Advances in Quantum Metrology

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In classical estimation theory, the central limit theorem implies that the statistical error in a measurement outcome can be reduced by an amount proportional to $n^{-1/2}$ by repeating the measures n times and then averaging. Using quantum effects, such as entanglement, it is often possible to do better, decreasing the error by an amount proportional to n^{-1}. Quantum metrology is the study of those quantum techniques that allow one to gain advantages over purely classical approaches. In this review, we analyze some of the most promising recent developments in this research field. Specifically, we deal with the developments of the theory and point out some of the new experiments. Then we look at one of the main new trends of the field, the analysis of how the theory must take into account the presence of noise and experimental imperfections.

Any measurement consists in three parts: the preparation of a probe, its interaction with the system to be measured, and the probe readout. This process is often plagued by statistical or systematic errors. The source of the former can be accidental (e.g. deriving from an insufficient control of the probes or of the measured system) or fundamental (e.g. deriving from the Heisenberg uncertainty relations). Whatever their origin, we can reduce their effect by repeating the measurement and averaging the resulting outcomes. This is a consequence of the central-limit theorem: given a large number n of independent measurement results (each having a standard deviation $\Delta\sigma$), their average will converge to a Gaussian distribution with standard deviation $\Delta\sigma/\sqrt{n}$, so that the error scales as $n^{-1/2}$. In quantum mechanics this behavior is referred to as "standard quantum limit" (SQL) and is associated with procedures which do not fully exploit the quantum nature of the system under investigation[1]. Notably it is possible to do better when one employs quantum effects, such as entanglement among the probing devices employed for the measurements, e.g. see Refs. [1–6]. Consequently, the SQL is not a fundamental quantum mechanical bound as it can be surpassed by using "non-classical" strategies. Nonetheless, through Heisenberg-like uncertainty relations, quantum mechanics still sets ultimate limits in precision which are typically referred to as "Heisenberg bounds". In Fig. 1 we present a simple example which may be useful to understand the quantum enhancement. Part of the emerging field of quantum technology [7], quantum metrology studies these bounds and the (quantum) strategies which allows us to attain them. More generally it deals with measurement and discrimination procedures that receive some kind of enhancement (in precision, efficiency, simplicity of implementation, etc.) through the use of quantum effects.

This paper aims to review some of the most recent development of the field. For a more historical perspective of quantum metrology, we refer the reader to [5]. We start by introducing some important results on quantum estimation theory which focus on the optimization of the probe readout. Then we report some recent findings obtained in the context of parameter estimation for channels, which allows also for the optimization of the probe preparation. It is only at this stage that Heisenberg-like scaling is obtained. Schemes based on filtering protocols and nonlinear effects will be also introduced. In the remaining of the paper we will then deal with the analysis of the typically very fragile quantum metrology protocols in the presence of noise.

Basics on Quantum Estimation for states

In its simplest version a typical quantum estimation problem [8–14] consists in recovering the value of a continuous parameter x (say the phase φ of Fig. 1) which is encoded into a fixed set of states ρ_x of a quantum system S. As in the example of Fig. 1, we can describe it as a two step process where we first perform a measurement on S, and then extrapolate the value of x with some data-processing of the measurement results. The measurement is described by a Positive Op-

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[1] In quantum optics, the $n^{-1/2}$ scaling is also indicated as "shot noise", since it is connected to the discrete nature of the radiation that can be heard as "shots" in a photon counter operating in Geiger mode.
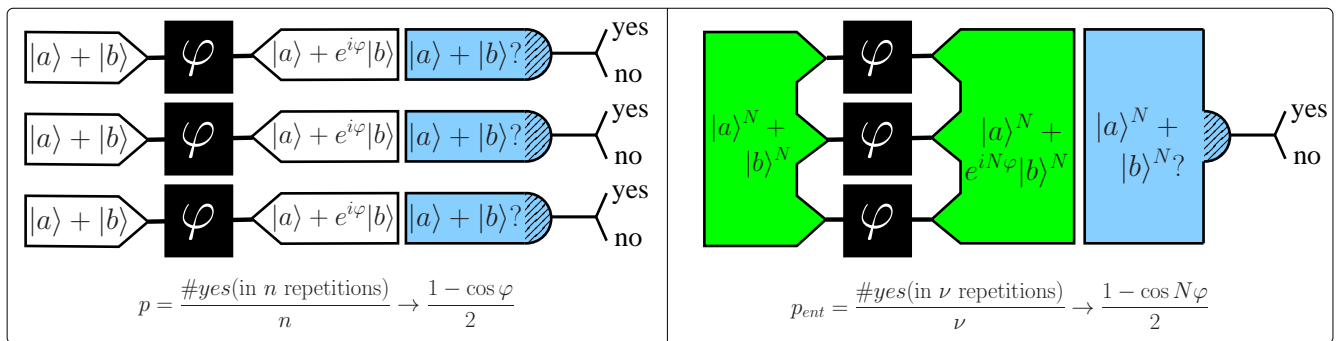
FIG. 1: Ramsey interferometry. Its aim is to measure an unknown relative phase φ which is picked up by two orthogonal states $|a\rangle$, $|b\rangle$ of an atomic probing system. This is a generic framework that encompasses many different interferometric measurements (e.g. frequency-standards, magnetometry, optical phase, etc.) [4]. **Left:** In a conventional setup the probe preparation consists in producing each atom in the superposition $|\psi_{in}\rangle = (|a\rangle + |b\rangle)/\sqrt{2}$, which yields the output state $|\psi_\varphi\rangle = (|a\rangle + e^{i\varphi}|b\rangle)/\sqrt{2}$ after the probing stage (black-boxes in the figure). The readout consists in checking whether $|\psi_\varphi\rangle$ is still in the initial state $|\psi_{in}\rangle$. This happens with probability $p = |\langle\psi_{in}|\psi_\varphi\rangle|^2 = (1 - \cos\varphi)/2$. Therefore by taking the ratio between the number of successes and the total number of readouts, we can recover the phase φ. If we repeat this measurement n times, the associated error on our estimation of φ, can then be evaluated by the standard deviation on the determination of p (i.e. $\Delta p = \sqrt{p(1-p)/n}$) and by using error propagation theory, obtaining the SQL scaling $\delta\varphi_n = \sqrt{\frac{p(1-p)}{n}}/|\frac{\partial p}{\partial\varphi}| = n^{-1/2}$. **Right:** Quantum enhanced setting: A simple quantum strategy consists in dividing the n probes into groups of N, prepared in an entangled state $(|a\rangle^{\otimes N} + |b\rangle^{\otimes N})/\sqrt{2}$. Since each of the N vectors $|b\rangle$ acquires a relative phase φ, the final state is $(|a\rangle^{\otimes N} + e^{iN\varphi}|b\rangle^{\otimes N})/\sqrt{2}$. The probability that this state is equal to the initial one is now $p_{ent} = (1 - \cos N\varphi)/2$. Furthermore since we have $\nu = n/N$ groups of probes, we can repeat this procedure ν times, obtaining an error $\delta\varphi_n = \sqrt{\frac{p_{ent}(1-p_{ent})}{\nu}}/|\frac{\partial p_{ent}}{\partial\varphi}| = (nN)^{-1/2}$, with a $N^{1/2}$ enhancement in precision with respect to previous case, namely, the Heisenberg bound for phase sensing [2, 6].

erator Valued Measurement (POVM) $\mathcal{E}^{(n)}$ of elements $\{E_y^{(n)}\}$ (where n is the number of copies of ρ_x we use). The conditional probability of getting the outcome y will be then computed as $p_n(y|x) = \text{Tr}[E_y^{(n)}\rho_x^{\otimes n}]$. The data-processing of the result y will yield our estimate z of the value of x and, in the most general case, will be characterized by assigning some conditional probabilities $p_{est}^{(n)}(z|y)$. Ideally, we would like to have z as close as possible to the parameter x; unfortunately, depending on the physics of the problem and, possibly, on the selected estimation strategy, there will often be some residual uncertainty in the determination of the parameter. This uncertainty is fully characterized by the probability $P_n(z|x) := \sum_y p_{est}^{(n)}(z|y)p_n(y|x)$, which describes the statistical dependence of z on the true value x. It allows us to define the Root Mean Square Error (RMSE) $\delta X_n := \sqrt{\sum_z[z-x]^2 P_n(z|x)}$ [10] which provides a good measure of the estimation accuracy (different measures are possible but will not be treated here, see e.g. Ref. [11]). Under reasonable assumptions on the asymptotic behavior of the estimation function (i.e. requiring it to be *asymptotically Locally Unbiased*[2]), δX_n

can be shown to obey to the so called Cramér-Rao (CR) bound [15], which implies

$$\delta X_n \geqslant 1/\sqrt{F_n(x)}, \tag{1}$$

where $F_n(x) := \sum_y \left(\frac{\partial p_n(y|x)}{\partial x}\right)^2/p_n(y|x)$ is the Fisher information associated with the selected POVM measurement.

Optimizing Eq. (1) with respect to all possible POVMs $\mathcal{E}^{(n)}$ one then gets the inequality [8–14]

$$\delta X_n \geqslant \frac{1}{\sqrt{\max_{\mathcal{E}^{(n)}} F_n(x)}} \geqslant \frac{1}{\sqrt{n\, J(\rho_x)}}. \tag{2}$$

The term on the right is the *quantum* Cramér-Rao (q-CR) bound. It is obtained by exploiting an upper bound of

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[2] This means that for $n \to \infty$ there exists a value of the parameter x for which the estimator tends to the correct value, and at the same time the derivative of the estimator at that point is unity [10]. All reasonable estimators satisfy this condition while the pathological ones, do not (e.g. trivial estimation procedures

that obtains the correct value of x by chance, as the case of a stopped clock which is correct twice per day). A generalization of Eq. (1) which applies to all estimation strategies without the assumption of locally unbiasedness was provided in [8, 9]. It is obtained by replacing δX_n on the lhs of Eq. (1) with its regularized version $\delta\tilde{X}_n = \sqrt{\sum_z[\tilde{z}(x)-x]^2 P_n(z|x)}$ in which $\tilde{z}(x)$ is the quantity z divided by $\frac{\partial}{\partial x}\sum_z z\, P_n(z|x)$. Even though the inequality for $\delta\tilde{X}_n$ refers to a larger number of estimation strategies, it is weaker since $\delta\tilde{X}_n \geqslant \delta X_n$. Note also that the regularization implies that $\delta\tilde{X}_n$ will diverge for the pathological estimators that would be excluded by the restrictions adopted in deriving (1).

$\max_{\mathcal{E}^{(n)}} F_n(x)$ in terms of the *quantum Fisher informa-tion* [3] $J(\rho_x)$. For instance, consider the case in which ρ_x are pure states of the form $|\psi_x\rangle = \exp[-iHx]|0\rangle$ with H an Hermitian operator and $|0\rangle$ a reference vector. Then, the q-CR bound takes the simple form of an uncertainty relation [8–11],

$$\delta X_n \geqslant \frac{1}{2\sqrt{n}\,\Delta H}\ , \qquad (3)$$

where $\Delta H := \sqrt{\langle (H - \langle H \rangle)^2 \rangle}$ is the spread of H on $|0\rangle$.

Three things are worth stressing at this point: *i)* the bound in Eq. (2) holds for *all* possible POVMs, including those which operate jointly on the n copies while exploiting entanglement resources; *ii)* the SQL scaling $n^{-1/2}$ on the rhs is a direct consequence of the additivity of the quantum Fisher information when applied to tensor states $\rho_x^{\otimes n}$, i.e. $J(\rho_x^{\otimes n}) = nJ(\rho_x)$; *iii)* in the asymptotic limit of large n, the q-CR bound is always achievable and the estimation strategy which attains it can be constructed via *local measurements* and adaptive estimators, a strategy that employs only Local Operations and Classical Communication (LOCC) [12, 13, 16–20]. This implies that entangled resources at the measurement stage are not necessary to achieve the q-CR bound. Local measures and some clever classical data-processing are sufficient. It also shows that the quantity $1/\sqrt{nJ(x)}$ has a clear operational meaning and can be used to quantify how hard the estimation problem is.

Finally, we briefly consider the multi-parameter case where x is a vector of random variables [10–14]. Also in this case one can construct an inequality in terms of the associated quantum Fisher information, but in general it will not coincide with the ultimate achievable bound. This inequality applies only to all n-body *separable* measurements [20] – i.e. POVMs which may act globally on $\rho_x^{\otimes n}$ but whose elements can be expressed as convex combination of tensor products of positive operators (which is a larger class than LOCC [21]). If, instead, we allow for generic joint measurements which exploit entangled resources in a non trivial fashion, then the bound may not hold and better performances are expected (even though the $n^{-1/2}$ scaling still holds).

Quantum parameter estimation for channels

The analysis of the previous section focused on scenarios in which the set of states $\rho_x^{\otimes n}$ is fixed by the estimation problem. Thus, it can only describe the last stage (probe readout) of a general measurement. Since (2) and (3)

obey a SQL scaling $n^{-1/2}$, to obtain a quantum metrology type enhancement, we need to consider the other two stages (i.e. probe preparation and interaction). For this purpose it is useful to describe the correspondence $x \to \rho_x$ in terms of a quantum channel Φ_x which produces ρ_x when acting on the initial input state ρ_0 of the probe, through the mapping $\Phi_x(\rho_0) = \rho_x$. Aim of this section is to characterize the best estimate of x which can be obtained for a fixed number n of applications of Φ_x, while optimizing with respect to the measurement, the estimation functions, *and* the choice of the initial state [22–37]. This captures the basic aspects of most quantum metrology applications [5].
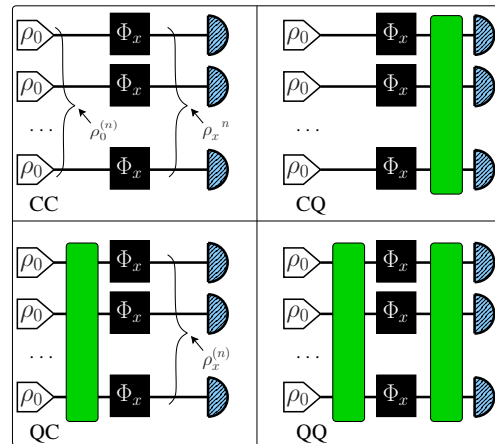

FIG. 2: Schematic representation of parallel estimation strategies. In the picture, a white wedge element represents an input probe entering the apparatus, a black box represents an application of the channel Φ_x, a blue semicircles describes a local measurement on the probe, while a green box represents an entangling operation among the probes. CC (classical-input, classical-output): these are those strategies where the probes are prepared in a separable state $\rho_0^{(n)}$ (not necessarily all probes must be in the same state as depicted), and LOCC measurements at the output. In this case, the averaging of the local results yields a decrease in the result's precision that at most scales as the SQL, i.e. $n^{-1/2}$. CQ (classical-input, quantum-output): entanglement among the probes is generated just before the detection. QC (quantum-input, classical-output): entanglement among the probes is generated before they are fed into the channel while no entanglement resource is employed at the detection stage which uses a LOCC strategy. QQ (quantum-input, quantum-output): the most general strategy where entanglement can be used both at the probe preparation and at the probe detection stages. By construction the QQ and CC strategies will always provide the best and the worst performances respectively. In the case of estimation of unitary channels, CQ has the same $n^{-1/2}$ yield of CC, whereas QC and QQ can achieve the Heisenberg bound n^{-1} [24]. Non-unitary channels have equal or worse performances (since any non-unitary map can be purified into a unitary), depending on their action on the probes [24, 25].

As shown in Figs. 2 and 3, we can employ different strategies depending on the choice for the initial state of the probes, for the measurements, and for the operation we can perform on the probed system (in parallel

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[3] This is defined as $J(\rho_x) := \text{Tr}[\mathcal{R}_{\rho_x}^{-1}(\rho_x')\rho_x \mathcal{R}_{\rho_x}^{-1}(\rho_x')]$, where $\rho_x' = \partial \rho_x / \partial x$, and where $\mathcal{R}_{\rho}^{-1}(O) := \sum_{j,k:\lambda_j+\lambda_k \neq 0} 2O_{jk}|j\rangle\langle k|/(\lambda_j + \lambda_k)$ is the symmetric logarithmic derivative written in the basis that diagonalizes $\rho_x = \sum_j \lambda_j |j\rangle\langle j|$ [10].

or sequentially) [24]. In full analogy to what is done in quantum channel communication theory, we can also consider *entanglement assisted* schemes in which the incoming probes are entangled with an external ancillary system A which is not effected by the channel Φ_x (this is the most general method of performing measurements). Accuracy enhancements via entanglement assisted schemes for parallel configurations have been derived for special classes of quantum channels Φ_x [28–32].
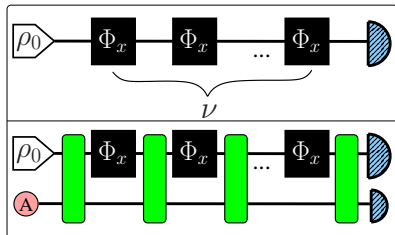

FIG. 3: **Right:** Sequential (or multi-round) scheme [24–26, 38, 39], where a single probe state samples the n black boxes sequentially. **Left:** Entanglement-assisted sequential scheme, where an external ancillary system A aids the estimation. These schemes have some advantages with respect to the parallel schemes as in principle do not require to use multi-partite entanglement among n different probes to achieve sub-shot-noise scaling (furthermore they can simulate any other schemes by properly tuning the intermediate controls). The most general estimation scheme will be a combination of the sequential and of the parallel schemes of Fig. 2. An optimization over the possible black-box dispositions was presented in [37].

Optimizing with respect to all available resources can be a hard problem. For the sake of simplicity in the following we will focus on the parallel strategies. In this context the conventional approach is to split the optimization into two stages [28–32]. Specifically one *first* fixes a (possibly entangled) input state $\rho_0^{(n)}$ and optimizes with respect to the measurements, and *then* minimizes the result with respect to the $\rho_0^{(n)}$. From the results of the previous section it follows that the accuracy of the first stage will not be larger than q-CR bound associated with the selected state. Therefore, the second optimization yields

$$\delta X_n \geqslant \min_{\rho_0^{(n)}} \frac{1}{\sqrt{J(\rho_x^{(n)})}} , \qquad (4)$$

where $J(\rho_x^{(n)})$ is the quantum Fisher information of the output state $\rho_x^{(n)}$. Since the quantity $J(\cdot)$ on the right is convex [28], the minimum is always achieved by choosing pure states. In addition, due to the additivity properties of $J(\cdot)$ it is clear that restricting the minimization in Eq. (4) to input states which are separable with respect to n probes, one can achieve at most a $n^{-1/2}$ scaling (SQL) [24]. Better scalings require entangled states $\rho_x^{(n)}$. To exemplify this, we focus on the special case in which the Φ_x perform random unitary rotations

e^{-ixH}. This is a paradigmatic model which, among others, includes the Ramsey setup of Fig. 1, and the Mach-Zehnder interferometer [4, 5, 40, 41], commonly used to detect an unknown phase shift (see below). Assume that the total number $n = \nu N$ of probes is split in $\nu \gg 1$ groups of N elements, where each group is prepared in the same entangled input state $|\psi_0^{(N)}\rangle$. The output state of all n probes after passing through the black-boxes is $|\psi_x^{(N)}\rangle^{\otimes\nu} := (e^{-iHx})^{\otimes n}|\psi_0^{(N)}\rangle^{\otimes\nu}$. The bound (4) can then be evaluated exploiting Eq. (3) and yields [24],

$$\delta X_n \geqslant \frac{c}{\sqrt{\nu}N} , \qquad (5)$$

where $1/c$ is the largest gap in the spectrum of the generator H of the transformation Φ_x. For $N > 1$ it presents a $N^{1/2}$-enhancement with respect to the standard-quantum-limit. The term on the right is the Heisenberg bound and can be attained by preparing the probes in each group into the entangled state $|\psi_0^{(N)}\rangle = (|\lambda_{max}\rangle^{\otimes N} + |\lambda_{min}\rangle^{\otimes N})/\sqrt{2}$, where $|\lambda_{max}\rangle, |\lambda_{min}\rangle$ are the eigenvectors associated to the maximum and minimum eigenvalues of H, respectively[4]. From the previous section it follows that the threshold (5) is asymptotically attainable for $\nu \gg 1$ by exploiting POVMs which act locally on the blocks and a maximum likelihood approach. Alternatively [24] it is also achievable via adaptive strategies based on POVM which acts locally on each probe (so that the yield of QC strategies coincides with the one of QQ strategies).

In contrast, the situation is more complicated for finite ν, specifically for $\nu = 1$ where Eq. (5) would yield the impressive n^{-1} scaling for the RMSE of the Heisenberg bound. For the special case in which Φ_x induces unitary transformations, this problem was analyzed in Refs. [25, 26] where a scaling of order $n^{-1}\log n$ was achieved by means of local adaptive strategies. In Ref. [25] it was also observed that if the channels Φ_x are *programmable*[5] then a sub-shot-noise scaling of the accuracy is not allowed. Examples for which the no-go theorem of [25] hold are provided by the class of classical channels (i.e. quantum channels Φ_x which cannot propagate quantum information) and by the class of depolarizing channels [30]: for these families of maps the

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[4] For the Ramsey setup $|\psi_0^{(N)}\rangle$ corresponds to the maximally entangled state $(|a\rangle^{\otimes N} + |b\rangle^{\otimes N})/\sqrt{2}$ of Fig. 1. In the atomic literature, where $|a\rangle$ and $|b\rangle$ refer to different atomic levels, $|\psi_0^{(N)}\rangle$ often is referred to as "spin-squeezed state" [42, 43] or "Schrödinger-cat state" [44–46]. It has been experimentally demonstrated using trapped ions [44–47], Bose-Einstein condensates [48], macroscopic atomic ensembles [49], and NMR [50].

[5] The map Φ_x is called programmable if one can express it in terms of a constant interaction with an external ancillary system B whose initial state encodes the dependence on the parameter x, i.e. $\Phi_x(\cdot) = \mathrm{Tr}_B\{V[\sigma_x \otimes (\cdot)]V^\dagger\}$ where V is a unitary transformation acting on B and S, σ_x is the state of B, and $\mathrm{Tr}_B\{\cdot\}$ is the partial trace on B [51].

best estimation accuracy of the parameter x will scale at most as $n^{-1/2}$. Recently it was also pointed out that for finite ν, the two-step optimization approach adopted in the derivation of Eq. (4) in general fails to provide the achievable bound [22, 52]. Instead one must adopt a min-max optimization scheme, minimizing the maximum RMSE, which always yields attainable bounds. It gives slightly worse performance than the two-step approach, but (apart from numerical prefactors) it maintains the same n^{-1} scaling (at least when the quantum channels Φ_x induce unitary transformations). In concluding, we also briefly mention that analogous n^{-1} scalings for specific problems of quantum channel estimation have been obtained by using error measures which are substantially different from the RMSE adopted here (e.g. from those results one would not be able to infer the exact n^{-1} scaling for the RMSE which follows from the q-CR bound). In particular Refs. [33–37] consider the problem of estimating unitary rotations in finite dimensional systems using a metric introduced by Holevo [11].

Applications in Quantum interferometry

The prototypical example of a quantum interferometric application of a quantum estimation procedure is provided by the Mach-Zehnder interferometer. In this setting two input optical modes are merged at a first 50-50 beam splitter, propagate along two paths of different length accumulating an (unknown) relative phase shift φ, and are then merged at a second 50-50 beam splitter. The goal is to recover the value of φ by measuring the signals emerging from the interferometer, while employing a limited amount of resources (i.e. by setting an upper limit N on either the maximum number or on the average number of the photons entering the interferometer at each experimental run). Indicating with a and b the annihilation operators associated with the two internal paths of the interferometer, the problem of recovering φ reduces to estimating a channel $\Phi_{x=\varphi}$ which induces a unitary rotation e^{-ixH} with $H = (a^\dagger a - b^\dagger b)/2$ being the effective system Hamiltonian, e.g. see Refs. [4, 40, 53]. With this identification the two-step optimization strategy that brought us to Eq. (5) can be used to set a lower bound on the RMSE. First consider the situation in which generic POVM measurements are performed on ν independent preparations of the interferometer. In this case Eq. (3) yields the following bound

$$\delta\varphi_\nu \geqslant \min_{|\Psi\rangle} \frac{1}{2\sqrt{\nu}\,\Delta H}\,,\qquad(6)$$

where the minimization is performed over the set of input states $|\Psi\rangle$ which satisfy the selected photon number constraint (i.e. the maximum number, or the average number constraint), and where ΔH is the associated energy spread. Under both constraints, the optimal input $|\Psi\rangle$ is provided by a state which at the level of the internal modes of the interferometer, can be expressed as

a "NOON" state [4, 54, 55], a superposition of the form $(|N,0\rangle + |0,N\rangle)/\sqrt{2}$, in which N photons are propagating along the first or the second optical path, e.g. see Refs. [56]. NOON states are the formal analogue of the spin-squeezed states that achieve the Heisenberg bound in a Ramsey configuration (see footnote 4 and Fig. 1): in fact, also the NOON states exhibit a special sensitivity with respect to the transformation which encodes the random variable φ (indeed NOON get transformed into output states $(e^{-i\varphi N/2}|N,0\rangle + e^{i\varphi N/2}|0,N\rangle)/\sqrt{2}$ in which the phase φ result effectively multiplied by a factor N). With this choice Eq. (6) yields a lower bound of the form

$$\delta\varphi_\nu \geqslant \frac{1}{\sqrt{\nu}\,N}\,,\qquad(7)$$

which, for any given N, is achievable in the limit of large ν, e.g. via maximum-likelihood estimation based on the photo-counting statistics at the outport ports of the interferometer. The bound (7) shows a $N^{-1/2}$ enhancement with respect to more standard estimation approaches where, for instance, the input ports of the interferometer are fed with coherent states of average photon number N (these procedures show a SQL scaling $\delta\varphi_\nu = 1/\sqrt{\nu\,N}$ in which, basically, *all* the νN photons contribute independently in the estimation process). For this reason Eq. (7) can be seen as the quantum optical counterpart of the Heisenberg bound of Eq. (5).

The attainability of the bound (7) requires some extra considerations. First of all it assumes the ability of creating NOON states. This is possible via some rather complicated optical schemes [57–61] which have only been implemented in highly refined, but post-selected experiments [61–66]. However states which possess a high fidelity with the NOON states, also for large values of N, can be simply obtained by mixing a squeezed vacuum and a coherent state at a beam splitter [67–69]. Introducing such states at the input of a Mach-Zehnder, a scaling N^{-1} in the average photon number of photons employed in a given experimental run can be achieved [1]. To do so however one needs to employ the proper estimation process [70]: a point which is not sufficiently stressed in literature where often suboptimal performances can be associated with a poor processing of the measurement outcomes. Analogous N^{-1} performances can also be achieved by employing different sources and/or by using estimator functions which are simpler than the maximum-likelihood approach. Typically these schemes are based on adaptive strategies where the parameter φ is pushed toward an optimal working point which guarantees higher performances, e.g. two recent proposals are Refs. [71, 72] while a list of older ones are in Refs. [4, 5]. Alternative schemes instead are based on sequential strategies of Fig. 3 where a *single* photon pulse probes recursively the phase shift φ by passing multiple times through the delay line [24–26, 38, 39].

The achievability of the bound (7) for finite values of ν is also non trivial [6, 41, 52, 53, 56, 70, 73–76]. Several numerical analysis supported the evidence that Heisenberg-

like N^{-1} scaling should be achievable in the limit of large N also for $\nu = 1$, where Eq. (7) would yield a scaling analogous to the strong Heisenberg bound (see the previous section). In particular Refs. [53, 70, 74, 75] studied the asymptotic behavior of the confidence of the error probability by adopting a Bayesian estimation strategy [41]. A recent work by Hayashi [52] however appears to settle the problem by showing that while for $\nu = 1$ the bound of Eq. (7) is not exactly achievable, one could still reach an asymptotic N^{-1} scaling for the RMSE by adopting a min-max optimization approach [22].

Filtering protocols

Instead of going through the trouble of creating the complex and fragile quantum states necessary for the quantum enhancements of quantum metrology, some protocols have been proposed that use easy to create and robust classical states, and then filter (post-select) the high-resolution states at the measurement stage. The basic idea is to employ retrodiction [77, 78]: once a high-resolution quantum state has been detected at the output, one can interpret the whole experiment as having employed such a state since the input. This is a consequence of the fact that the wave-function collapse can be placed at an arbitrary time between probe preparation and measurement [77, 79].

The filtering intrinsic in such protocols implies that part of the resources available at the onset are wasted: the system is sampled with many more resources than those actually employed for the parameter estimation. Moreover, since the system is sampled with classical states that present no quantum correlations, it is clear from the preceding sections that the SQL cannot be beaten: there is no increase in resolution over the optimal *classical* strategy that could employ *all* the resources (without filtering). For this reason, filtering protocols cannot be considered proper quantum metrology protocols, according to the definition we have given above. Nonetheless they can be extremely useful in the common case when the optimal classical strategy that employs all the resources that have sampled the system is impractical. Moreover, in practical situations efficiency considerations rarely play a role, whereas robustness to noise is paramount. Classical states are, by definition, the most robust ones. In addition, there are situations where filtering methods achieve tasks that would be impossible with purely classical strategies, and post-selecting on the high-resolution quantum states is often very simple.

In Ref. [78] the theory of filtering protocols for phase estimation is developed, and necessary conditions are given to distinguish super-sensitivity (i.e. the error in the estimation is lower than allowed using classical resources) from super-resolution (i.e. the error in the estimation is lower than what would be allowed by a classical procedure that uses only the resources that the filtering retains). Filtering protocols can achieve super-

resolution (with an appropriate measuring strategy) but cannot achieve the super-sensitivity of quantum metrology proper.

Beyond the Heisenberg bound: nonlinear estimation strategies

Several Authors [80–96] have recently considered the possibility of using nonlinear effects to go beyond the N^{-1} Heisenberg-like scalings in phase estimation problems. These new regimes have been called "super-Heisenberg" scalings in Ref. [87], but a proper accounting of the resources shows that they are fully compatible with the analysis presented in the previous sections, see e.g. Ref. [97]. Ultimately the idea of these proposals is to consider settings where the unitary transformation that "writes" the unknown parameter x into the probing signals, is characterized by many-body Hamiltonian generators which are no longer extensive functions of the number of probes employed in the estimation [83–92] or, for the optical implementations which yielded the inequality (6), in the photon number operator of the input signals [80–83, 93–96]. Consequently, in these setups, the mapping $(e^{-ixH})^{\otimes n}$ which acts on the input states $\rho_0^{(n)}$, gets replaced by a transformations of the form $e^{-ixH^{(n)}}$ which couples the probes non trivially. Accordingly the minimization of Eq. (6) is no longer forced to obey to the inequality (7). For instance, RMSE with scalings of the order $\propto N^{-k}$ can be obtained when using Hamiltonians that involves k-system interactions between the probes [85], while $\propto 2^{-N}$ scaling can be achieved by introducing an exponentially large number of coupling terms [83]. Proposed implementations include scattering in Bose condensates [87, 88], Duffing nonlinearity in nano-mechanical resonators [93], two-pass effective nonlinearity with an atomic ensemble [89], Kerr-like nonlinearities [80–82, 95, 96], and nonlinear quantum atom-light interfaces [92].

Quantum metrology with noise

The study of noisy quantum metrology is a special case of parameter estimation for channels, where the map Φ_x which describes the encoding of the unknown parameter x, contains also the description of the noise tampering with the process. Hence, even though very few general results are known, many results detailed in the previous sections can be used to characterize noise effects. Non trivial examples are known where a Heisenberg scaling can be retained even in the presence of noise [25]. On the other hand a simple application of the programmable criterion of Ref. [25] shows that even a small amount of depolarizing noise is sufficient to ruin any sub-shot-noise performances one could reach by entangling the probing systems. Typical quantum metrology protocols are indeed extremely sensitive to noise.

For instance, the incoherent loss of a single photon in a NOON state transforms it into a statistical mixture $(|N-1,0\rangle\langle N-1,0| + |0,N-1\rangle\langle 0,N-1|)/2$ which is useless for phase-sensing. Such extreme sensitivity to losses implies that this state cannot be used in any practical situation [68, 98, 99]. Is it then really possible to outperform classical strategies in practical phase sensing [100]?

Surprisingly, it has been shown that asymptotically it is possible to do so only by a constant factor [101, 102]: for any nonzero loss, for sufficiently high number of photons N the scaling of the optimal phase sensing is proportional to the scaling of the shot noise $\propto N^{-1/2}$. While this means that quantum approaches are useful in highly controlled environments [102] (such as for gravitational wave detection [1]), they only allow for very small enhancements in free-space target acquisition [102]. Nonetheless, the shot noise can be beaten [103] and the optimal states to do so in the presence of loss have been calculated numerically using various optimization techniques for fixed number of input photons [104, 105] and for photon-number detection [106]. A post-selected proof of principle experiment that employs some of these optimal states was recently performed [66]. Note that, for very low values of loss, NOON states retain their optimality [104, 106], and can be approximated by states that are easy to generate [67, 68]. Also, a very simple proposal based on parametric downconversion which can be realized without post-selection was proposed in [72]: it can achieve the Heisenberg bound for low loss and degrades gracefully with noise. In the case in which there is a large amount of loss *after* the sample has interacted with the light probe, even achieving the shot noise limit might become cumbersome (most of the photons that interacted with the phase shifter and contain phase information are lost). A simple strategy that amplifies the signal before detection and can asymptotically achieve the shot noise was experimentally tested in [107]. In contrast, the optimal states for the sequential (or multi-round) interferometry [38, 39] have not been obtained so far in the presence of noise [100]. However, as an alternative to the fragile two-mode states, some more robust single mode states were also analyzed, e.g. pure Gaussian states in the presence of phase diffusion [108], mixed Gaussian states in the presence of loss [109], or single mode variants of the two mode states [110]. In contrast to the two mode case (where the phase is the relative one between the two modes) here the phase is measured relative to a strong classical signal (using heterodyne or homodyne measurements) or similar strategies.

Historically, the first analysis of a quantum metrology protocol [2] in the presence of noise was performed in [111], where frequency measurements are analyzed in the presence of dephasing. Interestingly, a phase transition was shown: for any nonzero value of dephasing, the maximally entangled state suddenly ceases to present any advantage over a classically correlated state. (An analogous result for magnetometry exists [112].) We briefly present this result, as it is instructive of the subtleties encountered by quantum metrology when noise sources are considered.

Consider the Ramsey setup of Fig. 1 when a dephasing process acts on the atomic probes. For the conventional setup the probability of finding the probe in the initial state after the application of the phase shift becomes $p = (1 + e^{-\gamma t}\cos\varphi)/2$, where $\gamma \geqslant 0$ measures the dephasing rate and t is the time elapsed from the state preparation up to the measurement readout. Analogously, for the quantum setting we get $p_{ent} = (1 + e^{-\gamma Nt}\cos N\varphi)/2$, where the factor N in the exponent derives from the exponentially greater sensitivity of the entangled state to the dephasing. The corresponding RMSE (using the equations derived in Fig. 1) is $\delta\varphi_n = (\frac{e^{2\gamma t} - \cos^2\varphi}{n\sin^2\varphi})^{1/2}$, and $\delta\varphi_n = (\frac{e^{2\gamma Nt} - \cos^2 N\varphi}{\nu N^2 \sin^2 N\varphi})^{1/2}$, respectively. For $\gamma > 0$ they are both minimized for $t = 0$ (namely, we have to use a measurement procedure which is as fast as possible), and one recovers the typical $N^{1/2}$ enhancement of quantum metrology also in the presence of dephasing. However, if one wants to measure a frequency [2, 111] ω, then the phase factor is $\varphi = \omega t$. In this case, the errors of the separable and entangled procedure are respectively $\delta\omega_n = (\frac{e^{2\gamma t} - \cos^2\omega t}{nt^2 \sin^2\omega t})^{1/2}$ and $\delta\omega_n = (\frac{e^{2\gamma Nt} - \cos^2 N\omega t}{\nu N^2 t^2 \sin^2 N\omega t})^{1/2}$. For $\gamma = 0$, we have $\delta_n\omega = 1/t\sqrt{n}$ and $\delta\omega_n = 1/(tN\sqrt{\nu})$. The optimization over t says that our measurement must be as long as possible. Choosing the same t in the two cases, the entangled protocol is $\sqrt{N}$ more precise. Interestingly, for $\gamma > 0$ the two situations cannot be anymore optimized on the same t. For example, if the frequency is $\omega = \pi\gamma$, it is easy to see that one has to choose $t = (2\gamma)^{-1}$ to optimize the separable procedure and $t = (2N\gamma)^{-1}$ to optimize the entangled one. With these choices, one obtains for the two cases $\delta\omega_n = \sqrt{e(2\gamma)^2/n}$ and $\delta\omega_n = \sqrt{e(2\gamma)^2/\nu}$. We must assign equal resources to the two protocols, so we can repeat times the entangled procedure N times as it is N times faster. This recovers a factor $1/\sqrt{N}$ also for the entangled procedure, whose error then matches the separable one, $\delta\omega_n$. In [111, 113] it is shown that using non-maximally entangled states one can beat the separable procedure by $\sim 40\%$, which was proven to be the ultimate gain.

Summarizing, the estimation of the phase φ is unaffected by the presence of dephasing noise (as long as the measurement time t can be chosen short enough): the full $N^{-1/2}$ enhancement of quantum metrology is retained. On the contrary, the estimation of the frequency $\omega = \varphi/t$ is strongly affected by the same noise: basically all quantum enhancement is lost for any value of γ. This is connected to the fragility of the entangled resource in the presence of noise: the entangled procedure must be performed N times faster, which reduces the precision in the estimation of ω by a factor N, thus achieving the same final yield as the separable protocol. Note, however, that if the observation time t in atomic clocks is constrained by experimental issues (typically, by the local oscillator fluctuations), then entanglement allows a sub-shot noise scaling [114] which scales as $N^{-2/3}$ [115].

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