

# Generalized minimal output entropy conjecture for one-mode Gaussian channels : definitions and some exact results

V. Giovannetti,<sup>1</sup> A. S. Holevo,<sup>2</sup> S. Lloyd,<sup>3</sup> and L. Maccone<sup>3</sup>

<sup>1</sup>*NEST, Scuola Normale Superiore & CNR-INFM, Piazza dei Cavalieri 7, I-56126 Pisa, Italy*

<sup>2</sup>*Steklov Mathematical Institute, Gubkina 8, 119991 Moscow, Russia*

<sup>3</sup>*Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, MA, 02139, USA*

(Dated: October 4, 2010)

A formulation of the generalized minimal output entropy conjecture for Gaussian channels is presented. It asserts that, for states with fixed input entropy, the minimal value of the output entropy of the channel (i.e. the minimal output entropy increment for fixed input entropy) is achieved by Gaussian states. In the case of centered channels (i.e. channels which do not add squeezing to the input state) this implies that the minimum is obtained by thermal (Gibbs) inputs. The conjecture is proved to be valid in some special cases.

PACS numbers:

## I. INTRODUCTION

Several of the most difficult problems in quantum information theory [1, 2] deal with optimizations of non-linear cost functions. In particular, in close analogy to what is done in the classical theory [3], the efficiency of a communication line (quantum channel) is measured by maximizing an entropic functional over the set of possible channel inputs. Apart from some special cases, such optimizations are in general too complex to be performed explicitly. In an effort to simplify the analysis, several conjectures were proposed based either on physical intuition or on symmetries of the problem. The most known is the additivity conjecture recently disproved by Hastings [4]: it claimed that the minimal value of the von Neumann entropy at the output of a memoryless channel is achieved by separable input states or, equivalently [5], that its Holevo capacity [6] is additive. As a consequence we now know that the classical capacity [7] of a memoryless quantum channel (i.e. the maximum achievable rate of reliable classical communication) will be in general difficult to evaluate as it necessarily requires to perform a nontrivial regularization over infinitely many channel uses.

Of course it is still possible that special classes of quantum channels will obey additivity rules that would allow us to simplify their analysis. In particular it is generally believed that Bosonic Gaussian channels [8–10] should be one of such classes. As a matter of fact Bosonic Gaussian channels appear to have a preferred (simple) set of inputs states (the Gaussian states [11]) over which the optimization of the relevant entropic quantities can be performed simplifying the calculation [12]. Several results support such belief. In particular the capacity of lossy Gaussian channels was proved to be additive by explicitly computing its value [13]; for thermalizing channels the minimum values of the Rényi entropies were shown to be additive for integer orders and unconstrained input [14], and for arbitrary order under the constraint of Gaussian inputs [15]; finally the degradability and additivity of the

coherent information for some of those channels was established in Refs. [16–18]. Partially motivated by the above results few years ago a conjecture was proposed a solution of which would allow one to simplify the whole scenario, allowing for instance a direct computation of the classical capacity of some Gaussian channels. In particular it was suggested that the (unconstrained) minimum of the von Neumann [20] or Rényi [21] entropies for attenuators or additive Gaussian classical noise channels should be achieved by the vacuum state. Up to now all attempts to prove this apparently innocuous claim have failed, including an innovative approach that was recently presented [22] in which the original conjecture of Ref. [20] was generalized. The aim of this paper is to review the problem extending the conjecture to include all Gaussian channels, and to prove it in some special but not necessarily trivial cases. In our treatment of unbounded operators arising from Canonical Commutation Relations we focus on the aspects essential for physical calculations. A number of analytical complications related to infinite dimensionality and unboundedness unavoidably arises in connection with Bosonic systems and Gaussian states. A detailed treatment of related mathematical tools can be found in [23, 24] and the references therein.

The paper is organized as follows. In Sec. II the notations are introduced and the formulation of the conjecture is presented. For channels which admit semigroup structure we also introduce an infinitesimal version of the conjecture. In Sec. III we focus on a special class of channels and introduce a relatively simple argument based on subadditivity properties of the von Neumann entropy which allows one to prove the conjecture for some (lucky) cases. Sec. IV proves the conjecture for a class of degenerate Gaussian channels. Conclusion and remarks are given in Sec. V.

## II. THE CONJECTURE

Let  $\Phi$  be a linear, completely positive, trace preserving (LCPT) map (see, e.g. [2]) which transforms a (possibly

infinite dimensional) input system  $A$  to an output system  $B$ . Indicating with  $\mathfrak{S}(\mathcal{H}_A)$  the set of density matrices of the input space, we define  $\mathfrak{S}_{S_0}(\mathcal{H}_A)$  and  $\mathfrak{S}_{S_0}^+(\mathcal{H}_A)$  as the subsets of  $\mathfrak{S}(\mathcal{H}_A)$  formed, respectively, by states with entropy equal to  $S_0$  and by states with entropy larger than or equal to  $S_0$ , i.e.

$$\mathfrak{S}_{S_0}(\mathcal{H}_A) = \{\rho \in \mathfrak{S}(\mathcal{H}_A) : S(\rho) = S_0\}, \quad (1)$$

$$\mathfrak{S}_{S_0}^+(\mathcal{H}_A) = \{\rho \in \mathfrak{S}(\mathcal{H}_A) : S(\rho) \geq S_0\}, \quad (2)$$

with  $S(\rho) = -\text{Tr}[\rho \ln \rho]$  being the von Neumann entropy [2] of  $\rho$ . By the concavity of  $S$  one has that the  $\mathfrak{S}_{S_0}^+(\mathcal{H}_A)$  are convex sets which can be expressed as proper unions of the  $\mathfrak{S}_{S_0}(\mathcal{H}_A)$ , namely  $\mathfrak{S}_{S_0}^+(\mathcal{H}_A) = \bigcup_{S \geq S_0} \mathfrak{S}_S(\mathcal{H}_A)$ . Furthermore they form an ordered family under inclusion, i.e.

$$\mathfrak{S}_{S_0}^+(\mathcal{H}_A) \subset \mathfrak{S}_{S'_0}^+(\mathcal{H}_A), \quad \text{for all } S'_0 > S_0. \quad (3)$$

In particular for  $S_0 = 0$ ,  $\mathfrak{S}_0(\mathcal{H}_A)$  represents the set of pure state of the system while  $\mathfrak{S}_0^+(\mathcal{H}_A)$  coincides with the whole space  $\mathfrak{S}(\mathcal{H}_A)$ . We are interested in computing the minimum value that the output entropy  $S(\Phi(\rho))$  can take on the set  $\mathfrak{S}_{S_0}(\mathcal{H}_A)$ , i.e. the quantity

$$\mathcal{F}(\Phi; S_0) \equiv \inf_{\rho \in \mathfrak{S}_{S_0}(\mathcal{H}_A)} S(\Phi(\rho)). \quad (4)$$

Due to the concavity of  $S$  and the linearity of  $\Phi$ , such a minimum can also be expressed as a minimum over the larger set  $\mathfrak{S}_{S_0}^+(\mathcal{H}_A)$ , i.e.

$$\mathcal{F}(\Phi; S_0) = \inf_{\rho \in \mathfrak{S}_{S_0}^+(\mathcal{H}_A)} S(\Phi(\rho)). \quad (5)$$

For  $S_0 = 0$  the quantity (4) provides the (unconstrained) minimal output entropy of the channel which plays a fundamental role in quantum communication [1]. (In particular, its additivity property under successive uses of the channel was recently disproved in Ref. [4].) Moreover, in the special case in which  $\Phi$  represents an attenuator or additive Gaussian classical noise channels [8] operating on a single Bosonic mode, a conjecture was proposed in Ref. [20] which, if true, would allow one to compute in closed form its classical capacity [7] under the energy constraint. Specifically it was conjectured that the value of  $\mathcal{F}(\Phi; S_0 = 0)$  is attained by a Gaussian input state. In a recent attempt [22] to prove such a property, it was recently extended to include all values of  $S_0 > 0$  and a broader class of maps. Indeed consider a set of  $n$  input Bosonic modes and a Gaussian channel  $\Phi$  which maps them into  $m$  output modes. We remind that for these systems a state is said to be Gaussian if its symmetrically ordered characteristic function (or equivalently its Wigner distribution) corresponds to a Gaussian envelop [11], whereas a LCPT map is said to be Gaussian channel if, when acting on a Gaussian state of the input modes transforms it into an Gaussian state of the output modes [8–11]. It is claimed that:

**Conjecture (v1):** For all  $S_0 \geq 0$  the minimization in Eq. (4) is saturated by a Gaussian element of the set  $\mathfrak{S}_{S_0}(\mathcal{H}_A)$ , i.e.

$$\mathcal{F}(\Phi; S_0) = S(\Phi(\rho_0)), \quad (6)$$

with  $\rho_0 \in \mathfrak{S}_{S_0}(\mathcal{H}_A)$  a Gaussian state (notice that for all  $S_0$ , the sets  $\mathfrak{S}_{S_0}(\mathcal{H}_A)$  always admit at least one Gaussian element).

While in some simple cases the conjecture can be easily verified, in the general scenario it appears to be particularly challenging. In the following we will specify the analysis to the case of single-mode Gaussian channels ( $n = m = 1$ ) for which the canonical decomposition of  $\Phi$  applies [17, 18]. In particular, it is known that apart from the special cases which we will treat in Sec. IV, by making a proper choice of the canonical observables at the input and the output of the channel one can focus on *centered* Gaussian channels which respect the standard complex structure associated with the multiplication by  $i$  (these channels do not introduce squeezing or displacement). They have the property to induce the following transformation on the average photon expectation value,

$$\text{Tr}[\Phi(\rho)a^\dagger a] = \kappa^2 \text{Tr}[\rho a^\dagger a] + c, \quad (7)$$

where  $\kappa$  and  $c$  are constants which depend upon  $\Phi$ , where  $a, a^\dagger$  are the annihilation and creation operator of the system mode. Specifically attenuator channels are characterized by  $\kappa^2 \in [0, 1]$  and  $c = (1 - \kappa^2)N$  with  $N \geq 0$ , while amplifier channels are characterized by  $\kappa^2 \geq 1$  and  $c = (\kappa^2 - 1)(N + 1)$  where again  $N \geq 0$ , (class  $C$  of [17]). For additive Gaussian classical noise channel (class  $B_2$  of [17]) instead one has  $\kappa = 1$  and  $c = N$ . Finally for the weak conjugate of the amplifier channels (class  $D$  of [17]) one has  $\kappa^2 \geq 0$  and  $c = \kappa^2(N + 1) + N$ . Due to the property (7) one can refine the conjecture (v1) by saying that the state  $\rho_0 \in \mathfrak{S}_{S_0}(\mathcal{H})$  entering in Eq. (6) is the (thermal) Gibbs state,

$$\rho_0 = \frac{1}{N_0 + 1} \left( \frac{N_0}{N_0 + 1} \right)^{a^\dagger a}, \quad (8)$$

with  $N_0 = \text{Tr}[a^\dagger a \rho_0] > 0$  being the average photon number of  $\rho_0$  which allows to express the input entropy of  $\rho_0$  as

$$S_0 = g(N_0) \equiv (N_0 + 1) \ln(N_0 + 1) - N_0 \ln N_0, \quad (9)$$

(for  $N_0 = 0$  the the density matrix  $\rho_0$  must be identified with the vacuum state, while  $S_0 = 0$ ). By general properties of Gibbs states, see e.g. [24], we know that  $\rho_0$  is the element of  $\mathfrak{S}_{S_0}^+(\mathcal{H})$  which has minimal energy, i.e.

$$N_0 \leq \text{Tr}[a^\dagger a \rho], \quad \text{for all } \rho \in \mathfrak{S}_{S_0}^+(\mathcal{H}), \quad (10)$$

the identity applying only for  $\rho = \rho_0$ . Furthermore, for the channel under consideration  $\Phi$  will transform  $\rho_0$

into a new Gibbs state  $\rho'_0 = \Phi(\rho_0)$ , having mean photon number  $N'_0 = \kappa^2 N_0 + c$  with  $\kappa, c$  as in Eq. (7). Therefore the conjecture can be restated as follows:

**Conjecture (v2):** *For all  $S_0 \geq 0$  the minimization in Eq. (4) is saturated by the Gibbs state  $\rho_0$  of Eq. (8). Therefore it holds*

$$\mathcal{F}(\Phi; S_0) = S(\Phi(\rho_0)) = g(\kappa^2 N_0 + c), \quad (11)$$

or, equivalently, for all  $\rho \in \mathfrak{S}_{S_0}(\mathcal{H})$  one has

$$S(\Phi(\rho)) \geq g(\kappa^2 N_0 + c). \quad (12)$$

The inequality (12) can be cast in a different form by introducing the relative entropy  $S(\rho||\sigma)$ , see e.g. [24]. Indeed, for all  $\rho \in \mathfrak{S}_{S_0}(\mathcal{H})$  simple algebraic manipulations allows us to write

$$\begin{aligned} S(\Phi(\rho)) - g(\kappa^2 N_0 + c) \\ = \kappa^2 \frac{\ln\left(\frac{\kappa^2 N_0 + c + 1}{\kappa^2 N_0 + c}\right)}{\ln\left(\frac{N_0 + 1}{N_0}\right)} S(\rho||\rho_0) - S(\Phi(\rho)||\Phi(\rho_0)), \end{aligned} \quad (13)$$

where we used the fact that  $S(\rho) = S_0$ . This shows that a necessary and sufficient condition for the conjecture (12) is the inequality,

$$\kappa^2 \frac{\ln\left(\frac{\kappa^2 N_0 + c + 1}{\kappa^2 N_0 + c}\right)}{\ln\left(\frac{N_0 + 1}{N_0}\right)} S(\rho||\rho_0) \geq S(\Phi(\rho)||\Phi(\rho_0)), \quad (14)$$

which needs to apply to all input states  $\rho \in \mathfrak{S}_{S_0}(\mathcal{H})$ . It is worth reminding that the relative entropy is monotonically decreasing under the action of LCPT maps (see, e.g. [2]), that is  $S(\rho||\rho_0) \geq S(\Phi(\rho)||\Phi(\rho_0))$  for all  $\Phi, \rho$  and  $\rho_0$ . Therefore a sufficient condition to prove Eq. (14) would be  $\kappa^2 \ln\left(\frac{\kappa^2 N_0 + c + 1}{\kappa^2 N_0 + c}\right) \geq \ln\left(\frac{N_0 + 1}{N_0}\right)$ . Unfortunately however, for all values of  $\kappa, N_0$  and  $c$  as in Eq. (7) this inequality is always false (notice that  $c$  and  $k$  cannot be taken as independent variables). Incidentally this shows that proving Eq. (14) (and thus the conjecture) requires one to go beyond the monotonicity property of the relative entropy.

For those Gaussian channels  $\Phi$  which possess a semigroup structure [19] the conjecture can be rephrased in terms of a condition on the infinitesimal increments of the entropy. Specifically let  $\mathcal{L}$  be a Lindblad generator and let  $\{\Phi_t : t \geq 0\}$  be a one-parameter family of Gaussian LCPT maps which solve the equation

$$\frac{\partial}{\partial t} \Phi_t = \mathcal{L} \circ \Phi_t, \quad \Phi_0 = \mathcal{I}, \quad (15)$$

with  $\mathcal{I}$  being the identity channel and  $\circ$  being the composition of maps. For instance this property holds for attenuator, amplifier and additive Gaussian classical noise channels with

$$\mathcal{L} = \frac{\gamma_+}{2} \mathcal{L}_+ + \frac{\gamma_-}{2} \mathcal{L}_-, \quad (16)$$

where  $\gamma_{\pm}$  are positive parameters related to  $\kappa$  and  $c$  of Eq. (7) and where

$$\mathcal{L}_+(\cdot) = 2a^\dagger(\cdot)a - aa^\dagger(\cdot) - (\cdot)aa^\dagger, \quad (17)$$

$$\mathcal{L}_-(\cdot) = 2a(\cdot)a^\dagger - a^\dagger a(\cdot) - (\cdot)a^\dagger a. \quad (18)$$

In particular one can easily verify that attenuators form a semigroup satisfying Eq. (15) with  $\gamma_+ = N, \gamma_- = N+1$  when setting  $\kappa^2 = e^{-t}$ . Similarly for the amplifiers we have  $\gamma_+ = N+1, \gamma_- = N$  by choosing  $\kappa^2 = e^t$ , while for additive Gaussian classical noise channels we have  $\gamma_+ = \gamma_- = 1$  when setting  $N = t$ .

For each  $T \geq 0$  we can then express the output entropy of the map  $\Phi_{t=T}$  in the integral form

$$S(\Phi_T(\rho)) = S(\rho) - \int_0^T dt \operatorname{Tr}[\mathcal{L}(\Phi_t(\rho)) \ln \Phi_t(\rho)], \quad (19)$$

where we used the fact that  $\frac{\partial}{\partial t} S(\Phi_t(\rho)) = -\operatorname{Tr}[\mathcal{L}(\Phi_t(\rho)) \ln \Phi_t(\rho)]$ . Due to the identity (19) we can now present an infinitesimal version of the conjecture:

**Conjecture (v3):** *For all  $S_0 \geq 0$  define the quantity*

$$\mathcal{F}(\mathcal{L}; S_0) \equiv - \inf_{\rho \in \mathfrak{S}_{S_0}(\mathcal{H}_A)} \operatorname{Tr}[\mathcal{L}(\rho) \ln \rho], \quad (20)$$

then the minimization is saturated by the Gibbs state  $\rho_0 \in \mathfrak{S}_{S_0}(\mathcal{H}_A)$  defined in Eq. (8), i.e. it holds the identity

$$\begin{aligned} \mathcal{F}(\mathcal{L}; S_0) &= -\operatorname{Tr}[\mathcal{L}(\rho_0) \ln \rho_0] \\ &= [(\gamma_+ - \gamma_-)N_0 + \gamma_+] \ln\left(\frac{N_0 + 1}{N_0}\right). \end{aligned} \quad (21)$$

Clearly if the conjecture (v2) is true for all discrete channels  $\Phi_T$  and for all  $S_0$ , then also the infinitesimal version (v3) of the conjecture must be true. This follows simply by the fact that the functional  $\mathcal{F}(\mathcal{L}; S_0)$  can be expressed as the limit for  $dt \rightarrow 0$  of the minimum value that the finite entropy increments  $[S(\Phi_{dt}(\rho)) - S(\Phi_0(\rho))]/dt = [S(\Phi_{dt}(\rho)) - S(\rho)]/dt$  assume over the set  $\mathfrak{S}_{S_0}(\mathcal{H}_A)$ . Of course if (v2) is true then for all  $dt$  such minimum is achieved on  $\rho_0$ , validating Eq. (21). Analogously if the infinitesimal version (v3) is true for all  $S_0$ , then all the channels  $\Phi_T$  generated by  $\mathcal{L}$  will obey the finite version (v2) of the conjecture (to verify this simply use the fact that Gibbs states are mapped into Gibbs states).

Proving or disproving any of the above version of the conjecture is apparently a formidable task. In the following section we will then focus on some special (non trivial) cases for which some preliminary results can be derived.

### III. A SPECIAL CASE

As a special class of single mode Gaussian channel  $\Phi$  consider an attenuator channel  $\mathcal{E}_\eta^N$  which mixes via a beam-splitter (BS) of transmissivity  $\eta \in [0, 1]$  the input

state of the system with a (thermal) Gibbs environmental state  $\rho_E$  characterized by having  $N$  mean photon number and thus entropy

$$S(\rho_E) = g(N), \quad (22)$$

with  $g(\cdot)$  as in Eq. (9). In the language of Ref. [17] this map is attenuator belonging to the class C with  $\kappa^2 = \eta$  and  $c = (1 - \eta)N$  in Eq. (7). Alternatively, following the notation of Ref. [20], it can be expressed in terms of the following input-output transformation

$$\chi(\mu) \longrightarrow \chi'(\mu) = \chi(\sqrt{\eta}\mu) e^{-(1-\eta)(N+1/2)|\mu|^2}, \quad (23)$$

where  $\chi(\mu) = \text{Tr}[\rho D(\mu)]$  and  $\chi'(\mu) = \text{Tr}[\mathcal{E}_\eta^N(\rho) D(\mu)]$  are the symmetrically ordered characteristic function of the input and output state of the system, respectively ( $D(\mu) = \exp[\mu a^\dagger - \mu^* a]$  being the displacement operator of the mode). This channel maps the Gibbs state  $\rho_0$  into a new Gibbs state  $\rho'_0$  of average photon number  $N'_0 = \eta N_0 + (1 - \eta)N$  and of output entropy

$$S(\mathcal{E}_\eta^N(\rho_0)) = S(\rho'_0) = g(N'_0) = g(\eta N_0 + (1 - \eta)N). \quad (24)$$

According to the version **(v2)** of the conjecture then we should have  $\mathcal{F}(\mathcal{E}_\eta^N; S_0) = g(\eta N_0 + (1 - \eta)N)$ , or equivalently

$$S(\mathcal{E}_\eta^N(\rho)) \geq g(\eta N_0 + (1 - \eta)N), \quad (25)$$

for all  $\rho \in \mathfrak{S}_{S_0}(\mathcal{H})$ . Also proving this inequality is rather complicated. In the following we thus focus on the following (very) specific configuration where the input entropy  $S_0$  which defines the set  $\mathfrak{S}_{S_0}(\mathcal{H})$  of possible input states coincides with the entropy of the environment  $\rho_E$ . In particular due to Eqs. (9) and (22) this implies that  $\rho_E$  and  $\rho_0$  are indeed the same state, and thus

$$g(N_0) = g(N) \iff N = N_0. \quad (26)$$

Under this condition we first notice that  $\rho_0$  is the fixed point of the map  $\mathcal{E}_\eta^{N_0}$ , i.e.

$$\mathcal{E}_\eta^{N_0}(\rho_0) = \rho_0, \quad (27)$$

(this can be easily verified from Eq. (23) by reminding that the symmetrically ordered characteristic function of the Gibbs state  $\rho_0$  is  $\exp[-(N_0 + 1/2)|\mu|^2]$ ). Therefore proving the inequality (25) (and thus the conjecture **(v2)**) is now equivalent to showing that

$$S(\mathcal{E}_\eta^{N_0}(\rho)) \geq g(N_0), \quad (28)$$

holds for all  $\rho \in \mathfrak{S}_{S_0}(\mathcal{H})$ . Equivalently, this can also be rewritten as (see Eq. (14)),

$$\eta S(\rho || \rho_0) \geq S(\mathcal{E}_\eta^{N_0}(\rho) || \mathcal{E}_\eta^{N_0}(\rho_0)), \quad (29)$$

which should again apply to all  $\rho \in \mathfrak{S}_{S_0}(\mathcal{H})$ . Among the various properties of the channel  $\mathcal{E}_\eta^{N_0}$  we remind that

they form a semigroup under multiplication due to the properties [18, 20]

$$\mathcal{E}_{\eta_1}^{N_0} \circ \mathcal{E}_{\eta_2}^{N_0} = \mathcal{E}_{\eta_2 \eta_1}^{N_0}, \quad \mathcal{E}_{\eta=1}^{N_0} = \mathcal{I}. \quad (30)$$

Defining thus  $\Phi_t = \mathcal{E}_{\eta=e^{-t}}^{N_0}$  the Lindblad generator is easily derived as in Eq. (16) with  $\gamma_+ = N_0$  and  $\gamma_- = N_0 + 1$ . This allows us to rephrase the infinitesimal version **(v3)** of the conjecture as

$$\mathcal{F}(\mathcal{L}; S_0) = -\text{Tr}[\mathcal{L}(\rho_0) \ln \rho_0] = 0. \quad (31)$$

Interestingly enough even though for arbitrary values of  $\eta$  the inequality (28) is difficult to derive, there are some special case in which it simply follows by general consideration on von Neumann entropy. Specifically the following result can be shown:

**Theorem.** *For arbitrary positive values of  $N_0 \geq 0$ , the inequalities in Eqs. (28) and (29) hold for all  $\eta = 1/k$  with  $k$  integer.*

*Proof:* For  $k = 1$  the result is trivial. For  $k \geq 2$  it follows from the subadditivity of the von Neumann entropy. In particular consider first the case of  $k = 2$ . In this case we introduce a unitary representation [17, 18] of the channel  $\mathcal{E}_{1/2}^{N_0}$  constructed by mixing the input state  $\rho$  via a BS of transmissivity  $\eta = 1/2$  with the thermal environment  $\rho_E$ , i.e.

$$\mathcal{E}_{1/2}^{N_0}(\rho) = \text{Tr}_E[U_{1/2}^{(AE)}(\rho \otimes \rho_E)[U_{1/2}^{(AE)}]^\dagger], \quad (32)$$

where  $\text{Tr}_E$  is the partial trace over the environment and where  $U_\eta^{(AE)} = \exp[\arccos \sqrt{\eta}(a^\dagger b - ab^\dagger)]$  is the BS unitary coupling which connects  $A$  and  $E$  (here  $a$  and  $b$  stands for the annihilation operators of the two systems). In this case the weak complementary  $\tilde{\mathcal{E}}_{1/2}^{N_0}(\dots) = \text{Tr}_A[U_{1/2}^{(AE)}(\dots \otimes \rho_E)[U_{1/2}^{(AE)}]^\dagger]$  is known [18, 26] to be unitary equivalent to  $\mathcal{E}_{1/2}^{N_0}$  (here  $\text{Tr}_A$  indicates the partial trace over the system degree of freedom). Therefore by invoking the subadditivity of the von Neumann entropy we can write,

$$\begin{aligned} 2S(\mathcal{E}_{1/2}^{N_0}(\rho)) &= S(\mathcal{E}_{1/2}^{N_0}(\rho)) + S(\tilde{\mathcal{E}}_{1/2}^{N_0}(\rho)) \\ &\geq S(U_{1/2}^{(AE)}(\rho \otimes \rho_E)[U_{1/2}^{(AE)}]^\dagger) = S(\rho) + S(\rho_E) = 2g(N_0), \end{aligned}$$

which proves the thesis (in the last identity we used the fact that since  $\rho \in \mathfrak{S}_{S_0}(\mathcal{H})$  it has the same entropy  $S_0 = g(N_0)$  of  $\rho_E$ ). For  $k > 2$  we use a similar trick concatenating more BS transformations in series in order to obtain a set-up with  $k$  output ports and  $k$  inputs (one input for the state  $\rho$  and the remaining for  $k - 1$  copies of  $\rho_E$ ). Adjusting the transmissivities of the BS in such a way to guarantee that all of output ports have overall transmissivities  $1/k$  we can invoke the subadditivity to finally derive the inequality

$$S(\mathcal{E}_{1/k}^{N_0}(\rho)) \geq g(N_0), \quad (33)$$

which proves the thesis. More precisely the above construction consists in introducing  $k - 1$  copies of the state  $\rho_E$  and introducing the following  $k$  modes state,

$$\Omega_{AE_1 \dots E_{k-1}} = \rho \otimes \rho_{E_1} \otimes \rho_{E_2} \otimes \dots \otimes \rho_{E_{k-1}}, \quad (34)$$

which has entropy equal to  $kg(N_0)$  when  $\rho \in \mathfrak{S}_{S_0}(\mathcal{H}_A)$ . Consider then following unitary couplings

$$W = U_{\eta_{k-1}}^{AE_{k-1}} \dots U_{\eta_2}^{AE_2} U_{\eta_1}^{AE_1}, \quad (35)$$

where for  $j = 1, \dots, k-1$ ,  $U_{\eta_j}^{AE_j}$  is the BS unitary transformation of transmissivity  $\eta_j$  which couples  $A$  with the system  $E_j$ . The inequality (33) then can be obtained applying the subadditivity of von Neumann entropy to the state  $\Omega'_{AE_1 \dots E_k} = W \Omega_{AE_1 \dots E_k} W^\dagger$ , i.e. using the relation

$$S(\Omega'_{AE_1 \dots E_{k-1}}) \leq S(\Omega'_A) + \sum_{j=1}^{k-1} S(\Omega'_{E_j}), \quad (36)$$

where  $\Omega'_A$  is the reduced matrix of  $\Omega'_{AE_1 \dots E_k}$  associated with the system  $A$ , and where for all  $j = \{1, \dots, k-1\}$   $\Omega'_{E_j}$  is the reduced matrix of  $\Omega'_{AE_1 \dots E_k}$  associated with the system  $E_j$ . Indeed the left-hand side term of this expression coincides with the von Neumann entropy of  $\Omega_{AE_1 \dots E_k}$  hence

$$S(\Omega'_{AE_1 \dots E_{k-1}}) = kg(N_0). \quad (37)$$

On the other hand we notice that for reduced density operator of the subsystem  $A$  one has,

$$\begin{aligned} \Omega'_A &\equiv \text{Tr}_{E_1 \dots E_{k-1}} [\Omega'_{AE_1 \dots E_{k-1}}] \\ &= \mathcal{E}_{\eta_{k-1}}^{N_0} \circ \mathcal{E}_{\eta_{k-2}}^{N_0} \circ \dots \circ \mathcal{E}_{\eta_1}^{N_0}(\rho) = \mathcal{E}_{\bar{\eta}_k}^{N_0}(\rho), \end{aligned} \quad (38)$$

where we used the semigroup property (30) and defined  $\bar{\eta}_k \equiv \eta_{k-1} \eta_{k-2} \dots \eta_2 \eta_1$ . Similarly for the reduced density operator associated with the system  $E'_1$  we notice that

$$\begin{aligned} \Omega'_{E'_1} &\equiv \text{Tr}_{AE_2 E_3 \dots E_{k-1}} [\Omega'_{AE_1 \dots E_{k-1}}] \\ &= \text{Tr}_A [U_{\eta_1}^{(AE_1)}(\rho \otimes \rho_{E_1}) [U_{\eta_1}^{(AE_1)}]^\dagger] = \tilde{\mathcal{E}}_{\eta_1}^{N_0}(\rho), \end{aligned}$$

with  $\tilde{\mathcal{E}}_{\eta_1}^{N_0}$  being the weak-complementary of the channel  $\mathcal{E}_{\eta_1}^{N_0}$  under the unitary representation of Eq. (32). Apart from an irrelevant unitary rotation, this is known to be equivalent to the channel  $\mathcal{E}_{1-\eta_1}^{N_0}$  [18, 26]. Thus we can conclude that,

$$S(\Omega'_{E'_1}) = S(\tilde{\mathcal{E}}_{\eta_1}^{N_0}(\rho)) = S(\mathcal{E}_{1-\eta_1}^{N_0}(\rho)). \quad (39)$$

In a similar fashion we have that for arbitrary  $j = 1, 2, \dots, k-1$  the reduced density matrices of the subsystem  $E_j$  can be expressed as

$$\begin{aligned} \Omega'_{E_j} &= \tilde{\mathcal{E}}_{\eta_j}^{N_0} \circ \mathcal{E}_{\eta_{j-1}}^{N_0} \circ \mathcal{E}_{\eta_{j-2}}^{N_0} \circ \dots \circ \mathcal{E}_{\eta_1}^{N_0}(\rho) \\ &= \tilde{\mathcal{E}}_{\eta_j}^{N_0} \circ \mathcal{E}_{\eta_{j-1} \eta_{j-2} \dots \eta_1}^{N_0}(\rho), \end{aligned}$$

where again the semigroup property (30) was used to simplify the expression. Exploiting then the unitary equivalence between  $\tilde{\mathcal{E}}_{\eta_j}^{N_0}$  and  $\mathcal{E}_{1-\eta_j}^{N_0}$  we finally get

$$\begin{aligned} S(\Omega'_{E_j}) &= S(\tilde{\mathcal{E}}_{\eta_j}^{N_0} \circ \mathcal{E}_{\eta_{j-1} \eta_{j-2} \dots \eta_1}^{N_0}(\rho)) \\ &= S(\mathcal{E}_{1-\eta_j}^{N_0} \circ \mathcal{E}_{\eta_{j-1} \eta_{j-2} \dots \eta_1}^{N_0}(\rho)) = S(\mathcal{E}_{\bar{\eta}_j}^{N_0}(\rho)), \end{aligned} \quad (40)$$

with  $\bar{\eta}_j = (1-\eta_j) \eta_{j-1} \eta_{j-2} \dots \eta_1$ . Equation (36) can thus be rewritten as,

$$kg(N_0) \leq \sum_{j=1}^k S(\mathcal{E}_{\bar{\eta}_j}^{N_0}(\rho)). \quad (41)$$

To prove the thesis we need thus only to find  $\eta_j$  such that  $\bar{\eta}_j = 1/k$  for all  $j = 1, 2, \dots, k$ . To do so we take  $\eta_j = \frac{k-j}{k-j+1}$  for all  $j = 1, 2, \dots, k-1$ . With this choice the right-hand side term of Eq. (41) becomes  $\sum_{j=1}^k S(\mathcal{E}_{\bar{\eta}_j}^{N_0}(\rho)) = kS(\mathcal{E}_{1/k}^{N_0}(\rho))$  yielding Eq. (33). ■

#### IV. PROOF OF THE CONJECTURE FOR ONE MODE DEGENERATE GAUSSIAN CHANNELS

In this section we describe the solution of the conjecture (v1) in the cases of one mode degenerate Gaussian channels. In the canonical form of Ref. [17, 18] they correspond to the classes  $A_1, A_2, B_1$  and are formally characterized by the fact at least one of the two  $2 \times 2$  matrices that describe their action on the Weyl operator of the system is not invertible.

Channels belonging to the class  $A_1$  satisfy the equation

$$\chi(\mu) \longrightarrow \chi'(\mu) = \chi(0) e^{-(N+1/2)|\mu|^2}, \quad (42)$$

which maps any input state into fixed output Gaussian state (indeed they can be seen as limiting cases of attenuators channels (class  $C$ ) with zero beam splitter transmissivity). Hence for these channels, the output entropy is constant and the problem is trivial.

Case  $A_2$  corresponds to the equation

$$\chi(\mu) \longrightarrow \chi'(\mu) = \chi(-i\Im\mu) e^{-(N+1/2)|\mu|^2}, \quad (43)$$

where  $\Im\mu$  is the imaginary part of  $\mu$ . The channel is given explicitly by

$$\Phi(\rho) = \int e^{ixp} \rho_E e^{-ixp} P_\rho(dx), \quad (44)$$

where  $\rho_E$  is a Gibbs state of mean energy  $N$ ,  $p = i(a^\dagger - a)/\sqrt{2}$  is the momentum quadrature of the system, and  $P_\rho(dx) = \langle x|\rho|x\rangle dx$  is the probability distribution of the position operator  $q = (a^\dagger + a)/\sqrt{2}$  in the state  $\rho$  [27]. It is an entanglement-breaking channel [25] which describes position measurement followed by preparation of the state  $e^{ixp} \rho_E e^{-ixp}$  shifted by the outcome of the measurement  $x$ . By concavity of the entropy

$$S(\Phi(\rho)) \geq \int S(e^{ixp} \rho_E e^{-ixp}) P_\rho(dx) = S(\rho_E), \quad (45)$$

and in fact

$$\inf_{\rho \in \mathfrak{S}_{S_0}(\mathcal{H}_A)} S(\Phi(\rho)) = S(\rho_E). \quad (46)$$

To prove this, consider the input Gaussian states  $\rho_{\sigma_q, \sigma_p}$  with zero mean, variances  $Dq = \sigma_q^2$ ,  $Dp = \sigma_p^2$ , and uncorrelated  $q, p$ . The entropy of such states is equal to  $S(\rho_{\sigma_q, \sigma_p}) = g\left(\frac{\sigma_q \sigma_p}{\hbar} - \frac{1}{2}\right)$ . By fixing it equal to  $S_0$  and letting  $\sigma_q \rightarrow 0$ , we obtain  $S(\rho') \rightarrow S(\rho_E)$ .

Case  $B_1$  is described by the equation

$$\chi(\mu) \longrightarrow \chi'(\mu) = \chi(\mu) e^{-(1/2)|\Im\mu|^2}, \quad (47)$$

which corresponds to degenerate additive Gaussian classical noise (only in the component  $q$ , with variance  $1/2$ ). In other words

$$\Phi(\rho) = \int e^{ixp} \rho e^{-ixp} P(dx), \quad (48)$$

where  $P(dx) = dx \exp[-x^2/4]/\sqrt{4\pi}$  is a Gaussian noise distribution. Then similarly to the previous case,  $S(\rho') \geq S(\rho) = S_0$ . Moreover

$$\Phi(\rho_{\sigma_q, \sigma_p}) = \rho_{\sqrt{\sigma_q^2+1/2}, \sigma_p} \quad (49)$$

so fixing  $S(\rho_{\sigma_q, \sigma_p}) = S_0$  and letting  $\sigma_p \rightarrow 0$ , we obtain  $S(\rho') \rightarrow S_0$ . Thus

$$\inf_{\rho \in \mathfrak{S}_{S_0}(\mathcal{H}_A)} S(\Phi(\rho)) = S_0. \quad (50)$$

## V. CONCLUSION

In this work we discussed a generalized minimal output conjecture for Gaussian channels. For degenerate one-mode quantum channels it has been proved explicitly. For attenuator channels the conjecture was proved for some values of the transmissivity, under the assumption that the input entropy and the entropy of the thermal state environment coincide.

VG acknowledges support from the FIRB-IDEAS project under the contract RBID08B3FM. AH acknowledges the support of Scuola Normale Superiore di Pisa, which he was visiting while part of this work was done, and the RAS Program ‘‘Mathematical Control Theory’’. SL acknowledges the support NEC, DARPA, ONR, and the Keck Foundation.

- 
- [1] C. H. Bennett and P. W. Shor, IEEE Trans. Inf. Theory **44**, 2724 (1998).
- [2] D. Petz, *Quantum Information Theory and Quantum Statistics*, (Springer, Berlin Heidelberg 2008).
- [3] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley-Interscience, 1991).
- [4] M. B. Hastings, Nature Physics **5**, 255 (2009).
- [5] P. W. Shor, Commun. Math. Phys. **246**, 453 (2004).
- [6] A. S. Holevo, Probl. Inf. Transm. **9**, 177 (1973).
- [7] A. S. Holevo, IEEE Trans. Inf. Theory **44**, 269 (1998); P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters, Phys. Rev. A **54**, 1869 (1996); B. Schumacher and M. D. Westmoreland, *ibid.* **56**, 131 (1997).
- [8] A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001).
- [9] F. Caruso, J. Eisert, V. Giovannetti, and A. S. Holevo, New J. Phys. **10**, 083030 (2008).
- [10] J. Eisert and M. M. Wolf, Quantum Information with Continuous Variables of Atoms and Light, 23 (Imperial College Press, London, 2007).
- [11] A. S. Holevo, *Probabilistic Aspects of Quantum Theory* (Amsterdam: North-Holland, 1982).
- [12] M. M. Wolf, G. Giedke, and J. K. Cirac, Phys. Rev. Lett. **96**, 080502 (2006).
- [13] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, Phys. Rev. Lett. **92**, 027902 (2004).
- [14] V. Giovannetti and S. Lloyd, Phys. Rev. A **69** 062307 (2004).
- [15] A. Serafini, J. Eisert, and M. M. Wolf, Phys. Rev. A **71** 012320 (2005).
- [16] M. M. Wolf, D. Perez-Garcia, and G. Giedke, Phys. Rev. Lett. **98**, 130501 (2007).
- [17] A. S. Holevo, Problems of Information Transmission, **43**, 1 (2007); F. Caruso, V. Giovannetti, and A. S. Holevo, New J. Phys. **8** 310 (2006).
- [18] F. Caruso and V. Giovannetti, Phys. Rev. A **74**, 062307 (2006).
- [19] T. Heinosaari, A. S. Holevo, and M. M. Wolf, arXiv:0909.0408 [quant-ph].
- [20] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, and J. H. Shapiro, Phys. Rev. A **70**, 032315 (2004).
- [21] V. Giovannetti, S. Lloyd, L. Maccone, J. H. Shapiro, and B. J. Yen, Phys. Rev. A **70**, 022328 (2004).
- [22] This approach was presented on two papers that appeared on the quant-ph arXive and were subsequently withdrawn by the authors (for reference see S. Lloyd, V. Giovannetti, L. Maccone, N.J. Cerf, S. Guha, R. Garcia-Patron, S. Mitter, S. Pirandola, M.B. Ruskai, J.H. Shapiro, and H. Yuan, arXiv:0906.2758v2 [quant-ph]; S. Lloyd, V. Giovannetti, L. Maccone, S. Pirandola, and R. Garcia-Patron arXiv:0906.2762v2 [quant-ph]). An updated version of the first one has been recently posted on the archive: it clarifies why the proposed approach fails to provide a proof of the conjecture.
- [23] A. S. Holevo, arXiv:1004.0196 [quant-ph]; arXiv:1003.5765 [quant-ph].
- [24] A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978).
- [25] A. S. Holevo, Problems of Information Transmission, **44**, 3 (2008).
- [26] V. Giovannetti and R. Fazio, Phys. Rev. A **71**, 032314 (2005).
- [27] To verify that the operator (44) has the characteris-

tic function  $\chi'(\mu)$  in the right side of (43) use the consequence of the Canonical Commutation Relation  $\exp(-ixp)D(\mu)\exp(ixp) = \exp(-ix\sqrt{2}\Im\mu)D(\mu)$  and the

fact that  $\int \exp(-ix\sqrt{2}\Im\mu)|x\rangle\langle x|dx = D(-i\Im\mu)$ .