A Special Vortex-Vortex Scattering Process in Superconductors

Pauline Mc Carthy B.Sc.

Abstract

In this thesis we discuss the evidence for scattering at right angles of two vortices in a head-on collision. The evidence is given in terms of the approximate solutions of the equations of motion or the Euler-Lagrange equations

$$D_i D^i \phi + \frac{1}{2}\lambda\phi(|\phi|^2 - 1) = 0,$$
$$\partial_i F^{ij} + \frac{i}{2}(\phi^* D^j \phi - \phi(D^j \phi)^*) = 0$$

where $D_i = (\partial_i - iA_i)$ and $F_{ij} = \partial_i A_j - \partial_j A_i$ and $(A_i(x), \phi(x))$ describe the gauge potentials and Higgs fields respectively.

The case $\lambda = 1$ describes the case where there are no net forces on the vortices but we also extend the analysis to the case of a small net repulsive force between the corresponding static vortex configurations where $\lambda > 1$. The ordinary differential equations, which result from the ansatz for the approximate solutions, are solved by Taylor series at the origin and asymptotic series at infinity.

A Special Vortex-Vortex Scattering Process in Superconductors

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Submitted in fulfillment of an M.Sc. degree by research

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Declaration
I declare that this dissertation is entirely my own work and that it has not been submitted to any other University as an exercise for a degree.

Signed Pauline Mc Carthy

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Dedicated

To my mother.

Abstract

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Contents

Chapter 1

Introduction

In 1911 Heike Kamerlingh Onnes was surprised to find that mercury cooled by liquid helium to four degrees Kelvin lost all electrical restistance; this phenomenon is superconductivity. Many materials when cooled below a critical temperature T_c, (which is different for each material) exhibit this phenomenon. The superconducting state is characterized by three macroscopic properties. First, electric currents flow without resistence. Second, magnetic fields vanish inside the superconducting medium; this is known as "flux expulsion" or the Meissner effect. Third, no net energy is released in the transition from the normal state to the superconducting state.

The potential applications of superconductivity are vast and various, extending from the production of high-intensity magnetic fields to lossless power-transmission lines. The developement of practical superconductors has, however, been retarded, mainly because of the prodigous engineering challenges involved. Once these problems are overcome the envisioned applications are numerous, generators and motors, energy storage, magnetically levitating trains and magnetic-reasonance imaging being but a few. In many of the applications the behaviour of the superconductor in a magnetic field (an external field or one generated by the supercurrent) is important.

An important development of recent years has been the investigation of the dynamical behaviour of magnetic flux structures and the discovery of the intimate connection between flux motion and the transport properties of superconductors. Motion of the magnetic flux structure (vortex) can be induced experimentally, hence we consider the theoretical work on the scattering of vortices to be of particular importance. Futhermore vortices can be considered as soliton-like objects because of their stability. This is another reason to investigate their dynamics.

In Chapter 2 we give an overview of the theory of Ginzburg and Landau and outline the theory involved in solving the problem. In Chapter 3 we review the evidence for scattering, at right angles, of slowly moving vortices between which the nett force is zero. The ansatz chosen leads to ordinary differential equations which we solve in Chapter 4 using Taylor series at the origin and asymptotic series at infinity. In Chapter 5 we investigate the case for which the nett force between the static vortices is not zero and $\lambda \geq 1$. In Appendix A we include the derivation of the energy density while Appendix B contains all the computer programs and Numerical procedures which were used in preceding chapters.

Chapter 2

Vortices in the Ginzburg Landau Theory

2.1 Introduction

A phenomenological theory for dealing with superconductors has been developed by Ginzburg and Landau. This theory is based on Landau's theory of second order phase transitions in which the important concept of the order parameter was introduced. In superconductor phase transitions the order parameter is a complex quantity. Its absolute value $\mid \phi(\underline{r}) \mid$ is connected with the local density of superconducting electrons (which have combined to form Cooper pairs). The phase of the order parameter is needed for describing supercurrents. The free energy density is expanded in powers of $\mid \phi(\underline{r}) \mid^2$ and $\mid \nabla\phi(\underline{r}) \mid^2$, assuming ϕ and $\nabla\phi$ are small. The minimum energy is found from a variational method leading to a pair of coupled differential equations for $\phi(\underline{r})$ and the vector potential $\underline{A}(\underline{r})$, of the magnetic field into which the superconductor has been placed. The emerging theory is a gauge theory with gauge group $U(1)$. The space of its finite-energy solutions is topologically nontrivial. The topological nontrivial finite-energy solutions are flux tubes called vortices.

2.2 Free Energy and the Ginzburg-Landau Equations

In the simplest case, we assume the order parameter $\phi(\underline{r})$ to be constant and the local magnetic flux density $\underline{h}$ to be zero throughout the superconductor. For small values of $\phi(\underline{r})$ ie. $T \rightarrow T_c$, the free energy f can be expanded in the form

$$f = f_n + \alpha(T) \mid \phi \mid^2 + \frac{\beta(T)}{2} \mid \phi \mid^4 + \cdots \tag{2.1}$$

Stability of the system at the transition point (at which $\phi = 0$) requires f to attain a minimum for $\phi = 0$. Therefore, in the expansion of f only even powers of ϕ can appear. For the minimum of f to occur at finite values of $\mid \phi \mid^2$, we must have $\beta > 0$, otherwise the lowest value of f would be reached at arbitrarily large values of $\mid \phi \mid^2$. For $\alpha > 0$ the minimum occurs at $\mid \phi \mid^2 = 0$ corresponding to the normal state and the case $T > T_c$. On the other hand, for $\alpha < 0$ the minimum occurs at

$$\mid \phi \mid^2 = \mid \phi_o \mid^2 = \frac{-\alpha}{\beta} \tag{2.2}$$

corresponding to $T < T_c$. We note that α must change its sign at $T = T_c$, using the expansion $\alpha(T) = a(T - T_c)$, where $a > 0$ is a constant, (2.2) then reduces to

$$| \phi_o |^2 = \frac{a}{\beta(T)}(T_c - T), \tag{2.3}$$

representing a rather general result characteristic of a second order transition. Substituting (2.3) into (2.1) we can approximate very close to T_c

$$f = f_n + \alpha(T) | \phi_o |^2 = f_n - \frac{a^2}{\beta(T)}(T_c - T)^2$$

yielding

$$\frac{\partial(f_n - f)}{\partial T} = \frac{-2a^2}{\beta(T)}(T_c - T), \tag{2.4}$$

to first order in $(T_c - T)$. We see that for $T \to T_c$ we have $\frac{\partial(f_n - f)}{\partial T} \to 0$, indicating a phase transition of at least second order.

We now relax our assumptions, allowing spatial variations of the order parameter, however first still keeping $\underline{h} = 0$. To the free energy expansion of (2.1) we now add terms of the form, $(\frac{\partial\phi}{\partial x})^2, (\frac{\partial\phi}{\partial x})(\frac{\partial\phi}{\partial y})$, etc, the first significant terms being second order, since in the absence of a magnetic field the equilibrium corresponds to $\phi = const$. For spherical symmetry we have the expansion

$$f = f_n + \alpha(T) | \phi |^2 + \frac{\beta(T)}{2} | \phi |^4 + \gamma[(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2] + \cdots \tag{2.5}$$

with $r > 0$ for $T = T_c$. Equation (2.5) is the basis of Landau's general theory of second order phase transitions. Finally, we also need to include the presence of magnetic fields $\underline{h} = curl\underline{A}$. Then the free energy density can be expanded in the form

$$f = f_n + \alpha(T) | \phi |^2 + \frac{\beta(T)}{2} | \phi |^4 + \frac{1}{2m^*} | (\frac{\hbar}{i}\underline{\nabla} - \frac{e^*}{c}\underline{A})\phi |^2 + \frac{\underline{h}^2}{8\pi}. \tag{2.6}$$

Note that for $\phi = 0$ we have $f = f_n + \frac{\underline{h}^2}{8\pi}$, the free energy density of the normal state. Here m and e are the mass and charge of an electron respectively with $m^* = 2m$ and $e^* = 2e$.

The fourth term in the expansion of (2.6) becomes clearer by writing ϕ in the form

$$\phi = | \phi | e^{i\theta}.$$

It then becomes

$$\frac{1}{2m^*}[\hbar^2(\underline{\nabla} | \phi |)^2 + (\hbar\underline{\nabla}\theta - \frac{e^*}{c}\underline{A})^2 | \phi |^2]. \tag{2.7}$$

The first contribution represents the additional energy arising from gradients in the magnitude of the order parameter. The second contribution contains the kinetic energy density of the supercurrents, as we can see by identifying $| \phi |^2$ with n_s^* (the number density of Cooper pairs). The kinetic energy density is then $(\frac{1}{2})m^* v_s^2 n_s^*$, where the supercurrent velocity v_s is given by

$$m^* v_s = \underline{\rho_s} - \frac{e^*}{c}\underline{A} = \hbar\underline{\nabla}\theta - \frac{e^*}{c}\underline{A} \tag{2.8}$$

and $\underline{\rho_s}$ is the generalised particle momentum.

Having obtained the expression (2.6) for the free energy density, we must now find its minimum with respect to spatial variations of the order parameter $\phi(\underline{r})$ and the magnetic field distribution $\underline{A}(\underline{r})$. Following the standard variation procedure, one finds the Ginzburg Landau differential equations

$$\alpha\phi + \beta \mid \phi \mid^2 \phi + \frac{1}{2m^*}(\frac{\hbar}{i}\underline{\nabla} - \frac{e^*}{c}\underline{A})^2\phi = 0 \tag{2.9}$$

and the current

$$\underline{J_s} = \frac{e^*\hbar}{2m^*i}(\phi^*\underline{\nabla}\phi - \phi\underline{\nabla}\phi^*) - \frac{e^{2*}}{m^*c}\phi^*\phi\underline{A} \tag{2.10}$$

for the equations of motion

$$\partial_i(\partial^i A^j - \partial^j A^i) = J_s^j, \qquad i,j = 1,2,3,. \tag{2.11}$$

The variational procedure requires the introduction of the boundary condition on the magnetic potential of

$$(\frac{\hbar}{i}\underline{\nabla} - \frac{e^*}{c}\underline{A})^2\phi = 0 \tag{2.12}$$

The above theory provides a macroscopic description of the system described microscopically by the theory of Bardeen, Cooper and Schrieffer (BCS). In this theory the onset of superconductivity is due to the formation of bound electron pairs (Cooper pairs). With respect to small applied forces the electron pairs interact as a single entity, a particle with twice the charge of a single electron, therefore in the Ginzburg Landau theory we must take $m^* = 2m$ and $e^* = 2e$ where m is the mass of an electron and e is the charge of an electron.

2.3 The Abelian Higgs Model-A Gauge Theory

We now include time-dependence into the formulas (2.6), (2.9), (2.10) and (2.11) and discuss the resulting model.

First, to simplify (2.6) we add a constant, redefine the fields and write

$$f = \frac{1}{2}(D_i\phi)(D^i\phi) + \frac{1}{4}F_{ij}F^{ij} + \frac{\lambda}{8}(\phi\phi * -1)^2, \tag{2.13}$$

where the new fields ϕ and A_i are given in terms of the old fields ϕ^{old} and $\underline{A}^{old}$ in (2.9)- (2.11), as

$$\begin{aligned} \phi &= \sqrt{\frac{e^*\hbar}{m^*}}\phi^{old} \\ A_i &= \frac{e^*}{\hbar c}\underline{A}^{old} \end{aligned}$$

and

$$f = f^{old} + c, \qquad c = constant,$$

with $\lambda = \beta/2 = -4\alpha$.The covariant derivative $D_i\phi$ and the field F_{ij} are defined as

$$D_i\phi = (\partial_i - iA_i)\phi, \qquad F_{ij} = \partial_i A_j - \partial_j A_i.$$

Time dependence is introduced by considering an electric potential A_o as well as the magnetic potential $\vec{A}$ and x_o as the time coordinate. In terms of $(A_o, \vec{A})$ and $(x_o, \vec{x})$ the Lorentz invariant Lagrangian in Minkowski space corresponding to (2.13) reads

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)(D^\mu\phi)^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{8}(\phi\phi* - 1)^2. \tag{2.14}$$

The covariant derivative $D_\mu\phi$ and the fields $F_{\mu\nu}$ are

$$D_\mu\phi = (\partial_\mu - iA_\mu)\phi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mu,\nu = 0,1,2,3. \tag{2.15}$$

Indices are lowered and raised with the metric tensor $g = diag(+1,-1,-1,-1)$.

The variational techniques that were used in the previous section to derive (2.9) and (2.11) can also be used to derive the Euler Lagrange equations from (2.14). Using

$$\frac{\partial}{\partial x_\mu}(\frac{\partial\mathcal{L}}{\partial A_{\nu,\mu}}) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \tag{2.16}$$

and

$$\frac{\partial}{\partial x_\mu}(\frac{\partial\mathcal{L}}{\partial \phi_{,\mu}}) - \frac{\partial\mathcal{L}}{\partial \phi} = 0 \tag{2.17}$$

where $A_{\nu,\mu} = \frac{\partial A_\nu}{\partial x_\mu}$ and $\phi_{,\mu} = \frac{\partial\phi}{\partial x_\mu}$, we find the coupled differential equations

$$\frac{\partial}{\partial x_\mu}(F_{\mu\nu}) = \frac{-i}{2}(\phi D_\nu\phi^* - \phi^* D_\nu\phi), \tag{2.18}$$

$$D^\mu(D_\mu\phi) = \frac{-\lambda}{2}\phi(\phi\phi^* - 1). \tag{2.19}$$

For $A_o = 0$ and time independent fields, these equations reduce to the equations (2.9) and (2.11) of the Ginzburg-Landau theory. The theory given by the Lagrangian (2.14) is called the Abelian Higgs model.

We will now show that the Abelian Higgs model is a classical gauge field theory. A gauge theory is characterized by a group of symmetries but the symmetry group is not associated with any physical coordinate transformation in space-time. Gauge theory is based on an "internal" symmetry transformation under which the fields change. The properties of a gauge theory is gauge invariance ie. under a gauge transformation the equations of motion transform covariantly. If the original fields were solutions of the equations of motion so are the gauge transformed fields. The coordinate used to describe the internal symmetry is the phase of the wave function. The change of phase will not affect any observable quantity provided that the gauge transformation for the fields combine to leave the Lagrangian invariant and therefore also the equations of motion. Hence, a gauge transformation transforms the Higgs field ϕ in the Lagrangian (2.14) to

$$\phi'(x) = U(x)\phi(x) \tag{2.20}$$

where $U(x) = e^{-ig(x)}$. That means here U is an element of $U(1)$, the multiplicative group of complex numbers of unit modulus. Clearly, $\phi'\phi'^* = \phi\phi^*$ and the Higgs potential in (2.14) is invariant under gauge transformations of this kind. If we can achieve that $D'_\mu\phi' = UD_\mu\phi$, ie.,

$$(\partial_\mu - iA'_\mu)U\phi = U(\partial_\mu - iA_\mu)\phi \tag{2.21}$$

then obviously also $(D_\mu\phi)(D^\mu\phi)^* = (D'_\mu\phi')(D^{\mu\prime}\phi')^*$ is invariant. Condition (2.21) holds if

$$A'_\mu = UA_\mu U^{-1} - i(\partial_\mu U)U^{-1}. \tag{2.22}$$

for $U = e^{-ig(x)}$, the gauge transformation (2.22) reduces to

$$A'_\mu = A_\mu - \partial_\mu g, \tag{2.23}$$

which leads to

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \tag{2.24}$$

therefore the Lagrangian (2.14) is invariant under the gauge transformations (2.20) and (2.22), where $U = e^{-ig(x)} \in U(1)$. Hence the Lagrangian (2.14) gives a $U(1)$ gauge theory. For other gauge theories the definition of $F_{\mu\nu}$ is suitably modified so that the $F_{\mu\nu}F^{\mu\nu}$ term is invariant under the gauge transformation (2.22).

2.4 Other Features of the Abelian Higgs Model

The property of any gauge theory is the gauge invariance of the Lagrangian. The ground state, however, in many cases, like that of the Superconductor, is not gauge invariant. The mechanism by which the symmetry is broken in superconductors is called "Spontaneous Symmetry Breaking" because it does not require any explicit mass term in the Lagrangian to manifest itself. A mass term of the form $m^2A_\mu A^\mu$ in the Lagrangian would break its gauge invariance. We will now show that the ground state, the time-independent state of lowest energy into which the system eventually settles, is not gauge invariant.

First, it is always possible to gauge away A_0 by choosing $g(x_0, \vec{x})$ such that $A'_0 = A_0 - \partial_0 g = 0$. Then, for time- independent fields, the energy density reads

$$\mathcal{E} = \frac{1}{2}(D_i\phi)(D^i\phi)^* + \frac{1}{4}F_{ij}F^{ij} + \frac{\lambda}{8}(\phi\phi* - 1)^2. \tag{2.25}$$

The energy density is positive definite and zero for $A_i = 0, \partial_i\phi_0 = 0, (i = 1,2,3)$ and $| \phi_0 |= 1$. In the ground state, $\phi_0 = e^{i\varphi}$ holds and clearly ϕ_0 is not invariant under the gauge transformation (2.20). In fact, $\phi'_0 = e^{i(\varphi - g(x))} \neq \phi_0$ for $g(x) \neq 2\pi n$. We conclude that the theory given by the Lagrangian (2.14) has a gauge symmetry which is not displayed by the ground state. This phenomenon is called spontaneous symmetry breaking or hidden symmetry.

Since the ground state is given by $\phi_0 = e^{i\varphi}$, the physical fields, relative to the ground state, are A_i and $\eta = \phi - \phi_0$. In terms of η, the Lagrangian reads

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(D_\mu\eta)(D^\mu\eta)^* - iA_\mu\phi_0(\partial^\mu + iA^\mu)\eta^* \\ &+ iA^\mu\phi_0^*(\partial_\mu - iA_\mu)\eta + | \phi_0 |^2 A_\mu A^\mu \\ &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{8} | \eta |^2 + \eta\phi_0^* + \eta^*\phi_0)^2.\end{aligned} \tag{2.26}$$

The Lagrangian has acquired a mass term $| \phi_0 |^2 A_\mu A^\mu =: m_p^2 A_\mu A^\mu$ for the magnetic field, the photon field, which leads to a $m_p^2 A_\mu$ term in the equations of motion. The effect of this term is that the electromagnetic field becomes short ranged. This can be understood as follows. The solution to the equation

$$-\Delta\psi = \delta(\vec{x}), \qquad \vec{x}\epsilon\Re^3 \tag{2.27}$$

for a point source at the origin is $\psi = 1/(4\pi|x|)$, ie., the field falls off like $1/r$ and has long range. On the other hand, the equation

$$-\Delta\psi + m^2\psi = \delta(\vec{x}), \qquad \vec{x}\epsilon\Re^3 \tag{2.28}$$

for a point source at the origin with mass term, has the solution $\psi = e^{-mr}/(4\pi|x|)$, ie., the field falls off exponentially and has short range. Physically, for the superconductor, this means that the magnetic field cannot penetrate far into the superconductor, which is called flux expulsion or the Meissner effect.

The Higgs potential has a further consequence. The following discussion is based on Coleman [9]. For a rigorous detailed analysis see Jaffe and Taubes [11]. First, we restrict our attention to finite energy configurations since these are the only configurations which can be realized in an experiment. "Reasonable" finite energy configurations must go to a unimodular number at infinity. Otherwise $\lambda/8(|\ \phi\ |^2 -1)^2$ does not go to zero at infinity and the energy, the integral of the energy density, diverges. Second, we consider a superconductor in a long cylindrically symmetric magnetic field in the z-direction. Then to a good approximation, none of the physical quatities depend on z and we can write in two space dimensions. In two space dimensions, the above condition on the energy leads to a a map from the circle at infinity S^1 in $\Re^2$ to the circle of unimodular numbers S^1 in C:

$$\phi(r,\theta) \longrightarrow \phi_\infty(\theta) = e^{i\varphi(\theta)}, \qquad r \to \infty. \tag{2.29}$$

Clearly the continous maps ϕ_∞ fall into different classes depending on the number of times ϕ_∞ winds around S^1 while going around S^1_∞. The winding number is given by

$$N = \int_0^{2\pi} \frac{d\theta}{2\pi}\frac{d\varphi}{d\theta}. \tag{2.30}$$

N can also be written in terms of the field strength F_{12} as

$$N = \frac{1}{2\pi}\int d^2x\, F_{12}. \tag{2.31}$$

This can be explained as follows: if the energy is to be finite, then as $r \to \infty, |\phi| \to 1$ and $(\partial_i - iA_i)\phi \to 0$. Thus asymptotically

$$\begin{aligned} \phi &\approx e^{i\varphi(\theta)}, \\ A_i &\approx \partial_i\varphi(\theta). \end{aligned} \tag{2.32}$$

Since ϕ must be single-valued and continuous, φ must satisfy

$$\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi N \tag{2.33}$$

for some integer N. Continuous variations of the fields, subject only to the constraint of finite energy, cannot change N; it is a topological invariant. From (2.32) and (2.33) it follows that

$$\begin{aligned} N &= -\frac{i}{2\pi}\oint_c d\ln\phi \\ &= \frac{1}{2\pi}\oint_c d\vec{\ell}.\vec{A} \\ &= \frac{1}{2\pi}\int d^2x F_{12}, \end{aligned} \tag{2.34}$$

using Green's theorem, where the line integrals are to be taken around a contour at infinity. Equation (2.31) shows that for $N \neq 0, F_{12}$ goes like $1/r^2$ at infinity, and of course, is independent of z, which means it describes a flux tube, a vortex.

Futhermore, for $\lambda = 1$, if we use integration by parts to rewrite (2.13) we find

$$
\begin{aligned}
E = \int \mathcal{E} d^2x = \frac{1}{2}\int d^2x \ \ [& \ (\partial_1\phi_1 + A_1\phi_2) \mp (\partial_2\phi_2 - A_2\phi_1)]^2 \\
+ & \ [(\partial_2\phi_1 + A_2\phi_2) \pm (\partial_1\phi_2 - A_1\phi_1)]^2 \\
+ & \ [F_{12} \pm \frac{1}{2}(\phi_1^2 + \phi_2^2 - 1)]^2 \\
\pm & \int d^2x F_{12},
\end{aligned}
\tag{2.35}
$$

where ϕ_1 and ϕ_2 are the real and imaginary parts of the scalar field ϕ. The integrand in the first integral is positive semi-definite while the second integral is simply a multiple of the winding number N. Taking the upper or lower sign according to whether N is positive or negative yields

$$
E \geq \mid N \mid \pi \tag{2.36}
$$

with equality if

$$
\begin{aligned}
(D_1 \pm iD_2)\phi &= 0 \\
F_{12} &= \mp\frac{1}{2}(\phi^*\phi - 1).
\end{aligned}
\tag{2.37}
$$

These equations are known as the Bogomolny equations for vortices and have solutions for all N. They form a pair of coupled first order differential equations and their solutions solve (2.18) and (2.19), the equations of motion, for $\lambda = 1$.

Chapter 3

A 90^0 Scattering Process

3.1 Introduction

In this chapter we consider, for $\lambda = 1$, a special scattering process of vortices inside a superconductor. To do this we look for approximations to the gauge potentials and the Higgs field $(A_\mu(t,x), \phi(t,x))$ which have finite energy given by (2.13) and satisfy the equations of motion (2.18), (2.19).

The approximations considered here are of the form

$$\phi = \overset{\circ}{\phi} + \tilde{\phi}, \tag{3.1}$$

and

$$\begin{aligned} A_i &= \overset{\circ}{A}_i + \tilde{A}_i, \\ A_0 &= \overset{\circ}{A}_0 = 0, \end{aligned} \tag{3.2}$$

where $(\overset{\circ}{A}_i, \overset{\circ}{\phi})$ is the static solution for two vortices sitting on top of each other, and the perturbations on the static case $(\tilde{A}_i, \tilde{\phi})$ are represented by $(tB_i, t\xi_i)$. These are small so that the equations for (B_i, ξ_i) can be linearized. In the following, the static solution, the assumption that $(tB_i, t\xi_i)$ are small and the solution of the equations for (B_i, ξ_i) will be discussed. Our discussion is based on work by Ruback [15] and Weinberg [16]. We discuss the scattering process from shortly before to shortly after the collision in terms of the differential equations only. This will make it possible to discuss scattering away from the Bogomolny limit in Chapter 5.

3.2 The Static Solution

Consider the gauge potential $A_0 = 0, A_i(r,\theta)$ and the Higgs field $\phi(r,\theta)$. It has been shown by Plohr [14], that to find n vortices superimposed at the origin the solution can be written in the form

$$\begin{aligned} \phi &= e^{in\theta} f(r), \\ A_i(r,\theta) &= \frac{-\epsilon_{ij} x_j n a(r)}{r^2}. \end{aligned} \tag{3.3}$$

We know that (A_i, ϕ) satisfy the equations (2.18), (2.19) if they are solutions to the Bogomolny equations (2.37). Substition of (3.3) into (2.37) yields

$$
\begin{aligned}
|\phi|^2 &= f^2, \\
D_i\phi &= [\frac{f'x_i}{r} - i(nf - naf)\frac{\epsilon_{ij}x_j}{r^2}]e^{in\theta}, \\
F_{12} &= \frac{\epsilon_{12}n}{r}a'.
\end{aligned} \tag{3.4}
$$

and therefore

$$
\begin{aligned}
f' &= \pm\frac{nf}{r}(1-a), \\
na' &= \mp\frac{r}{2}(f^2-1).
\end{aligned} \tag{3.5}
$$

Here we take the upper sign if $n \geq 0$ and the lower if $n < 0$. To show the eqs. (3.5) have finite energy solutions, we argue as follows: Consider the time-independent Euler-Lagrange equations

$$
\begin{aligned}
D_iD^i\phi + \frac{1}{2}\lambda\phi(|\phi|^2 - 1) &= 0, \\
\partial_i F^{ij} + (i/2)(\overline{\phi}D^j\phi - \phi\overline{D^j\phi}) &= 0.
\end{aligned} \tag{3.6}
$$

where we sum over the spatial indices only. The ansatz (3.3) yields

$$
\begin{aligned}
D_iD^i\phi &= (rf')' - \frac{n^2 f}{r}(a-1)^2, \\
\partial_i F^{ij} &= \hat{x}_i n\epsilon_{ij}(\frac{a'}{r})', && (3.7) \\
\overline{\phi}D^j\phi - \phi\overline{D_j\phi} &= x_i n\epsilon_{ij}\frac{f^2}{r}(a-1). && (3.8)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
(rf')' - \frac{n^2 f}{r}(a-1)^2 - \frac{r}{2}\lambda f(f^2 - 1 &= 0, \\
(\frac{a'}{r})' - \frac{f^2}{r}(a-1) &= 0.
\end{aligned} \tag{3.9}
$$

Plohr [14] has shown that there are functions (f, a) which minimize the energy

$$
E = \int [(1/2)(D_i\phi)(D_i\phi)^* + (1/4)(F_{ij})^2 + (\lambda/8)(\phi\phi^* - 1)^2]d^2x \tag{3.10}
$$

and thus solve the corresponding Euler-Lagrange equations (3.9). On the other hand, it can be seen that solutions of (3.5) satisfy (3.9). And Jaffe and Taubes [11] have shown that all finite energy solutions of (3.9) are solutions of (3.5). This establishes the existence of a finite energy solution of (3.5). For $n = 2$, this is our configuration $(\mathring{A}_\mu, \mathring{\phi})$.

3.3 The Approximations

If we substitute the fields (3.1) and (3.2) into the equations (2.18), (2.19),use the fact that $(\mathring{A}_\mu, \mathring{\phi})$ solve the time-independent equations and keep only linear terms in $(\tilde{A}_i, \tilde{\phi})$ we find that they become

$$\mathring{D}_\mu \overset{\circ\mu}{D} \tilde{\phi} - 2i\tilde{A}_\mu \overset{\circ\mu}{D} \mathring{\phi} - i\,\mathring{\phi}\, \partial_\mu \tilde{A}^\mu + \frac{1}{2}\tilde{\phi}(|\mathring{\phi}|^2 - 1) + \frac{1}{2}\,\mathring{\phi}\,(\mathring{\phi}\,\bar{\tilde{\phi}} + \overline{\mathring{\phi}}\tilde{\phi}) = 0,$$

and

$$\partial_\mu \tilde{F^{\mu\nu}} + \tilde{A^\nu}\,|\mathring{\phi}|^2 + \frac{i}{2}[\bar{\tilde{\phi}}\,\overset{\circ\nu}{D}\,\mathring{\phi} - \tilde{\phi}\overline{\overset{\circ\nu}{D}\,\mathring{\phi}} + \overline{\mathring{\phi}}\,\overset{\circ\nu}{D}\,\tilde{\phi} - \mathring{\phi}\,\overline{\overset{\circ\nu}{D}\,\tilde{\phi}}] = 0, \tag{3.11}$$

where

$$\begin{aligned} \overset{\circ\nu}{D}\,\mathring{\phi} &= \mathring{\phi}\,(\partial^\nu - i\,\overset{\circ\nu}{A}), \\ \tilde{F^{\mu\nu}} &= \partial^\nu \tilde{A^\mu} - \partial^\mu \tilde{A^\nu}. \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \tilde{\phi}(t, r, \theta) &= t\xi(r, \theta), \\ \tilde{A}_i(t, r, \theta) &= tB_i(r, \theta) \end{aligned} \tag{3.13}$$

where $t \in (-\epsilon, \epsilon), \epsilon \ll 1$ and $\xi(r,\theta) = \xi_1 + i\xi_2$ This means that we are studying the scattering process only from the time shortly before the position of superposition of the two vortices until a time shortly after. Thus, we obtain

$$-\,\mathring{D}_i \mathring{D}_i\,\xi + i\,\mathring{\phi}\,\partial_i \tilde{B}_i + \frac{1}{2}\xi(|\mathring{\phi}|^2 - 1) + \frac{1}{2}\,\mathring{\phi}\,(\mathring{\phi}\,\bar{\xi} + \overline{\mathring{\phi}}\xi) = 0. \tag{3.14}$$

The sign changes in the above are due to the form of the metric as outlined in Chapter 2. The second set of equations (3.11) become for $\nu = j$

$$\partial_i F^{ij} - B^j\,|\mathring{\phi}|^2 - \frac{i}{2}[\bar{\xi}\,\overset{\circ j}{D}\,\mathring{\phi} - \xi\overline{\overset{\circ j}{D}\,\phi 0} + \overline{\mathring{\phi}}\,\overset{\circ j}{D}\,\xi - \mathring{\phi}\,\overline{\overset{\circ j}{D}\,\xi}] = 0 \tag{3.15}$$

because

$$\partial_0 \tilde{F^{0j}} = 0 \tag{3.16}$$

by definition above and for $\nu = 0$ (3.11) becomes

$$(\partial_i B_i + \frac{i}{2}[\overline{\mathring{\phi}}\xi - \mathring{\phi}\,\bar{\xi}]) = 0. \tag{3.17}$$

Equations (3.14), (3.15) are also obtained by substituting (3.1), (3.2) and (3.13) into the time-independent equations (3.6). Solutions of the Bogomolny eqs. (2.37) solve eqs. (3.6) ,so if we put (3.1), (3.2) and (3.13) into eqs. (2.37) keep only the terms linear in (B_i, ξ_i) and solve the equations, we have solved (3.14) and (3.15).

To solve (3.17) , we write (B_i, ξ_i) in the form

$$\begin{aligned} \xi_1 &= n\cos n\theta f(r)h_1(r,\theta) - n\sin n\theta f(r)h_2(r,\theta), \\ \xi_2 &= n\sin n\theta f(r)h_1(r,\theta) + n\cos n\theta f(r)h_2(r,\theta), \\ B_1 &= \frac{n}{r}[-\sin\theta b(r,\theta) + \cos\theta c(r,\theta)], \\ B_2 &= \frac{n}{r}[\cos\theta b(r,\theta) + \sin\theta c(r,\theta)]. \end{aligned} \tag{3.18}$$

Where the perturbed fields take the form

$$\begin{aligned} \phi &= \overset{\circ}{\phi} + t\xi, \\ A_i &= \overset{\circ}{A}_i + tB_i. \end{aligned} \tag{3.19}$$

Substituting (3.18) into (3.17) we find that

$$\begin{aligned} \partial_i B_i &= \frac{2}{r}\frac{\partial c}{\partial r} + \frac{2}{r^2}\frac{\partial b}{\partial \theta}, \\ \frac{i}{2}[\overline{\overset{\circ}{\phi}}\xi - \overset{\circ}{\phi}\,\overline{\xi}] &= -2f^2 h_2. \end{aligned}$$

Therefore (3.17) becomes

$$\frac{1}{r}\frac{\partial c}{\partial r} + \frac{1}{r^2}\frac{\partial b}{\partial \theta} - f^2 h_2 = 0. \tag{3.20}$$

If we substitute the perturbed fields (3.19) into the Bogomolny eqs. (2.37) and use the fact that $(\overset{\circ}{A}, \overset{\circ}{\phi})$ are solutions to the unperturbed case we find that eqs. (2.37) become

$$\overset{\circ}{D}_1 \xi \pm \overset{\circ}{D}_2 \xi - i(B_1 \pm iB_2)\overset{\circ}{\phi} = 0, \tag{3.21}$$

$$\tilde{F}_{12} + \frac{1}{2}(\xi\overline{\overset{\circ}{\phi}} + \overline{\xi}\,\overset{\circ}{\phi}) = 0 \tag{3.22}$$

where

$$\begin{aligned} \xi &= 2f(\cos 2\theta + i\sin 2\theta)(h_1 + ih_2), \\ B_i &= \frac{2}{r}(-b\epsilon_{ij}\hat{x}_j + c\hat{x}_i), \\ \tilde{F}_{12} &= \partial_1 B_2 - \partial_2 B_1 \end{aligned}$$

and

$$\partial_i = \hat{x}_i\partial_r - \epsilon_{ij}\frac{\hat{x}_j}{r}\partial_\theta.$$

Substition of (3.18) into (3.21) gives

$$\begin{aligned} \overset{\circ}{D}_1 \xi &= e^{2i\theta}[2\hat{x}_1(f'h + f\frac{\partial h}{\partial r}) + 2i\hat{x}_2(\frac{-2fh(1-a)}{r} + i\frac{f}{r}\frac{\partial h}{\partial \theta})], \\ i\,\overset{\circ}{D}_2 \xi &= e^{2i\theta}[2i\hat{x}_2(f'h + f\frac{\partial h}{\partial r}) + 2\hat{x}_1(\frac{-2fh(1-a)}{r} + i\frac{f}{r}\frac{\partial h}{\partial \theta})], \\ -iB_1\,\overset{\circ}{\phi} &= \frac{-2i}{r}(-\hat{x}_2 b + \hat{x}_1 c)e^{2i\theta} f, \\ B_2\,\overset{\circ}{\phi} &= \frac{2}{r}(\hat{x}_1 b + \hat{x}_2 c)e^{2i\theta} f. \end{aligned} \tag{3.23}$$

Equation (3.21) must be separated into real and imaginary parts and one must also remember that $h = h_1 + ih_2$ then we get

$$\frac{\partial h_1}{\partial r} - \frac{1}{r}\frac{\partial h_2}{\partial \theta} + \frac{b}{r} = 0, \tag{3.24}$$

$$\frac{\partial h_1}{\partial \theta} + r\frac{\partial h_2}{\partial r} - c = 0. \tag{3.25}$$

The second of the perturbed Bogomolny equations (3.22) must now be calculated. It can be seen that

$$\begin{aligned} \tilde{F}_{12} &= \frac{2}{r}\frac{\partial b}{\partial r} - \frac{2}{r^2}\frac{\partial c}{\partial \theta}, \\ \frac{1}{2}(\xi \overline{\mathring{\phi}} + \bar{\xi}\, \mathring{\phi}) &= 0. \end{aligned}$$

Therefore (3.22) becomes

$$\frac{1}{r}\frac{\partial b}{\partial r} - \frac{1}{r^2}\frac{\partial c}{\partial \theta} + f^2 h_1 = 0, \tag{3.26}$$

The four equations (3.20), (3.24), (3.25), (3.26) are the equations for the four unknown functions (b, c, h_1, h_2). Solutions to these equations will describe the type of motion and scattering to be found in superconductors.

3.4 Translational Motion

Consider equations (3.24), (3.25), (3.26). If we substitute (3.24) and (3.25) into (3.26) we find

$$-\frac{1}{r}(\frac{d}{dr}r\frac{\partial h_1}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 h_1}{\partial \theta^2} + f^2 h_1 = 0 \tag{3.27}$$

and if we substitute those same two equations into (3.20) we find

$$-\frac{1}{r}(\frac{d}{dr}r\frac{\partial h_2}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 h_2}{\partial \theta^2} + f^2 h_2 = 0 \tag{3.28}$$

which we can write as

$$-\frac{1}{r}(\frac{d}{dr}r\frac{\partial h}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} + f^2 h = 0 \tag{3.29}$$

for $h = h_1 + ih_2$. If we now Fourier expand h,

$$h(r,\theta) = \sum_{k=0}^{\infty} h_k^1 \cos k\theta + h_k^2 \sin k\theta. \tag{3.30}$$

we obtain

$$-\frac{1}{r}(\frac{d}{dr}r\frac{dh_k^{(i)}}{dr}) + (\frac{k^2}{r^2} + f^2)h_k^{(i)} = 0 \tag{3.31}$$

for $i = 1, 2; k = 0, 1, 2 \ldots.$

Solutions of this equation will behave like $C_1 r^{-k} + C_2 r^k$ at the origin and like $C_3 e^{-r} + C_4 e^r$ as $r \to \infty$. The perturbation of ϕ must be non-singular. It is clear from (3.5) that $f(r)$ has an n^{th} order zero at the origin. Therefore $h(r)$ may be as singular as r^{-n}. It can thus be seen from the solutions of (3.31) and (3.5), in order that $\tilde{\phi}$ be non-singular, $k \leq n$. Thus for $k \leq n$ we can always obtain an acceptable solution to (3.31) by choosing the proper behaviour as $r \to \infty$.

In our case we only consider the $n = 2$ vortex solution, in which case we can find solutions if the Fourier expansion for $h(r,\theta)$ contains $\cos\theta, \sin\theta, \cos 2\theta$ and $\sin 2\theta$ terms. If in (3.30) we take only the case $k = 1$ and set all other $h_k^i = 0$ we are then left with a single term fourier expansion namely

$$h(r,\theta) = h_1^1 \cos\theta + h_1^2 \sin\theta. \tag{3.32}$$

Now if we also set $h_1^1 = ih_1^2 = \frac{f'}{f}$, which is a solution of (3.31), and then multiply across by $\alpha + i\beta$ we find that the function $h(r,\theta)$ becomes

$$h(r,\theta) = h_1 + ih_2 = \frac{f'}{f}(\alpha + i\beta)e^{-i\theta}. \tag{3.33}$$

Our aim now is to show that perturbations of the form (3.19), where $h(r,\theta)$ is of the form (3.33) and b, c can be calculated from equations (3.20), (3.24), (3.25) and (3.26), describe translational motion of the vortices (ie. the vortices move together in the same direction.) To do this we first show that the guage invariant quantity $|\ \phi\ |^2$ is the same after translation as it is with addition of the perturbation (3.33).

Consider a translation of the form

$$\begin{aligned} f &: \Re \to \Re \ : \ x \to x + \gamma_1, \\ g &: \Re \to \Re \ : \ y \to y + \gamma_2 \end{aligned} \tag{3.34}$$

and a gauge transformation (2.20) which does not change the physics. If we apply a translation and a gauge transformation to the given Higgs field we get

$$\overset{\circ}{\phi}(x,y) \to e^{i\chi(x,y)}\ \overset{\circ}{\phi}(x+\gamma_1, y+\gamma_2). \tag{3.35}$$

If we now write

$$\overset{\circ}{\phi}(r,\theta) = e^{i2\theta}f(r) \tag{3.36}$$

then

$$\overset{\circ}{\phi}(x+\gamma_1, y+\gamma_2) = e^{i2\arctan\left(\frac{y+\gamma_2}{x+\gamma_1}\right)} f\left(\sqrt{(x+\gamma_1)^2 + (y+\gamma_2)^2}\ \right). \tag{3.37}$$

Since we consider small deviations only we can also expand $\overset{\circ}{\phi}(x+\gamma_1, y+\gamma_2)$ in a Taylor expansion. To first order, this corresponds to

$$\overset{\circ}{\phi}(x+\gamma_1, y+\gamma_2) = \overset{\circ}{\phi}(x,y) + \partial_x\ \overset{\circ}{\phi}(x,y).\gamma_1 + \partial_y\ \overset{\circ}{\phi}(x,y).\gamma_2 + \ldots \tag{3.38}$$

where ∂_x and ∂_y are the partial derivatives with respect to x and y respectively. As

$$\begin{aligned} \overset{\circ}{\phi}(x,y) &= e^{i2\arctan\left(\frac{y}{x}\right)} f(\sqrt{x^2+y^2}); \\ \partial_x\ \overset{\circ}{\phi}(x,y) &= e^{i2\theta}f(r)\frac{-2iy}{x^2+y^2} + e^{i2\theta}f'(r)\frac{x}{r}; \end{aligned}$$

and

$$\partial_y\ \overset{\circ}{\phi}(x,y) = e^{i2\theta}f(r)\frac{2ix}{x^2+y^2} + e^{i2\theta}f'(r)\frac{y}{r},$$

by substitution into (3.38) we find that

$$\begin{aligned} \overset{\circ}{\phi}(x+\gamma_1, y+\gamma_2) &= e^{i2\theta}f(r)[1 + \frac{f'}{f}(\cos\theta\gamma_1 + \sin\theta\gamma_2) \\ &\quad + \frac{2i}{r}(-\sin\theta\gamma_1 + \cos\theta\gamma_2) + \ldots]. \end{aligned} \tag{3.39}$$

If we now consider the form of the perturbed field ϕ as in (3.19) we see that it takes the form

$$\phi(x,y) = e^{i2\theta}f + 2e^{i2\theta}f'(r)[t\alpha\cos\theta + t\beta\sin\theta + i(t\beta\cos\theta - t\alpha\sin\theta)]. \tag{3.40}$$

After inspection of the gauge invariant quantities

$$|\overset{\circ}{\phi}(x+\gamma_1, y+\gamma_2)|^2 = f^2[1+\frac{2f'}{f}(\gamma_1\cos\theta+\gamma_2\sin\theta] \tag{3.41}$$

and

$$|\phi(x,y)|^2 = f^2[1+\frac{4f't}{f}(\alpha\cos\theta+\beta\sin\theta)] \tag{3.42}$$

we see that if $2\alpha t = \gamma_1, 2\beta t = \gamma_2$ then the above equations are equal. From this result, we determine the gauge transformation, ie. we find a function $\chi(x,y)$ such that

$$\phi' = e^{-i\chi(x,y)}\phi. \tag{3.43}$$

If

$$\chi(x,y) = \epsilon_{ij}\hat{x}_i\gamma_j(\frac{f'}{f}-\frac{2}{r}), \tag{3.44}$$

$$\hat{x}_i = \frac{x_i}{r} \tag{3.45}$$

and we Taylor expand $e^{-i\epsilon_{ij}\hat{x}_i\gamma_j(\frac{f'}{f}-\frac{2}{r})}$, (3.43) becomes

$$\phi' = [1-i\epsilon_{ij}\hat{x}_i\gamma_j(\frac{f'}{f}-\frac{2}{r})]\phi, \tag{3.46}$$

$$\phi' = \phi - i\epsilon_{ij}\hat{x}_i\gamma_j(\frac{2f'}{f}-\frac{4}{r})\overset{\circ}{\phi}. \tag{3.47}$$

because of the γ_j there are no higher terms other than $\overset{\circ}{\phi}$ because $\gamma_j\alpha$ or $\gamma_j\beta$ represent quadratic terms. If (3.40) is substituted into the above we find that

$$\begin{aligned}\phi' &= e^{i2\theta}f(r)[1+\frac{f'}{f}(\cos\theta\gamma_1+\sin\theta\gamma_2)+\frac{2i}{r}(-\sin\theta\gamma_1+\cos\theta\gamma_2)],\\ \phi' &= e^{-i\epsilon_{ij}\hat{x}_i\gamma_j(\frac{2f'}{f}-\frac{4}{r})}\phi,\\ \phi' &= \overset{\circ}{\phi}(x+\gamma_1, y+\gamma_2).\end{aligned} \tag{3.48}$$

From all the above we can see that with the introduction of a gauge transformation and a transformation of the form (3.34) the perturbed field ϕ and the field $\overset{\circ}{\phi}(x+\gamma_1, y+\gamma_2)$ are the same up to gauge transformation. Now we also have to prove that for the same gauge the gauge potential $A_i(x,y)$ is the same as the translated gauge potential up to gauge transformation. To do this we need to prove that

$$A_i(x,y) = \overset{\circ}{A}_i(x+\gamma_1, y+\gamma_2) + \partial_i\chi(x,y) \tag{3.49}$$

where

$$\begin{aligned}A_i(x,y) &= \overset{\circ}{A}_i + tB_i,\\ &= -\epsilon_{ij}x_j\frac{na}{r^2} - \epsilon_{ij}\gamma_j\frac{na'}{r}.\end{aligned} \tag{3.50}$$

The gauge potential $A_i(x,y)$ after the spatial transformation takes the form

$$\overset{\circ}{A}_i(x+\gamma_1, y+\gamma_2) = -n\epsilon_{ij}(x_j+\gamma_j)\frac{a(\sqrt{(x+\gamma_1)^2+(y+\gamma_2)^2})}{((x+\gamma_1)^2+(y+\gamma_2)^2)}. \tag{3.51}$$

To simplify this we Taylor expand to get

$$\begin{aligned}\overset{\circ}{A}_i (x+\gamma_1, y+\gamma_2) &= \overset{\circ}{A}_i + \gamma_1 D_1 \overset{\circ}{A}_i + \gamma_2 D_2 \overset{\circ}{A}_i, \\ &= -2\epsilon_{ij} x_j \frac{a}{r^2} - 2\epsilon_{ij}\gamma_j \frac{a}{r^2} - \epsilon_{ij} x_j \gamma_k x_k (\frac{2a'}{r^3} - \frac{4a}{r^4}). \quad (3.52)\end{aligned}$$

The second term in (3.49) becomes after expansion

$$\partial_i \chi(x,y) = \epsilon_{ij}\gamma_j \frac{-2a}{r^2} - \epsilon_{jk} x_j x_i \gamma_k (\frac{2a'}{r^3} - \frac{4a}{r^4}). \quad (3.53)$$

On addition of (3.52) and (3.53) we find that

$$\begin{aligned}\overset{\circ}{A}_i (x+\gamma_1, y+\gamma_2) + \partial_i \chi(x,y) &= -2\epsilon_{ij} x_j \frac{a}{r^2} - 4\epsilon_{ij}\gamma_j \frac{a}{r^2} \\ &- (\frac{2a'}{r^3} - \frac{4a}{r^4})(\epsilon_{ij} x_j x_k \gamma_k + \epsilon_{jk} x_i x_j \gamma_k). \quad (3.54)\end{aligned}$$

But $\epsilon_{ij} x_j x_k \gamma_k + \epsilon_{jk} x_i x_j \gamma_k = r^2 \epsilon_{ij}\gamma_j$ therefore it can be seen from (3.50)and (3.54) that

$$A_i(x,y) = \overset{\circ}{A}_i (x+\gamma_1, y+\gamma_2) + \partial_i \chi. \quad (3.55)$$

The proof is now complete. We have shown that up to gauge transformation the Higgs field and the gauge potential as in (3.19) using the perturbation (3.33) describe translational motion.

3.5 90^0 Scattering

Up to now we have considered the perturbation which described translational motion but by far the more interesting of the two modes is the splitting of the vortices and their subsequent scattering at right angles. This time we consider the $k = 2$ terms in (3.30) in the special form

$$h(r,\theta) = k(r)(A+iB)e^{-2i\theta}. \quad (3.56)$$

On substitution of the above into (3.20), (3.24), (3.25) and (3.26) we can calculate

$$\begin{aligned}b(r,\theta) &= -(B\sin 2\theta + A\cos 2\theta)(2k + rk'), \\ c(r,\theta) &= (B\cos 2\theta - A\sin 2\theta)(2k + rk'), \quad (3.57)\end{aligned}$$

where $k(r)$ satisfies (3.31) for $k = 2$. The perturbation to the original system is

$$\begin{aligned}\phi &= \overset{\circ}{\phi} + t\xi_1 + it\xi_2, \\ A_i &= \overset{\circ}{A}_i + tB_i, \quad (3.58)\end{aligned}$$

therefore using (3.18), (3.56) and (3.57) it can easily be seen that

$$\begin{aligned}\xi_1 + i\xi_2 &= 2e^{i2\theta} f(r) h(r,\theta), \\ &= 2kf(A+iB) \quad (3.59)\end{aligned}$$

and

$$\begin{aligned}B_1 + iB_2 &= \frac{n}{r} e^{i2\theta}(c+ib), \\ &= \frac{-2i}{r} e^{-i\theta}(2k + rk')(A+iB). \quad (3.60)\end{aligned}$$

In the following, we consider the case $A = 1, B = 0$. In the case $A = 0, B = 1$ the analysis and the results are analogous. As shown by Jaffe and Taubes [11] the topological positions of the vortices are given by the zero's of the Higgs field $| \phi |^2 = 0$, for the unperturbed case $\phi = e^{i2\theta} f(r), | \phi |^2 = f^2(r) \Rightarrow | \phi |^2 = 0$ when $r = 0$ indicating that both vortices lie together at the origin. To return to the case at hand consider

$$\begin{aligned} \phi &= e^{i2\theta} f + 2kft, \\ | \phi |^2 &= f^2(1 + 4kt\cos 2\theta + 4t^2k^2) = 0, \\ &= H(r, \theta), \end{aligned}$$

where $H(r,\theta)$ is some surface described by r and θ. The point, or line, of intersection between $H(r,\theta)$ and the (r,θ) plane indicates the position of the vortices. The field is zero when

$$1 + 4kt\cos 2\theta + 4t^2k^2 = 0. \tag{3.61}$$

Take $t < 0$ ie. pre-scattering

$$1 - 4kt\cos 2\theta + 4t^2k^2 = 0 \tag{3.62}$$

consider the case $\theta = \pm\frac{\pi}{2}$

$$\begin{aligned} \Rightarrow \cos 2\theta &= -1, \\ 4t^2k^2 + 4tk + 1 &= 0, \\ (2tk + 1)^2 &= 0, \\ \Rightarrow k &= \frac{-1}{2t}. \end{aligned}$$

We can see that the intersection of $H(r,\theta)$ with the (r,θ) plane is always positive, as it is a square. Therefore only the points $\theta = \pm\pi/2, r = k^{-1}(-1/2t)$, and not lines, of intersection are allowable. We will now try the solution of $H(r,\theta)$ when $t > 0$,

$$4t^2k^2 + 4kt\cos 2\theta + 1 = 0$$

This is only possible if $\theta = 0$ or $\theta = \pi$. Then,

$$\begin{aligned} 4t^2k^2 + 4tk + 1 &= 0, \\ (2tk + 1)^2 &= 0, \\ k &= \frac{-1}{2t}. \end{aligned}$$

For the incoming vortices ($t < 0$) the zeros of the Higgs field are at $\theta = \pm\pi/2, r = k^{-1}(-1/2t)$, for the outgoing vortices, they are at $\theta = 0$ and $\theta = \pi, r = k^{-1}(-1/2t)$. That k^{-1} exists will be shown later. That this is evidence of 90^o scattering can be seen as follows: microscopicly there is a current of superpairs flowing around a vortex, sustained by and sustaining the magnetic flux. This configuration can only be smooth if there are no Cooper pairs at the centre of the flux tube. Hence, the zeros of the Higgs field give the locations of the centers of the vortices. Furthermore, as Fig. 3.1 illustrates head-on collision can be considered as the limit of a sequence (and of its mirror image) of collision with nonzero impact parameter. This leads to a left-right symmetry in a head-on collision which rules out scattering at angles other than $0^o, 90^o$ and 180^o. (If there is any deflection at any impact parameter, as presumed in Fig. 3.1, one also would not expect 180^o scattering.) The above arguments

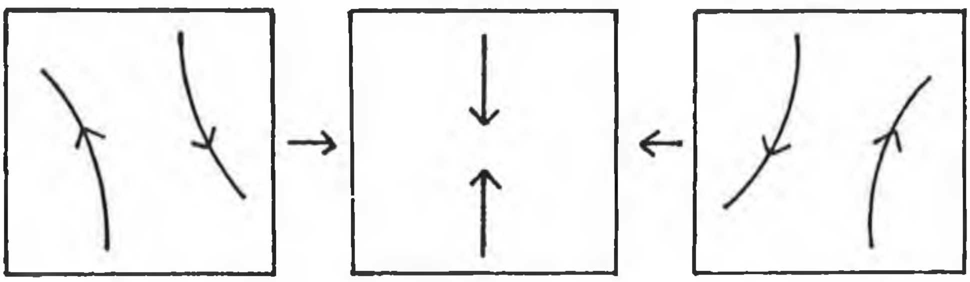

Figure 3.1: Head-on collision as the limit of collisions with nonzero impact parameter.

clearly discriminate in favour of 90^o scattering against 0^o and 180^o scattering. To understand better what happens during the collision we study the energy density. The energy is given by

$$E = \int [\frac{1}{2} D_i \phi \bar{D_i \phi} + \frac{1}{4} F_{ij}^2 + \frac{\lambda}{8}(\phi\phi^* - 1)^2] d^2x \tag{3.63}$$

and the energy density is given by

$$\mathcal{E}(r,\theta) = \frac{1}{2} D_i \phi \bar{D_i \phi} + \frac{1}{4} F_{ij}^2 + \frac{\lambda}{8}(\phi\phi^* - 1)^2 \tag{3.64}$$

where i, j are summed over spatial indices $1, 2$ only. Concentrating on the more interesting mode, as indicated at the beginning of this section, the remainder of this thesis will be confined to scattering whose perturbations take the following forms:

$$\begin{aligned} \xi_1 + i\xi_2 &= 2k(r) f(r), \\ B_1 + iB_2 &= \frac{-2i}{r} e^{-i\theta}(2k + rk'). \end{aligned}$$

The perturbed gauge potentials and Higgs fields become

$$\begin{aligned} A_i &= -\epsilon_{ij} x_j \frac{na}{r^2} + tB_i, \\ \phi &= e^{i2\theta} f(r) + t\xi_1 + it\xi_2. \end{aligned} \tag{3.65}$$

B_i can be written using summation notation and the Pauli spin matrix σ given by

$$\sigma = \begin{pmatrix} 01 \\ 10 \end{pmatrix}. \tag{3.66}$$

This means that (3.65) becomes

$$\begin{aligned} A_i &= -\epsilon_{ij} x_j \frac{na}{r^2} - 2\sigma_{ik} \hat{x}_k t(k' + \frac{2k}{r}), \\ \phi &= e^{i2\theta} f(r) + 2kft, \end{aligned} \tag{3.67}$$

if we set $A = 1$.

The calculation for the energy density is long and has been included in Appendix A and only the result is included here

$$\begin{aligned} \mathcal{E}(r,\theta) &= (\frac{2f}{r})^2 (1 - a^2) + 8(\frac{akft}{r})^2 + 16(\frac{f}{r})^2 akt(a-1)\cos 2\theta \\ &+ 2t^2(k'f + \frac{2kf}{r}(1-a))^2 + 2t^2 f^2 (k' + \frac{2k}{r})^2 (1 + 4kt\cos 2\theta + (2kt)^2) \\ &- \frac{8}{r} f^2 t \cos 2\theta (k' + \frac{2k}{r})[(a-1) + tka\cos 2\theta + tk(a-1)\cos 2\theta + 2t^2k^2 a] \\ &- 4t^2 f^2 k'(k' + \frac{2k}{r})\sin^2 2\theta + \frac{1}{4}(f^2 - 1)^2 + f^2 kt\cos 2\theta (f^2 + 2f^2 kt \cos 2\theta - 1) \\ &+ \frac{1}{8}(f^2 + 4kf^2 t\cos 2\theta + (2kft)^2 - 1)^2. \end{aligned} \tag{3.68}$$

Given the form of the energy density, we can check the finiteness of the energy by investigating each term individually. In the next section, we will show that

$$f \approx r^2, \quad , a \approx r^2, \quad k \approx r^{-2} \quad as \;\; r \to 0 \tag{3.69}$$

and

$$f \approx e^{-r}, \quad ,a \approx e^{-r}, \quad k \approx e^{-r} \quad as \quad r \to \infty. \tag{3.70}$$

If we examine both cases as $r \to \infty$ and $r \to 0$ it can easily be seen that the energy density is indeed finite. In the case where $r \to \infty$ all the terms in (3.68) die exponentially fast therefore never become infinite, however in the case $r \to 0$ the leading behaviour of $f(r)$ always compensates for the $k(r)$ terms. Consider for example terms of the form $k' + 2k/r$ if we substitute in the approximate values for k and k' we find that they exactly cancel each other and all combination terms involving f and k combine in such a way that they are finite. So we can conclude that the energy density never does become infinite. The asymptotic behaviour of k also shows that k^{-1} exists. For large r, k is strictly monotonic decreasing. Assume that this is not the case for all $r > 0$. Then, there exists a point r_0 with $k(r_0) > 0, k'(r_0) = 0$ and $k''(r_0) < 0$. This would be inconsistent with (3.31) and (3.56). Therefore, k is strictly monotonic decreasing on $(0, \infty)$ as r increases and k^{-1} exists.

Finally we study the potential energy density in the collision process. The Kinetic energy density is radially symmetric and does not alter our argument. The potential energy density was graphed using the numerical results found for $f(r), a(r)$ and $k(r)$ in Appendix B. Then a simple driver program was written in Fortran to calculate the potential energy density and plot it as a function of x and y. The situation

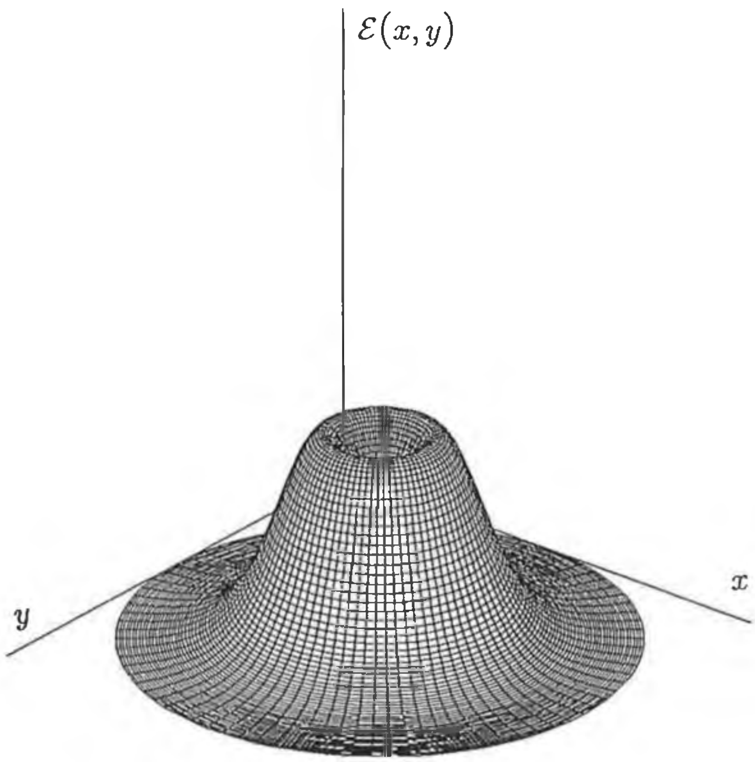

Figure 3.2: Static solution with both vortices situated at the origin, $t = 0$

depicted in Fig 3.2 is the static solution where both vortices lie at the origin. This plot shows that there is a local minimum at the center and a maximum lies in a ring around the axis, so that the vortex is mainly concentrated in a toroidal region.

Fig 3.3 shows the pre-scattering case where $t = -\frac{1}{2}$ and the vortices are about to collide. The view in this plot is not directly along the x axis (this is just so that the two vortices can be distinguished).

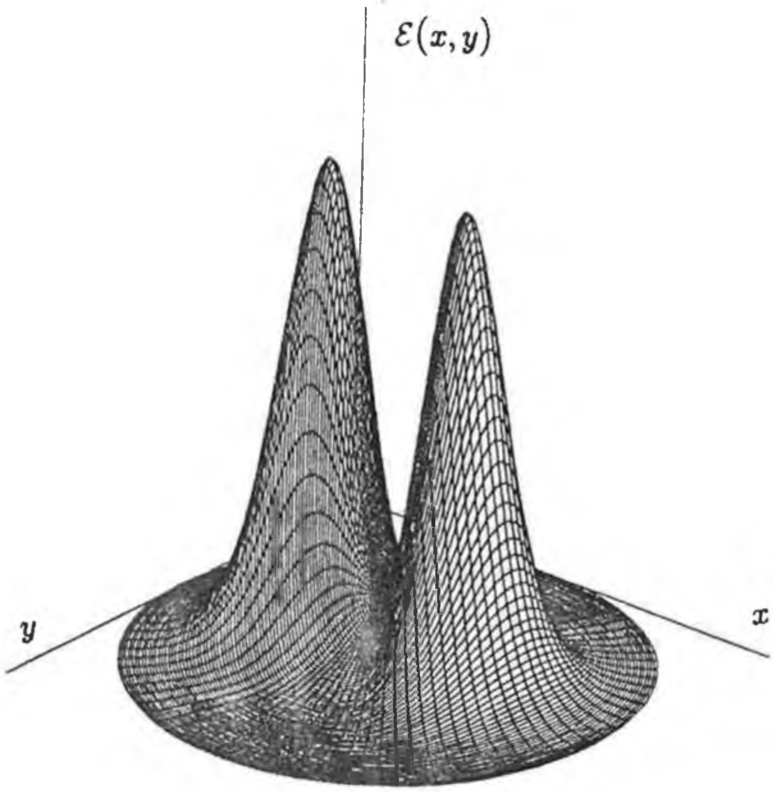

Figure 3.3: Pre-scattering with $t = \frac{-1}{2}$

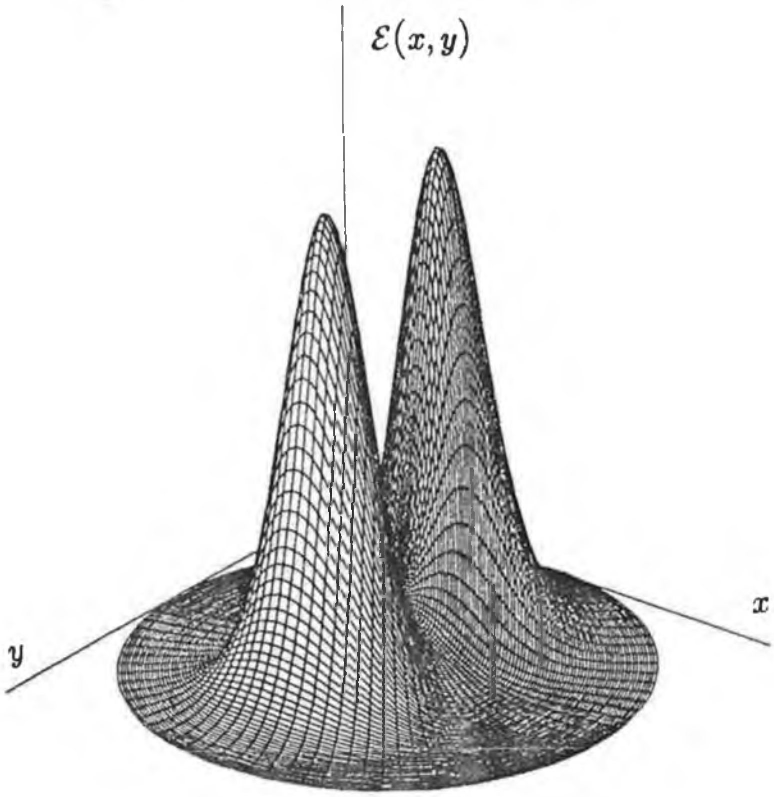

Figure 3.4: Post-scattering with $t = \frac{+1}{2}$

Fig 3.4 depicts the position of the vortices after the collision. A comparison between Fig 3.3 and Fig 3.4 does indeed show that the vortices scatter at right angles substantiating evidence discussed earlier.

There are at least still two problems which have to be addressed. First, a solution for $t \in (-\epsilon, \epsilon)$, is not a scattering solution. However, we can take the configuration for $t = 0$ as initial data of a solution for $t \in (-\infty, \infty)$ which we know exists [8]. For $t \in (-\epsilon, \epsilon), \epsilon \ll 1$, the linearization which leads to equations (3.11) should be justified and the solutions we discussed should be an approximation for $t \in (-\epsilon, \epsilon)$ to the scattering solution for $t \in (-\infty, \infty)$. The second problem is concerned with the experimental realization of the 90^o scattering process. We have given evidence for 90^o scattering, by presenting special approximate solutions, which require special initial data. However, since the parameter space for static vortices is 4-dimensional and we have found a 4- parameter family of approximate solutions (3.33) and (3.56), which all describe 90^o scattering possibly with a spatial translation, we expect 90^o scattering for slowly moving vortices for all initial data which lead to a collision.

Chapter 4

Series Solutions

4.1 Introduction

In Chapter 3 we found two first order coupled differential equations for $f(r)$ and $a(r)$, and a second order differential equation for $k(r)$. In this Chapter we will investigate the series solutions of these equations at zero and infinity. In the vicinity of zero we use Taylor series and at infinity we use asymptotic power series. The results obtained in this Chapter are then used in Appendix B to aid in the numerical investigation of the respective functions.

4.2 The Taylor Expansions at Zero

Consider the Bogomolny equations for the $n = 2$ case

$$f' = \frac{2f}{r}(1-a), \tag{4.1}$$

$$a' = \frac{-r}{4}(f^2-1). \tag{4.2}$$

Taylor series take the form

$$f = \sum_{n=1}^{\infty} f_n r^n = f_1 r + f_2 r^2 + f_3 r^3 + \cdots,$$

$$a = \sum_{n=1}^{\infty} a_n r^n = a_1 r + a_2 r^2 + a_3 r^3 + \cdots.$$

Substition of the above into (4.1) and (4.2) and solving for the respective coefficients we find that

$$\begin{array}{ll} f_1 = 0, & a_1 = 0, \\ f_2 = f_2, & a_2 = \frac{1}{8}, \\ f_3 = 0, & a_3 = 0, \\ f_4 = -a_2 f_2, & a_4 = 0, \\ f_5 = 0, & a_5 = 0, \\ f_6 = 2a_2^2 f_2, & a_6 = \frac{-f_2^2}{24}. \end{array} \tag{4.3}$$

The Taylor series for the solutions about zero for the functions $f(r)$ and $a(r)$ are therefore

$$f(r) = f_2 r^2 - \frac{1}{8} f_2 r^4 + \frac{1}{128} f_2 r^6 + \cdots, \tag{4.4}$$

and

$$a(r) = \frac{1}{8}r^2 - \frac{1}{24}f_2^2r^6 + \frac{1}{256}f_2^2r^8 + \cdots,$$

where f_2 is unknown.

From the equations it can easily be seen that all odd powers of r seem to be lost. To investigate wheither this is true for all higher powers of r consider

$$\begin{aligned} f(r) &= \sum_{n=1}^{N} f_n r^{2n} + F r^{2N+1} + o(r^{2N+2}), \\ a(r) &= \sum_{n=1}^{N} a_n r^{2n}, \end{aligned}$$

substituting these into (4.1), we find that

$$\begin{aligned} \sum_{n=1}^{N} 2n f_n r^{2n} + (2N+1)F r^{2N+1} &= 2\sum_{n=1}^{N} f_n r^{2n} + F r^{2N+1} \\ &- 2\sum_{n=1}^{N} \sum_{N,n_1,n_2 \geq 1}^{n} f_{n_1} a_{n_2} \delta_{n,n_1+n_2} \\ &+ F a_{n_2} \delta_{2n,2N+2n_2+1} \end{aligned} \tag{4.5}$$

but by definition

$$\delta_{2n,2N+2n_2+1} = 0$$

always and by comparison of coefficients in (4.5) we see that

$$\begin{aligned} (2N-1)F &= 0, \\ \Rightarrow F &= 0. \end{aligned}$$

Therefore there are no odd terms in the expansion of $f(r)$. If we assume on the other hand that

$$f(r) = \sum_{n=1}^{N+1} f_n r^{2n}$$

and

$$a(r) = \sum_{n=1}^{N} a_n r^{2n} + A r^{2N+1} + o(r^{2N+2}).$$

Substituting these into (4.2) we obtain

$$4\sum_{n=1}^{N} 2n a_n r^{2n-1} + (2N+1)A r^{2N} = -\sum_{n=1}^{N} \sum_{n_1,n_2 \geq 1}^{n} f_{n_1} f_{n_2} \delta_{n,n_1+n_2+1}. \tag{4.6}$$

Equating coefficients reveals

$$\begin{aligned} (2N+1)A &= 0, \\ \Rightarrow A &= 0 \end{aligned}$$

and therefore we see that there are no odd terms in the expansion around zero of $a(r)$.

We can then write the general forms of the series in the form

$$f(r) = \sum_{n=1}^{\infty} f_n r^{2n}, \qquad a(r) = \sum_{n=1}^{\infty} a_n r^{2n}, \tag{4.7}$$

respectively. To find a general expression for the n^{th} coefficient of the Taylor series we proceed as follows: substitute (4.7) into (4.1) and (4.2) to find that

$$\sum_{n=1}^{\infty} 2n f_n r^{2n} = 2 \sum_{n=1}^{\infty} f_n r^{2n} (1 - \sum_{n=1}^{\infty} a_n r^{2n}). \tag{4.8}$$

Simplifying this we find that

$$f_n = \frac{1}{1-n} \sum_{m=1}^{n-1} f_m a_{n-m}, \qquad n > 1,$$

and

$$4 \sum_{n=1}^{\infty} 2n a_n r^{2n-1} = - \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} f_m f_{n-m} r^{2n+1}$$

which reduces to

$$a_n = \frac{-1}{8n} \sum_{m=1}^{n-2} f_m f_{n-m-1}, \qquad n > 1.$$

These represent the recursion relations for the coefficients of $f(r)$ and $a(r)$ respectively, where f_2 is an arbitrary constant and $a_2 = 1/8$.

After finding the Taylor expansions and the recursion relations for $f(r)$ and $a(r)$ we will now consider the second order equation for k(r)

$$r^2 k'' + r k' - k(4 + r^2 f^2) = 0. \tag{4.9}$$

Using the result for f, we see that the solution near zero of the equation behaves like

$$k(r) = c_1 r^{-2} + c_2 r^2$$

leaving us reason to believe that yet again only even terms of the Taylor expansion survive. Proceeding as before we know that

$$f(r) = \sum_{n=1}^{N+1} f_n r^{2n}$$

and we assume that

$$k(r) = \sum_{n=-1}^{N} k_n r^{2n} + K r^{2N+1} + o(r^{2N+2}).$$

Then we have

$$k'(r) = \sum_{n=-1}^{N} 2nk_n r^{2n-1} + (2N+1)Kr^{2N},$$
$$k''(r) = \sum_{n=-1}^{N} 2n(2n-1)k_n r^{2n-2} + 2N(2N+1)Kr^{2N-1}.$$

If we substitute these into (4.9) we find that

$$\sum_{n=-1}^{N} (4n^2-4)k_n r^{2n} + (4N^2+4N-3)Kr^{2N+1} = \sum_{n=-1}^{N} \Big[\sum_{n_1,n_2,n_3\geq 1} f_{n_1} f_{n_2} k_{n_3} \delta_{n,n_1+n_2+n_3+1} + \sum_{N,n_1,n_2\geq 1} f_{n_1} f_{n_2} K \delta_{2n,2n_1+2n_2+2N+1} \Big]$$

but by definition

$$\delta_{2n,2n_1+2n_2+2N+1} = 0$$

always, therefore comparing coefficients gives

$$(4N^2+4N-3)K = 0,$$
$$\Rightarrow K = 0.$$

Again we have shown that no odd terms exist in the Taylor expansion at zero. We now need to find a general recursion relation for the coefficients in the expansion of $k(r)$. Following the same procedure as for $f(r)$ and $a(r)$ and using (4.9) we find that

$$k_n = \frac{1}{(4n^2-4)} \sum_{n_1,n_2,n_3\geq 1} f_{n_1} f_{n_2} k_{n_3} \delta_{n,n_1+n_2+n_3+1}. \tag{4.10}$$

4.3 Convergence of The Series Solutions at Zero

To prove the convergence of the Taylor series at zero we now show by induction that

$$|f_k| \leq \frac{M^k}{(k+1)^2}, \qquad k \leq n-1; \tag{4.11}$$
$$|a_k| \leq \frac{M^k}{(k+1)^2}, \qquad k \leq n-1; \tag{4.12}$$
$$|k_k| \leq \frac{M^k}{(k+1)^2}, \qquad k \leq n-1, \tag{4.13}$$

hold for sufficiently large k and $M \geq 1$. For this purpose consider

$$f_n = \frac{1}{(1-n)} \sum_{m_1,m_2\geq 1}^{n-1} f_{m_1} a_{m_2} \delta_{n,m_1+m_2}.$$

Taking the absolute value and using equations (4.11) and (4.12) we find that

$$|f_n| = \left| \frac{1}{1-n} \right| \sum_{m_1,m_2\geq 1}^{n-1} \frac{M^{m_1}}{(m_1+1)^2} \frac{M^{n-m_1}}{(n-m_1+1)^2}$$

where $n = m_1 + m_2$ from δ_{n,m_1+m_2}. Therefore

$$| f_n |=| \frac{M^n}{1-n} | \sum_{m_1=1}^{n-1} \frac{1}{(m_1+1)^2} \frac{1}{(n-m_1+1)^2}. \tag{4.14}$$

To complete the proof by induction consider the integral

$$\int_{1/2}^{n-1/2} \frac{dx}{(1+x)^2(n-x+1)^2}.$$

We need to show that the tangents to the above integrand always fall below the curve and then it follows that

$$\sum_{m_1=1}^{n-1} \frac{1}{(m_1+1)^2(n-m_1+1)^2} \leq \int_{1/2}^{n-1/2} \frac{dx}{(1+x)^2(n-x+1)^2}. \tag{4.15}$$

In order that we might do this consider

$$y = \frac{1}{(1+x)^2(n-x+1)^2}.$$

Then

$$\frac{dy}{dx} = -2[\frac{1}{(n-x+1)^2(1+x)^3} - \frac{1}{(1+x)^2(n-x+1)^3}].$$

Equating this to zero we find the critical point to lie at $x = \frac{n}{2}$. Using the second derivative test it can easily be seen that this is a minimum point. If we examine the curve over the respective interval we see that it is symmetrical about the point $x = \frac{n}{2}$, that the first derivative is increasing and that the second derivative is always greater than zero. Then we can say that the tangent always falls below the curve and the inequality (4.14) holds. To calculate the integral use partial fractions. We find that

$$\sum_{m_1=1}^{n-1} \frac{1}{(m_1+1)^2(n-m_1+1)^2} \leq \frac{4}{(n+2)^2}[\frac{1}{3} - \frac{1}{(2n+1)} + \frac{1}{n+2} \ln \frac{2n+1}{3}].$$

Therefore from (4.14), (4.15) and (4.11) we see that

$$\begin{aligned} | f_n | &\leq | \frac{M^n}{(1-n)} | \frac{4}{(n+2)^2}[\frac{1}{3} - \frac{1}{(2n+1)} + \frac{1}{n+2} \ln \frac{2n+1}{3}], \\ | f_n | &\leq \frac{M^n}{(1-n)(n+2)^2} o(1), \\ &\leq \frac{M^n}{(n+1)^2}. \end{aligned}$$

This proves the inequality (4.11) and convergence of the series $f(r)$ for $r \leq \frac{1}{\sqrt{M}}$. Similarly for a_n we find that

$$\begin{aligned} | a_n | &\leq | \frac{M^n}{-8n} | \sum_{m=1}^{n-2} | f_{m_1} || f_{m_2} | \delta_{n,m_1+m_2+1}, \\ | a_n | &\leq | \frac{M^{n-1}}{-8n} | \sum_{m_1=1}^{n-2} \frac{1}{(m_1+1)^2} \frac{1}{(n-m_1)^2}, \\ &\leq | \frac{M^{n-1}}{-8n} | \frac{1}{(n+1)^2} o(1), \\ &\leq \frac{M^n}{(n+1)^2}. \end{aligned}$$

This proves the inequality (4.12) and convergence of the series $a(r)$ for $r \leq \frac{1}{\sqrt{M}}$.

It now remains to calculate the convergence of the series for k(r). From (4.10) we have

$$k_n = \frac{1}{(4n^2-4)} \sum_{n_1,n_2,n_3 \geq 1} f_{n_1} f_{n_2} k_{n_3} \delta_{n,n_1+n_2+n_3+1}.$$

By taking the absolute value of both sides and separating the summation we find that

$$| k_n |=| \frac{1}{4(n^2-1)} | \sum_{m_3 \geq 1} \frac{M^{m_3}}{(m_3+1)^2} [\sum_{m_1 \geq 1} \frac{M^{m_1}}{(m_1+1)^2} \frac{M^{n-m_1-m_3-1}}{(n-m_1-m_3)^2}]$$

but this simplifies to

$$| k_n |=| \frac{1}{4(n^2-1)} | \sum_{m_3 \geq 1} \frac{M^{m_3}}{(m_3+1)^2} \frac{M^{n-m_3-1}}{(n-m_3)^2}.$$

Therefore if we use (4.15) we see as in the cases for $f(r)$ and $a(r)$ above that

$$\begin{aligned} | k_n | &= | \frac{M^{n-1}}{4(n^2-1)(n+1)^2} | o(1) \\ &\leq \frac{M^n}{(n+1)^2}. \end{aligned}$$

Therefore all the coefficients of the Taylor expansions at zero are convergent, it now remains to investigate the behaviour of the functions at infinity.

4.4 Asymptotic Power Series

Consider yet again the Bogomolny equations (4.1) and (4.2) but this time note the boundary conditions at infinity:

$$\begin{aligned} \lim_{r \to \infty} f(r) &= 1, \\ \lim_{r \to \infty} a(r) &= 1. \end{aligned}$$

From Plohr [14] we see that the solutions to (4.1) and (4.2) for large r are

$$\begin{aligned} f(r) &= 1 - \alpha k_0(r)(1+o(e^{-r})) + \frac{\beta^2}{3}[k_1(r)]^2, \\ &= 1 + f_1(r)e^{-r} + F_2(r) \end{aligned}$$

and

$$\begin{aligned} a(r) &= 1 - \frac{\beta}{2} r k_1(r)(1+o(e^{-r})), \\ &= 1 + a_1(r)e^{-r} + A_2(r) \end{aligned}$$

where k_μ is a Bessel function of order μ which is subdominant at infinity. This means that $f_1(r)$ and $a_1(r)$ are polynomially bounded and that $F_2(r)$ and $A_2(r)$ approach zero faster than $r^m e^{-r}$ for any power of m.

Consider now the second order differential equation for k(r)

$$r^2 k'' + rk' - (r^2 f^2 + 4)k = 0. \tag{4.16}$$

To leading order k(r) satisfies

$$r^2k'' + rk' - (r^2+4)k = 0. \tag{4.17}$$

This is a modified Bessel equation of order two but can be easily transformed to a Bessel equation using the transformation

$$z = ir, \qquad w(z) = k(r).$$

From Abramovitz and Stegun [1] (4.17) then becomes

$$z^2w'' + zw' + (z^2-4)w = 0.$$

This is a Bessel equation of order two with a general solution of the form

$$w(z) = c_1J_2(z) + c_2Y_2(z)$$

where $J_2(z)$ is a Bessel function of the first kind and Y_2 is a Bessel function of the second kind. From Olver [13] $w(r)$ can also be written as a linear combination of Hankel functions as

$$\begin{aligned} H_2^{(1)} &= J_2(z) + iY_2(z) \sim \sqrt{\frac{2}{\pi z}}e^{i(z-\frac{5\pi}{4})}, \\ H_2^{(2)} &= J_2(z) - iY_2(z) \sim \sqrt{\frac{2}{\pi z}}e^{-i(z-\frac{5\pi}{4})}, \\ \Rightarrow k(r) &= AH_2^{(1)} + BH_2^{(2)}. \end{aligned}$$

However in order that we maintain finite energy $B = 0$ and $k(r)$ takes the form

$$k(r) = k_1(r)e^{-r} + K_2(r)$$

where $k_1(r)$ is polynomially bounded and $K_2(r)$ approaches zero faster than r^me^{-r} for any power of m.

Using the induction hypothesis we assume that

$$f(r) = \sum_{k=0}^{n-1} f_ke^{-kr} + F_n(r), \qquad f_0 = 1, \tag{4.18}$$

$$a(r) = \sum_{k=0}^{n-1} a_ke^{-kr} + A_n(r), \qquad a_0 = 1, \tag{4.19}$$

$$k(r) = \sum_{k=0}^{n-1} k_ke^{-kr} + K_n(r), \tag{4.20}$$

where f_k, a_k and k_k are polynomially bounded and F_n, A_n and K_n all approach zero faster than $r^me^{-(n-1)r}$ for any m. We now need to prove that $F_n(r)$ behaves like f_ne^{-r} and similarly for $A_n(r)$ and $K_n(r)$.

If we now change the variables such that $f = 1 + F$ and $a = 1 + A$ and substitute (4.18) and (4.19) into (4.1), (4.2) then simplifying we find that (4.1) becomes

$$F' = -\frac{2}{r}A(1+F) \tag{4.21}$$

and (4.2) becomes

$$A' = -\frac{r}{4}F(2+F). \tag{4.22}$$

To decouple the differential equations, differentiate (4.21) and substitute in (4.22) to eliminate the A' term. Then we get

$$F'' + \frac{1}{r}F' - F = \frac{3}{2}F^2 + \frac{4}{r^2}A^2 + \frac{1}{2}F^3 + \frac{4}{r^2}A^2 F. \tag{4.23}$$

Consider firstly the homogeneous equation

$$F'' + \frac{1}{r}F' - F = 0 \tag{4.24}$$

which is a Bessel equation of order zero the solution of which can be written as

$$F = AH_0^{(1)}(ir) + BH_0^{(2)}(ir)$$

where B is zero to maintain finite energy. Hence by simplifying the Hankel functions F can be written in the form

$$F = f_1(r)e^{-r} + F_2(r).$$

If we substitute (4.18) and (4.19) into (4.23) keeping only leading terms we find that (4.23) becomes

$$\begin{aligned} F_n'' + \frac{1}{r}F_n' - F_n &= [\frac{3}{2}\sum_{n_1,n_2=1}^{\infty} f_{n_1} f_{n_2}\delta_{n,n_1+n_2} \\ &+ \frac{4}{r^2}\sum_{n_1,n_2=1}^{\infty} a_{n_1}a_{n_2}\delta_{n,n_1+n_2} + \frac{1}{2}\sum_{n_1,n_2,n_3=1}^{\infty} f_{n_1}f_{n_2}f_{n_3}\delta_{n,n_1+n_2+n_3} \\ &+ \frac{4}{r^2}\sum_{n_1,n_2,n_3=1}^{\infty} a_{n_1}a_{n_2}f_{n_3}\delta_{n,n_1+n_2+n_3}]e^{-nr}, \\ &=: \alpha_n(r)e^{-nr}. \end{aligned} \tag{4.25}$$

If we now substitute (4.18) and (4.19) into (4.22) we find that (4.22) becomes

$$A_n' = -\frac{r}{2}f_n(r)e^{-nr} - \frac{r}{4}\sum_{n_1,n_2=1}^{\infty} f_{n_1}f_{n_2}\delta_{n,n_1+n_2}e^{-nr}, \tag{4.26}$$

$$=: \beta_n(r)e^{-nr}. \tag{4.27}$$

Analogously the second order equation for $k(r)$ (4.16) can be rewritten using the same change of variables as for the $f(r)$ equation to find that it becomes

$$K'' + \frac{1}{r}K' - (1 - \frac{4}{r^2})K = 2FK + F^2K. \tag{4.28}$$

If we substitute (4.18) and (4.20) into this equation it reduces to

$$\begin{aligned} K'' + \frac{1}{r}K' - (1 - \frac{4}{r^2})K &= [2\sum_{n_1,n_2=1}^{\infty} f_{n_1}k_{n_2}\delta_{n,n_1+n_2} \\ &+ \sum_{n_1,n_2,n_3=1}^{\infty} f_{n_1}f_{n_2}k_{n_3}\delta_{n,n_1+n_2+n_3}]e^{-nr}, \\ &=: \gamma_n(r)e^{-nr}. \end{aligned} \tag{4.29}$$

In order to find the full solution to (4.25) consider the Green's function

$$g(r,\rho) = \begin{cases} a_1 r H_0^{(1)}(ir) + a_2 r H_0^{(2)}(ir), & (0 < r \leq \rho) \\ b_1 r H_0^{(1)}(ir) + b_2 r H_0^{(2)}(ir), & (\rho < r < \infty) \end{cases}$$

In order that this Green's function describe the situation at hand three conditions must hold, firstly

$$\lim_{r\to\infty} g(r,\rho) = 0 \Rightarrow b_1 = b_2 = 0.$$

Secondly the Green's function must be continuous ie.

$$\begin{aligned} g(\rho^+,\rho) &= g(\rho^-,\rho), \\ \Rightarrow a_1 &= aH_0^{(2)}(i\rho), \\ a_2 &= -aH_0^{(1)}(i\rho). \end{aligned} \tag{4.30}$$

Therefore

$$g(r,\rho) = a[H_0^{(2)}(i\rho)H_0^{(1)}(ir) - H_0^{(1)}(i\rho)H_0^{(2)}(ir)], \quad (0 < r < \rho) \tag{4.31}$$

and finally

$$\begin{aligned} -\frac{d}{dr}(g(\rho^-,\rho)) &= 1, \\ \Rightarrow -ia[H_0^{(2)}(i\rho)H_0^{(1)\prime}(i\rho) - H_0^{(1)}(i\rho)H_0^{(2)\prime}(i\rho)] &= 1. \end{aligned} \tag{4.32}$$

From Abramovitz and Stegun [1] we know that

$$H_0^{(n)\prime}(i\rho) = -H_1^{(n)}(i\rho),$$

and substituting this into (4.32) we find that

$$ia[H_0^{(2)}(i\rho)H_1^{(1)}(i\rho) - H_0^{(1)}(i\rho)H_1^{(2)}(i\rho)] = 1. \tag{4.33}$$

Since the determinant

$$H_1^{(1)}(i\rho)H_0^{(2)}(i\rho) - H_0^{(1)}(i\rho)H_1^{(2)}(i\rho) = \frac{-4}{\pi\rho}$$

is non-zero (Abramovitz and Stegun [1]) this system is solvable and we find that

$$a = \frac{i\pi\rho}{4}.$$

Now we can say that

$$F_n(r) = \int_{r_0}^{\infty} g(r,\rho)\alpha_n(\rho)e^{-(n)\rho}d\rho \tag{4.34}$$

and expanding out the Green's function we find that (4.34) becomes

$$F_n(r) = \frac{i\pi}{4}\int_r^{\infty} \rho[H_0^{(2)}(i\rho)H_0^{(1)}(ir) - H_0^{(1)}(i\rho)H_0^{(2)}(i\rho)]\alpha_n(\rho)e^{-n\rho}d\rho. \tag{4.35}$$

Similarly for K_n we find the Green's function of the form

$$g(r,\rho) = H(r-\rho)\frac{i\pi\rho}{4}[H_2^{(2)}(i\rho)H_2^{(1)}(ir) - H_2^{(1)}(i\rho)H_2^{(2)}(ir)] \tag{4.36}$$

and therefore we can write K_n in the form

$$K_n(r) = \frac{i\pi}{4}\int_r^{\infty} \rho[H_2^{(2)}(i\rho)H_2^{(1)}(ir) - H_2^{(1)}(i\rho)H_2^{(2)}(ir)]\gamma_n(\rho)e^{-n\rho}d\rho \tag{4.37}$$

where $\gamma_n(\rho)$ is given in (4.29). We have now found in terms of Green's functions the form of the three series. All that remains to do is to investigate whether or not these series converge at infinity.

4.5 Convergence of the Asymtotic Power Series

To prove the convergence of the asymptotic series at infinity, we assume there exist numbers c_0, M and R such that

$$sup_{r>R} \mid rf_n(r)e^{-\frac{nr}{2}} \mid < \frac{c_0 M^n}{(n+1)^2}, \tag{4.38}$$

$$sup_{r>R} \mid r^2 a_n(r)e^{-\frac{nr}{2}} \mid < \frac{c_0 M^n}{(n+1)^2}, \tag{4.39}$$

$$sup_{r>R} \mid rk_n(r)e^{-\frac{nr}{2}} \mid < \frac{c_0 M^n}{(n+1)^2}, \tag{4.40}$$

for large enough n. For this purpose consider $g(r,\rho)$ and $\alpha_n(\rho)$ as calculated in (4.31) and (4.25) respectively. Then

$$sup_{r>R} \mid rf_n(r)e^{-\frac{nr}{2}} \mid \;=\; sup_{r>R} \mid rF_n(r)e^{\frac{nr}{2}} \mid \tag{4.41}$$

by definition and using (4.25) and (4.31) we find that

$$\begin{aligned} sup_{r>R}|rF_n(r)e^{\frac{nr}{2}}| <\; & sup_{r>R}\frac{\pi}{4}|\int_r^\infty [H_0^{(2)}(i\rho)H_0^{(1)}(ir) - H_0^{(1)}(i\rho)H_0^{(2)}(ir)] \\ & e^{-\frac{n(\rho-r)}{2}}\rho^2\alpha_n(\rho)e^{-\frac{n\rho}{2}}d\rho|, \qquad (4.42) \\ <\; & sup_{r>R}\frac{\pi}{4}|r^2\alpha_n(r)e^{-\frac{nr}{2}}| \\ & sup_{r>R}\int_r^\infty |H_0^{(2)}(i\rho)H_0^{(1)}(ir) - H_0^{(1)}(i\rho)H_0^{(2)}(ir)| \\ & e^{-\frac{n(\rho-r)}{2}}d\rho. \qquad (4.43) \end{aligned}$$

Taking R large enough, we can bound $|H_\mu^{(1,2)}(ir)|$ by $ce^{(\mp r)}/\sqrt{r}$, (4.43) then reduces to

$$sup_{r>R}\frac{\pi}{4}|r^2\alpha_n(r)e^{-\frac{nr}{2}}|.sup_{r>R}\int_r^\infty (e^{\rho-r}+e^{r-\rho})e^{-\frac{n(\rho-r)}{2}}d\rho. \tag{4.44}$$

Calculating the integral in the above equation we find that

$$\begin{aligned} \int_r^\infty (e^{\rho-r}+e^{r-\rho})e^{-\frac{n(\rho-r)}{2}}d\rho &= [\frac{-2}{n-2}e^{-\frac{(n-2)(\rho-r)}{2}} - \frac{2}{n+2}e^{-\frac{(n+2)(\rho-r)}{2}}]_r^\infty, \\ &= -\frac{4n}{n^2-4}. \qquad (4.45) \end{aligned}$$

Then we can say that

$$sup_{r>R}|rf_n(r)e^{-\frac{nr}{2}}| \;<\; \frac{\pi n}{n^2-4}sup_{r>R}|r^2\alpha_n(r)e^{-\frac{nr}{2}}|. \tag{4.46}$$

Using the induction hypothesis and substituting for $\alpha_n(r)$ we find that (4.46) becomes

$$\begin{aligned} \frac{\pi n}{n^2-4}sup_{r>R}|r^2\alpha_n(r)e^{-\frac{nr}{2}}| < & \frac{\pi n}{n^2-4}[\frac{11}{2}M^n\sum_{n_1=1}^{n-1}\frac{1}{(n_1+1)^2}\frac{1}{(n-n_1+1)^2} \\ & +\frac{9}{2}M^n\sum_{n_1=1}^{n-2}\frac{1}{(n_1+1)^2}\sum_{n_2=1}^{n-n_1-2}\frac{1}{(n_2+1)^2}\frac{1}{(n-n_1-n_2+1)^2}], \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\pi n}{n^2-4}[\frac{M^n}{(n+2)^2}o(1) \\
&\qquad + M^n \sum_{n_1=1}^{n-2} \frac{1}{(n_1+1)^2}\frac{1}{(n-n_1+2)^2}o(1)], \\
&\leq \frac{\pi n M^n}{n^2-4}[\frac{1}{(n+2)^2}o(1) + \frac{1}{(n+3)^2}o(1)], \\
&< \frac{M^n}{(n+1)^2}. \qquad (4.47)
\end{aligned}
$$

This proves the inequality (4.38) and convergence for $r > R$ and $r > 2\log M$. Similarly for k_n we find that

$$
\begin{aligned}
sup_{r>R}|rk_n(r)e^{-\frac{nr}{2}}| &= sup_{r>R}|rK_n e^{\frac{nr}{2}}|, \qquad (4.48) \\
&\leq sup_{r>R}|r^2\gamma_n(r)e^{-\frac{nr}{2}}|\frac{\pi}{4}\int_r^\infty (e^{\rho-r}+e^{r-\rho})e^{-\frac{n(\rho-r)}{2}}d\rho (4.49)
\end{aligned}
$$

where $\gamma_n(r)$ is given in equation (4.29) and the Green's function for $k_n(r)$ is similar to that for f_n. Proceeding in exactly the same way as we did for f_n we find that (4.49) is less than or equal to

$$
\begin{aligned}
sup_{r>R}|rK_n e^{\frac{nr}{2}}| &\leq \frac{\pi n}{n^2-4}[2M^n \sum_{n_1=1}^{n-1} \frac{1}{(n_1+1)^2}\frac{1}{n-n_1+1)^2} \\
&+ M^n \sum_{n_1=1}^{n-2} \frac{1}{(n_1+1)^2}\sum_{n_2=1}^{n-n_1-2}\frac{1}{(n_2+1)^2}\frac{1}{(n-n_1-n_2+1)^2}],(4.50) \\
&\leq \frac{\pi n M^n}{n^2-4}[\frac{1}{(n+2)^2}o(1) + \frac{1}{(n+3)^2}o(1)], \qquad (4.51) \\
&< \frac{M^n}{(n+1)^2}. \qquad (4.52)
\end{aligned}
$$

This proves the inequality (4.40) and convergence for $r > R$ and $r > 2\log M$. It now remains to calculate the convergence of the series for $a(r)$. From (4.39) we have

$$
\begin{aligned}
sup_{r>R}|ra_n(r)e^{-\frac{nr}{2}}| &= sup_{r>R}|rA_n(r)e^{\frac{nr}{2}}|, \qquad (4.53) \\
&\leq sup_{r>R}\int_r^\infty |\rho\beta_n(\rho)e^{-\frac{n\rho}{2}}|e^{-\frac{n(\rho-r)}{2}}d\rho, \qquad (4.54) \\
&\leq sup_{r>R}|r\beta_n(r)e^{-\frac{nr}{2}}|\left[-\frac{2}{n}e^{-\frac{n(\rho-r)}{2}}\right]_r^\infty, \qquad (4.55) \\
&= \frac{2}{n}sup_{r>R}|r\beta_n(r)e^{-\frac{nr}{2}}| \qquad (4.56)
\end{aligned}
$$

where β_n is given by (4.27). We notice that the term $r\beta_n(r)$ contains the term $r^2 f_n(r)$ which cannot be controlled by the inequality (4.38). Therefore, we make a change of variables of the form

$$
A_n = r\tilde{A_n}, \qquad a_n = r\tilde{a_n},
$$

this means that equation (4.26) becomes

$$
\begin{aligned}
\tilde{A_n}' + \frac{1}{r}\tilde{A_n} &= \left\{-\frac{1}{2}f_n - \frac{1}{4}\sum_{n_1,n_2=1}^{\infty} f_{n_1}f_{n_2}\delta_{n,n_1+n_2}\right\}e^{-nr}, \qquad (4.57) \\
&=: \beta_n e^{-nr}. \qquad (4.58)
\end{aligned}
$$

The Green's function for this equation is $g(r,\rho) = -H(\rho - r)\frac{\rho}{r}$. The solution of the above equation is then of the form

$$\bar{A}_n(r) = -\frac{1}{r}\int_r^\infty \rho\beta_n(\rho)e^{-n\rho}d\rho. \tag{4.59}$$

To prove the convergence of this consider

$$sup_{r>R}|r\bar{a}_n(r)e^{-\frac{nr}{2}}| = sup_{r>R}|r\bar{A}_n(r)e\frac{nr}{2}|, \tag{4.60}$$

$$\leq sup_{r>R}|r\beta_n(r)e^{-\frac{nr}{2}}|\int_r^\infty e^{-\frac{n(\rho-r)}{2}}d\rho, \tag{4.61}$$

$$< \frac{2}{n}\left\{\frac{1}{2}\frac{M^n}{(n+1)^2} + \frac{M^n}{4}\sum_{n_1=1}^{n-1}\frac{1}{(n_1+1)^2}\frac{1}{(n-n_1+1)^2}\right\} \tag{4.62}$$

$$< \frac{M^n}{(n+1)^2}. \tag{4.63}$$

Therefore all the coefficients of the asymptotic series at infinity are convergent for some $r > R$ and $r > 2\log M$.

Chapter 5

The Case $\lambda \geq 1$

The theoretical predictions for the scattering of soliton-like objects are very exciting. For static vortices the only degrees of freedom are the positions of the vortices and any unusual behaviour would hence be due to their soliton-like nature. Left-right symmetry in a head-on collision would only allow scattering at an angle of $0^o, 90^o$ or 180^o as shown in Chapter 3. For slowly moving vortices at the point between type I and type II superconductivity (where $\lambda = 1$) we have shown that the vortices do indeed scatter at right angles. If the repulsion between the vortices increases and they cannot come close anymore, we would expect to see a switch over to back scattering at a certain value of the repulsion. There is numerical evidence that for fixed repulsion an increase in the velocity can bring the vortices close enough together again to produce scattering at right angles. In this thesis we now change the strength of repulsion.

Consider the equations of motion

$$\begin{aligned} D_\mu D^\mu \phi + \frac{1}{2}\lambda\phi(|\phi|^2 - 1) = 0, \\ \partial_\mu F^{\mu\nu} + \frac{i}{2}(\phi^* D^\nu \phi - \phi(D^\nu \phi)^*) = 0 \end{aligned} \tag{5.1}$$

and the fields

$$\begin{aligned} \phi = \mathring{\phi} + \tilde{\phi}, \\ A_i = \mathring{A}_i + \tilde{A}_i \end{aligned} \tag{5.2}$$

where $(\mathring{A}_i, \mathring{\phi})$ is the static solution for two vortices sitting on top of each other and $(\tilde{A}_i, \tilde{\phi})$ are the perturbations on the static solution. When $\lambda = 1$ all internal forces balance so to introduce a small repulsive force consider $\lambda = 1 + \tilde{\lambda}, \tilde{\lambda} \ll 1$. If we substitute (5.2) and λ into (5.1), use the fact that $(\mathring{A}_i, \mathring{\phi})$ solve the time-independent equations and keep only terms linear in the perturbation we find the equations of motion become

$$\begin{aligned} \mathring{D}_i \mathring{D}^i \tilde{\phi} - 2i\tilde{A}_i \mathring{D}^i \mathring{\phi} - i \mathring{\phi}\, \partial_i \tilde{A}^i + \frac{1}{2}\tilde{\phi}(| \mathring{\phi} |^2 - 1) + \frac{1}{2} \mathring{\phi} (\mathring{\phi} \tilde{\phi}^* + \mathring{\phi}^* \tilde{\phi}) \\ + \frac{1}{2}\tilde{\lambda} \mathring{\phi} (| \mathring{\phi} |^2 - 1) = 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \frac{i}{2}[\tilde{\phi}^* \mathring{D}_j \mathring{\phi} - \tilde{\phi}(\mathring{D}_j \mathring{\phi})^* + \mathring{\phi}^* \mathring{D}_j \tilde{\phi} - \mathring{\phi} (\mathring{D}_j \tilde{\phi})^*] \\ + \partial^i \tilde{F}_{ij} + \tilde{A}_j | \mathring{\phi} |^2 + = 0, \end{aligned} \tag{5.4}$$

$$\partial^i \partial_0 \tilde{A}_i + \tfrac{i}{2} [\mathring{\phi}^* \partial_0 \tilde{\phi} - \mathring{\phi}\, \partial_0 \tilde{\phi}^*] = 0, \tag{5.5}$$

where $\overset{\circ}{D}_i = \partial_i - i \overset{\circ}{A}_i$ and $\tilde{F}_{ij} = \partial_i \tilde{A}_j - \partial_j \tilde{A}_i$ and

$$\begin{aligned} \tilde{\phi}(t, \vec{x}) &= \tilde{\lambda}\varphi(\vec{x}) + t\xi(\vec{x}) \\ \tilde{A}_i(t, \vec{x}) &= \tilde{\lambda} a_i(\vec{x}) + t\beta(\vec{x}) \end{aligned} \tag{5.6}$$

where $(\overset{\circ}{\phi} + \tilde{\lambda}\varphi, \tilde{A}_i + \tilde{\lambda} a_i)$ satifies the static equations of motion linearized in $\tilde{\lambda}$. Hence $(\tilde{\lambda}\varphi, \tilde{\lambda} a_i)$ is a solution of the inhomogeneous system of equations (5.5) and again we have a 4- parameter family of solutions which is what is required for 90^o scattering. The homogeneous system is the one which had to be solved in the case $\lambda = 1$. Therefore, also in the case $1 < \lambda = 1 + \tilde{\lambda}, \tilde{\lambda} \ll 1$, we find the approximate solutions which describe 90^o scattering. This is important because in an experiment $\lambda = 1$ can never be exactly realised. Our argument shows that if the net repulsion is small enough, slowly moving vortices can overcome it and will scatter at right angle. Here, slowly moving means slow enough for the approximation to apply, which is a very indirect way of quantifying the velocity.

Chapter 6

Conclusions

We used results of Weinberg [16] and Ruback [15] to construct approximate solutions to the partial differential equations which describe vortex-vortex scattering. Together with a simple argument, which rules out scattering at angles other than $0^\circ, 90^\circ$ or 180°, this provides further analytical evidence for 90° scattering. Our method makes it possible to extend the analysis to the case of a small net repulsive force between the corresponding static vortex configurations. We have also studied the ordinary differential equations, which result from the ansatz for the approximate solution. These equations are solved by Taylor series at the origin and by Asymptotic series at infinity.

Appendix A

The Energy Density

The potential energy density is given by

$$\mathcal{E}(r,\theta) = \frac{1}{2}D_i\phi\overline{D_i\phi} + \frac{1}{4}F_{ij}^2 + \frac{\lambda}{8}(\phi\phi^* - 1)^2$$

where i, j are summed over the spatial indices only. The Kinetic energy density is

$$\mathcal{E}_{kin}(r,\theta)\frac{1}{2}D_0\phi\overline{D_0\phi} + \frac{1}{2}F_{0i}F_{0i}.$$

In this section we calculate the energy density for the perturbed state that describes 90^0 scattering ie. we consider ϕ and A_i of the form

$$\begin{aligned} \phi(r,\theta) &= e^{in\theta}f(r) + 2tk(r)f(r), \\ A_i(r,\theta) &= -\epsilon_{ij}\hat{x}_j\frac{na}{r} - 2\sigma_{ik}\hat{x}_k t(k' + \frac{2k}{r}) \end{aligned}$$

as can be seen in (3.67).To proceed with the calculation

$$D_0\phi = \partial_0\phi = 2kf \tag{A.1}$$

and

$$\begin{aligned} D_i\phi &= (\partial_i - iA_i)\phi, \\ &= -\epsilon_{ij}\frac{\hat{x_j}}{r}\partial_\theta + \hat{x}_i\partial_r + i\epsilon_{ij}\hat{x_j}\frac{na}{r} + i2\sigma_{ik}\hat{x}_k t(k' + \frac{2k}{r})(e^{in\theta}f + 2tkf) \end{aligned}$$

where $\partial_i = -\epsilon_{ij}\frac{\hat{x_j}}{r}\partial_\theta + \hat{x}_i\partial_r$ and for convenience we write $f(r), a(r)$ and $k(r)$ as f, a and k respectively. Then

$$\begin{aligned} D_i\phi &= i\epsilon_{ij}\hat{x_j}[e^{in\theta}\frac{nf(a-1)}{r} + \frac{2natkf}{r}] + \hat{x}_i(e^{in\theta}f' + 2k'ft + 2kf't) \\ &+ i2\sigma_{ik}\hat{x}_k t(k' + \frac{2k}{r})(e^{in\theta}f + 2kft), \end{aligned}$$

where $\overline{D_i\phi}$ is the complex conjugate of $D_i\phi$. Then

$$\begin{aligned} D_i\phi\overline{D_i\phi} &= (\frac{nf}{r})^2 + (\frac{2tankf}{r})^2 + 4ka(a-1)t(\frac{nf}{r})^2\cos n\theta \\ &+ 4tf'(k'f + f'k)\cos n\theta + t^2(2k'f + 2f'k)^2 \\ &+ 4(k' + \frac{2k}{r})^2 t^2(f^2 + 4tkf^2 + (2kft)^2) - 16\hat{x_1}\hat{x_2}(k' + \frac{2k}{r})tk'f^2\sin n\theta \\ &+ 4(\hat{x_2^2} - \hat{x_1^2})(k' + \frac{2k}{r})tf^2\frac{1}{r}[(nf(a-1))^2 - 2knt\cos n\theta + 4nak^2t^2 \\ &+ 4kanf^2t\cos n\theta] \end{aligned}$$

where

$$
\begin{aligned}
-i\epsilon_{ij}\hat{x_j}.i\epsilon_{ik}\hat{x_k} &= \hat{x_2} + \hat{x_1} = 1, \\
\sigma_{ij}\hat{x_j}\sigma_{ik}\hat{x_k} &= 1, \\
\epsilon_{ij}\hat{x_j}\sigma_{ik}\hat{x_k} &= \hat{x_2^2} - \hat{x_1^2}, \\
\hat{x_i}\epsilon_{ij}\hat{x_j} &= 0, \\
\hat{x_i}\hat{x_i} &= 1, \\
\hat{x_i}\sigma_{ik}\hat{x_k} &= 2\hat{x_1}\hat{x_2}.
\end{aligned}
$$

The second term of the energy density is calculated as follows:

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

where

$$A_i(r,\theta) = -\epsilon_{ij}\hat{x_j}\frac{na}{r} - 2\sigma_{ik}\hat{x_k}t(k' + \frac{2k}{r})$$

and

$$A_j(r,\theta) = -\epsilon_{ji}\hat{x_i}\frac{na}{r} - 2\sigma_{jk}\hat{x_k}t(k' + \frac{2k}{r}).$$

Therefore

$$
\begin{aligned}
F_{ij} &= -\epsilon_{ji}\frac{na}{r^2} - \epsilon_{jk}x_k\hat{x_i}n(\frac{a}{r^2})' - 2\sigma_{ji}t(\frac{k'}{r} + \frac{2k}{r^2}) \\
&- 2\sigma_{jk}x_k\hat{x_i}(\frac{k'}{r} + \frac{2k}{r^2})'t - \{-\epsilon_{ij}\frac{na}{r^2} - \epsilon_{ik}x_k\hat{x_j}n(\frac{a}{r^2})' \\
&- 2\sigma_{ij}t(\frac{k'}{r} + \frac{2k}{r^2}) - 2\sigma_{ik}x_k\hat{x_j}(\frac{k'}{r} + \frac{2k}{r^2})'t.
\end{aligned}
$$

This becomes

$$F_{ij} = \epsilon_{ij}\frac{na'}{r} - 2\epsilon_{ij}r(\hat{x_1^2} - \hat{x_2^2})(\frac{k'}{r} + \frac{2k}{r^2})'t$$

with

$$
\begin{aligned}
-\epsilon_{ji} &= \epsilon_{ij}, \\
\epsilon_{ij}\epsilon_{ij} &= 2, \\
(\epsilon_{ik}\hat{x_j} - \epsilon_{jk}\hat{x_i})x_k &= \epsilon_{ij}\hat{x_k}x_k = \epsilon_{ij}r, \\
(\delta_{jk}\hat{x_i} - \delta_{ik}\hat{x_j})x_k &= \epsilon_{ij}(\hat{x_1^2} - \hat{x_2^2})r.
\end{aligned}
$$

Now we can say that

$$\frac{1}{4}F_{ij}^2 = \frac{1}{2}(\frac{na'}{r})^2 - na'(\hat{x_1^2} - \hat{x_2^2})(\frac{k'}{r} + \frac{2k}{r^2})'t + r^2(\hat{x_1}^2 - \hat{x_2}^2)((\frac{k'}{r} + \frac{2k}{r^2})')^2 t^2$$

and

$$\frac{1}{2}F_{0i}^2 = 2(k' + \frac{2k}{r})^2. \tag{A.2}$$

The final term in the expansion for the energy density is

$$\frac{1}{8}(\phi\phi^* - 1)^2 = \frac{1}{8}(f^2 + 4kf^2 t\cos 2\theta + (2kft)^2 - 1)^2.$$

Before we consider the full form of the energy density consider the substitutions we can make i.e.

$$\begin{aligned}
f' &= \frac{nf}{r}(1-a), \\
a' &= -\frac{r}{2n}(f^2-1), \\
(\frac{k'}{r}+\frac{2k}{r^2})' &= \frac{1}{r}(k''+\frac{1}{r}k'-\frac{4}{r^2}k), \\
&= \frac{1}{r}f^2k, \\
\hat{x_1}^2-\hat{x_2}^2 &= \cos 2\theta
\end{aligned}$$

and also $n = 2$ everywhere as we are only considering the $n = 2$ case. The final answer for the potential energy density is then given by

$$\begin{aligned}
\mathcal{E}(r,\theta) &= \frac{4f(1-a)^2}{r} + 8(\frac{akft}{r})^2 + 16akt(\frac{f}{r})^2(a-1)\cos 2\theta \\
&+ 2t^2(k'f+\frac{2kf}{r}(1-a))^2 + 2t^2f^2(k'+\frac{2k}{r})^2(1+4kt\cos 2\theta+(2kt)^2) \\
&- \frac{8}{r}\cos 2\theta t f^2(k'+\frac{2k}{r})[(a-1)+2kat\cos 2\theta - tk\cos 2\theta + 2ak^2t^2] \\
&- 4t^2k'f^2(k'+\frac{2k}{r})\sin^2 2\theta + \frac{1}{8}(f^2-1)^2 + f^2kt\cos 2\theta(2\cos 2\theta f^2kt + f^2 - 1) \\
&+ \frac{1}{8}(f^2+4ktf^2\cos 2\theta + (2kft)^2-1)^2.
\end{aligned}$$

This equation is used in both Chapter 3 and Appendix B. In Chapter 3 the finiteness of the potential energy density is demonstrated for this particular perturbation and in Appendix B it is used to calculate the potential energy density using the numerical results obtained for the functions f, a and k respectively.

The Kinetic energy density is

$$\mathcal{E}_{kin}(r,\theta) = 2k^2f^2 + 2(k'+\frac{2k}{r})^2. \tag{A.3}$$

We see that $\mathcal{E}_{kin}$ is independant of θ. That is why only the potential energy density was studied in Chapter 3. We also have to show that the addition of $\mathcal{E}_{kin}$ can be considered as a small perturbation of the configuration (3.3) of two static vortices. This is the case, because (4.9) for k is linear, and we can always multiply any solution k by a small parameter such that $\mathcal{E}_{kin}$ is much smaller than the energy density for the configuration (3.3) of two static vortices.

Appendix B

Numerical Analysis

B.1 Introduction

The mathematical problem can be considered in three stages. Firstly we must solve

$$f' = \frac{2}{r}(1-a), \tag{B.1}$$
$$a' = -\frac{r}{4}(f^2-1) \tag{B.2}$$

subject to

$$\lim_{r\to 0} f(r) = \lim_{r\to 0} a(r) = 0, \tag{B.3}$$
$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} a(r) = 1. \tag{B.4}$$

Then we substitute $f(r)$ into

$$-k'' - \frac{k'}{r} + (f^2 + \frac{4}{r^2})k = 0 \tag{B.5}$$

where

$$k(r) \to \infty \quad as \quad r \to 0, \tag{B.6}$$
$$\lim_{r\to\infty} k(r) = 0. \tag{B.7}$$

We need both $k(r)$ and $k'(r)$ for the final stage which is to find the approximate energy-density for the interval $0 < r < \infty, 0 \leq \theta < 2\pi$. To do this we use (3.68), the equation for the energy-density as calculated in Appendix A.

Problems (B.1) (B.2) and (B.5) cannot be solved analytically over the full domain however perturbation approximations can be obtained at the origin and infinity. These proved critical for the numerical problem solution and are detailed in Chapter 4 with the series solutions.

B.2 The Numerical Problem

(B.1) and (B.2) with boundary conditions (B.3) define an initial value problem (IVP). Substitution of the boundary conditions into (B.1) and (B.2) shows that $f' = a' = 0$ at the boundary points, which implies the boundary points are fixed points and so any IVP integrator will never move away from either set of initial conditions and so would never solve them. The next obvious approach is to treat both equations as a boundary value problem (BVP). It is because of the $\frac{1}{r}$ in (B.1) that this is difficult near $r = 0$.

To get over the latter we can use the Taylor series expansion near $r = 0$ and match these up with the numerics at $r = r_\epsilon$, some small value near zero. The problem with this is that we do not know α, the unknown coefficient in the Taylor expansion for $f(r)$ about $r = 0$.

It turns out that the second order problem for $a(r)$ is easily solved over $r_\epsilon \leq r < r_\infty$, where we use

$$\begin{aligned} a(r_\epsilon) &= \frac{r_\epsilon^2}{8}, \\ a(r_\infty) &= 1 \end{aligned} \tag{B.8}$$

as boundary conditions. Since

$$a(r) = \frac{r^2}{8} + O(r^6) \tag{B.9}$$

we choose r_ϵ such that $r_\epsilon^2 \gg r_\epsilon^6$ so that $a(r) = \frac{r^2}{8}$ is an accurate approximation. In the program $r_\epsilon = 10^{-8}$.

The idea is to solve the second order problem for $a(r)$ and then to fix r at some small value ($r = 0.1$ in fact) to get $a_N := a(0.1)$. We then solve

$$0 = T(\alpha) = a_N - a_T^\alpha(0.1), \tag{B.10}$$

where $a_T^\alpha(r)$ is the Taylor series approximation to $a(r)$, which depends on α. The program uses the NAG finite difference routine D02RAF to solve the second order problem for $a(r)$, and a simple bisection method to find the value of α which solves (B.10). Once we have α we can produce values for $f(r)$ and $a(r)$ near the origin. The program uses the Taylor series values over $0 \leq r \leq 0.1$.

Next the program solves (B.1) and (B.2) over $0.1 < r \leq 100$ using the collocation package COLSYS. This gives us back functional representations for $f(r)$ and $a(r)$ over the range, and combined with the Taylor series we have $f(r)$ and $a(r)$ over $0 \leq r \leq 100$ which is effectively $0 \leq r \leq \infty$.

The final stage is to solve the second order problem for $k(r)$ and write out a range of values for $r, k(r), f(r), a(r), k'(r)$ to a file to be used later in the numerical calculation of the potential energy density.

The easiest way to find $k(r)$ is to use $k(r) = e^{-r}$ near infinity to move away from the fixed point at $k(\infty) = 1$. In the program infinity is approximated by 34 as e^{-35} is practically zero. We use double and quadruple precission where necessary in the programs to achieve the required precission. The program uses the NAG routine D02QBK integrator to integrate back from 34 to 0. Between $r = 0$ and $r = 5$ we output values as described above at intervals of 0.1. The interval $[0, 5]$ was chosen after some initial experimentation. Once we have all the required values the program E-DENSITY generates the points $(x, y, E(x, y))$ for a 3D graphics program.

B.3 Program Listings

B.3.1 SOLUTION

```
      program solve
c             -----
c
      implicit none
c
      real*16 alpha,a1val,a2val,f1val,f2val
      real*8 fspace,r0,rinf,r,av1,av2,fv
      integer ispace
c
      dimension fspace(100000),ispace(6000)
c
      common /alpha/alpha
      common /colspace/fspace,ispace
      common /soln/a1val,a2val,f1val,f2val
      common /rbegin/r0
      common /rend/rinf
c
c     r0 is chosen so that r**2 >> r**6
c     then A(r)=r**2/8                  - See Taylor series for A(r)
c
      r0=1.0d-8
c
c     Here infinity is approximated by 100.0 (!!!). This is justified
c     by observing the values for A and A', which are identically 1.0 and 0.0,
c     from the output of D02RAF, at 100.0.
c
      rinf=100.0d0
c
      call get_alpha
      call collocate_fa
      call get_k
c
      end
c
c*****************************************************************************
c
      subroutine get_alpha
c                ---------
c
      implicit none
c
      real*16 alpha,r,a1val0,a2val0
      real*8 dr,da1val,da2val
c
      real*16 hdelta,halpha,p,f1p,f2p,a,fa,b,fb,
     &a1val,a2val,f1val,f2val,fun1,fun2,ff1,ff2,hf1,hf2
c
      common /alpha/alpha
      common /soln/a1val,a2val,f1val,f2val
c
      fun1()=a1val0-a1val
      fun2()=a2val0-a2val
c
c At r=0.1:
c      alpha= 2.361459634210712q-1
c
c      A(1) = 1.249997680814110D-03   A(2) = 2.499986093580511D-02
c      F(1) = 2.358509655613953D-03   F(2) = 4.711123048028515D-02
c
      dr=0.1d0
      call get_a_val(dr,da1val,da2val)
c
```

```
      r=qext(dr)
      a1val0=qext(da1val)
      a2val0=qext(da2val)
c
c Use first terms from Taylor series to get initial estimate for ALPHA
      alpha=qsqrt(-24.0q0*(a1val0-r*r/8.0q0)/r**6)
c
      hdelta=1.0q-18
      halpha=0.01q0
c
      a=alpha-halpha
      b=alpha+halpha
      call taylor_fa(r,a)
      fa=fun1()
      call taylor_fa(r,b)
      fb=fun1()
c
      p=1.0q0
      f1p=1.0q0
      f2p=1.0q0
c
      hf1=1.0q0
      hf2=1.0q0
c
      write(*,'(/1x,a)')'          Alpha               A1VAL0-A1VAL
     & A2VAL0-A2VAL    F1-Diff      F2-Diff'
c
      do while((qabs(f1p).gt.hdelta).or.(qabs(f2p).gt.hdelta).or.
     &         (qabs(hf1).gt.hdelta).or.(qabs(hf2).gt.hdelta))
         ff1=f1val
         ff2=f2val
c
         p=(a+b)/2.0q0
         call taylor_fa(r,p)
c
         f1p=fun1()
         f2p=fun2()
         hf1=f1val-ff1
         hf2=f2val-ff2
c
         if ((f1p*fa).gt.0.0q0) then
            a=p
            fa=f1p
         else
            b=p
            fb=f1p
         end if
         write(*,'(1x,1pd25.18,4(3x,1pd10.3))')p,f1p,f2p,hf1,hf2
      end do
c
      alpha=p
c
      write(*,'(/1x,a,1pd11.4)')'At r =',r
      write(*,'(2(1x,a,1pd22.15),a)')'A(1) =',a1val0,'  A(2) =',a2val0,
     &' - From D02RAF'
      write(*,'(/2(1x,a,1pd22.15))')'A(1) =',a1val,'  A(2) =',a2val
      write(*,'(2(1x,a,1pd22.15))')'F(1) =',f1val,'  F(2) =',f2val
c
      end
c
c*******************************************************************************
c
```

```
      subroutine taylor_fa(r,alpha)
c                -------------------
c
      implicit none
c
      real*16 r,alpha
c
      real*16 f,a,fv,av,fvd,avd,hf,ha,hfd,had,sum
      integer i,n,it
c
      common /soln/av,avd,fv,fvd
c
      dimension f(0:1000),a(0:1000)
c
      f(0)=0.0q0
      f(1)=0.0q0
      f(2)=alpha
      f(3)=0.0q0
      f(4)=-f(2)/8.0q0
c
      a(0)=0.0q0
      a(1)=0.0q0
      a(2)=1.0q0/8.0q0
      a(3)=0.0q0
      a(4)=0.0q0
c
      fv=f(2)*r*r+f(4)*r**4
      av=a(2)*r*r
c
      fvd=2.0q0*f(2)*r+4.0q0*f(4)*r**3
      avd=2.0q0*a(2)*r
c
      n=4
c
      ha=1.0q0
      hf=1.0q0
c
      it=0
c
      do while((n.lt.1000).and.(it.lt.20))
         n=n+2
         sum=0.0q0
         do i=2,(n-4)
            sum=sum+f(i)*f(n-i-2)
         end do
         a(n)=-sum/(4.0q0*qfloat(n))
c
         sum=0.0q0
         do i=2,(n-2)
            sum=sum+f(i)*a(n-i)
         end do
         f(n)=-2.0q0*sum/(qfloat(n)-2.0q0)
c
         hf=f(n)*r**n
         ha=a(n)*r**n
c
         hfd=qext(n)*f(n)*r**(n-1)
         had=qext(n)*a(n)*r**(n-1)
c
         fv=fv+hf
         av=av+ha
c
```

```
         fvd=fvd+hfd
         avd=avd+had
c
         if ((qabs(ha).lt.qabs(av)*1.0q-22).and.
     &                        (qabs(had).lt.qabs(avd)*1.0q-22).and.
     &       (qabs(hf).lt.qabs(fv)*1.0q-22).and.
     &                        (qabs(hfd).lt.qabs(fvd)*1.0q-22)) then
            it=it+1
         else
            it=0
         end if
      end do
c
      end
c
c*******************************************************************************
c
      subroutine get_a_val(rcheck,a1val,a2val)
c                -----------------------------
c
      implicit none
c
      real*8 rcheck,a1val,a2val
c
      real*8 rbreak,deleps,tol,r0,h,hmesh,abt(2),work(310100),
     &r(10000),y(2,10000),rinf
      integer i,ifail,ijac,init,j,liwork,mnp,n,np,numbeg,nummix,lwork
      integer iwork(60100)
c
      common /rbegin/r0
      common /rend/rinf
c
      external afcn,abcs,d02gaz,d02gay,d02gax
c
      tol=1.0e-18
      lwork=310100
      liwork=60100
      mnp=10000
      n=2
      np=1002
      numbeg=1
      nummix=0
c
      rbreak=20.0d0
c
      r(1)=0.0d0
      r(np)=rinf
c
      h=(rbreak-r(1))/dfloat(np/2-1)
      hmesh=h
c
      do i=1,np/2
         r(i)=r(1)+dfloat(i-1)*h
         if (r(i).lt.dsqrt(8.0d0)) then
            y(1,i)=r(i)*r(i)/8.0d0
            y(2,i)=2.0d0*r(i)/8.0d0
         else
            y(1,i)=1.0d0
            y(2,i)=0.0d0
         end if
      end do
c
```

```
      h=(r(np)-rbreak)/dfloat(np/2)
c
      do j=1,(np/2+1)
         i=j-1+np/2
         r(i)=rbreak+dfloat(j-1)*h
         y(1,i)=1.0d0
         y(2,i)=0.0d0
      end do
c
      do i=1,np
         if (r(i).ge.rcheck) then
            if (dabs(r(i)-rcheck).gt.hmesh/4.0d0) then
               do j=np,i,-1
                  r(j+1)=r(j)
               end do
               np=np+1
            end if
            r(i)=rcheck
            h=dabs(r(i))*1.0d-12
            goto 1
         end if
      end do
c
 1    r(1)=r0
c
      init=1
      deleps=0.0d0
      ijac=0
      ifail=111
c
      call d02raf(n,mnp,np,numbeg,nummix,tol,init,r,y,
     &2,abt,afcn,abcs,ijac,d02gaz,d02gay,deleps,d02gaz,d02gax,
     &work,lwork,iwork,liwork,ifail)
c
      do i=1,np
         if (dabs(r(i)-rcheck).lt.h) then
            a1val=y(1,i)
            a2val=y(2,i)
            goto 2
         end if
      end do
c
 2    return
c
      end
c
c**********************************************************************************
c
      subroutine afcn(r,eps,y,f,n)
c                -----------------
c
      real*8 f(n),y(n)
      real*8 eps,r
      integer n
c
      f(1)=y(2)
      f(2)=5.0d0*y(2)/r-4.0d0*y(1)*y(2)/r+y(1)-1.0d0
c
      return
      end
c
c**********************************************************************************
```

```
c
      subroutine abcs(eps,ya,yb,bc,n)
c                ---------------------
c
      real*8 eps
      real*8 ya(n),yb(n),bc(n)
      integer n
c
      real*8 r0
c
      common /rbegin/r0
c
      bc(1)=ya(1)-r0*r0/8.0d0
      bc(2)=yb(1)-1.0d0
c
      return
      end
c
c*******************************************************************************
c
      subroutine collocate_fa
c                ------------
c
      implicit none
c
      real*8 fspace,aleft,aright,zeta(2),tol(2),z(2),u(2),err(2),
     &rbreak,f1bc,a1bc,rinf
      integer i,ncomp,ispace,m(2),ipar(11),ltol(2),iflag
c
      dimension fspace(100000),ispace(6000)
c
      common /colspace/fspace,ispace
      common /fa_bcval/f1bc,a1bc
      common /rend/rinf
c
      external fafcn,jacfa,fabcs,fajacbcs,dummy
c
      rbreak=0.1d0
      call get_fa(rbreak,f1bc,a1bc,.true.)
c
      ncomp=2
      m(1)=1
      m(2)=1
      aleft=rbreak
      aright=rinf
      zeta(1)=aleft
      zeta(2)=aright
      do 10 i=1,11
 10      ipar(i)=0
      ipar(1)=1
      ipar(2)=7
      ipar(3)=50
      ipar(4)=2
      ipar(5)=100000
      ipar(6)=6000
      ipar(7)=-1
      do 20 i=1,2
         ltol(i)=i
 20      tol(i)=1.0d-17
      call colsys(ncomp,m,aleft,aright,zeta,ipar,ltol,tol,
     &dummy,ispace,fspace,iflag,fafcn,jacfa,fabcs,fajacbcs,dummy)
c
```

```
      if (iflag.ne.1) then
         write (*,'(/1x,a,i2)')'COLSYS: The value of iflag is ',iflag
         stop
      endif
c
      return
      end
c
c***********************************************************************
c
      subroutine dummy
c                ----- 
c
      end
c
c***********************************************************************
c
      subroutine fafcn(r,z,f)
c                ------------
c
      real*8 r,z(2),f(2)
c
      f(1)=2.0d0*z(1)*(1.0d0-z(2))/r
      f(2)=-r*(z(1)*z(1)-1.0d0)/4.0d0
c
      return
      end
c
c***********************************************************************
c
      subroutine jacfa(r,z,df)
c                -------------
c
      real*8 z(2),df(2,2),r
c
      df(1,1)=2.0d0*(1-z(2))/r
      df(1,2)=-2.0d0*z(1)/r
      df(2,1)=-z(1)*r/2.0d0
      df(2,2)=0.0d0
c
      return
      end
c
c***********************************************************************
c
      subroutine fabcs(i,z,g)
c                ------------
c
      implicit none
c
      integer i
      real*8 z(2),g
c
      real*8 f1bc,a1bc
c
      common /fa_bcval/f1bc,a1bc
c
      goto(1,2),i
c
 1    g=z(1)-f1bc
      return
c
```

```
 2    g=z(2)-1.0d0
      return
c
      end
c
c**********************************************************************************
c
      subroutine fajacbcs(i,z,dg)
c                ----------------
c
      implicit none
c
      real*8 z(2),dg(2)
      integer i
c
      integer j
c
      do 10 j=1,2
 10      dg(j)=0.0d0
c
      goto(1,2),i
c
 1    dg(1)=1.0d0
      return
c
 2    dg(2)=1.0d0
      return
c
      end
c
c**********************************************************************************
c
      subroutine get_fa(r,fv,av,usetaylor)
c                ---------------------------
c
      implicit none
c
      real*8 r,fv,av
c
      real*16 alpha,a1val,a2val,f1val,f2val
      real*8 fa(2),fspace
      integer ispace
      logical*1 usetaylor
c
      dimension fspace(100000),ispace(6000)
c
      common /soln/a1val,a2val,f1val,f2val
      common /colspace/fspace,ispace
      common /alpha/alpha
c
      if ((r.le.0.1d0).or.usetaylor) then
         call taylor_fa(qext(r),alpha)
         fv=dbleq(f1val)
         av=dbleq(a1val)
      else
         call appsln(r,fa,fspace,ispace)
         fv=fa(1)
         av=fa(2)
      end if
c
      return
      end
```

```
c
c******************************************************************************
c
      subroutine get_k
c                -----
c
      implicit none
c
      real*8 tol,r,rend,fa(2),cin(7),y(2),f(2),cout(16),comm(5),
     &const(5),w(2,22),pw(2,2),fspace,fv,av,rlast,rstart
      integer i,ifail,j,mped,n,nout,iw,iw1,ispace
c
      dimension fspace(100000),ispace(6000)
c
      common /colspace/fspace,ispace
c
      external kfcn,dummy
c
      rstart=5.0d0
c
      open(unit=1,status='unknown',file='rkfa.dat')
c
c     Here infinity is approximated by 34.0 (!!!!!!). This is necessary since
c     if r>=35.0 exp(-r) is effectively zero (10**(-16)), which causes problems
c     for the integrator.
c
      r=34.0d0
      y(1)=dexp(-r)
      y(2)=-dexp(-r)
c
      n=2
      iw=2
      iw1=22
      mped=0
      tol=1.0d-18
      rend=1.0d-4
      ifail=1
c
      do i=1,5
         cin(i)=0.0d0
      end do
      do i=1,5
         comm(i)=0.0d0
         const(i)=0.0d0
      end do
      cin(1)=1.0d0
      comm(4)=1.0d0
c
 10   call d02qbf(r,rend,n,y,cin,tol,kfcn,comm,const,cout,mped,
     &dummy,pw,w,iw,iw1,ifail)
c
      if (r.gt.rstart) then
         rlast=r
         ifail=1
         goto 10
      end if
c
      if ((rend.lt.r).and.(ifail.eq.0)) then
         if (dabs(r-rlast).ge.0.01d0) then
            rlast=r
            call get_fa(r,fv,av,.false.)
            write(1,'(5(1x,1pd18.11))')r,y(1),fv,av,y(2)
```

```
            ifail=1
            goto 10
         else
            ifail=1
            goto 10
         end if
      end if
c
      if (ifail.ne.0) then
         write(*,'(1x,a,i2)')'IFAIL from D02QBF is ',ifail
         write(*,'(1x,a,1pd11.4)')'The value of cin(1) is ',cin(1)
         stop
      end if
c
      return
      end
c
c*****************************************************************************
c
      subroutine kfcn(r,k,f)
c                ------------
c
      implicit none
c
      real*8 r,k(2),f(2)
c
      real*8 fv,av
c
      call get_fa(r,fv,av,.false.)
c
      f(1)=k(2)
      f(2)=-k(2)/r+(fv*fv+4.0d0/(r*r))*k(1)
c
      return
      end
c
c*****************************************************************************
```

B.3.2 E-DENSITY

```
      program edensity
c
      implicit none
c
      real*8 eden,q,pi,twopi,f,a,k1,k2,r,density,hq,t
      integer j,nq,itr
      logical*1 use
c
      pi=4.0d0*datan(1.0d0)
c
      open(unit=3,status='old',file='rkfa.dat',readonly)
      open(unit=4,status='unknown',file='test12.dat')
c
      write(4,'(2(1x,i2))')83,83
c
      twopi=2.0d0*pi
      nq=83
      hq=twopi/dfloat(nq-1)
c
      use=.false.
      itr=5
```

```fortran
      do while(.not.use)
         use=.true.
         read(3,*,end=1)r,k1,f,a,k2
         use=.false.
1        itr=itr+1
         if ((itr.eq.6).or.use) then
            itr=0
            t=1.0d0/2.0d0
            do j=1,nq
               q=dfloat(j-1)*hq
               density=eden(r,q,f,a,k1,k2,t)
               write(4,'(3(1x,1pd14.7))')r*dcos(q),r*dsin(q),density
            end do
         end if
      end do
c
      stop
      end
c
c*****************************************************************************
c
      real*8 function eden(r,q,f,a,k1,k2,t)
c
      implicit none
c
      real*8 r,q,f,a,k1,k2,t
c
      real*8 term(10)
      integer i
c
      term(1)=4.0d0*f*f*(1.0d0-a)**2/(r*r)
      term(2)=8.0d0*(k1*a*f*t/r)**2
      term(3)=2.0d0*t*t*(k2*f+2.0d0*k1*f*(1.0d0-a)/r)**2
      term(4)=-4.0d0*(k2+2.0d0*k1/r)*t*t*k2*f*f*dsin(2.0d0*q)
     &*dsin(2.0d0*q)
      term(5)=2.0d0*t*t*(k2+2.0d0*k1/r)**2*(f*f+4.0d0*f*f*k1
     &*t*dcos(2.0d0*q)+4.0d0*k1*k1*f*f*t*t)
      term(6)=dcos(2.0d0*q)*f*f*k1*t*(2.0d0*dcos(2.0d0*q)*f*f
     &*k1*t+f*f-1.0d0)
      term(7)=-8.0d0*dcos(2.0d0*q)*f*f*t*(k2+2.0d0*k1/r)*
     &(2.0d0*a*k1*k1*t*t/r-
     &k1*t*dcos(2.0d0*q)/r+2.0d0*k1*a*t*dcos(2.0d0*q)/r+(a-1.0d0)/r)
      term(8)=(f*f-1.0d0)**2/8.0d0
      term(9)=(f*f+4.0d0*f*f*k1*t*dcos(2.0d0*q)+(2.0d0*t*f*k1)**2
     &-1.0d0)**2/8.0d0
      term(10)=16.0d0*k1*a*f*f*dcos(2.0d0*q)*(a-1.0d0)*t/(r*r)
c
      eden=0.0d0
      do i=1,10
         eden=eden+term(i)
      end do
c
      return
      end
c
c*****************************************************************************
```

Bibliography

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