SCHWARZ DOMAIN DECOMPOSITION METHODS FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Applied Mathematical Sciences is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Contents

Abstract

We study iterative numerical methods, based on Schwarz-iterative techniques and Shishkin meshes, for reaction-diffusion and convection-diffusion problems. We introduce the criteria of (ε, N)-uniform convergent numerical approximations. We examine the convergence of the numerical approximations with respect to the dimension of the discrete problem and the number of iterations. It is shown that the techniques used to design an (ε, N)-uniform numerical method for reaction-diffusion problems are not applicable to convection-diffusion problems. A systematic analysis of several variants of Schwarz, including overlapping and non-overlapping methods using different boundary conditions, was undertaken for one dimensional convection-diffusion problems. The convergence behaviour and the iteration counts were examined. Unlike the reaction-diffusion problem, it is shown that the methods using uniform meshes in each subomain do not meet all the $(\varepsilon, N)-$uniform convergence criteria. In the case of the convection-diffusion problems, it is demonstrated analytically and numerically that these iterative methods are convergent and have low computational cost for small values of the singular perturbation parameter ε. We feel it is of importance that the methods can be extended to higher dimensions with sufficient ease. As an example of this, we extend a non-overlapping method to a two dimensional convection-diffusion problem. The analysis of this method illustrates an appropriate domain structure and the need for sharp bounds on the partial derivatives. Finally, it is shown that an overlapping Schwarz method, using uniform subdomains, can be used to produce (ε, N)-uniform convergence for a time dependent problem with parabolic boundary layers. Numerical results are presented for the methods studied.

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Chapter 1

Introduction

The subject of this thesis is an investigation of Schwarz domain decomposition methods applied to singularly perturbed differential equations.

It is our purpose to analyse the convergence properties of Classical Schwarz iterative methods used in conjunction with appropriate Shishkin fitted meshes.

In this Introduction, we give a short overview of this field of study and a brief summary of the dissertations main findings and content.

1.1 Domain decomposition methods for differential equations

With the arrival of supercomputers and parallel computing, domain decomposition methods for partial differential equations has become an area of increasing interest in recent years. Underlying this surge in interest is the need to develop parallel algorithms for the large scale problems arising in physics and engineering.

The earliest know domain decomposition method is believed to have been discovered in 1869 by Hermann Amandus Schwarz, [26]. Schwarz devised the method for elliptic equations, to establish the existence of harmonic functions on regions with nonsmooth boundaries.

The Schwarz algorithm partitioned the solution domain into two overlapping regions, on which he produced a sequence of functions, defined on the union of the subdomains, converging to the harmonic function satisfying the given boundary conditions.

Today, the Schwarz algorithm provides an effective platform for numerically solving partial differential equations. A discussion of domain decomposition methods for partial differential equations is given in the books [23], [33], and also in the proceedings of the International Symposium on Domain Decomposition Methods for Partial Differential Equations [10].

When devising numerical methods for singularly perturbed problems, difficulties can arise, which often depend on the geometry of the domain. Therefore, it is of interest to reduce the solution of the original problem to that of the set of problems in subdomains with simpler geometries; also the numerical method can be locally adapted to any singularity in the solution that arises in a specific subdomain.

1.2 Singularly perturbed model problems

Singularly perturbed partial differential equations with a small parameter (denoted here by ε) multiplying the highest derivative arise in many areas of engineering, for example, in the modeling of semi-conductor devices, heat transfer, and computational fluid dynamics (see, for example, Morton [20]).

There is extensive information on numerical methods for singularly perturbed problems in the literature [30],[18],[6],[25] and [20]. In this section, we introduce four characteristic model problems. Let $\Omega = (0,1)$ and $G = \Omega \times (0,T]$. Consider the following four problems.

1. One-dimensional convection-diffusion

$$\begin{aligned} -\varepsilon u'' + au' &= f \quad \text{on} \quad \Omega \\ u(0) = \gamma_0, \qquad & u(1) = \gamma_1 \\ a > \alpha > 0 \quad \text{in} \quad & \bar{\Omega} \quad \text{and} \quad 0 < \varepsilon \leq 1 \end{aligned} \tag{1.1}$$

2. One-dimensional reaction-diffusion

$$\begin{aligned} -\varepsilon u'' + bu &= f \quad \text{on} \quad \Omega \\ u(0) = \gamma_0, \qquad & u(1) = \gamma_1 \\ b > \beta > 0 \quad \text{in} \quad & \bar{\Omega} \quad \text{and} \quad 0 < \varepsilon \leq 1 \end{aligned} \tag{1.2}$$

3. Time dependent convection-diffusion

$$\begin{aligned} -\varepsilon u'' + au' + d\dot{u} &= f \quad \text{on} \quad G \\ u(0,t) = \gamma_0(t), \quad u(1,t) &= \gamma_1(t), \quad u(x,0) = \varphi(x) \\ d > \delta > 0, \quad a > \alpha > 0 \quad \text{on} \quad & \bar{G} \quad \text{and} \quad 0 < \varepsilon \leq 1 \end{aligned} \tag{1.3}$$

4. Time dependent reaction-diffusion

$$\begin{aligned} -\varepsilon u'' + bu + d\dot{u} &= f \quad \text{on} \quad G \\ u(0,t) = \gamma_0(t), \quad u(1,t) &= \gamma_1(t), \quad u(x,0) = \varphi(x) \\ d > \delta > 0, \quad b > \beta > 0 \quad \text{on} \quad &\bar{G} \quad \text{and} \quad 0 < \varepsilon \leq 1 \end{aligned} \tag{1.4}$$

where the functions $a, b, d, f, \gamma_0, \gamma_1, \varphi$ comply with the assumption of sufficient smoothness and the functions $\varphi, \gamma_0, \gamma_1$ are sufficiently compatible to guarantee the solutions of (1.1)-(1.4) are smooth. Note that (1.3) and (1.4) are the time-dependent analogies of (1.1) and (1.2). Note also that α, β, δ are constants.

Singularly perturbed problems are characterized by the perturbation parameter ε. For small values of ε, steep gradients appear in the solution of these problems. In the problems we are investigating these gradients appear in the boundary region, and are called boundary layers.

Problem (1.1) is called the convection-diffusion problem and is characterized by the existence of a first derivative term. Only one initial condition may be imposed on the limiting (reduced) solution of (1.1),

$$au_0' = f,$$

when ε is set to 0. The radical difference between the solution of (1.1) and the solution of its reduced problem will mean a boundary layer appears near $x = 1$. Problem (1.2) is called the reaction-diffusion problem. It is characterized by the absence of a first derivative term. No boundary conditions can be imposed on the reduced problem and boundary layers will appear at $x = 0$ and $x = 1$.

Problems (1.3) and (1.4) are the time dependent counterparts of (1.1) and (1.2) respectively, in that, (1.1) and (1.2) are steady state problems associated with $\dot{u} = 0$

$(du/dt = 0)$. The boundary layers which arise on the lateral sides of the rectangle G, are determined by the characteristics of the reduced solution $(\varepsilon = 0)$. Away from the corners of the domain a boundary layer is of either regular or parabolic type. A layer arising in a corner region is known as a corner layer. In Problem (1.3), these characteristics are not parallel to the boundary and a regular boundary layer appears near the wall at $x = 1$. The characteristics in Problem (1.4) are parallel to the lateral sides of the rectangle, G and layers of parabolic type arise along these sides.

1.3 Non-iterative numerical methods for singularly perturbed problems

For a singular perturbation problem, an appropriate norm for studying the convergence behaviour of numerical solutions is the maximum norm, which is defined by

$$||\psi||_{\bar{\Omega}} = \max_{x \in \bar{\Omega}} |\psi(x)|.$$

The associated seminorms, defined for each integer $k \geq 0$, are

$$|\psi|_k = ||\psi^{(k)}||.$$

In designing a numerical method to approximate the solution of a singular perturbation problem we are interested in obtaining approximations which converge to the true solution independently of the parameter ε. This is known as ε-uniform convergence and can be formally defined for (1.1) and (1.2) as follows.

Definition 1.3.1 *Suppose that*

$$\bar{\Omega}^N = \{x_i\}_0^N, \quad 0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$$

is a set of mesh points and $U_\varepsilon^N(x_i)$ *is an approximation to* $u_\varepsilon(x_i)$. *Let* $\bar{U}_\varepsilon^N$ *be an interpolant of* U_ε^N. *Then the sequence of functions* $\{\bar{U}_\varepsilon^N\}_{N=2}^{\infty}$ *converges* **ε-uniformly** *to* u_ε *if there exists* N_0 *such that for all* $N \geq N_0$

$$\sup_{0<\varepsilon\leq 1} ||\bar{U}_\varepsilon^N - u_\varepsilon||_{\bar{\Omega}} \leq CN^{-p}, \quad p > 0$$

and C, p *and* N_0 *are independent of* ε *and* N.

An analogous definition can be given for (1.3) and (1.4).

Classical numerical methods can satisfy an error bound of the form

$$||\bar{U} - u_\varepsilon||_{\bar{\Omega}} \leq C(\varepsilon)N^{-p}, \quad p > 0,$$

where $C(\varepsilon)$ is unbounded as $\varepsilon \to 0$. Such a method is convergent but not ε-uniformly convergent.

There have been two common approaches taken to obtain ε-uniform convergence, *Fitted Operator Methods* and *Fitted Mesh Methods*. The former uses special fitted operators on a uniform mesh and, while achieving some success, a powerful negative result by Shishkin [28] states that no fitted operator method using a uniform mesh can be guaranteed to give ε-uniform convergence when applied to a class of parabolic problems of the form (1.4). The fitted mesh methods use standard finite difference operators applied on a suitable fitted mesh. Shishkin has developed comprehensive theory to support fitted mesh methods which demonstrate ε-uniform convergence for a large class of singularly perturbed problems (we refer to these meshes as the Shishkin meshes), see Shishkin [30]. The Shishkin mesh can also be extended to higher dimensions and is easy to implement. This mesh is piecewise uniform with specially defined transition points separating course and fine mesh regions, in which appropriate proportions of the mesh points are placed.

1.4 Iterative numerical methods for singularly perturbed problems

In this thesis, we apply some of the theory for Shishkin meshes to the Schwarz decomposition method. The Schwarz domain decomposition iterative procedure partitions the solution domain into overlapping or non-overlapping subdomains and, on each of these, the problem is solved separately using an appropriate algorithm, where some appropriate interface conditions transfer solution values between subdomains at each iteration. The main advantages of using a domain decomposition method for a singularly perturbed problem are as follows;

1. An appropriately chosen algorithm can be implemented on each subdomain.
2. The division of the solution domain makes it possible to deal with complex domain structures.

When an iterative numerical method is employed, both the discretization error and the iterations should be examined as functions of the small parameter ε. We now give the definition of (ε, N)-uniform convergence.

Definition 1.4.1 *Suppose that*

$$\bar{\Omega}^N = \{x_i\}_0^N, \quad 0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$$

is a set of mesh points and $U_\varepsilon^{N,k}(x_i)$ is an approximation to $u_\varepsilon(x_i)$, generated by some iterative process. Let $\bar{U}_\varepsilon^{N,k}$ be an interpolant of $U_\varepsilon^{N,k}$. Then the functions $\{\bar{U}_\varepsilon^{N,k}\}_{N=2,k=1}^{\infty}$ converge <u>(ε, N)**-uniformly**</u> *to u_ε if there exists N_0 such that for all*

$N \geq N_0$

$$\sup_{0<\varepsilon\leq 1} \|\bar{U}_\varepsilon^{N,k} - u_\varepsilon\|_{\bar{\Omega}} \leq CN^{-p} + Cq^k, \quad p > 0,\, 0 \leq q < 1,$$

and C, p, q and N_0 are independent of ε and N. Here k denotes the iteration parameter.

We will say that an iterative method is ε-uniform if C, p and N_0 are independent of ε and N and q is independent of ε. Certain Schwarz methods will be seen to be ε-uniform but not (ε, N)-uniform. That is, C, p, q and N are independent of ε but $q \rightarrow 1$ as $N \rightarrow \infty$. Some other methods may be neither ε-uniform nor (ε, N)-uniform.

When designing a Schwarz method we request that our method fulfils the following criteria.

1. Simplicity of the method and possible extension to higher dimensions.
2. (ε, N)-uniform convergence.

With respect to the simplicity of the method, it is preferable to use uniform meshes where possible. Of course, it is imperative that the method produces ε-uniform approximations (Definition 1.3.1), and for a method to be (ε, N)-uniformly convergent we stipulate in Definition 1.4.1, that an iterative method should be computationally economic, that is the number of iterations required for convergence is independent of N and ε. These criterion will be used in accessing the effectiveness of a numerical method in subsequent chapters.

In the context of iterative methods, Garbey [7] and Garbey and Kaper [8] examined discrete Schwarz methods for singularly perturbed problems. In their methods, the number of mesh points is inversely proportional to the size of the singular perturbation parameter ε, and so these methods are not ε-uniform. Boglaev [1] examined a non-overlapping Schwarz method for a time-dependent singularly perturbed analogue of Problem (1.1), using a standard finite difference operator on a special piecewise-uniform mesh. However, this method is not ε-uniform, as the restriction $\varepsilon N \leq 1$ is imposed on the method. Boglaev and Sirotkin [2] and Farrell et al. [4] examined Schwarz methods for singularly perturbed semi-linear analogous of Problem (1.2), using a complicated fitted finite difference operator with special non-uniform meshes on the subdomains. In their methods, strong restrictions are placed on the distribution of the nodes that do not permit the use of a uniform mesh in each subdomain.

In Chapter 2, the Schwarz approach to the reaction-diffusion problem (1.2) is examined and it is shown that the numerical solution of an overlapping Schwarz method, based on a standard finite difference operator with a uniform mesh in each subdomain, converges (ε, N)-uniformly to the exact solution when the position of the subdomains is chosen using Shishkin transition points. The appropriate decomposition of the solution and bounds on derivatives are given, and the continuous and discrete Schwarz methods are examined. Numerical results are presented which agree with the theoretical error results.

In chapter 3, the Schwarz approach to the convection-diffusion problem (1.1) is examined. Both Mathew [16] and Nataf and Rogier [21] examined the theoretical convergence properties of the continuous, but not the discrete, Schwarz methods for singularly perturbed problems. Numerical computations are presented in this thesis which conclusively show that the overlapping Schwarz method with uniform meshes fails to produce (ε, N)-uniform convergent approximations. In fact, the numerical results show that, when the Shishkin transition points are used, the error contained in the approximations is unacceptably large, for small values of ε. This is surprising and, we feel, an important result which highlights a difference in Problems (1.1) and (1.2) when using Schwarz methods, and which also reveals completely different convergence behaviour for the continuous and discrete methods.

In chapter 4, we present some alternative Schwarz methods which are designed to address the difficulties encountered in the Schwarz approach to the convection-diffusion problem. These include using special meshes, non-overlapping subdomains and non-iterative algorithms. Each method is introduced, and its advantages, drawbacks and possible applications discussed. We present theoretical results for the convergence behaviour and numerical computations which agree with the theoretical error estimates. The methods are judged using criterion of an efficient Schwarz domain decomposition method, that is we require the numerical solutions to be (ε, N)-uniformly convergent and it would be preferable to use uniform meshes in each subdomain.

In chapter 5, we extend one of the methods, introduced in Chapter 4, to a two dimensional convection-diffusion problem with regular boundary layers . The two dimensional problem contains extra complexity. The decomposition of the solution contains more layer components than the one dimensional case. The subdomains interface along edges, and in the analysis of this method, it becomes imperative to use the Shishkin bounds on partial derivatives and mixed partial derivatives. An appropriate placement of the subdomains is also determined. Numerical computations are presented.

In chapter 6, we analyse a Schwarz overlapping method with uniform subdomains applied to the parabolic problem (1.4). A general result for this method is presented in Shishkin [31]. Here, we analysis the method using similar techniques to those applied in the previous chapters and verify that (ε, N)-uniform convergence can be achieved. This result confirms that this class of equations does not present the difficulties seen in the Schwarz approach to the convection-diffusion class. The negative result of Shishkin, that no fitted operator method on a uniform mesh can be guaranteed to converge uniformly in ε, makes this an interesting result because by using a Schwarz approach we can retrieve uniformity of the meshes and retain the (ε, N)-uniform convergence.

Notation: Throughout this thesis, the letter C denotes a generic constant that is independent of the singular perturbation parameter ε, the discretization parameter N and the Schwarz iteration counter k.

Chapter 2

A Schwarz method for reaction-diffusion problems

2.1 Introduction

In this chapter, we present an overlapping Schwarz method based on a standard finite difference operator using uniform meshes in each subdomain. This method is similar to the one proposed in Shishkin [30], where various theoretical results were announced. However, no detailed proofs, no consideration of numerical results and no iteration counts were provided in [30]. Here, we put forward a detailed proof of the parameter-uniform convergence of the method and numerical results, validating this theoretical result, are presented. Iteration counts are presented and their dependence on ε is discussed. The material in this chapter has appeared in [15].

On $\Omega = (0,1)$, we consider the following class of singularly perturbed reaction–

diffusion problems

$$L_\varepsilon u_\varepsilon(x) \equiv -\varepsilon u_\varepsilon''(x) + b(x)u_\varepsilon(x) = f(x), \quad x \in \Omega, \tag{2.1a}$$

$$u_\varepsilon(0) = A, \quad u_\varepsilon(1) = B, \tag{2.1b}$$

$$b(x) > \beta > 0 \qquad \text{for all } x \in \bar{\Omega}, \tag{2.1c}$$

where the functions satisfy $b,\ f \in C^2(\overline{\Omega})$ and the singular perturbation parameter ε satisfies $0 < \varepsilon \leq 1$.

An outline of this chapter is as follows. In Section 2.2 the solution is decomposed into smooth and singular components. Parameter explicit bounds on derivatives are given. In Section 2.3, the continuous Schwarz method is introduced and error bounds are given. In Section 2.4, the discrete Schwarz method is described and bounds on the difference between the discrete Schwarz iterates and the continuous solution are derived. This leads to the main theoretical result of the paper: a parameter–uniform error estimate in the maximum norm of the discrete Schwarz iterates. In Section 2.5 results of a series of numerical experiments are presented which demonstrate the theoretical estimates derived in the earlier sections.

2.2 The continuous problem

The reduced problem corresponding to (2.1) is the problem $b(x)v_0(x) = f(x)$ whose solution $v_0(x) = f(x)/b(x)$ cannot be made to satisfy arbitrary preassigned boundary conditions at the boundary points $\{0, 1\}$ of Ω. Thus, in general, u_ε exhibits boundary layer behaviour at these points, the width of the boundary layers being $0(\sqrt{\varepsilon})$ (see, for example, [3] or [18]). It is well known that L_ε satisfies the following

Comparison Principle. *Let $a, b \in \overline{\Omega}$ and assume that $\psi(a) \geq 0$ and $\psi(b) \geq 0$. Then $L_\varepsilon \psi(x) \geq 0$ for all $x \in (a, b)$ implies that $\psi(x) \geq 0$ for all $x \in [a, b] \subset \overline{\Omega}$.*

The uniqueness and stability of the solution u_ε of (2.1) are immediate consequences of this comparison principle.

We state without proof the following lemma which gives classical ε–explicit bounds on the derivatives of the solution of Problem 2.1. A proof of this lemma is given in [18].

Lemma 2.2.1 *Let ψ_ε be the solution of the problem*

$$L_\varepsilon \psi_\varepsilon = f, \quad x \in (a, b) \subset \Omega,$$

where $\psi_\varepsilon(a), \psi_\varepsilon(b)$ are given and $|\psi_\varepsilon(a)|, |\psi_\varepsilon(b)| \leq C$. Then, for all k, $0 \leq k \leq 4$,

$$|\psi_\varepsilon^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k/2}\left(e^{-(x-a)\sqrt{\beta/\varepsilon}} + e^{-(b-x)\sqrt{\beta/\varepsilon}}\right)\right), \quad x \in [a, b],$$

where C is a constant independent of ε.

In what follows, we make extensive use of the following decomposition of the solution u_ε. We write $u_\varepsilon = v_\varepsilon + w_l + w_r$ where the smooth component v_ε and singular components w_l, w_r are defined to be the solutions of the problems

$$\begin{aligned} L_\varepsilon v_\varepsilon &= f, \;\; v_\varepsilon(0) = f(0)/b(0), \;\; v_\varepsilon(1) = f(1)/b(1), \\ L_\varepsilon w_l &= 0, \;\; w_l(0) = u_\varepsilon(0) - v_\varepsilon(0), \;\; w_l(1) = 0, \\ L_\varepsilon w_r &= 0, \;\; w_r(0) = 0, \;\; w_r(1) = u_\varepsilon(1) - v_\varepsilon(1). \end{aligned}$$

This decomposition enables us to establish non–classical sharper ε–explicit bounds on the derivatives of the solution of Problem 2.1. These are contained in the following

Lemma 2.2.2 *[17] The solution u_ε of Problem 2.1 can be written in the form*

$$u_\varepsilon = v_\varepsilon + w_l + w_r,$$

where, for all k, $0 \leq k \leq 4$,

$$|v_\varepsilon|_k \leq C(1 + \varepsilon^{1-k/2}),$$

and, for all $x \in \bar{\Omega}$,

$$|w_l^{(k)}(x)| \leq C\varepsilon^{-k/2} e^{-x\sqrt{\beta/\varepsilon}}, \quad |w_r^{(k)}(x)| \leq C\varepsilon^{-k/2} e^{-(1-x)\sqrt{\beta/\varepsilon}},$$

where C is a constant independent of ε.

2.3 Continuous Schwarz method

We now describe a continuous Schwarz method for Problem 2.1, which is an iterative process generating a sequence of iterates $u_\varepsilon^{[k]}$, which converge as $k \to \infty$ to the exact solution u_ε. First, we introduce three overlapping subdomains of Ω

$$\Omega_c = (\sigma, 1-\sigma), \quad \Omega_l = (0, 2\sigma), \quad \Omega_r = (1-2\sigma, 1),$$

where the subdomain parameter σ is an appropriate constant, specified in Section 2.4, which satisfies

$$0 < \sigma \leq 0.25.$$

Then for each integer $k \geq 0$, the Schwarz iterates $u_\varepsilon^{[k]}$ are then defined as follows. For $k = 0$ we put

$$u_\varepsilon^{[0]}(x) \equiv 0, \ 0 < x < 1, \quad u_\varepsilon^{[0]}(0) = u_\varepsilon(0), \ u_\varepsilon^{[0]}(1) = u_\varepsilon(1).$$

and for all $k \geq 1$

$$u_\varepsilon^{[k]} = \begin{cases} u_c^{[k]} \text{ in } \overline{\Omega}_c\ , \\ u_i^{[k]} \text{ in } \overline{\Omega}_i \setminus \Omega_c, \quad i = l, r, \end{cases}$$

where the $u_i^{[k]}$ are the solutions of the problems

$$L_\varepsilon u_i^{[k]} = f \quad \text{in } \Omega_i, \quad u_i^{[k]} = u_\varepsilon^{[k-1]} \quad \text{on } \partial\Omega_i, \quad i = l, r$$

and

$$L_\varepsilon u_c^{[k]} = f \quad \text{in } \Omega_c, u_c^{[k]}(\sigma) = u_l^{[k]}(\sigma), \quad u_c^{[k]}(1-\sigma) = u_r^{[k]}(1-\sigma).$$

The parameter–uniform convergence of these Schwarz iterates to u_ε is established in the following lemma. This is a well–known result (see, for example, [4], [18], [16]).

Lemma 2.3.1 *For all $k \geq 1$*

$$\|u_\varepsilon^{[k]} - u_\varepsilon\|_{\overline{\Omega}} \leq Cq^k\ ,$$

where C is a constant independent of k and ε and

$$q = e^{-\sigma\sqrt{\beta/\varepsilon}} < 1.$$

2.4 Discrete Schwarz method

The discrete Schwarz method is obtained from the continuous Schwarz method by using a uniform mesh Ω_i^N, $i = c, l, r$ on each subdomain Ω_i and replacing the differential operator L_ε by the standard centred finite difference operator L_ε^N. For any mesh function Z, on a uniform mesh with N subintervals L_ε^N is given by

$$L_\varepsilon^N Z_i = -\varepsilon\delta^2 Z_i + b(x_i)Z_i, \quad \delta^2 Z_i = N^2(Z_{i+1} - 2Z_i + Z_{i-1}).$$

For each $j,\ 0 \le j \le 2$ at each point of Ω_j associated with any mesh function Z defined on Ω_j^N, we define the piecewise linear interpolant $\bar{Z}_j$.

Method 2.1 *The sequence of discrete Schwarz iterates* $\bar{U}_\varepsilon^{[k]}$ *is defined by*

$$\bar{U}_\varepsilon^{[0]}(x) \equiv 0, 0 < x < 1, \quad U_\varepsilon^{[0]}(0) = u_\varepsilon(0),\ U_\varepsilon^{[0]}(1) = u_\varepsilon(1).$$

For $k \geq 1$ *the iterates* $\bar{U}_\varepsilon^{[k]}$ *are defined by*

$$\bar{U}_\varepsilon^{[k]} = \begin{cases} \bar{U}_c^{[k]} \ in\ \overline{\Omega}_c\ , \\ \bar{U}_i^{[k]} \ in\ \overline{\Omega}_i \setminus \Omega_c, \quad i = l, r\ , \end{cases}$$

where the $\bar{U}_i^{[k]}$ *are the solutions of the problems*

$$\begin{aligned} L_\varepsilon^N U_i^{[k]} &= f \quad \text{in } \Omega_i^N, \quad U_i^{[k]} = \bar{U}_\varepsilon^{[k-1]} \quad \text{on } \partial\Omega_i^N, \quad i = l, r\ , \\ L_\varepsilon^N U_c^{[k]} &= f \quad \text{in } \Omega_c^N, \quad U_c^{[k]}(\sigma) = \bar{U}_l^{[k]}(\sigma), \quad U_c^{[k]}(1-\sigma) = \bar{U}_r^{[k]}(1-\sigma)\ , \end{aligned}$$

and $\bar{U}$ *is the linear interpolant of* U.

Note that the centred finite difference operator L_ε^N satisfies the following discrete comparison principle in $\bar{\Omega}_j^N,\ j = l, c, r$.

Discrete Comparison Principle. *Assume that* $\Psi_0 \geq 0$ *and* $\Psi_N \geq 0$. *Then* $L_\varepsilon^N \Psi_i \geq 0$ *for all* $x_i \in \Omega_j^N,\ j = l, c, r$ *implies that* $\Psi_i \geq 0$ *for all* $x_i \in \bar{\Omega}_j^N,\ j = l, c, r$.

An immediate consequence of this is the following parameter–uniform stability result for L_ε^N. Let Z_i be any mesh function on $\Omega_j^N,\ j = l, c, r$. Then for all $i,\ 0 \le i \le N$,

$$|Z_i| \le (1/\beta) \max_{1 \le j \le N-1} |L_\varepsilon^N Z_j| + \max\{Z_0, Z_N\}.$$

In order that the convergence properties of the discrete Schwarz method are parameter–uniform, we take the subdomain parameter σ to be

$$\sigma = \min\{1/4, 2\sqrt{\varepsilon/\beta} \ln N\}. \tag{2.2}$$

The discrete Schwarz iterates are now decomposed in an analogous way to u_ε. Thus we write

$$U_\varepsilon^{[k]} = V_\varepsilon^{[k]} + W_l^{[k]} + W_r^{[k]}.$$

Each term of $U_\varepsilon^{[k]}$ in the sequence of discrete Schwarz approximations is decomposed as follows,

$$\bar{U}_\varepsilon^{[k]} = \bar{V}_\varepsilon^{[k]} + \bar{W}_l^{[k]} + \bar{W}_r^{[k]} = \begin{cases} \bar{V}_c^{[k]} + \bar{W}_{l,c}^{[k]} + \bar{W}_{r,c}^{[k]} \text{ in } \overline{\Omega}_c \ , \\ \bar{V}_i^{[k]} + \bar{W}_{l,i}^{[k]} + \bar{W}_{r,i}^{[k]} \text{ in } \overline{\Omega}_i \setminus \Omega_c \quad i = l, r \ , \end{cases}$$

where

$$\begin{aligned} L_\varepsilon^N V_i^{[k]} &= f \quad \text{in } \Omega_i^N, \quad V_i^{[k]} = \bar{V}_\varepsilon^{[k-1]} \quad \text{on } \partial\Omega_i^N, \quad i = l, r \ , \\ L_\varepsilon^N V_c^{[k]} &= f \quad \text{in } \Omega_c^N, \ V_c^{[k]}(\sigma) = \bar{V}_l^{[k]}(\sigma), \quad V_c^{[k]}(1-\sigma) = \bar{V}_r^{[k]}(1-\sigma) \ , \end{aligned}$$

and for W_l

$$\begin{aligned} L_\varepsilon^N W_{l,i}^{[k]} &= 0 \quad \text{in } \Omega_i^N, \quad W_{l,i}^{[k]} = \bar{W}_l^{[k-1]} \quad \text{on } \partial\Omega_i^N, \quad i = l, r \ , \\ L_\varepsilon^N W_{l,c}^{[k]} &= 0 \quad \text{in } \Omega_c^N, \ W_{l,c}^{[k]}(\sigma) = \bar{W}_{l,l}^{[k]}(\sigma), \quad W_{l,c}^{[k]}(1-\sigma) = \bar{W}_{l,r}^{[k]}(1-\sigma) \ , \end{aligned}$$

and for W_r

$$\begin{aligned} L_\varepsilon^N W_{r,i}^{[k]} &= 0 \quad \text{in } \Omega_i^N, \quad W_{r,i}^{[k]} = \bar{W}_r^{[k-1]} \quad \text{on } \partial\Omega_i, \quad i = l, r \ , \\ L_\varepsilon^N W_{r,c}^{[k]} &= 0 \quad \text{in } \Omega_c^N, \ W_{r,c}^{[k]}(\sigma) = \bar{W}_{r,l}^{[k]}(\sigma), \quad W_{r,c}^{[k]}(1-\sigma) = \bar{W}_{r,r}^{[k]}(1-\sigma). \end{aligned}$$

The sequences are started by taking

$$\begin{aligned} \bar{V}_\varepsilon^{[0]}(x) &\equiv 0, 0 < x < 1, \quad V_\varepsilon^{[0]}(0) = V_\varepsilon(0), \ V_\varepsilon^{[0]}(1) = V_\varepsilon(1) \ , \\ \bar{W}_l^{[0]}(x) &\equiv 0, 0 < x < 1, \quad W_l^{[0]}(0) = W_l(0), \ W_l^{[0]}(1) = 0 \ , \\ \bar{W}_r^{[0]}(x) &\equiv 0, 0 < x < 1, \quad W_r^{[0]}(0) = 0, \ W_r^{[0]}(1) = W_r(1). \end{aligned}$$

In the following two lemmas parameter–uniform error estimates of the iterates are established. The first lemma concerns the smooth components.

Lemma 2.4.1 *Let v_ε and $\bar{V}^{[k]}$ denote the regular components of u_ε and $\bar{U}^{[k]}$ respectively and let σ be chosen as in (2.2). Then, for all $k \geq 1$,*

$$\|v_\varepsilon - \bar{V}_\varepsilon^{[k]}\|_{\bar{\Omega}} \leq CN^{-2} + C2^{-k} \ ,$$

where C is a constant independent of k, N and ε.

Proof. Note that $(v_\varepsilon - V_l^{[1]})(0) = 0$ and $|(v_\varepsilon - V_l^{[1]})(2\sigma)| = |v_\varepsilon(2\sigma)| \leq C_0$. For $x_i \in \Omega_l^N$

$$\begin{aligned} |L_\varepsilon^N(v_\varepsilon - V_l^{[1]})(x_i)| &= |(L_\varepsilon^N - L_\varepsilon)(v_\varepsilon(x_i))| \leq C\varepsilon(2\sigma)^2 N^{-2}|v_\varepsilon|_4 \\ &\leq C_1 N^{-2}. \end{aligned}$$

Here we have used the following standard local truncation error estimate for $z \in C^4(x_{i-1}, x_i)$ and $x_{i+1} - x_i = x_i - x_{i-1} = CN^{-1}$, then

$$|\delta^2 z - z''| \leq (CN)^{-2}|z|_{4,(x_{i-1},x_{i+1})}$$

and Lemma 2.2.2. Consider the two mesh functions

$$\frac{C_0 x_i}{2\sigma} + \frac{C_1}{\beta}N^{-2} \pm (v_\varepsilon - V_l^{[1]})(x_i).$$

Then, from the discrete minimum principle, we get, for all $x_i \in \Omega_l^N$,

$$|(v_\varepsilon - V_l^{[1]})(x_i)| \leq \frac{C_0 x_i}{2\sigma} + \frac{C_1}{\beta}N^{-2}.$$

Likewise, for all $x_i \in \Omega_r^N$,

$$|(v_\varepsilon - V_r^{[1]})(x_i)| \leq \frac{C_0(1 - x_i)}{2\sigma} + \frac{C_1}{\beta}N^{-2}.$$

For all $x_i \in \Omega_c^N$, we obtain

$$|L_\varepsilon^N(v_\varepsilon - V_c^{[1]})(x_i)| = |(L_\varepsilon^N - L_\varepsilon)(v_\varepsilon(x_i))| \leq C\varepsilon N^{-2}|v_\varepsilon|_4 \leq C_1 N^{-2},$$

with

$$|(v_\varepsilon - V_c^{[1]})(\sigma)| = |(v_\varepsilon - V_l^{[1]})(\sigma)| \le \frac{C_0}{2} + C_1 N^{-2} ,$$

and

$$|(v_\varepsilon - V_c^{[1]})(1-\sigma)| = |(v_\varepsilon - V_r^{[1]})(1-\sigma)| \le \frac{C_0}{2} + C_1 N^{-2}.$$

Hence

$$|(v_\varepsilon - V_c^{[1]})(x_i)| \le C_1 N^{-2} + \frac{C_0}{2}, \quad x_i \in \Omega_c^N.$$

Consider now the second iteration. Observe that $(v_\varepsilon - V_l^{[2]})(0) = 0$ and $|(v_\varepsilon - V_l^{[2]})(2\sigma)| \le C_1 N^{-2} + \frac{C_0}{2}$. For $x_i \in \Omega_l^N$

$$|L_\varepsilon^N(v_\varepsilon - V_l^{[2]})(x_i)| \le C_1 N^{-2}.$$

Consider the two mesh functions

$$\left(\frac{C_0}{2}\right)\frac{x_i}{2\sigma} + \frac{C_1}{\beta}N^{-2} \pm (v_\varepsilon - V_l^{[2]})(x_i).$$

Then, from the discrete minimum principle, we get, for all $x_i \in \Omega_l^N$,

$$|(v_\varepsilon - V_l^{[2]})(x_i)| \le \left(\frac{C_0}{2}\right)\frac{x_i}{2\sigma} + \frac{C_1}{\beta}N^{-2}.$$

and for all $x_i \in \Omega_r^N$,

$$|(v_\varepsilon - V_r^{[2]})(x_i)| \le \left(\frac{C_0}{2}\right)\frac{(1-x_i)}{2\sigma} + \frac{C_1}{\beta}N^{-2}.$$

Hence

$$|(v_\varepsilon - V_c^{[2]})(\sigma)| \le \frac{C_1}{\beta}N^{-2} + \frac{C_0}{4} \quad \text{and} \quad |(v_\varepsilon - V_c^{[2]})(1-\sigma)| \le \frac{C_1}{\beta}N^{-2} + \frac{C_0}{4} ,$$

and thus

$$|(v_\varepsilon - V_c^{[2]})(x_i)| \le \frac{C_1}{\beta}N^{-2} + \frac{C_0}{4}, \quad x_i \in \Omega_c^N.$$

We now use the standard interpolation error estimate for linear interpolation. That is, if $z \in C^2(x_{i-1}, x_i)$ and $\bar{z}$ is the linear interpolant then

$$\|z - \bar{z}\|_{\infty,(x_{i-1},x_i)} \leq C(x_i - x_{i-1})^2 \|z''\|_{\infty,(x_{i-1},x_i)} ,$$

which leads to

$$\begin{aligned} \|v_\varepsilon - \bar{V}_l^{[k]}\|_{\bar{\Omega}_l} &\leq \|\bar{V}_l^{[k]} - \bar{v}_\varepsilon\| + \|\bar{v}_\varepsilon - v_\varepsilon\| \\ &\leq \|\bar{V}_l^{[k]} - \bar{v}_\varepsilon\| + CN^{-2}(2\sigma)^2|v_\varepsilon|_2 \\ &\leq C_0 2^{-k} + \frac{C_1}{\beta}N^{-2} + C_2 N^{-2}. \end{aligned}$$

This completes the proof. $\diamond$

The next lemma gives error estimates for the singular components.

Lemma 2.4.2 *Let w_l, w_r and $\bar{W}_l^{[k]}$, $\bar{W}_r^{[k]}$ denote the singular components of u_ε and $\bar{U}^{[k]}$ respectively and let σ be chosen as in (2.2). Then, for all $k \geq 1$ we have*

$$(i)\ \|w_l - \bar{W}_l^{[k]}\|_{\bar{\Omega}} \leq C(N^{-1}\ln N)^2, \quad (ii)\ \|w_r - \bar{W}_r^{[k]}\|_{\bar{\Omega}} \leq C(N^{-1}\ln N)^2,$$

where C is a constant independent of k, N and ε.

Proof. We give the proof of (i); the proof of (ii) is analogous. Consider first the case when $\sigma < 1/4$. For $x_i \in \Omega_l^N$,

$$\begin{aligned} |L_\varepsilon^N(w_l - W_{l,l}^{[1]})(x_i)| &= |(L_\varepsilon^N - L_\varepsilon)(w_l(x_i))| \leq C\varepsilon(2\sigma)^2 N^{-2}|w_l|_4 \\ &\leq C(2\sigma)^2 N^{-2}\varepsilon^{-1} \leq C(N^{-1}\ln N)^2. \end{aligned}$$

Hence, by the discrete minimum principle,

$$\|w_l - W_{l,l}^{[1]}\|_{\bar{\Omega}_l^N} \leq C(N^{-1}\ln N)^2.$$

Therefore

$$\|w_l - \bar{W}_{l,l}^{[1]}\|_{\bar{\Omega}_l} \le C(N^{-1}\ln N)^2 + CN^{-2}(2\sigma)^2|w_l|_2 \le C(N^{-1}\ln N)^2.$$

Likewise

$$\|w_l - \bar{W}_{l,r}^{[1]}\|_{\bar{\Omega}_r} \le C(N^{-1}\ln N)^2.$$

Note that

$$\begin{aligned} L_\varepsilon^N(W_{l,c}^{[1]}) = 0 \quad &, \qquad \text{on } \Omega_c^N \ , \\ |W_{l,c}^{[1]}(\sigma)| = |\bar{W}_{l,l}^{[1]}(\sigma)| \quad &\le \quad |w_l(\sigma)| + C(N^{-1}\ln N)^2 \ , \\ |W_{l,c}^{[1]}(1-\sigma)| = |\bar{W}_{l,r}^{[1]}(1-\sigma)| \quad &\le \quad |w_l(1-\sigma)| + C(N^{-1}\ln N)^2 \ , \end{aligned}$$

and we conclude that

$$|W_{l,c}^{[1]}|_{\bar{\Omega}_c^N} \le C(N^{-1}\ln N)^2.$$

Hence

$$\|w_l - \bar{W}_{l,c}^{[1]}\|_{\bar{\Omega}_c} \le \|w_l - \bar{w}_l\|_{\bar{\Omega}_c} + C(N^{-1}\ln N)^2.$$

Note that for any function z we have

$$\begin{aligned} |z - \bar{z}|_{(x_{i-1},x_i)} \quad &= \quad |\int_{x_{i-1}}^{x} z'(t)\ dt - (\int_{x_{i-1}}^{x_i} z'(t)\ dt)(x - x_{i-1})/(x_i - x_{i-1})| \\ &\le \quad |\int_{x_{i-1}}^{x_i} z'(t)\ dt| \ , \end{aligned}$$

and so, using Lemma 2.2.2, we have

$$\begin{aligned} \|w_l - \bar{w}_l\|_{(x_{i-1},x_i)} \quad &\le \quad |\int_{x_{i-1}}^{x_i} w_l'(t)\ dt| \le e^{-x_{i-1}\sqrt{\beta/\varepsilon}} \\ &\le \quad e^{-\sigma\sqrt{\beta/\varepsilon}} \le C(N^{-1}\ln N)^2 \quad \text{for } x_{i-1} \ge \sigma. \end{aligned}$$

We conclude that

$$\|w_l - \bar{W}_{l,c}^{[1]}\|_{\bar{\Omega}_c} \le C(N^{-1}\ln N)^2.$$

The proof is completed by an induction argument. For the case of $\sigma = 1/4$, use the argument in the previous lemma and note that $\sigma^2/\varepsilon \leq C(\ln N)^2$. $\diamond$

Combining this with Lemma 2.4.1 immediately yields the main theoretical result of the chapter, which is contained in

Theorem 2.4.1 *Let u_ε be the solution of Problem 2.1 and let $\{\bar{U}_\varepsilon^{[k]}\}$ be the set of discrete Schwarz iterates with σ chosen as in (2.2). Then, for all $k \geq 1$*

$$||u_\varepsilon - \bar{U}_\varepsilon^{[k]}||_{\bar{\Omega}} \leq C(N^{-1}\ln N)^2 + C2^{-k} ,$$

where C is a constant independent of k, N and ε.

2.5 Numerical results

Numerical results are presented in this section, which confirm the theoretical estimates established in the previous section. The discrete Schwarz method described in Section 2.1 is applied to two problems from Problem 2.1. For notational reasons, it is helpful to introduce the piecewise-uniform mesh $\overline{\Omega}_\varepsilon^N$ associated with the overlapping subdomains by

$$\overline{\Omega}_\varepsilon^N \equiv \overline{\Omega}_c^N \cup (\overline{\Omega}_l^N \setminus \overline{\Omega}_c) \cup (\overline{\Omega}_r^N \setminus \overline{\Omega}_c). \tag{2.3}$$

For both examples, the stopping criterion for the Schwarz iterations is taken to be

$$\max_{x_i \in \overline{\Omega}_\varepsilon^N} |U_\varepsilon^{[k]}(x_i) - U_\varepsilon^{[k-1]}(x_i)| \leq 10^{-8}.$$

Our first problem is the constant coefficient problem

$$-\varepsilon u_\varepsilon''(x) + u_\varepsilon(x) = 0, \quad x \in \Omega , \tag{2.4a}$$

$$u_\varepsilon(0) = u_\varepsilon(1) = 1. \tag{2.4b}$$

Its exact solution in closed form is easy to find, which means that the exact pointwise errors can be calculated. The discrete Schwarz iterates are computed on a sequence of meshes with $N = 8, 16, ...1024$ for $\varepsilon = 2^{-2p}$, $p = 0, 1, 2, ...29$. Estimates of the global error

$$||\bar{U}_\varepsilon^N - u_\varepsilon||_{\bar{\Omega}}$$

are obtained by evaluating

$$E_{\varepsilon,global}^N = \max_{x_i \in \Omega^*} |\bar{U}_\varepsilon^N(x_i) - u_\varepsilon(x_i)| \ ,$$

where $\Omega^* = \Omega_l^* \cup \Omega_c^* \cup \Omega_r^*$ and

$$\begin{aligned}
\Omega_l^* &= \{x_i | x_i = i\varepsilon/4096, 0 \leq i \leq 4096\} \ , \\
\Omega_c^* &= \{x_i | x_i = \varepsilon + i(1-2\varepsilon)/4096, 0 \leq i \leq 4096\} \ , \\
\Omega_r^* &= \{x_i | x_i = 1 - \varepsilon + i\varepsilon/4096, 0 \leq i \leq 4096\},
\end{aligned}$$

where U_ε^N is the final Schwarz iterate. We normally omit the superscript k on the final iterate and write simply U_ε^N. Note that Ω^* depends on ε, but not on N.

Estimates of the parameter–uniform global pointwise error are obtained from

$$E_{global}^N = \max_\varepsilon E_{\varepsilon,global}^N.$$

and estimates of the parameter-uniform order of convergence are computed for each N from

$$p_{global}^N = \log_2\Big(\frac{E_{global}^N}{E_{global}^{2N}}\Big).$$

The values of $E_{\varepsilon,global}^N, E_{global}^N$ and p_{global}^N for the discrete Schwarz method applied to problem (2.4) are given in Table 2.1. In this and all subsequent tables, the dots indicate that the intermediate computed values are essentially the same as the given values. It is clear from Table 2.1 that this method is parameter–uniform for prob-

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	1.90e-03	4.77e-04	1.19e-04	2.99e-05	7.47e-06	1.86e-06	4.59e-07	1.08e-07
2^{-2}	6.07e-03	1.55e-03	3.91e-04	9.83e-05	2.46e-05	6.16e-06	1.54e-06	3.81e-07
2^{-4}	1.42e-02	3.79e-03	9.71e-04	2.47e-04	6.21e-05	1.56e-05	3.90e-06	9.68e-07
2^{-6}	2.34e-02	6.82e-03	1.82e-03	4.72e-04	1.20e-04	3.01e-05	7.47e-06	1.90e-06
2^{-8}	2.61e-02	7.88e-03	2.28e-03	6.13e-04	1.58e-04	4.03e-05	1.01e-05	2.37e-06
2^{-10}	8.46e-02	2.60e-02	7.16e-03	1.87e-03	4.78e-04	1.21e-04	3.04e-05	7.61e-06
2^{-12}	8.99e-02	4.62e-02	2.00e-02	7.16e-03	1.87e-03	4.78e-04	1.21e-04	3.04e-05
2^{-14}	8.99e-02	4.62e-02	2.00e-02	7.70e-03	2.71e-03	9.02e-04	2.79e-04	8.80e-05
2^{-16}	8.99e-02	4.60e-02	1.98e-02	7.71e-03	2.69e-03	8.59e-04	2.62e-04	8.68e-05
2^{-18}	8.99e-02	4.57e-02	1.95e-02	7.70e-03	2.22e-03	8.55e-04	2.62e-04	7.29e-05
2^{-20}	8.99e-02	4.33e-02	1.95e-02	2.70e-03	2.24e-03	5.08e-04	1.74e-04	6.48e-05
2^{-22}	8.99e-02	3.75e-02	9.78e-03	2.72e-03	1.73e-03	4.96e-04	1.71e-04	6.48e-05
2^{-24}	2.49e-02	2.81e-02	9.78e-03	2.73e-03	1.38e-03	4.58e-04	1.45e-04	1.87e-05
2^{-26}	1.87e-02	9.21e-03	5.20e-03	1.76e-03	4.92e-04	1.23e-04	3.27e-05	1.39e-05
2^{-28}	1.87e-02	4.33e-03	1.01e-03	3.51e-04	1.19e-04	3.23e-05	8.73e-06	3.55e-06
2^{-30}	1.88e-02	4.34e-03	1.02e-03	2.46e-04	6.00e-05	1.46e-05	3.50e-06	7.78e-07
2^{-32}	1.88e-02	4.35e-03	1.03e-03	2.48e-04	6.07e-05	1.49e-05	3.60e-06	8.10e-07
2^{-34}	1.88e-02	4.35e-03	1.02e-03	2.47e-04	6.03e-05	1.47e-05	3.53e-06	8.19e-07
2^{-36}	1.88e-02	4.34e-03	1.02e-03	2.47e-04	6.00e-05	1.45e-05	3.44e-06	7.67e-07
2^{-38}	1.88e-02	4.34e-03	1.02e-03	2.46e-04	5.98e-05	1.44e-05	3.39e-06	7.41e-07
2^{-40}	1.88e-02	4.34e-03	1.02e-03	2.46e-04	5.98e-05	1.44e-05	3.37e-06	7.28e-07
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-58}	1.88e-02	4.34e-03	1.02e-03	2.46e-04	5.97e-05	1.44e-05	3.34e-06	7.16e-07
E^N_{global}	8.99e-02	4.62e-02	2.00e-02	7.71e-03	2.71e-03	9.02e-04	2.79e-04	8.80e-05
p^N_{global}	9.59e-01	1.21e+00	1.38e+00	1.51e+00	1.59e+00	1.69e+00	1.66e+00	1.67e+00

Table 2.1: Computed global maximum pointwise errors $E^N_{\varepsilon,global}$, E^N_{global} and parameter-uniform orders of convergence p^N_{global} for Method 2.1 applied to problem (2.4) for various values of ε and N.

lem (2.4). The computed double mesh order of convergence corresponding to an asymptotic convergence rate is $(N^{-1}\ln N)^2$ is

$$p^N_{asym} = \log_2\Big(\frac{(N^{-1}\ln N)^2}{(2N)^{-2}(\ln(2N))^2}\Big) = 2\Big(1-\log_2(\frac{\ln 2N}{\ln N})\Big),$$

which correspond closely to the computed orders of convergence given in the last row of Table 2.1.

Our second problem is the variable coefficient problem

$$-\varepsilon u''_\varepsilon(x) + (1+x^2)u_\varepsilon(x) = x^3, \quad x \in \Omega\ , \tag{2.5a}$$

$$u_\varepsilon(0) = u_\varepsilon(1) = 1. \tag{2.5b}$$

In this case, the exact solution is not used to estimate the numerical errors. Instead, the nodal errors and orders of convergence are estimated using the double mesh principle modified in accordance with the parameter–robust definition (see Farrell and Hegarty [5] for the nodal double mesh principle). The double mesh differences are defined by

$$D^N_\varepsilon \equiv \max_{x_i\in\overline{\Omega}^N_\varepsilon} |U^N_\varepsilon(x_i) - \bar{U}^{2N}_\varepsilon(x_i)|\ ,$$

and the parameter–uniform differences are defined by

$$D^N = \max_\varepsilon D^N_\varepsilon.$$

From these the parameter–uniform order of convergence is computed from

$$p^N = \log_2(\frac{D^N}{D^{2N}}).$$

The numerical errors are then estimated by using the Schwarz solution on the finest available mesh, corresponding to $N = 4096$, as an approximation to the exact solution in the expression for the error. The corresponding computed maximum pointwise error is taken to be

$$E^N_{\varepsilon,nodal} = \max_{x_i\in\overline{\Omega}^N_\varepsilon} |U^N_\varepsilon(x_i) - \bar{U}^{4096}_\varepsilon(x_i)|\ ,$$

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	9.01e-04	2.25e-04	5.62e-05	1.41e-05	3.51e-06	8.75e-07	2.16e-07	5.14e-08
2^{-2}	2.58e-03	6.52e-04	1.62e-04	4.06e-05	1.01e-05	2.53e-06	6.24e-07	1.48e-07
2^{-4}	4.83e-03	1.25e-03	3.07e-04	7.73e-05	1.92e-05	4.80e-06	1.19e-06	2.82e-07
2^{-6}	5.18e-03	1.47e-03	3.60e-04	9.10e-05	2.26e-05	5.66e-06	1.40e-06	3.33e-07
2^{-8}	4.42e-03	1.51e-03	3.85e-04	9.71e-05	2.43e-05	6.07e-06	1.50e-06	3.57e-07
2^{-10}	1.41e-02	3.74e-03	9.71e-04	2.45e-04	6.14e-05	1.53e-05	3.78e-06	9.01e-07
2^{-12}	1.51e-02	6.87e-03	2.80e-03	9.58e-04	2.41e-04	6.04e-05	1.49e-05	3.56e-06
2^{-14}	1.51e-02	6.82e-03	2.80e-03	1.03e-03	3.51e-04	1.14e-04	3.54e-05	1.00e-05
2^{-16}	1.51e-02	6.80e-03	2.80e-03	1.03e-03	3.49e-04	1.14e-04	3.55e-05	1.03e-05
2^{-18}	1.51e-02	6.79e-03	2.80e-03	1.03e-03	3.49e-04	1.14e-04	3.54e-05	1.02e-05
2^{-20}	1.51e-02	6.79e-03	2.80e-03	1.03e-03	3.49e-04	1.14e-04	3.54e-05	1.02e-05
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-58}	1.51e-02	6.78e-03	2.80e-03	1.03e-03	3.49e-04	1.14e-04	3.54e-05	1.02e-05
E^N_{nodal}	1.51e-02	6.87e-03	2.80e-03	1.03e-03	3.51e-04	1.14e-04	3.55e-05	1.03e-05
p^N	7.81e-01	1.06e+00	1.29e+00	1.43e+00	1.35e+00	1.55e+00	1.64e+00	1.64e+00

Table 2.2: Computed nodal maximum pointwise errors E_ε^N, E^N and parameter–uniform order of convergence p^N for Method 2.1 applied to problem (2.5) for various values of ε and N.

and, for each N we define the computed parameter–uniform pointwise error by

$$E^N_{nodal} = \max_{\varepsilon} E^N_{\varepsilon,nodal}.$$

Values of $E^N_{\varepsilon,nodal}$, E^N_{nodal} and p^N for the discrete Schwarz method applied to problem (2.5) are given in Table 2.2. They show experimentally that the method is parameter–uniform for problem (2.5). Iteration counts for various values of ε and N for the discrete Schwarz method applied to problem (2.5) are given in Table 2.3. We see that these iteration counts are essentially independent of N and decrease with decreasing values of ε.

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	25	25	25	25	25	25	25	25
2^{-2}	21	21	21	21	21	21	21	21
2^{-4}	15	15	15	15	15	15	15	15
2^{-6}	10	10	10	10	10	10	10	10
2^{-8}	6	6	6	6	6	6	6	6
2^{-10}	4	4	4	4	4	4	4	4
2^{-12}	3	3	3	3	3	3	3	3
2^{-14}	4	3	3	3	3	3	2	2
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
2^{-58}	4	3	3	3	3	3	2	2

Table 2.3: Iteration count for Method 2.1 applied to problem (2.5) for various values of ε and N.

2.6 Conclusions

In this chapter, a one-dimensional singularly perturbed reaction–diffusion problem was examined. It was shown that a suitably designed discrete Schwarz method, Method 2.1, gives approximations which converge (ε, N)-uniformly to the exact solution. This parameter-uniform convergence was shown to be essentially second order. Numerical results were presented, which show that, for a small value of the parameter ε, only a few iterations are required and that the number of iterations is independent of the number of mesh points used.

Chapter 3

Overlapping Schwarz methods with uniform meshes applied to convection-diffusion problems

3.1 Introduction

In this chapter, we investigate a Schwarz approach to the convection-diffusion class of problems, which is analogous to the parameter-robust method for reaction-diffusion problems discussed in chapter 2. This Schwarz method partitions the solution domain into two overlapping subdomains, in each of which a uniform mesh is placed, and uses simple Dirichlet conditions at the interfaces. A prelimenary version of the material in this chapter has appeared in [14].

On $\Omega = (0, 1)$, we consider the following class of singularly perturbed convection-diffusion problems

$$L_\varepsilon u_\varepsilon(x) \equiv -\varepsilon u_\varepsilon''(x) + a(x)u_\varepsilon'(x) = f(x), \quad x \in \Omega \tag{3.1a}$$

$$u_\varepsilon(0) = A, \quad u_\varepsilon(1) = B \tag{3.1b}$$

where the functions $a, f \in C^2(\Omega)$ and the singular perturbation parameter ε satisfies $0 < \varepsilon \leq 1$. It is also assumed that a satisfies the condition

$$a(x) > \alpha > 0 \quad \text{for all} \quad x \in \bar{\Omega}. \tag{3.1c}$$

For Problem 3.1, the Shishkin piecewise uniform fitted mesh method, introduced in [27], uses the transition point

$$\tau = \min\{1/3, \frac{\varepsilon}{\alpha} \ln N\}, \tag{3.2}$$

between fine and course mesh regions. We aim to incorporate the theory of this method into a Schwarz domain decomposition approach and we recall that, this approach was successful in the reaction-diffusion case described in Chapter 2.

In this chapter, two iterative methods are examined. The first uses subdomain interfaces positions which are independent of ε. We establish, by means of numerical experiments that, using arbitrary fixed interface positions, this method does not produce ε-uniform convergent approximations for Problem 3.1. The second method uses an ε-dependent overlap. However, we also demonstrate numerically that this discrete Schwarz method, which uses uniform meshes on overlapping subdomains, positioned using the parameter τ, also fails to produce ε-uniform error approximations.

We now briefly introduce the important aspects of this second method, which are discussed in detail in the course of this Chapter. In this method the solution domain, Ω is divided into two overlapping subdomains $\Omega_0 = (0, \xi^+)$ and $\Omega_1 = (\xi^-, 1)$. On each of these subdomains a uniform mesh is introduced, which we denote by Ω_0^N and Ω_1^N. The interior points ξ^-, ξ^+ are chosen to be

$$\xi^- = 1 - \tau, \qquad \xi^+ = 1 - 2\tau,$$

where τ is the Shishkin transition point given in (3.2). Therefore, the mesh size in the fine mesh subdomain Ω_1^N is refined sufficiently to accurately determine the large gradients present in the boundary layers. Dirichlet boundary conditions are applied at the interior interface points ξ^+ and ξ^-.

The success of the Schwarz iterative technique depends on how the initial error, introduced at the interior boundary points, is propagated during the iteration process. For this method the reduction in the initial error is shown to take place when the problem is solved on Ω_0 (using maximum principle and appropriate barrier functions). In the continuous Schwarz method the reduction is exponential over Ω_0 and therefore can be shown to be independent of the size of the overlap. Hence, the approximate solutions converge to the true solution independently of ε.

However, problems arise in the discretization of this Schwarz method. For small values of ε, the width of the interval $(\xi^-, 1)$ reduces and there may be no interior mesh points in Ω_0^N that are also present in the overlap region $\bar{\Omega}_0^N \cap \bar{\Omega}_1^N$. Therefore, this discrete method is unable to accurately determine the exponential error reduction achieved by the continuous method. In fact, in the discrete method, the error function behaves linearly, since the linear interpolant of the pointwise solution is used as a global approximation. The reduction in the error at each iteration is a function of the width of the overlap. Therefore, the number of iterations required by the method is inversely proportional to the size of the overlap, which is order ε. We also show that, an increase in the error occurs when solution values are passed between subdomains. This unwanted increase corrupts the discrete approximation. This is very bad news for this discrete method, since in order to capture the steep gradients the overlapping region must reduce but this contradicts the conditions necessary for convergence. This method therefore fails for small values of the parameter ε and large values of N. Also, the iteration numbers increase with the number of mesh points N and with decreasing values of ε. Consequently, this is not an appropriate method for this class of singularly perturbed convection-diffusion problems.

We now give an outline of the material in this chapter. In Section 3.2, we specify the decomposition of the solution of Problem 3.1 into its smooth and singular components. Bounds on the derivatives of the solution are given. We then describe, in Section 3.3, the continuous Schwarz method, stating the appropriate convergence results. In Section 3.4, we present numerical results for a classical discrete Schwarz approach, with overlap dependent on ε, to Problem 3.1. Then in Section 3.5, we describe the discrete Schwarz method and detail the problems which arise in a theoretical analysis of this method. Finally, in Section 3.5.1 we present numerical results which demonstrate the failure of this approach.

3.2 The continuous problem

The convection diffusion problem is non-self adjoint and only one initial condition may be imposed on the reduced solution of Problem 3.1,

$$\begin{cases} \text{Find } v_0 \in C^1(\bar{\Omega}) \text{ such that } v_0(0) = u_0 \\ \text{and, for all } x \in \Omega, \quad a(x)v_0'(x) = f(x) \end{cases}$$

Thus, in general, u_ε exhibits boundary layer behaviour at $x = 1$, the width of the boundary layers being $O(\varepsilon)$ (see, for example, [3] or [18]). It is well know that L_ε satisfies the following comparison principle.

Comparison Priniple *Let* $a, b \in \overline{\Omega}$ *and assume that* $\psi(a) \geq 0$ *and* $\psi(b) \geq 0$. *Then* $L_\varepsilon \psi(x) \geq 0$ *for all* $x \in (a, b)$ *implies that* $\psi(x) \geq 0$ *for all* $x \in [a, b] \subset \overline{\Omega}$.

The uniqueness and stability of the solution of Problem 3.1 are immediate consequences of the comparison principle. We state without proof the classical ε-explicit estimates on the derivatives of the solution of Problem 3.1. A proof of this lemma is given in [18].

Lemma 3.2.1 *Let* u_ε *be the solution of Problem 3.1. Then, for* $0 \leq k \leq 3$

$$|u_\varepsilon^{(k)}(x)| \leq C(1 + \varepsilon^{-k} e^{-\alpha(1-x)/\varepsilon}) \quad \textit{for all } x \in \Omega$$

where C *is a constant independent of* ε.

Throughout this chapter, we make extensive use of the following decomposition of the solution u_ε. We write $u_\varepsilon = v_\varepsilon + w_\varepsilon$ where the smooth component v_ε and singular component w_ε are defined to be the solutions of the problems

$$\begin{aligned} L_\varepsilon v_\varepsilon &= f, \quad v_\varepsilon(0) = u_0 - w_\varepsilon(0), \quad v_\varepsilon(1) = u_1 - w_\varepsilon(1) \\ L_\varepsilon w_\varepsilon &= 0, \quad w_\varepsilon(0) = w_\varepsilon(1) e^{-\alpha/\varepsilon} \end{aligned}$$

where $w_\varepsilon(1)$ is chosen so that the first and second derivatives of v_ε are bounded uniformly in ε. This decomposition enables us to establish non-classical ε-explicit bounds on the derivatives of the solution of Problem 3.1. These are contained in the following lemma with is derived in [18].

Lemma 3.2.2 *[18] The solution* u_ε *of Problem 3.1 can be written in the form*

$$u_\varepsilon = v_\varepsilon + w_\varepsilon,$$

where, for all k, $0 \le k \le 3$, *and all* $x \in \Omega$,

$$|v_\varepsilon^{(k)}(x)| \le C(1 + \varepsilon^{-(k-2)}e^{-\alpha(1-x)/\varepsilon})$$

and, for all $x \in \Omega$,

$$|w_\varepsilon^{(k)}(x)| \le C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon}$$

for some constant C *independent of* ε.

3.3 Continuous Schwarz method

We now describe the continuous iterative Schwarz method for Problem 3.1. This process generates a sequence of iterates $u_\varepsilon^{[k]}$, which converge as $k \to \infty$ to the exact solution u_ε. First we introduce the two subdomains of $\Omega = (0, 1)$, as shown in Fig. 3.1,

$$\Omega_0 = (0, \xi^+), \quad \Omega_1 = (\xi^-, 1)$$

where

$$0 < \xi^- < \xi^+ < 1.$$

The iterative process is defined as follows.

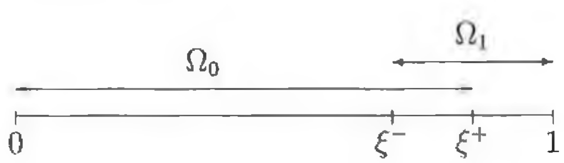

Figure 3.1: The subdomains Ω_0 and Ω_1 for the continuous overlapping Schwarz method

$$u_\varepsilon^{[0]}(x) \equiv 0,\ 0 < x < 1, \quad u_\varepsilon^{[0]}(0) = u_\varepsilon(0),\ u_\varepsilon^{[0]}(1) = u_\varepsilon(1).$$

For $k \geq 1$ the iterates $u_\varepsilon^{[k]}$ are defined by

$$u_\varepsilon^{[k]}(x) = \begin{cases} u_0^{[k]}(x), & x \in \bar{\Omega}_0 \setminus \bar{\Omega}_1 \\ u_1^{[k]}(x), & x \in \bar{\Omega}_1 \end{cases}$$

where

$$L_\varepsilon u_0^{[k]} = f \quad \text{in } \Omega_0, \quad u_0^{[k]}(\xi^+) = u_1^{[k-1]}(\xi^+), \quad u_0^{[k]}(0) = u_\varepsilon(0)$$

and

$$L_\varepsilon u_1^{[k]} = f \quad \text{in } \Omega_0, \quad u_1^{[k]}(\xi^-) = u_0^{[k]}(\xi^-), \quad u_1^{[k]}(1) = u_\varepsilon(1).$$

The following lemma establishes the parameter-robust convergence of these Schwarz iterates to the exact solution. The proof of this is given in [18]. A similar result for a continuous Schwarz approach is discussed in [7].

Lemma 3.3.1 *[18] Let u_ε be the solution of Problem 3.1 and let $\{u_\varepsilon^{[k]}\}_{k=1}^\infty$ be the sequence of Schwarz iterates. Then, for all $k \geq 1$,*

$$||u_\varepsilon^{[k]} - u_\varepsilon||_{\bar{\Omega}} \leq C q_1^k,$$

where C is independent of k and ε and

$$q_1 = e^{-\alpha(\xi^+ - \xi^-)/\varepsilon} < 1.$$

3.4 Classical discrete Schwarz approach with overlapping subdomains with interface positions independent of ε and N

In this section, we present a discrete classical Schwarz approach applied to a convection-diffusion equation. The numerical experiments are implemented on a pure layer problem (3.4), whose solution only contains the singular component w_ε. We choose this problem because any observations made on the convergence behaviour are due only to the singularly perturbed nature of the problem and not because of a complicated v_ε component. We will see later in Chapter 4, that it is necessary to take contrary precautions when examining methods that work well for a problem whose solution contains only w_ε terms.

We now describe a Schwarz method which uses simple Dirichlet boundary conditions at some arbitary fixed interface points, denoted by a and b.

Method 3.1 *Let the solution domain Ω be partitioned into the two overlapping subdomains*

$$\Omega_0 = (0, a) \quad \textit{and} \quad \Omega_1 = (b, 1), \quad \textit{where} \quad 0 < b < a < 1,$$

and $\bar{\Omega}_0^N = \{x_i\}_0^N$ be a uniform mesh on Ω_0 with $x_i = ia/N$ and $\bar{\Omega}_1^N = \{x_i\}_0^N$ be a uniform mesh on Ω_1 with $x_i = b + i(1-b)/N$. The exact solution u_ε is approximated by the limit $\bar{U}_\varepsilon$ of a sequence of discrete Schwarz iterates $\{\bar{U}_\varepsilon^{[k]}\}_{k=0}^{\infty}$, which are defined as follows. For each $k \geq 1$,

$$\bar{U}_\varepsilon^{[k]}(x) = \begin{cases} \bar{U}_0^{[k]}(x), & x \in \bar{\Omega}_0 \setminus \bar{\Omega}_1 \\ \bar{U}_1^{[k]}(x), & x \in \bar{\Omega}_1 \end{cases}$$

where $\bar{U}_i^{[k]}$ is the linear interpolant of $U_i^{[k]}$. Then for $k = 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[1]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[1]}(0) = u_0, \quad U_0^{[1]}(a) = 0, \\ L_\varepsilon^N U_1^{[1]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[1]}(b) = \bar{U}_0^{[1]}(b), \quad U_0^{[1]}(1) = u_1, \end{aligned}$$

and for $k > 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[k]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[k]}(0) = u_0, \quad U_0^{[k]}(a) = \bar{U}_1^{[k-1]}(a), \\ L_\varepsilon^N U_1^{[k]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[k]}(b) = \bar{U}_0^{[k]}(b), \quad U_0^{[k]}(1) = u_1. \end{aligned}$$

Analogous definitions can be made for the iterate components $V_\varepsilon^{[k]}$ and $W_\varepsilon^{[k]}$, where

$$U_\varepsilon^{[k]} = V_\varepsilon^{[k]} + W_\varepsilon^{[k]}.$$

The differential operator L_ε is replaced with the standard upwind difference operator L_ε^N defined for any mesh function Z_i by

$$L_\varepsilon^N Z_i = -\varepsilon \delta^2 Z_i + a(x_i) D^- Z_i, \tag{3.3a}$$

where

$$\delta^2 Z_i = \left(\frac{D^- Z_{i+1} - D^- Z_i}{(x_{i+1} - x_{i-1})/2} \right), \quad D^- Z_i = \left(\frac{Z_i - Z_{i-1}}{x_i - x_{i-1}} \right). \tag{3.3b}$$

For the purpose of the results tabulated in this section we choose the constants a and b to be

$$a = \frac{2}{3}, \qquad b = \frac{1}{3}.$$

The example problem is the constant coefficient problem

$$-\varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) = 0, \quad x \in \Omega \tag{3.4a}$$

$$u_\varepsilon(0) = 0, \quad u_\varepsilon(1) = 1. \tag{3.4b}$$

whose exact solution in closed form is easy to find. We define the uniform mesh $\bar{\Omega}_u^N$ associated with the overlapping subdomains by

$$\bar{\Omega}_u^N = \bar{\Omega}_1^N \cup (\bar{\Omega}_0^N \setminus \bar{\Omega}_1^N). \tag{3.5}$$

The stopping criterion for the Schwarz iterations is taken to be

$$\max_{x_i \in \overline{\Omega}_u^N} |U_\varepsilon^{[k]}(x_i) - U_\varepsilon^{[k-1]}(x_i)| \leq 10^{-9}. \tag{3.6}$$

The values presented in Table 3.1 are the maximum nodal pointwise errors, $E_{\varepsilon,nodal}^N$ defined by

$$E_{\varepsilon,nodal}^N = \max_{x_i \in \bar{\Omega}_u^N} |U_\varepsilon^N(x_i) - u_\varepsilon(x_i)|, \tag{3.7}$$

where U_ε^N is the final Schwarz iterate. We normally omit the superscript k on the final iterate and write simply U_ε^N. For a specific value of the mesh parameter N, the value in bold font in the appropriate column in Table 3.1 depicts the maximum nodal error, for all values of ε. The method is not ε-uniform. This type of behaviour for a classical non-iterative method on a uniform mesh method is discussed in detail in [6].

We can show that the numerical solution of Method 3.1 for Problem 3.1 converges to the solution of the numerical approximations generated by a non-iterative numerical method consisting of an upwinded finite difference operator on a uniform mesh.

Let U_ε be the numerical approximation for the solution of Problem 3.1, using the standard upwind difference operator L_ε^N, defined by (3.3), on a uniform mesh. For $k = 1$, on the subdomain $\bar{\Omega}_0^N$,

$$|(U_0^{[1]} - U_\varepsilon)(0)| = 0, \quad |(U_0^{[1]} - U_\varepsilon)(a)| = |-U_\varepsilon(a)|,$$

and $L_\varepsilon^N(U_0^{[1]} - U_\varepsilon) = 0$ on Ω_0^N. We choose the barrier function,

$$\Phi(x_i) = \frac{|U_\varepsilon(a)|x_i}{a},$$

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	4.78e-03	2.45e-03	1.24e-03	6.25e-04	3.14e-04	1.57e-04	7.86e-05	3.93e-05
2^{-1}	1.66e-02	8.71e-03	4.46e-03	2.26e-03	1.14e-03	5.71e-04	2.86e-04	1.43e-04
2^{-2}	4.69e-02	2.52e-02	1.31e-02	6.69e-03	3.38e-03	1.70e-03	8.51e-04	4.26e-04
2^{-3}	9.53e-02	5.36e-02	2.85e-02	1.47e-02	7.50e-03	3.78e-03	1.90e-03	9.52e-04
2^{-4}	1.65e-01	9.64e-02	5.40e-02	2.87e-02	1.48e-02	7.53e-03	3.80e-03	1.91e-03
2^{-5}	**2.03e-01**	1.65e-01	9.64e-02	5.40e-02	2.87e-02	1.48e-02	7.53e-03	3.80e-03
2^{-6}	1.53e-01	**2.03e-01**	1.65e-01	9.64e-02	5.40e-02	2.87e-02	1.48e-02	7.53e-03
2^{-7}	8.57e-02	1.53e-01	**2.03e-01**	1.65e-01	9.64e-02	5.40e-02	2.87e-02	1.48e-02
2^{-8}	4.48e-02	8.57e-02	1.53e-01	**2.03e-01**	1.65e-01	9.64e-02	5.40e-02	2.87e-02
2^{-9}	2.29e-02	4.48e-02	8.57e-02	1.53e-01	**2.03e-01**	1.65e-01	9.64e-02	5.40e-02
2^{-10}	1.16e-02	2.29e-02	4.48e-02	8.57e-02	1.53e-01	**2.03e-01**	1.65e-01	9.64e-02
2^{-11}	5.83e-03	1.16e-02	2.29e-02	4.48e-02	8.57e-02	1.53e-01	**2.03e-01**	1.65e-01
2^{-12}	2.92e-03	5.83e-03	1.16e-02	2.29e-02	4.48e-02	8.57e-02	1.53e-01	**2.03e-01**
2^{-13}	1.46e-03	2.92e-03	5.83e-03	1.16e-02	2.29e-02	4.48e-02	8.57e-02	1.53e-01
2^{-14}	7.32e-04	1.46e-03	2.92e-03	5.83e-03	1.16e-02	2.29e-02	4.48e-02	8.57e-02
2^{-15}	3.66e-04	7.32e-04	1.46e-03	2.92e-03	5.83e-03	1.16e-02	2.29e-02	4.48e-02
2^{-16}	1.83e-04	3.66e-04	7.32e-04	1.46e-03	2.92e-03	5.83e-03	1.16e-02	2.29e-02
2^{-17}	9.15e-05	1.83e-04	3.66e-04	7.32e-04	1.46e-03	2.92e-03	5.83e-03	1.16e-02
2^{-18}	4.58e-05	9.15e-05	1.83e-04	3.66e-04	7.32e-04	1.46e-03	2.92e-03	5.83e-03
2^{-19}	2.29e-05	4.58e-05	9.15e-05	1.83e-04	3.66e-04	7.32e-04	1.46e-03	2.92e-03
2^{-20}	1.14e-05	2.29e-05	4.58e-05	9.15e-05	1.83e-04	3.66e-04	7.32e-04	1.46e-03

Table 3.1: Maximum pointwise nodal errors $E^N_{\varepsilon,nodal}$ on $\bar{\Omega}^N_u$ for various values of ε and N for Method 3.1 applied to problem (3.4)

and by the discrete maximum principle for L_ε^N on Ω_0^N,

$$|(U_0^{[1]} - U_\varepsilon)(x_i)| \leq \frac{|U_\varepsilon(a)|x_i}{a} \quad \text{in} \quad \bar{\Omega}_1^N.$$

Now, on the subdomain $\bar{\Omega}_1^N$,

$$\begin{aligned} |(U_1^{[1]} - U_\varepsilon)(b)| &= |(U_0^{[1]} - U_\varepsilon)(b)| \\ &\leq |U_\varepsilon(a)|\frac{b}{a}, \\ |(U_1^{[1]} - U_\varepsilon)(1)| &= 0. \end{aligned}$$

By the maximum principle it follows that

$$|(U_1^{[1]} - U_\varepsilon)(x_i)| \leq |U_\varepsilon(a)|\frac{b}{a} \quad \text{in} \quad \bar{\Omega}_1^N,$$

and so,

$$|(U_\varepsilon^{[1]} - U_\varepsilon)(x_i)| \leq |U_\varepsilon(a)|\frac{b}{a} \quad \text{in} \quad \bar{\Omega}_\varepsilon^N.$$

By induction it then follows that,

$$|(U_\varepsilon^{[k]} - U_\varepsilon)(x_i)| \leq |U_\varepsilon(a)| \left(\frac{b}{a}\right)^k \quad \text{in} \quad \bar{\Omega}_u^N.$$

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	16	16	16	16	16	16	16	16
2^{-1}	15	15	15	15	15	15	15	15
2^{-2}	13	13	12	12	12	12	12	12
2^{-3}	10	9	8	8	8	8	8	8
2^{-4}	7	6	5	5	5	4	4	4
2^{-5}	4	4	3	3	3	3	2	2
2^{-6}	3	2	2	2	2	2	2	2
2^{-7}	3	2	2	2	2	2	2	2
2^{-8}	2	2	2	2	2	2	2	2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-20}	2	2	2	2	2	2	2	2

Table 3.2: Iteration counts for Method 3.1 applied to problem (3.4)

Therefore, the discrete Schwarz method converges to the solution of upwinding on a uniform mesh, if the subdomain interfaces positions are independent of ε. The rate of convergence is controlled by $q = b/a$. We also note that if the overlap is dependent on ε, for example $b = 1 - 2\tau$ and $a = 1 - \tau$, where $\tau = \frac{\varepsilon}{\alpha} \ln N$ then

$$\lim_{\varepsilon \to 0} \frac{b}{a} = \lim_{\varepsilon \to 0} \frac{1 - 2\tau}{1 - \tau} = 1,$$

and the method ceases to converge, which will have important consequences in the next section.

3.5 Discrete Schwarz method with ε dependent overlap

It would be hoped that the continuous Schwarz method is the limiting case for the corresponding discrete Schwarz method and that the convergence results for the discrete Schwarz method could be motivated by the continuous approach.

To avoid the convergence behaviour highlighted in Table 3.1 and to obtain ε-uniform approximations, we combine the Shishkin fitted mesh and the Schwarz iterative method.

We define the overlapping subdomains

$$\Omega_0 = (0, \xi^+), \qquad \Omega_1 = (\xi^-, 1),$$

where the constants ξ^+ and ξ^- satisfy

$$\xi^+ = 1 - \tau, \qquad \xi^- = 1 - 2\tau, \tag{3.8}$$

and τ is the Shishkin transition point defined by,

$$\tau = \min\{1/3, \frac{\varepsilon}{\alpha} \ln N\}. \tag{3.9}$$

The discrete Schwarz iteratives are defined as follows.

Method 3.2 *For each $k \geq 1$,*

$$\bar{U}_{\varepsilon}^{[k]}(x) = \begin{cases} \bar{U}_0^{[k]}(x) & x \in \bar{\Omega}_0 \setminus \bar{\Omega}_1 \\ \bar{U}_1^{[k]}(x) & x \in \bar{\Omega}_1 \end{cases}$$

where $\bar{U}_i^{[k]}$ is the linear interpolant of $\bar{U}_i^{[k]}$. Then for $k = 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[1]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[1]}(0) = u_0, \quad U_0^{[1]}(\xi^+) = 0, \\ L_\varepsilon^N U_1^{[1]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[1]}(\xi^-) = \bar{U}_0^{[1]}(\xi^-), \quad U_0^{[1]}(1) = u_1, \end{aligned}$$

and for $k > 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[k]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[k]}(0) = u_0, \quad U_0^{[k]}(\xi^+) = \bar{U}_1^{[k-1]}(\xi^+), \\ L_\varepsilon^N U_1^{[k]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[k]}(\xi^-) = \bar{U}_0^{[k]}(\xi^-), \quad U_0^{[k]}(1) = u_1. \end{aligned}$$

The difference operator L_ε^N is given by (3.3). We note here that for the continuous Schwarz method, from Lemma 3.3.1, the choice of ξ^- and ξ^+ stated above in (3.8) and (3.9) yields the convergence estimate, for all $k \geq 1$,

$$||u_\varepsilon^{[k]} - u_\varepsilon||_{\bar{\Omega}} \leq Cq^k, \quad q = \max\{e^{-\frac{\alpha}{3\varepsilon}}, N^{-1}\}$$

Thus, it is clear that the solution of the continuous Schwarz method converges independently of the width of the overlapping region.

In any theoretical analysis, based on comparison principle arguments, in both the continuous and the discrete methods, we observe that it is not possible to introduce a decreasing function as an upper bound on the approximation error in either subdomain. Therefore, it is only possible to show that any error reduction must take place when the method is applied in the subdomain Ω_0. This immediately indicates difficulties in the discrete method since the size of the error depends on the width of the overlap and here, the overlap region decreases with respect to Ω_0, as ε becomes smaller. Hence, we expect the iteration numbers to rise for small values of ε. The effect of this becomes more pronounced for very small values of ε since, as the width of the subdomain Ω_1^N reduces, there may be no grid nodes common to Ω_0^N and the overlap region. As a consequence, the interface condition $U_1^{[k]}(\xi^-) = \bar{U}_0^{[k]}(\xi^-)$ means that in this case the error reduction at $x = \xi^-$ is given by a linear interpolant. Therefore the error reduction which has been shown to be exponential in the continuous case is now only being interpreted as a linear reduction. This would imply the method requires a very large number of iterations.

However, as we will see in the numerical results, not only does the discrete method require an unacceptable large number of iterations for small values of ε, but also the approximations do not converge to the correct solution. To understand why this is we must examine the $V_\varepsilon^{[k]}$ and $W_\varepsilon^{[k]}$ components of the Schwarz iterates separately. The $V_\varepsilon^{[k]}$ component, although requiring large iterations, can be shown to be first order ε-uniform convergent using arguments similar to those in Lemma 2.4.1. It is with the singular component $W_\varepsilon^{[k]}$ that problems arise. This is now discussed.

In Ω_0^N, for $k \geq 1$, we may use the triangle inequality

$$|(W_0^{[k]} - w_\varepsilon)(x_i)| \leq |W_0^{[k]}(x_i)| + |w_\varepsilon(x_i)|$$

and together with maximum principle arguments, similar to those in Lemma 2.4.2, obtain bounds for $|W_0^{[k]}(x_i)|$ and $|w_\varepsilon(x_i)|$ separately. For simplicity we use the notation

$$||(W_0^{[k]} - w_\varepsilon)(x_i)|| = E_0.$$

However, in Ω_1^N, the solution component w_ε is not small, recall $|w_\varepsilon| \leq Ce^{-\alpha(1-x)/\varepsilon}$, and so we must examine $||W_1^{[k]} - w_\varepsilon||$. It is clear that,

$$\begin{aligned} |(W_1^{[k]} - w_\varepsilon)(1)| &= 0, \\ |(W_1^{[k]} - w_\varepsilon)(\xi^-)| &= |(\bar{W}_0^{[k]} - w_\varepsilon)(\xi^-)| \leq E_0. \end{aligned}$$

Now, following arguments similar to those given in [18],

$$|L_\varepsilon^N(W_1^{[k]} - w_\varepsilon)(x)| \leq C\varepsilon^{-2}N^{-1}\tau.$$

We choose a barrier function ϕ_i,

$$\phi_i = (x_i - (1-2\tau))C\varepsilon^{-2}\tau N^{-1} + E_0\Psi_i.$$

It can be seen that,

$$\begin{aligned} \phi_i \pm (W_1^{[k]} - w_\varepsilon)(\xi^-) &\geq 0, \quad \text{if} \quad \Psi_i(\xi^-) \geq 1, \\ \phi_i \pm (W_1^{[k]} - w_\varepsilon)(1) &\geq 0, \quad \text{if} \quad \Psi_i(1) \geq 0, \\ L_\varepsilon^N(\phi_i \pm (W_1^{[k]} - w_\varepsilon))(x_i) &\geq 0, \quad \text{if} \quad L_\varepsilon^N\Psi_i \geq 0. \end{aligned}$$

Therefore, using the discrete maximum principle $\Psi_i \geq 1$, and we obtain,

$$\begin{aligned} |(W_1^{[k]} - w_\varepsilon)(x_i)| &\leq (x_i - (1-2\tau))C\varepsilon^{-2}\tau N^{-1} + E_0 \\ &\leq 2CN^{-1}(\ln N)^2 + E_0. \end{aligned}$$

Consequently, for the $(k+1)^{\text{th}}$ iteration in Ω_0^N;

$$\begin{aligned} |(W_0^{[k+1]} - w_\varepsilon)(\xi^+)| &= |(W_1^{[k]} - w_\varepsilon)(\xi^+)| \\ &\leq (1-\tau-(1-2\tau))C\varepsilon^{-2}\tau N^{-1} \\ &= CN^{-1}(\ln N)^2 + E_0. \end{aligned}$$

In other words, an increase in the error contained in the approximating solution, with an upper bound of $CN^{-1}(\ln N)^2$, occurs once the method is applied in Ω_1.

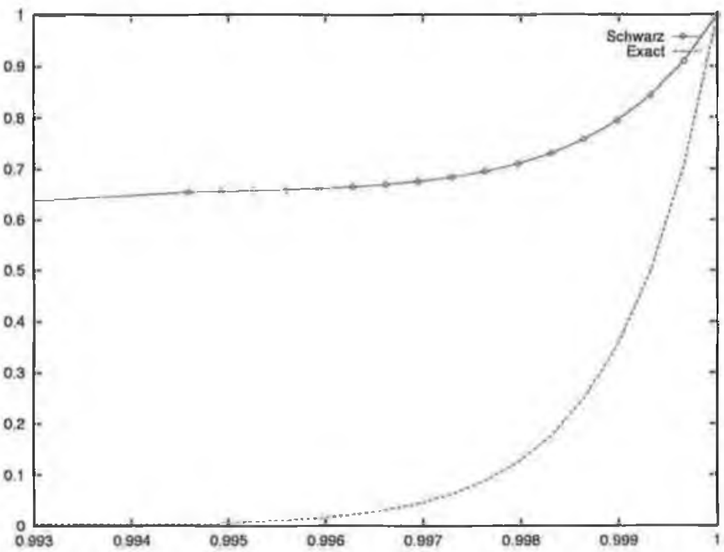

Figure 3.2: Comparison of the exact solution of problem (3.4) and the numerical solution given by Method 3.2 with $N = 16$ and $\varepsilon = 2^{-10}$

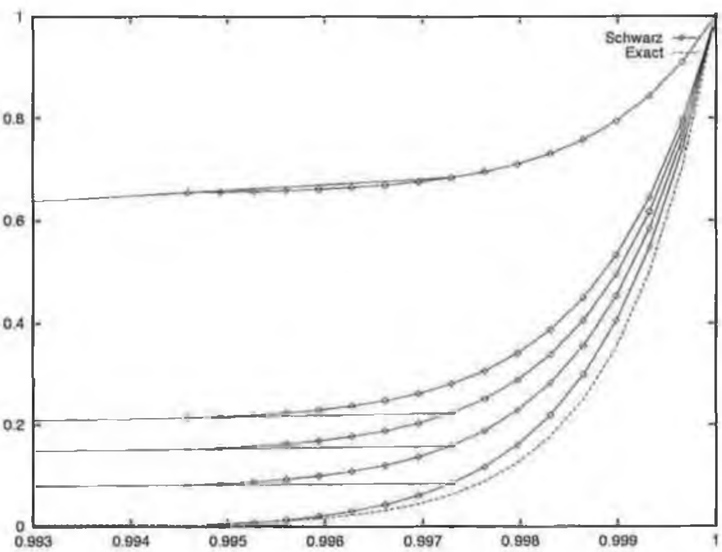

Figure 3.3: Comparison of the exact solution of problem (3.4) and the numerical solution at various iterations, $k = 1, 2, 3$, and 140, given by Method 3.2 with $N = 16$ and $\varepsilon = 2^{-10}$(within the layer region)

The above argument does not guarantee an increase, it simply suggests the possibility of such an increase. However, it can be seen in Fig. 3.2, that the numerical solution generated by Method 3.2 bears no relation to the exact solution near the boundary point $x = 1$. The accumulation of the error during the iterative process can be seen in Fig. 3.3, where the solutions of the initial four and final iterations in each subdomain are compared to the exact solution. It is clear from Fig. 3.3, that when Method 3.2 is implemented in the layer domain, $\Omega_1^N = (0.994585, 1)$ an increase in the error occurs and since the first interior mesh point in $\Omega_0^N = (0, 0.997292)$ does not lie within the overlap region, Fig. 3.4, the linear interpolant fails to sufficiently reduce this error when the method is solved in Ω_0^N.

Also, in the next section numerical results are presented which strongly indicate that this increase does occur when interface values are passed to Ω_1 and this is then passed back to the solution in Ω_0 during the iterative process. Thus results in a build up

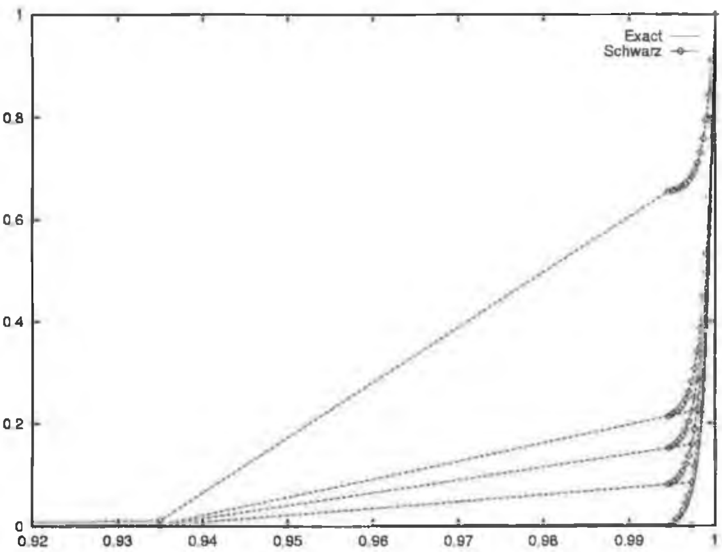

Figure 3.4: Comparison of the exact solution of problem (3.4) and the numerical solution at various iterations given by Method 3.2 with $N = 16$ and $\varepsilon = 2^{-10}$ (including last interior mesh point in Ω_0^N)

of error when the overlap region is small and is the cause for this discrete method converging to the wrong solution.

3.5.1 Numerical results

In this section, numerical results are given which demonstrate the convergence behaviour of the discrete method, discussed in the previous section. These computations are carried out on the model problem (3.4). The stopping criterion for the Schwarz iterations is given by (3.6). We define the piecewise uniform mesh $\bar{\Omega}_\varepsilon^N$ associated with the overlapping subdomains by

$$\bar{\Omega}_\varepsilon^N = \bar{\Omega}_1^N \cup (\bar{\Omega}_\varepsilon^N \setminus \bar{\Omega}_1^N). \tag{3.10}$$

For each ε and N, the maximum nodal error, $E_{\varepsilon,nodal}^N$, is computed using (3.7) and presented in Table 3.3. From this table it is immediately obvious that although the errors reduce for fixed ε, they are unacceptably large for small ε and in fact tend to 100% error. The iteration counts for this method, given in Table 3.4, also increase in proportion to N and ε^{-1}, illustrating the poor efficiency of this method. These numerical results indicate that the method proposed in Miller et al. [18] does not produce satisfactory approximations for the convection-diffusion Problem 3.1.

Note that the nodal errors in Table 3.3 are being measured at different mesh points to those in Table 3.1. In fact the global error for Method 3.1 is slightly worse than for Method 3.2.

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	4.78e-03	2.45e-03	1.24e-03	6.25e-04	3.14e-04	1.57e-04	7.86e-05	3.93e-05
2^{-2}	4.69e-02	2.52e-02	1.31e-02	6.69e-03	3.38e-03	1.70e-03	8.51e-04	4.26e-04
2^{-4}	9.97e-02	5.89e-02	3.69e-02	2.27e-02	1.35e-02	7.53e-03	3.80e-03	1.91e-03
2^{-6}	3.23e-01	8.96e-02	4.03e-02	2.32e-02	1.36e-02	7.83e-03	4.44e-03	2.48e-03
2^{-8}	6.84e-01	2.99e-01	7.86e-02	2.54e-02	1.40e-02	7.88e-03	4.44e-03	2.48e-03
2^{-10}	8.90e-01	6.51e-01	2.61e-01	6.51e-02	1.86e-02	8.26e-03	4.48e-03	2.48e-03
2^{-12}	9.59e-01	8.82e-01	6.00e-01	2.23e-01	5.38e-02	1.40e-02	5.03e-03	2.53e-03
2^{-14}	9.78e-01	9.65e-01	8.58e-01	5.44e-01	1.91e-01	4.56e-02	1.10e-02	3.47e-03
2^{-16}	9.83e-01	9.88e-01	9.60e-01	8.28e-01	4.91e-01	1.65e-01	3.98e-02	9.18e-03
2^{-18}	9.84e-01	9.94e-01	9.89e-01	9.51e-01	7.95e-01	4.46e-01	1.46e-01	3.56e-02
2^{-20}	9.84e-01	9.96e-01	9.96e-01	9.87e-01	9.40e-01	7.63e-01	4.08e-01	1.31e-01
2^{-22}	9.84e-01	9.96e-01	9.98e-01	9.97e-01	9.84e-01	9.28e-01	7.34e-01	3.76e-01
2^{-24}	9.84e-01	9.96e-01	9.99e-01	9.99e-01	9.96e-01	9.81e-01	9.17e-01	7.07e-01
2^{-26}	9.84e-01	9.96e-01	9.99e-01	1.00e+00	9.99e-01	9.95e-01	9.78e-01	9.06e-01
2^{-28}	9.84e-01	9.96e-01	9.99e-01	1.00e+00	1.00e+00	9.99e-01	9.94e-01	9.75e-01

Table 3.3: Maximum nodal pointwise errors $E^N_{\varepsilon,nodal}$ on $\bar{\Omega}^N_\varepsilon$ for Method 3.2 applied to problem (3.4)

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	16	16	16	16	16	16	16	16
2^{-2}	13	13	12	12	12	12	12	12
2^{-4}	16	10	7	6	5	4	4	4
2^{-6}	44	21	11	7	6	5	4	4
2^{-8}	82	70	34	12	8	6	4	4
2^{-10}	103	140	116	57	22	9	6	4
2^{-12}	110	186	254	197	97	40	14	7
2^{-14}	112	203	358	466	339	165	70	28
2^{-16}	112	207	399	705	856	585	283	123
2^{-18}	112	208	411	808	1381	1561	1017	489
2^{-20}	112	209	414	838	1629	2669	2828	1774
2^{-22}	112	209	415	846	1706	3243	5085	5097
2^{-24}	112	209	415	848	1727	3427	6351	9574
2^{-26}	112	209	415	849	1732	3476	6773	12266
2^{-28}	112	209	415	849	1733	3489	6887	13194

Table 3.4: Computed iteration counts for Method 3.2 applied to problem (3.4)

3.6 Conclusions

In this chapter, a one-dimensional convection-diffusion problem was examined. It was shown that the solution of a discrete overlapping Schwarz method with uniform meshes converges to the solution of upwinding on a uniform mesh, if the interface points are independent of ε, and is therefore not an ε-uniform method. A second method, which uses uniform meshes on overlapping subdomains partitioned using the Shishkin parameter, $\tau = \min\{1/3, \frac{\varepsilon}{\alpha} \ln N\}$, has an ε-dependent overlap and it is shown numerically that this method also fails to produce ε-uniform approximations.

Chapter 4

Alternative Schwarz methods for convection-diffusion problems

4.1 Introduction

In this chapter, we focus on alternative discrete Schwarz methods for convection-diffusion problems. It is our intention to investigate the convergence properties of these methods and to shed further light on the complexities in the Schwarz approach to convection-diffusion problems.

Firstly, we review the main difficulties in a Schwarz approach to convection-diffusion type problems. These will be the governing factors when designing an alternative method.

1. It is not possible to use uniform overlapping subdomains and obtain ε-uniform convergent approximations.

2. When applying maximum principle arguments, as we have seen in Chapter 3, it is necessary to consider any error reduction taking place when the problem is solved outside the layer region.

3. The transfer of solution values between subdomains is also difficult since, it is observed, when using the Maximum Principle arguments, that an increase in the approximation error arises when solution values are passed from the layer region. This additional error can then accumulate during the iteration process.

4. It is desirable for the numerical solutions to be (ε, N)-uniformly convergent and this appears difficult to achieve.

We now introduce five methods which, in the course of this chapter, are examined both theoretically and numerically. They are developed to try and overcome the difficulties stated above and attempt to comply with the criteria of an optimal Schwarz domain decomposition method.

In section 4.2, we discuss an overlapping method in which a piecewise uniform mesh is fitted on the subdomain containing the boundary layer. We derive error bounds for the regular and singular components of the Schwarz iterates separately, and combine these estimates to give the convergence behaviour of the method. Theoretical analysis and numerical experiments show the approximations are ε-uniformly convergent. This method would therefore be considered to be a significant improvement on those discussed in Chapter 3. However, it is prevented from being the perfect method for the following two reasons. Firstly, since the width of the overlap is $O(N^{-1})$, the number of required iterations increases with the number of mesh points N and, although not significant when ε is very small, this becomes a problem when both N and ε are large. Secondly, the method does not contain uniform meshes in both subdomains. Ultimately, we would wish to be able to extend a method, designed using a one-dimensional model problem, to higher dimensions and although this method worked well we feel the analysis would not extend easily to higher dimensions.

In section 4.3, we consider a non-overlapping Schwarz method (Method 4.2) which uses uniform meshes in both subdomains. In the design of this method, we sought to retain the convergence properties of the previous method while simplifying the algorithm by using uniform meshes. We use very simple Dirichlet boundary conditions at the interface of the two non-overlapping subdomains to pass information during the iteration process. The analysis, using the decomposition of the Schwarz iterates and maximum principle techniques, shows that the error estimate contains an $O(\varepsilon)$ term. The method, therefore, fails to generate accurate approximations when ε is large, specifically, when ε is greater than N^{-1}. Also, like the previous method, the iterations increase with N. Overall, we feel that when the perturbation parameter ε is small this is a satisfactory method, and is easily implemented. In Chapter 5, we extend it to a two-dimensional convection-diffusion problem.

We address the problem of large iteration numbers in section 4.4 by investigating a non-overlapping, non-iterative method which contains a Neumann condition at the interface. This algorithm is motivated by Method 4.2, in which, the chosen Dirichlet boundary conditions can be considered to mimic a Neumann condition on the interface of the subdomain outside the layer. Here, no information is passed and so the algorithm does not iterate. The convergence of this Method is analysed and we find it exhibits equivalent behaviour to Method 4.2. This is illustrated by numerical results, which verify this method produces identical approximations to Method 4.2. This is not surprising since both methods use essentially the same interface conditions. The main advantage of this method, for small values of ε, is that no iterations are required to obtain satisfactory approximations. Therefore, we feel this method outperforms Method 4.2.

A major drawback to Methods 4.2 and 4.3, for some fixed value of ε, is that when the number of mesh points is increased to order ε^{-1} or greater the method will fail. In many practical applications, where $\varepsilon << N^{-1}$, this does not arise and it is well known that where N is large, typically $N >> \varepsilon^{-1}$, it becomes possible to use a classical approach. However, we feel it is important to develop a single method which produces accurate approximations for all values of ε. We believe that, for a non-overlapping discrete Classical Schwarz method with uniform meshes, other "Dirichlet" type interface conditions would adapt to large values of ε and achieve convergent approximations comparable to those attained by Method 4.1, but would not improve on the iteration behaviour of Method 4.2. We discuss, in Section 4.5, a non-overlapping method with a Neumann interface condition which, at each iteration, uses a difference approximation of the first derivative calculated from previous iteration values. This method is described and numerical results show that the method converges for both large and small values of ε. No theory is presented here for this method. Much work has been carried out on special types of interface conditions such as Robin, Neumann-Neumann and Neumann-Dirichlet type conditions (see, for example, [21], [22],[13],[12], [35], [9] and [24]). We hope, in the future, to investigate some of these non-classical types of interface conditions more thoroughly.

The final method in this chapter, introduced in Section 4.6, produces (ε, N)-uniformly convergent numerical approximations. The width of the overlap region is fixed as a proportion of the subdomain positioned outside the boundary layer and a Shishkin mesh is fitted on the subdomain containing the layer. At first, this Schwarz method appears to have no advantages over the fitted Shishkin mesh, and this is the case in one-dimension. However, a problem involving a complex domain structure in higher dimensions, in which a fitted mesh may not be viable, may require this type of Schwarz method. Numerical results are presented which verify the convergence behaviour of this method.

It is our suggestion, in this chapter, that a type of trade-off must be agreed upon in designing a Schwarz method for convection-diffusion equations. We do not believe it is possible for a Classical Schwarz method to have uniform meshes and (ε, N)-uniform convergence. However, depending on what is desired of the method, it is possible to attain some of these important attributes.

4.1.1 Preliminaries

The Shishkin decomposition and bounds on derivatives, as stated in Chapter 3, are necessary for the analysis of the Schwarz methods investigated in this chapter. In the following methods we use the Shishkin transition point, between fine and coarse mesh regions, to be

$$\tau = \min\{1/3, \frac{\varepsilon}{\alpha} \ln N\}.$$

We consider only the case, $\tau = \varepsilon/\alpha \ln N < 1/3$, when examining the methods theoretically. When implementing these methods for values of ε and N, such that $\varepsilon/\alpha \ln N \geq 1/3$, we use an algorithm which is appropriate for non-singularly perturbed problems and, for each of the methods, this algorithm is described in the relevant numerical section.

In Methods 4.1 to 4.4, we make use of the following notation. The constants ξ^+, ξ^- are given by

$$\xi^+ = 1 - \tau, \qquad \xi^- = \frac{(N-1)}{N}(1-\tau).$$

The numerical experiments presented in this chapter are performed using the following model problem,

$$-\varepsilon u'' + u' = x, \quad u(0) = u(1) = 0. \tag{4.1}$$

The methods were also tested on

$$-\varepsilon u'' + (1 + x^3)u' = x^2, \quad u(0) = u(1) = 0. \tag{4.2}$$

producing equivalent behaviour.

4.2 An overlapping Schwarz method using a special mesh in one subdomain

4.2.1 Introduction

This Schwarz domain decomposition method consists of partitioning the solution domain into two overlapping subdomains, one positioned outside the layer region and discretised by a uniform mesh, and the other containing the layer region, is discretised with a special piecewise uniform mesh. The piecewise uniform mesh is designed so that the width of the overlapping region is $O(N^{-1})$ and therefore, for any fixed N, it does not reduce for decreasing values of ε. That is, the overlapping region is held sufficiently large with respect to ε. The method is analyzed in Lemmas 4.2.1 to 4.2.5 by considering the components of the decomposed Schwarz iterates separately and the main convergence result for this method is then given in Theorem 4.2.1. Numerical results are given in section 4.2.3 which verify the theoretical estimates.

The continuous analogue of this method would be very similar to that outlined in Chapter 3 and therefore we do not repeat it here. Also the same Shishkin decomposition, outlined in Chapter 3, applies to this method and will be used throughout the analysis in this section.

4.2.2 Discrete Schwarz method

We now formally describe the method.

Method 4.1 *The exact solution u_ε is approximated by the limit $\bar{U}_\varepsilon$ of a sequence of* *discrete Schwarz iterates* $\{\bar{U}_\varepsilon^{[k]}\}_{k=0}^{\infty}$, *which are defined as follows. For each* $k \geq 1$,

$$\bar{U}_\varepsilon^{[k]}(x) = \begin{cases} \bar{U}_0^{[k]}(x), & x \in \bar{\Omega}_0 \setminus \bar{\Omega}_1 \\ \bar{U}_1^{[k]}(x), & x \in \bar{\Omega}_1 \end{cases}$$

where $\bar{U}_i^{[k]}$ *is the linear interpolant of* $U_i^{[k]}$. *Let* $\bar{\Omega}_0^N = \{x_i\}_0^N$ *be a uniform mesh on* Ω_0 *with* $x_i = i\xi^+/N$ *and* $\bar{\Omega}_1^N = \{x_i\}_0^{N+1}$ *be a piecewise uniform mesh on* Ω_1 *with* $x_0 = \xi^-, x_i = \xi^+ + i(1-\xi^+)/N$, *as shown in Fig. 4.1. Then for* $k = 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[1]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[1]}(0) = u_0, \quad U_0^{[1]}(\xi^+) = 0, \\ L_\varepsilon^N U_1^{[1]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[1]}(\xi^-) = U_0^{[1]}(\xi^-), \quad U_0^{[1]}(1) = u_1, \end{aligned}$$

and for $k > 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[k]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[k]}(0) = u_0, \quad U_0^{[k]}(\xi^+) = U_1^{[k-1]}(\xi^+), \\ L_\varepsilon^N U_1^{[k]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[k]}(\xi^-) = U_0^{[k]}(\xi^-), \quad U_0^{[k]}(1) = u_1. \end{aligned}$$

Analogous definitions can be made for the iterate components $V_\varepsilon^{[k]}$ and $W_\varepsilon^{[k]}$ where

$$U_\varepsilon^{[k]} = V_\varepsilon^{[k]} + W_\varepsilon^{[k]}.$$

The differential operator L_ε is replaced with the standard upwind finite difference operator L_ε^N defined for any mesh function Z_i by

$$L_\varepsilon^N Z_i = -\varepsilon\delta^2 Z_i + a(x_i)D^- Z_i$$

where

$$\delta^2 Z_i = \left(\frac{D^- Z_{i+1} - D^- Z_i}{(x_{i+1} - x_{i-1})/2}\right), \quad D^- Z_i = \left(\frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}\right).$$

Remark 4.2.1 *Any mesh function* ψ_i *on a uniform mesh* $\Omega_s^N = \{x_i = i\xi^+/N, i = 1,..,N-1\}$, *which satisfies the difference equation*

$$L_\varepsilon^N \psi_i = -\varepsilon\delta^2\psi_i + \alpha D^-\psi_i = 0, \quad on \quad \Omega_s^N$$

is given explicitly by

$$\psi_i = \psi_N - (\psi_N - \psi_0)\left(\frac{1-\lambda^{-N+i}}{1-\lambda^{-N}}\right)$$

where

$$\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau), \quad \tau = \min\{1/3, \frac{\varepsilon}{\alpha}\ln N\}$$

and the boundary values ψ_0 *and* ψ_N *are known. Hence, the solution evaluated at the mesh point* $\xi^- = (1-\tau)(N-1)/N$ *is given by*

$$\begin{aligned} \psi_{N-1} &= \psi_0 q + \psi_N(1-q), \quad where \\ q &= \frac{1-\lambda^{-1}}{1-\lambda^{-N}}. \end{aligned}$$

In the following lemma, we derive an estimate for the error contained in the smooth Schwarz component $V_\varepsilon^{[k]}$.

Lemma 4.2.1 *For all* $k \geq 1$

$$||(V_\varepsilon^{[k]} - v_\varepsilon)(x_i)|| \leq CN^{-1} + C\lambda^{-k}$$

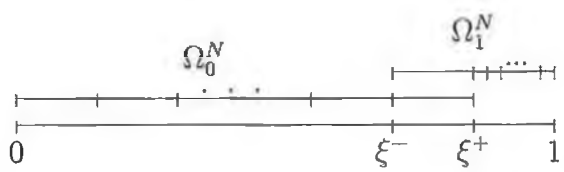

Figure 4.1: The discretised overlapping subdomains Ω_0^N and Ω_1^N for Method 4.1

where

$$\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau).$$

Proof. Here we use maximum principle arguments with selected barrier functions to obtain the appropriate error estimates. The proof will be by induction.

Firstly in Ω_0, using the estimates $|v_\varepsilon| \le C$, it follows that,

$$|(V_0^{[1]} - v_\varepsilon)(0)| = 0, \quad |(V_0^{[1]} - v_\varepsilon)(\xi^+)| \le C.$$

By applying classical arguments it can be shown that the local truncation error satisfies

$$\begin{aligned} L_\varepsilon^N(V_0^{[1]} - v_\varepsilon)(x_i) &= (L_\varepsilon - L_\varepsilon^N)v_\varepsilon(x_i) \\ &= -\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right)v_\varepsilon(x_i) + a(x_i)\left(\frac{d}{dx} - D^-\right)v_\varepsilon(x_i), \end{aligned}$$

and it follows that,

$$|L_\varepsilon^N(V_0^{[1]} - v_\varepsilon)|(x_i) \le C(x_{i+1} - x_{i-1})(\varepsilon|v_\varepsilon|_3 + |v_\varepsilon|_2). \tag{4.3}$$

Now, using the estimates on $|v_\varepsilon|_3$ and $|v_\varepsilon|_2$, gives

$$|L_\varepsilon^N(V_0^{[1]} - v_\varepsilon)(x_i)| \le CN^{-1},$$

where $x_{i+1} - x_{i-1} = 2\xi^+/N < 2/N$ and $|v_\varepsilon|_k < C\varepsilon^{2-k}$, $0 \le k \le 3$.

Note the following choice of barrier function,

$$\phi_i = C\left(1 - \left(\frac{1-\lambda^{-N+i}}{1-\lambda^{-N}}\right)\right) + CN^{-1}x_i,$$

where $\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau)$.

It can be seen that the inequalities,

$$\begin{aligned}
\left(\phi_0 \pm (V_0^{[1]} - v_\varepsilon)(0)\right) &= 0, \\
\left(\phi_N \pm (V_0^{[1]} - v_\varepsilon)(\xi^+)\right) &\geq 0, \\
L_\varepsilon^N\left(\phi_i \pm (V_0^{[1]} - v_\varepsilon)\right)(x_i) &= \left(L_\varepsilon^N \phi_i \pm L_\varepsilon^N (V_0^{[1]} - v_\varepsilon)\right)(x_i) \\
&\geq CN^{-1} \pm CN^{-1} \\
&\geq 0,
\end{aligned}$$

are satisfied. Therefore, by applying the discrete maximum principle for L_ε^N on Ω_0^N, we obtain the following estimate,

$$|\left(V_0^{[1]} - v_\varepsilon)\right)(x_i)| \leq C\left(1 - \left(\frac{1-\lambda^{-N+i}}{1-\lambda^{-N}}\right)\right) + CN^{-1}x_i, \tag{4.4}$$

for all $x_i \in \bar{\Omega}_0^N$. Now on Ω_1, using the boundary conditions and (4.4) we obtain

$$\begin{aligned}
|(V_1^{[1]} - v_\varepsilon)(\xi^-)| &= |(V_0^{[1]} - v_\varepsilon)(\xi^-)| \\
&\leq C\left(1 - \left(\frac{1-\lambda^{-1}}{1-\lambda^{-N}}\right)\right) + CN^{-1}(\xi^-), \\
|(V_1^{[1]} - v_\varepsilon)(1)| &= 0.
\end{aligned}$$

Applying arguments similar to those used on Ω_0^N, it follows that the local truncation error in Ω_1^N also satisfies

$$|L_\varepsilon^N (V_1^{[1]} - v_\varepsilon)(x_i)| \leq CN^{-1},$$

since $(x_{i+1} - x_{i-1}) \leq 2N^{-1}$ in Ω_1. Here, we choose the barrier function, ϕ_i, to be

$$\phi_i = C(1-q) + CN^{-1}x_i, \quad \text{where} \quad q = \frac{1-\lambda^{-1}}{1-\lambda^{-N}}.$$

It can be seen that,

$$\begin{aligned}
\left(\phi_0 \pm (V_1^{[1]} - v_\varepsilon)(\xi^-)\right) &\geq 0, \\
\left(\phi_{N+1} \pm (V_1^{[1]} - v_\varepsilon)(1)\right) &\geq 0, \\
L_\varepsilon^N\left(\phi_i \pm (V_1^{[1]} - v_\varepsilon)\right)(x_i) &\geq 0.
\end{aligned}$$

Therefore, the discrete maximum principle can be applied to give the estimate

$$|(V_1^{[1]} - v_\varepsilon)(x_i)| \le C(1-q) + CN^{-1}x_i, \quad \text{in} \quad \bar{\Omega}_1^N. \tag{4.5}$$

Now, combining (4.4) and (4.5), then gives

$$|(V_\varepsilon^{[1]} - v_\varepsilon)(x_i)| \le C(1-q) + CN^{-1}x_i$$

in $\bar{\Omega}^N$. Next, we assume the induction hypothesis

$$|(V_\varepsilon^{[k]} - v_\varepsilon)(x_i)| \le C(1-q)^k + CN^{-1}x_i, \quad \text{in} \quad \bar{\Omega}_1^N.$$

Considering the $(k+1)^{\text{th}}$ Schwarz iterative in Ω_0, we see that

$$\begin{aligned}
|(V_0^{[k+1]} - v_\varepsilon)(0)| &= 0, \\
|(V_0^{[k+1]} - v_\varepsilon)(\xi^+)| &= |(V_1^{[k]} - v_\varepsilon)(\xi^+)|, \\
&\le C(1-q)^k + CN^{-1}(\xi^+).
\end{aligned}$$

We choose the barrier function, ϕ_i, to be

$$\phi_i = C(1-q)^k \left(1 - \left(\frac{1-\lambda^{-N+i}}{1-\lambda^{-N}}\right)\right) + CN^{-1}x_i,$$

and, as before, the appropriate inequalities,

$$\begin{aligned}
\left(\phi_0 \pm (V_0^{[k+1]} - v_\varepsilon)(0)\right) &= 0, \\
\left(\phi_N \pm (V_0^{[k+1]} - v_\varepsilon)(\xi^+)\right) &\ge 0, \\
L_\varepsilon^N\left(\phi_i \pm (V_0^{[k+1]} - v_\varepsilon\right)(x_i) &\ge 0
\end{aligned}$$

are satisfied. Now, applying maximum principle arguments on $\bar{\Omega}_0^N$, then yields

$$|(V_0^{[k+1]} - v_\varepsilon)(x_i)| \le C(1-q)^k \left(1 - \left(\frac{1-\lambda^{-N+i}}{1-\lambda^{-N}}\right)\right) + CN^{-1}x_i. \tag{4.6}$$

Now, in Ω_1^N,

$$\begin{aligned}
|(V_1^{[k+1]} - v_\varepsilon)(1)| &= 0, \\
|(V_1^{[k+1]} - v_\varepsilon)(\xi^-)| &= |(V_0^{[k+1]} - v_\varepsilon)(\xi^-)| \\
&\le C(1-q)^k \left(1 - \left(\frac{1-\lambda^{-1}}{1-\lambda^{-N}}\right)\right) + CN^{-1}(\xi^-) \\
&= C(1-q)^{k+1} + CN^{-1}(\xi^-).
\end{aligned}$$

Here, we choose ϕ_i to be

$$\phi_i = C(1-q)^{k+1} + CN^{-1}x_i,$$

and the inequalities

$$\begin{aligned}
\left(\phi_0 \pm (V_1^{[k+1]} - v_\varepsilon)(\xi^-)\right) &\geq 0, \\
\left(\phi_{N+1} \pm (V_1^{[k+1]} - v_\varepsilon)(1)\right) &\geq 0, \\
L_\varepsilon^N\left(\phi_i \pm (V_1^{[k+1]} - v_\varepsilon)\right)(x_i) &\geq 0, \quad x_i \in \Omega_1^N,
\end{aligned}$$

are satisfied. So, from the discrete maximum principle,

$$|(V_1^{[k+1]} - v_\varepsilon)(x_i)| \leq C(1-q)^{k+1} + CN^{-1}x_i. \tag{4.7}$$

Combining the estimates (4.6) and (4.7), then gives

$$|(V_\varepsilon^{[k+1]} - v_\varepsilon)(x_i)| \leq C(1-q)^{k+1} + CN^{-1}x_i, \quad \forall x_i \in \bar{\Omega}^N.$$

Finally, under the assumptions of this method, $1 - \tau > 0$, and it is clear that

$$\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau) > 1.$$

Therefore, the term $1 - q$ can be bounded as follows,

$$q = \frac{1-\lambda^{-1}}{1-\lambda^{-N}} > 1 - \lambda^{-1}$$

and hence,

$$1 - q < \lambda^{-1}$$

which concludes this lemma. ◇

The remaining lemmas of this section are concerned with obtaining error estimates for the discrete singular component, W_ε. Lemma 4.2.2 explicitly derives the solution of the singular component of the Schwarz iterates, $W_\varepsilon^{[k]}$ at the first mesh point in Ω_1^N, $x_i = \xi^+$.

Lemma 4.2.2 *For $k \geq 1$, the explicit solution to the finite difference scheme*

$$-\varepsilon\delta^2 W_1^{[k]}(x_i) + \alpha D^- W_1^{[k]}(x_i) = 0, \quad in \quad \bar{\Omega}_1^N,$$

at $x_1 = \xi^+$ is given by

$$W_1^{[k]}(x_1) = pW_1^{[k]}(x_{N+1}) + (1-p)W_1^{[k]}(x_0),$$

where

$$p = \left(\frac{1}{1 + \frac{\tau}{1-\tau}\left(1 + \frac{\alpha}{2N\varepsilon}\right)\left(\frac{\mu^N - 1}{\mu - 1}\right)}\right),$$

with $\mu = 1 + \ln N/N$ and the boundary values $W_1^{[k]}(x_0)$ and $W_1^{[k]}(x_{N+1})$ are known.

Proof. On the interval $[1-\tau, 1]$ the mesh is uniform and, $\forall \quad 1 \leq i \leq N+1$,

$$W_1^{[k]}(x_i) = W_1^{[k]}(x_{N+1}) - (W_1^{[k]}(x_{N+1}) - W_1^{[k]}(x_1))\left(\frac{\mu^{N+1} - \mu^i}{\mu^{N+1} - \mu}\right), \tag{4.8}$$

with

$$\mu = 1 + \frac{\alpha h}{\varepsilon}, \quad h = \frac{\tau}{N} \quad \text{and hence, } \mu = 1 + \frac{\ln N}{N}.$$

Now, at $i = 1$, $W_1^{[k]}(x_1)$ satisfies the finite difference scheme

$$-\frac{\varepsilon}{\bar{H}}\left(\left(\frac{W_1^{[k]}(x_2) - W_1^{[k]}(x_1)}{h}\right) - \left(\frac{W_1^{[k]}(x_1) - W_1^{[k]}(x_0)}{H}\right)\right) + \alpha\left(\frac{W_1^{[k]}(x_1) - W_1^{[k]}(x_0)}{H}\right) = 0, \tag{4.9}$$

where

$$H = \frac{1-\tau}{N}, \quad \bar{H} = \frac{h+H}{2}.$$

From (4.8) it can be seen that the term $W_1^{[k]}(x_2)$ satisfies

$$W_1^{[k]}(x_2) = W_1^{[k]}(x_{N+1}) - (W_1^{[k]}(x_{N+1}) - W_1^{[k]}(x_1))\left(\frac{\mu^{N+1} - \mu^2}{\mu^{N+1} - \mu}\right). \tag{4.10}$$

Letting,

$$\bar{\mu} = \left(\frac{\mu^{N+1}-\mu^2}{\mu^{N+1}-\mu}\right) = \left(\frac{\mu^N-\mu}{\mu^N-1}\right),$$

and combining (4.9) and (4.10), then gives

$$\frac{-\varepsilon}{\bar{H}}\Bigg(\bigg(\frac{W_1^{[k]}(x_{N+1}) - (W_1^{[k]}(x_{N+1}) - W_1^{[k]}(x_1))\bar{\mu} - W_1^{[k]}(x_1)}{h}\bigg) - \bigg(\frac{W_1^{[k]}(x_1) - W_1^{[k]}(x_0)}{H}\bigg)\Bigg) + \frac{\alpha}{\bar{H}}(W_1^{[k]}(x_1) - W_1^{[k]}(x_0)) = 0. \quad (4.11)$$

Now, it remains to simplify (4.11) into terms containing $W_1^{[k]}(x_{N+1})$ and $W_1^{[k]}(x_0)$. It follows that,

$$W_1^{[k]}(x_{N+1})(1-\bar{\mu}) - W_1^{[k]}(x_1)\bigg((1-\bar{\mu}) + \frac{h}{H}\Big(1+\frac{\alpha\bar{H}}{\varepsilon}\Big)\bigg) + W_1^{[k]}(x_0)\bigg(\frac{h}{H}\Big(1+\frac{\alpha\bar{H}}{\varepsilon}\Big)\bigg) = 0.$$

Using the following notation,

$$h = \frac{\tau}{N} \quad , H = \frac{1-\tau}{N}, \quad \bar{H} = \frac{h+H}{2} = \frac{1}{2N},$$

then yields,

$$W_1^{[k]}(x_1) = pW_1^{[k]}(x_{N+1}) + (1-p)W_1^{[k]}(x_0), \quad p = \left(\frac{1}{1+\frac{\tau}{1-\tau}(1+\frac{\alpha}{2N\varepsilon})(\frac{\mu^N-1}{\mu-1})}\right),$$

which completes this lemma. ◇

In Lemma 4.2.3 we derive estimates for the coefficient p, defined in Lemma 4.2.2.

Lemma 4.2.3 *For $k \geq 1$, the coefficient p in the equation*

$$W_1^{[k]}(\xi^+) = pW_1^{[k]}(1) + (1-p)W_1^{[k]}(\xi^-)$$

satisfies the inequalities

$$0 < p < (1+\frac{N}{4})^{-1}.$$

Proof. Noting $\mu = 1 + \frac{\ln N}{N}$ and using $\frac{\mu^N - 1}{\mu - 1} > \frac{N^2}{2\ln N}$ (we use an argument as in Miller et al. [18] Pg.31) implies that

$$p < \left(\frac{1}{1 + \frac{\tau}{1-\tau}\left(1 + \frac{\alpha}{2N\varepsilon}\right)\left(\frac{N^2}{2\ln N}\right)} \right).$$

Also, $\tau = \frac{\varepsilon}{\alpha} \ln N$ and $(1 - \tau) < 1$, and so

$$\begin{aligned} p &< \left(\frac{1}{1 + \frac{\varepsilon}{\alpha}\ln N\left(1 + \frac{\alpha}{2N\varepsilon}\right)\left(\frac{N^2}{2\ln N}\right)} \right) \\ &= \frac{1}{1 + \frac{\varepsilon N^2}{2\alpha} + \frac{N}{4}} \\ &< \left(1 + \frac{N}{4}\right)^{-1}, \quad \forall \quad \varepsilon \geq 0. \end{aligned} \tag{4.12}$$

◇

Note that

$$\lim_{\varepsilon \to 0} p = \frac{2}{1 + \mu^N} \quad \text{and} \quad \lim_{N\to\infty} \lim_{\varepsilon \to 0} p = 0.$$

In Lemma 4.2.4 we obtain an upperbound for the quantity p/q for all ε, where $q = (1 - \lambda^{-1})/(1 - \lambda^{-N})$ is given by (4.3).

Lemma 4.2.4 *For $N \geq 4$,*

$$\frac{p}{q} < \left(1 - \frac{\varepsilon}{\alpha}\ln N\right)^{-1}\left(1 + \frac{N}{4}\right)^{-1},$$

where p is as given in Lemma 4.2.2 and

$$q = \frac{1 - \lambda^{-1}}{1 - \lambda^{-N}}, \quad \lambda = 1 + \frac{\alpha}{\varepsilon N}(1 - \tau).$$

Proof. The estimate for the coefficient p is given by the inequality (4.12), and in Lemma 4.2.1 we derived the factor q.

$$\begin{aligned} p &< \frac{1}{1 + \frac{\varepsilon N^2}{2\alpha} + \frac{N}{4}}, \\ q &= \frac{1 - \lambda^{-1}}{1 - \lambda^{-N}} > 1 - \lambda^{-1}. \end{aligned}$$

It is clear that an initial upperbound for p/q is given by

$$\frac{p}{q} < \frac{\left(1+\frac{\varepsilon N^2}{2\alpha}+\frac{N}{4}\right)^{-1}}{1-\left(1+\frac{\alpha}{\varepsilon N}-\frac{\ln N}{N}\right)^{-1}} = \frac{1+\frac{\alpha}{\varepsilon N}\left(1-\frac{\varepsilon}{\alpha}\ln N\right)}{\left(\frac{\alpha}{\varepsilon N}+\frac{N}{2}+\frac{\alpha}{4\varepsilon}\right)\left(1-\frac{\varepsilon}{\alpha}\ln N\right)}.$$

Note, in this method $\tau = \frac{\varepsilon}{\alpha}\ln N$ satisfies the inequality $\tau < 1/3$. Consequently, we obtain the inequality,

$$\frac{p}{q} < \frac{1}{1-\frac{\varepsilon}{\alpha}\ln N}\left(\frac{1+\frac{\alpha}{\varepsilon N}}{\frac{\alpha}{\varepsilon N}+\frac{N}{2}+\frac{\alpha}{4\varepsilon}}\right).$$

Furthermore, it is now possible to show that the term p/q is $O(N^{-1})$ since

$$\frac{p}{q} < \frac{1}{(1-\frac{\varepsilon}{\alpha}\ln N)}\left(1+\frac{N}{4}\right)^{-1}.$$

This is true for $1+N/4 < N/2$,. This concludes this lemma. $\diamond$

In the next lemma we incorporate Lemmas 4.2.1 and 4.2.4, to give a ε-uniform estimate for the error between the discrete Schwarz singular component, $W_\varepsilon^{[k]}$ and its continuous counterpart, w_ε.

Lemma 4.2.5 *For all* $k \geq 1$,

$$||(W_\varepsilon^{[k]} - w_\varepsilon)(x_i)||_{\bar{\Omega}^N} \leq CN^{-1}(\ln N)^2 + CN^{-1}\left(1-\frac{\varepsilon}{\alpha}\ln N\right)^{-1}.$$

Proof. Firstly, we consider the Schwarz component, $W_0^{[k]}$, on the interval $[0, \xi^+]$. Recall, for $0 \leq x \leq \xi^+$, that we can write

$$|w_\varepsilon(x)| \leq Ce^{-\alpha(1-x)/\varepsilon} \leq Ce^{-\alpha(1-\xi^+)/\varepsilon} = Ce^{-\alpha\tau/\varepsilon} = CN^{-1}. \qquad (4.13)$$

Note that,

$$W_0^{[k]}(0) = w_\varepsilon(0), \quad W_0^{[k]}(\xi^+) = W_1^{[k-1]}(\xi^+), \quad L_\varepsilon^N W_0^{[k]} = 0 \text{ in } \Omega_0^N.$$

The maximum principle for L_ε^N on Ω^N, and (4.13), then give

$$|W_0^{[k]}(\xi^+)| \leq \max\{CN^{-1}, |W_1^{[k-1]}(\xi^+)|\}. \tag{4.14}$$

Now, we obtain an explicit expressions for $W_1^{[k]}(\xi^+)$, using Lemmas 4.2.2, 4.2.3 and 4.2.4, and derive an error estimate for $||W_\varepsilon^{[k]} - w_\varepsilon||$ in the interval $[0, \xi^+]$.

On Ω_0, $\forall k \geq 1$,

$$W_0^{[k]}(x_i) = W_0^{[k]}(\xi^+) - (W_0^{[k]}(\xi^+) - w_\varepsilon(0))\left(\frac{1-\lambda^{-N+i}}{1-\lambda^{-N}}\right),$$

where

$$\lambda = 1 + \frac{\alpha}{\varepsilon N} - \frac{\ln N}{N},$$

and the iterative scheme, as stated in Section 4.2.2, gives for $W_0^{[k]}(\xi^+)$,

$$W_0^{[1]}(\xi^+) = 0, \quad W_0^{[k]}(\xi^+) = W_1^{[k-1]}(\xi^+). \quad k > 1$$

Therefore we have,

$$\begin{aligned} W_0^{[k]}(\xi^-) &= w_\varepsilon(0)q, \quad k = 1, \\ W_0^{[k]}(\xi^-) &= w_\varepsilon(0)q + W_1^{[k-1]}(\xi^+)(1-q), \quad k > 1, \end{aligned}$$

with

$$q = \frac{1-\lambda^{-1}}{1-\lambda^{-N}}.$$

On Ω_1, from Lemma 4.2.2, it can be seen that

$$W_1^{[k]}(\xi^+) = pw_\varepsilon(1) + (1-p)W_0^{[k]}(\xi^-).$$

Hence, the iterative process gives

$$
\begin{aligned}
W_0^{[1]}(\xi^-) &= w_\varepsilon(0)q, \\
W_1^{[1]}(\xi^+) &= (w_\varepsilon(0)q)(1-p) + pw_\varepsilon(1), \\
W_0^{[2]}(\xi^-) &= w_\varepsilon(0)q + (w_\varepsilon(0)q(1-p) + pw_\varepsilon(1))(1-q), \\
W_1^{[2]}(\xi^+) &= \big(w_\varepsilon(0)q + (w_\varepsilon(0)q(1-p) + pw_\varepsilon(1))(1-q)\big)(1-p) + pw_\varepsilon(1), \\
W_0^{[3]}(\xi^-) &= w_\varepsilon(0)q + \big(w_\varepsilon(0)q(1-p) + (w_\varepsilon(0)q(1-p) + pw_\varepsilon(1))(1-q)(1-p) \\
&\quad + pw_\varepsilon(1)\big)(1-q), \\
W_1^{[3]}(\xi^+) &= w_\varepsilon(0)q(1-p) + w_\varepsilon(0)(1-p)^2 q(1-q) + (w_\varepsilon(0)q(1-p) \\
&\quad + pw_\varepsilon(1))(1-q)^2(1-p)^2 + (1-p)pw_\varepsilon(1)(1-q) + pw_\varepsilon(1),
\end{aligned}
$$

and after k iterations,

$$
\begin{aligned}
W_1^{[k]}(\xi^+) &= w_\varepsilon(0)q(1-p)\big(1 + (1-p)(1-q) + (1-p)^2(1-q)^2 + \cdots \\
&\quad + (1-p)^{k-2}(1-q)^{k-2}\big) + \big(w_\varepsilon(0)q(1-p) + pw_\varepsilon(1)\big)(1-p)^{k-1}(1-q)^{k-1} \\
&\quad + pw_\varepsilon(1)\big(1 + (1-p)(1-q) + (1-p)^2(1-q)^2 + \cdots \\
&\quad + (1-p)^{k-2}(1-q)^{k-2}\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
W_1^{[k]}(\xi^+) &= w_\varepsilon(0)q(1-p)\left(\frac{1 - \big((1-p)(1-q)\big)^{k-1}}{1 - \big((1-p)(1-q)\big)}\right) + w_\varepsilon(0)q(1-p) \\
&\quad \times \big((1-p)^{k-1}(1-q)^{k-1}\big) + w_\varepsilon(1)\frac{p}{p + (1-p)q}\big(1 - \big((1-p)(1-q)\big)^{k-1}
\end{aligned}
$$

Now, $\dfrac{q(1-p)}{p + q(1-p)} < 1$, and $\dfrac{p}{p + (1-p)q} < p/q$. It then follows that,

$$
\begin{aligned}
|W_1^{[k]}(\xi^+)| &< |w_\varepsilon(0)| + |w_\varepsilon(1)|\frac{p}{q}\big(1 - \big((1-p)(1-q)\big)^{k-1}\big) \\
&< CN^{-2} + C\frac{p}{q}
\end{aligned}
$$

since $|w_\varepsilon(0)| < CN^{-2}$ and $|w_\varepsilon(1)| < C$. Hence, in the interval $[0, \xi^+]$ an appropriate estimate for the error in the Schwarz iterative is given by

$$\begin{aligned} ||(W_\varepsilon^{[k]} - w_\varepsilon)(x_i)|| &\leq |W_\varepsilon^{[k]}(\xi^+)| + |w_\varepsilon(\xi^+)| \\ &\leq C\frac{p}{q} + CN^{-1} \\ &\leq CN^{-1}\left(1 - \frac{\varepsilon}{\alpha}\ln N\right)^{-1}. \end{aligned} \quad (4.15)$$

Now consider the interval $[\xi^+, 1]$,

$$\begin{aligned} |(W_1^{[k]} - w_1)(\xi^+)| &\leq CN^{-1}\left(1 - \frac{\varepsilon}{\alpha}\ln N\right)^{-1}, \\ |(W_1^{[k]} - w_1)(\xi^+)| &= 0, \qquad \forall k \geq 1, \end{aligned}$$

and

$$|L_\varepsilon^N(W_1^{[k]} - w_1)(x_i)| \leq C\varepsilon^{-2}\tau N^{-1}, \quad \forall x_i \in \Omega_1^N.$$

We choose the barrier function, $\phi_i = (x_i - \xi^+)C\varepsilon^{-2}\tau N^{-1} + CN^{-1}\left(1 - \frac{\varepsilon}{\alpha}\ln N\right)^{-1}$ and it can be seen that,

$$\begin{aligned} \left(\phi_0 \pm (W_1^{[k]} - w_1)(\xi^+)\right) &\geq 0, \\ \left(\phi_{N+1} \pm (W_1^{[k]} - w_1)(1)\right) &\geq 0, \\ \mathrm{L}_\varepsilon^N\left(\phi_i \pm (W_1^{[k]} - w_1)\right)(x_i) &\geq 0, \end{aligned}$$

Therefore, applying discrete maximum principle arguments gives,

$$\begin{aligned} |(W_1^{[k]} - w_1)(x_i)| \leq \phi_i &\leq C\varepsilon^{-2}N^{-1}\tau^2 + CN^{-1}\left(1 - \frac{\varepsilon}{\alpha}\ln N\right)^{-1} \\ &\leq CN^{-1}(\ln N)^2 + CN^{-1}\left(1 - \frac{\varepsilon}{\alpha}\ln N\right)^{-1}, \end{aligned} \quad (4.16)$$

for some C, independent of the perturbation parameter ε. And so, using the estimates (4.15) and (4.16), we get

$$||(W_\varepsilon^{[k]} - w_\varepsilon)(x_i)|| \leq CN^{-1}(\ln N)^2 + CN^{-1}\left(1 - \frac{\varepsilon}{\alpha}\ln N\right)^{-1},$$

in the domain $\bar{\Omega}^N$ and this concludes the proof. ◇

The following corollary to Lemmas 4.2.2 and 4.2.5 gives an error estimate for the continuous and discrete Schwarz iterates.

Corollary 4.2.1 *For all $k \geq 1$,*

$$||(U_\varepsilon^{[k]} - u_\varepsilon)(x_i)||_{\bar{\Omega}^N} \leq CN^{-1}(\ln N)^2 + CN^{-1}(1-\tau)^{-1} + C\lambda^{-k},$$

where C is a constant independent of k, N and ε, $\tau = \frac{\varepsilon}{\alpha}\ln N < 1/3$, and $\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau)$.

It can also be shown, see Chapter 3 in [6], that the piecewise linear interpolant $\bar{U}_\varepsilon^{[k]}$ retains the above error estimate and so we can now state the main theoretical result of this section.

Theorem 4.2.1 *For all $k \geq 1$,*

$$||\bar{U}_\varepsilon^{[k]} - u_\varepsilon||_{\bar{\Omega}} \leq CN^{-1}(\ln N)^2 + CN^{-1}(1-\tau)^{-1} + C\lambda^{-k},$$

where C is a constant independent of k,N and ε, $\tau = \frac{\varepsilon}{\alpha}\ln N < 1/3$, and $\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau)$.

4.2.3 Numerical results

In this section we present numerical results for Method 4.1. We note that when applied to problem (4.2), this method exhibits equivalent convergent behaviour to that when applied to problem (4.1), and so we only include here tabulated results for problem (4.1).

For notational reasons, we define the piecewise uniform mesh $\bar{\Omega}_\varepsilon^N$ associated with the overlapping subdomains by

$$\bar{\Omega}_\varepsilon^N = \bar{\Omega}_1^N \cup (\bar{\Omega}_0^N \setminus \bar{\Omega}_1^N).$$

The stopping criterion for the Schwarz iterations is taken to be

$$\max_{x_i \in \bar{\Omega}_\varepsilon^N} |U_\varepsilon^{[k]}(x_i) - U_\varepsilon^{[k-1]}(x_i)| \leq 10^{-9},$$

and we use the notation $\bar{U}_\varepsilon^N$ for the interpolant of the final Schwarz iterate. The exact solution in its closed form is easy to find for problem (4.1), which means that the exact pointwise errors can be calculated. The discrete Schwarz iterates are computed on a sequence of meshes with $N = 8, 16, ..2048$ for $\varepsilon = 2^{-p}$, $p = 1, 2, ..30$. Estimates of the global error

$$||\bar{U}_\varepsilon^N - u_\varepsilon||_{\bar{\Omega}}$$

are obtained by evaluating

$$E_{\varepsilon,global}^N = \max_{x_i \in \Omega^*} |\bar{U}_\varepsilon^N(x_i) - u_\varepsilon(x_i)| \ ,$$

where $\Omega^* = \Omega_0^* \cup \Omega_1^*$ and

$$\begin{aligned} \Omega_0^* &= \{x_i | x_i = i(1-\tau)/4096, 0 \leq i \leq 4096\} \\ \Omega_c^* &= \{x_i | x_i = 1 - \tau + i\tau/4096, 0 \leq i \leq 4096\}. \end{aligned}$$

Estimates of the orders of convergence are computed for each N and ε from

$$p_{\varepsilon,global}^N = \log_2\Big(\frac{E_{\varepsilon,global}^N}{E_{\varepsilon,global}^{2N}}\Big).$$

The values of $E_{\varepsilon,global}^N$ for problem (4.1) are given in Table 4.1, and the corresponding rates $p_{\varepsilon,global}^N$ are given in Table 4.3.

For values of ε and N such that $\frac{\varepsilon}{\alpha} \ln N \geq 1/3$, we choose the following Schwarz approach. Let $\tau = 1 - 1/N$ and $\xi^+ = 1 - \tau = 1/N$, $\xi^- = \frac{1}{2N}$. Using these new interface conditions we apply the same approach described in Subsection 4.2.2. In this method we have moved the interface position so that the overlapping region is now half of the interval $[0, \xi^+]$ and so this method iterates efficiently while it is also seen to be convergent. In Tables 4.1 to 4.3, the figures relating to this approach are located above the horizontal lines seen in the body of the tables.

We now consider the values of ε and N such that $\frac{\varepsilon}{\alpha} \ln N < 1/3$, located below the horizontal lines, that is the singularly perturbed case. In Table 4.1, we observe that Method 4.1 produces approximations which converge to the true solution with a first rate of convergence, illustrated in Table 4.3. Therefore, allowing the change over between methods we can say this Schwarz approach produces ε-uniform error convergence. This verifies the theoretical result stated in Theorem 4.2.1.

However, a significant drawback to this method is observed in Table 4.2. As predicted by the analysis the iteration counts increase as the mesh dimension N is increased. This effect is more pronounced when τ is close to 1/3, when both N and ε are large, and we see the iteration counts almost doubling in size. For very small values of ε, we observe in Table 4.2, that the iterations remain small even as N is increased. This is explained by the dominance of ε over N in the reduction factor λ^{-1} given in Theorem 4.2.1.

4.2.4 Conclusions

We conclude that the approximate solutions of Method 4.1 converge ε- uniformly to the solution of Problem 3.1, and is therefore, a suitable alternative to Method 3.2. However, the iteration numbers, although small for small values of ε, are proportional to N when ε is large. Also, Method 4.1 does not have the advantage of using uniform

	Number of Intervals N in each subdomain								
ε	8	16	32	64	128	256	512	1024	2048
2^{-0}	1.22e-02	7.09e-03	3.77e-03	1.94e-03	9.81e-04	4.93e-04	2.47e-04	1.24e-04	6.20e-05
2^{-1}	2.23e-02	1.25e-02	6.54e-03	3.34e-03	1.69e-03	8.49e-04	4.26e-04	2.13e-04	1.07e-04
2^{-2}	4.29e-02	2.39e-02	1.26e-02	6.42e-03	3.24e-03	1.63e-03	8.17e-04	4.09e-04	2.05e-04
2^{-3}	*1.98e-02*	4.20e-02	2.22e-02	1.14e-02	5.75e-03	2.89e-03	1.45e-03	7.25e-04	3.63e-04
2^{-4}	2.43e-02	1.27e-02	7.05e-03	4.43e-03	*2.76e-03*	5.47e-03	2.75e-03	1.38e-03	6.90e-04
2^{-5}	3.58e-02	1.90e-02	9.70e-03	4.83e-03	2.38e-03	1.38e-03	8.20e-04	4.79e-04	2.75e-04
2^{-6}	4.56e-02	2.29e-02	1.19e-02	6.01e-03	3.00e-03	1.49e-03	7.40e-04	4.21e-04	2.42e-04
2^{-7}	6.07e-02	2.54e-02	1.32e-02	6.71e-03	3.38e-03	1.69e-03	8.42e-04	4.20e-04	2.25e-04
2^{-8}	7.20e-02	2.93e-02	1.39e-02	7.10e-03	3.59e-03	1.80e-03	9.01e-04	4.50e-04	2.25e-04
2^{-9}	7.95e-02	3.54e-02	1.43e-02	7.31e-03	3.70e-03	1.86e-03	9.34e-04	4.67e-04	2.33e-04
2^{-10}	8.41e-02	3.94e-02	1.65e-02	7.45e-03	3.77e-03	1.90e-03	9.52e-04	4.76e-04	2.38e-04
2^{-11}	8.69e-02	4.19e-02	1.85e-02	7.52e-03	3.80e-03	1.92e-03	9.61e-04	4.81e-04	2.47e-04
2^{-12}	8.85e-02	4.33e-02	1.97e-02	8.49e-03	3.83e-03	1.92e-03	1.02e-03	5.28e-04	2.70e-04
2^{-13}	8.94e-02	4.41e-02	2.04e-02	9.12e-03	3.92e-03	1.98e-03	1.07e-03	5.67e-04	2.94e-04
2^{-14}	8.99e-02	4.45e-02	2.08e-02	9.45e-03	4.24e-03	2.02e-03	1.10e-03	5.94e-04	3.14e-04
2^{-15}	9.02e-02	4.49e-02	2.11e-02	9.62e-03	4.34e-03	2.04e-03	1.12e-03	6.11e-04	3.27e-04
2^{-16}	9.03e-02	4.49e-02	2.11e-02	9.64e-03	4.35e-03	2.05e-03	1.13e-03	6.20e-04	3.35e-04
2^{-17}	9.03e-02	4.49e-02	2.11e-02	9.65e-03	4.36e-03	2.05e-03	1.14e-03	6.24e-04	3.40e-04
2^{-18}	9.03e-02	4.49e-02	2.11e-02	9.65e-03	4.36e-03	2.06e-03	1.14e-03	6.27e-04	3.42e-04
2^{-19}	9.03e-02	4.49e-02	2.11e-02	9.66e-03	4.36e-03	2.06e-03	1.14e-03	6.28e-04	3.43e-04
2^{-20}	9.03e-02	4.49e-02	2.11e-02	9.66e-03	4.37e-03	2.06e-03	1.14e-03	6.29e-04	3.44e-04
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-30}	9.03e-02	4.49e-02	2.11e-02	9.66e-03	4.37e-03	2.06e-03	1.14e-03	6.29e-04	3.44e-04

Table 4.1: Computed global errors $E^N_{\varepsilon,global}$ for various values of ε and N for Method 4.1 applied to problem (4.1)

	Number of Intervals N in each subdomain								
ε	8	16	32	64	128	256	512	1024	2048
2^{-0}	22	21	19	17	15	13	11	9	7
2^{-1}	22	21	20	18	16	14	12	10	8
2^{-2}	21	21	20	18	16	14	12	10	8
2^{-3}	**31**	19	19	17	15	13	11	9	7
2^{-4}	**19**	**32**	**56**	**105**	**204**	12	10	8	6
2^{-5}	**14**	**20**	**31**	**53**	**95**	**178**	**341**	**659**	**1280**
2^{-6}	**11**	**14**	**20**	**31**	**51**	**91**	**167**	**314**	**598**
2^{-7}	9	**11**	**15**	**20**	**31**	**50**	**88**	**161**	**301**
2^{-8}	8	9	**11**	**15**	**20**	**31**	**50**	**87**	**158**
2^{-9}	7	8	9	**11**	**15**	**20**	**31**	**50**	**86**
2^{-10}	6	7	8	9	**12**	**15**	**20**	**30**	**49**
2^{-11}	6	6	7	8	10	**12**	**15**	**20**	**30**
2^{-12}	6	6	7	7	8	10	**12**	**15**	**20**
2^{-13}	5	6	6	7	7	8	10	**12**	**15**
2^{-14}	5	5	6	6	7	7	8	10	**12**
2^{-15}	5	5	5	6	6	7	7	8	10
2^{-16}	5	5	5	5	6	6	7	7	8
2^{-17}	4	5	5	5	5	6	6	7	7
2^{-18}	4	5	5	5	5	5	6	6	7
2^{-19}	4	4	5	5	5	5	5	6	6
2^{-20}	4	4	4	5	5	5	5	5	6
2^{-21}	4	4	4	4	5	5	5	5	5
2^{-22}	4	4	4	4	4	5	5	5	5
2^{-23}	4	4	4	4	4	4	5	5	5
2^{-24}	4	4	4	4	4	4	4	5	5
2^{-25}	4	4	4	4	4	4	4	4	5
2^{-26}	4	4	4	4	4	4	4	4	4
2^{-27}	4	4	4	4	4	4	4	4	4
2^{-28}	4	4	4	4	4	4	4	4	4
2^{-29}	4	4	4	4	4	4	4	4	4
2^{-30}	4	4	4	4	4	4	4	4	4

Table 4.2: Iteration counts for Method 4.1 applied to problem (4.1)

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	0.78	0.91	0.96	0.98	0.99	1.00	1.00	1.00
2^{-1}	0.84	0.93	0.97	0.98	0.99	1.00	1.00	1.00
2^{-2}	0.84	0.93	0.97	0.98	0.99	1.00	1.00	1.00
2^{-3}	*-1.08*	0.92	0.97	0.98	0.99	1.00	1.00	1.00
2^{-4}	0.94	0.85	0.67	0.69	*-0.99*	0.99	1.00	1.00
2^{-5}	0.91	0.97	1.00	1.02	0.79	0.75	0.78	0.80
2^{-6}	0.99	0.95	0.98	1.00	1.01	1.01	0.81	0.80
2^{-7}	1.26	0.95	0.97	0.99	1.00	1.00	1.01	0.90
2^{-8}	1.30	1.08	0.97	0.99	0.99	1.00	1.00	1.00
2^{-9}	1.17	1.30	0.97	0.98	0.99	1.00	1.00	1.00
2^{-10}	1.10	1.26	1.14	0.98	0.99	1.00	1.00	1.00
2^{-11}	1.05	1.18	1.30	0.98	0.99	1.00	1.00	0.96
2^{-12}	1.03	1.13	1.22	1.15	0.99	0.92	0.95	0.97
2^{-13}	1.02	1.11	1.16	1.22	0.98	0.89	0.92	0.95
2^{-14}	1.01	1.10	1.13	1.16	1.07	0.87	0.89	0.92
2^{-15}	1.01	1.09	1.13	1.15	1.09	0.86	0.88	0.90
2^{-16}	1.01	1.09	1.13	1.15	1.09	0.86	0.87	0.89
2^{-17}	1.01	1.09	1.13	1.15	1.08	0.85	0.86	0.88
2^{-18}	1.01	1.09	1.13	1.15	1.08	0.85	0.86	0.87
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-30}	1.01	1.09	1.13	1.15	1.08	0.85	0.86	0.87

Table 4.3: Computed convergence rates $p^N_{\varepsilon,global}$ for various values of ε and N for Method 4.1 applied to problem (4.1)

meshes in both subdomains.

4.3 A non-overlapping Schwarz method with Dirichlet interface conditions

4.3.1 Introduction

In this Schwarz method, the solution domain is partitioned into two non-overlapping subdomains using the Shishkin transition point, τ. As in the previous method, the subdomain containing the boundary layer reduces for decreasing values of ε thus, refining the mesh step size, while the course mesh subdomain is positioned outside the layer. During the iteration process, the discrete solution value at the last interior point in the coarse mesh subdomain is passed, using a Dirichlet boundary condition, to the layer subdomain. In this way, values are passed from the smooth region into the layer region but not back. The method is analysed, using a decomposition of the discrete Schwarz iterates, in Lemmas 4.3.1 and 4.3.2, and these results are combined in Theorem 4.3.1 to give a general estimate of the approximating error.

4.3.2 Discrete Schwarz method

We now formally describe the method.

Method 4.2 *The exact solution u_ε is approximated by the limit $\bar{U}_\varepsilon$ of a sequence of discrete Schwarz iterates $\{\bar{U}_\varepsilon^{[k]}\}_{k=0}^{\infty}$, which are defined as follows. For each $k \geq 1$,*

$$\bar{U}_\varepsilon^{[k]}(x) = \begin{cases} \bar{U}_0^{[k]}(x) & x \in \bar{\Omega}_0 \\ \bar{U}_1^{[k]}(x) & x \in \bar{\Omega}_1 \end{cases}$$

where $\bar{U}_i^{[k]}$ is the linear interpolant of $U_i^{[k]}$. Let $\bar{\Omega}_0^N = \{x_i\}_0^N$ be a uniform mesh on Ω_0

with $x_i = i\xi^+/N$ *and* $\bar{\Omega}_1^N = \{x_i\}_0^N$ *be a uniform mesh on* Ω_1 *with* $x_i = \xi^+ + i(1-\xi^+)/N$, *as shown in Fig 4.2. Then for* $k = 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[1]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[1]}(0) = u_0, \quad U_0^{[1]}(\xi^+) = 0, \\ L_\varepsilon^N U_1^{[1]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[1]}(\xi^+) = U_0^{[1]}(\xi^-), \quad U_0^{[1]}(1) = u_1, \end{aligned}$$

and for $k > 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[k]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[k]}(0) = u_0, \quad U_0^{[k]}(\xi^+) = U_1^{[k-1]}(\xi^+), \\ L_\varepsilon^N U_1^{[k]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[k]}(\xi^+) = U_0^{[k]}(\xi^-), \quad U_0^{[k]}(1) = u_1. \end{aligned}$$

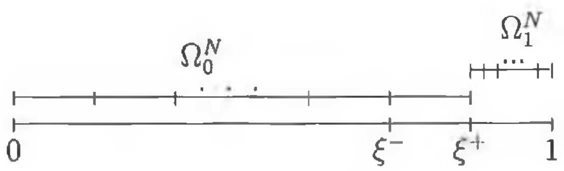

Figure 4.2: The discretised non-overlapping subdomains Ω_0^N and Ω_1^N for Method 4.2

In the following lemma we derive an estimate for the error contained in the smooth Schwarz component $V_\varepsilon^{[k]}$.

Lemma 4.3.1 *Let* $V_\varepsilon^{[k]}$ *denote the discrete approximation to* v_ε, *the singular component of the solution* u_ε. *For all* $k \geq 1$

$$||(V_\varepsilon^{[k]} - v_\varepsilon)(x_i)|| \leq CN^{-1} + C\lambda^{-k} + C\varepsilon.$$

Proof. The discrete problem is first solved on the discretised subdomain Ω_0^N using the initial boundary condition $V_0^{[1]} = 0$.

For $x_i \in \Omega_0^N$,

$$|(V_0^{[1]} - v_\varepsilon)(0)| = 0, \quad |(V_0^{[1]} - v_\varepsilon)(\xi^+)| \leq C.$$

Classical arguments together with estimates for v_ε'' and v_ε''' leads, as before, to the following local truncation estimate

$$|L_\varepsilon^N(V_0^{[1]} - v_\varepsilon)(x_i)| \leq CN^{-1}.$$

Then, as in Lemma 4.2.1, the barrier function

$$\Phi_i = C\left(1 - \left(\frac{1 - \lambda^{-N+i}}{1 - \lambda^{-N}}\right)\right) + CN^{-1}x_i,$$

is defined, and it follows by maximum principle arguments that

$$|(V_0^{[1]} - v_\varepsilon)(x_i)| \leq C\left(1 - \left(\frac{1 - \lambda^{-N+i}}{1 - \lambda^{-N}}\right)\right) + CN^{-1}x_i, \quad x_i \in \Omega_0^N. \tag{4.17}$$

The discrete problem is then solved on the subdomain Ω_1^N, which contains the layer region. The boundary value at the interface point $x_0 = \xi^+$ is given by $V_1^{[1]}(\xi^+) = V_0^{[1]}(\xi^-)$. On the boundary $\partial\Omega_1^N$,

$$\begin{aligned}
|(V_1^{[1]} - v_\varepsilon)(1)| &= 0. \\
|(V_1^{[1]} - v_\varepsilon)(\xi^+)| &= |V_0^{[1]}(\xi^-) - v_\varepsilon(\xi^+)| \\
&\leq |V_0^{[1]}(\xi^-) - v_\varepsilon(\xi^-)| + |v_\varepsilon(\xi^+) - v_\varepsilon(\xi^-)| \\
&\leq C\Big(1 - \Big(\frac{1 - \lambda^{-1}}{1 - \lambda^{-N}}\Big)\Big) + CN^{-1}(\xi^-) \\
&\quad + |v_\varepsilon(\xi^+) - v_\varepsilon(\xi^-)| \\
&\leq C(1 - q) + m + CN^{-1}(\xi^-),
\end{aligned}$$

where we use the notation $m = |v_\varepsilon(\xi^+) - v_\varepsilon(\xi^-)|$, and as in (4.3), $q = \left(\frac{1-\lambda^{-1}}{1-\lambda^{-N}}\right)$. The barrier function, Φ_i is introduced on Ω_1^N,

$$\Phi_i = C(1 - q) + m + CN^{-1}x_i,$$

and it can be seen that,

$$
\begin{aligned}
L_\varepsilon^N\Big(\Phi_i \pm (V_1^{[1]} - v_\varepsilon)\Big)(x_i) &\geq 0, \\
\big(\Phi_i \pm (V_1^{[1]} - v_\varepsilon)\big)(\xi^+) &\geq 0, \\
\big(\Phi_i \pm (V_1^{[1]} - v_\varepsilon)\big)(1) &\geq 0.
\end{aligned}
$$

The discrete maximum principle for L_ε^N on Ω_0^N then gives,

$$
|(V_1^{[1]} - v_\varepsilon)(x_i)| \leq C(1-q) + m + CN^{-1}(x_i), \quad \forall x_i \in \bar{\Omega}_1^N. \tag{4.18}
$$

Hence, from the estimates given by the inequalities (4.17) and (4.18), we can conclude that

$$
|(V_\varepsilon^{[1]} - v_\varepsilon)(x_i)| \leq C(1-q) + m + CN^{-1}(x_i), \quad \forall x_i \in \bar{\Omega}^N.
$$

Noting that this is an iterative method, and that we have seen in chapter 3 an error can be introduced during the iteration process, we have included the second iteration in this proof. The proof is then concluded by induction.

For all $x_i \in \Omega_0^N$,

$$
\begin{aligned}
|(V_0^{[2]} - v_\varepsilon)(0)| &= 0, \\
|(V_0^{[2]} - v_\varepsilon)(\xi^+)| &= |(V_1^{[1]} - v_\varepsilon)(\xi^+)| \\
&\leq C(1-q) + m + CN^{-1}(\xi^+).
\end{aligned}
$$

The barrier function, Φ_i, is defined to be

$$
\Phi_i = (C(1-q)+m)\Big(1 - \Big(\frac{1-\lambda^{-N+i}}{1-\lambda^{-N}}\Big)\Big) + CN^{-1}x_i,
$$

and by similar arguments as before, it follows that

$$
|(V_0^{[2]} - v_\varepsilon)(x_i)| \leq (C(1-q)+m)\Big(1 - \Big(\frac{1-\lambda^{-N+i}}{1-\lambda^{-N}}\Big)\Big) + CN^{-1}x_i. \tag{4.19}
$$

The discrete problem is now solved on Ω_1^N, with

$$\begin{aligned}
|(V_0^{[2]} - v_\varepsilon)(\xi^+)| &\leq |(V_1^{[2]}(\xi^+) - v_\varepsilon(\xi^-)| + |v_\varepsilon(\xi^+) - v_\varepsilon(\xi^-)| \\
&\leq \Big(C(1-q) + m\Big)(1-q) + m + CN^{-1}x_i, \\
|(V_1^{[2]} - v_\varepsilon)(1)| &= 0.
\end{aligned}$$

As before, using a barrier function together with discrete maximum principle arguments gives an appropriate estimate,

$$|(V_1^{[2]} - v_\varepsilon)(x_i)| \leq (C(1-q) + m)(1-q) + m + CN^{-1}(\xi^-), \quad \forall x_i \in \bar{\Omega}_1^N. \tag{4.20}$$

Combining (4.19) and (4.20) then gives,

$$|(V_\varepsilon^{[2]} - v_\varepsilon)(x_i)| \leq C(1-q)^2 + m(1-q) + m + CN^{-1}(\xi^-), \quad \forall x_i \in \bar{\Omega}^N.$$

By induction, it can be shown that for the k^{th} iteration,

$$\begin{aligned}
|(V_\varepsilon^{[k]} - v_\varepsilon)(x_i)| &\leq C(1-q)^k + m(1-q)^{k-1} + m(1-q)^{k-2} \\
&+ \ldots + m(1-q) + m + CN^{-1}(\xi^-) \quad \text{in } \bar{\Omega}^N \\
&= C(1-q)^k + \frac{m}{q}(1-(1-q)^k) + CN^{-1}x_i \\
&\leq C\lambda^{-k} + \frac{m}{q} + CN^{-1}, \quad \text{since} \quad 1-q < \lambda^{-1}.
\end{aligned}$$

Therefore, we can say that the inequality,

$$|(V_\varepsilon^{[k]} - v_\varepsilon)(x_i)| \leq C\lambda^{-k} + \frac{m}{q} + CN^{-1}x_i \tag{4.21}$$

provides an estimate for the approximation error contained in the smooth Schwarz iterate, $V_\varepsilon^{[k]}$, where $m = |v_\varepsilon(\xi^+) - v_\varepsilon(\xi^-)|$. Now, it remains to determine in what way the term $\frac{m}{q}$ depends on the parameters ε and N.

An upper bound for m is obtained from the Mean Value Theorem,

$$v_\varepsilon(\xi^+) = v_\varepsilon(\xi^-) + (\xi^+ - \xi^-)\frac{dv}{dx}(\mu), \quad \text{where } \xi^- < \mu < \xi^+.$$

Using the bounds on the derivatives of v_ε then gives

$$\begin{aligned} |v_\varepsilon(\xi^+) - v_\varepsilon(\xi^-)| &\leq (\xi^+ - \xi^-)\left|\frac{dv}{dx}(\mu)\right| \\ &\leq C\frac{(1-\tau)}{N}. \end{aligned}$$

Noting, $q = \dfrac{1-\lambda^{-1}}{1-\lambda^{-N}}$, $\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau)$, and the restriction $\frac{\varepsilon}{\alpha}\ln N < 1$ guarantees $\lambda > 1$. It follows that,

$$\begin{aligned} q > 1 - \lambda^{-1} &= 1 - \frac{1}{1 + \dfrac{\alpha}{\varepsilon N}(1-\tau)} \\ &= \frac{\alpha(1-\tau)}{\varepsilon N + \alpha(1-\tau)}. \end{aligned}$$

Thus, an upper bound for the term $\frac{m}{q}$, is then given by

$$\frac{m}{q} < \frac{\dfrac{C(1-\tau)}{N}}{\dfrac{\alpha(1-\tau)}{\varepsilon N + \alpha(1-\tau)}} = C\Big(\frac{\varepsilon N + \alpha(1-\tau)}{N}\Big),$$

and we have

$$\begin{aligned} \frac{m}{q} &< C\Big(\frac{\varepsilon N + \alpha(1-\tau)}{N}\Big) \\ &= C\varepsilon + \frac{\alpha(1-\tau)}{N} \\ &\leq C\varepsilon + CN^{-1}. \end{aligned} \tag{4.22}$$

Combining (4.21) and (4.22) concludes this lemma. ◇

Now, in the following lemma we derive error estimates for the approximation of the singular Schwarz component, $W_\varepsilon^{[k]}$, to the solution component, w_ε.

Lemma 4.3.2 *Let $W_\varepsilon^{[k]}$ denote the discrete approximation to w_ε, the singular component of the solution u_ε. Then, for all $k \geq 1$*

$$|(W_\varepsilon^{[k]} - w_\varepsilon)(x_i)| \leq CN^{-1}(\ln N)^2$$

where C is a constant independent of k, N and ε.

Proof. From the bounds on w_ε, stated in the decomposition of the solution, it is clear that

$$|w_\varepsilon(x)| \le CN^{-1} \quad \text{in } \bar{\Omega}_0.$$

Now, on the boundary, $\partial\Omega_0$ we have the conditions $W_0^{[1]}(\xi^+) = 0$ and $W_0^{[1]}(0) = w_\varepsilon(0)$, and from the decomposition of the Schwarz iterates, $L_\varepsilon^N W_0^{[1]} = 0$. Therefore, by applying the discrete maximum principle we can see that,

$$|W_0^{[1]}(x)| \le |w_\varepsilon(0)| \le CN^{-1} \quad \text{in } \bar{\Omega}_0^N.$$

Consequently, using the triangle inequality then gives a estimate for the error contained in the $W_0^{[1]}$ iterate,

$$|(W_0^{[1]} - w_\varepsilon)(x_i)| \le CN^{-1}, \quad \forall x_i \in \bar{\Omega}_0^N. \tag{4.23}$$

In the subdomain Ω_1^N, the interface condition $W_1^{[1]}(\xi^+) = W_0^{[1]}(\xi^-)$, gives

$$\begin{aligned}
|(W_1^{[1]} - w_\varepsilon)(\xi^+)| &= |W_0^{[1]}(\xi^-) - w_\varepsilon(\xi^+)| \\
&\le |W_0^{[1]}(\xi^-)| + |w_\varepsilon(\xi^+)| \\
&\le CN^{-1} + CN^{-1}, \\
|(W_1^{[1]} - w_\varepsilon)(1)| &= 0.
\end{aligned}$$

As in Lemma 4.2.5, an upper bound for the local truncation, for $x_i \in (\xi^+, 1)$, is given by

$$|L_\varepsilon^N (W_1^{[1]} - w_\varepsilon)(x_i)| \le C\varepsilon^{-2}\tau N^{-1}.$$

Now, we define an appropriate barrier function,

$$\Phi = (x_i - \xi^+)C\varepsilon^{-2}\tau N^{-1} + CN^{-1},$$

and using discrete maximum principle for L_ε^N on Ω_1^N, it follows that

$$|(W_1^{[1]} - w_\varepsilon)(x_i)| \le CN^{-1}(\ln N)^2 \qquad \text{in } \bar{\Omega}_1^N. \tag{4.24}$$

Hence, combining (4.23) and (4.24) the estimate

$$|(W_\varepsilon^{[1]} - w_\varepsilon)(x_i)| \leq CN^{-1}(\ln N)^2$$

holds, for all $x_i \in \bar{\Omega}^N$.

Again, we continue with the 2nd iteration and the proof is concluded by mathematical induction. Recall, for $x \in \bar{\Omega}_0$,

$$|w_\varepsilon(x)| \leq CN^{-1} \quad \text{in } \Omega_0.$$

Furthermore, at $\partial\Omega_0^N = \{0, \xi^+\}$ it can easily be seen from the previous iteration that

$$|W_0^{[2]}(\xi^+)| = |W_1^{[1]}(\xi^+)| = |W_0^{[1]}(\xi^-)| \leq CN^{-1},$$

and also

$$|W_0^{[2]}(0)| \leq |w_\varepsilon(0)| \leq CN^{-1}.$$

Now, from the decomposition of the Schwarz iterates, $W_0^{[2]}(x_i)$ satisfies

$$L_\varepsilon^N W_0^{[2]} = 0 \quad \forall x_i \in \Omega_0^N.$$

Therefore, using the discrete maximum principle for L_ε^N on Ω_0 then gives, for all $x_i \in \bar{\Omega}_0^N$,

$$|(W_0^{[2]} - w_\varepsilon)(x_i)| \leq CN^{-1}, \tag{4.25}$$

and in using similar arguments to those used in the previous iteration yields,

$$|(W_1^{[2]} - w_\varepsilon)(x_i)| \leq CN^{-1}(\ln N)^2. \tag{4.26}$$

Finally, combining (4.25) and (4.26) gives

$$|(W_\varepsilon^{[2]} - w_\varepsilon)(x_i)| \leq CN^{-1}(\ln N)^2, \quad \forall x_i \in \bar{\Omega}^N,$$

and proceeding by mathematical induction then provides the general result for the k^{th} iterative,

$$|(W_\varepsilon^{[k]} - w_\varepsilon)(x_i)| \leq CN^{-1}(\ln N)^2, \quad \forall x_i \in \bar{\Omega}^N,$$

which concludes the proof of this lemma. $\diamond$

In the following corollary to Lemmas 4.3.1 and 4.3.2 we give an approximation error estimate for the discrete Schwarz iterates, $U^{[k]}$, by considering the triangle inequality

$$|U_\varepsilon^{[k]} - u_\varepsilon| \leq |V_\varepsilon^{[k]} - v_\varepsilon| + |W_\varepsilon^{[k]} - w_\varepsilon|.$$

Corollary 4.3.1 *For all $k \geq 1$*

$$||(U_\varepsilon^{[k]} - u_\varepsilon)(x_i)|| \leq CN^{-1}(\ln N)^2 + C\varepsilon + C\lambda^{-k},$$

where C is a constant independent of k and ε, $\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau)$ and $\tau = \frac{\varepsilon}{\alpha}\ln N < 1/3$.

It is known that the piecewise linear interpolant $\bar{U}_\varepsilon^{[k]}$ retains the above error estimate (see [6]). This corollary combined with Lemma 3 provides the main theoretical result of this section.

Theorem 4.3.1 *For all $k \geq 1$,*

$$||(\bar{U}_\varepsilon^{[k]} - u_\varepsilon)|| \leq CN^{-1}(\ln N)^2 + C\varepsilon + C\lambda^{-k},$$

where C is a constant independent of k and ε, $\lambda = 1 + \frac{\alpha}{\varepsilon N}(1-\tau)$ and $\tau = \frac{\varepsilon}{\alpha}\ln N < 1/3$.

4.3.3 Numerical results

Numerical results are presented in this section which confirm the theoretical estimates derived in the previous section. The stopping criterion for the Schwarz iterations, the meshes $\bar{\Omega}^N_\varepsilon$ and Ω^*, and the notation for the errors, $E^N_{\varepsilon,global}$, $p^N_{\varepsilon,global}$ are as given in Section 4.2.3. As in the previous method, the results obtained for problem (4.2) show the same convergent behaviour as observed for problem (4.1), and so, we only tabulate results for the latter problem.

When ε and N are such that $\frac{\varepsilon}{\alpha}\ln N > 1/3$, we let $\xi^+ = 1/N$ and $\xi^- = 1/(2N)$ and use Method 4.2, thus creating a non-singularly perturbed approach. The values associated with this approach appear in the tables above the horizontal lines and, as expected, these approximations are first order convergent and the number of required iterations is small.

In Tables 4.4 and 4.6, we observe that the approximating error for this method is bounded above by ε, as predicted by the theoretical results. This diagonal effect is illustrated by the emphasized error values in Table 4.4 and the bolded rate values in Table 4.6. The iteration numbers, presented in Table 4.5, increase with N and are of the same order as those presented by the previous method in Table 4.2

4.3.4 Conclusions

We conclude that, although Method 4.2 is not ε-uniform convergent, it produces almost first order approximations for $\varepsilon \leq N^{-1}$, and the iteration counts are small for small values of ε.

ε	Number of Intervals N in each subdomain								
	8	16	32	64	128	256	512	1024	2048
2^{-0}	1.63e-02	8.12e-03	4.02e-03	2.00e-03	9.96e-04	4.97e-04	2.48e-04	1.24e-04	6.20e-05
2^{-1}	2.68e-02	1.35e-02	6.80e-03	3.41e-03	1.71e-03	8.53e-04	4.27e-04	2.13e-04	1.07e-04
2^{-2}	4.76e-02	2.49e-02	1.28e-02	6.48e-03	3.26e-03	1.63e-03	8.18e-04	4.09e-04	2.05e-04
2^{-3}	*7.41e-02*	4.26e-02	2.24e-02	1.14e-02	5.76e-03	2.89e-03	1.45e-03	7.25e-04	3.63e-04
2^{-4}	*4.43e-02*	*4.62e-02*	*4.67e-02*	*4.64e-02*	4.53e-02	5.47e-03	2.75e-03	1.38e-03	6.90e-04
2^{-5}	*3.55e-02*	*2.82e-02*	*2.75e-02*	*2.68e-02*	*2.64e-02*	*2.62e-02*	*2.58e-02*	*2.53e-02*	*2.47e-02*
2^{-6}	4.22e-02	*2.28e-02*	*1.73e-02*	*1.56e-02*	*1.47e-02*	*1.43e-02*	*1.42e-02*	*1.41e-02*	*1.40e-02*
2^{-7}	4.69e-02	2.54e-02	*1.31e-02*	*1.02e-02*	*8.63e-03*	*7.80e-03*	*7.48e-03*	*7.40e-03*	*7.38e-03*
2^{-8}	5.12e-02	2.65e-02	1.39e-02	*7.62e-03*	*5.79e-03*	*4.66e-03*	*4.08e-03*	*3.85e-03*	*3.80e-03*
2^{-9}	5.55e-02	2.75e-02	1.43e-02	7.31e-03	*4.46e-03*	*3.21e-03*	*2.48e-03*	*2.11e-03*	*1.96e-03*
2^{-10}	5.83e-02	2.83e-02	1.45e-02	7.46e-03	3.84e-03	*2.54e-03*	*1.75e-03*	*1.31e-03*	*1.09e-03*
2^{-11}	6.01e-02	2.92e-02	1.47e-02	7.52e-03	3.80e-03	2.22e-03	*1.41e-03*	*9.43e-04*	*6.86e-04*
2^{-12}	6.11e-02	3.00e-02	1.49e-02	7.55e-03	3.83e-03	2.06e-03	1.25e-03	*7.73e-04*	*5.04e-04*
2^{-13}	6.17e-02	3.05e-02	1.50e-02	7.59e-03	3.84e-03	1.99e-03	1.17e-03	6.93e-04	*4.19e-04*
2^{-14}	6.20e-02	3.08e-02	1.52e-02	7.63e-03	3.84e-03	1.95e-03	1.13e-03	6.53e-04	3.79e-04
2^{-15}	6.23e-02	3.10e-02	1.54e-02	7.66e-03	3.85e-03	1.94e-03	1.11e-03	6.34e-04	3.59e-04
2^{-16}	6.23e-02	3.10e-02	1.54e-02	7.67e-03	3.86e-03	1.94e-03	1.10e-03	6.24e-04	3.50e-04
2^{-17}	6.23e-02	3.10e-02	1.54e-02	7.68e-03	3.87e-03	1.94e-03	1.10e-03	6.19e-04	3.45e-04
2^{-18}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.87e-03	1.94e-03	1.10e-03	6.17e-04	3.42e-04
2^{-19}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.87e-03	1.94e-03	1.10e-03	6.16e-04	3.41e-04
2^{-20}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.87e-03	1.94e-03	1.10e-03	6.15e-04	3.41e-04
2^{-21}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.88e-03	1.95e-03	1.10e-03	6.15e-04	3.40e-04
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-30}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.88e-03	1.95e-03	1.09e-03	6.15e-04	3.40e-04

Table 4.4: Computed global errors $E^N_{\varepsilon,global}$ for various values of ε and N for Method 4.2 applied to problem (4.1)

	Number of Intervals N in each subdomain								
ε	8	16	32	64	128	256	512	1024	2048
2^{-0}	19	16	13	10	7	4	2	2	2
2^{-1}	19	17	14	11	8	5	2	2	2
2^{-2}	18	17	15	12	9	6	3	2	2
2^{-3}	**35**	17	15	13	10	7	4	2	2
2^{-4}	**21**	**34**	**58**	**106**	**203**	8	5	2	2
2^{-5}	**15**	**21**	**32**	**54**	**96**	**178**	**340**	**656**	**1274**
2^{-6}	**12**	**15**	**21**	**32**	**52**	**91**	**167**	**313**	**597**
2^{-7}	**10**	**12**	**15**	**21**	**31**	**51**	**88**	**161**	**300**
2^{-8}	8	**10**	**12**	**15**	**21**	**31**	**50**	**87**	**158**
2^{-9}	7	8	**10**	**12**	**15**	**20**	**31**	**50**	**86**
2^{-10}	7	7	8	**10**	**12**	**15**	**20**	**31**	**49**
2^{-11}	6	7	7	8	**10**	**12**	**15**	**20**	**30**
2^{-12}	6	6	7	7	8	**10**	**12**	**15**	**20**
2^{-13}	5	6	6	7	7	8	**10**	**12**	**15**
2^{-14}	5	5	6	6	7	7	8	**10**	**12**
2^{-15}	5	5	5	6	6	7	7	8	**10**
2^{-16}	5	5	5	5	6	6	7	7	8
2^{-17}	5	5	5	5	5	6	6	7	7
2^{-18}	4	5	5	5	5	5	6	6	7
2^{-19}	4	4	5	5	5	5	5	6	6
2^{-20}	4	4	4	5	5	5	5	5	6
2^{-21}	4	4	4	4	5	5	5	5	5
2^{-22}	4	4	4	4	4	5	5	5	5
2^{-23}	4	4	4	4	4	4	5	5	5
2^{-24}	4	4	4	4	4	4	4	5	5
2^{-25}	4	4	4	4	4	4	4	4	5
2^{-26}	4	4	4	4	4	4	4	4	4
2^{-27}	4	4	4	4	4	4	4	4	4
2^{-28}	4	4	4	4	4	4	4	4	4
2^{-29}	4	4	4	4	4	4	4	4	4
2^{-30}	4	4	4	4	4	4	4	4	4

Table 4.5: Iteration counts for Method 4.2 applied to problem (4.1)

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	1.01	1.01	1.01	1.01	1.00	1.00	1.00	1.00
2^{-1}	0.98	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2^{-2}	0.93	0.96	0.98	0.99	1.00	1.00	1.00	1.00
2^{-3}	*0.80*	0.93	0.97	0.99	0.99	1.00	1.00	1.00
2^{-4}	**-0.06**	**-0.01**	**0.01**	**0.03**	*3.05*	0.99	1.00	1.00
2^{-5}	**0.34**	**0.04**	**0.04**	**0.02**	**0.01**	**0.02**	**0.03**	**0.03**
2^{-6}	0.89	**0.39**	**0.15**	**0.09**	**0.04**	**0.01**	**0.01**	**0.01**
2^{-7}	0.89	0.95	**0.36**	**0.24**	**0.15**	**0.06**	**0.01**	**0.00**
2^{-8}	0.95	0.93	0.87	**0.40**	**0.31**	**0.19**	**0.08**	**0.02**
2^{-9}	1.01	0.94	0.97	0.71	**0.48**	**0.37**	**0.23**	**0.10**
2^{-10}	1.04	0.96	0.96	0.96	0.60	**0.54**	**0.42**	**0.27**
2^{-11}	1.04	0.99	0.97	0.98	0.78	0.65	**0.58**	**0.46**
2^{-12}	1.02	1.01	0.98	0.98	0.89	0.72	0.69	**0.62**
2^{-13}	1.01	1.03	0.98	0.98	0.95	0.76	0.76	0.72
2^{-14}	1.01	1.02	0.99	0.99	0.98	0.78	0.79	0.78
2^{-15}	1.01	1.01	1.01	0.99	0.99	0.80	0.81	0.82
2^{-16}	1.01	1.01	1.00	0.99	1.00	0.81	0.82	0.84
2^{-17}	1.01	1.01	1.00	0.99	1.00	0.82	0.83	0.85
2^{-18}	1.01	1.01	1.00	0.99	1.00	0.82	0.83	0.85
2^{-19}	1.01	1.01	1.00	0.99	0.99	0.83	0.83	0.85
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-30}	1.01	1.01	1.00	0.99	0.99	0.83	0.83	0.85

Table 4.6: Computed convergence rates $p^N_{\varepsilon,global}$ for various values of ε and N for Method 4.2 applied to problem (4.1)

4.4 A non-overlapping, non-iterative Schwarz method

4.4.1 Introduction

This method uses uniform meshes on two non-overlapping subdomains and incorporates the Shishkin transition point τ, when fixing the position of the interface. We define a Neumann condition on the interior boundary of the subdomain outside the layer region. As a result, this method does not iterate since no solution values are passed between the subdomains.

4.4.2 Discrete Schwarz method

We now formally describe the method.

Method 4.3 *The exact solution u_ε is approximated by $\bar{U}_\varepsilon$ which is defined as follows.*

$$\bar{U}_\varepsilon(x) = \begin{cases} \bar{U}_0(x), & x \in \bar{\Omega}_0 \\ \bar{U}_1(x), & x \in \bar{\Omega}_1 \end{cases}$$

where $\bar{U}_i^{[k]}$ is the linear interpolant of $U_i^{[k]}$. Let $\bar{\Omega}_0^N = \{x_i\}_0^N$ be a uniform mesh on Ω_0 with $x_i = i\xi^+/N$ and $\bar{\Omega}_1^N = \{x_i\}_0^N$ be a piecewise uniform mesh on Ω_1 with $x_i = \xi^+ + i(1-\xi^+)/N$.

$$\begin{aligned} L_\varepsilon^N U_0 &= f \quad \text{in } \Omega_0^N, \quad U_0(0) = u_0, \quad D^- U_0(\xi^+) = 0, \\ L_\varepsilon^N U_1 &= f \quad \text{in } \Omega_1^N, \quad U_1(\xi^+) = U_0(\xi^+), \quad U_0(1) = u_1, \end{aligned}$$

Now, we state the following maximum principles which are required in the subsequent analysis in this section.

Comparison Principle. [6] *Let $a, b \in \Omega$ and assume that $\psi(a) \geq 0$, $\psi'(b) \geq 0$ and $L_\varepsilon \psi(x) \geq 0$ in Ω, then $\psi(x) \geq 0$ for all $x \in [a, b] \subset \bar{\Omega}$.*

Discrete Comparison Principle. [6] *Let Ω^N be some mesh of dimension N. If ϕ is some mesh function defined on Ω^N such that $\phi(x_0) \geq 0$, $D^-\phi(x_N) \geq 0$ and $L_\varepsilon^N \phi \geq 0 \quad in \quad \Omega^N$, then $\phi \geq 0 \quad in \quad \bar{\Omega}^N$.*

To obtain estimates for the approximation error in this method we first consider $|V_0 - v_\varepsilon|$ in the interval $[0, \xi^+]$. Using the triangle inequality we can write

$$|V_0 - v_\varepsilon| \leq |V_0 - \tilde{v}_0| + |v_\varepsilon - \tilde{v}_0|$$

where $\tilde{v}_0$ is defined by

$$L_\varepsilon \tilde{v}_0 = -\varepsilon \tilde{v}_0{}'' + a(x)\tilde{v}_0' = f, \tag{4.27a}$$

$$\tilde{v}_0(0) = 0, \quad \tilde{v}_0{}'(\xi^+) = 0, \tag{4.27b}$$

for all $x \in \bar{\Omega}_0 = [0, \xi^+]$. By this we mean, on the subdomain $\Omega_0 = (0, \xi^+)$, $\tilde{v}_0$ is the smooth component of some function $\tilde{u}_0$ defined by

$$L_\varepsilon \tilde{u}_0 = f, \quad \tilde{u}_0(0) = u_\varepsilon(0), \quad \tilde{u}_0'(\xi^+) = 0.$$

In this first lemma we examine the behaviour of $|v_\varepsilon - \tilde{v}_0|$ in the domain Ω_0.

Lemma 4.4.1 *Let $\tilde{v}_0$ be the solution of (4.27) and v_ε the smooth component of u_ε, given in Lemma 3.2.2, then*

$$|v_\varepsilon - \tilde{v}_0| \leq C\varepsilon, \quad \textit{for all} \quad x \in [0, \xi^+],$$

where C is a constant independent of ε and N.

Proof. The component v_ε satisfies

$$-\varepsilon v_\varepsilon'' + a v_\varepsilon' = f,$$

where $v_\varepsilon(0), v_\varepsilon(1)$ are specified according to the Shishkin decomposition. Let $\bar{e} = v_\varepsilon - \bar{v}_0$. Then $\bar{e}$ satisfies the equation,

$$L_\varepsilon \bar{e} = -\varepsilon \bar{e}'' + a(x)\bar{e}' = 0, \tag{4.28a}$$

$$\bar{e}(0) = 0, \quad \bar{e}'(\xi^+) = v_\varepsilon'(\xi^+). \tag{4.28b}$$

We consider the following constant coefficient equation,

$$L_\varepsilon \psi = -\varepsilon \psi'' + \alpha \psi' = 0, \tag{4.29a}$$

$$\psi(0) = 0, \quad \psi'(\xi^+) = |v_\varepsilon'(\xi^+)|. \tag{4.29b}$$

Now, using the maximum principle, we can show that the solution of (4.29) is an upper bound for the solution of (4.28). First observe that,

$$(\psi - \bar{e})(0) = 0, \quad (\psi' - \bar{e}')(\xi^+) \geq 0.$$

Also,

$$\begin{aligned} L_\varepsilon(\psi - \bar{e}) &= -\varepsilon \psi'' + a\psi' \\ &= -\varepsilon + (a - \alpha)\psi' + \alpha\psi' \\ &= (a - \alpha)\psi', \end{aligned}$$

and so, it remains to show that $\psi' \geq 0$. To achieve this we can solve (4.29) explicitly. The function ψ is given by,

$$\psi = \frac{\varepsilon}{\alpha} e^{-\alpha(\xi^+ - x)/\varepsilon} |v_\varepsilon'(\xi^+)| - \frac{\varepsilon}{\alpha} e^{-\alpha(\xi^+)/\varepsilon} |v_\varepsilon'(\xi^+)|.$$

Thus

$$\psi'(x) = e^{-\alpha(\xi^+ - x)/\varepsilon} |v_\varepsilon(\xi^+)| \geq 0,$$

and from the maximum principle it follows, that

$$\begin{aligned}|\tilde{e}(x)| \le |\psi(x)| &\le \frac{\varepsilon}{\alpha}e^{-\alpha(\xi^+-x)/\varepsilon}|v_\varepsilon'(\xi^+)| + \frac{\varepsilon}{\alpha}e^{-\alpha(\xi^+)/\varepsilon}|v_\varepsilon'(\xi^+)| \\ &\le C\frac{\varepsilon}{\alpha} \quad \text{since} \quad |v_\varepsilon|_k < C\varepsilon^{2-k}, 0 \le k \le 3.\end{aligned}$$

This concludes this lemma. ◇

In this next lemma we obtain an estimate for $|V_\varepsilon - v_\varepsilon|$ in the solution domain $\bar{\Omega}^N$.

Lemma 4.4.2 *Let V_ε be the smooth component of the Schwarz approximation then,*

$$|(V_\varepsilon - v_\varepsilon)(x_i)| \le CN^{-1} + C\varepsilon, \quad \textit{for all } x_i \in \bar{\Omega}^N$$

where C is independent of both ε and N.

Proof. At the boundary points, $\partial\Omega_0 = \{0, \xi^+\}$,

$$\begin{aligned}(V_0 - \tilde{v}_0)(0) &= 0 \\ D^-(V_\varepsilon - \tilde{v}_0)(\xi^+) &= \tilde{v}_0'(\xi^+) - D^-\tilde{v}_0(\xi^+) \\ &= \frac{1}{\xi^+ - \xi^-}\int_{\xi^-}^{\xi^+}(s - \xi^-)\tilde{v}_0''(s)ds \\ &\le CN^{-1}, \quad \text{where} \quad \xi^+ - \xi^- = \frac{1-\tau}{N}.\end{aligned}$$

As before,

$$|L_\varepsilon^N(V_0 - \tilde{v}_0)(x_i)| \le CN^{-1}.$$

With the barrier function $\phi(x_i) = CN^{-1}x_i \pm (V_0 - \tilde{v}_0)(x_i)$, and the maximum principle it follows that

$$|(V_0 - \tilde{v}_0)(x_i)| \le CN^{-1}.$$

Combining this with the estimate derived in Lemma 4.4.1 then gives,

$$|(V_0 - v_\varepsilon)(x_i)| \le CN^{-1} + C\frac{\varepsilon}{\alpha}, \quad \forall x_i \in \Omega_0^N. \tag{4.30}$$

On the interval $[\xi^+, 1]$, from (4.30) and the boundary conditions we observe that

$$\begin{aligned} |(V_1 - v_\varepsilon)(\xi^+)| &= |(V_0 - v_\varepsilon)(\xi^+)| \leq CN^{-1} + C\frac{\varepsilon}{\alpha}, \\ |(V_1 - v_\varepsilon)(1)| &= 0. \end{aligned}$$

Also,

$$L_\varepsilon^N (V_1 - v_\varepsilon)(x_i) \leq CN^{-1}.$$

Therefore, applying the barrier function $\phi_i = C(N^{-1} + \varepsilon/\alpha)$ and the discrete maximum principle gives the estimate

$$|(V_1 - v_\varepsilon)(x_i)| \leq CN^{-1} + C\frac{\varepsilon}{\alpha}, \quad \forall x_i \in \Omega_1^N.$$

And so, combining (4.30) and (4.31) gives

$$|(V_1 - v_\varepsilon)(x_i)| \leq CN^{-1} + C\frac{\varepsilon}{\alpha}, \quad \forall x_i \in \bar{\Omega}^N,$$

which completes this proof. ◇

Now we concentrate on the W_ε term and, in the following lemmas, we derive estimate for $|W_\varepsilon - w_\varepsilon|$, the error in the approximation W_ε to the solution component w_ε. Firstly, in the interval $[0, \xi^+]$ we can write the following triangle inequality,

$$|W_0 - w_\varepsilon| \leq |W_0 - \tilde{w}_0| + |w_\varepsilon - \tilde{w}_0|,$$

where $\tilde{w}_0$, the singular component of $\tilde{u}_0$, is defined by

$$L_\varepsilon \tilde{w}_0 = -\varepsilon \tilde{w}_0{}'' + a(x)\tilde{w}_0' = 0, \tag{4.31a}$$

$$\tilde{w}_0(0) = 0, \quad \tilde{w}_0{}'(\xi^+) = 0, \tag{4.31b}$$

for all $x \in \bar{\Omega} = [0, \xi^+]$. In the following lemma we give an estimate for $|w_\varepsilon - \tilde{w}_0|$ in $[0, \xi^+]$. This is then used in Lemma 4.4.4 where we bound $|W_\varepsilon - w_\varepsilon|$ on $\bar{\Omega} = [0, 1]$.

Lemma 4.4.3 *Let $\tilde{w}_0$ be the solution of (4.31) and w_ε the smooth component of u_ε, as given in Lemma 3.2.2, then*

$$\|w_\varepsilon - \tilde{w}_0\| \leq CN^{-1}, \quad \textit{for all} \quad x \in [0, \xi^+],$$

where C is a constant independent of ε and N

Proof. The arguments in this lemma are very similar to those applied in Lemma 4.4.1, so to avoid repetition we only include an outline of the necessary steps here. From the decomposition we see that the solution component w_ε satisfies

$$-\varepsilon w_\varepsilon'' + a w_\varepsilon' = f,$$

where $w_\varepsilon(0)$ and $w_\varepsilon(1)$ are specified according to the decomposition. Now, letting $\tilde{e} = w_\varepsilon - \tilde{w}_\varepsilon$, we can see that $\tilde{e}$ satisfies a differential problem akin to (4.28), with $\tilde{e}'(\xi^+) = w_\varepsilon'(\xi^+)$. Using an analogous maximum principle argument, it can be deduced that $|\tilde{e}(x)|$ is bounded above by $|\psi(x)|$, where $\psi'(\xi^+) = |w_\varepsilon'(\xi^+)|$ and $\psi(x)$ is given by

$$\psi = \frac{\varepsilon}{\alpha} e^{-\alpha(\xi^+ - x)/\varepsilon} |w_\varepsilon'(\xi^+)| - \frac{\varepsilon}{\alpha} e^{-\alpha(\xi^+)/\varepsilon} |w_\varepsilon'(\xi^+)|.$$

Hence we can now conclude that

$$\begin{aligned} |w_\varepsilon - \tilde{w}_0| &= |\tilde{e}(x)| \leq |\psi(x)| \leq \frac{\varepsilon}{\alpha} e^{-\alpha(\xi^+ - x)/\varepsilon} |w_\varepsilon'(\xi^+)| - \frac{\varepsilon}{\alpha} e^{-\alpha(\xi^+)/\varepsilon} |w_\varepsilon'(\xi^+)| \\ &\leq CN^{-1} \end{aligned}$$

since $|w_\varepsilon'| \leq C\varepsilon^{-1} e^{-\alpha(1-x)/\varepsilon}$ and $\varepsilon |w_\varepsilon'(\xi^+)| \leq e^{-\alpha(1-\xi^+)/\varepsilon} = CN^{-1}$. This completes this lemma. $\diamond$

Now, we derive, using Lemma 4.4.3, an estimate for the error in the singular component, $|W_\varepsilon - w_\varepsilon|$, of the Schwarz iterate in this method.

Lemma 4.4.4 *Let W_ε be the singular component of the Schwarz iterate, then*

$$\|(W_\varepsilon - w_\varepsilon)(x_i)\| \leq CN^{-1}(\ln N)^2, \quad \forall x_i \in \bar{\Omega}^N,$$

where C is independent of both ε and N.

Proof. Recall, in Lemma 4.4.3, we obtained an estimate for $|w_\varepsilon - \tilde{w}_0|$ on $\bar{\Omega}_0$. Here we first bound $|W_0 - \tilde{w}_0|$ on $\bar{\Omega}_0^N$ on Ω_0 and complete the analysis by deriving and estimate for $|W_1 - w_\varepsilon|$ in $\bar{\Omega}_1^N$. Consider the term $|\tilde{w}_0|$ on the subdomain $\Omega_0^N = (0, \xi^+)$, it is clear that $\tilde{w}_0$ is given by

$$\begin{aligned} |\tilde{w}_0(0)| &= |w_\varepsilon(0)| \leq CN^{-2}, \\ |\tilde{w}_0'(\xi^+)| &= 0, \\ L\tilde{w}_0(x) &= 0, \end{aligned}$$

and so, applying the maximum principle yields,

$$|\tilde{w}_0(x)| \leq CN^{-2} \quad \text{in} \quad \bar{\Omega}_0.$$

The discrete Schwarz component $W_0(x)$, satisfies

$$\begin{aligned} |\bar{W}_0(0)| &= |w_\varepsilon(0)| \leq CN^{-2}, \\ |D^- W_0(\xi^+)| &= 0, \\ L_\varepsilon^N \bar{W}_0(x) &= 0. \end{aligned}$$

Therefore, by the discrete maximum principle it follows that

$$|W_0(x)| \leq CN^{-2} \quad \text{in} \quad \bar{\Omega}_0^N.$$

Hence,

$$|(W_0 - \tilde{w})| \leq |W_0(x)| + |\tilde{w}_0(x)| \leq CN^{-1}, \quad \forall x_i \in \bar{\Omega}_0^N. \tag{4.32}$$

Then, using the result derived in Lemma 4.4.3 and (4.32) we obtain the estimate

$$|(W_0 - w_\varepsilon)| \leq |W_0(x) - \tilde{w}_0(x)| + |w_\varepsilon - \tilde{w}_0| \leq CN^{-1}, \quad \forall\, x_i \in \bar{\Omega}_0^N.$$

Now, on the subdomain Ω_1^N, it is clear, using the Dirichlet interface condition $W_1(\xi^+) = W_0(\xi^+)$, that

$$\begin{aligned} |(W_1 - w_\varepsilon)(\xi^+)| &\leq |(W_0 - w_\varepsilon)(\xi^+)| \leq CN^{-1}, \\ |(W_1 - w_\varepsilon)(1)| &= 0, \\ \text{and as before,} \qquad & |L_\varepsilon^N(W_1 - w_\varepsilon)(x_i)| \leq CN^{-1}\varepsilon^2\tau. \end{aligned}$$

Therefore, using arguments analogous to those given in Lemma 4.2.5 it follows that

$$|(W_1 - w_\varepsilon)| \leq CN^{-1}(\ln N)^2, \quad \forall\, x_i \in \bar{\Omega}_1^N.$$

We can conclude that

$$|(W_\varepsilon - w_\varepsilon)| \leq CN^{-1}(\ln N)^2, \quad \forall\, x_i \in \bar{\Omega}^N.$$

This completes this lemma. ◇

The following corollary to Lemmas 4.4.2 and 4.4.4 gives an error estimate for the continuous and discrete Schwarz iterates.

Corollary 4.4.1 *Assume $\tau < 1/3$. Then,*

$$||(U_\varepsilon - u_\varepsilon)(x_i)|| \leq CN^{-1}(\ln N)^2 + C\varepsilon,$$

where C is a constant independent of N and ε.

It is known that the piecewise linear interpolant $\bar{U}_\varepsilon^{[k]}$ retains the above error estimate (see [6]) and in the following lemma we state the main theoretical result of this section.

Theorem 4.4.1 *Assume $\tau < 1/3$. Then,*

$$||\bar{U}_\varepsilon - u_\varepsilon||_{\bar{\Omega}} \leq CN^{-1}(\ln N)^2 + C\varepsilon,$$

where C is a constant independent of N and ε.

4.4.3 Numerical results

In the case of $\frac{\varepsilon}{\alpha} \ln N \geq 1/3$, we apply Method 4.3, using the interface position $\xi^+ = 1/N$. Note, from the values located above the horizontal lines in Tables 4.7 and 4.8, it is clear that this non-iterative method is first order convergent when the problem is not singularly perturbed.

For $\frac{\varepsilon}{\alpha} \ln N < 1/3$, the numerical computations presented in this section demonstrate that this method generates equivalent errors, given in Table 4.7, and therefore equivalent computed error rates, given in Table 4.8, to those produced by Method 4.2. These experiments verify the theoretical estimate stated in Theorem 4.4.1 and illustrate that this non-iterative method is as accurate as the previously described Method 4.2, without incurring the computational cost of large iterations.

4.4.4 Conclusions

We conclude that Method 4.3 is not ε-uniform, but does generate first order accurate numerical approximations when the assumption $\varepsilon \leq N^{-1}$ is made. Therefore, for small values of ε, this Schwarz approach is a suitable and computationally efficient method.

	Number of Intervals N in each subdomain								
ε	8	16	32	64	128	256	512	1024	2048
2^{-0}	1.61e-02	8.08e-03	4.02e-03	2.00e-03	9.96e-04	4.97e-04	2.48e-04	1.24e-04	6.20e-05
2^{-1}	2.64e-02	1.35e-02	6.79e-03	3.41e-03	1.71e-03	8.53e-04	4.27e-04	2.13e-04	1.07e-04
2^{-2}	4.70e-02	2.48e-02	1.28e-02	6.48e-03	3.26e-03	1.63e-03	8.18e-04	4.09e-04	2.05e-04
2^{-3}	7.41e-02	4.25e-02	2.24e-02	1.14e-02	5.76e-03	2.89e-03	1.45e-03	7.25e-04	3.63e-04
2^{-4}	4.43e-02	4.62e-02	4.67e-02	4.64e-02	4.53e-02	5.47e-03	2.75e-03	1.38e-03	6.90e-04
2^{-5}	3.55e-02	2.82e-02	2.75e-02	2.68e-02	2.64e-02	2.62e-02	2.58e-02	2.53e-02	2.47e-02
2^{-6}	4.22e-02	2.28e-02	1.73e-02	1.56e-02	1.47e-02	1.43e-02	1.42e-02	1.41e-02	1.40e-02
2^{-7}	4.69e-02	2.54e-02	1.31e-02	1.02e-02	8.63e-03	7.80e-03	7.48e-03	7.40e-03	7.38e-03
2^{-8}	5.12e-02	2.65e-02	1.39e-02	7.62e-03	5.79e-03	4.66e-03	4.08e-03	3.85e-03	3.79e-03
2^{-9}	5.55e-02	2.75e-02	1.43e-02	7.31e-03	4.46e-03	3.21e-03	2.48e-03	2.11e-03	1.96e-03
2^{-10}	5.83e-02	2.83e-02	1.45e-02	7.46e-03	3.84e-03	2.54e-03	1.75e-03	1.31e-03	1.09e-03
2^{-11}	6.01e-02	2.92e-02	1.47e-02	7.52e-03	3.80e-03	2.22e-03	1.41e-03	9.43e-04	6.86e-04
2^{-12}	6.11e-02	3.00e-02	1.49e-02	7.55e-03	3.83e-03	2.06e-03	1.25e-03	7.73e-04	5.04e-04
2^{-13}	6.17e-02	3.05e-02	1.50e-02	7.59e-03	3.84e-03	1.99e-03	1.17e-03	6.93e-04	4.19e-04
2^{-14}	6.20e-02	3.08e-02	1.52e-02	7.63e-03	3.84e-03	1.95e-03	1.13e-03	6.53e-04	3.79e-04
2^{-15}	6.23e-02	3.10e-02	1.54e-02	7.66e-03	3.85e-03	1.94e-03	1.11e-03	6.34e-04	3.59e-04
2^{-16}	6.23e-02	3.10e-02	1.54e-02	7.67e-03	3.86e-03	1.94e-03	1.10e-03	6.24e-04	3.50e-04
2^{-17}	6.23e-02	3.10e-02	1.54e-02	7.68e-03	3.87e-03	1.94e-03	1.10e-03	6.19e-04	3.45e-04
2^{-18}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.87e-03	1.94e-03	1.10e-03	6.17e-04	3.42e-04
2^{-19}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.87e-03	1.94e-03	1.10e-03	6.16e-04	3.41e-04
2^{-20}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.87e-03	1.94e-03	1.10e-03	6.15e-04	3.41e-04
2^{-21}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.88e-03	1.95e-03	1.10e-03	6.15e-04	3.40e-04
2^{-22}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.88e-03	1.95e-03	1.09e-03	6.15e-04	3.40e-04
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-30}	6.23e-02	3.10e-02	1.54e-02	7.69e-03	3.88e-03	1.95e-03	1.09e-03	6.15e-04	3.40e-04

Table 4.7: Computed global errors $E^N_{\varepsilon,global}$ for various values of ε and N for Method 4.3 applied to problem (4.1)

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	0.99	1.01	1.01	1.00	1.00	1.00	1.00	1.00
2^{-1}	0.97	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2^{-2}	0.92	0.96	0.98	0.99	1.00	1.00	1.00	1.00
2^{-3}	*0.80*	0.93	0.97	0.99	0.99	1.00	1.00	1.00
2^{-4}	**-0.06**	**-0.01**	**0.01**	*0.03*	3.05	0.99	1.00	1.00
2^{-5}	**0.34**	**0.04**	**0.04**	**0.02**	**0.01**	**0.02**	**0.03**	**0.03**
2^{-6}	0.89	**0.39**	**0.15**	**0.09**	**0.04**	**0.01**	**0.01**	**0.01**
2^{-7}	0.89	0.95	**0.36**	**0.24**	**0.15**	**0.06**	**0.01**	**0.00**
2^{-8}	0.95	0.93	0.87	**0.40**	**0.31**	**0.19**	**0.08**	**0.02**
2^{-9}	1.01	0.94	0.97	0.71	**0.48**	**0.37**	**0.23**	**0.10**
2^{-10}	1.04	0.96	0.96	0.96	0.60	**0.54**	**0.42**	**0.27**
2^{-11}	1.04	0.99	0.97	0.98	0.78	0.65	**0.58**	**0.46**
2^{-12}	1.02	1.01	0.98	0.98	0.89	0.72	0.69	**0.62**
2^{-13}	1.01	1.03	0.98	0.98	0.95	0.76	0.76	0.72
2^{-14}	1.01	1.02	0.99	0.99	0.98	0.78	0.79	0.78
2^{-15}	1.01	1.01	1.01	0.99	0.99	0.80	0.81	0.82
2^{-16}	1.01	1.01	1.00	0.99	1.00	0.81	0.82	0.84
2^{-17}	1.01	1.01	1.00	0.99	1.00	0.82	0.83	0.85
2^{-18}	1.01	1.01	1.00	0.99	1.00	0.82	0.83	0.85
2^{-19}	1.01	1.01	1.00	0.99	0.99	0.83	0.83	0.85
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-30}	1.01	1.01	1.00	0.99	0.99	0.83	0.83	0.85

Table 4.8: Computed convergence rates $p^N_{\varepsilon,global}$ for various values of ε and N for Method 4.3 applied to problem (4.1)

4.5 A non-overlapping, iterative Schwarz method with Neumann interface conditions

4.5.1 Introduction

In this section, we discuss a non-overlapping Schwarz method with Neumann-type interface conditions. Research has been carried out on various interface conditions for Schwarz methods; Nataf and Rogier [21], Rodrigue and Reiter [24], Tallec and Tidriri [35] and Otto and Lube [22] examine general interface conditions for the Schwarz methods at the continuous level.

Nataf and Rodgier [21], considered a continuous Schwarz algorithm and proposed that by replacing the Dirichlet interface conditions with more general boundary conditions, the efficiency of the algorithm is increased. However, we have seen, in Chapter 3, that the convergence behaviour of the discrete Schwarz method is radically different, for convection-diffusion problems, to that of a continuous method. Nevertheless, non-overlapping methods using mixed interface conditions appear to have advantages for singularly perturbed methods. Gastaldi et al. [9] and Lube et al. [13] present convergence results for Schwarz methods with mixed interface conditions in the context of Finite Element formulations. However, our interest lies in the pointwise norm which is not a natural norm for finite element methods.

In the previous section, we observed that a Schwarz approach using the simplest type of Neumann condition at the interface of the non-overlapping subdomains, produces accurate numerical approximations for small ε without any iterations. However, it was noted that using $D^-U_0(\xi^+) = 0$ does not agree with the solution component v_ε so long as $v_\varepsilon'(\xi^+) \neq 0$. Therefore, this would not be an accurate assumption to make if convergent approximations were required for all values of ε such that $\frac{\varepsilon}{\alpha}\ln N < 1/3$.

In Chapter 3, for a Schwarz method with uniform meshes, it was shown that the interface positions cannot be fixed to be independent of ε and N (see, Method 3.1), and the width of the overlap must be independent of ε (see, Method 3.2). An appropriate choice of overlap, in Method 4.1, was of order N^{-1}. We therefore feel that, for the convection-diffusion Problem 3.1, a non-overlapping Classical Schwarz method which uses some interface conditions may be ε-uniformly convergent, but for large values of N, the iterations will become large.

We describe and investigate numerically a Schwarz method which uses Neumann-Dirichlet type interface conditions, and demonstrate that although this method is convergent for larger values of ε, the computational costs, which are the iteration numbers, are of the same order as those in Methods 4.1 and 4.2.

4.5.2 Discrete Schwarz method

We now formally describe the method.

Method 4.4 *The exact solution u_ε is approximated by the limit $\bar{U}_\varepsilon$ of a sequence of discrete Schwarz iterates $\{\bar{U}_\varepsilon^{[k]}\}_{k=0}^{\infty}$, which are defined as follows. For each $k \geq 1$,*

$$\bar{U}_\varepsilon^{[k]}(x) = \begin{cases} \bar{U}_0^{[k]}(x), & x \in \bar{\Omega}_0 \\ \bar{U}_1^{[k]}(x), & x \in \bar{\Omega}_1 \end{cases}$$

where $\bar{U}_i^{[k]}$ is the linear interpolant of $U_i^{[k]}$. Let $\bar{\Omega}_0^N = \{x_i\}_0^N$ be the uniform mesh on Ω_0 with $x_i = i\xi^+/N$ and $\bar{\Omega}_1^N = \{x_i\}_0^N$ be the uniform mesh on Ω_1 with $x_i = i*(1-\xi^+)/N$.

Then for $k = 1$,

$$\begin{aligned} L_\varepsilon^N U_0^{[1]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[1]}(0) = u_0, \quad U_0^{[1]}(\xi^+) = 0, \\ L_\varepsilon^N U_1^{[1]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[1]}(\xi^+) = U_0^{[1]}(\xi^-), \quad U_0^{[1]}(1) = u_1, \end{aligned}$$

and for $k > 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[k]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[k]}(0) = u_0, \quad D^+U_0^{[k]}(\xi^-) = D^-U_0^{[k-1]}(\xi^-), \\ L_\varepsilon^N U_1^{[k]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[k]}(\xi^+) = U_0^{[k]}(\xi^+), \quad U_0^{[k]}(1) = u_1. \end{aligned}$$

4.5.3 Numerical results

In the computations presented in the following tables, we applied the non-iterative approach, described in Section 4.4.3, for the case $\frac{\varepsilon}{\alpha} \ln N \geq 1/3$.

In Table 4.9, we see the approximations given by this method converge to the true solution of problem (4.1) for larger values of ε than observed in Method 4.3. We observe the iteration numbers close to the horizontal lines in Table 4.10 are large and, as expected, these then decrease as one moves down the table. We note that, in Table 4.9, for $\varepsilon = 2^{-5}$ the method fails to be first order convergent for $N = 512, 1024, 2048$, as is illustrated by the emphasized error values. It is not clear why this occurred and further theoretical investigations would be required to determine the exact behaviour of this method, but we can remark that this did not occur when the algorithm was applied to problem (4.2).

	Number of Intervals N in each subdomain								
ε	8	16	32	64	128	256	512	1024	2048
2^{-0}	1.61e-02	8.08e-03	4.02e-03	2.00e-03	9.96e-04	4.97e-04	2.48e-04	1.24e-04	6.20e-05
2^{-1}	2.64e-02	1.35e-02	6.79e-03	3.41e-03	1.71e-03	8.53e-04	4.27e-04	2.13e-04	1.07e-04
2^{-2}	4.70e-02	2.48e-02	1.28e-02	6.48e-03	3.26e-03	1.63e-03	8.18e-04	4.09e-04	2.05e-04
2^{-3}	*6.54e-02*	4.25e-02	2.24e-02	1.14e-02	5.76e-03	2.89e-03	1.45e-03	7.25e-04	3.63e-04
2^{-4}	8.83e-02	4.35e-02	1.96e-02	7.58e-03	*2.53e-03*	5.47e-03	2.75e-03	1.38e-03	6.90e-04
2^{-5}	9.91e-02	5.15e-02	2.55e-02	1.22e-02	5.53e-03	2.22e-03	*7.98e-04*	*7.75e-04*	*7.86e-04*
2^{-6}	1.04e-01	5.51e-02	2.81e-02	1.40e-02	6.84e-03	3.28e-03	1.50e-03	6.20e-04	2.40e-04
2^{-7}	1.07e-01	5.69e-02	2.92e-02	1.47e-02	7.35e-03	3.64e-03	1.78e-03	8.56e-04	3.95e-04
2^{-8}	1.08e-01	5.78e-02	2.97e-02	1.51e-02	7.56e-03	3.78e-03	1.88e-03	9.31e-04	4.57e-04
2^{-9}	1.09e-01	5.82e-02	3.00e-02	1.52e-02	7.66e-03	3.84e-03	1.92e-03	9.57e-04	4.76e-04
2^{-10}	1.09e-01	5.84e-02	3.01e-02	1.53e-02	7.71e-03	3.87e-03	1.93e-03	9.67e-04	4.83e-04
2^{-11}	1.09e-01	5.85e-02	3.02e-02	1.53e-02	7.73e-03	3.88e-03	1.94e-03	9.72e-04	4.86e-04
2^{-12}	1.09e-01	5.85e-02	3.02e-02	1.54e-02	7.74e-03	3.88e-03	1.95e-03	9.74e-04	4.87e-04
2^{-13}	1.09e-01	5.86e-02	3.03e-02	1.54e-02	7.75e-03	3.89e-03	1.95e-03	9.75e-04	4.88e-04
2^{-14}	1.09e-01	5.86e-02	3.03e-02	1.54e-02	7.75e-03	3.89e-03	1.95e-03	9.75e-04	4.88e-04
2^{-15}	1.09e-01	5.86e-02	3.03e-02	1.54e-02	7.75e-03	3.89e-03	1.95e-03	9.75e-04	4.88e-04
2^{-16}	1.09e-01	5.86e-02	3.03e-02	1.54e-02	7.75e-03	3.89e-03	1.95e-03	9.75e-04	4.88e-04
2^{-17}	1.09e-01	5.86e-02	3.03e-02	1.54e-02	7.75e-03	3.89e-03	1.95e-03	9.76e-04	4.88e-04
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-30}	1.09e-01	5.86e-02	3.03e-02	1.54e-02	7.75e-03	3.89e-03	1.95e-03	9.76e-04	4.88e-04

Table 4.9: Computed global errors $E^N_{\varepsilon,global}$ for various values of ε and N for Method 4.4 applied to problem (4.1)

	Number of Intervals N in each subdomain								
ε	8	16	32	64	128	256	512	1024	2048
2^{-0}	2	2	2	2	2	2	2	2	2
2^{-1}	2	2	2	2	2	2	2	2	2
2^{-2}	2	2	2	2	2	2	2	2	2
2^{-3}	35	2	2	2	2	2	2	2	2
2^{-4}	21	32	53	96	184	2	2	2	2
2^{-5}	14	20	29	48	83	153	289	553	1064
2^{-6}	11	14	19	27	44	75	136	253	476
2^{-7}	9	11	13	18	26	41	70	125	230
2^{-8}	8	9	10	13	17	25	39	66	116
2^{-9}	7	8	9	10	12	16	24	37	62
2^{-10}	6	7	7	8	10	12	16	22	35
2^{-11}	6	6	7	7	8	9	12	15	21
2^{-12}	5	6	6	6	7	8	9	11	14
2^{-13}	5	5	6	6	6	7	8	9	11
2^{-14}	5	5	5	5	6	6	7	7	8
2^{-15}	5	5	5	5	5	6	6	6	7
2^{-16}	5	5	5	5	5	5	5	6	6
2^{-17}	4	4	5	5	5	5	5	5	6
2^{-18}	4	4	4	4	5	5	5	5	5
2^{-19}	4	4	4	4	4	4	5	5	5
2^{-20}	4	4	4	4	4	4	4	4	5
2^{-21}	4	4	4	4	4	4	4	4	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-29}	4	4	4	4	4	4	4	4	4
2^{-30}	3	3	3	3	3	3	3	3	3

Table 4.10: Iteration counts for Method 4.4 applied to problem (4.1)

4.5.4 Conclusions

From the numerical results, it appears that Method 4.4 is ε-uniformly convergent. However, the iteration counts increase with N, and for large N, are almost doubling. We feel these results are an indication of the convergence and iteration profile expected for a non-overlapping Schwarz method using uniform meshes and general interface conditions, applied to the class of convection-diffusion problems.

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	0.99	1.01	1.01	1.00	1.00	1.00	1.00	1.00
2^{-1}	0.97	0.99	1.00	1.00	1.00	1.00	1.00	1.00
2^{-2}	0.92	0.96	0.98	0.99	1.00	1.00	1.00	1.00
2^{-3}	*0.62*	0.93	0.97	0.99	0.99	1.00	1.00	1.00
2^{-4}	1.02	1.15	1.37	1.58	*-1.11*	0.99	1.00	1.00
2^{-5}	0.95	1.01	1.06	1.14	1.32	1.48	*0.04*	*-0.02*
2^{-6}	0.92	0.98	1.01	1.03	1.06	1.13	1.28	1.37
2^{-7}	0.91	0.96	0.99	1.00	1.01	1.03	1.06	1.11
2^{-8}	0.90	0.96	0.98	0.99	1.00	1.01	1.01	1.03
2^{-9}	0.90	0.95	0.98	0.99	1.00	1.00	1.00	1.01
2^{-10}	0.90	0.95	0.98	0.99	1.00	1.00	1.00	1.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-30}	0.90	0.95	0.98	0.99	0.99	1.00	1.00	1.00

Table 4.11: Computed convergence rates $p^N_{\varepsilon,global}$ for various values of ε and N for Method 4.4 applied to problem (4.1)

4.6 A parameter-robust Schwarz method

4.6.1 Introduction

We now describe an overlapping Schwarz method which produces (ε, N)-uniform approximations to the solution of Problem 3.1. In this technique, the width of the overlapping region is a fixed proportion of the width of the subdomain, Ω_0, positioned outside the layer. A Shishkin mesh is fitted onto the subdomain, Ω_1, containing the layer and, as in previous methods, the subdomain Ω_0 is discretised using a uniform mesh. To avoid repetition, the theoretical analysis of this method is not included in this section, since this would require incorporating results, given in [18], for the Shishkin mesh into a Schwarz argument similar to those discussed in Chapters 3 and in previous sections of this chapter.

The convergence properties of this method are demonstrated using numerical computations in Section 4.6.3. Although this Schwarz method has no real advantages over a fitted Shishkin mesh in one dimension, in higher dimensions, where the solution domain may have a complex structure, this technique could be useful.

4.6.2 Discrete Schwarz method

We now formally describe the method.

Method 4.5 *Introduce the overlapping subdomains*

$$\Omega_0 = (0, \xi^+), \quad \Omega_1 = (\xi^-, 1)$$

where the constants ξ^+ *and* ξ^- *are given by*

$$\xi^+ = 1 - \tau \quad and \quad \xi^- = \frac{1-\tau}{2}.$$

The exact solution u_ε *is approximated by the limit* $\bar{U}_\varepsilon$ *of a sequence of discrete Schwarz iterates* $\{\bar{U}_\varepsilon^{[k]}\}_{k=0}^{\infty}$*, which are defined as follows. For each* $k \geq 1$,

$$\bar{U}_\varepsilon^{[k]}(x) = \begin{cases} \bar{U}_0^{[k]}(x), & x \in \bar{\Omega}_0 \setminus \bar{\Omega}_1 \\ \bar{U}_1^{[k]}(x), & x \in \bar{\Omega}_1 \end{cases}$$

where $\bar{U}_i^{[k]}$ *is the linear interpolant of* $U_i^{[k]}$*. Let* $\bar{\Omega}_0^N = \{x_i\}_0^N$ *be the uniform mesh on* Ω_0 *with* $x_i = i\xi^+/N$ *and* $\bar{\Omega}_1^N = \{x_i\}_0^{N+1}$ *be the piecewise uniform mesh on* Ω_1 *defined by*

$$x_i - x_{i-1} = \begin{cases} \frac{1-\tau}{N} & for \quad 0 < i \leq \frac{N}{2} \\ \frac{2\tau}{N} & for \quad \frac{N}{2} < i \leq N \end{cases},$$

as shown in Fig 4.3. Then for $k > 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[1]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[1]}(0) = u_0, \quad U_0^{[1]}(\xi^+) = 0, \\ L_\varepsilon^N U_1^{[1]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[1]}(\xi^-) = \bar{U}_0^{[1]}(\xi^-), \quad U_0^{[1]}(1) = u_1, \end{aligned}$$

and for $k \geq 1$

$$\begin{aligned} L_\varepsilon^N U_0^{[k]} &= f \quad \text{in } \Omega_0^N, \quad U_0^{[k]}(0) = u_0, \quad U_0^{[k]}(\xi^+) = U_1^{[k-1]}(\xi^+), \\ L_\varepsilon^N U_1^{[k]} &= f \quad \text{in } \Omega_1^N, \quad U_1^{[k]}(\xi^-) = \bar{U}_0^{[k]}(\xi^-), \quad U_0^{[k]}(1) = u_1. \end{aligned}$$

4.6.3 Numerical results

In Table 4.12, we observe this method produces (ε, N)-uniform approximations and the computed rates, given in Table 4.14, show this method is first order convergent.

The iterations numbers are presented in Table 4.13, and we note that, as a consequence of the overlap being a fixed proportion of $\Omega_0 = (0, 1-\tau)$, these iteration numbers do not increase with N.

4.6.4 Conclusions

The numerical solution of Method 4.5 converges (ε, N)-uniformly to Problem 3.1. A computational drawback to this method is that it does not have uniform meshes in both subdomains and, in one dimension, it has no advantage over using a fitted Shishkin mesh. However, in higher dimensions, for a problem with complex domain geometries, this type of Schwarz approach may be necessary.

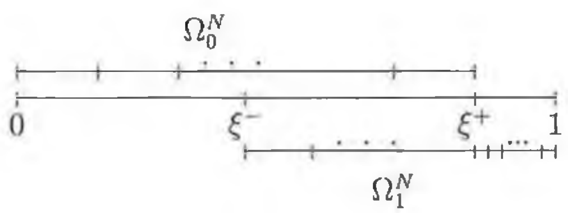

Figure 4.3: The discretised overlapping subdomains Ω_0^N and Ω_1^N for Method 4.5

	Number of Intervals N in each subdomain								
ε	8	16	32	64	128	256	512	1024	2048
2^{-0}	2.92e-03	1.35e-03	6.52e-04	3.20e-04	1.58e-04	7.88e-05	3.93e-05	1.96e-05	9.81e-06
2^{-1}	9.16e-03	4.53e-03	2.24e-03	1.12e-03	5.57e-04	2.78e-04	1.39e-04	6.95e-05	3.48e-05
2^{-2}	2.46e-02	1.26e-02	6.32e-03	3.18e-03	1.59e-03	7.97e-04	3.99e-04	1.99e-04	9.97e-05
2^{-3}	4.16e-02	2.72e-02	1.37e-02	6.93e-03	3.48e-03	1.74e-03	8.73e-04	4.37e-04	2.19e-04
2^{-4}	4.27e-02	2.70e-02	1.69e-02	1.05e-02	6.35e-03	3.57e-03	1.79e-03	8.96e-04	4.48e-04
2^{-5}	4.61e-02	2.80e-02	1.63e-02	9.75e-03	5.80e-03	3.41e-03	1.97e-03	1.12e-03	6.28e-04
2^{-6}	6.30e-02	3.05e-02	1.72e-02	9.73e-03	5.58e-03	3.22e-03	1.85e-03	1.05e-03	5.89e-04
2^{-7}	8.04e-02	3.33e-02	1.88e-02	1.03e-02	5.68e-03	3.18e-03	1.80e-03	1.01e-03	5.67e-04
2^{-8}	9.35e-02	4.02e-02	2.03e-02	1.11e-02	6.01e-03	3.28e-03	1.81e-03	1.00e-03	5.58e-04
2^{-9}	1.02e-01	4.72e-02	2.13e-02	1.18e-02	6.42e-03	3.46e-03	1.87e-03	1.02e-03	5.59e-04
2^{-10}	1.08e-01	5.19e-02	2.20e-02	1.23e-02	6.78e-03	3.66e-03	1.96e-03	1.05e-03	5.68e-04
2^{-11}	1.11e-01	5.48e-02	2.39e-02	1.27e-02	7.03e-03	3.84e-03	2.06e-03	1.10e-03	5.87e-04
2^{-12}	1.13e-01	5.65e-02	2.54e-02	1.29e-02	7.18e-03	3.95e-03	2.15e-03	1.15e-03	6.11e-04
2^{-13}	1.14e-01	5.74e-02	2.62e-02	1.30e-02	7.26e-03	4.03e-03	2.21e-03	1.19e-03	6.36e-04
2^{-14}	1.14e-01	5.79e-02	2.66e-02	1.30e-02	7.31e-03	4.07e-03	2.24e-03	1.22e-03	6.57e-04
2^{-15}	1.15e-01	5.83e-02	2.69e-02	1.30e-02	7.33e-03	4.09e-03	2.26e-03	1.24e-03	6.71e-04
2^{-16}	1.15e-01	5.83e-02	2.69e-02	1.30e-02	7.34e-03	4.10e-03	2.27e-03	1.25e-03	6.79e-04
2^{-17}	1.15e-01	5.84e-02	2.69e-02	1.30e-02	7.34e-03	4.10e-03	2.27e-03	1.25e-03	6.84e-04
2^{-18}	1.15e-01	5.84e-02	2.69e-02	1.30e-02	7.35e-03	4.10e-03	2.28e-03	1.25e-03	6.86e-04
2^{-19}	1.15e-01	5.84e-02	2.69e-02	1.30e-02	7.35e-03	4.11e-03	2.28e-03	1.26e-03	6.87e-04
2^{-20}	1.15e-01	5.84e-02	2.69e-02	1.30e-02	7.35e-03	4.11e-03	2.28e-03	1.26e-03	6.88e-04
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-30}	1.15e-01	5.84e-02	2.69e-02	1.31e-02	7.35e-03	4.11e-03	2.28e-03	1.26e-03	6.88e-04

Table 4.12: Computed global errors $E^N_{\varepsilon,global}$ for various values of ε and N for Method 4.5 applied to problem (4.1)

	Number of Intervals N in each subdomain								
ε	8	16	32	64	128	256	512	1024	2048
2^{-0}	14	14	14	14	14	14	14	14	14
2^{-1}	14	14	14	14	14	14	14	14	14
2^{-2}	13	12	12	12	12	12	12	12	12
2^{-3}	9	9	9	9	9	8	8	8	8
2^{-4}	6	6	5	5	5	5	5	5	5
2^{-5}	5	4	3	3	3	3	3	3	3
2^{-6}	4	3	3	2	2	2	2	2	2
2^{-7}	3	3	2	2	2	2	2	2	2
2^{-8}	3	2	2	2	2	2	2	2	2
2^{-9}	3	2	2	2	2	2	2	2	2
2^{-10}	3	2	2	2	2	2	2	2	2
2^{-11}	2	2	2	2	2	2	2	2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-30}	2	2	2	2	2	2	2	2	2

Table 4.13: Iteration counts for Method 4.5 applied to problem (4.1)

	Number of Intervals N in each subdomain							
ε	8	16	32	64	128	256	512	1024
2^{-0}	1.11	1.05	1.03	1.01	1.01	1.00	1.00	1.00
2^{-1}	1.02	1.01	1.01	1.00	1.00	1.00	1.00	1.00
2^{-2}	0.97	0.99	0.99	1.00	1.00	1.00	1.00	1.00
2^{-3}	0.62	0.99	0.98	0.99	1.00	1.00	1.00	1.00
2^{-4}	0.66	0.68	0.68	0.73	0.83	1.00	1.00	1.00
2^{-5}	0.72	0.78	0.74	0.75	0.77	0.79	0.81	0.84
2^{-6}	1.05	0.82	0.82	0.80	0.79	0.80	0.82	0.83
2^{-7}	1.27	0.82	0.87	0.85	0.84	0.82	0.83	0.84
2^{-8}	1.22	0.99	0.87	0.89	0.87	0.86	0.85	0.85
2^{-9}	1.11	1.15	0.85	0.88	0.89	0.89	0.88	0.87
2^{-10}	1.05	1.24	0.83	0.87	0.89	0.90	0.90	0.89
2^{-11}	1.02	1.19	0.92	0.85	0.87	0.89	0.91	0.91
2^{-12}	1.00	1.15	0.98	0.84	0.86	0.88	0.90	0.91
2^{-13}	0.99	1.13	1.01	0.83	0.85	0.87	0.89	0.91
2^{-14}	0.98	1.12	1.03	0.83	0.85	0.86	0.88	0.89
2^{-15}	0.98	1.12	1.05	0.83	0.84	0.85	0.87	0.88
2^{-16}	0.98	1.12	1.05	0.83	0.84	0.85	0.86	0.88
2^{-17}	0.98	1.12	1.05	0.83	0.84	0.85	0.86	0.87
2^{-18}	0.97	1.12	1.05	0.83	0.84	0.85	0.86	0.87
2^{-19}	0.97	1.12	1.05	0.83	0.84	0.85	0.86	0.87
2^{-20}	0.97	1.12	1.05	0.83	0.84	0.85	0.86	0.87
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-30}	0.97	1.12	1.05	0.83	0.84	0.85	0.86	0.87

Table 4.14: Computed convergence rates $p^N_{\varepsilon,global}$ for various values of ε and N for Method 4.5 applied to problem (4.1)

Chapter 5

A non-overlapping Schwarz method for two-dimensional convection-diffusion problems

5.1 Introduction

In this chapter, we discuss a discrete Schwarz method which is designed to produce accurate numerical approximations to a two-dimensional convection-diffusion problem with regular boundary layers for small values of the singular perturbation parameter. Our objective is to develop an appropriate domain decomposition for a non-overlapping Schwarz approach and to extend a one dimensional Schwarz method to two dimensions. To these aims, we choose to extend Method 4.2.

We consider the singularly perturbed linear convection-diffusion equation with variable coefficients on the unit square $\Omega = (0,1)^2$.

$$-\varepsilon\Delta u_\varepsilon + \vec{a}\cdot\nabla u_\varepsilon = f \quad \text{on} \quad \Omega = (0,1)\times(0,1) \tag{5.1a}$$

$$u = g \quad \text{on} \quad \partial\Omega \tag{5.1b}$$

$$\vec{a} = (a_1, a_2), \quad a_1 > \alpha_1 > 0, \quad a_2 > \alpha_2 > 0 \quad \text{on} \quad \bar{\Omega} \tag{5.1c}$$

where $a_1, a_2, f \in C^3(\bar{\Omega})$ and $0 < \varepsilon \leq 1$. We will assume that f and g are sufficiently compatible at the four corners. For small values of ε, regular boundary layers appear along the boundaries at $x = 1$ and $y = 1$. It is well known that if one uses a monotone finite difference operator on an appropriately fitted mesh [6], the piecewise bilinear interpolant of the discrete solution satisfies $||\bar{U}_\varepsilon^N - u_\varepsilon|| \leq CN^{-1}\ln N$, where C is a constant independent of ε. Motivated by this result for a fitted mesh method, we choose the domain interface positions using the Shishkin transition points τ_1, τ_2 given by

$$\tau_1 = \min\{1/3, \frac{\varepsilon}{\alpha_1}\ln N\}, \quad \tau_2 = \min\{1/3, \frac{\varepsilon}{\alpha_2}\ln N\}, \tag{5.2}$$

and the analysis, presented in this chapter, substantiates that an appropriate domain decomposition for this problem consists of the four domains Ω_a, Ω_b, Ω_c and Ω_d, illustrated in Fig. 5.1. In order to avoid repetition, we consider only the discrete two-dimensional analogue of Method 4.2 and study the case $\frac{\varepsilon}{\alpha_1}\ln N < 1/3$ and $\frac{\varepsilon}{\alpha_2}\ln N < 1/3$. The layout of this chapter is as follows. In Section 5.2, we specify the decomposition of the solution of (5.1) into its smooth and singular components, and bounds on the derivatives of these components are stated. Then, in Section 5.3, we describe the discrete Schwarz method and theoretically analyse the convergence behaviour of this method. Finally, in Section 5.4, we present numerical results which agree with the theoretical error estimates derived in Section 5.3.

5.2 The continuous problem

In Shishkin [30], the solution u_ε of Problem 5.1 is decomposed into its smooth and singular components v_ε and w_ε respectively,

$$u_\varepsilon = v_\varepsilon + w_\varepsilon.$$

The smooth component, v_ε is defined by

$$\begin{aligned} -\varepsilon\Delta v_\varepsilon + \vec{a}\cdot\nabla v_\varepsilon &= f, \qquad (x,y)\in\Omega, && (5.3a)\\ v_\varepsilon &= g \quad \text{on } \partial\Omega_{in}, && (5.3b)\\ v_\varepsilon &= h \quad \text{on } \partial\Omega_{0ut}, && (5.3c)\end{aligned}$$

where h is chosen so that the first and second derivatives of v_ε are bounded independently of ε at all points in Ω. The singular component w_ε satisfies the homogeneous

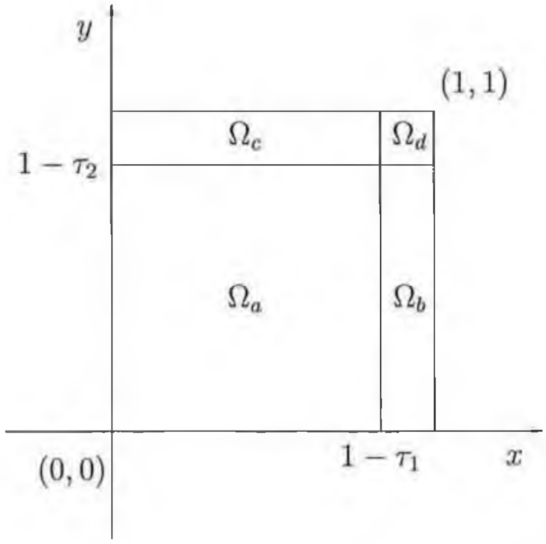

Figure 5.1: The non-overlapping domain structure for Method 5.1

differential equation

$$
\begin{aligned}
-\varepsilon\Delta w_\varepsilon + \vec{a}\cdot\nabla w_\varepsilon &= 0, \qquad (x,y)\in\Omega, &(5.4a)\\
w_\varepsilon &= 0 \quad \text{on } \partial\Omega_{in}, &(5.4b)\\
w_\varepsilon &= g - v_\varepsilon \quad \text{on } \partial\Omega_{out}. &(5.4c)
\end{aligned}
$$

The inflow and outflow boundaries, $\partial\Omega_{in}$ and $\partial\Omega_{out}$ respectively, are shown in Fig. 5.2. Bounds on the derivatives of the components v_ε and w_ε are given in the following

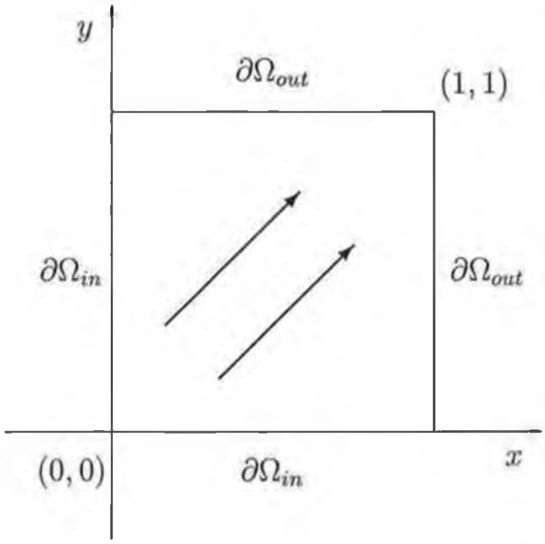

Figure 5.2: The inflow $\partial\Omega_{in}$ and outflow $\partial\Omega_{out}$ boundaries for Problem 5.1

lemma.

Lemma 5.2.1 *[30](pg.205). The solution of u_ε of (5.1) has the decomposition*

$$u_\varepsilon = v_\varepsilon + w_\varepsilon$$

where, for all k, $0 \le k \le 3$, and all $(x,y)\in\bar{\Omega}$, the smooth component v_ε satisfies

$$|v_\varepsilon|_k \le C(1+\varepsilon^{2-k})$$

and the singular component w_ε can be further decomposed into the sum

$$w_\varepsilon = w_1 + w_2 + w_{1,2}$$

where, for all $0 \le k_1, k_2 \le 3$,

$$\left|\frac{\partial^k w_1}{\partial x^{k_1}\partial y^{k_2}}\right| \le C(\varepsilon^{-k_1} + \varepsilon^{1-k})e^{-\alpha_1(1-x)/\varepsilon}, \quad k = k_1 + k_2,$$

$$\left|\frac{\partial^k w_2}{\partial x^{k_1}\partial y^{k_2}}\right| \le C(\varepsilon^{-k_2} + \varepsilon^{1-k})e^{-\alpha_2(1-y)/\varepsilon}, \quad k = k_1 + k_2,$$

$$\left|\frac{\partial^k w_{1,2}}{\partial x^{k_1}\partial y^{k_2}}\right| \le C\varepsilon^{-k}\min\{e^{-\alpha_1(1-x)/\varepsilon}, e^{-\alpha_1(1-y)/\varepsilon}\}, \quad k = k_1 + k_2,$$

for some constant C *independent of* ε.

Remark. Note the extra positive power of ε in the derivatives orthogonal to the layer direction. For example,

$$\left|\frac{\partial^2 w_1}{\partial y^2}\right| \le C\varepsilon^{-1}, \quad \left|\frac{\partial^2 w_1}{\partial x^2}\right| \le C\varepsilon^{-2}.$$

5.3 Discrete Schwarz method

We extend Method 4.2 to the two dimensional Problem 5.1. The solution domain $\Omega = (0,1)^2$ is partitioned into four non-overlapping subdomains Ω_a, Ω_b, Ω_c and Ω_d defined by

$$\begin{aligned}\Omega_a &= (0, 1-\tau_1) \times (0, 1-\tau_2), \quad \Omega_b = (1-\tau_1, 1) \times (0, 1-\tau_2),\\ \Omega_c &= (0, 1-\tau_1) \times (1-\tau_2, 1), \quad \Omega_d = (1-\tau_1, 1) \times (1-\tau_2, 1),\end{aligned}$$

where the transition parameters τ_1, τ_2 are given by (5.2) and the interior interfaces Γ_i are denoted by $\Gamma_i = \partial\Omega_i^N \setminus \partial\Omega^N$, $i = a, b, c, d$ and let $\Gamma = \bigcup_i \Gamma_k$ be the interior boundary. We use the notation:

$$\xi_1^+ = 1 - \tau_1, \quad \xi_2^+ = 1 - \tau_2, \quad \xi_1^- = \frac{N-1}{N}(1-\tau_1), \quad \xi_2^- = \frac{N-1}{N}(1-\tau_2).$$

Method 5.1 *For each* $k \geq 1$, $\bar{U}_\varepsilon^{[k]}(x,y) = \bar{U}_i^{[k]}(x,y), \quad (x,y) \in \bar{\Omega}_i$, $i = a,b,c,d$ *where* $\bar{U}_i^{[k]}$ *is the bilinear interpolant of* $U_i^{[k]}$*. Let* $\Omega_i^N = \{(x_j, y_k)\}_{j,k=1}^N$ *be a uniform mesh on* Ω_i*. On* $\bar{\Omega} \setminus \Omega, \quad U_i^{[k]} = g, \quad \forall\ k \geq 1$*. Then for* $k = 1$

$$
\begin{aligned}
L_\varepsilon^N U_a^{[1]} &= f \text{ in } \Omega_a^N,\ U_a^{[1]}(x_i, y_j) = \Psi \text{ on } \Gamma_a,\\
L_\varepsilon^N U_b^{[1]} &= f \text{ in } \Omega_b^N,\ U_b^{[1]}(x_i, \xi_2^+) = \Psi,\ U_b^{[1]}(\xi_1^+, y_j) = U_a^{[1]}(\xi_1^-, y_j),\\
L_\varepsilon^N U_c^{[1]} &= f \text{ in } \Omega_c^N,\ U_c^{[1]}(\xi_1^+, y_j) = \Psi,\ U_c^{[1]}(x_i, \xi_2^+) = U_a^{[1]}(x_i, \xi_2^-),\\
L_\varepsilon^N U_d^{[1]} &= f \text{ in } \Omega_d^N,\ U_d^{[1]}(\xi_1^+, y_j) = U_c^{[1]}(\xi_1^-, y_j),\ U_d^{[1]}(x_i, \xi_2^+) = U_b^{[1]}(x_i, \xi_2^-).
\end{aligned}
$$

Then for $k > 1$,

$$
\begin{aligned}
L_\varepsilon^N U_a^{[k]} &= f \text{ in } \Omega_a^N,\ U_a^{[k]}(\xi_1^+, y_j) = U_b^{[k-1]}(\xi_1^+, y_j),\ U_a^{[k]}(x_i, \xi_2^+) = U_c^{[k-1]}(x_i, \xi_2^+),\\
L_\varepsilon^N U_b^{[k]} &= f \text{ in } \Omega_b^N,\ U_b^{[k]}(x_i, \xi_2^+) = U_d^{[k-1]}(x_i, \xi_2^+),\ U_b^{[k]}(\xi_1^+, y_j) = U_a^{[k]}(\xi_1^-, y_j),\\
L_\varepsilon^N U_c^{[k]} &= f \text{ in } \Omega_c^N,\ U_c^{[k]}(\xi_1^+, y_j) = U_d^{[k-1]}(\xi_1^+, y_j),\ U_c^{[k]}(x_i, \xi_2^+) = U_a^{[k]}(x_i, \xi_2^-),\\
L_\varepsilon^N U_d^{[k]} &= f \text{ in } \Omega_d^N,\ U_d^{[k]}(\xi_1^+, y_j) = U_c^{[k]}(\xi_1^-, y_j),\ U_d^{[k]}(x_i, \xi_2^+) = U_b^{[k]}(x_i, \xi_2^-),
\end{aligned}
$$

where Ψ *is some arbitrary function with sufficient smoothness and* $g = \Psi$ *on* Γ_i, $i = a,b,c,d$*. For example,* $\Psi(\xi_1^+, \xi_2^+) = 0$ *and then use linear interpolant along* Γ *to specify* Ψ *on the interior boundary.*

The finite difference operator is defined by

$$L_\varepsilon^N = -\varepsilon(\delta_x^2 + \delta_y^2) + a_1 D_x^- + a_2 D_y^-,$$

where for any mesh function Z,

$$\delta_x^2 Z(x_i, y_j) = \frac{(D_x^+ - D_x^-)Z(x_i, y_j)}{(x_{i+1} - x_{i-1})/2}$$

with

$$D_x^+ Z(x_i, x_j) = \frac{Z(x_{i+1}, y_j) - Z(x_i, x_j)}{x_{i+1} - x_i}, \quad D_x^- Z(x_i, y_j) = \frac{Z(x_i, y_j) - Z(x_{i-1}, y_j)}{x_i - x_{i-1}}.$$

It satisfies the following discrete comparison principle on each $\bar{\Omega}_i^N$, $i = a, b, c, d$.

Discrete Comparison Principle. *Assume that the mesh function Z satisfies $Z \geq 0$ on $\partial\Omega_i^N$. Then $L_\varepsilon^N Z \geq 0$ on Ω_i^N implies that $Z \geq 0$ at each point on $\bar{\Omega}_i^N$.*

The Schwarz iterates $U_\varepsilon^{[k]}$ are now decomposed in an analogous way to u_ε. Thus we write,

$$U_\varepsilon^{[k]} = V_\varepsilon^{[k]} + W_1^{[k]} + W_2^{[k]} + W_{12}^{[k]}.$$

Each term of $U_\varepsilon^{[k]}$ in the sequence of discrete Schwarz approximations is decomposed as follows,

$$U_\varepsilon^{[k]} = V_\varepsilon^{[k]} + W_1^{[k]} + W_2^{[k]} + W_{12}^{[k]} = V_i^{[k]} + W_{1,i}^{[k]} + W_{2,i}^{[k]} + W_{12,i}^{[k]} \quad \text{in} \quad \bar{\Omega}_i \quad i = a, b, c, d,$$

where

$$\begin{aligned} L_\varepsilon^N V_\varepsilon^{[1]} &= f, \quad V_\varepsilon^{[1]}(\xi_1^+, \xi_2^+) = 0, \\ V_\varepsilon^{[1]} &= v_\varepsilon \quad \text{on } \partial\Omega, \quad V_\varepsilon^{[1]} \quad \text{is linear along } \Gamma, \end{aligned}$$

and

$$\begin{aligned} L_\varepsilon^N W_{i,\varepsilon}^{[1]} &= 0, \quad W_{i,\varepsilon}^{[1]}(\xi_1^+, \xi_2^+) = 0, \\ W_{i,\varepsilon}^{[1]} &= w_i \quad \text{on } \partial\Omega, \quad W_{i,\varepsilon}^{[1]} \quad \text{is linear along } \Gamma. \end{aligned}$$

For $k > 1$, the components $V_\varepsilon^{[k]}, W_1^{[k]}, W_2^{[k]}, W_{12}^{[k]}$ are defined by an analogous decomposition and the equations satisfied by $U_\varepsilon^{[k]}$, in Method 5.1.

In the following lemma we derive an estimate of the error contained in the regular component of the discrete Schwarz iterate $V_\varepsilon^{[k]}$.

Lemma 5.3.1 *Assume* $\tau_i < 1/3$. *For* $k \geq 1$,

$$|(V_\varepsilon^{[k]} - v_\varepsilon)(x_i, y_j)| \leq C(N^{-1} + \lambda_1^{-k} + \lambda_2^{-k} + \varepsilon)$$

where,

$$\lambda_1 = 1 + \frac{\alpha_1(1-\tau_1)}{\varepsilon N}, \quad \lambda_2 = 1 + \frac{\alpha_2(1-\tau_2)}{\varepsilon N}.$$

Proof. The discrete problem is first solved on the subdomain Ω_a^N using some arbitrarily assigned boundary conditions Ψ on the interior boundary Γ_a. On the boundary $\partial\Omega_a^N$,

$$|(V_{\varepsilon,a}^{[1]} - v_\varepsilon)(x_i, y_j)| \;=\; 0 \qquad \text{on } \partial\Omega_a^N \backslash \Gamma_a$$

$$|(V_{\varepsilon,a}^{[1]} - v_\varepsilon)(\xi_1^+, y_j)| \leq C_1 \qquad |(V_{\varepsilon,a}^{[1]} - v_\varepsilon)(x_i, \xi_2^+)| \leq C_2 \qquad \text{on } \Gamma_a$$

since $|v_\varepsilon| \leq C$.

An estimate of the truncation error is given by the following classical argument,

$$\begin{aligned}
L_\varepsilon^N(V_{\varepsilon,a}^{[1]} - v_\varepsilon) &= (L_\varepsilon - L_\varepsilon^N)v_\varepsilon \\
&= -\varepsilon\Big(\frac{\partial^2}{\partial x^2} - \delta_x^2\Big)v_\varepsilon + a_1(x,y)\Big(\frac{\partial}{\partial x} - D_x^-\Big)v_\varepsilon \\
&\quad - \varepsilon\Big(\frac{\partial^2}{\partial y^2} - \delta_y^2\Big)v_\varepsilon + a_2(x,y)\Big(\frac{\partial}{\partial y} - D_y^-\Big)v_\varepsilon, \\
|L_\varepsilon^N(V_{\varepsilon,a}^{[1]} - v_\varepsilon)(x_i, y_j)| &\leq C(x_{i+1} - x_{i-1})(\varepsilon|v_\varepsilon|_3 + |v_\varepsilon|_2) \\
&\quad + C(y_{j+1} - y_{j-1})(\varepsilon|v_\varepsilon|_3 + |v_\varepsilon|_2).
\end{aligned}$$

Noting that $x_{i+1} - x_{i-1} \leq 2N^{-1}$ and $y_{j+1} - y_{j-1} \leq 2N^{-1}$, and the appropriate bounds on the second and third derivatives of v_ε yields,

$$|L_\varepsilon^N(V_{\varepsilon,a}^{[1]} - v_\varepsilon)(x_i, y_j)| \leq CN^{-1}.$$

We introduce the following barrier function, $\Phi_{i,j}$ which is an upper bound on the solution of $L_\varepsilon^N\Psi = 0$ on Ω_a^N with boundary conditions $\Psi = 0$ on $\partial\Omega_a^N \setminus \Gamma_a$ and

$|\Psi| \leq C$ on Γ_a,

$$\Phi_{i,j} = C(x_i + y_j)N^{-1} + C_1\Big(1 - \Big(\frac{1-\lambda_1^{-N+i}}{1-\lambda_1^{-N}}\Big)\Big) + C_2\Big(1 - \Big(\frac{1-\lambda_2^{-N+j}}{1-\lambda_2^{-N}}\Big)\Big),$$

where $\lambda_1 = 1 + \frac{\alpha_1(1-\tau_1)}{\varepsilon N}$ and $\lambda_2 = 1 + \frac{\alpha_2(1-\tau_2)}{\varepsilon N}$. It can easily be verified that the inequalities

$$\begin{aligned}
(\Phi_{i,j} \pm (V_{\varepsilon,a}^{[1]} - v_\varepsilon))(x_i, y_j) &\geq 0 \quad \text{on } \partial\Omega_a^N, \\
L_\varepsilon^N\big(\Phi_{i,j} \pm (V_{\varepsilon,a}^{[1]} - v_\varepsilon)(x_i, y_j)\big) &\geq 0 \quad \text{on } \Omega_a^N,
\end{aligned}$$

are satisfied. The discrete maximum principle for L_ε^N on Ω_a^N then gives

$$\begin{aligned}
|(V_{\varepsilon,a}^{[1]} - v_\varepsilon)(x_i, y_j)| \leq CN^{-1}(x_i + y_j) + C_1\Big(1 - \Big(\frac{1-\lambda_1^{-N+i}}{1-\lambda_1^{-N}}\Big)\Big) \\
+ C_2\Big(1 - \Big(\frac{1-\lambda_2^{-N+j}}{1-\lambda_2^{-N}}\Big)\Big).
\end{aligned}$$

Now, the discrete problem is solved in Ω_b^N where the interface values are passed from the discrete solution on Ω_a^N using the interface condition $V_{\varepsilon,b}^{[1]}(\xi_1^+, y_j) = V_{\varepsilon,a}^{[1]}(\xi_1^-, y_j)$. On the boundary $\partial\Omega_b^N$,

$$\begin{aligned}
|(V_{\varepsilon,b}^{[1]} - v_\varepsilon)(x_i, y_j)| &= 0 \quad \text{on } \partial\Omega_b^N \backslash \Gamma_b, \\
|(V_{\varepsilon,b}^{[1]} - v_\varepsilon)(x_i, \xi_2^+)| &\leq C, \\
|(V_{\varepsilon,b}^{[1]} - v_\varepsilon)(\xi_1^+, y_j)| &= |V_{\varepsilon,a}^{[1]}(\xi_1^-, y_j) - v_\varepsilon(\xi_1^+, y_j)| \\
&\leq |V_{\varepsilon,a}^{[1]}(\xi_1^-, y_j) - v_\varepsilon(\xi_1^-, y_j)| + |v_\varepsilon(\xi_1^+, y_j) - v_\varepsilon(\xi_1^-, y_j)| \\
&\leq CN^{-1}(\xi_1^+ + y_j) + C_1\Big(1 - \Big(\frac{1-\lambda_1^{-1}}{1-\lambda_1^{-N}}\Big)\Big) \\
&\quad + C_2\Big(1 - \Big(\frac{1-\lambda_2^{-N+j}}{1-\lambda_2^{-N}}\Big)\Big) + m_1 \\
&= C(\xi_1^+ + y_j)N^{-1} + C_1(1 - q_1) + C_2\Big(1 - \Big(\frac{1-\lambda_2^{-N+j}}{1-\lambda_2^{-N}}\Big)\Big) \\
&\quad + m_1,
\end{aligned}$$

where we use the notation

$$m_1 = \max_{0 \leq y_j \leq 1} |v_\varepsilon(\xi_1^+, y_j) - v_\varepsilon(\xi_1^-, y_j)|, \quad q_1 = \Big(\frac{1-\lambda_1^{-1}}{1-\lambda_1^{-N}}\Big).$$

Here, an appropriate barrier function is,

$$\Phi_{i,j} = C(x_i + y_j)N^{-1} + C_1(1 - q_1) + C_2\Big(1 - \Big(\frac{1 - \lambda_2^{-N+j}}{1 - \lambda_2^{-N}}\Big)\Big) + m_1.$$

As before, the discrete maximum principle then gives the error estimate

$$\begin{aligned} |(V_{\varepsilon,b}^{[1]} - v_\varepsilon)(x_i, y_j)| &\leq C(x_i + y_j)N^{-1} + C_1(1 - q_1) \\ &\quad + C_2\Big(1 - \Big(\frac{1 - \lambda_2^{-N+j}}{1 - \lambda_2^{-N}}\Big)\Big) + m_1, \end{aligned}$$

for all $(x_i, y_j) \in \bar{\Omega}_b^N$. In an analogous result for Ω_c^N, we see that

$$\begin{aligned} |(V_{\varepsilon,c}^{[1]} - v_\varepsilon)(x_i, y_j)| &\leq C(x_i + y_j)N^{-1} + C_1\Big(1 - \Big(\frac{1 - \lambda_1^{-N+i}}{1 - \lambda_1^{-N}}\Big)\Big) \\ &\quad + C_2(1 - q_2) + m_2 \end{aligned}$$

where,

$$m_2 = \max_{0 \leq x_i \leq 1} |v_\varepsilon(x_i, \xi_2^+) - v_\varepsilon(x_i, \xi_2^-)|, \quad q_2 = \Big(\frac{1 - \lambda_2^{-N+j}}{1 - \lambda_2^{-N}}\Big).$$

The discrete problem is then solved in the corner region Ω_d^N with interface conditions, $V_{\varepsilon,d}^{[1]}(x_i, \xi_2^+) = V_{\varepsilon,b}^{[1]}(x_i, \xi_2^-)$ and $V_{\varepsilon,d}^{[1]}(\xi_1^+, y_j) = V_{\varepsilon,c}^{[1]}(\xi_1^-, y_j)$. On the boundary $\partial\Omega_d^N$,

$$\begin{aligned} |(V_{\varepsilon,d}^{[1]} - v_\varepsilon)(x_i, \xi_2^+)| &= |V_{\varepsilon,b}^{[1]}(x_i, \xi_2^-) - v_\varepsilon(x_i, \xi_2^+)| \\ &\leq |V_{\varepsilon,b}^{[1]}(x_i, \xi_2^-) - v_\varepsilon(x_i, \xi_2^-)| + |v_\varepsilon(x_i, \xi_2^+) - v_\varepsilon(x_i, \xi_2^-)| \\ &\leq CN^{-1}(x_i + y_j) + C_1(1 - q_1) \\ &\quad + C_2\Big(1 - \Big(\frac{1 - \lambda_2^{-1}}{1 - \lambda_2^{-N}}\Big)\Big) + m_1 + m_2 \\ &= C(x_i + y_j)N^{-1} + C_1(1 - q_1) + C_2(1 - q_2) \\ &\quad + m_1 + m_2 \end{aligned}$$

and similarly,

$$\begin{aligned} |(V_{\varepsilon,d}^{[1]} - v_\varepsilon)(\xi_1^+, y_j)| &\leq C(x_i + y_j)N^{-1} + C_1(1 - q_1) + C_2(1 - q_2) \\ &\quad + m_1 + m_2 \end{aligned}$$

The barrier function $\Phi_{i,j}$,

$$\Phi_{i,j} = C(x_i + y_j)N^{-1} + C_1(1-q_1) + C_2(1-q_2) + m_1 + m_2$$

is chosen and applying the discrete maximum principle to $\Phi_{i,j} \pm (V_{\varepsilon,d}^{[1]} - v_\varepsilon)(x_i, y_j)$ gives,

$$|(V_{\varepsilon,d}^{[1]} - v_\varepsilon)(x_i, y_j)| \leq C(x_i + y_j)N^{-1} + C_1(1-q_1) + C_2(1-q_2) + m_1 + m_2.$$

Thus combining the estimates derived in each subdomain Ω_i^N, $\quad i = a, b, c, d$ then yields

$$|(V_\varepsilon^{[1]} - v_\varepsilon)(x_i, y_j)| \leq C(x_i + y_j)N^{-1} + C_1(1-q_1) + C_2(1-q_2) + m_1 + m_2,$$

$\forall (x_i, y_j) \in \bar{\Omega}^N$. By repeating the above analysis for the second iterative and then by induction we obtain the following estimate for the k^{th} iterate,

$$\begin{aligned}
|(V_\varepsilon^{[k]} - v_\varepsilon)(x_i, y_j)| &\leq C(x_i + y_j)N^{-1} + C_1(1-q_1)^k + C_2(1-q_2)^k \\
&\quad + m_1(1-q_1)^{k-1} + \ldots + m_1 + m_2(1-q_2)^{k-1} + \ldots + m_2 \\
&\leq C((x_i + y_j)N^{-1} + (1-q_1)^k + (1-q_2)^k + \frac{m_1}{1-(1-q_1)} \\
&\quad + \frac{m_2}{1-(1-q_2)}.
\end{aligned}$$

Using the assumptions of the method, $0 < 1 - q_1 < \lambda_1^{-1} < 1$ and $0 < 1 - q_2 < \lambda_2^{-1} < 1$. Now, from the Mean Value Theorem it is easy to see that for fixed y_j,

$$v_\varepsilon(\xi_1^+, y_j) = v_\varepsilon(\xi_1^-, y_j) + (\xi_1^+ - \xi_1^-)\frac{\partial v}{\partial x}(\zeta, y_j)$$

where $\xi_1^- < \zeta < \xi_1^+$ and $|v_\varepsilon|_l \leq C\varepsilon^{2-l}$, $\; 0 \leq l \leq 3$. Hence it follows that,

$$\begin{aligned}
|m_1| &= |v_\varepsilon(\xi_1^+, y_j) - v_\varepsilon(\xi_1^-, y_j)| \\
&\leq (\xi_1^+ - \xi_1^-)|\frac{\partial v}{\partial x}(\zeta, y_j)| \\
&\leq CN^{-1}
\end{aligned}$$

also, $|m_2| \leq CN^{-1}$. Hence, using similar arguments to those in Lemma 4.3.1 yields,

$$\frac{m_1}{1-(1-q_1)} < C\varepsilon, \qquad \frac{m_2}{1-(1-q_2)} < C\varepsilon,$$

and so,

$$|(V_\varepsilon^{[k]} - v_\varepsilon)(x_i, y_j)| \leq C(N^{-1} + \lambda_1^{-k} + \lambda_2^{-k} + \varepsilon).$$

This completes the proof.$\diamond$

In the remaining lemmas of this section we concentrate on obtaining error bounds for the components of discrete Schwarz approximation, $W_1^{[k]}$, $W_2^{[k]}$ and $W_{12}^{[k]}$, which are estimates of the solution layer components w_1, w_2 and $w_{1,2}$ respectively. Lemma 5.3.2 considers the error in $W_1^{[k]}$. Note, we use the notation $W_1^{[k]} = W_{1,i}^{[k]}, \quad (x_i, y_j) \in \bar{\Omega}_i^N, i = a, b, c, d$.

Lemma 5.3.2 *Assume $\tau_i < 1/3$. For $k \geq 1$,*

$$|(W_1^{[k]} - w_1)(x_i, y_j)| \ \leq \ CN^{-1}(\ln N)^2 + C(\lambda_2)^{-k} + C\varepsilon$$

where

$$\lambda_2 = 1 + \frac{\alpha_2}{\varepsilon N}(1 - \tau_2).$$

Proof. First consider $(x, y) \in \Omega_a$, which is outside the boundary layers, then from Lemma 5.2.1, we see that $|w_1(x, y)| \leq Ce^{-\alpha(1-x)/\varepsilon} \leq Ce^{-\alpha(1-\xi_1^+)/\varepsilon} = CN^{-1}$. Now on Γ_a, $W_{1,a}^{[1]}$ is defined to be

$$\begin{aligned} W_{1,a}^{[1]}(\xi_1^+, y_j) &= w_1(\xi_1^+, 0) - \frac{w_1(\xi_1^+, 0)y_j}{\xi_2^+} \\ W_{1,a}^{[1]}(x_i, \xi_2^+) &= w_1(0, \xi_2^+) - \frac{w_1(0, \xi_2^+)x_i}{\xi_1^+}. \end{aligned}$$

and on $\partial\Omega_a^N \backslash \Gamma_a$, $W_{1,a}^{[1]}(x_i, y_j) = w_1(x_i, y_j)$. Now, $W_{1,a}^{[1]}$ satisfies the homogeneous equation,

$$L_\varepsilon^N(W_{1,a}^{[1]}) = 0 \quad \text{on } \Omega_a^N.$$

Therefore, the discrete maximum principle for L_ε^N on Ω_a^N yields the following estimate,

$$\begin{aligned} |W_{1,a}^{[1]}(x_i, y_j)| &\leq |w_1(x_i, y_j)| \\ &\leq CN^{-1}. \end{aligned}$$

And so, for all $(x_i, y_j) \in \bar{\Omega}_a^N$,

$$|(W_{1,a}^{[1]} - w_1)(x_i, y_j)| \leq CN^{-1}.$$

For $(x, y) \in \Omega_b$, which is inside the boundary layer w_1 term contains large gradients for $\varepsilon << 1$. On the boundary $\partial\Omega_b^N$,

$$\begin{aligned} |W_{1,b}^{[1]}(\xi_1^+, y_j)| &= |W_{1,a}^{[1]}(\xi_1^+, y_j)| \leq CN^{-1}, \\ |W_{1,b}^{[1]}(x_i, \xi_2^+)| &= |W_{1,a}^{[1]}(\xi_1^-, \xi_2^+) + \frac{w_1(1, \xi_2^+) - W_{1,a}^{[1]}(\xi_1^-, \xi_2^+)}{\tau_1}(x_i - \xi_1^+)| \\ &\leq C, \\ |W_{1,b}^{[1]}(x_i, y_j)| &= |w_1(x_i, y_j)| \quad \text{on } \partial\Omega_b^N \setminus \Gamma_b. \end{aligned}$$

Now, the local truncation argument combined with estimates of the partial derivatives of w_ε yield,

$$\begin{aligned} |L_\varepsilon^N(W_{1,b}^{[1]} - w_1)| &\leq C\Big| - \varepsilon(x_{i+1} - x_{i-1})\Big|\frac{\partial^3 w_1}{\partial x^3}\Big| - \varepsilon(y_{j+1} - y_{j-1})\Big|\frac{\partial^3 w_1}{\partial y^3}\Big| \\ &\quad + a_1(x, y)(x_i - x_{i-1})\Big|\frac{\partial^2 w_1}{\partial x^2}\Big| + a_2(x, y)(y_j - y_{j-1})\Big|\frac{\partial^2 w_1}{\partial y^2}\Big| \\ &\leq C\big[\varepsilon(2\tau_1 N^{-1})\varepsilon^{-3} + \varepsilon(1 - \tau_2)N^{-1}\varepsilon^{-2} + a_1 2\tau_1 N^{-1}\varepsilon^{-2} \\ &\quad + a_2(1 - \tau_2)N^{-1}\varepsilon^{-1}\big] \\ &\leq C\varepsilon^{-2}\tau_1 N^{-1} + C\varepsilon^{-1}N^{-1} \\ &\leq C\varepsilon^{-2}\tau_1 N^{-1}, \qquad \text{since} \quad \tau_1 > \varepsilon. \end{aligned}$$

Note that here we have used the sharp bounds on the derivatives given in Lemma 5.2.1. Consider the barrier function $\Phi_{i,j}$,

$$\Phi_{i,j} = (x_i - \xi_1^+)C\varepsilon^{-2}\tau_1 N^{-1} + C\left(1 - \left(\frac{1-\lambda_2^{-N+j}}{1-\lambda_2^{-N}}\right)\right) + CN^{-1},$$

then by the discrete maximum principle for L_ε^N on Ω_b^N,

$$|(W_{1,b}^{[1]} - w_1)(x_i, y_j)| \leq CN^{-1}(\ln N)^2 + C\left(1 - \left(\frac{1-\lambda_2^{-N+j}}{1-\lambda_2^{-N}}\right)\right),$$

$\forall (x_i, y_j) \in \bar{\Omega}_b^N$. The discrete problem is now solved on the subdomain Ω_c^N. As in Ω_a^N, the term w_1 satisfies the inequality, $|w_1| \leq CN^{-1}, \quad \forall (x, y) \in \Omega_c$. On the boundary $\partial\Omega_c^N$,

$$\begin{aligned}
|(W_{1,c}^{[1]} - w_1)(x_i, \xi_2^+)| &= |(W_{1,a}^{[1]}(x_i, \xi_2^-) - w_1(x_i, \xi_2^+)| \\
&\leq CN^{-1}, \\
|(W_{1,c}^{[1]} - w_1)(x_i, \xi_2^+)| &= |W_{1,a}^{[1]}(\xi_1^+, \xi_2^-) \\
&\quad + \frac{w_1(\xi_1^+, 1) - W_{1,a}^{[1]}(\xi_1^+, \xi_2^-)}{\tau_2} \times (y_j - \xi_2^+)| \\
&\leq CN^{-1}
\end{aligned}$$

and on $\partial\Omega_c^N \setminus \Gamma_c$, $|(W_{1,c}^{[1]} - w_1)(x_i, y_j)| = 0$. Applying similar methods as in $\bar{\Omega}_a^N$ it follows that,

$$|(W_{1,c}^{[1]} - w_1)(x_i, y_j)| \leq CN^{-1}, \quad \forall (x_i, y_j) \in \Omega_c^N.$$

Finally, the discrete problem is solved on the corner region Ω_d^N to complete the first iteration. On the boundary $\partial\Omega_d^N$,

$$\begin{aligned}
|(W_{1,d}^{[1]} - w_1)(\xi_1^+, y_j)| &= |(W_1^{[1]}(\xi_1^-, y_j) - w_1(\xi_1^+, y_j)| \\
&\leq CN^{-1}, \\
|(W_{1,d}^{[1]} - w_1)(x_i, \xi_2^+)| &= |(W_1^{[1]}(x_i, \xi_2^-) - w_1(x_i, \xi_2^+)| \\
&\leq |(W_{1,b}^{[1]}(x_i, \xi_2^-) - w_1(x_i, \xi_2^-)| + |(w_1(x_i, \xi_2^+) - w_1(x_i, \xi_2^-)| \\
&\leq CN^{-1}(\ln N)^2 + C(1 - q_2) + s_2,
\end{aligned}$$

where $s_2 = |(w_1(x_i, \xi_2^+) - w_1(x_i, \xi_2^-)|$, $q_2 = \frac{1-\lambda_2^{-1}}{1-\lambda_2^{-N}}$ and $\lambda_2 = 1 + \frac{\alpha_2(1-\tau)}{\varepsilon N}$. Here the barrier function is chosen to be,

$$\Phi_{i,j} = (x_i - \xi_1^+)C\varepsilon^{-2}\tau_1 N^{-1} + C(1-q_2) + CN^{-1}(\ln N)^2 + s_2$$

and the discrete maximum principle for L_ε^N on Ω_d^N then gives the estimate

$$|(W_{1,d}^{[1]} - w_1)(x_i, y_j)| \le C(1-q_2) + CN^{-1}(\ln N)^2 + s_2.$$

To fully understand the effect of the iteration process on the error estimates in the first iteration we continue with the second iteration and the proof is then completed by induction. An outline of the second iteration is now given. As before in Ω_a^N,

$$|(W_{1,a}^{[2]} - w_1)(x_i, y_j)| \le CN^{-1}.$$

Now on the boundary $\partial\Omega_b^N$,

$$|(W_{1,b}^{[2]}(\xi_1^+, y_j)| = |W_{1,a}^{[2]}(\xi_1^-, y_j)| \le CN^{-1},$$

which implies that

$$|(W_{1,b}^{[2]} - w_1)(\xi_1^+, y_j)| \le CN^{-1},$$

and

$$\begin{aligned} |(W_{1,b}^{[2]} - w_1)(x_i, \xi_2^+)| &= |(W_{1,d}^{[1]} - w_1)(x_i, \xi_2^+)| \\ &\le CN^{-1}(\ln N)^2 + C(1-q_2) + s_2. \end{aligned}$$

As before, we apply the truncation error argument,

$$|L_\varepsilon^N (W_{1,b}^{[2]} - w_1)(x_i, y_j)| \le C\varepsilon^{-2}\tau_1 N^{-1},$$

where Φ is defined to be the barrier function,

$$\Phi_{i,j} = (x_i - \xi_1^+)C\varepsilon^{-2}\tau_1 N^{-1} + \Big(C(1-q_2) + CN^{-1}(\ln N)^2 + s_2\Big) \times \left(1 - \left(\frac{1-\lambda_2^{N+j}}{1-\lambda_2^{-N}}\right)\right),$$

and the discrete maximum principle for L_ε^N on Ω_b^N then gives the estimate,

$$|(W_{1,b}^{[2]} - w_1)(x_i, y_j)| \leq (x_i - \xi_1^+)C\varepsilon^{-2}\tau_1 N^{-1} + \Big(C(1-q_2) + CN^{-1}(\ln N)^2 + s_2\Big) \times \left(1 - \left(\frac{1-\lambda_2^{N+j}}{1-\lambda_2^{-N}}\right)\right) \quad \text{in} \quad \bar{\Omega}_b^N.$$

Now on the boundary $\partial\Omega_c^N$,

$$|(W_{1,c}^{[2]})(\xi_1^+, y_j)| = |(W_{1,a}^{[2]})(\xi^-, y_j)| \leq CN^{-1}$$

and as before,

$$|(W_{1,c}^{[2]} - w_1)(x_i, y_j)| \leq CN^{-1}, \quad \text{in} \quad \bar{\Omega}_c^N.$$

On the boundary $\partial\Omega_d^N$,

$$\begin{aligned}
|(W_{1,d}^{[2]})(\xi_1^+, y_j)| &= |(W_{1,c}^{[2]})(\xi_1^-, y_j)| \leq CN^{-1}, \\
|(W_{1,d}^{[2]} - w_1)(\xi_1^+, y_j)| &\leq CN^{-1}, \\
|(W_{1,d}^{[2]} - w_1)(x_i, \xi_2^+)| &= |W_{1,b}^{[2]}(x_i, \xi_2^-) - w_1(x_i, \xi_2^+)| \\
&\leq |W_{1,b}^{[2]}(x_i, \xi_2^-) - w_1(x_i, \xi_2^-)| + |w_1(x_i, \xi_2^+) - w_1(x_i, \xi_2^-)| \\
&\leq CN^{-1}(\ln N)^2 + C(N^{-1}(\ln N)^2 + (1-q_2) + s_2) \\
&\quad \times (1-q_2) + s_2 \\
&= CN^{-1}(\ln N)^2 + CN^{-1}(\ln N)^2(1-q_2) \\
&\quad + C(1-q_2)^2 + s_2(1-q_2) + s_2.
\end{aligned}$$

Here, the barrier function $\Phi_{i,j}$ is chosen to be

$$\begin{aligned}
\Phi_{i,j} &= C\varepsilon^{-2}\tau_1 N^{-1}(x_i - \xi_1^+) + CN^{-1}(\ln N)^2 + CN^{-1}(\ln N)^2(1-q_2) \\
&\quad + C(1-q_2)^2 + s_2(1-q_2) + s_2
\end{aligned}$$

and the discrete maximum principle then gives the estimate,

$$\begin{aligned}
|(W_{1,d}^{[2]} - w_1)(x_i, y_j)| &\leq 2CN^{-1}(\ln N)^2 + CN^{-1}(\ln N)^2(1-q_2) \\
&\quad + C(1-q_2)^2 + s_2(1-q_2) + s_2.
\end{aligned}$$

The following error estimate for the k^{th} iterative is obtained by induction.

$$\begin{aligned}|(W_1^{[k]} - w_1)(x_i, y_j)| &\leq 2CN^{-1}(\ln N)^2 \\ &\quad + CN^{-1}(\ln N)^2(1-q_2)^{k-1} + C(1-q_2)^k \\ &\quad + \frac{s_2}{q_2}[1-(1-q_2)^k]\end{aligned}$$

As before, $1 - q_2 < \lambda_2^{-1}$, and now it remains to bound

$$s_2 = \left|w_1(x_i, \xi_2^+) - w_1(x_i, \xi_2^-)\right|, \quad \xi_1^+ < x_i < 1,$$

in the layer region. By the Mean Value Theorem,

$$w_1(x_i, \xi_2^+) = w_1(x_i, \xi_2^-) + \frac{\partial w_1}{\partial y}(x_i, \zeta)(\xi_2^+ - \xi_2^-), \quad \zeta \in (\xi_2^-, \xi_2^+)$$

where from Lemma 5.2.1, $|\frac{\partial^k w_1}{\partial^{k_1} x \partial^{k_2} y}| \leq C(\varepsilon^{-k_1} + \varepsilon^{1-k})e^{-\alpha_1(1-x)/\varepsilon}$ and so, s_2 satisfies

$$\begin{aligned}s_2 = |(w_1(x_i, \xi_2^+) - w_1(x_i, \xi_2^-)| &\leq |\frac{\partial w_1}{\partial y}(x_i, \zeta)(\xi_2^+ - \xi_2^-)|\frac{\xi_2^+}{N}| \\ &\leq CN^{-1}e^{-\alpha_1(1-x)/\varepsilon} \\ &\leq CN^{-1},\end{aligned}$$

where $\zeta \in (\xi_2^+, \xi_2^-)$. Note again have we used the sharp bounds (Lemma 5.2.1). This completes the proof. ◇

Now, in an analogous result we state the error estimate for the Schwarz component $W_2^{[k]}$.

Lemma 5.3.3 *Assume* $\tau_i < 1/3$. *For* $k \geq 1$,

$$|(W_2^{[k]} - w_2)(x_i, y_j)| \leq CN^{-1}(\ln N)^2 + C(\lambda_1)^{-k} + C\varepsilon$$

where

$$\lambda_1 = 1 + \frac{\alpha_1}{\varepsilon N}(1 - \tau_1).$$

Proof. The proof is analogous to Lemma 5.3.2.

Now, we obtain a bound for the error in the Schwarz estimate $W_{12}^{[k]}$ of the corner layer component $w_{1,2}$.

Lemma 5.3.4 *Assume $\tau_i < 1/3$. For $k \geq 1$,*

$$|(W_{12}^{[k]} - w_{12})(x_i, y_j)| \leq CN^{-1}(\ln N)^2$$

Proof. Firstly in Ω_a^N, the component w_{12} is "small" away from the corner region,

$$\begin{aligned} |w_{12}(x,y)| &\leq C\min\{e^{-\alpha_1(1-x)/\varepsilon}, e^{-\alpha_2(1-y)/\varepsilon}\} \\ &\leq CN^{-1}, \qquad (x,y) \in \Omega \setminus \bar{\Omega}_d. \end{aligned}$$

From the discrete comparison principle we see that $W_{12,a}^{[1]}(x_i, y_j)$ is bounded above by $|w_{12}|$ in Ω_a^N, thus

$$|W_{12,a}^{[1]}(x_i, y_j)| \leq CN^{-1}, \quad \forall (x_i, y_j) \in \bar{\Omega}_a^N.$$

In the subdomain Ω_b^N,

$$\begin{aligned} |W_{12,b}^{[1]}(\xi_1^+, y_j)| &= |W_{12,a}^{[1]}(\xi_1^-, y_j)| \\ &\leq CN^{-1} \\ |W_{12,b}^{[1]}(x_i, \xi_2^+)| &= \left| W_{12,a}^{[1]}(\xi_1^-, \xi_2^+) + \frac{w_{12}(1,\xi_2^+) - W_{12,a}^{[1]}(\xi_1^-, \xi_2^+)(x_i - \xi_1^+)}{\tau_1} \right| \\ &\leq CN^{-1} \end{aligned}$$

where, $|w_{12}(1, \xi_2^+)| \leq C\min\{1, e^{-\alpha\tau_2/\varepsilon}\} = CN^{-1}$. By discrete maximum principle of L_ε^N on Ω_b^N, $W_{12,b}^{[1]}(x_i, y_j) \leq CN^{-1}$, in $\bar{\Omega}_b^N$ and by the same argument $W_{12,c}^{[1]}(x_i, y_j) \leq$

CN^{-1} in $\bar{\Omega}_c^N$. On the boundary $\partial\Omega_d^N$,

$$\begin{aligned}
|(W_{12,d}^{[1]} - w_{12})(\xi_1^+, y_j)| &\leq |W_{12,c}^{[1]}(\xi_1^-, y_j)| + |w_{12}(\xi_1^+, y_j)| \\
&\leq CN^{-1}, \\
|(W_{12,d}^{[1]} - w_{12})(x_i, \xi_2^+)| &\leq |W_{12,c}^{[1]}(x, \xi_2^-)| + |w_{12}(x_i, \xi_2^+)| \\
&\leq CN^{-1}.
\end{aligned}$$

The truncation error estimate for $|L_\varepsilon^N(W_{12,b}^{[1]} - w_{12})|$ gives,

$$\begin{aligned}
|L_\varepsilon^N(W_{12,d}^{[1]} - w_{12})| &\leq C\Big| - \varepsilon(x_{i+1} - x_{i-1})\Big|\frac{\partial^3 w_{12}}{\partial x^3}\Big| - \varepsilon(y_{j+1} - y_{j-1})\Big|\frac{\partial^3 w_{12}}{\partial y^3}\Big| \\
&\quad + a_1(x,y)(x_i - x_{i-1})\Big|\frac{\partial^2 w_{12}}{\partial x^2}\Big| + a_2(x,y)(y_j - y_{j-1})\Big|\frac{\partial^2 w_{12}}{\partial y^2}\Big|\Big| \\
&\leq C\big[\varepsilon 2\tau_1 N^{-1}\varepsilon^{-3} + \varepsilon 2\tau_2 N^{-1}\varepsilon^{-3} \\
&\quad + a_1(x,y)\tau_1 N^{-1}\varepsilon^{-2} + a_2(x,y)\tau_2 N^{-1}\varepsilon^{-2}\big] \\
&\leq C\varepsilon^{-2}\tau_1 N^{-1} + C\varepsilon^{-2}\tau_2 N^{-1}.
\end{aligned}$$

Now we consider the barrier function $\Phi_{i,j}$,

$$\Phi_{i,j} = C\varepsilon^{-2}\tau_1 N^{-1}(x_i - \xi_1^+) + C\varepsilon^{-2}\tau_2 N^{-1}(y_j - \xi_2^+) + CN^{-1}.$$

The discrete maximum principle for L_ε^N on Ω_d^N then gives,

$$|(W_{12,d}^{[1]} - w_{12})(x_i, y_j)| \leq CN^{-1}(\ln N)^2, \quad \forall (x_i, y_j) \in \bar{\Omega}_d^N,$$

and by induction it can be shown that,

$$|(W_{12}^{[k]} - w_{12})(x_i, y_j)| \leq CN^{-1}(\ln N)^2, \quad \forall (x_i, y_j) \in \bar{\Omega}^N,$$

which concludes this lemma.◇

In the following theorem we combine the results from Lemmas 5.3.1 to 5.3.4, together with a result known (see, for example, Stynes and O'Riordan [34]) for the bilinear interpolant.

Theorem 5.3.1 *Assume $\tau_i < 1/3$. For $k \geq 1$,*

$$\|(\bar{U}_\varepsilon^{[k]} - u_\varepsilon)(x,y)\| \leq C(N^{-1}(\ln N)^2 + \lambda_1^{-k} + \lambda_2^{-k} + \varepsilon)$$

where $\lambda_i = 1 + \frac{\alpha_i(1-\tau_i)}{\varepsilon N} \quad i = 1,2.$

This theorem reveals a natural extension of Method 4.2 to two dimensions.

5.4 Numerical experiments

Numerical computations are carried out on the following model problem for a sequences of meshes Ω_i^N, $i = a, b, c, d$ corresponding to $N = 4, 8, 16, 32, 64, 128$.

$$\varepsilon\Delta + (2 + x^2 y)u_x + (1 + xy)u_y = x^2 + y^3 + \cos(x + 2y) \tag{5.5a}$$

with boundary conditions:

$$u(x,0) = 0 \quad u(x,1) = \begin{cases} 4x(1-x) & x < 1/2 \\ 1, & x \geq 1/2 \end{cases} \tag{5.5b}$$

$$u(0,y) = 0 \quad u(1,y) = \begin{cases} 8(y - 2y^2) & x < 1/4 \\ 1, & x \geq 1/4 \end{cases} \tag{5.5c}$$

In the numerical experiments the initial arbitrary mesh function Ψ is chosen to be the linear interpolant from the boundary values where $\Psi(\xi_1^+, \xi_2^+) = 0$.

In Figure 5.3, the numerical solution $\bar{U}_\varepsilon^{16}$, with $N = 16$ intervals in each subdomain and $\varepsilon = 0.001$, is shown. In Table 5.1, the required iteration counts are given for a tolerance level of

$$\max_{(x_i,y_j)\in\bar{\Omega}_\varepsilon^N} |U_\varepsilon^{[k]}(x_i,y_j) - U_\varepsilon^{[k-1]}(x_i,y_j)| \leq 10^{-8}.$$

The computed orders of convergence presented in Table 5.3 are computed using the double mesh principle (see [6]),

$$p_\varepsilon^N = \log_2\left(\frac{D_\varepsilon^N}{D_\varepsilon^{2N}}\right) \quad \text{where} \quad D_\varepsilon^N = \max_{(x_i,y_j)\in\bar{\Omega}_\varepsilon^N} |U_\varepsilon^N(x_i,y_j) - \bar{U}_\varepsilon^{2N}(x_i,y_j)|,$$

and the differences D_ε^N, are shown in Table 5.2. The numerical errors are then estimated by using the solution of a Shishkin fitted mesh on the finest available mesh, corresponding to $N = 256$, as an approximation to the exact solution. The corresponding computed maximum pointwise error is taken to be

$$E_{\varepsilon,nodal}^N = \max_{x_i\in\bar{\Omega}_\varepsilon^N} |U_\varepsilon^N(x_i,y_j) - \bar{U}_\varepsilon^{256}(x_i,y_j)|.$$

Values of $E_{\varepsilon,nodal}^N$ are given in Table 5.4.

We note that, since the boundary functions in problem (5.5) are non-trivial, and Method 5.1 uses the interface conditions $U_\varepsilon^{[k]}(\xi_1^+,y_j) = U_\varepsilon^{[k]}(\xi_1^-,y_j)$, $U_\varepsilon^{[k]}(x_i,\xi_2^+) = U_\varepsilon^{[k]}(x_i,\xi_2^-)$, for small values of N, the error in the numerical solutions is large. However, the computed orders of convergence, in Table 5.3, show that Method 5.1 is first

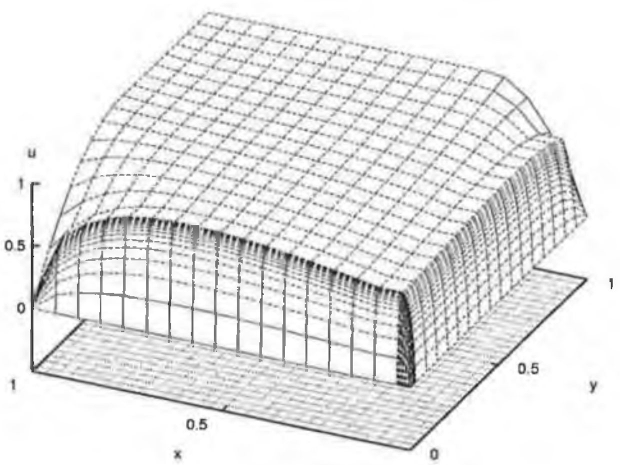

Figure 5.3: Numerical solution generated by Method 5.1 applied to problem (5.5) with $N = 16$ and $\varepsilon = 0.001$

order convergent, for $\varepsilon < N^{-1}$, where the error is bounded above by $C\varepsilon$ for $\tau < 1/5$. Also, in Table 5.1, the iterations become small for small values of ε.

Note. The linear system, resulting from the difference scheme, was solved using the Gauss-Seidel iterative solver, with a tolerance of 10^{-13}.

	Number of Intervals N in each subdomain					
ε	4	8	16	32	64	128
2^{-4}	11	16	26	45	84	163
2^{-5}	9	12	17	27	45	81
2^{-6}	7	9	12	18	28	46
2^{-7}	6	8	10	13	18	28
2^{-8}	6	7	8	10	13	18
2^{-9}	5	6	7	8	10	13
2^{-10}	5	5	6	7	8	10
2^{-11}	4	5	5	6	7	8
2^{-12}	4	4	5	5	6	7
2^{-13}	4	4	4	5	5	6
2^{-14}	4	4	4	4	5	5
2^{-15}	4	5	4	4	4	5
2^{-16}	3	4	6	4	4	4
2^{-17}	3	3	4	6	11	22
2^{-18}	3	3	3	4	6	10
2^{-19}	3	3	3	3	4	5

Table 5.1: Computed iterations for Method 5.1 applied to problem (5.5)

5.5 Conclusions

In this chapter, a two dimensional convection-diffusion Problem 5.1 with regular boundary layers was examined. It was shown that a one dimensional non-overlapping Schwarz method with uniform meshes, Method 4.2, can be extended to a corresponding two dimensional method, Method 5.1. Theoretical analysis showed that the con-

	Number of Intervals N in each subdomain				
ε	4	8	16	32	64
2^{-4}	1.47E-01	7.16E-02	5.78E-02	4.04E-02	2.38E-02
2^{-5}	1.53E-01	1.31E-01	1.05E-01	7.27E-02	4.29E-02
2^{-6}	1.60E-01	1.52E-01	1.29E-01	9.45E-02	6.35E-02
2^{-7}	1.79E-01	1.68E-01	1.39E-01	9.68E-02	6.26E-02
2^{-8}	1.91E-01	1.81E-01	1.46E-01	9.70E-02	5.86E-02
2^{-9}	1.97E-01	1.89E-01	1.52E-01	9.81E-02	5.64E-02
2^{-10}	2.01E-01	1.94E-01	1.56E-01	9.97E-02	5.59E-02
2^{-11}	2.03E-01	1.96E-01	1.58E-01	1.01E-01	5.61E-02
2^{-12}	2.04E-01	1.98E-01	1.60E-01	1.02E-01	5.64E-02
2^{-13}	2.04E-01	1.98E-01	1.60E-01	1.02E-01	5.67E-02
2^{-14}	2.05E-01	1.99E-01	1.61E-01	1.03E-01	5.69E-02
2^{-15}	2.05E-01	1.99E-01	1.61E-01	1.03E-01	5.69E-02
2^{-16}	2.05E-01	1.99E-01	1.61E-01	1.03E-01	5.70E-02
2^{-17}	2.05E-01	1.99E-01	1.61E-01	1.03E-01	5.70E-02
2^{-18}	2.05E-01	1.99E-01	1.61E-01	1.03E-01	5.70E-02
2^{-19}	2.05E-01	1.99E-01	1.61E-01	1.03E-01	5.70E-02

Table 5.2: Computed differences D_ε^N for Method 5.1 applied to problem (5.5)

	Number of Intervals N in each subdomain			
ε	4	8	16	32
2^{-4}	1.04	0.31	0.52	0.76
2^{-5}	0.23	0.31	0.53	0.76
2^{-6}	0.08	0.24	0.45	0.57
2^{-7}	0.09	0.28	0.52	0.63
2^{-8}	0.08	0.31	0.59	0.73
2^{-9}	0.06	0.32	0.63	0.80
2^{-10}	0.05	0.31	0.65	0.84
2^{-11}	0.05	0.31	0.65	0.85
2^{-12}	0.04	0.31	0.65	0.85
⋮	⋮	⋮	⋮	⋮
2^{-19}	0.04	0.31	0.65	0.85

Table 5.3: Computed orders of convergence p_ε^N for Method 5.1 applied to problem (5.5)

vergence properties of Method 4.2 are retained by Method 5.1. Numerical experiments were presented, which showed that, for small values of ε, only a few iterations are required. Therefore, for small values of ε, Method 5.1 is a applicable Schwarz method for Problem 5.1.

	Number of Intervals N in each subdomain				
ε	4	8	16	32	64
2^{-4}	2.53E-01	1.38E-01	8.28E-02	4.99E-02	3.55E-02
2^{-5}	2.77E-01	2.38E-01	1.66E-01	9.76E-02	5.99E-02
2^{-6}	3.38E-01	3.31E-01	2.53E-01	1.60E-01	1.02E-01
2^{-7}	3.73E-01	3.87E-01	3.00E-01	1.88E-01	1.12E-01
2^{-8}	3.92E-01	4.18E-01	3.24E-01	2.01E-01	1.13E-01
2^{-9}	4.03E-01	4.35E-01	3.37E-01	2.08E-01	1.15E-01
2^{-10}	4.08E-01	4.43E-01	3.44E-01	2.11E-01	1.16E-01
2^{-11}	4.11E-01	4.48E-01	3.48E-01	2.13E-01	1.16E-01
2^{-12}	4.12E-01	4.50E-01	3.49E-01	2.14E-01	1.17E-01
2^{-13}	4.13E-01	4.51E-01	3.50E-01	2.14E-01	1.17E-01
2^{-14}	4.13E-01	4.52E-01	3.51E-01	2.14E-01	1.17E-01
2^{-15}	4.14E-01	4.52E-01	3.51E-01	2.14E-01	1.17E-01
⋮	⋮	⋮	⋮	⋮	⋮
2^{-19}	4.14E-01	4.52E-01	3.51E-01	2.15E-01	1.17E-01

Table 5.4: Computed nodal maximum pointwise error E_ε^N for Method 5.1 applied to problem (5.5)

Chapter 6

An overlapping Schwarz approach for parabolic boundary layers

6.1 Introduction.

In this chapter, we describe and analyse a discrete overlapping Schwarz approach to problems with parabolic boundary layers. Consider the following elliptic problem.

$$-\varepsilon\Delta u + a_1 u_x = f \quad \text{on} \quad \Omega = (0,1)\times(0,1) \tag{6.1a}$$

$$u = g \quad \text{on} \quad \partial\Omega \tag{6.1b}$$

$$a_1 \geq \alpha_1 > 0 \quad \text{on} \quad \bar{\Omega} \tag{6.1c}$$

The characteristics of the reduced problem become parallel to the x-coordinate axis and parabolic boundary layers appear along the bottom and top boundaries at $y = 0$ and $y = 1$ respectively, as shown in Fig. 6.1. Therefore, when designing a Schwarz approach to Problem 6.1 we require a technique for parabolic boundary layers.

In this chapter, we construct and analyse an appropriate Schwarz approach for a singularly perturbed parabolic problem, whose solution has parabolic boundary layers. At the end of the chapter, we examine numerically the natural extension of this method to an elliptic singularly perturbed problem whose solution contains parabolic boundary layers.

We consider the following class of time-dependent problems.

$$L_\varepsilon u_\varepsilon(x,t) = -\varepsilon \frac{\partial^2 u_\varepsilon(x,t)}{\partial x^2} + b(x,t)u_\varepsilon(x,t) + d(x,t)\frac{\partial u_\varepsilon(x,t)}{\partial t} = f(x,t),$$

$$\text{for} \quad (x,t) \in G, \quad (6.2a)$$

$$u_\varepsilon = \phi \text{ on } \Gamma, \quad (6.2b)$$

$$d(x,t) > \delta > 0, \quad b(x,t) \geq \beta \geq 0 \text{ in } \bar{G}, \quad (6.2c)$$

where $G = (0,1) \times (0,T]$ and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where Γ_l, Γ_r and Γ_b are the left, right and bottom boundaries respectively. Assume that b, d, f, ϕ are sufficiently smooth and compatible at the corners. A negative result in Shishkin [28] (see also Miller et al. [18]) stated that, for the class of problems containing Problem 6.2, when a parabolic boundar layer is present, a fitted finite difference operator on a uniform mesh will not generate ε-uniform numerical approximations. In Miller et al. [19], a fitted mesh method, using a piecewise uniform mesh condensing in the boundary layers, is shown to be ε-uniform for this problem. The Schwarz decomposition method discussed here is proposed in Shishkin [31] and Shishkin and Vabishchevich [32]. However, no detailed proofs, no consideration of numerical results and no iteration counts were provided in [32], [31].

In Section 6.2, we present the decomposition of the solution of (6.2) and give the bounds on the derivatives of the solution components. Then, in Section 6.3, the discrete Schwarz method, Method 6.1, is introduced and in Lemmas 6.3.1 and 6.3.2 we derive parameter-robust estimates for the error contained in the Schwarz approximations to the regular and singular solution components. These results are combined in Theorem 6.3.1 to give the main theoretical result of this chapter. That is, the numerical solutions of Method 6.1 converge (ε, N)-uniformly to the exact solution of (6.2). Finally, in Section 6.4, we numerically examine an elliptic problem with degenerate boundary layers. It is known that, for this class of differential equations, the underlying nature of the complex boundary layer structure is related to the parabolic boundary layers appearing in Problem 6.2. We numerically show that the natural extension of Method 6.1, Method 6.2, is parameter-uniform for this type of elliptic problem.

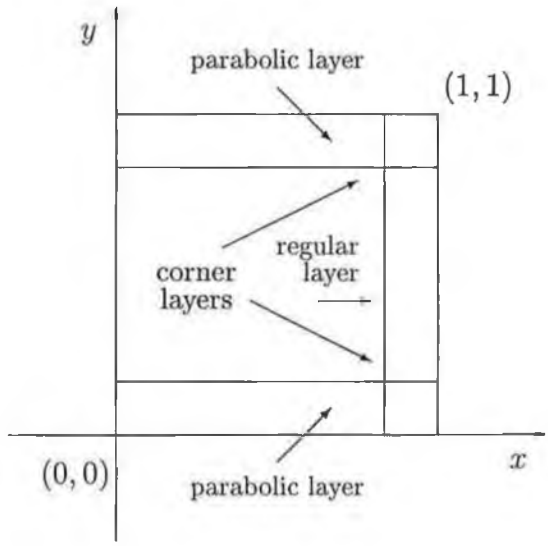

Figure 6.1: Regular and parabolic boundary layers of Problem 6.1.

6.2 The continuous problem

The solution u_ε of (6.2) is decomposed into a sum of its regular component $v_\varepsilon(x,t)$ and its singular components $w_l(x,t)$ and $w_r(x,t)$,

$$u_\varepsilon = v_\varepsilon + w_l + w_r.$$

The appropriate bounds on the derivatives of these components is given in the following lemma.

Lemma 6.2.1 *[19] The solution u_ε of Problem 6.2 has the decomposition*

$$u_\varepsilon = v_\varepsilon + w_l + w_r$$

where, for all non-negative integers i, j such that $0 \le i + 2j \le 4$,

$$\left\| \frac{\partial^{i+j} v_\varepsilon}{\partial x^i \partial t^j} \right\|_G \le C(1 + \varepsilon^{1-i/2}),$$

and for all $(x,t) \in G$,

$$\left| \frac{\partial^{i+j} w_l(x,t)}{\partial x^i \partial t^j} \right| \le C\varepsilon^{-i/2} e^{-x/\sqrt{\varepsilon}}$$

and

$$\left| \frac{\partial^{i+j} w_r(x,t)}{\partial x^i \partial t^j} \right| \le C\varepsilon^{-i/2} e^{-(1-x)/\sqrt{\varepsilon}}.$$

6.3 Discrete Schwarz method

The method we apply to Problem 6.2 is a time-dependent analogue of Method 2.1. The solution domain G is partitioned into three overlapping subregions G_l, G_r and G_c, defined by

$$G_l = (0, 2\sigma) \times (0, T], \quad G_r = (1 - 2\sigma) \times (0, T], \quad G_c = (\sigma, 1 - \sigma) \times (0, T],$$

and the transition parameter

$$\sigma = \min\{1/4, 2\sqrt{\varepsilon}\ln N\},$$

as shown in Fig. 6.2. The finite difference operator is

$$L_\varepsilon^N = -\varepsilon\delta_x^2 + bI + dD_t^-,$$

where for any mesh function Ψ,

$$\delta_x^2\Psi_{i,j} = \frac{(D_x^+ - D_x^-)\Psi_{i,j}}{(x_{i+1} - x_{i-1})/2}$$

with

$$D_x^+\Psi_{i,j} = \frac{\Psi_{i+1,j} - \Psi_{i,j}}{x_{i+1} - x_i}, \quad D_x^-\Psi_{i,j} = \frac{\Psi_{i,j} - \Psi_{i-1,j}}{x_i - x_{i-1}}$$

and an analogous definition for D_t^-.

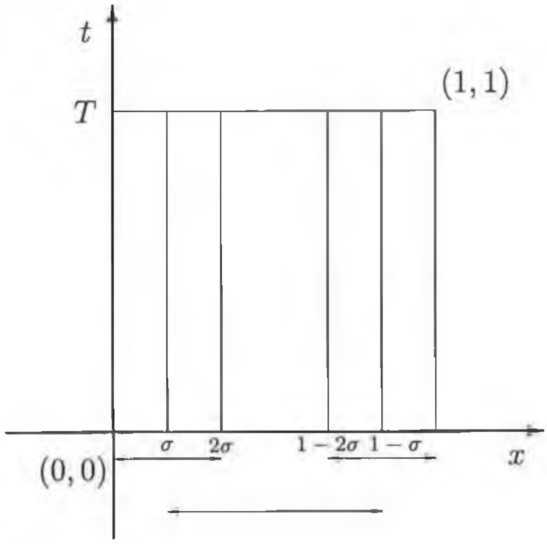

Figure 6.2: The structure of the subdomains for Method 6.1 applied to Problem 6.2.

Method 6.1 *The sequence of discrete Schwarz iterates* $\bar{U}_\varepsilon^{[k]}$ *is defined by*

$$\bar{U}_\varepsilon^{[k]} = \begin{cases} \bar{U}_c^{[k]} \ \text{in } \overline{G}_c\ , \\ \bar{U}_i^{[k]} \ \text{in } \overline{G}_i \setminus G_c, \quad i = l, r\ , \end{cases}$$

where the $\bar{U}_i^{[k]}$ are the solutions of the following problems. On the boundary ∂G^N, $U_i^{[k]} = u_\varepsilon$. For $k = 1$,

$$\begin{aligned}
L_\varepsilon^N U_l^{[1]} &= f \quad \text{in } G_l^N, \quad U_l^{[1]}(2\sigma, t_j) = u_\varepsilon(2\sigma, 0), \\
L_\varepsilon^N U_r^{[1]} &= f \quad \text{in } G_r^N, \quad U_r^{[1]}(1-2\sigma, t_j) = u_\varepsilon(1-2\sigma, 0), \\
L_\varepsilon^N U_c^{[1]} &= f \quad \text{in } G_c^N, \quad U_c^{[1]}(\sigma, t_j) = \bar{U}_l^{[1]}(\sigma, t_j), \quad U_c^{[1]}(1-\sigma, t_j) = \bar{U}_r^{[1]}(1-\sigma, t_j),
\end{aligned}$$

and for $k > 1$,

$$\begin{aligned}
L_\varepsilon^N U_l^{[k]} &= f \quad \text{in } G_l^N, \quad U_l^{[k]}(2\sigma, t_j) = \bar{U}_c^{[k-1]}(2\sigma, t_j), \\
L_\varepsilon^N U_r^{[k]} &= f \quad \text{in } G_r^N, \quad U_r^{[k]}(1-2\sigma, t_j) = \bar{U}_c^{[k-1]}(1-2\sigma, t_j), \\
L_\varepsilon^N U_c^{[k]} &= f \quad \text{in } G_c^N, \quad U_c^{[k]}(\sigma, t_j) = \bar{U}_l^{[k]}(\sigma, t_j), \quad U_c^{[k]}(1-\sigma, t_j) = \bar{U}_r^{[k]}(1-\sigma, t_j),
\end{aligned}$$

and $\bar{U}$ is the bilinear interpolant of U and G_i^N are uniform rectangular meshes on G_i.

The discrete Schwarz iterates are decomposed in an analogous way to u_ε. Thus we write

$$\bar{U}_\varepsilon^{[k]} = \bar{V}_\varepsilon^{[k]} + \bar{W}_l^{[k]} + \bar{W}_r^{[k]} = \begin{cases} \bar{V}_c^{[k]} + \bar{W}_{l,c}^{[k]} + \bar{W}_{r,c}^{[k]} \text{ in } \overline{G}_c \ , \\ \bar{V}_i^{[k]} + \bar{W}_{l,i}^{[k]} + \bar{W}_{r,i}^{[k]} \text{ in } \overline{G}_i \setminus G_c \quad i = l, r \ , \end{cases}$$

The finite difference operator L_ε^N satisfies the following discrete comparison principle in G_i^N, $i = l, r, c$.

Discrete Comparison Principle *Assume that the mesh function Ψ satisfies $\Psi \geq 0$ on ∂G_i^N. Then $L_\varepsilon^N \Psi \geq 0$ on G_i^N implies that $\Psi \geq 0$ at each point in $\bar{G}_i^N$.*

Note. The piecewise bilinear interpolant, $\bar{Z}$ of the mesh function Z defined on the mesh G^N to the domain $\bar{G}$ retains the ε-uniform error estimates established at every point in $\bar{G}^N$. This can be established using arguments given in for example, [18].

In the following lemma we obtain a bound for the error in the regular Schwarz component $V_\varepsilon^{[k]}$, which approximates the solution component v_ε.

Lemma 6.3.1 *For all $k \geq 1$*

$$||(V_\varepsilon^{[k]} - v_\varepsilon)(x_i, t_j)|| \leq C(N_x^{-2} + N_t^{-1}) + Cq^k$$

where $q \leq 1/2$.

Proof. On the boundary ∂G_l, we note the arbitrary initial condition along $(2\sigma, t_j)$, is chosen to be $V^{[1]}(2\sigma, t_j) = v_\varepsilon(2\sigma, 0)$ and,

$$|(V_l^{[1]} - v_\varepsilon)(0, t_j)| = 0, \qquad |(V_l^{[1]} - v_\varepsilon)(x_i, 0)| = 0,$$

$$\begin{aligned} |(V_l^{[1]} - v_\varepsilon)(2\sigma, t_j)| &\leq |v_\varepsilon(2\sigma, 0)| + |v_\varepsilon(2\sigma, t_j)| \\ &\leq C, \end{aligned}$$

since $|v_\varepsilon(x_i, t_j)| \leq C$. The classical truncation error estimate and appropriate bounds on the partial derivatives, given in Lemma 6.2.1, then gives

$$\begin{aligned} |L_\varepsilon^N (V_l^{[1]} - v_\varepsilon)(x_i, y_j)| &= |(L_\varepsilon^N - L_\varepsilon) v_\varepsilon(x_i, y_j)| \\ &\leq \frac{\varepsilon}{12}(x_i - x_{i-1})^2 ||\frac{\partial^4 v_\varepsilon}{\partial x^4}|| + \frac{d(x_i, t_j)}{2}(t_j - t_{j-1})||\frac{\partial^2 v_\varepsilon}{\partial t^2}|| \\ &\leq C(x_i - x_{i-1})^2 + C(t_j - t_{j-1}) \\ &\leq C(N_x^{-2} + N_t^{-1}). \end{aligned}$$

The following barrier function, $\Phi(x_i, y_j)$ is selected,

$$\Phi(x_i, y_j) = \frac{C_1 x_i}{2\sigma} + C_1(N_x^{-2} + N_t^{-1}) t_j,$$

where the constant C_1 is chosen so that

$$\begin{aligned} L_\varepsilon^N(\Phi \pm (V_l^{[1]} - v_\varepsilon))(x_i, t_j) &\geq 0 \quad \text{on } G_l^N, \\ (\Phi \pm (V_l^{[1]} - v_\varepsilon)) &\geq 0 \quad \text{on } \partial G_l. \end{aligned}$$

The discrete maximum principle for L_ε^N on G_l^N then gives,

$$|(V_l^{[1]} - v_\varepsilon))(x_i, t_j)| \leq \Phi_{i,j} = \frac{Cx_i}{2\sigma} + C_1(N_x^{-2} + N_t^{-1})t_j \quad \text{in } \bar{G}_l,$$

and in a analogous result,

$$|(V_r^{[1]} - v_\varepsilon))(x_i, t_j)| \leq \Phi_{i,j} = \frac{C(1-x_i)}{2\sigma} + C_1(N_x^{-2} + N_t^{-1})t_j \quad \text{in } \bar{G}_r.$$

Now, the discrete problem is solved in G_c, and the interface conditions satisfy,

$$\begin{aligned}
|(V_c^{[1]} - v_\varepsilon))(\sigma, t_j)| &= |(\bar{V}_l^{[1]} - v_\varepsilon))(\sigma, t_j)| \\
&\leq \frac{C(\sigma)}{2\sigma} + C_1(N_x^{-2} + N_t^{-1})t_j \\
&\leq C(1/2) + C_1(N_x^{-2} + N_t^{-1})t_j, \\
|(V_c^{[1]} - v_\varepsilon))(1-\sigma, t_j)| &= |(\bar{V}_r^{[1]} - v_\varepsilon))(1-\sigma, t_j)| \\
&\leq \frac{C(1-(1-\sigma))}{2\sigma} + C_1(N_x^{-2} + N_t^{-1})t_j \\
&\leq C(1/2) + C_1(N_x^{-2} + N_t^{-1})t_j.
\end{aligned}$$

And as before, the truncation error is given by

$$L_\varepsilon^N(V_l^{[1]} - v_\varepsilon)(x_i, y_j)| \leq C(N_x^{-2} + N_t^{-1}).$$

We choose the barrier function to be $\Phi_{i,j} = C(1/2) + C_1(N_x^{-2} + N_t^{-1})t_j$ and so, it follows by the discrete maximum principle that

$$|(V_c^{[1]} - v_\varepsilon))(x_i, t_j)| \leq C(1/2) + C_1(N_x^{-2} + N_t^{-1})t_j \quad \text{in } \bar{G}_c.$$

Then combining estimates from $\bar{G}_l$, $\bar{G}_r$ and $\bar{G}_c$, gives

$$|(V_\varepsilon^{[1]} - v_\varepsilon))(x_i, t_j)| \leq C(1/2) + C_1(N_x^{-2} + N_t^{-1})t_j \quad \text{in } \bar{G},$$

and, by mathematical induction we obtain the estimate,

$$|(\bar{V}_\varepsilon^{[k]} - v_\varepsilon))(x_i, t_j)| \leq C(1/2)^k + C_1(N_x^{-2} + N_t^{-1}).$$

This completes the proof. ◇

In the following lemma we derive an error estimate for $W_l^{[k]}$, which is an approximation of the solution component w_l.

Lemma 6.3.2 *For all* $k \geq 1$

$$||(W_l^{[k]} - w_l)(x_i, t_j)|| \leq C_1 \left((N_x^{-1} \ln N_x)^2 + N_t^{-1}\right)$$

Proof. First, we recall from the bounds on derivatives, given in Lemma 6.2.1, that $|w_l| \leq Ce^{-x/\sqrt{\varepsilon}}$. The interface conditions on the subdomain $\bar{G}_l$ satisfy,

$$\begin{aligned} |(W_{l,l}^{[1]} - w_l)(0, t_j)| &= 0, \quad |(W_{l,l}^{[1]} - w_l)(x_i, 0)| = 0 \\ |(W_{l,l}^{[1]} - w_l)(2\sigma, t_j)| &= |w_l(2\sigma, 0) - w_l(2\sigma, t_j)| \\ &\leq |w_l(2\sigma, 0)| + |w_l(2\sigma, t_j)| \\ &\leq 2Ce^{-2\sigma/\sqrt{\varepsilon}} = e^{-4 \ln N_x} = CN_x^{-4}. \end{aligned}$$

The truncation error is given by,

$$\begin{aligned} |L_\varepsilon^N (W_{l,l}^{[1]} - w_l)(x_i, y_j)| &= |(L_\varepsilon^N - L_\varepsilon) w_l(x_i, y_j)| \\ &\leq \frac{\varepsilon}{12}(x_i - x_{i-1})^2 ||\frac{\partial^4 w_l}{\partial x^4}|| + \frac{d(x_i, t_j)}{2}(t_j - t_{j-1}) ||\frac{\partial^2 w_l}{\partial t^2}|| \\ &\leq \varepsilon^2 (N_x^{-1} \ln N_x)^2 C\varepsilon^{-2} e^{-x/\sqrt{\varepsilon}} + CN_t^{-1} \\ &= C(N_x^{-1} \ln N_x)^2 + CN_t^{-1}, \end{aligned}$$

and we choose the barrier function $\Phi(x_i, t_j) = C((N_x^{-1} \ln N_x)^2 + N_t^{-1})t_j + CN_x^{-2}$. Then by the discrete maximum principle for L_ε^N on G_l gives,

$$|(W_{l,l}^{[1]} - w_l)(x_i, t_j)| \leq C(N_x^{-1} \ln N)^2 + CN_t^{-1} \quad \text{in } \bar{G}_l,$$

and in an analogous result $|(W_{l,r}^{[1]} - w_l)(x_i, t_j)| \leq C(N_x^{-1} \ln N)^2 + CN_t^{-1}$ in $\bar{G}_r$. In G_c, the solution component w_l is "small", since $|w_l| \leq e^{-\sigma/\sqrt{\varepsilon}} = CN_x^{-2}$, and we will use $|(W_{l,r}^{[1]} - w_l)(x_i, t_j)| \leq |W_{l,l}^{[1]}| + |w_l|$, and obtain bounds for $|W_{l,l}^{[1]}|$ and $|w_l|$ separately.

Now, at the interface points,

$$\begin{aligned}
|W^{[1]}_{l,c}(\sigma,t_j)| &= |\bar{W}^{[1]}_{l,l}(\sigma,t_j)| \\
&\le |(\bar{W}^{[1]}_{l,l} - w_l)(\sigma,t_j)| + |w_l(\sigma,t_j)| \\
&\le C_1((N_x^{-1}\ln N_x)^2 + N_t^{-1})t_j + CN_x^{-2}.
\end{aligned}$$

$$\begin{aligned}
|W^{[1]}_{l,c}(1-\sigma,t_j)| &= |\bar{W}^{[1]}_{l,r}(1-\sigma,t_j)| \\
&\le |(\bar{W}^{[1]}_{l,r} - w_l)(1-\sigma,t_j)| + |w_l(1-\sigma,t_j)| \\
&\le (C_1(N_x^{-1}\ln N_x)^2 + N_t^{-1})t_j + CN_x^{-2}.
\end{aligned}$$

$$|W^{[1]}_{l,c}(x_i,0)| = |w_l(x_i,0)| \le CN_x^{-2}, \quad x_i \ge 2\sigma.$$

Also, $W^{[1]}_{1,c}$ satisfies the homogeneous difference equation,

$$L^N_\varepsilon(W^{[1]}_{l,c}(x_i,t_j)) = 0,$$

thus the discrete maximum principle gives

$$|W^{[1]}_{l,c}(x_i,t_j)| \le C_1((N_x^{-1}\ln N_x)^2 + N_t^{-1})t_j + CN_x^{-2},$$

and therefore the following estimate holds in $\bar{G}_c$,

$$|(W^{[1]}_{l,c} - w_l)(x_i,t_j)| \le C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1} + CN_x^{-2} + CN_x^{-2}.$$

Then combining the estimates from the subdomains $\bar{G}_l$, $\bar{G}_l$ and $\bar{G}_l$ it follows in $\bar{G}^N$ that,

$$|(W^{[1]}_l - w_l)(x_i,t_j)| \le C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1} + 2CN_x^{-2}.$$

We continue with the second iteration and the proof is then completed by induction. An outline of the second iteration is now given. In $\bar{G}_l$, the boundary conditions are given by

$$|(W^{[2]}_{l,l} - w_l)(0,t_j)| = 0, \qquad |(W^{[2]}_{l,l} - w_l)(x_i,0)| = 0,$$

$$\begin{aligned}
|(W^{[2]}_{l,l} - w_l)(2\sigma,t_j)| &= |(W^{[1]}_l - w_l)(2\sigma,t_j)| \\
&\le C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + 2CN_x^{-2}.
\end{aligned}$$

We choose the barrier function $\Phi_{i,j}$ to be,

$$\Phi_{i,j} = C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN_x^{-2})\frac{x_i}{2\sigma},$$

and using the discrete maximum principle then yields,

$$|(W_{l,l}^{[2]} - w_l)(x_i,t_j)| \leq C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN_x^{-2})\frac{x_i}{2\sigma} \quad \text{in } \bar{G}_l.$$

And, applying analogous arguments for $W_r^{[2]}$ gives,

$$|(W_{l,r}^{[2]} - w_l)(x_i,t_j)| \leq C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN_x^{-2})\frac{(1-x_i)}{2\sigma} \quad \text{in } \bar{G}_l.$$

In G_c, the interface conditions now satisfy,

$$\begin{aligned}
|(W_{l,c}^{[2]} - w_l)(\sigma,t_j)| &= |(\bar{W}_{l,l}^{[2]} - w_l)(\sigma,t_j)| \\
&\leq C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN_x^{-2})(\frac{1}{2}).
\end{aligned}$$

Using the triangle inequality,

$$|(W_{l,c}^{[2]}(\sigma,t_j))| \leq C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN_x^{-2})(\frac{1}{2}) + CN^{-2},$$

and also, on the interface $(1-\sigma,t_j)$,

$$|(W_{l,c}^{[2]}(1-\sigma,t_j))| \leq C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN_x^{-2})(\frac{1}{2}) + CN^{-2}.$$

Hence, using the discrete maximum principle we obtain,

$$|(W_{l,c}^{[2]}(x_i,t_j))| \leq C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN_x^{-2})(\frac{1}{2}) + CN^{-2},$$

and again, using the triangle inequality then gives,

$$|(W_{i,c}^{[2]} - w_l)(x_i,t_j))| \leq C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN_x^{-2})(\frac{1}{2}) + 2CN^{-2}.$$

This argument is concluded by mathematical induction

$$\begin{aligned}
|(W_{l,c}^{[k]} - w_l)(x_i,t_j))| &\leq C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) \\
&\quad + 2CN^{-2}\big[(1/2)^k + (1/2)^{k-1} + \ldots + 1\big] \\
&= C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + (2CN^{-2})\Big[\frac{1-(1/2)^{k-1}}{1-1/2}\Big] \\
&= C_1(N_x^{-1}\ln N_x)^2 + CN_t^{-1}(t_j) + 2CN^{-2}\big[1-(1/2)^{k-1}\big]
\end{aligned}$$

for all $(x, y) \in \bar{G}^N$. $\diamond$

Remark. The analysis of the Schwarz component $W_r^{[k]}$ is analogous to that in Lemma 6.3.2.

In the following theorem we combine the estimates given in Lemma 6.3.1 and Lemma 6.3.2, to provide an (ε, N)-uniform error estimate for the bilinear interpolant, $\bar{U}_\varepsilon^{[k]}$, for the solution of Method 6.1.

Theorem 6.3.1 *For all $k \geq 1$*

$$\|(\bar{U}_\varepsilon^{[k]} - u_\varepsilon)\|_{\bar{G}} \leq C\left((N_x^{-1} \ln N_x)^2 + N_t^{-1} + q^k\right)$$

where $q \leq 1/2$.

6.4 Numerical experiments

Numerical computations are carried out on the following two dimensional elliptic problem with degenerate parabolic layers, for a sequences of meshes Ω_i^N, $i = a, b, c, d$ corresponding to $N = 4, 8, 16, 32, 64, 128$.

$$\varepsilon \Delta u + y^\alpha (1-y)^\beta u_x = (1-x) y^\alpha (1-y)^\beta, \quad (x,y) \in (0,1)^2 \tag{6.3a}$$

$$u(x,0) = x, \quad u(x,1) = x^2, \tag{6.3b}$$

$$u(1,y) = 1, \quad u_x(0,y) = 8. \tag{6.3c}$$

The characteristics of the corresponding reduced problem are parallel to the x-coordinate axis and so, for small values of the parameter ε, layers appear in the solution close to the boundaries at $y = 0$ and $y = 1$. A solution of a differential equation of the type

$$\varepsilon u_{xx} + \varepsilon u_{yy} + a u_x = f,$$

can be considered to contain similar layer behaviour to the solution of Problem 6.2 near the sides $y = 0$ and $y = 1$, since for $\varepsilon << 1$, εu_{xx} is "small away" from the side $x = 0$ in the layer region, and the resulting solution is comparable to the solution of the equation,

$$\varepsilon u_{yy} + a u_x = f, \quad (y, x) \in (0, 1) \times (a, 1)$$

This, in turn, can be thought of as a parabolic differential equation, where $-x$ is behaving like a time variable. However, in problem (6.3) the coefficient of the x-derivative is equal to zero on the boundaries $y = 0$ and $y = 1$ and along these boundaries the parabolic equations are degenerate. Theoretical results for both non-degenerate and degenerate parabolic boundary layers are presented in Shishkin [29] and the convergence of a Shishkin fitted mesh method was examined computationally in Hegarty et al [11]. Here, we present computations which demonstrate that the natural extension of Method 6.1 can be applied to problem (6.3) to produce parameter-uniform numerical approximations. We note, that in problem (6.3) a weak layer, associated with the Neumann boundary condition at $x = 1$, is present in the solution near the boundary at $x = 1$. For this weak layer a uniform mesh is sufficient. The transition parameters [29] for this problem are

$$\begin{aligned} \sigma_1 &= \min\{1/4, m\varepsilon^{\frac{1}{2+\alpha}} \ln(N_2)\}, \\ \sigma_2 &= \min\{1/4, m\varepsilon^{\frac{1}{2+\beta}} \ln(N_2). \end{aligned}$$

We choose the subdomains

$$\Omega_b = (0, 1) \times (0, 2\sigma_1), \quad \Omega_c = (0, 1) \times (\sigma_1, 1 - \sigma_2) \quad \Omega_t = (0, 1) \times (1 - 2\sigma_2, 1),$$

as shown Fig. 6.3. We define the method which will be applied to problem (6.3).

Method 6.2 *The sequence of discrete Schwarz iterates $\bar{U}_\varepsilon^{[k]}$ is defined by*

$$\bar{U}_\varepsilon^{[k]} = \begin{cases} \bar{U}_c^{[k]} \text{ in } \overline{\Omega}_c \ , \\ \bar{U}_i^{[k]} \text{ in } \overline{\Omega}_i \setminus \Omega_c, \quad i = b, t \ , \end{cases}$$

where the $\bar{U}_i^{[k]}$ are the solutions of the following problems. On the boundary $\partial\Omega^N$, $U_\varepsilon^{[k]} = u_\varepsilon$. For $k = 1$,

$$\begin{aligned} L_\varepsilon^N U_b^{[1]} &= f \quad \text{in } \Omega_b^N, \quad U_b^{[1]}(x_i, 2\sigma_1) = \Psi, \\ L_\varepsilon^N U_t^{[1]} &= f \quad \text{in } \Omega_t^N, \quad U_t^{[1]}(x_i, 1-2\sigma_2) = \Psi, \\ L_\varepsilon^N U_c^{[1]} &= f \quad \text{in } \Omega_c^N, \ U_c^{[1]}(x_i, \sigma_1) = \bar{U}_b^{[1]}(x_i, \sigma_1), \ U_c^{[1]}(x_i, 1-\sigma_2) = \bar{U}_t^{[1]}(x_i, 1-\sigma_2), \end{aligned}$$

and for $k > 1$,

$$\begin{aligned} L_\varepsilon^N U_b^{[k]} &= f \quad \text{in } \Omega_b^N, \quad U_b^{[k]}(x_i, 2\sigma_1) = \bar{U}_c^{[k-1]}(x_i, 2\sigma_1), \\ L_\varepsilon^N U_t^{[k]} &= f \quad \text{in } \Omega_t^N, \quad U_t^{[k]}(x_i, 1-2\sigma_2) = \bar{U}_c^{[k-1]}(x_i, 1-2\sigma_2), \\ L_\varepsilon^N U_c^{[k]} &= f \quad \text{in } \Omega_c^N, \ U_c^{[k]}(x_i, \sigma_1) = \bar{U}_b^{[k]}(x_i, \sigma_1), \ U_c^{[k]}(x_i, 1-\sigma_2) = \bar{U}_t^{[k]}(x_i, 1-\sigma_2), \end{aligned}$$

and $\bar{U}$ is the bilinear interpolant of U and Ω_i^N are uniform rectangular meshes on Ω_i.

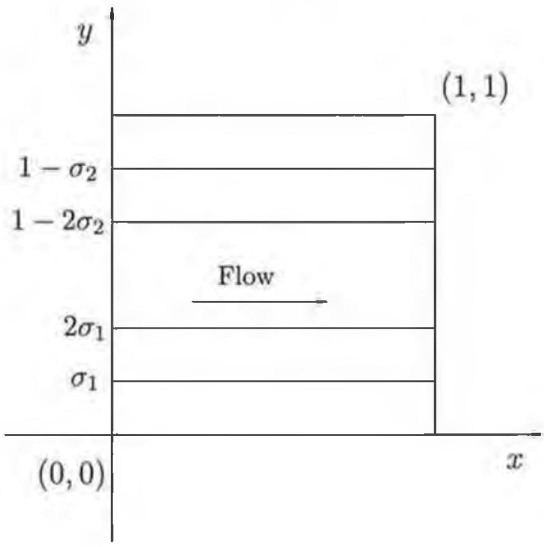

Figure 6.3: The structure of the subdomains for Method 6.2 applied to problem (6.3)

Ψ is some arbitrary function with sufficient smoothness, for example, $U_b^{[1]}(x_i, 2\sigma_1) = \Psi$ where Ψ is the linear interpolant of the boundary values $u_\varepsilon(0, 2\sigma_1)$, $u_\varepsilon(1, 2\sigma_1)$.

Figures 6.4, 6.5 and 6.6 show the numerical solution for Method 6.2, for different values of α and β, for $\varepsilon = 2^{-20}$, with $N = 8$ intervals in each subdomain. We see in these figures, that the method adapts to the different solution behaviour in the layer regions, for different values of α and β. The orders of convergence presented

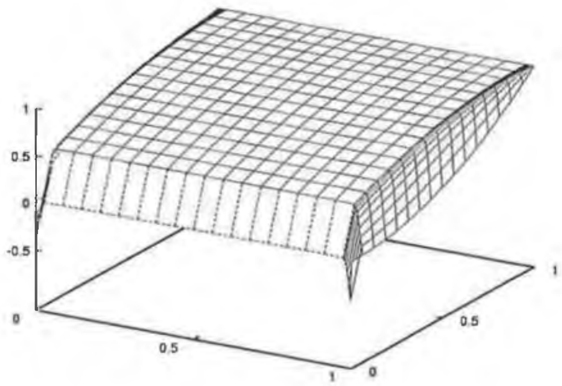

Figure 6.4: Numerical solution generated by Method 6.2 applied to problem (6.3) with Ω_i^8, $\alpha = \beta = 1.0$ and $\varepsilon = 2^{-20}$

in Table 6.3 are computed using the double mesh principle (see [6]),

$$p_\varepsilon^N = \log_2\left(\frac{D_\varepsilon^N}{D_\varepsilon^{2N}}\right) \quad \text{where} \quad D_\varepsilon^N = \max_{(x_i,y_j)\in\bar{\Omega}_i^N} |U_\varepsilon^N(x_i, y_j) - \bar{U}_\varepsilon^{2N}(x_i, y_j)|,$$

and the differences D_ε^N are shown in Table 6.2. In Table 6.1, the required iteration counts are given for a tolerance level of

$$\max_{x_i\in\bar{\Omega}_\varepsilon^N} |U_\varepsilon^{[k]}(x_i, y_j) - U_\varepsilon^{[k-1]}(x_i, y_j)| \leq 10^{-7}.$$

They show experimentally that the numerical solutions generated by Method 6.2 converge (ε, N)-uniformly to the solution of problem (6.3).

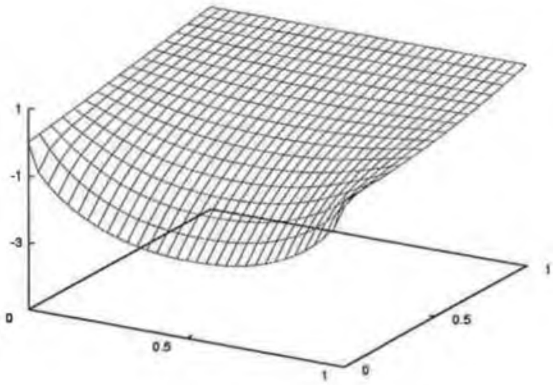

Figure 6.5: Numerical solution generated by Method 6.2 applied to problem (6.3) with Ω_i^8, $\alpha = \beta = 10$ and $\varepsilon = 2^{-20}$

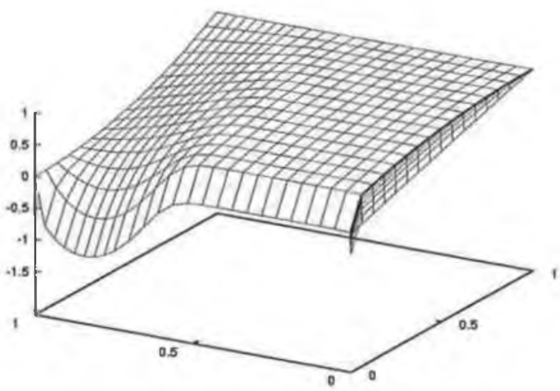

Figure 6.6: Numerical solution generated by Method 6.2 applied to problem (6.3) with Ω_i^8, $\alpha = 0.1$, $\beta = 10$ and $\varepsilon = 2^{-20}$

Remarks. The linear system, resulting from the difference scheme, was solved using a Gauss-Seidel iterative solver, with a tolerance of 10^{-7}. We choose the stopping criterion for the Schwarz iterates to be 10^{-6} because for values of ε close to 1, round-off error occurs and the number of iterations required to meet a lower tolerance level increases.

	Number of Intervals N in each subdomain					
ε	4	8	16	32	64	128
2^{-0}	17	18	18	18	17	17
2^{-2}	16	17	17	17	17	16
2^{-4}	13	13	14	14	14	13
2^{-6}	8	7	7	7	7	8
2^{-8}	5	4	4	4	4	4
2^{-10}	5	3	3	3	3	3
2^{-12}	6	3	3	2	2	3
2^{-14}	6	4	3	2	2	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-28}	6	4	3	2	2	3

Table 6.1: Iterations counts for Method 6.2 applied to problem (6.3)

6.5 Conclusions

In this chapter, an analogous Schwarz approach to Method 2.1 was shown to be (ε, N)-uniformly convergent to Problem 6.1. This parameter-uniform convergence was shown theoretically to be essentially second order in space and first order in time. Numerical results where presented for Method 6.2, a natural extension of Method 6.1 to elliptic singularly perturbed problem whose solution contains parabolic boundary layers, which experimentally show that, for this class of problems this Schwarz approach is parameter-uniform.

	Number of Intervals N in each subdomain				
ε	4	8	16	32	64
2^{-0}	6.80E-01	3.50E-01	1.78E-01	9.00E-02	4.53E-02
2^{-2}	6.95E-01	3.55E-01	1.79E-01	9.03E-02	4.53E-02
2^{-4}	7.43E-01	3.71E-01	1.84E-01	9.16E-02	4.57E-02
2^{-6}	8.84E-01	4.45E-01	2.23E-01	1.12E-01	5.61E-02
2^{-8}	9.37E-01	4.69E-01	2.34E-01	1.17E-01	5.86E-02
2^{-10}	9.38E-01	4.69E-01	2.35E-01	1.17E-01	5.86E-02
2^{-12}	9.38E-01	4.70E-01	2.35E-01	1.17E-01	5.87E-02
2^{-14}	9.38E-01	4.69E-01	2.35E-01	1.17E-01	5.87E-02
2^{-16}	9.38E-01	4.69E-01	2.36E-01	1.19E-01	5.89E-02
2^{-18}	9.38E-01	4.69E-01	2.37E-01	1.20E-01	6.03E-02
2^{-20}	9.38E-01	4.69E-01	2.37E-01	1.21E-01	6.09E-02
2^{-22}	9.38E-01	4.69E-01	2.37E-01	1.21E-01	6.11E-02
2^{-24}	9.38E-01	4.69E-01	2.38E-01	1.21E-01	6.12E-02
2^{-26}	9.38E-01	4.69E-01	2.38E-01	1.21E-01	6.13E-02
2^{-28}	9.38E-01	4.69E-01	2.38E-01	1.21E-01	6.13E-02

Table 6.2: Computed differences D_{ε}^{N} for Method 6.2 applied to problem (6.3)

	Number of Intervals N in each subdomain			
ε	8	16	32	64
2^{-0}	0.96	0.97	0.99	0.99
2^{-2}	0.97	0.98	0.99	0.99
2^{-4}	1.00	1.01	1.01	1.00
2^{-6}	0.99	0.99	1.00	1.00
2^{-8}	1.00	1.00	1.00	1.00
2^{-10}	1.00	1.00	1.00	1.00
2^{-12}	1.00	1.00	1.00	1.00
2^{-14}	1.00	1.00	1.00	1.00
2^{-16}	1.00	0.99	0.99	1.01
2^{-18}	1.00	0.98	0.98	1.00
2^{-20}	1.00	0.98	0.97	0.99
2^{-22}	1.00	0.98	0.97	0.98
2^{-24}	1.00	0.98	0.97	0.98
2^{-26}	1.00	0.98	0.97	0.98
2^{-28}	1.00	0.98	0.97	0.98

Table 6.3: Computed orders of convergence p_ε^N for Method 6.2 applied to problem (6.3)

Bibliography

[1] I. P. Boglaev. On a domain decomposition algorithm for a singularly perturbed reaction-diffusion problem. *J. Comput. Appli. Math.*, 98(2):213–232, 1998.

[2] I. P. Boglaev and V.V Sirotkin. Computational method for a singularly perturbation problem via domain decomposition and its parallel implementation. *Applied Math. and Comput.*, 56(1):71–95, 1993.

[3] E.P. Doolan, J.J.H. Miller, and W.H.A. Schilders. *Uniform Numerical Methods for Problems with Initial and Boundary Layers.* Boole Press, Dublin, 1980.

[4] P.A. Farrell, I.G. Boglaev, and V.V. Sirotkin. Parallel domain decomposition methods for semi-linear singularly perturbed differential equations. *Comput. Fluid Dynamic Journal*, 2(4):423–433, January 1994.

[5] P.A. Farrell and A. F. Hegarty. On the determination of the order of uniform convergence. *Procs. 13th IMACS World Congress on Computational and Applied Mathematics, eds. R. Vichnevetsky and J.J.H. Miller*, pages 501–502, 1991.

[6] P.A. Farrell, A.F. Hegarty, J.J.H Miller, E. O'Riordan, and G.I. Shishkin. *Robust Computational Techniques for Boundary Layers.* Chapman and Hall/CRC Press, Boca Raton, U.S.A., 2000.

[7] M. Garbey. A Schwarz alternating procedure for singular perturbation problems. *J. Sci. Comput.*, 17(5):1175–1201, September 1996.

[8] M. Garbey and H.G. Kaper. Heterogeneous domain decomposition for singularly perturbed elliptic boundary value problems. *SIAM J. Numer. Anal.*, 34(4):1513–1544, 1997.

[9] F. Gastaldi, L. Gastaldi, and A. Quarteroni. Adaptive domain decomposition methods for advection dominated equations. *East-West J. Numer. Math.*, 4(3):165–206, 1996.

[10] R. Glowinski, G.H. Golub, G.A. Meurant, and J. Periaux. *First international Symposium on Domain Decomposition Methods for Partial Differential Equations.* SIAM, Philadelphia, 1988.

[11] A.F. Hegarty, J.J.H. Miller, E. O'Riordan, and G.I. Shishkin. Special numerical methods for convection-dominated laminar flows at arbitary Reynolds number. *East-West J. Numer. Math.*, 2(1):65–73, 2000.

[12] G. Lube, L. Muller, and F. C. Otto. A non-overlapping domain decomposition method for the advection-diffusion problem. *Computing*, 64:49–68, 2000.

[13] G. Lube, F.C. Otto, and H. Muller. A non-overlapping domain decomposition method for parabolic initial-boundary value problems. *Appli. Numer. Math.*, 28:359–369, 1998.

[14] H. MacMullen, J.J.H Miller, E. O'Riordan, and G.I. Shishkin. Schwarz iterative methods for convection-diffusion problems with boundary layers. *Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems,eds. J.J.H. Miller and G.I Shishkin and L.G Vulkov, Nova, New York,*, to appear.

[15] H. MacMullen, J.J.H Miller, E. O'Riordan, and G.I. Shishkin. A second order parameter-uniform overlapping Schwarz method for reaction-diffusion problems with boundary layers. *J. Comput. Appli. Math*, (to appear).

[16] T.P. Mathew. Uniform convergence of the Schwarz alternating method for solving singularly perturbed advection-diffusion equations. *SIAM J. Numer. Anal.*, 35(4):1663–1683, August 1998.

[17] J.J.H. Miller, E. O'Riordan, and G.I. Shishkin. Fitted mesh methods for the singular perturbated reaction diffusion problem. *Procs. V-th International Colloquium on Numerical Analysis, ed. E. Minchev, Academic Publications, Plovdiv, Bulgaria*, pages 99–105, 1996.

[18] J.J.H. Miller, E. O'Riordan, and G.I. Shishkin. *Fitted Numerical Methods for Singular Perturbation Problems.* World Scientific Publishing Co., Singapore, 1996.

[19] J.J.H. Miller, E. O'Riordan, G.I. Shishkin, and L.P. Shishkina. Fitted mesh methods for problems with parabolic boundary layers. *Mathematical proceedings of the Royal Irish Academy*, 98A(2):173–190, 1998.

[20] K. W. Morton. *Numerical Solution of Convection-Diffusion Problems.* Chapman & Hall, London, 1996.

[21] F. Nataf and F. Rogier. Factorization of the convection-diffusion operator and the Schwarz algorithm. *Math. Models Meth. Appli. Sci.*, 67(1):67–93, 1995.

[22] F.-C. Otto and G. Lube. A nonoverlapping domain decomposition method for the Oseen equations. *Math. Models and Meth. Appli. Sci.*, 8(6):1091–1117, 1998.

[23] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations.* Oxford University Press, New York, 1999.

[24] G. Rodrigue and E. Reiter. A domain decomposition method for boundary layer problems. *Second International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 226–234, 1989.

[25] H.-G. Roos, M. Stynes, and L. Tobiska. *Numerical Methods for Singularly Perturbed Differential Equations.* Springer-Verlag, Berlin, 1996.

[26] H.A. Schwarz. Uber einige Abbildungsaufgaben. *J. Reine Angew. Math.*, 70:105–120, 1869.

[27] G. I. Shishkin. A difference scheme for a singularly perturbed equation of parabolic type with a discontinuous initial condition. *Soviet Math. Dokl.*, 37:792–796, 1988.

[28] G.I. Shishkin. Approximation of solutions of singularly perturbed boundary value problems with a parabolic boundary layer. *USSR Comput. Maths. Math. Phys*, 29(4):1–10, 1989.

[29] G.I. Shishkin. Grid approximation of singularly perturbed parabolic equations degenerating on the boundary. *J. Vychisl. Mat. Fis. (in Russian)*, 31(10):1498–1511, 1991.

[30] G.I. Shishkin. *Discrete approximation of singularly perturbed elliptic and parabolic equations.* Russian Academy of Sciences, Ural Section, Ekaterinburg, 1992.

[31] G.I. Shishkin. Acceleration of the process of the numerical solution to singularly perturbed boundary value problems for parabolic equations on the basis of parallel computations. *Russ. J. Numer. Anal. Math. Modeling*, 12(3):271–291, 1997.

[32] G.I. Shishkin and P.N. Vabishchevich. Parallel domain decomposition methods with the overlapping of subdomains for parabolic problems. *Math. Models and Methods in Applied Sciences*, 6(8):1169–1185, 1996.

[33] B. Smith, P. Bjørstad, and W. Gropp. *Domain Decomposition.* Cambridge University Press, Cambridge, 1996.

[34] M. Stynes and E. O'Riordan. A uniformly convergent Galerkin method on a Shishkin mesh for a convection-diffusion problem. *J. Math. Anal. App.*, 214:36–54, 1997.

[35] P. Le Tallec and M.D Tidriri. Convergence analysis of domain decomposition algorithms with full overlapping for the advection-diffusion problems. *Math. of Computation*, 8(226):585–606, April 1999.