2)} & \cdots & & \\ & & & & & & & \Phi_{(4,-3)} & \cdots & & \end{array} \tag{1.20}$$

The string fields $\Phi_{(g,p)}$ with $g \leq 0$ on the left side are the "fields," and the $\Phi_{(g,p)}$ with $g \geq 2$ on the right side are the "antifields." Note the gap at $g = 1$. Collecting all the fields in Φ_- and all antifields in Φ_+ as

$$\Phi_- = \sum_{g=0}^{\infty} \sum_{p=0}^{g} \Phi_{(-g,p)}\,, \qquad \Phi_+ = \sum_{g=2}^{\infty} \sum_{p=1}^{g-1} \Phi_{(g,-p)}\,, \tag{1.21}$$

[3] In open bosonic string field theory the string fields for fields and those for antifields are of the same (odd) Grassmann parity.

we can show that the free master action S implied by (1.18) and by our identification of fields and antifields takes the form:

$$S \;=\; -\frac{1}{2}\,\langle\Phi_-|\,Q\eta_0\,|\Phi_-\rangle+\langle\Phi_+|\,(Q+\eta_0)\,|\Phi_-\rangle\;. \tag{1.22}$$

The master equation $\{S, S\} = 0$, where $\{\cdot\,,\,\cdot\}$ is the BV antibracket, will be shown to be satisfied.

2 Gauge fixing of the free theory

In this section we perform gauge fixing of the free open superstring field theory using the Faddeev-Popov method. We first review the procedure in the free open bosonic string field theory, and then we extend it to open superstring field theory. We also demonstrate that the resulting gauge-fixed action coincides with that of Witten's superstring field theory in Siegel gauge after integrating out auxiliary fields.

2.1 Open bosonic string field theory

The gauge-invariant action of the free theory is given by

$$S_0=-\frac{1}{2}\langle\Psi_1|Q|\Psi_1\rangle\,, \tag{2.1}$$

where Ψ_1 is the open string field. It is Grassmann odd and carries ghost number one, as indicated by the subscript. The BRST operator Q is BPZ odd: $Q^\star = -Q$. This action is invariant under the following gauge transformation:

$$\delta_\epsilon\Psi_1=Q\epsilon_0\,, \tag{2.2}$$

where ϵ_0 is a Grassmann-even string field of ghost number zero.

The Faddeev-Popov method consists of adding two terms to the gauge-invariant action. The first term is given by

$$\mathcal{L}_{\rm GF}=\lambda^i F_i(\phi)\,, \tag{2.3}$$

where $F_i(\phi) = 0$ are the gauge-fixing conditions on the field ϕ and λ^i are the corresponding Lagrange multiplier fields. The second term is the Faddeev-Popov term given by

$$\mathcal{L}_{\rm FP}=b^i\left(c^\alpha\frac{\delta}{\delta\epsilon^\alpha}\right)\delta_\epsilon F_i(\phi)\,. \tag{2.4}$$

It is obtained from $\mathcal{L}_{\rm GF}$ by changing λ^i to the antighost fields b^i and by changing $F_i(\phi)$ to its gauge transformation $\delta_\epsilon F_i(\phi)$ with the gauge parameters ϵ^α replaced by the ghost fields c^α. The sum of the two terms $\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$ is then BRST exact: $\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP} = -\delta_B(\,b^i F_i(\phi)\,)$ under the convention $\delta_B b^i = -\lambda^i$.

Let us apply this procedure to the free theory of open bosonic string field theory and choose the Siegel gauge condition

$$b_0\Psi_1=0 \tag{2.5}$$

for gauge fixing. Note that b_0 is BPZ even: $b_0^\star = b_0$. It is convenient to decompose Ψ_1 into two subsectors according to the zero modes b_0 and c_0 as follows:

$$\Psi_1 = \Psi_1^- + c_0 \Psi_1^c \,, \tag{2.6}$$

where Ψ_1^- and Ψ_1^c are both annihilated by b_0. The superscript '$-$' indicates the sector without c_0 and the superscript 'c' indicates the sector with c_0, although c_0 has been removed in Ψ_1^c. Therefore Ψ_1^c is Grassmann even and carries ghost number zero, and so the subscript, which is carried over from Ψ_1, does not coincide with the ghost number of Ψ_1^c. The operator c_0 we used in the decomposition is BPZ odd: $c_0^\star = -c_0$. Using this decomposition, the Siegel gauge condition can be stated as

$$\Psi_1^c = 0 \,. \tag{2.7}$$

The gauge-fixing term $S_{\rm GF}$ implementing this condition can be written as

$$S_{\rm GF} = -\langle N | c_0 | \Psi_1^c \rangle \,, \tag{2.8}$$

where the Lagrange multiplier field N is annihilated by b_0. Note that the insertion of c_0 is necessary for the inner product to be nonvanishing. The ghost number of N is two and component fields playing the role of Lagrange multiplier fields have to be Grassmann even, so the string field N is Grassmann even. This term can be equivalently written as

$$S_{\rm GF} = -\langle N_2 | \Psi_1 \rangle \,, \tag{2.9}$$

with the constraint

$$b_0 N_2 = 0 \,. \tag{2.10}$$

This can be seen by decomposing N_2 before imposing the constraint as

$$N_2 = N_2^- + c_0 N_2^c \,, \tag{2.11}$$

where N_2^- and N_2^c are annihilated by b_0. The inner product $\langle N_2 | \Psi_1 \rangle$ is then given by

$$\langle N_2 | \Psi_1 \rangle = \langle N_2^- | c_0 | \Psi_1^c \rangle + \langle N_2^c | c_0 | \Psi_1^- \rangle \,. \tag{2.12}$$

The constraint $b_0 N_2 = 0$ eliminates N_2^c, and the remaining field N_2^- is identified with the Lagrange multiplier field N. The string field N_2 is Grassmann even and carries ghost number two.

Another way to derive $S_{\rm GF}$ is to use the form $b_0 \Psi_1 = 0$ for the gauge-fixing condition and write

$$S_{\rm GF} = \langle \widetilde{N}_3 | b_0 | \Psi_1 \rangle \,. \tag{2.13}$$

We then redefine the Lagrange multiplier as

$$N_2 = b_0 \widetilde{N}_3 \,. \tag{2.14}$$

The resulting field N_2 is subject to the constraint $b_0 N_2 = 0$. Since $\{b_0, c_0\} = 1$, any solution N_2 to this constraint can be written as $N_2 = \{b_0, c_0\} N_2 = b_0 c_0 N_2 = b_0 \widetilde{N}_3$ with $\widetilde{N}_3 = c_0 N_2$. Therefore, N_2 obtained from the redefinition $N_2 = b_0 \widetilde{N}_3$ is equivalent to N_2 with the constraint $b_0 N_2 = 0$.

The Faddeev-Popov term S_{FP} can be obtained from S_{GF} by changing N_2 to the Grassmann-odd antighost field Ψ_2 of ghost number two and by changing Ψ_1 to its gauge transformation $Q\epsilon_0$ with ϵ_0 replaced by the Grassmann-odd ghost field Ψ_0 of ghost number zero. We have

$$S_{\mathrm{FP}} = -\langle \Psi_2 | Q | \Psi_0 \rangle \,, \tag{2.15}$$

with the constraint

$$b_0 \Psi_2 = 0 \,, \tag{2.16}$$

which is inherited from $b_0 N_2 = 0$. After integrating out N_2, the total action we obtain is

$$S_0 + S_1 = -\frac{1}{2}\langle \Psi_1 | Q | \Psi_1 \rangle - \langle \Psi_2 | Q | \Psi_0 \rangle \,, \tag{2.17}$$

with

$$b_0 \Psi_1 = 0 \,, \qquad b_0 \Psi_2 = 0 \,. \tag{2.18}$$

This action $S_0 + S_1$ is invariant under the following gauge transformation:

$$\delta_\epsilon \Psi_0 = Q \epsilon_{-1} \,. \tag{2.19}$$

We can choose

$$b_0 \Psi_0 = 0 \tag{2.20}$$

for gauge fixing. Repeating the same Faddeev-Popov procedure, we obtain

$$S_0 + S_1 + S_2 = -\frac{1}{2}\langle \Psi_1 | Q | \Psi_1 \rangle - \langle \Psi_2 | Q | \Psi_0 \rangle - \langle \Psi_3 | Q | \Psi_{-1} \rangle \,, \tag{2.21}$$

with

$$b_0 \Psi_1 = 0 \,, \quad b_0 \Psi_2 = 0 \,, \qquad b_0 \Psi_0 = 0 \,, \quad b_0 \Psi_3 = 0 \,, \tag{2.22}$$

where Ψ_3 has ghost number three and Ψ_{-1} has ghost number minus one.

The action $S_0 + S_1 + S_2$ is invariant under $\delta_\epsilon \Psi_{-1} = Q \epsilon_{-2}$. In this way the gauge-fixing procedure continues, and at the end we obtain

$$S = \sum_{n=0}^{\infty} S_n \,, \tag{2.23}$$

where

$$S_0 = -\frac{1}{2}\langle \Psi_1 | Q | \Psi_1 \rangle \,, \qquad S_n = -\langle \Psi_{n+1} | Q | \Psi_{-n+1} \rangle \quad \text{for} \quad n \geq 1 \tag{2.24}$$

with

$$b_0 \Psi_n = 0 \,, \quad \forall n \,. \tag{2.25}$$

The action S can also be written compactly as

$$S = -\frac{1}{2}\langle \Psi | Q | \Psi \rangle \quad \text{with} \quad \Psi = \sum_{n=-\infty}^{\infty} \Psi_n \,, \quad b_0 \Psi = 0 \,. \tag{2.26}$$

2.2 Open superstring field theory

Let us now perform gauge fixing of the free open superstring field theory. We denote a string field of ghost number g and picture number p by $\Phi_{(g,p)}$. The gauge-invariant action of the free theory is given by

$$S_0 = -\frac{1}{2}\langle \Phi_{(0,0)} | Q\eta_0 | \Phi_{(0,0)} \rangle \,, \tag{2.27}$$

where η_0 is the zero mode of the superconformal ghost carrying ghost number one and picture number minus one. It is therefore BPZ odd: $\eta_0^\star = -\eta_0$. The action S_0 is invariant under the following gauge transformations:

$$\delta_\epsilon \Phi_{(0,0)} = Q\epsilon_{(-1,0)} + \eta_0 \epsilon_{(-1,1)} \,. \tag{2.28}$$

We can choose $\epsilon_{(-1,1)}$ appropriately such that the condition

$$\xi_0 \Phi_{(0,0)} = 0 \tag{2.29}$$

on $\Phi_{(0,0)}$ is satisfied. The operator ξ_0 we used in the gauge-fixing condition is BPZ even: $\xi_0^\star = \xi_0$. Because $\{\eta_0, \xi_0\} = 1$, a field $\Phi_{(0,0)}$ satisfying (2.29) can be written as

$$\Phi_{(0,0)} = \xi_0 \widehat{\Phi}_{(1,-1)} \,, \tag{2.30}$$

where $\widehat{\Phi}_{(1,-1)}$ carrying ghost number one and picture number minus one is in the small Hilbert space, namely, it is annihilated by η_0. The equation of motion

$$Q\eta_0 \Phi_{(0,0)} = 0 \tag{2.31}$$

in the large Hilbert space reduces to

$$Q\widehat{\Phi}_{(1,-1)} = 0 \,. \tag{2.32}$$

Since this is the familiar equation of motion in the small Hilbert space, we know that we can choose the condition

$$b_0 \widehat{\Phi}_{(1,-1)} = 0 \tag{2.33}$$

to fix the remaining gauge symmetry. This gauge-fixing condition can be stated for the original field $\Phi_{(0,0)}$ as $b_0\Phi_{(0,0)} = 0$ when $\xi_0\Phi_{(0,0)} = 0$ is imposed. The condition $b_0\Phi_{(0,0)} = 0$ can be satisfied by appropriately choosing $\epsilon_{(-1,0)}$, and it is compatible with $\xi_0\Phi_{(0,0)} = 0$ by adjusting $\epsilon_{(-1,1)}$. To summarize, we can choose

$$b_0 \Phi_{(0,0)} = 0 \,, \quad \xi_0 \Phi_{(0,0)} = 0 \tag{2.34}$$

as the gauge-fixing conditions on $\Phi_{(0,0)}$.

It is convenient to decompose $\Phi_{(g,p)}$ into four subsectors according to the zero modes b_0, c_0, η_0, and ξ_0 as follows:

$$\Phi_{(g,p)} = \Phi^{--}_{(g,p)} + c_0 \Phi^{c-}_{(g,p)} + \xi_0 \Phi^{-\xi}_{(g,p)} + c_0 \xi_0 \Phi^{c\xi}_{(g,p)} \,, \tag{2.35}$$

where $\Phi^{--}_{(g,p)}$, $\Phi^{c-}_{(g,p)}$, $\Phi^{-\xi}_{(g,p)}$, and $\Phi^{c\xi}_{(g,p)}$ are all annihilated by both b_0 and η_0. Note that the subscript (g,p) is carried over from $\Phi_{(g,p)}$ and does not indicate the ghost and picture numbers of the fields in the subsectors. The ghost and picture numbers (g,p) are (g,p) for $\Phi^{--}_{(g,p)}$, $(g-1,p)$ for $\Phi^{c-}_{(g,p)}$, $(g+1,p-1)$ for $\Phi^{-\xi}_{(g,p)}$, and $(g,p-1)$ for $\Phi^{c\xi}_{(g,p)}$. Using this notation, the gauge-fixing conditions (2.34) can be stated as

$$\Phi^{--}_{(0,0)} = 0\,, \quad \Phi^{c-}_{(0,0)} = 0\,, \quad \Phi^{c\xi}_{(0,0)} = 0\,. \tag{2.36}$$

The gauge-fixing term $S_{\rm GF}$ implementing these conditions can be written as

$$S_{\rm GF} = -\,\langle N^{c\xi}_{(2,-1)}|c_0\xi_0|\Phi^{--}_{(0,0)}\rangle + \langle N^{-\xi}_{(2,-1)}|c_0\xi_0|\Phi^{c-}_{(0,0)}\rangle + \langle N^{--}_{(2,-1)}|c_0\xi_0|\Phi^{c\xi}_{(0,0)}\rangle\,, \tag{2.37}$$

where the Lagrange multiplier fields $N^{c\xi}_{(2,-1)}$, $N^{-\xi}_{(2,-1)}$, and $N^{--}_{(2,-1)}$ are all annihilated by b_0 and η_0. Note that the insertion of $c_0\xi_0$ to each term is necessary for the inner product to be nonvanishing, as can be seen from (1.3). This term can be equivalently written as

$$S_{\rm GF} = \langle N_{(2,-1)}|\Phi_{(0,0)}\rangle \tag{2.38}$$

with the constraint

$$b_0\xi_0 N_{(2,-1)} = 0\,. \tag{2.39}$$

This can be seen by writing $N_{(2,-1)}$ before imposing the constraint as

$$N_{(2,-1)} = N^{--}_{(2,-1)} + c_0 N^{c-}_{(2,-1)} + \xi_0 N^{-\xi}_{(2,-1)} + c_0\xi_0 N^{c\xi}_{(2,-1)}\,, \tag{2.40}$$

where $N^{--}_{(2,-1)}$, $N^{c-}_{(2,-1)}$, $N^{-\xi}_{(2,-1)}$, and $N^{c\xi}_{(2,-1)}$ are all annihilated by both b_0 and η_0. The constraint $b_0\xi_0 N_{(2,-1)} = 0$ eliminates $N^{c-}_{(2,-1)}$, and the remaining fields $N^{c\xi}_{(2,-1)}$, $N^{-\xi}_{(2,-1)}$, and $N^{--}_{(2,-1)}$ implement $\Phi^{--}_{(0,0)} = 0$, $\Phi^{c-}_{(0,0)} = 0$, and $\Phi^{c\xi}_{(0,0)} = 0$.

Another way to derive $S_{\rm GF}$ is to use the form $b_0\Phi_{(0,0)} = \xi_0\Phi_{(0,0)} = 0$ for the gauge-fixing conditions and write

$$S_{\rm GF} = -\,\langle \widetilde{N}_{(3,-1)}|b_0|\Phi_{(0,0)}\rangle - \langle \widetilde{N}_{(3,-2)}|\xi_0|\Phi_{(0,0)}\rangle\,. \tag{2.41}$$

We then redefine the Lagrange multiplier as

$$N_{(2,-1)} = b_0\widetilde{N}_{(3,-1)} + \xi_0\widetilde{N}_{(3,-2)}\,. \tag{2.42}$$

The resulting field $N_{(2,-1)}$ is subject to the constraint $b_0\xi_0 N_{(2,-1)} = 0$, and the solution to the constraint can be written as $N_{(2,-1)} = b_0\widetilde{N}_{(3,-1)} + \xi_0\widetilde{N}_{(3,-2)}$. This time, however, $\widetilde{N}_{(3,-1)}$ and $\widetilde{N}_{(3,-2)}$ are not uniquely determined for a given solution. Comparing this with the decomposition (2.40), we find that $N^{--}_{(2,-1)}$ is in the part $b_0\widetilde{N}_{(3,-1)}$ and $c_0\xi_0 N^{c\xi}_{(2,-1)}$ is in the part $\xi_0\widetilde{N}_{(3,-2)}$, but $\xi_0 N^{-\xi}_{(2,-1)}$ can be in either part. This ambiguity is related to the fact that a part of $\widetilde{N}_{(3,-1)}$ and a part of $\widetilde{N}_{(3,-2)}$ impose the same constraint $\Phi^{c-}_{(0,0)} = 0$. More specifically, if we write $\widetilde{N}_{(3,-1)}$ and $\widetilde{N}_{(3,-2)}$ as

$$\widetilde{N}_{(3,-1)} = c_0\widetilde{N}^{c-}_{(3,-1)} + c_0\xi_0\widetilde{N}^{c\xi}_{(3,-1)}\,, \quad \widetilde{N}_{(3,-2)} = \widetilde{N}^{--}_{(3,-2)} + c_0\widetilde{N}^{c-}_{(3,-2)} \tag{2.43}$$

with $\widetilde{N}^{c-}_{(3,-1)}$, $\widetilde{N}^{c\xi}_{(3,-1)}$, $\widetilde{N}^{--}_{(3,-2)}$, and $\widetilde{N}^{c-}_{(3,-2)}$ all annihilated by both b_0 and η_0, both $\widetilde{N}^{c\xi}_{(3,-1)}$ and $\widetilde{N}^{--}_{(3,-2)}$ impose the condition $\Phi^{c-}_{(0,0)} = 0$. So we should be careful if we use $\widetilde{N}_{(3,-1)}$ and $\widetilde{N}_{(3,-2)}$ as Lagrange multiplier fields. No such issues arise if we use $N_{(2,-1)}$ with the constraint $b_0\xi_0 N_{(2,-1)} = 0$ as the Lagrange multiplier field.

The Faddeev-Popov term $S_{\rm FP}$ can be obtained from $S_{\rm GF}$ by changing $N_{(2,-1)}$ to the Grassmann-odd antighost field $\Phi_{(2,-1)}$ and by changing $\Phi_{(0,0)}$ to its gauge transformations $Q\epsilon_{(-1,0)}+\eta_0\epsilon_{(-1,1)}$ with $\epsilon_{(-1,0)}$ and $\epsilon_{(-1,1)}$ replaced by the Grassmann-even ghost fields $\Phi_{(-1,0)}$ and $\Phi_{(-1,1)}$, respectively. We have

$$S_{\rm FP} = \langle \Phi_{(2,-1)} | \Big(Q|\Phi_{(-1,0)}\rangle + \eta_0 |\Phi_{(-1,1)}\rangle \Big) \tag{2.44}$$

with the constraint

$$b_0\xi_0\Phi_{(2,-1)} = 0\,, \tag{2.45}$$

which is inherited from $b_0\xi_0 N_{(2,-1)} = 0$. After integrating out $N_{(2,-1)}$, the total action we obtain is

$$S_0 + S_1 = -\frac{1}{2}\langle \Phi_{(0,0)}|Q\eta_0|\Phi_{(0,0)}\rangle + \langle \Phi_{(2,-1)} | \Big(Q|\Phi_{(-1,0)}\rangle + \eta_0 |\Phi_{(-1,1)}\rangle \Big) \tag{2.46}$$

with

$$b_0\Phi_{(0,0)} = 0\,, \quad \xi_0\Phi_{(0,0)} = 0\,, \quad b_0\xi_0\Phi_{(2,-1)} = 0\,. \tag{2.47}$$

The action S_1 can be written in the following form:

$$S_1 = \langle\, \Phi_{(2,-1)} |\, \begin{pmatrix} Q & \eta_0 \end{pmatrix} \begin{pmatrix} \Phi_{(-1,0)} \\ \Phi_{(-1,1)} \end{pmatrix} \rangle\,. \tag{2.48}$$

This action $S_0 + S_1$ is invariant under the following gauge transformations:

$$\begin{aligned} \delta_\epsilon \Phi_{(-1,0)} &= Q\epsilon_{(-2,0)} + \eta_0\epsilon_{(-2,1)}\,, \\ \delta_\epsilon \Phi_{(-1,1)} &= Q\epsilon_{(-2,1)} + \eta_0\epsilon_{(-2,2)}\,, \end{aligned} \tag{2.49}$$

which can also be written as

$$\delta_\epsilon \begin{pmatrix} \Phi_{(-1,0)} \\ \Phi_{(-1,1)} \end{pmatrix} = \begin{pmatrix} Q & \eta_0 & 0 \\ 0 & Q & \eta_0 \end{pmatrix} \begin{pmatrix} \epsilon_{(-2,0)} \\ \epsilon_{(-2,1)} \\ \epsilon_{(-2,2)} \end{pmatrix}\,. \tag{2.50}$$

We can choose $\epsilon_{(-2,1)}$ appropriately such that the condition

$$\xi_0\Phi_{(-1,0)} = 0 \tag{2.51}$$

on $\Phi_{(-1,0)}$ is satisfied. Moreover, we can choose $\epsilon_{(-2,2)}$ appropriately such that the condition

$$\xi_0\Phi_{(-1,1)} = 0 \tag{2.52}$$

on $\Phi_{(-1,1)}$ is satisfied. Then $\Phi_{(-1,0)}$ satisfying (2.51) can be written as

$$\Phi_{(-1,0)} = \xi_0 \widehat{\Phi}_{(0,-1)}\,, \tag{2.53}$$

where $\widehat{\Phi}_{(0,-1)}$ is in the small Hilbert space. We can then choose the condition

$$b_0 \widehat{\Phi}_{(0,-1)} = 0\,, \tag{2.54}$$

to fix the remaining gauge symmetry. This gauge-fixing condition can be stated for the original field $\Phi_{(-1,0)}$ as $b_0\Phi_{(-1,0)} = 0$ when $\xi_0\Phi_{(-1,0)} = 0$ is imposed. The condition $b_0\Phi_{(-1,0)} = 0$ can be satisfied by appropriately choosing $\epsilon_{(-2,0)}$, and it is compatible with $\xi_0\Phi_{(-1,0)} = 0$ and $\xi_0\Phi_{(-1,1)} = 0$ by adjusting $\epsilon_{(-2,1)}$ and $\epsilon_{(-2,2)}$. To summarize, we can choose

$$b_0\Phi_{(-1,0)} = 0\,, \quad \xi_0\Phi_{(-1,0)} = 0\,, \quad \xi_0\Phi_{(-1,1)} = 0\,, \tag{2.55}$$

as the gauge-fixing conditions on $\Phi_{(-1,0)}$ and $\Phi_{(-1,1)}$.

Each of $\Phi_{(-1,0)}$ and $\Phi_{(-1,1)}$ can be decomposed into four subsectors as before so that we have eight subsectors in total. It is straightforward to see that the conditions (2.55) eliminate five of the eight subsectors and three subsectors remain, which match with the three remaining subsectors of $\Phi_{(2,-1)}$ after imposing the constraint $b_0\xi_0\Phi_{(2,-1)} = 0$. We can thus invert the kinetic term S_1 to obtain the propagator, as we explicitly do in the next section. It is also straightforward to see that the five conditions can be implemented by the Lagrange multiplier fields $N_{(3,-1)}$ and $N_{(3,-2)}$ as

$$S_{\rm GF} = \langle N_{(3,-1)} | \Phi_{(-1,0)} \rangle + \langle N_{(3,-2)} | \Phi_{(-1,1)} \rangle\,, \tag{2.56}$$

with the constraints

$$b_0\xi_0 N_{(3,-1)} = 0\,, \quad \xi_0 N_{(3,-2)} = 0\,. \tag{2.57}$$

This can be verified by decomposing each of $N_{(3,-1)}$ and $N_{(3,-2)}$ into four subsectors. The corresponding Faddeev-Popov term is then given by

$$S_{\rm FP} = \langle \Phi_{(3,-1)} | \Big(Q|\Phi_{(-2,0)}\rangle + \eta_0 |\Phi_{(-2,1)}\rangle \Big) + \langle \Phi_{(3,-2)} | \Big(Q|\Phi_{(-2,1)}\rangle + \eta_0 |\Phi_{(-2,2)}\rangle \Big) \tag{2.58}$$

with

$$b_0\xi_0 \Phi_{(3,-1)} = 0\,, \quad \xi_0 \Phi_{(3,-2)} = 0\,. \tag{2.59}$$

After integrating out $N_{(3,-1)}$ and $N_{(3,-2)}$, the total action we obtain is

$$\begin{aligned} S_0 + S_1 + S_2 = & -\frac{1}{2}\langle \Phi_{(0,0)} | Q\eta_0 | \Phi_{(0,0)} \rangle + \langle \Phi_{(2,-1)} | \Big(Q|\Phi_{(-1,0)}\rangle + \eta_0 |\Phi_{(-1,1)}\rangle \Big) \\ & + \langle \Phi_{(3,-1)} | \Big(Q|\Phi_{(-2,0)}\rangle + \eta_0 |\Phi_{(-2,1)}\rangle \Big) + \langle \Phi_{(3,-2)} | \Big(Q|\Phi_{(-2,1)}\rangle + \eta_0 |\Phi_{(-2,2)}\rangle \Big) \end{aligned} \tag{2.60}$$

with

$$\begin{aligned} & b_0\Phi_{(0,0)} = 0\,, \quad \xi_0\Phi_{(0,0)} = 0\,, && b_0\xi_0\Phi_{(2,-1)} = 0\,, \\ & b_0\Phi_{(-1,0)} = 0\,, \quad \xi_0\Phi_{(-1,0)} = 0\,, \quad \xi_0\Phi_{(-1,1)} = 0\,, && b_0\xi_0\Phi_{(3,-1)} = 0\,, \quad \xi_0\Phi_{(3,-2)} = 0\,. \end{aligned} \tag{2.61}$$

The action S_2 can be written in the following form:

$$S_2 = \Big\langle \begin{pmatrix} \Phi_{(3,-1)} & \Phi_{(3,-2)} \end{pmatrix} \Big| \begin{pmatrix} Q & \eta_0 & 0 \\ 0 & Q & \eta_0 \end{pmatrix} \begin{pmatrix} \Phi_{(-2,0)} \\ \Phi_{(-2,1)} \\ \Phi_{(-2,2)} \end{pmatrix} \Big\rangle . \tag{2.62}$$

The action $S_0 + S_1 + S_2$ is invariant under the following gauge transformations:

$$\delta_\epsilon \begin{pmatrix} \Phi_{(-2,0)} \\ \Phi_{(-2,1)} \\ \Phi_{(-2,2)} \end{pmatrix} = \begin{pmatrix} Q & \eta_0 & 0 & 0 \\ 0 & Q & \eta_0 & 0 \\ 0 & 0 & Q & \eta_0 \end{pmatrix} \begin{pmatrix} \epsilon_{(-3,0)} \\ \epsilon_{(-3,1)} \\ \epsilon_{(-3,2)} \\ \epsilon_{(-3,3)} \end{pmatrix} . \tag{2.63}$$

It is straightforward to show that we can impose the conditions

$$b_0 \Phi_{(-2,0)} = \xi_0 \Phi_{(-2,0)} = 0 \,, \quad \xi_0 \Phi_{(-2,1)} = 0 \,, \quad \xi_0 \Phi_{(-2,2)} = 0 \tag{2.64}$$

for gauge fixing. In this way the gauge-fixing procedure continues, and at the end we obtain

$$S = \sum_{n=0}^{\infty} S_n \,, \tag{2.65}$$

where S_n for $n \geq 1$ is

$$S_n = \Bigg\langle \begin{pmatrix} \Phi_{(n+1,-1)} & \Phi_{(n+1,-2)} & \cdots & \Phi_{(n+1,-n)} \end{pmatrix} \Bigg| \begin{pmatrix} Q & \eta_0 & 0 & \cdots & 0 & 0 \\ 0 & Q & \eta_0 & \cdots & 0 & 0 \\ 0 & 0 & Q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q & \eta_0 \end{pmatrix} \begin{pmatrix} \Phi_{(-n,0)} \\ \Phi_{(-n,1)} \\ \vdots \\ \Phi_{(-n,n)} \end{pmatrix} \Bigg\rangle \tag{2.66}$$

with

$$\begin{gathered} b_0 \Phi_{(-n,0)} = \xi_0 \Phi_{(-n,0)} = 0 \,, \qquad b_0 \xi_0 \Phi_{(n+1,-1)} = 0 \,, \\ \xi_0 \begin{pmatrix} \Phi_{(-n,1)} \\ \Phi_{(-n,2)} \\ \vdots \\ \Phi_{(-n,n)} \end{pmatrix} = 0 \,, \qquad \xi_0 \begin{pmatrix} \Phi_{(n+1,-2)} \\ \Phi_{(n+1,-3)} \\ \vdots \\ \Phi_{(n+1,-n)} \end{pmatrix} = 0 \,. \end{gathered} \tag{2.67}$$

2.3 Comparison with Witten's superstring field theory in Siegel gauge

We have seen that string fields of various ghost and picture numbers appear in the process of gauge fixing, and we imposed various conditions on these string fields. While those features may look exotic, we will demonstrate that the gauge-fixed action of the free superstring field theory in the Berkovits formulation derived in the preceding subsection describes the conventional physics by showing that it reduces to the gauge-fixed action of the free superstring field theory in the Witten formulation using Siegel gauge after eliminating auxiliary fields.

The gauge-invariant action of Witten's superstring field theory is given by

$$\widetilde{S}_0 = -\frac{1}{2} \langle\langle \Psi_{(1,-1)} | Q | \Psi_{(1,-1)} \rangle\rangle \,, \tag{2.68}$$

where $\langle\langle A|B \rangle\rangle$ is the BPZ inner product of A and B in the small Hilbert space, which is related to $\langle A|B \rangle$ in the large Hilbert space as $\langle\langle A|B \rangle\rangle = (-1)^A \langle A|\xi_0|B \rangle$ up to an overall sign depending on a convention. Here $(-1)^A = 1$ when A is Grassmann even and $(-1)^A = -1$ when A is Grassmann odd. Gauge fixing in Siegel gauge is completely parallel to that in the bosonic string, and the gauge-fixed action is given by

$$\widetilde{S} = \sum_{n=0}^{\infty} \widetilde{S}_n \,, \tag{2.69}$$

where

$$\widetilde{S}_0 = -\frac{1}{2} \langle\langle \Psi_{(1,-1)} | Q | \Psi_{(1,-1)} \rangle\rangle \,, \qquad \widetilde{S}_n = - \langle\langle \Psi_{(n+1,-1)} | Q | \Psi_{(-n+1,-1)} \rangle\rangle \quad \text{for} \quad n \geq 1 \tag{2.70}$$

with

$$b_0 \Psi_{(n,-1)} = 0 \,, \quad \forall \, n \,. \tag{2.71}$$

As in the bosonic case, the action $\widetilde{S}$ can also be written compactly as

$$\widetilde{S} = -\frac{1}{2} \langle\langle \Psi | Q | \Psi \rangle\rangle \quad \text{with} \quad \Psi = \sum_{n=-\infty}^{\infty} \Psi_{(n,-1)} \,, \quad b_0 \Psi = 0 \,. \tag{2.72}$$

Since Ψ is annihilated by b_0, we need c_0 from Q for the inner product to be nonvanishing. Using $\{Q, b_0\} = L_0$, we see that the gauge-fixed action reduces to

$$\widetilde{S} = -\frac{1}{2} \langle\langle \Psi | c_0 L_0 | \Psi \rangle\rangle \,. \tag{2.73}$$

Similarly, $\widetilde{S}_n$ reduces to

$$\widetilde{S}_0 = -\frac{1}{2} \langle\langle \Psi_{(1,-1)} | c_0 L_0 | \Psi_{(1,-1)} \rangle\rangle \,, \qquad \widetilde{S}_n = - \langle\langle \Psi_{(n+1,-1)} | c_0 L_0 | \Psi_{(-n+1,-1)} \rangle\rangle \quad \text{for} \quad n \geq 1 \,. \tag{2.74}$$

We will compare this with the gauge-fixed action derived in the preceding subsection. Let us start with S_0. Under the gauge-fixing conditions (2.34) the string field $\Phi_{(0,0)}$ reduces to

$$\Phi_{(0,0)} = \xi_0 \Phi^{-\xi}_{(0,0)} \,. \tag{2.75}$$

Then the action S_0 reduces to

$$S_0 = \frac{1}{2} \langle \Phi^{-\xi}_{(0,0)} | \, \xi_0 Q \eta_0 \xi_0 \, | \Phi^{-\xi}_{(0,0)} \rangle = \frac{1}{2} \langle \Phi^{-\xi}_{(0,0)} | \xi_0 Q | \Phi^{-\xi}_{(0,0)} \rangle = \frac{1}{2} \langle \Phi^{-\xi}_{(0,0)} | \xi_0 c_0 L_0 | \Phi^{-\xi}_{(0,0)} \rangle \,. \tag{2.76}$$

This coincides with

$$\widetilde{S}_0 = -\frac{1}{2} \langle\langle \Psi_{(1,-1)} | c_0 L_0 | \Psi_{(1,-1)} \rangle\rangle \tag{2.77}$$

in Witten's theory under the identification

$$\Psi_{(1,-1)} = \Phi^{-\xi}_{(0,0)} \,. \tag{2.78}$$

Let us next consider S_1. Under the gauge-fixing conditions (2.55), $\Phi_{(-1,0)}$ and $\Phi_{(-1,1)}$ reduce to

$$\Phi_{(-1,0)} = \xi_0 \Phi^{-\xi}_{(-1,0)} \,, \qquad \Phi_{(-1,1)} = \xi_0 \Phi^{-\xi}_{(-1,1)} + c_0 \xi_0 \Phi^{c\xi}_{(-1,1)} \,. \tag{2.79}$$

Then the action S_1 reduces to

$$\begin{aligned} S_1 &= \langle \Phi_{(2,-1)} | Q \xi_0 | \Phi^{-\xi}_{(-1,0)} \rangle + \langle \Phi_{(2,-1)} | \eta_0 \xi_0 | \Phi^{-\xi}_{(-1,1)} \rangle + \langle \Phi_{(2,-1)} | \eta_0 c_0 \xi_0 | \Phi^{c\xi}_{(-1,1)} \rangle \\ &= \langle \Phi_{(2,-1)} | Q \xi_0 | \Phi^{-\xi}_{(-1,0)} \rangle + \langle \Phi_{(2,-1)} | \Phi^{-\xi}_{(-1,1)} \rangle - \langle \Phi_{(2,-1)} | c_0 | \Phi^{c\xi}_{(-1,1)} \rangle \,. \end{aligned} \tag{2.80}$$

The antighost field $\Phi_{(2,-1)}$ with the constraint $b_0 \xi_0 \Phi_{(2,-1)} = 0$ can be decomposed as

$$\Phi_{(2,-1)} = \Phi^{--}_{(2,-1)} + \xi_0 \Phi^{-\xi}_{(2,-1)} + c_0 \xi_0 \Phi^{c\xi}_{(2,-1)} \,, \tag{2.81}$$

and the last two terms on the right-hand side of (2.80) reduce to

$$\langle \Phi_{(2,-1)} | \Phi^{-\xi}_{(-1,1)} \rangle = \, - \, \langle \Phi^{c\xi}_{(2,-1)} | c_0 \xi_0 | \Phi^{-\xi}_{(-1,1)} \rangle \,, \quad \langle \Phi_{(2,-1)} | c_0 | \Phi^{c\xi}_{(-1,1)} \rangle = \, - \, \langle \Phi^{-\xi}_{(2,-1)} | c_0 \xi_0 | \Phi^{c\xi}_{(-1,1)} \rangle \,. \tag{2.82}$$

Since $\Phi^{-\xi}_{(-1,1)}$ and $\Phi^{c\xi}_{(-1,1)}$ only appear in these terms, these fields act as Lagrange multiplier fields imposing

$$\Phi^{c\xi}_{(2,-1)} = 0 \,, \qquad \Phi^{-\xi}_{(2,-1)} = 0 \,. \tag{2.83}$$

After integrating out $\Phi^{-\xi}_{(-1,1)}$ and $\Phi^{c\xi}_{(-1,1)}$, the action S_1 therefore reduces to

$$S_1 = \langle \Phi^{--}_{(2,-1)} | Q \xi_0 | \Phi^{-\xi}_{(-1,0)} \rangle = \langle \Phi^{--}_{(2,-1)} | c_0 L_0 \xi_0 | \Phi^{-\xi}_{(-1,0)} \rangle \,. \tag{2.84}$$

This coincides with

$$\widetilde{S}_1 = - \langle\langle \Psi_{(2,-1)} | c_0 L_0 | \Psi_{(0,-1)} \rangle\rangle \tag{2.85}$$

in Witten's theory under the identification

$$\Psi_{(2,-1)} = - \Phi^{--}_{(2,-1)} \,, \qquad \Psi_{(0,-1)} = \Phi^{-\xi}_{(-1,0)} \,. \tag{2.86}$$

We can similarly show that S_n with $n \geq 1$ reduces to

$$S_n = \langle \Phi^{--}_{(n+1,-1)} | Q \xi_0 | \Phi^{-\xi}_{(-n,0)} \rangle = \langle \Phi^{--}_{(n+1,-1)} | c_0 L_0 \xi_0 | \Phi^{-\xi}_{(-n,0)} \rangle \tag{2.87}$$

and coincides with

$$\widetilde{S}_n = - \langle\langle \Psi_{(n+1,-1)} | c_0 L_0 | \Psi_{(-n+1,-1)} \rangle\rangle \tag{2.88}$$

in Witten's theory under the identification

$$\Psi_{(n+1,-1)} = - \Phi^{--}_{(n+1,-1)} \,, \qquad \Psi_{(-n+1,-1)} = \Phi^{-\xi}_{(-n,0)} \qquad \text{for} \quad n \geq 1 \,. \tag{2.89}$$

We have thus shown that the gauge-fixed action derived in the preceding subsection coincides with that of Witten's superstring field theory in Siegel gauge after integrating out auxiliary fields.

While the kinetic term of Witten's superstring field theory is consistent, there are problems in the construction of the cubic interaction term using the picture-changing operator. On the other hand, interaction terms can be constructed without using picture-changing operators in the Berkovits formulation. We have confirmed that both theories describe the same physics in the free case, and we expect a regular extension to the interacting theory in the Berkovits formulation.

2.4 Various gauge-fixing conditions

In subsection 2.2, we have seen that the completely gauge-fixed action in the WZW-type open superstring field theory is given by the sum (2.65) of the original action S_0 and all the Faddeev-Popov terms with the gauge-fixing conditions (2.67). As we will see later in section 4, the action (2.65) is precisely the solution to the (classical) master equation in the BV formalism if we identify antighosts with antifields.[4] From this point of view S is a universal quantity and different gauge-fixed actions can be obtained simply by imposing different conditions on Φ's. In this subsection, we list some gauge-fixing conditions different from (2.67). For further generalization and for the validity of the gauge-fixing conditions, see [24].

Let us first mention a one-parameter extension of (2.67):

$$\begin{aligned} b_0\Phi_{(-n,0)} &= 0 \quad (n \geq 0)\,, \\ \xi_0\Phi_{(-n,m)} + \alpha b_0\Phi_{(-n,m+1)} &= 0 \quad (0 \leq m \leq n-1)\,, \\ \xi_0\Phi_{(-n,n)} &= 0 \quad (n \geq 0)\,, \\ \\ b_0\xi_0\Phi_{(n+1,-1)} &= 0 \quad (n \geq 1)\,, \\ \alpha b_0\Phi_{(n+1,-m)} + \xi_0\Phi_{(n+1,-(m+1))} &= 0 \quad (1 \leq m \leq n-1)\,. \end{aligned} \tag{2.90}$$

The previous condition corresponds to the case in which the parameter α is zero. Unlike (2.67), the above set of equations includes linear combinations of Φ's.

Another interesting class of gauge-fixing conditions is obtained when we use the zero mode d_0 of the operator $d = [Q, b\xi]$, instead of ξ_0:

$$\begin{aligned} b_0\Phi_{(-n,0)} &= 0 \quad (n \geq 0)\,, \\ d_0\Phi_{(-n,m)} + \alpha b_0\Phi_{(-n,m+1)} &= 0 \quad (0 \leq m \leq n-1)\,, \\ d_0\Phi_{(-n,n)} &= 0 \quad (n \geq 0)\,, \\ \\ b_0 d_0\Phi_{(n+1,-1)} &= 0 \quad (n \geq 1)\,, \\ \alpha b_0\Phi_{(n+1,-m)} + d_0\Phi_{(n+1,-(m+1))} &= 0 \quad (1 \leq m \leq n-1)\,. \end{aligned} \tag{2.91}$$

[4] In the language of the BV formalism we have chosen a gauge-fixing fermion such that antifields of minimal-sector fields are identified with antighosts. In this paper we consider only such gauge-fixing conditions, and thus we will not distinguish antifields and antighosts.

The operator d is identical to the generator $\widetilde{G}^-$ of the twisted $N=2$ superconformal algebra investigated by Berkovits and Vafa [19]. Because d is a counterpart of b, it seems natural to adopt the symmetric gauge, in which $\alpha = 1$.

In the next section we will calculate propagators, mainly considering the gauge (2.90) with $\alpha = 0$, which is identical to (2.67), and the gauge (2.91) with $\alpha = 1$.

3 Calculation of propagators

Let us derive propagators under the gauge-fixing conditions proposed in the preceding section. For this purpose, we introduce source terms of the form

$$S_0^J = \big\langle \Phi_{(0,0)} \,|\, J_{(2,-1)} \big\rangle \,, \tag{3.1a}$$

$$\begin{aligned} S_n^J &= \sum_{m=0}^{n} \big\langle \Phi_{(-n,m)} \,|\, J_{(n+2,-m-1)} \big\rangle + \sum_{m=1}^{n} \big\langle \Phi_{(n+1,-m)} \,|\, J_{(-n+1,m-1)} \big\rangle \\ &= \left\langle \begin{pmatrix} \Phi_{(-n,0)} & \cdots & \Phi_{(-n,n)} \end{pmatrix} \middle| \begin{pmatrix} J_{(n+2,-1)} \\ \vdots \\ J_{(n+2,-(n+1))} \end{pmatrix} \right\rangle \\ &\quad + \left\langle \begin{pmatrix} \Phi_{(n+1,-1)} & \cdots & \Phi_{(n+1,-n)} \end{pmatrix} \middle| \begin{pmatrix} J_{(-(n-1),0)} \\ \vdots \\ J_{(-(n-1),n-1)} \end{pmatrix} \right\rangle \qquad (n \geq 1)\,, \end{aligned} \tag{3.1b}$$

and consider the action

$$S_n[J] = S_n + S_n^J \quad (n \geq 0)\,. \tag{3.2}$$

Here J's of positive (non-positive) ghost number are Grassmann-even (Grassmann-odd) sources. Each source is coupled with a Φ of the same Grassmann parity. Note that Φ's are subject to their gauge-fixing conditions, but sources are free from any constraints. The actions S_0 and S_n, for $n \geq 1$, were defined in (2.27) and (2.66), respectively. Starting from the action (3.2), we can calculate propagators as follows. First we solve the equations of motion of the Φ's derived from $S_n[J]$ in order to find a stationary point. Then we put the solution back into $S_n[J]$, to obtain a quadratic form of J's, from which propagators can be read off.

3.1 Propagators for gauge fixing with b_0 and ξ_0

Let us first apply the above-mentioned procedure to the gauge (2.67). To calculate the propagator of $\Phi_{(0,0)}$ we start from the action

$$S_0[J] = -\frac{1}{2} \big\langle \Phi_{(0,0)} \,|\, Q\eta_0 \,|\, \Phi_{(0,0)} \big\rangle + \big\langle \Phi_{(0,0)} \,|\, J_{(2,-1)} \big\rangle \,. \tag{3.3}$$

Gauge-fixing conditions for the field $\Phi_{(0,0)}$ are of the form

$$b_0\Phi_{(0,0)} = \xi_0\Phi_{(0,0)} = 0\,, \tag{3.4}$$

which leads to

$$\Phi_{(0,0)} = \{b_0, c_0\}\{\xi_0, \eta_0\}\,\Phi_{(0,0)} = b_0c_0\xi_0\eta_0\,\Phi_{(0,0)}\,. \tag{3.5}$$

Thus, the equation of motion derived from $S_0[J]$ is

$$\eta_0\xi_0\,c_0b_0\Big(\,Q\eta_0\,\Phi_{(0,0)} - J_{(2,-1)}\Big) = 0\,. \tag{3.6}$$

This can be solved easily. Using the identity

$$Q\eta_0\frac{\xi_0b_0}{L_0} = 1 - \frac{b_0Q}{L_0} - \xi_0\eta_0 + \frac{b_0Q}{L_0}\xi_0\eta_0\,, \tag{3.7}$$

we find that the solution is given by

$$\Phi_{(0,0)} = \frac{\xi_0b_0}{L_0}\,J_{(2,-1)}\,. \tag{3.8}$$

Note that this solution is consistent with the conditions (3.4). Evaluating the action (3.3) for this solution determines the propagator of $\Phi_{(0,0)}$:

$$S_0[J] \;=\; \frac{1}{2}\Big\langle J_{(2,-1)}\Big|\,\frac{\xi_0b_0}{L_0}\,\Big|\,J_{(2,-1)}\Big\rangle\,. \tag{3.9}$$

Next, let us consider ghost propagators. The action $S_1[J]$ takes the form

$$\begin{aligned} S_1[J] \;=\;& \langle\Phi_{(2,-1)}|\,\Big(Q|\Phi_{(-1,0)}\rangle + \eta_0|\Phi_{(-1,1)}\rangle\Big) \\ &+ \langle\Phi_{(2,-1)}\,|\,J_{(0,0)}\rangle + \langle\Phi_{(-1,0)}\,|\,J_{(3,-1)}\rangle + \langle\Phi_{(-1,1)}\,|\,J_{(3,-2)}\rangle\,. \end{aligned} \tag{3.10}$$

The gauge-fixing conditions at this step are

$$\begin{aligned} b_0\Phi_{(-1,0)} = \xi_0\Phi_{(-1,0)} \;&=\; 0\,, \\ \xi_0\Phi_{(-1,1)} \;&=\; 0\,, \\ b_0\xi_0\,\Phi_{(2,-1)} \;&=\; 0\,. \end{aligned} \tag{3.11}$$

Under these conditions, we have

$$\begin{aligned} &\Phi_{(-1,0)} = b_0c_0\xi_0\eta_0\Phi_{(-1,0)}\,, \\ &\Phi_{(-1,1)} = \xi_0\eta_0\Phi_{(-1,1)}\,, \\ &\Phi_{(2,-1)} = \{b_0, c_0\}\{\xi_0, \eta_0\}\,\Phi_{(2,-1)} = (b_0c_0 + c_0b_0\xi_0\eta_0)\,\Phi_{(2,-1)}\,. \end{aligned} \tag{3.12}$$

Therefore, the equations of motion are

$$\begin{aligned} (c_0b_0 + b_0c_0\eta_0\xi_0)\,\big(Q\Phi_{(-1,0)} + \eta_0\Phi_{(-1,1)} + J_{(0,0)}\big) &= 0\,, \\ \eta_0\xi_0c_0b_0\,\big(Q\Phi_{(2,-1)} + J_{(3,-1)}\big) &= 0\,, \\ \eta_0\xi_0\,\big(\eta_0\Phi_{(2,-1)} + J_{(3,-2)}\big) &= 0\,. \end{aligned} \tag{3.13}$$

Let us find a solution compatible with the conditions (3.11). This can be readily achieved by the use of the zero mode decomposition (2.35) of Φ's. The solution is given by

$$\begin{aligned}\Phi_{(-1,0)} &= -\frac{b_0}{L_0}\,\xi_0\eta_0\,J_{(0,0)}\,,\\ \Phi_{(-1,1)} &= \Big(-\xi_0+\frac{b_0}{L_0}\,\xi_0\eta_0X_0\Big)J_{(0,0)}\,,\\ \Phi_{(2,-1)} &= -\frac{b_0}{L_0}\,\eta_0\xi_0J_{(3,-1)}+\Big(-\xi_0+\frac{b_0}{L_0}\,\eta_0\xi_0X_0\Big)\,J_{(3,-2)}\,,\end{aligned} \tag{3.14}$$

where X_0 is the zero mode of the picture-changing operator $X=\{Q,\xi\}$. When the equations for $\Phi_{(-1,0)}$ and $\Phi_{(-1,1)}$ hold, the action $S_1[J]$ reduces to

$$S_1[J]=\langle\Phi_{(2,-1)}\,|\,J_{(0,0)}\rangle=-\langle J_{(0,0)}\,|\,\Phi_{(2,-1)}\rangle\,. \tag{3.15}$$

Substituting the solution (3.14) into (3.15), we immediately obtain

$$S_1[J] \;=\; \Big\langle J_{(0,0)}\,\Big|\,\frac{1}{L_0}b_0\eta_0\xi_0\,\Big|\,J_{(3,-1)}\Big\rangle+\;\Big\langle J_{(0,0)}\,\Big|\,\Big(1-\frac{1}{L_0}\,b_0\eta_0X_0\Big)\,\xi_0\,\Big|\,J_{(3,-2)}\Big\rangle\,. \tag{3.16}$$

On the other hand, when the equation of motion of $\Phi_{(2,-1)}$ holds, the action becomes

$$S_1[J]=\langle\Phi_{(-1,0)}\,|\,J_{(3,-1)}\rangle+\langle\Phi_{(-1,1)}\,|\,J_{(3,-2)}\rangle\,. \tag{3.17}$$

Needless to say, plugging the solution (3.14) into (3.17) gives the same result as in (3.16). The above expression can be rewritten by using a one-by-two propagator matrix:

$$S_1[J] \;=\; \Big\langle J_{(0,0)}\,\Big|\,\begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}J_{(3,-1)}\\ J_{(3,-2)}\end{pmatrix}\Big\rangle\,, \tag{3.18}$$

with

$$A\equiv\frac{1}{L_0}b_0\eta_0\xi_0\,,\quad B\equiv\Big(1-\frac{1}{L_0}\,b_0\eta_0X_0\Big)\,\xi_0\,. \tag{3.19}$$

We emphasize that the propagator includes the zero mode of the picture-changing operator.

We can continue the calculation in this manner. The action $S_2[J]$ takes the form

$$\begin{aligned}S_2[J] \;=\;& \langle\Phi_{(3,-1)}|\,\Big(Q|\Phi_{(-2,0)}\rangle+\eta_0|\Phi_{(-2,1)}\rangle\Big)+\langle\Phi_{(3,-1)}\,|\,J_{(-1,0)}\rangle\\ &+\langle\Phi_{(3,-2)}|\,\Big(Q|\Phi_{(-2,1)}\rangle+\eta_0|\Phi_{(-2,2)}\rangle\Big)+\langle\Phi_{(3,-2)}\,|\,J_{(-1,1)}\rangle\\ &+\langle\Phi_{(-2,0)}\,|\,J_{(4,-1)}\rangle+\langle\Phi_{(-2,1)}\,|\,J_{(4,-2)}\rangle+\langle\Phi_{(-2,2)}\,|\,J_{(4,-3)}\rangle\,,\end{aligned} \tag{3.20}$$

and the gauge-fixing conditions are given by

$$\begin{aligned}b_0\Phi_{(-2,0)}=\xi_0\Phi_{(-2,0)}&=0\,,\\ \xi_0\Phi_{(-2,1)}&=0\,,\\ \xi_0\Phi_{(-2,2)}&=0\,,\\ b_0\xi_0\Phi_{(3,-1)}&=0\,,\\ \xi_0\Phi_{(3,-2)}&=0\,.\end{aligned} \tag{3.21}$$

When the equations of motion for $\Phi_{(-2,0)}$, $\Phi_{(-2,1)}$, and $\Phi_{(-2,2)}$ are satisfied, the action $S_2[J]$ reduces to

$$S_2[J] = \langle \Phi_{(3,-1)} \,|\, J_{(-1,0)} \rangle + \langle \Phi_{(3,-2)} \,|\, J_{(-1,1)} \rangle = -\langle J_{(-1,0)} \,|\, \Phi_{(3,-1)} \rangle - \langle J_{(-1,1)} \,|\, \Phi_{(3,-2)} \rangle \,. \tag{3.22}$$

Substituting into (3.22) the solution of the equations of motion

$$\begin{aligned} \Phi_{(3,-1)} &= -\big(A\, J_{(4,-1)} + B\, J_{(4,-2)} + (-X_0) B\, J_{(4,-3)}\big)\,, \\ \Phi_{(3,-2)} &= \qquad\qquad\qquad\qquad\qquad\qquad\quad -\xi_0\, J_{(4,-3)}\,, \end{aligned} \tag{3.23}$$

we obtain

$$S_2[J] = \left\langle \begin{pmatrix} J_{(-1,0)} & J_{(-1,1)} \end{pmatrix} \,\middle|\, \begin{pmatrix} A & B & (-X_0)B \\ 0 & 0 & \xi_0 \end{pmatrix} \begin{pmatrix} J_{(4,-1)} \\ J_{(4,-2)} \\ J_{(4,-3)} \end{pmatrix} \right\rangle . \tag{3.24}$$

At the next step, the propagator matrix is given by

$$\begin{pmatrix} A & B & (-X_0)B & (-X_0)^2 B \\ 0 & 0 & \xi_0 & (-X_0)\xi_0 \\ 0 & 0 & 0 & \xi_0 \end{pmatrix}, \tag{3.25}$$

and at the n-th step we obtain the $n \times (n+1)$ matrix

$$\begin{pmatrix} A & B & (-X_0)B & (-X_0)^2 B & \dots & (-X_0)^{n-1} B \\ 0 & 0 & \xi_0 & (-X_0)\xi_0 & \dots & (-X_0)^{n-2}\xi_0 \\ 0 & 0 & 0 & \xi_0 & \dots & (-X_0)^{n-3}\xi_0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \xi_0 \end{pmatrix} . \tag{3.26}$$

The propagators in the gauge (2.90) with $\alpha \neq 0$ can be calculated in the same manner. The result, however, is a little complicated. To see this, we calculate the first-step ghost propagator. (Note that since the condition on $\Phi_{(0,0)}$ does not include α, the propagator of $\Phi_{(0,0)}$ is independent of the parameter.) We start with the gauge-fixing conditions below:

$$\begin{aligned} b_0 \Phi_{(-1,0)} &= 0\,, \\ \xi_0 \Phi_{(-1,0)} + \alpha\, b_0 \Phi_{(-1,1)} &= 0\,, \\ \xi_0 \Phi_{(-1,1)} &= 0\,, \\ b_0 \xi_0\, \Phi_{(2,-1)} &= 0\,. \end{aligned} \tag{3.27}$$

This time, the solution to the equations of motion derived from (3.10) is

$$\begin{aligned} \Phi_{(-1,0)} &= -\Big(\frac{\alpha}{1+\alpha\, L_0}\, b_0 \eta_0 \xi_0 + \frac{b_0}{L_0}\, \xi_0 \eta_0\Big) J_{(0,0)}\,, \\ \Phi_{(-1,1)} &= \Big(-\xi_0 + \frac{b_0}{L_0}\, \xi_0 \eta_0 X_0 + \frac{\alpha}{1+\alpha\, L_0}\, \xi_0 Q b_0 \eta_0 \xi_0\Big) J_{(0,0)}\,, \\ \Phi_{(2,-1)} &= -\Big(\frac{\alpha}{1+\alpha\, L_0}\, b_0 \xi_0 \eta_0 + \frac{b_0}{L_0}\, \eta_0 \xi_0\Big) J_{(3,-1)} \\ &\quad + \Big(-\xi_0 + \frac{b_0}{L_0}\, \eta_0 \xi_0 X_0 + \frac{\alpha}{1+\alpha\, L_0}\, \xi_0 \eta_0 b_0 Q \xi_0\Big) J_{(3,-2)}\,. \end{aligned} \tag{3.28}$$

The action evaluated for the sources is given by

$$S_1[J] \;=\; \Big\langle J_{(0,0)} \,\Big|\, \begin{pmatrix} A_\alpha & B_\alpha \end{pmatrix} \begin{pmatrix} J_{(3,-1)} \\ J_{(3,-2)} \end{pmatrix} \Big\rangle \,, \tag{3.29}$$

with

$$A_\alpha \equiv \frac{\alpha}{1+\alpha\, L_0}\, b_0\xi_0\eta_0 + \frac{b_0}{L_0}\,\eta_0\xi_0 \,, \quad B_\alpha \equiv \Big(1 - \frac{b_0}{L_0}\,\eta_0 X_0 - \frac{\alpha}{1+\alpha\, L_0}\,\xi_0\eta_0 b_0 Q\Big)\xi_0 \,. \tag{3.30}$$

When $\alpha = 0$, the expression (3.29) indeed reduces to the form (3.18).

3.2 Propagators for gauge fixing with b_0 and d_0

Thus far we have calculated propagators in the gauge (2.90), focusing on the $\alpha = 0$ case. These propagators include the zero mode of the picture-changing operator, which originates from the anti-commutation relation

$$\{Q, \xi_0\} = X_0 \,. \tag{3.31}$$

If instead of ξ_0 we use an operator whose anticommutator with Q vanishes, we expect that propagators are dramatically simplified. This is indeed the case: the operator d_0, the zero mode of $d = [Q, b\xi]$, provides us with simpler propagators. It satisfies the following algebraic relations:

$$d_0^2 = \{b_0, d_0\} = 0 \,, \quad \{Q, d_0\} = 0 \,, \quad \{\eta_0, d_0\} = L_0 \,. \tag{3.32}$$

In this subsection we investigate propagators in the gauge (2.91), concentrating on the symmetric case $\alpha = 1$.

First we consider $\Phi_{(0,0)}$, whose gauge-fixing conditions are

$$b_0\Phi_{(0,0)} = d_0\Phi_{(0,0)} = 0 \,. \tag{3.33}$$

In addition to the source $J_{(2,-1)}$, we introduce the Lagrange multipliers $\lambda_{(3,-1)}$ and $\lambda_{(3,-2)}$, and consider the action $S_0[J] + S_0^\lambda$ with

$$S_0^\lambda = \langle \Phi_{(0,0)} |\, b_0 \,| \lambda_{(3,-1)} \rangle + \langle \Phi_{(0,0)} |\, d_0 \,| \lambda_{(3,-2)} \rangle \;. \tag{3.34}$$

The equation of motion is

$$-\, Q\eta_0\, \Phi_{(0,0)} + J_{(2,-1)} + b_0\lambda_{(3,-1)} + d_0\lambda_{(3,-2)} = 0 \tag{3.35}$$

supplemented by the gauge-fixing conditions (3.33). We claim that

$$\Phi_{(0,0)} = -\frac{b_0}{L_0}\,\frac{d_0}{L_0}\, J_{(2,-1)} \,. \tag{3.36}$$

This follows quickly from the identity

$$Q\eta_0\, \frac{b_0}{L_0}\,\frac{d_0}{L_0} \;=\; -1 + \frac{d_0}{L_0}\,\eta_0 + \frac{b_0}{L_0}\,Q + \frac{b_0}{L_0}\,\frac{d_0}{L_0}\,Q\eta_0 \,, \tag{3.37}$$

acting on $J_{(2,-1)}$:

$$-\,Q\eta_0\,\Phi_{(0,0)} \;=\; -J_{(2,-1)} + \frac{d_0}{L_0}\,\eta_0\,J_{(2,-1)} + \frac{b_0}{L_0}\,Q\,J_{(2,-1)} + \frac{b_0}{L_0}\,\frac{d_0}{L_0}\,Q\eta_0\,J_{(2,-1)}\,. \tag{3.38}$$

Note that all terms on the right-hand side, except for the first, simply determine the values of the Lagrange multipliers in (3.35). Such values are not needed in the evaluation of the action since the solution satisfies the gauge-fixing conditions. Evaluating the action for this solution gives

$$S_0[J] \;=\; \frac{1}{2}\,\big\langle J_{(2,-1)}\big|\,\frac{d_0}{L_0}\,\frac{b_0}{L_0}\,\big|J_{(2,-1)}\big\rangle\,. \tag{3.39}$$

For the next step we have the gauge-fixing conditions

$$\begin{aligned} b_0 d_0\,\Phi_{(2,-1)} \;&=\; 0\,,\\ b_0\Phi_{(-1,0)} \;&=\; 0\,,\\ d_0\Phi_{(-1,1)} \;&=\; 0\,,\\ d_0\Phi_{(-1,0)} + b_0\Phi_{(-1,1)} \;&=\; 0\,. \end{aligned} \tag{3.40}$$

We implement the first and last gauge conditions with Lagrange multipliers. The relevant action is then $S_1[J] + S_1^{\lambda}$ with

$$S_1^{\lambda} = -\,\big\langle \lambda_{(4,-2)}\big|\,\Big(d_0\,\big|\Phi_{(-1,0)}\big\rangle + b_0\,\big|\Phi_{(-1,1)}\big\rangle\Big) + \big\langle \Phi_{(2,-1)}\big|\,b_0 d_0\,\big|\lambda_{(2,-1)}\big\rangle\,. \tag{3.41}$$

Note that both $J_{(0,0)}$ and $\lambda_{(4,-2)}$ are Grassmann odd. The gauge-fixed equations of motion are (recall that d_0 and b_0 are BPZ even, while η_0 and Q are BPZ odd)

$$\begin{aligned} Q\Phi_{(-1,0)} + \eta_0\Phi_{(-1,1)} + J_{(0,0)} + b_0 d_0\lambda_{(2,-1)} \;&=\; 0\,,\\ c_0 b_0\,\Big(Q\Phi_{(2,-1)} + J_{(3,-1)} + d_0\,\lambda_{(4,-2)}\Big) \;&=\; 0\,,\\ f_0 d_0\,\Big(\eta_0\Phi_{(2,-1)} + J_{(3,-2)} + b_0\,\lambda_{(4,-2)}\Big) \;&=\; 0\,, \end{aligned} \tag{3.42}$$

where f_0 is an operator satisfying $\{d_0, f_0\} = 1$.[5] In the last equation one may view $J_{(3,-2)} + b_0\,\lambda_{(4,-2)}$ as a source and solve the equation by writing

$$\Phi_{(2,-1)} = -\frac{d_0}{L_0}J_{(3,-2)} \;-\; \frac{d_0}{L_0}b_0\lambda_{(4,-2)} \;-\; \frac{b_0}{L_0}\eta_0\,\frac{d_0}{L_0}J_{(3,-1)}\,, \tag{3.43}$$

where the last term has been included with view of the second equation and does not disturb the third due to the η_0 factor it includes. Substitution into the second equation with some simplification yields

$$c_0 b_0\,\Big(\frac{d_0}{L_0}QJ_{(3,-2)} + \frac{d_0}{L_0}\eta_0\,J_{(3,-1)} + 2d_0\,\lambda_{(4,-2)}\Big) \;=\; 0\,. \tag{3.44}$$

The equation works out if the Lagrange multiplier is given by

$$\lambda_{(4,-2)} \;=\; -\frac{1}{2L_0}\big(QJ_{(3,-2)} + \eta_0 J_{(3,-1)}\big). \tag{3.45}$$

[5] For a concrete expression of f_0, see appendix A of [24].

Inserting (3.45) back in (3.43), we now have the solution for $\Phi_{(2,-1)}$. We find

$$\Phi_{(2,-1)} \;=\; -\frac{b_0}{L_0}\,J_{(3,-1)} \;-\; \frac{d_0}{L_0}\,J_{(3,-2)} \;+\; \frac{1}{2}\frac{d_0}{L_0}\frac{b_0}{L_0}QJ_{(3,-2)} \;+\; \frac{1}{2}\frac{b_0}{L_0}\frac{d_0}{L_0}\eta_0 J_{(3,-1)}\,. \tag{3.46}$$

A small rearrangement yields

$$\Phi_{(2,-1)} \;=\; -\frac{1}{2}\Big(\frac{b_0}{L_0}+\frac{b_0}{L_0}\eta_0\frac{d_0}{L_0}\Big)\,J_{(3,-1)} - \frac{1}{2}\Big(\frac{d_0}{L_0}+\frac{d_0}{L_0}Q\frac{b_0}{L_0}\Big)\,J_{(3,-2)}\,. \tag{3.47}$$

When the equations for $\Phi_{(-1,0)}$ and $\Phi_{(-1,1)}$ and the gauge-fixing conditions hold, the action is given by

$$S_1[J] = \big\langle \Phi_{(2,-1)} \big|\, J_{(0,0)} \big\rangle = -\big\langle J_{(0,0)} \big|\, \Phi_{(2,-1)} \big\rangle\,. \tag{3.48}$$

Its evaluation immediately gives

$$S_1[J] \;=\; \big\langle J_{(0,0)}\big|\,\frac{1}{2}\Big(\frac{b_0}{L_0}+\frac{b_0}{L_0}\eta_0\frac{d_0}{L_0}\Big)\;\big|J_{(3,-1)}\big\rangle + \;\big\langle J_{(0,0)}\big|\,\frac{1}{2}\Big(\frac{d_0}{L_0}+\frac{d_0}{L_0}Q\frac{b_0}{L_0}\Big)\;\big|J_{(3,-2)}\big\rangle\,. \tag{3.49}$$

This answer can be rewritten by using a one-by-two propagator matrix:

$$S_1[J] \;=\; \Big\langle J_{(0,0)}\Big|\,\begin{pmatrix} \tfrac{1}{2}\big(\tfrac{b_0}{L_0}+\tfrac{b_0}{L_0}\eta_0\tfrac{d_0}{L_0}\big) & \tfrac{1}{2}\big(\tfrac{d_0}{L_0}+\tfrac{d_0}{L_0}Q\tfrac{b_0}{L_0}\big)\end{pmatrix}\begin{pmatrix} J_{(3,-1)} \\ J_{(3,-2)}\end{pmatrix}\Big\rangle\,. \tag{3.50}$$

When the equation of motion of $\Phi_{(2,-1)}$ and the gauge-fixing conditions hold, the action reduces to

$$S_1[J] = \big\langle \Phi_{(-1,0)} \big|\, J_{(3,-1)} \big\rangle + \big\langle \Phi_{(-1,1)} \big|\, J_{(3,-2)} \big\rangle\,. \tag{3.51}$$

We can thus read the values of the fields, as bras. After BPZ conjugation we obtain

$$\begin{aligned} \Phi_{(-1,0)} \;&=\; -\frac{1}{2}\Big(\frac{b_0}{L_0}+\frac{d_0}{L_0}\eta_0\frac{b_0}{L_0}\Big)\,J_{(0,0)}\,, \\ \Phi_{(-1,1)} \;&=\; -\frac{1}{2}\Big(\frac{d_0}{L_0}+\frac{b_0}{L_0}Q\frac{d_0}{L_0}\Big)\,J_{(0,0)}\,. \end{aligned} \tag{3.52}$$

In the next step we have to deal with three fields and two antifields. We have the gauge-fixing conditions

$$\begin{aligned} b_0\,\Phi_{(3,-1)} + d_0\Phi_{(3,-2)} \;&=\; 0\,, \\ b_0\Phi_{(-2,0)} \;&=\; 0\,, \\ d_0\Phi_{(-2,0)} + b_0\Phi_{(-2,1)} \;&=\; 0\,, \\ d_0\Phi_{(-2,1)} + b_0\Phi_{(-2,2)} \;&=\; 0\,, \\ d_0\Phi_{(-2,2)} \;&=\; 0\,. \end{aligned} \tag{3.53}$$

The relevant action is $S_2[J] + S_2^\lambda$ with

$$\begin{aligned} S_2^\lambda = &-\big\langle \lambda_{(5,-2)}\big|\,\Big(d_0\,\big|\Phi_{(-2,0)}\big\rangle + b_0\,\big|\Phi_{(-2,1)}\big\rangle\Big) - \big\langle \lambda_{(5,-3)}\big|\,\Big(d_0\,\big|\Phi_{(-2,1)}\big\rangle + b_0\,\big|\Phi_{(-2,2)}\big\rangle\Big) \\ &- \big\langle \lambda_{(0,0)}\big|\,\Big(d_0\,\big|\Phi_{(3,-2)}\big\rangle + b_0\,\big|\Phi_{(3,-1)}\big\rangle\Big). \end{aligned} \tag{3.54}$$

The equations of motion obtained by varying the fields,

$$
\begin{aligned}
c_0 b_0 \Big(Q\Phi_{(3,-1)} + d_0\, \lambda_{(5,-2)} + J_{(4,-1)} \Big) &= 0\,, \\
Q\Phi_{(3,-2)} + \eta_0 \Phi_{(3,-1)} + b_0\, \lambda_{(5,-2)} + d_0\, \lambda_{(5,-3)} + J_{(4,-2)} &= 0\,, \\
f_0 d_0 \Big(\eta_0 \Phi_{(3,-2)} + b_0\, \lambda_{(5,-3)} + J_{(4,-3)} \Big) &= 0\,,
\end{aligned}
\tag{3.55}
$$

are simpler to solve than those obtained by varying the antifields. By solving the equations for the antifields we can determine the Lagrange multipliers:

$$
\begin{aligned}
\lambda_{(5,-2)} &= -\frac{1}{2L_0}\big(QJ_{(4,-2)} + \eta_0 J_{(4,-1)} \big)\,, \\
\lambda_{(5,-3)} &= -\frac{1}{2L_0}\big(QJ_{(4,-3)} + \eta_0 J_{(4,-2)} \big)\,.
\end{aligned}
\tag{3.56}
$$

The antifields are then given by

$$
\begin{aligned}
\Phi_{(3,-1)} &= -\frac{1}{2}\Big(\frac{b_0}{L_0} + \frac{b_0}{L_0}\eta_0 \frac{d_0}{L_0} \Big)\, J_{(4,-1)} - \frac{1}{2}\frac{d_0}{L_0}\, J_{(4,-2)}\,, \\
\Phi_{(3,-2)} &= \qquad\qquad\qquad\qquad - \frac{1}{2}\frac{b_0}{L_0}\, J_{(4,-2)} \quad - \quad \frac{1}{2}\Big(\frac{d_0}{L_0} + \frac{d_0}{L_0} Q \frac{b_0}{L_0} \Big)\, J_{(4,-3)}\,.
\end{aligned}
\tag{3.57}
$$

Thus the action takes the form

$$
S_2[J] \;=\; \Big\langle \big(J_{(-1,0)}\;\; J_{(-1,1)} \big) \Big| \begin{pmatrix} \frac{1}{2}\big(\frac{b_0}{L_0} + \frac{b_0}{L_0}\eta_0\frac{d_0}{L_0}\big) & \frac{1}{2}\frac{d_0}{L_0} & 0 \\ 0 & \frac{1}{2}\frac{b_0}{L_0} & \frac{1}{2}\big(\frac{d_0}{L_0} + \frac{d_0}{L_0}Q\frac{b_0}{L_0}\big) \end{pmatrix} \begin{pmatrix} J_{(4,-1)} \\ J_{(4,-2)} \\ J_{(4,-3)} \end{pmatrix} \Big\rangle. \tag{3.58}
$$

The full pattern is now clear. The full action $S[J]$ written in terms of propagators and bilinear in sources takes the form

$$
S[J] \;=\; \frac{1}{2}\big\langle J_{2,-1} \big| \,\frac{d_0}{L_0}\,\frac{b_0}{L_0}\, \big| J_{(2,-1)} \big\rangle + \sum_{n=0}^{\infty} S_{n+1}[J]\,, \tag{3.59}
$$

where $S_{n+1}[J]$ is the term coupling the sources of the $n+1$ antifields at ghost number $n+2$, to the sources of the $n+2$ fields at ghost number $-(n+1)$:

$$
S_{n+1}[J] \;=\; \Big\langle \Big(J_{(-n,0)}\;\; J_{(-n,1)}\;\; \cdots\;\; J_{(-n,n)} \Big) \Big| \mathcal{P}_{n+1,n+2} \begin{pmatrix} J_{(n+3,-1)} \\ J_{(n+3,-2)} \\ \vdots \\ J_{(n+3,-(n+2))} \end{pmatrix} \Big\rangle. \tag{3.60}
$$

Here the propagator matrix $\mathcal{P}_{n+1,n+2}$ has $n+1$ rows and $n+2$ columns. Its general form is the extension of our results in (3.50) and (3.58):

$$
\mathcal{P}_{n+1,n+2} = \frac{1}{2}\begin{pmatrix}
\frac{b_0}{L_0}+\frac{b_0}{L_0}\eta_0\frac{d_0}{L_0} & \frac{d_0}{L_0} & 0 & \cdots & 0 & 0 & 0 \\
0 & \frac{b_0}{L_0} & \frac{d_0}{L_0} & \ddots & \vdots & \vdots & \vdots \\
\vdots & 0 & \frac{b_0}{L_0} & \ddots & 0 & \vdots & \vdots \\
\vdots & \vdots & 0 & \ddots & \frac{d_0}{L_0} & 0 & \vdots \\
\vdots & \vdots & \vdots & \ddots & \frac{b_0}{L_0} & \frac{d_0}{L_0} & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{b_0}{L_0} & \frac{d_0}{L_0}+\frac{d_0}{L_0}Q\frac{b_0}{L_0}
\end{pmatrix} \quad (n \geq 0)\,. \tag{3.61}
$$

We can readily obtain the result with a general $\alpha\,(\neq -1)$ as well. The propagator matrices are given by

$$
\mathcal{P}_{1,2} = \begin{pmatrix} P_b & P_d \end{pmatrix}, \quad \mathcal{P}_{2,3} = \begin{pmatrix} P_b & \frac{d_0}{(\alpha+1)L_0} & 0 \\ 0 & \frac{\alpha b_0}{(\alpha+1)L_0} & P_d \end{pmatrix}, \tag{3.62a}
$$

$$
\mathcal{P}_{n+1,n+2} = \begin{pmatrix}
P_b & \frac{d_0}{(\alpha+1)L_0} & 0 & \cdots & 0 & 0 & 0 \\
0 & \frac{\alpha b_0}{(\alpha+1)L_0} & \frac{d_0}{(\alpha+1)L_0} & \ddots & \vdots & \vdots & \vdots \\
\vdots & 0 & \frac{\alpha b_0}{(\alpha+1)L_0} & \ddots & 0 & \vdots & \vdots \\
\vdots & \vdots & 0 & \ddots & \frac{d_0}{(\alpha+1)L_0} & 0 & \vdots \\
\vdots & \vdots & \vdots & \ddots & \frac{\alpha b_0}{(\alpha+1)L_0} & \frac{d_0}{(\alpha+1)L_0} & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{\alpha b_0}{(\alpha+1)L_0} & P_d
\end{pmatrix} \quad (n \geq 0)\,, \tag{3.62b}
$$

with

$$
P_b = \left(\alpha + \frac{\eta_0 d_0}{L_0}\right)\frac{b_0}{(\alpha+1)L_0}\,, \quad P_d = \left(1+\alpha\frac{Q b_0}{L_0}\right)\frac{d_0}{(\alpha+1)L_0}\,. \tag{3.63}
$$

Note that the case in which $\alpha = -1$ is exceptional: the propagators diverge, which means that gauge fixing is not complete. See [24] for details. When $\alpha = 0$, the above propagators correspond to those obtained from (3.26) by the replacement

$$
\xi_0 \longrightarrow \frac{d_0}{L_0}\,, \quad X_0 \longrightarrow 0\,. \tag{3.64}
$$

4 Verifying the master equation for the free action

Dynamical fields in string field theory are component fields. The BV formalism is defined in terms of these component fields, but it is convenient to recast it in terms of string fields. In this section we present the BV formalism of open superstring field theory in terms of string fields. We then show that the free action (1.22) satisfies the master equation.

The classical master equation is given by

$$\{S, S\} = 0\,, \tag{4.1}$$

and the antibracket is defined by

$$\{A, B\} = \sum_k \Big(\frac{\partial_R A}{\partial \phi_k} \frac{\partial_L B}{\partial \phi_k^*} - \frac{\partial_R A}{\partial \phi_k^*} \frac{\partial_L B}{\partial \phi_k} \Big)\,, \tag{4.2}$$

where ϕ_k forms a complete basis of fields and ϕ_k^* are the associated antifields. The Grassmann parity of a field can be arbitrary but the corresponding antifield has the opposite parity. In our case the fields are the component fields of Φ_- and the antifields are the component fields of Φ_+, with $\Phi_\pm$ defined in (1.21). With a slight abuse of language we will call Φ_- the string field and Φ_+ the string antifield.

Let us expand Φ_- and Φ_+ in terms of their component fields f and a with indices g, p, and r as follows:

$$\text{String field (even)} \qquad \Phi_- \;=\; \sum_{(g,p)\in\Delta_-} \sum_r f^r_{g,p} \Phi^r_{g,p}\,, \tag{4.3a}$$

$$\text{String antifield (odd)} \qquad \Phi_+ \;=\; \sum_{(g,p)\in\Delta_+} \sum_r a^r_{g,p} \Phi^r_{g,p}\,. \tag{4.3b}$$

We took f and a from the initials of "fields" and "antifields." For each pair (g, p) of the world-sheet ghost number g and picture number p, we chose a complete basis of states $\Phi^r_{g,p}$ labelled by r such that[6]

$$\langle\, \Phi^r_{g,p} \,|\, \Phi^{r'}_{g',p'} \rangle \;=\; \delta_{g+g',2}\, \delta_{p+p',-1}\, \delta_{r,r'} \qquad\qquad g \leq 0\,. \tag{4.4}$$

Since the Grassmann parity of $\Phi^r_{g,p}$ is $(-1)^g$, we have

$$\langle\, \Phi^r_{g,p} \,|\, \Phi^{r'}_{g',p'} \rangle \;=\; (-1)^g\, \delta_{g+g',2}\, \delta_{p+p',-1}\, \delta_{r,r'} \qquad\qquad g \geq 2\,, \tag{4.5}$$

which follows from

$$\langle A|\, B\rangle = (-1)^{AB} \,\langle B|\, A\rangle\,, \tag{4.6}$$

with $(-1)^{gg'} = (-1)^{g(2-g)} = (-1)^{-g^2} = (-1)^g$. Here and in what follows a string field in the exponent of (-1) represents its Grassmann parity: it is 0 mod 2 for a Grassmann-even string field and 1 mod 2 for a Grassmann-odd string field. While the states $\Phi^r_{g,p}$ carry ghost and picture numbers, the component fields $f^r_{g,p}$ and $a^r_{g,p}$ do no carry these numbers and their subscripts g and p refer to the states that multiply them. We also introduced the lattice Δ_- defined by the collection of pairs (g, p) that appear in Φ_- and the lattice Δ_+ defined by the collection of pairs (g, p) that appear in Φ_+.

[6] We do not need to consider states with $g = 1$, since they do not appear in the expansion (4.3).

As we mentioned in the introduction, $f^r_{g,p}$ and $a^r_{2-g,-1-p}$ should be paired in the BV formalism:

$$\text{Field-antifield pairing:} \quad f^r_{g,p} \quad \longleftrightarrow \quad a^r_{2-g,-1-p}\,. \tag{4.7}$$

The Grassmann parity of $f^r_{g,p}$ is $(-1)^g$ and that of $a^r_{g,p}$ is $-(-1)^g$. The Grassmann parity of $a^r_{2-g,-1-p}$ is indeed opposite to that of $f^r_{g,p}$, which is paired with $a^r_{2-g,-1-p}$. It then follows that Φ_- is Grassmann even and Φ_+ is Grassmann odd. Note that $a^r_{g,p}$ and $\Phi^r_{g,p}$ in Φ_+ commute, while $f^r_{g,p}\,\Phi^r_{g,p} = (-1)^g\,\Phi^r_{g,p}\,f^r_{g,p}$ in Φ_-. The antibracket (4.2) is thus defined by

$$\{A,B\} = \sum_{(g,p)\in\Delta_-}\sum_r\Big(\frac{\partial_R A}{\partial f^r_{g,p}}\,\frac{\partial_L B}{\partial a^r_{2-g,-1-p}} - \frac{\partial_R A}{\partial a^r_{2-g,-1-p}}\,\frac{\partial_L B}{\partial f^r_{g,p}}\Big)\,. \tag{4.8}$$

Our goal is to rewrite this antibracket (4.8) directly in terms of string fields and string antifields.

In previous sections we used the notation $\langle\, A\, B\,\rangle$ or $\langle\, A\,|B\,\rangle$ for the BPZ inner product of string fields A and B. When more than two string fields are involved, it is convenient to introduce the integration symbol as follows:[7]

$$\int A\star B = \langle\, A\, B\,\rangle = \langle\, A\,|B\,\rangle\,. \tag{4.9}$$

The relation (4.6) is generalized to

$$\int A_1\star A_2\star\ldots\star A_n = (-1)^{A_1(A_2+\ldots+A_n)}\int A_2\star\ldots\star A_n\star A_1\,. \tag{4.10}$$

The BPZ inner products of states in the basis (4.4) and (4.5) are translated into

$$\int \Phi^r_{g,p}\star\Phi^{r'}_{g',p'} \;=\; \delta_{g+g',2}\,\delta_{p+p',-1}\,\delta_{r,r'} \qquad g\le 0\,, \tag{4.11a}$$

$$\int \Phi^r_{g,p}\star\Phi^{r'}_{g',p'} \;=\; (-1)^g\,\delta_{g+g',2}\,\delta_{p+p',-1}\,\delta_{r,r'} \qquad g\ge 2\,. \tag{4.11b}$$

We are interested in evaluating $\{A,B\}$ where A and B depend on fields and antifields only through $\Phi_\pm$. Let us first consider $\{\Phi_-,\Phi_+\}$. This takes value in a tensor product of two Hilbert spaces of the string field. We therefore introduce a space number label and write it as $\{\Phi^{(1)}_-,\Phi^{(2)}_+\}$. We see that only the first term on the right-hand side of (4.8) contributes and find

$$\{\Phi^{(1)}_-,\Phi^{(2)}_+\} = \sum_{(g,p)\in\Delta_-}\sum_r\frac{\partial_R\Phi^{(1)}_-}{\partial f^r_{g,p}}\frac{\partial_L\Phi^{(2)}_+}{\partial a^r_{2-g,-1-p}} = \sum_{(g,p)\in\Delta_-}\sum_r(-1)^g\,\Phi^{r(1)}_{g,p}\,\Phi^{r(2)}_{2-g,-1-p}\,, \tag{4.12}$$

where the expansion (4.3) was used and the sign factor $(-1)^g$ came from the right derivative that must go through the state $\Phi^r_{g,p}$ to get to the component field $f^r_{g,p}$. An important property of $\{\Phi^{(1)}_-,\Phi^{(2)}_+\}$ is that it acts as the projector $\mathcal{P}_{\Delta_+}$ to the subspace defined by the lattice Δ_+ in the following sense:

$$\int_1 X^{(1)}\star_1\{\Phi^{(1)}_-,\Phi^{(2)}_+\} \;=\; (\mathcal{P}_{\Delta_+}X)^{(2)}\,, \tag{4.13}$$

[7]We use the symbol $\star$ to denote the star product in this section.

where the subscripts attached to the integration symbol and the star product represent the space number label. To see this, insert (4.12) into the left-hand side of (4.13)

$$\int_1 X^{(1)} \star_1 \sum_{(g,p)\in\Delta_-} \sum_r (-1)^g \, \Phi^{r(1)}_{g,p} \, \Phi^{r(2)}_{2-g,-1-p} \; = \; \sum_{(g,p)\in\Delta_-} \sum_r (-1)^g \int_1 \big(X^{(1)} \star_1 \, \Phi^{r(1)}_{g,p} \big) \Phi^{r(2)}_{2-g,-1-p} \,, \tag{4.14}$$

and expand $X^{(1)}$ in a complete basis of ghost and picture numbers,

$$X^{(1)} \; = \; \sum_{g',p'=-\infty}^{\infty} \sum_{r'} X^{r'}_{g',p'} \Phi^{r'(1)}_{g',p'} \,. \tag{4.15}$$

In (4.14) this expression is contracted with states in the subspace defined by Δ_-. Therefore, in (4.15) only states in the subspace defined by Δ_+ give nonvanishing contributions and the right-hand side of (4.14) becomes

$$\sum_{(g',p')\in\Delta_+} \sum_{r'} X^{r'}_{g',p'} \sum_{(g,p)\in\Delta_-} \sum_r (-1)^g \int_1 \big(\Phi^{r'(1)}_{g',p'} \star_1 \, \Phi^{r(1)}_{g,p} \big) \Phi^{r(2)}_{2-g,-1-p} \,. \tag{4.16}$$

Using the second equation in (4.11), we find

$$\sum_{(g',p')\in\Delta_+} \sum_{r'} X^{r'}_{g',p'} \sum_{(g,p)\in\Delta_-} \sum_r (-1)^g (-1)^{g'} \delta_{r,r'} \delta_{g+g',2} \, \delta_{p+p',-1} \Phi^{r(2)}_{2-g,-1-p} = \sum_{(g',p')\in\Delta_+} \sum_{r'} X^{r'}_{g',p'} \Phi^{r'(2)}_{g',p'} \,. \tag{4.17}$$

The right-hand side is the string field $X^{(1)}$ copied into the state space 2, with a projection to the subspace defined by Δ_+. We have thus shown the relation (4.13). Similarly, one can prove

$$\int_2 \{\Phi^{(1)}_-, \Phi^{(2)}_+\} \star_2 X^{(2)} \; = \; (\mathcal{P}_{\Delta_-} X)^{(1)} \,, \tag{4.18}$$

where $\mathcal{P}_{\Delta_-}$ is the projector to the subspace defined by Δ_-. We can also evaluate $\{\Phi^{(1)}_+, \Phi^{(2)}_-\}$ to obtain

$$\{\Phi^{(1)}_+, \Phi^{(2)}_-\} = - \sum_{(g,p)\in\Delta_-} \sum_r \Phi^{r(1)}_{2-g,-1-p} \, \Phi^{r(2)}_{g,p} \,, \tag{4.19}$$

$$\int_1 X^{(1)} \star_1 \{\Phi^{(1)}_+, \Phi^{(2)}_-\} \; = \; -(\mathcal{P}_{\Delta_-} X)^{(2)} \,, \qquad \int_2 \{\Phi^{(1)}_+, \Phi^{(2)}_-\} \star_2 X^{(2)} \; = \; -(\mathcal{P}_{\Delta_+} X)^{(1)} \,. \tag{4.20}$$

Let us next consider $\{A, \Phi_+\}$ where A is given by an integral of a product of Φ_+ and Φ_-. It is useful to define variational derivatives $\frac{\delta_R A}{\delta \Phi_-}$ and $\frac{\delta_R A}{\delta \Phi_+}$ by

$$\delta A = \int \left(\frac{\delta_R A}{\delta \Phi_-} \star \delta\Phi_- + \frac{\delta_R A}{\delta \Phi_+} \star \delta\Phi_+ \right) . \tag{4.21}$$

We also define $\frac{\delta_L A}{\delta \Phi_-}$ and $\frac{\delta_L A}{\delta \Phi_+}$ by

$$\delta A = \int \left(\delta\Phi_- \star \frac{\delta_L A}{\delta \Phi_-} + \delta\Phi_+ \star \frac{\delta_L A}{\delta \Phi_+} \right) , \tag{4.22}$$

which are related to $\frac{\delta_R A}{\delta\Phi_-}$ and $\frac{\delta_R A}{\delta\Phi_+}$ as follows:

$$\frac{\delta_R A}{\delta\Phi_+} = -(-1)^A \frac{\delta_L A}{\delta\Phi_+}\,, \tag{4.23a}$$

$$\frac{\delta_R A}{\delta\Phi_-} = \frac{\delta_L A}{\delta\Phi_-}\,. \tag{4.23b}$$

It is important to note that $\frac{\delta_R A}{\delta\Phi_-}$ and $\frac{\delta_L A}{\delta\Phi_-}$ are string fields in the subspace defined by Δ_+ since they are contracted with the variation $\delta\Phi_-$ in the subspace defined by Δ_-. Similarly, $\frac{\delta_R A}{\delta\Phi_+}$ and $\frac{\delta_L A}{\delta\Phi_+}$ are string fields in the subspace defined by Δ_- since they are contracted with the variation $\delta\Phi_+$ in the subspace defined by Δ_+. These can be expressed as follows:

$$\mathcal{P}_{\Delta_+}\frac{\delta_R A}{\delta\Phi_-} = \frac{\delta_R A}{\delta\Phi_-}\,, \quad \mathcal{P}_{\Delta_+}\frac{\delta_L A}{\delta\Phi_-} = \frac{\delta_L A}{\delta\Phi_-}\,, \quad \mathcal{P}_{\Delta_-}\frac{\delta_R A}{\delta\Phi_+} = \frac{\delta_R A}{\delta\Phi_+}\,, \quad \mathcal{P}_{\Delta_-}\frac{\delta_L A}{\delta\Phi_+} = \frac{\delta_L A}{\delta\Phi_+}\,. \tag{4.24}$$

We can now write

$$\frac{\partial_R A}{\partial f^r_{g,p}} = \int \frac{\delta_R A}{\delta\Phi_-} \star \frac{\partial_R \Phi_-}{\partial f^r_{g,p}} = (-1)^g \int \frac{\delta_R A}{\delta\Phi_-} \star \Phi^r_{g,p}\,. \tag{4.25}$$

We then obtain

$$\begin{aligned}\{A, \Phi^{(2)}_+\} &= \sum_{(g,p)\in\Delta_-}\sum_r \frac{\partial_R A}{\partial f^r_{g,p}}\frac{\partial_L \Phi^{(2)}_+}{\partial a^r_{2-g,-1-p}} = \int_1 \Big(\frac{\delta_R A}{\delta\Phi_-}\Big)^{(1)} \star_1 \{\Phi^{(1)}_-, \Phi^{(2)}_+\} \\ &= \Big(\mathcal{P}_{\Delta_+}\frac{\delta_R A}{\delta\Phi_-}\Big)^{(2)} = \Big(\frac{\delta_R A}{\delta\Phi_-}\Big)^{(2)}\,,\end{aligned} \tag{4.26}$$

where we used (4.12), (4.13), and (4.24). Deleting the space number label, we can write the relation as follows:

$$\{A, \Phi_+\} = \frac{\delta_R A}{\delta\Phi_-}\,. \tag{4.27}$$

Similarly, one can derive

$$\begin{aligned}\frac{\partial_R A}{\partial a^r_{g,p}} &= \int \frac{\delta_R A}{\delta\Phi_+} \star \frac{\partial_R \Phi_+}{\partial a^r_{g,p}} = \int \frac{\delta_R A}{\delta\Phi_+} \star \Phi^r_{g,p}\,, \\ \frac{\partial_L A}{\partial f^r_{g,p}} &= \int \frac{\partial_L \Phi_-}{\partial f^r_{g,p}} \star \frac{\delta_L A}{\delta\Phi_-} = \int \Phi^r_{g,p} \star \frac{\delta_L A}{\delta\Phi_-}\,, \\ \frac{\partial_L A}{\partial a^r_{g,p}} &= \int \frac{\partial_L \Phi_+}{\partial a^r_{g,p}} \star \frac{\delta_L A}{\delta\Phi_+} = \int \Phi^r_{g,p} \star \frac{\delta_L A}{\delta\Phi_+}\,,\end{aligned} \tag{4.28}$$

as well as the relations

$$\{A, \Phi_-\} = -\frac{\delta_R A}{\delta\Phi_+}\,, \tag{4.29a}$$

$$\{\Phi_+, A\} = -\frac{\delta_L A}{\delta\Phi_-}\,, \tag{4.29b}$$

$$\{\Phi_-, A\} = \frac{\delta_L A}{\delta\Phi_+}\,. \tag{4.29c}$$

We can see, using (4.23), that these relations are consistent with the familiar property of the antibracket:

$$\{A,B\} = (-1)^{A+B+AB}\{B,A\}\,. \tag{4.30}$$

Finally, let us consider $\{A,B\}$ where both A and B are integrals of products made of Φ_+ and Φ_-. In this case, we begin with (4.8) and use (4.25), (4.28), (4.12), and (4.19) to show that

$$\{A,B\} = \int_1\int_2 \left(\left(\frac{\delta_R A}{\delta\Phi_-}\right)^{(1)} \star_1 \{\Phi_-^{(1)}, \Phi_+^{(2)}\} \star_2 \left(\frac{\delta_L B}{\delta\Phi_+}\right)^{(2)} + \left(\frac{\delta_R A}{\delta\Phi_+}\right)^{(1)} \star_1 \{\Phi_+^{(1)}, \Phi_-^{(2)}\} \star_2 \left(\frac{\delta_L B}{\delta\Phi_-}\right)^{(2)}\right). \tag{4.31}$$

Using (4.13) or (4.18), (4.20), and (4.24), we obtain

$$\{A,B\} = \int \left(\frac{\delta_R A}{\delta\Phi_-} \star \frac{\delta_L B}{\delta\Phi_+} - \frac{\delta_R A}{\delta\Phi_+} \star \frac{\delta_L B}{\delta\Phi_-}\right). \tag{4.32}$$

This is the final expression of the antibracket. The expression (4.8) in terms of component fields has now been written in terms of the string field and the string antifield.

Let us evaluate the antibracket $\{S,S\}$ for the free action (1.22):

$$S = \int \Big(-\frac{1}{2}\Phi_- \star Q\eta_0\Phi_- + \Phi_+ \star (Q+\eta_0)\Phi_- \Big)\,. \tag{4.33}$$

Since

$$\frac{\delta_R S}{\delta\Phi_-} = \mathcal{P}_{\Delta_+}\Big(-Q\eta_0\Phi_- + (Q+\eta_0)\Phi_+ \Big)\,, \qquad \frac{\delta_L S}{\delta\Phi_+} = \mathcal{P}_{\Delta_-}\Big((Q+\eta_0)\Phi_- \Big)\,, \tag{4.34}$$

we find

$$\frac{1}{2}\{S,S\} = \int \frac{\delta_R S}{\delta\Phi_-} \star \frac{\delta_L S}{\delta\Phi_+} = \int \mathcal{P}_{\Delta_+}\Big(-Q\eta_0\Phi_- + (Q+\eta_0)\Phi_+ \Big) \star \Big((Q+\eta_0)\Phi_- \Big)\,. \tag{4.35}$$

Here we dropped the projector $\mathcal{P}_{\Delta_-}$ because it is automatically enforced by the other projector $\mathcal{P}_{\Delta_+}$ through the BPZ contraction. Since the only string field in Φ_- such that $Q\eta_0\Phi_-$ is in the subspace defined by Δ_+ is $\Phi_{(0,0)}$ and the action of Q or η_0 takes string fields in the subspace defined by Δ_+ to string fields in the subspace defined by Δ_+, we have

$$\begin{aligned} \frac{1}{2}\{S,S\} &= \int \mathcal{P}_{\Delta_+}\Big(-Q\eta_0\Phi_- + (Q+\eta_0)\Phi_+ \Big) \star \Big((Q+\eta_0)\Phi_- \Big) \\ &= \int \Big(-Q\eta_0\Phi_{(0,0)} + (Q+\eta_0)\Phi_+ \Big) \star \Big((Q+\eta_0)\Phi_- \Big)\,. \end{aligned} \tag{4.36}$$

Using (1.2), we conclude that the antibracket $\{S,S\}$ vanishes for the free action (4.33).

5 Conclusions and outlook

This paper is the first part of our report on the gauge structure and quantization of the Neveu-Schwarz superstring field theory [14]. A preliminary report of our study of these issues was presented by one of the present authors (S.T.) at the SFT 2010 conference in Kyoto [25]. In this paper we concentrated on the free theory and studied a class of gauge-fixing conditions and associated propagators. We also constructed the free master action and proved that it satisfies the classical master equation. One could further examine a larger class of gauge-fixing conditions and associated propagators. This research direction is described in [24], which appears concurrently with this paper.

The next problem is to find the full non-linear master action for the interacting case. The result of our study (in which we were joined by Berkovits) will appear soon in [26]. It turned out that it is a difficult problem, and we have not been able to obtain a complete form for the master action. One can think of several approaches to this problem. In [27], it was shown that a partial gauge fixing of the cubic democratic theory [28] leads to the theory studied here. If this partial gauge fixing could be extended to the BV level, it could be used in order to infer the full BV master action we are after. A similar approach, which is presumably simpler, could be to use another cubic theory constructed in such a way as to be equivalent to the theory we consider here [29]. Being cubic, its BV structure should be simple. If the relation between the theories could be extended to the BV level, the master action could be fully written. A very different approach towards the construction of a master action is to gauge fix some relatively trivial degrees of freedom in a way that leads to a simplified set of ghosts and antifields. Such an approach was studied by Berkovits [30]. We did not discuss at all in this work the so-called modified cubic theory [17, 18], but we note that a quantization of this theory has been proposed very recently in [31, 32].[8]

The complete open superstring field theory includes both Neveu-Schwarz and Ramond sectors. In the current discussion we ignored the Ramond sector. Its inclusion is certainly an important goal. It might be possible to generalize the current construction to the Ramond sector using ideas from [34, 35, 28, 27]. We leave the incorporation of the Ramond sector for future work.

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[8]A discussion of the gauge structure of the modified theory can be found in [33].

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References

[1] E. Fuchs and M. Kroyter, "Analytical solutions of open string field theory," Phys. Rept. **502**, 89 (2011) [arXiv:0807.4722].

[2] L. Bonora, C. Maccaferri and D. D. Tolla, "Relevant deformations in open string field theory: a simple solution for lumps," JHEP **1111**, 107 (2011) [arXiv:1009.4158].

[3] M. Kiermaier, Y. Okawa and P. Soler, "Solutions from boundary condition changing operators in open string field theory," JHEP **1103**, 122 (2011) [arXiv:1009.6185].

[4] T. Erler and C. Maccaferri, "Comments on lumps from RG flows," JHEP **1111**, 092 (2011) [arXiv:1105.6057].

[5] T. Noumi and Y. Okawa, "Solutions from boundary condition changing operators in open superstring field theory," JHEP **1112**, 034 (2011) [arXiv:1108.5317].

[6] E. Witten, "Non-commutative geometry and string field theory," Nucl. Phys. B **268**, 253 (1986).

[7] C. B. Thorn, "Perturbation theory for quantized string fields," Nucl. Phys. B **287**, 61 (1987).

[8] M. Bochicchio, "Gauge fixing for the field theory of the bosonic string," Phys. Lett. B **193**, 31 (1987).

[9] M. Bochicchio, "String field theory in the Siegel gauge," Phys. Lett. B **188**, 330 (1987).

[10] I. A. Batalin and G. A. Vilkovisky, "Gauge algebra and quantization," Phys. Lett. B **102**, 27 (1981).

[11] I. A. Batalin and G. A. Vilkovisky, "Quantization of gauge theories with linearly dependent generators," Phys. Rev. D **28**, 2567 (1983).

[12] C. B. Thorn, "String field theory," Phys. Rept. **175**, 1 (1989).

[13] B. Zwiebach, "Closed string field theory: Quantum action and the B-V master equation," Nucl. Phys. B **390**, 33 (1993) [hep-th/9206084].

[14] N. Berkovits, "Super-Poincaré invariant superstring field theory," Nucl. Phys. B **450**, 90 (1995) [Erratum-ibid. B **459**, 439 (1996)] [hep-th/9503099].

[15] D. Friedan, E. J. Martinec and S. H. Shenker, "Conformal invariance, supersymmetry and string theory," Nucl. Phys. B **271**, 93 (1986).

[16] E. Witten, "Interacting field theory of open superstrings," Nucl. Phys. B **276**, 291 (1986).

[17] C. R. Preitschopf, C. B. Thorn and S. A. Yost, "Superstring field theory," Nucl. Phys. B **337**, 363 (1990).

[18] I. Y. Arefeva, P. B. Medvedev and A. P. Zubarev, "New representation for string field solves the consistence problem for open superstring field," Nucl. Phys. B **341**, 464 (1990).

[19] N. Berkovits and C. Vafa, "$N = 4$ Topological strings," Nucl. Phys. B **433**, 123 (1995) [hep-th/9407190].

[20] N. Berkovits and C. T. Echevarria, "Four point amplitude from open superstring field theory," Phys. Lett. **B478**, 343 (2000) [hep-th/9912120].

[21] M. Henneaux, "Lectures on the antifield-BRST formalism for gauge theories," Nucl. Phys. Proc. Suppl. **18A**, 47 (1990).

[22] M. Henneaux and C. Teitelboim, "Quantization of gauge systems," Princeton University Press, USA (1992).

[23] J. Gomis, J. Paris and S. Samuel, "Antibracket, antifields and gauge theory quantization," Phys. Rept. **259**, 1 (1995) [hep-th/9412228].

[24] S. Torii, "Validity of gauge-fixing conditions and the structure of propagators in open superstring field theory," arXiv:1201.1762.

[25] S. Torii, "Gauge fixing of open superstring field theory in the Berkovits non-polynomial formulation," Prog. Theor. Phys. Suppl. **188**, 272 (2011) [arXiv:1201.1763].

[26] N. Berkovits, M. Kroyter, Y. Okawa, M. Schnabl, S. Torii and B. Zwiebach, "Open superstring field theory II: approaches to the BV master action," *to appear.*

[27] M. Kroyter, "Democratic superstring field theory: gauge fixing," JHEP **1103**, 081 (2011) [arXiv:1010.1662].

[28] M. Kroyter, "Superstring field theory in the democratic picture," arXiv:0911.2962.

[29] P. A. Grassi and M. Schnabl, "Chern-Simons formulation of WZW-like string field theory," *to appear.*

[30] N. Berkovits, "Constrained BV description of string field theory," arXiv:1201.1769.

[31] M. Kohriki, I. Kishimoto, T. Kugo, H. Kunitomo and M. Murata, "Gauge-fixing problem in modified cubic superstring field theory," Prog. Theor. Phys. Suppl. **188**, 263 (2011).

[32] M. Kohriki, T. Kugo and H. Kunitomo, "Gauge fixing of modified cubic open superstring field theory," arXiv:1111.4912.

[33] M. Kroyter, "Superstring field theory equivalence: Ramond sector," JHEP **0910**, 044 (2009) [arXiv:0905.1168].

[34] N. Berkovits, "The Ramond sector of open superstring field theory," JHEP **0111**, 047 (2001) [hep-th/0109100].

[35] Y. Michishita, "A covariant action with a constraint and Feynman rules for fermions in open superstring field theory," JHEP **0501**, 012 (2005) [hep-th/0412215].