Dimension of Scrambled Sets and The Dynamics of Tridiagonal Competitive-Cooperative System

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List of Publications

This thesis consists of an introductory part and the following four papers:

[A] C. Fang, W. Huang, Y. Yi and P. Zhang, *Dimensions of stable sets and scrambled sets in positive finite entropy systems*, Ergodic Theory Dynam. Systems 32(2012), No. 2, 599-628.

[B] C. Fang, M. Gyllenberg and Y. Wang, *Floquet theory for tridiagonal competitive-cooperative systems and the dynamics of time-recurrent systems*, SIAM J. Math. Anal. Vol. 45(2013), No. 4, 2477-2498.

[C] C. Fang, M. Gyllenberg and Y. Wang, *Non-hyperbolic minimal sets for tridiagonal competitive-cooperative systems*

[D] C. Fang, M. Gyllenberg and S. Liu, *Characterization of dominated splitting and its relation to hyperbolicity*

The author played a central role in the research for and writing of draft versions of the above papers, except for section 6 of the paper [A].

1 Overview

In nature, a number of problems can be described by processes evolving with time, which usually give rise to rather simple-looking difference or differential equations. This may happen, for example, when one wishes to predict the future behavior of several interacting species based on their present states and a given environment. Mathematically, this means investigating the asymptotic behavior of an orbit $\{f^t(x)\}$, where $x \in X$, with X the set of all possible states for these interaction species, and $t \in \mathbb{Z}$ or $\mathbb{R}$ denotes time. Here f is the rule describing how these species interact with each other and with the environment to produce new states in the near future. An ideal result we usually wish for for such problems is essential stability of the iteration, namely, the sequence $\{f^t(x)\}$ converges to a fixed point, indicating that the species will eventually stabilize at some equilibrium states, independently of their initial state. If this fails, we would hope for a weaker property: predictability. This means even if the present state is estimated within some reasonable error, the orbit starting from this fake state remains close to the true orbit. Unfortunately, it often happens that neither of these wishes is satisfied. One usually observes various sorts of chaotic behavior, all of which are predictable from the dynamics of $\{f^t\}$.

In this thesis, our first goal is to provide estimates for the dimension of chaotic sets in a positive entropy system. We then study a concrete model from biology, that is, the tridiagonal competitive-cooperative system. The dynamics for the mentioned system is proved to be simple if the experiment is carried out in an environment which is independent of time or an environment which changes periodically with time. In these situations, no chaotic phenomena are produced, also the essential stability and predictability are achieved. However, if the environment changes aperiodically, then the dynamics becomes quite complicated, and chaotic phenomena may ocurr. In this thesis, we study the dynamics of the tridiagonal competitive-cooperative system in the latter case. The following is an overview of these works.

The word "chaos" related to a map was first introduced by Li and Yorke in their influential paper [41], to describe the complexity of a system, although without a formal definition. Today, one can say that there are as many views of chaos as there are authors, see [79, Chapter 10] and references therein for more detail and their relationships. Generally speaking, these definitions can be divided into four classes. The first class is based on the instability of trajectories or sensitive dependence on initial conditions, such as Devaney chaos. The second class is Li-Yorke chaos, see [41]. Intuitively, this means there are many different states in a dynamical system and any two of them can approach each other for some sequence of moments in the time evolution, and for some other sequence of moments they remain separated. The third class is based on the concept of entropy. Measure-theoretic entropy was first introduced by Kolmogorov [35] for a transformation f preserving a probability measure. Adler, Konheim and McAndrew [1] then introduced the notion of topological entropy for a topological dynamical system. These two concepts are related to each other through the so-called variational principle [76, Chapter 8]. Both of them measure the asymptotic growth in information obtained through iterating f and relate to the rate at which points are being dispersed. Since a zero entropy system has some kind of certainty, many researchers consider systems with positive entropy to be chaotic. The fourth class is defined by recurrence and mixing properties.

In this thesis, we focus on the Li-Yorke definition of chaos [41], whose central idea is the existence of a scrambled set with uncountable cardinality. By a *topological dynamical system (TDS)* we mean a pair (X,T) where X is a compact metric space endowed with the metric d, and T is a continuous map from X to itself. When T is a homeomorphism, the TDS (X,T) is called *invertible.* A pair of points $(x,y) \in X \times X$ is said to be a *Li-Yorke pair* if

$$\liminf_{n\to+\infty} d(T^n x, T^n y) = 0. \tag{1.1}$$

and

$$\limsup_{n\to+\infty} d(T^n x, T^n y) > 0 \tag{1.2}$$

A subset S of X containing at least two points is called a *scrambled set* of the system, if, for any $x \neq y \in S$, (x,y) is a Li-Yorke pair. A TDS (X,T) is then called *Li-Yorke chaotic* if it has an uncountable scrambled set.

There are many topological dynamical systems that are Li-Yorke chaotic. For example, the classical result by Li and Yorke says that any continuous self-mapping on the interval $[0,1]$ with a period-three point is Li-Yorke chaotic. Huang and Ye [32] proved that there are "many" compacta, including some countable compacta, the Cantor set and continua of arbitrary dimension, admitting completely scrambled homeomorphisms. Another famous result is given by Blanchard et al [5] and says that positive entropy implies Li-Yorke chaos.

The dynamics of systems without Li-Yorke pairs, i.e., where every pair of points is either asymptotic or distal, is comparatively simple. Here a pair $(x, y) \in X \times X$ is said to be *asymptotic* if $\lim_{n\to\infty} d(T^n x, T^n y) = 0$, and one for which $\liminf_{n\to\infty} d(T^n x, T^n y) > 0$ is said to be *distal*. For example, these systems should be zero-entropy. They are minimal when transitive and their adherence semigroup is minimal, see [5]. Note also that the set of asymptotic pairs of any TDS is nonempty, see [6, 33]. Therefore, to investigate the structure of a dynamical system, especially to evaluate its complexity, the size of scrambled sets and the closure of stable sets of the system is definitely one of the most important aspects. In general, the size can be examined from the measure-theoretic point of view, from the topological point of view, or from the dimension point of view. We point out that one can also measure the size of a set from the Bowen dimension entropy point of view. Bowen dimension entropy is a notion of conjugacy that is a cross between the topological and measure theoretic ones. It was first introduced by Bowen [11] and then developed by Pesin and Pitskel [54]. So, to understand the complexity of a dynamical system, we always ask: do the chaotic set and the closure of stable set of this system have positive measure; can they have non-empty interiors; are their Hausdorff dimensions and Bowen dimension entropies bigger than zero? Unfortunately, the first two properties cannot hold in general, see [7, 14, 69, 70] and references therein for counterexamples and explanations. So we will consider the size of these sets from the dimensional point of view.

In paper [A], we estimate the Bowen dimension entropy and Hausdorff dimension of the closure of stable sets and of scrambled sets in a TDS with positive finite entropy. The second part of this thesis, including papers [B-D], is dedicated to the study of the dynamics of tridiagonal competitive-cooperative systems. These carry no Li-Yorke pairs when they are time-independent or time-periodic. In fact, when the system is autonomous, Smillie [71] showed that all solutions must converge to equilibria or diverge in a strong sense. Smith [73] studied the time-periodic system and proved that every bounded solution is asymptotic to a periodic solution whose period is the same as the system. This implies that all pairs are asymptotic or distal in these two cases. However, the behavior of this system becomes quite complicated as verified by [77] when we consider the nonautonomous case. A natural question arises: what kind of properties can one assume to guarantee the nonexistence of Li-Yorke pairs when the system is nonautonomous? We of course prefer properties that have some biological meaning. For example, stability or unstability. Further, since the competitive-cooperative relationship plays an important role in the real world, we are still happy to know the dynamics of this system even if the above properties are not satisfied. Actually, dynamical systems that describe the interaction of $n \geq 2$ species have been the object of intensive study ever since the seminal work by Lotka, Volterra and Kolmogorov [34, 38, 75]. So whether from the mathematical sense or from the biological sense, we are full of interest to know the whole dynamics of the tridiagonal competitive-cooperative system.

In paper [B] we show that any minimal set of the system which is hyperbolic (include uniformly stable, uniformly unstable, etc.) admits no *proximal pair*, i.e. the formula (1.1) does not hold. Together with the results in [77] and more specific analysis we prove any ω-limit set of this system is a 1-cover of its base. Biologically, this means the species in this model which want to survive should evolve simultaneously with the environment. In paper [C], we study the structure of non-hyperbolic minimal sets of the system. More precisely, we show the dynamics on the set is conjugate to a scalar product flow when the set has central dimension one. When the central dimension is bigger than one, we propose a conjecture, saying that the dynamics of the non-hyperbolic minimal set should be conjugate to a k-dimensional skew-product flow, where k is just the central dimension of this minimal set. By the Floquet theory established for tridiagonal competitive-cooperative systems in [B], we know that every invariant set of this system admits an exponential separation (we use a similar conception "dominated splitting" in differentiable dynamical systems, see [D]). To understand the dynamics of any invariant set of the tridiagonal competitive-cooperative system, we characterize exponential separation in several equivalent expressions in the paper [D] and find an interesting connection between hyperbolicity and exponential splitting under the frame of cocyle. Simply put, we prove the difference between exponential separation and hyperbolicity is only a functional torsion. This work is also meaningful in the setting of differentiable dynamical systems.

The rest of the introductory part is organized as follows. In the next section, we describe the methods, outline the proofs and summarize the main results of papers [A-D]. In Section 3, we give a short note on new progress concerning the results in paper [A]. Finally, in the Section 4, we propose two conjectures about the dynamics of the tridiagonal competitive-cooperative system.

2 Methods and Summary

2.1 Dimensions of stable sets and scrambled sets

Consider a TDS (X,T) with finite positive entropy. We already know that the stable sets and scrambled sets exist, see [5, 33]. To estimate their dimensions, we first give a lower bound for their Bowen dimension entropy with the entropy of the system, then study the relationship between Bowen dimension entropy and Hausdorff dimension and use this to give a lower bound of their Hausdorff dimensions in terms of entropy of the system and the exponential rate of decay of Lebesgue numbers, see also Section 3.

In order to make the argument more flexible, we divide this process into two big steps. In the first step we deal with the invertible case, then handle the non-invertible case in the second step. Afterwards, we apply similar ideas to C^1 self-maps of a Riemannian manifold with positive entropy and give a lower bound for the Hausdorff dimension of the closure of stable sets and scrambled sets in terms of the metric entropy and Lyapunov exponents.

Step 1. In this step, we build up the key lemma [A, Theorem 3.3] which describes the relationship of entropies in a factor map, which will be useful in the following steps 3, 4 and 6.

Let (X,T) and (Y,S) be two TDSs. A continuous map $\pi : (X,T) \to (Y,S)$ is called a *homomorphism* if it is onto and $\pi T = S\pi$. Then (X,T) is said to be an *extension* of (Y,S) and (Y,S) is said to be a *factor* of (X,T). Such a map is referred to as a *factor map*. The result [A, Theorem 3.3] says that the Bowen entropy of any subset in the extension system can be bounded above by the sum of Bowen entropy of its image in the factor system and the topological entropy of the fibers.

Bowen [9, Theorem 17] was the first to study the relation between the topological entropies of the extension system and its factor system. We also would like to point out that these results were generalized recently by Downarowicz and Zhang [19].

Step 2. In case the TDS (X,T) is a zero-dimensional invertible system and carries an ergodic measure such that the measure-theoretical entropy is positive, we estimate the Bowen dimension entropy of the closure of stable sets in terms of the given entropy. Further, assuming the continuum hypothesis we show there exists a strongly scrambled set for both T and T^{-1} in the closure of the stable set and prove that the Bowen dimension entropy of this scrambled set is bigger than the measure-theoretical entropy.

Note that in this step we did not assume that the system has positive finite entropy. In fact, positive finite entropy is used to guarantee the existence of a zero-dimensional principal extension for an invertible TDS, see [A, Lemma 4.7 and Remark 4.8]. The zero-dimension condition is technically needed during the proof so that there exists a sequence of finite clopen partions with the diameters of these partitions tending to zero. We also would like to point out that the continuum hypothesis is only used in the proof of the abstract result [A, Lemma 4.3]. We conjecture this result is true even without the continuum hypothesis. If this is true, then all of the results in [A] hold without this assumption.

Step 3. We work under the supposition that the TDS (X,T) is invertible and has finite positive entropy in this step. Therefore, the system (X, T) has zero-dimensional principal extension (Z, R) with R invertible and the above results in step 2 hold for TDS (Z, R). Then we use the key result built up in step 1 and properties of the principal extension to estimate the lower bound of Bowen dimension entropy of the closure of stable sets and scrambled sets in terms of entropy. Here an extension $\pi : (Z, R) \to (X, T)$ of two TDSs is said to be a *principal extension* if $h_\mu(R) = h_{\pi\mu}(T)$ for every invariant probability measure μ of the TDS (Z, R).

Step 4. In this step, we consider a general TDS (X, T) with finite positive entropy. If T is non-invertible, we first lift this TDS to its natural extension, which is invertible. Note that the natural extension has the same entropy as the original system. The results obtained in step 3 hold for this natural extension. By taking advantage of the result from step 1 again, we can estimate the Bowen dimension entropy of the closure of the stable sets and scrambled sets of the system (X, T).

Step 5. Given a TDS (X, T), we investigate in this step the relationship between the Bowen dimension entropy and Hausdorff dimension of a set. Suppose the map T is Lipschitz continuous with Lipschitz constant $L > 1$, then we can estimate a lower bound for the Hausdorff dimension of a set in terms of its Bowen dimension entropy and $\log L$ as in [A, Lemma 5.1]. Assume further the TDS (X, T) has positive finite entropy. By making use of results in step 4 and the variational principle for entropy, one can give a lower bound for the Hausdorff dimension of the closure of stable sets and scrambled sets in terms of topological entropy and L.

In fact, the above results still hold if we replace $\log L$ with the exponential rate of decay of Lebesgue numbers, which is introduced by Sun in [74]. I will explain this idea in detail in Section 3.

We also give an example [A, Theorem 5.8] in symbolic dynamics to show that the lower bounds can be achieved.

Step 6. Note that a C^1 transformation of a compact manifold has finite entropy, see [9] or [76, Theorem 7.15]. In this final step, we apply similar ideas to those of steps 2 to 5 to a C^1 self-map on a Riemannian manifold with positive entropy and estimate the lower bound of the Bowen dimension entropy and Hausdorff dimension of stable sets and scrambled sets in terms of the metric entropy and Lyapunov exponent.

2.2 Floquet theory for linear strongly tridiagonal competitive-cooperative system

In this subsection we build up the Floquet theory for linear strongly tridiagonal competitive-cooperative system, which is motivated by the work of Chow, Lu and Mallet-Paret [16, 17] for scalar parabolic equations. It is one part of the work in [B]. This theory plays a key role in the following subsections.

Consider the following families of ODEs:

$$\begin{aligned}
\dot{x}_1 &= b_{11}(y\cdot t)x_1 + b_{12}(y\cdot t)x_2,\\
\dot{x}_i &= b_{i,i-1}(y\cdot t)x_{i-1} + b_{ii}(y\cdot t)x_i + b_{i,i+1}(y\cdot t)x_{i+1},\ 2\le i\le n-1, \qquad (2.1)\\
\dot{x}_n &= b_{n,n-1}(y\cdot t)x_{n-1} + b_{nn}(y\cdot t)x_n,
\end{aligned}$$

where $y \in Y$ and $y\cdot t$ defines a flow on the compact metric space Y. Assume the $(n\times n)$-matrix-valued function $B(\cdot) = \{b_{ij}(\cdot)\}$ is continuously defined on Y and the subtridiagonal elements satisfy $b_{i,i+1}(\cdot)$, $b_{i+1,i}(\cdot) \ge \varepsilon_0 > 0$, for some positive constant ε_0 and all $1 \le i \le n-1$. Then the system (2.1) is called a *strongly linear tridiagonal competitive-cooperative system*.

We express the Floquet theory of system (2.1) in the language of vector bundles [4] and exponential separation [46, 52, 55].

Fix $y \in Y$, equation (2.1) has a y-independent discrete Lyapunov function σ defined on an open dense subset of $\mathbb{R}^n$ which can take only finite many values $\{0,\cdots,n-1\}$, see [71–73]. As showed in [B], see also [77], this property severely constricts the dynamics of a strongly tridiagonal competitive-cooperative system, including (2.1). For example, for each value of σ there exist an unique solution $x_m(t,y)$ (up to a constant) of (2.1) such that the σ-value along this solution is constant [B, Proposition 2.3]. Further, we show these solutions are continuously dependent on $y \in Y$ and form a frame base of the product space $\mathbb{R}^n \times Y$, see [B, Proposition 2.4 and Proposition 3.1].

Naturally, they form n-subbundles of $\mathbb{R}^n \times Y$, which are the so-called *Floquet bundles* of (2.1). The Floquet theory for the system (2.1) says these subbundles are in fact exponentially separated, see [B, Theorem 3.4], which is much finer than the well-known Sacker-Sell spectral bundles [59] for (2.1), see [B, Corollary 3.8].

We end this subsection by pointing out that similar methods can be used for other systems, for example, systems studied in [73], to get similar results.

2.3 Structure of hyperbolic ω-limit sets

Now we apply the Floquet theory obtained in the previous subsection to investigate the dynamics of *strongly tridiagonal competitive-cooperative system.* This is concerned with systems of ODEs of the form

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2), \\ \dot{x}_i &= f_i(t, x_{i-1}, x_i, x_{i+1}), \quad 2 \le i \le n-1, \\ \dot{x}_n &= f_n(t, x_{n-1}, x_n), \end{aligned} \tag{2.2}$$

where the nonlinearity $f = (f_1, f_2, \cdots, f_n)$ is defined on $\mathbb{R} \times \mathbb{R}^n$ and is C^1-admissible. We assume further that the variable x_{i+1} forces $\dot{x}_i$ and that x_i forces $\dot{x}_{i+1}$ strictly monotone in the same fashion (see [B, Assumption (**F**)]) and that the hull of f, denoted by $H(f)$, is minimal, see [B] for exact definitions.

As we mentioned in the overview section, when the system (2.2) is time-independent or time-periodic, then all of its bounded solutions converge to some equilibria or periodic solutions respectively, see [71, 73]. These results imply that the system (2.2) exhibits no Li-Yorke pairs in these two cases. On the other hand, if this system is time-dependent, the dynamics becomes more complicated. In fact, as showed in [77] that every minimal set should be almost 1-cover [B, Definition 4.3] of $H(f)$ and each ω-limit set of the system contains at most two minimal sets. We shall prove that any ω-limit set that is hyperbolic (which is biologically meaningful since similar set always exist after a small perturbation of the environment) admits no Li-Yorke pair. More precisely, we prove that a hyperbolic ω-limit set of system (2.2) is exactly a 1-cover [B, Theorem 4.6] of $H(f)$. As a consequence, when the coefficient

f is time almost-periodic, then this result implies the bounded solutions of system (2.2) are asymptotic to some almost-periodic solutions if the system's ω-limit set is hyperbolic [B, Remark 4.7].

We employ techniques from skew-product flows and dynamical systems to study these problems. Discrete skew-product flows originate in ergodic theory, see [3, 21, 53], which won't be used here. The study of continuous skew-product flows arise in the study of ODEs, especially for nonautonomous differential equations, see [47, 56–58, 62–64, 68]. Applied to our system, one can embed (2.2) into a skew-product flow $\Pi : \mathbb{R}\times\mathbb{R}^n\times H(f) \to \mathbb{R}^n\times H(f)$ by $\Pi(t, x_0, g) = (x(t, x_0, g), g\cdot t)$, where $x(t, x_0, g)$ is the solution of $\dot{x} = g(t, x)$ with initial condition $x(0, x_0, g) = x_0$ and $g\cdot t$ is the time shift flow on $H(f)$. Therefore, to study the dynamics of (2.2), we only need to investigate the structure of the ω-limit or invariant set Y of Π in $\mathbb{R}^n \times H(f)$. For this purpose, we follow the usual ideas and study first the linearized system of (2.2) along an invariant set Y, which is actually studied in the previous subsection. Then one can use the obtained Floquet theory for the linearized system to investigate the properties of the original nonlinear system.

We end this subsection by giving the definition of hyperbolicity. The invariant set $Y \subset \mathbb{R}^n\times H(f)$ is said to be *hyperbolic* if the Sarker-Sell spectrum [59] of the linearized system (2.1) does not contain zero. Hyperbolicity includes many interesting phenomena, for example, uniform stability, uniform unstability. Otherwise, the invariant set Y is called *non-hyperbolic.*

2.4 Structure of non-hyperbolic minimal sets

In this subsection, we investigate the structure of a non-hyperbolic minimal set Y of the skew-product flow induced by system (2.2). This is the subject of paper [C].

To interpret the ideas of this paper, we need to introduce a concept from dynamical systems. Let (X, f^t) and (Z, g^t) be two TDSs. If there exists a homeomorphism $h : X \to Y$ such that $h \circ f^t = g^t \circ h$ for all $t \in \mathbb{R}$, then we say system (X, f^t) is *topologically conjugate* to system (Z, g^t). Speaking informally, topological conjugation is a "change of coordinates" in the topological sense. Remember that Y is originally a subset of $\mathbb{R}^n \times H(f)$.

The idea in [C] is to show that the dynamics on Y is actually topologically conjugate to some dynamics in a lower dimension. In the paper, we conjecture and partially prove that this lower dimension is just the central dimension of Y. More precisely, we prove in the case the central dimension is one, then the dynamics on Y is topologically conjugate to a scalar skew-product flow. Recall that the conjecture also true when the central dimension is zero, i.e., when Y is hyperbolic, see [B]. We would also like to point out that this kind of result cannot hold in general for a dynamical system, see [13, 18].

The main reason we have these results and conjecture is due to the σ-values-description for invariant manifolds and for the differences of every two points of Y from the same fiber, see [C, Proposition 3.2 and Proposition 4.4]. Intuitively, when the central dimension is one, it follows that the sign of first coordinate of these differences is not zero. Therefore, one can give an order for the points in the same fiber. Further, one can prove their dynamics are actually topologically conjugated.

2.5 Relationships between hyperbolicity and dominated splitting/exponential separation

Consider system (2.2). It follows from the Section 2.2 or [B], that the skew-product flow induced by the linearized system (2.1) over an invariant set of system (2.2) admits the finest exponential separation/dominated splitting, see [B, Theorem 3.4]. Meanwhile, we have made clear the structure of a special class of ω-limit set of system (2.2) whose linearized system (2.1) admits an exponential dichotomy/hyperbolicity, see Section 2.3. If we compare the definitions of exponential separation/dominated splitting and hyperbolicity, then one could find they are quite similar and easily find hyperbolicity implies exponential separation. This observation promotes us to study their relationship from the reverse direction. We give several equivalent characterizations of dominated splitting in terms of reducibility and lower and upper functions in [D], and finally lead to an interesting relation from dominated splitting to hyperbolicity.

We present these results in [D] in the frame of dynamical systems, because we also want to use the obtained results to reprove the classical result of

Hirsch, Pugh and Shub [30] about the existence of plaque families when the system admits dominated splitting over some invariant set, which is part of our future work. These results are also true for skew-product flows.

3 Notes to the paper [A]

In this notes, our goal is to replace the Lipschitz condition on the map T in the main theorems of [A] with the exponential rate of decay of Lebesgue numbers of open covers, which is first studied by [74] recently.

The following ideas are suggested and discussed with Peng Sun.

Let (X,T) be a TDS and denote the Lebesgue number of an open cover $\mathcal{U}$ of X by $\delta(\mathcal{U})$. Write $\delta_n(T,\mathcal{U}) = \delta(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})$ and let $h_L^+(T,\mathcal{U}) = \limsup_{n\to\infty} -\frac{1}{n}\log\delta_n(T,\mathcal{U})$. Then *the rate of decay of Lebesgue numbers* of open covers $h_L^+(T)$ is defined as follows:

$$h_L^+(T) = \sup h_L^+(T,\mathcal{U}),$$

where the supremum is taken over all open covers of X.

The following results can be obtained by a similar argument to the one in [74], see also the Remark and Theorem 5.1 of the same reference.

Theorem 3.1. *Let (X,T) be a TDS and $Y \subset X$. Then*

$$H_d(Y)h_L^+(T) \geq h_{top}^B(T|Y).$$

Theorem 3.2. *If T is Lipschitz with constant L, then $h_L^+(T) \leq \max\{\log L, 0\}$.*

Therefore, by using [A, Theorem 4.10] and Theorem 3.1, we have the following result.

Theorem 3.3. *Let (X,T) be a TDS with metric d satisfying $h_{top}(T) < \infty$. If μ is a T-invariant ergodic measure with $h_\mu(T) > 0$, then*

$$H_d(\overline{W^s(x,T)}) \geq \frac{h_\mu(T)}{h_L^+(T)} \quad \text{for } \mu - a.e\ x \in X.$$

Moreover, under the continuum hypothesis, for μ-a.e. $x \in X$ there exists a scrambled set $S_x \subset \overline{W^s(x,T)}$ for T satisfying $H_d(S_x) \geq h_\mu(T)/h_L^+(T)$

4 Conjectures

We propose two conjectures in this section. The first one is about the dynamics of a minimal invariant set of system (2.2). The other one is concerned with the dynamics of 2-D tridiagonal competitive-cooperative systems.

Conjecture 4.1. *Let Y be a minimal invariant set of system* (2.2)*. Denote N^c the central dimension of the linearized system* (2.1) *over Y. Then the dynamics on Y is topologically conjugate to a subflow in $\mathbb{R}^{N^c} \times H(f)$.*

As we mentioned in the previous sections, this conjecture is true when the central dimension is zero or one, see [B, C].

Let $\Pi : \mathbb{R} \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n \times H(f)$ be the skew-product flow generated by system (2.2). Denote L the limit set of Π, that is the minimum closed invariant set that contains the ω- and α-limit set of any orbit of Π.

Conjecture 4.2. *Consider system* (2.2)*. Then the limit set L can be decomposed into $L = I \cup \widetilde{L} \cup R$ such that*

(1). *I is a disjoint union of finitely many normally hyperbolic arcs or simple closed curves.*

(2). *R is a finite union of subflows which are topologically conjugated to scalar skew-product flows.*

(3). *$\widetilde{L}$ is hyperbolic set.*

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc., 114(1965), 309-319.

[2] S. Ahmad, B. Granadosb and A. Tineo, *On tridiagonal predator-prey systems and a conjecture*, Nonlinear Analysis: Real World Applications 11(2010), 1878-1881.

[3] H. Anzai, *Ergodic skew product transformations on the torus*, Osaka Math. J. 3(1995), 83-99.

[4] M. Atiyah, *K-theory*, 2nd ed., Benjamin, New York, 1967.

[5] F. Blanchard, E. Glasner, S. Kolyada and A. Maass, *On Li-Yorke Pairs*, J. Reine Angew. Math. 547(2002) 51-68.

[6] F. Blanchard, B. Host and S. Ruette, *Asymptotic pairs in positive-entropy systems*, Ergod. Th. and Dynam. Sys. 22(2002) 671-686.

[7] F. Blanchard, W. Huang and L. Snoha, *Topological size of scrambled sets*, Colloq. Math. 110(2008) 293-361.

[8] C. Bonatti, S. Crovisier, *Central manifolds for partially hyperbolic set without strongly unstable connections*. In preparation.

[9] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153(1971) 401-414.

[10] R. Bowen, *Entropy-expansive maps*, Trans. Amer. Math. Soc. 164(1972) 323-331.

[11] R. Bowen, *Topological Entropy for Noncompact Sets*, Trans. Amer. Math. Soc. 184(1973) 125-136.

[12] M. Boyle and T. Downarowicz, *The entropy theory of symbolic extensions*, Invent. Math. 156(2004) 119-161.

[13] I. U. Bronshtein and V. F. Chernii, *Linear extensions satisfying Perron's condition*, I. Differential Equations 14(1978), 1234-1243.

[14] A. M. Bruckner and T. Hu, *On scrambled sets and chaotic functions*, Trans. Amer. Math. Soc. 301(1987) 289-297.

[15] J. Buzzi, *Intrinsic ergodicity of smooth interval maps*, Israel J. Math. 100(1997) 125-161.

[16] S.-N. Chow, K. Lu and J. Mallet-Paret, *Floquet theory for parabolic differential equations*, J. Differential Equations 109(1994), no.1, 147-200.

[17] S.-N. Chow, K. Lu and J. Mallet-Paret, *Floquet bundles for scalar parabolic equations* Arch. Rational Mech. Anal. 129(1995), no.3, 245-304.

[18] S. Crovisier, E. Pujals, *Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms*, arXiv:1011.3836.

[19] T. Downarowicz and G. Zhang, *Modeling potential as fiber entropy and pressure as entropy*, preprint.

[20] G. Fusco and W. M. Oliva, *Jacobi matrices and transversality*, Proc. R. Soc. Edinburgh Sect. A 109(1988), no. 3-4, 231-143.

[21] H. Furstenberg, *The structure of distal flows*, Amer. J. Math. 85(1963), 477-515.

[22] E. Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs 101, American Mathematical Society, Providence, RI, 2003.

[23] M. Gyllenberg, *Dynamics of structured populations*, Julkaisu No. 23, ISBN 951-754-127-9, 1987.

[24] J. K. Hale, *Ordinary differential equations*, Krieger Publishing Company, 1980

[25] J. K. Hale and A. S. Somolinos, *Competition for fluctuating nutrient*, J. Math. Biol. 18(1983), no.3, 255-280.

[26] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, 840, Springer-Verlag, Berlin-New York, 1981.

[27] G. Hetzer and W. Shen, *Convergence in Almost Periodic Competition Diffusion Systems*, J. Math. Anal. Appl. 262(2001), 307-338.

[28] G. Hetzer and W. Shen, *Uniform persistence, coexistence, and extinction in almost periodic/nonautonomous competition diffusion system*, SIAM J. Math. Anal. 34(2002), no.1, 204-227.

[29] M. W. Hirsch, *Differential equations and convergence almost everywhere in strongly monotone flows* Contemp. Math., 17(1983), 267-285.

[30] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant Maniflods*, Lecture Notes in Mathematics 583, 1977.

[31] W. Huang, *Stable sets and ϵ-stable sets in positive entropy systems*, Commun. Math. Phys. 297, 535-557(2008).

[32] W. Huang and X. Ye, *Homeomorphisms with the whole compacta being scrambled sets*, Ergod. Th. and Dynam. Sys., 21(2001), 77-91.

[33] W. Huang and X. Ye, *Devaney's chaos or 2-scattering implies Li-Yorke's chaos*, Topology Appl. 117(2002), no.3, 259-272.

[34] A. Kolmogorov, *Sulla teoria di Volterra della lotta per l'istenzia*, Ist. Ital. Attuari, **7**(1936), 74-80.

[35] A. N. Kolmogorov, *A new metric invariant of transient dynamical systems and automorphisms of Lebesgue spaces*, Dokl. Akad. Sci. SSSR., 119(1958) 861-864 (Russian).

[36] F. Ledrappier, *A variational principle for the topological conditional entropy*, Lecture Notes in Math. 729(1979) 78-88.

[37] E. Lindenstrauss, *Mean dimension, small entropy factors and an embedding theorem*, Publ. Math. I.H.E.S. 89(1999) 227-262.

[38] A. J. Lotka, *Elements of Physical Biology*, Williams & Wilkins, Baltimore, 1925.

[39] F. Ledrappier and P. Walters, *A relativised variational principle for continuous transformations*, J. London Math. Soc. 16(1977) 568-576.

[40] E. Lindenstrauss and B. Weiss, *Mean topological dimension*, Israel J. Math. 115 (2000) 1-24.

[41] T.-Y. Li and J. A. Yorke, *Period three implies chaos*, Amer. Math. Monthly 82(1975) 985-992.

[42] J. Mallet-Paret and G. Sell, *Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions*, J. Differential Equations 125(1996), 385-440.

[43] J. Mallet-Paret and H. L. Smith, *The Poincaré-Bendixson theorem for monotone cyclic feedback systems*, J. Dynam. Differential Equations 2(1990), no.4, 367-421.

[44] H. Matano, *Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math 29(1982), no.2, 401-441.

[45] J. Mierczyński, *The C^1 property of carrying simplices for a class of competitive systems of ODEs*, J. Differential Equations 111(1994), no.2, 385-409.

[46] J. Mierczyński, J and W. Shen, *Exponential separation and principal Lyapunov exponent/spectrum for random/nonautonomous parabolic equations* J. Differential Equations 191(2003), no.1, 175-205.

[47] R. K. Miller, *Almost periodic differential equations as dynamical systems with applications to the existence of A.P. solutions*, J. Differential Equations 1(1965), 337-345.

[48] M. Misiurewicz, *Topological conditional entropy*, Studia Math. 55 (1976)175-200.

[49] M. Misiurewicz, *On Bowen's definition of topological entropy*, Discrete Contin. Dyn. Syst., Ser. A 10(2004) 827-833.

[50] P. de Mottoni and A. Schiaffino, *Competition systems with periodic coefficients: a geometric approach*, J. Math. Biol. 11(1981), no.3, 319-335.

[51] K. Nickel, *Gestaltaussagen über Lösungen parabolischer Differentialgleichungen*, J. Reine Angew. Math. 211(1962), 78-94.

[52] K. J. Palmer, *Exponential separation, exponential dichotomy and spectral theory for linear systems of ordinary differential equations*, J. Differential Equations 46(1982), no.3, 324-345.

[53] W. Parry and P. Walters, *Minimal skew-product homeomorphisms and coalescence*, Compositio Math. 22(1970), 283-288.

[54] Ya. B. Pesin and B. S. Pitskel, *Topological pressure and the variational principle for noncompact sets* Funktsional. Anal. i Prilozhen. 18(1984) 50-63.

[55] P. Poláčik, *Parabolic equations: Asymptotic behavior and dynamics on invariant manifolds* Handbook of Dynamical Systems, Vol.2, pp. 835-883, North-Holland, Amsterdam, 2002.

[56] R. J. Sacker and G. R. Sell, *Skew-product flows, finite extensions of minimal transformation groups and almost periodic differential equations*, Bull. Amer. Math. Soc. 79(1973), 802-805.

[57] R. J. Sacker and G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems I*, J. Differential Equations, 15(1974), 429-458.

[58] R. J. Sacker and G. R. Sell, *Lifting properties in skew-product flows with applications to differential equations*, American mathematical society, Vol. 11, No. 190, 1977.

[59] R. J. Sacker and G. R. Sell, *A spectral theory for linear differential systems*, J. Differential Equations 27(1978), no.3, 320-358.

[60] R. J. Sacker and G. R. Sell, *Dichotomies for linear evolutionary equations in Banach spaces*, J. Differential Equations 113(1994), no.1, 17-57.

[61] J. F. Selgrade, *Isolated invariant sets for flows on vector bundles*, Trans. Amer. Math. Soc. 203(1975), 359-390.

[62] G. R. Sell, *Nonautonomous differential equations and topological dynamics. I. The basic theory*, Amer. Math. Soc. 127(1967), 241-262.

[63] G. R. Sell, *Nonautonomous differential equations and topological dynamics. II. Limiting equations*, Trans. Amer. Math. Soc. 127(1967) 263-283.

[64] G. R. Sell, *Topological dynamics and ordinary differential euqations*, Van Nostrand Reinhold Co., Van Nostrand Reinhold Mathematical Studies, No.33, 1971.

[65] W. Shen and Y. Wang, *Carrying simplices in nonautonomous and random competitive Kolmogorov systems*, J. Differential Equations 245(2008), no.1, 1-29.

[66] W. Shen and Y. Yi, *Asymptotic almost periodicity of scalar parabolic equations with almost periodic time dependence*, J. Differential Equations 122(1995), no.2, 373-397.

[67] W. Shen and Y. Yi, *Dynamics of almost periodic scalar parabolic equations*, J. Differential Equations 122(1995), no.1, 114-136.

[68] W. Shen and Y. Yi, *Almost automorphic and almost periodic dynamics in skew-product semiflows*, Mem. Amer. Math. Soc., 136(1998), no. 647.

[69] J. Smítal, *A chaotic function with some extremal properties*, Proc. Amer. Math. Soc. 87(1983), no. 1, 54-56.

[70] J. Smítal, *A chaotic function with a scrambled set of positive Lebesgue measure*, Proc. Amer. Math. Soc. 92(1984), no. 1, 50-54.

[71] J. Smillie, *Competitive and cooperative tridiagonal systems of differential equations*, SIAM J. Math. Anal., 15(1984), no.3, 530-534.

[72] H. L. Smith, *A discrete Lyapunov function for a class of linear differential equations*, Pacific J. Math. 144(1990), no. 2, 345-360.

[73] H. L. Smith, *Periodic tridiagonal competitive and cooperative systems of differential equations*, SIAM J. Math. Anal. 22(1991), no. 4, 1102-1109.

[74] P. Sun, *Exponential decay of Lebesgue numbers*, Discrete Contin. Dyn. Syst. 32 (2012), no. 10, 3773-3785.

[75] V. Volterra, *Leçons sur la théorie mathématique de la lutte pour la vie*, Gauthier-Villars, Paris, 1931.

[76] P. Walters, *An Introducion to Ergodic Theory*, Graduate Texts in Mathematics 79 (Springer-Verlag, New York-Berlin, 1981).

[77] Y. Wang, *Dynamics of nonautonomous tridiagonal competitive-cooperative system of differential equaitons*, Nonlinearity 20(2007), no.4, 831-843.

[78] J. C. Xiong, F. Tan and J. Lu, *Dependent sets of a family of relations of full measure on probability space*, Science in china Ser. A 50 (2007) 475-484.

[79] X. Ye, W. Huang and S. Shao, *A brief introduction to topological dynamical systems (in Chinese)*, Science press, Beijing, 2008.

[80] Y. Yi, *Stability of integral manifold and orbital attraction of quasi-periodic motions*, J. Differential Equations 103(1993), no.2, 278-322.

[81] L. S. Young, *Dimension, entropy and Lyapunov exponents*, Ergod. Th. and Dynam. Sys. 2 (1982) 109-124.