On a generalisation of trapezoidal words

Amy Glen

School of Chemical & Mathematical Sciences
Murdoch University, Perth, Australia

amy.glen@gmail.com
http://amyglen.wordpress.com

Joint work with Florence Levé (Université de Picardie – Jules Verne).

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Words

By a *word*, I mean a finite or infinite sequence of symbols (*letters*) taken from a non-empty finite set $\mathcal{A}$ (*alphabet*).

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- $(001)^\infty = 001001001001001001001001001\cdots$
- $110011110001101110111100110111001011111101\cdots$
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- $[1; 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots] = \sqrt{3}$

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- The extent to which a word exhibits strong regularity properties is generally inversely proportional to its "complexity".

  Basic measure: number of distinct blocks (factors) of each length occurring in the word.

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Conjecture: $C_{\boldsymbol{x}}(n) = 2^n$ for all n as it is believed $\sqrt{2}$ is *normal* in base 2.

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- Numerous equivalent definitions & characterisations . . .

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An infinite word $\boldsymbol{w}$ is Sturmian if and only if

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What do such words look like? And how can we construct them?

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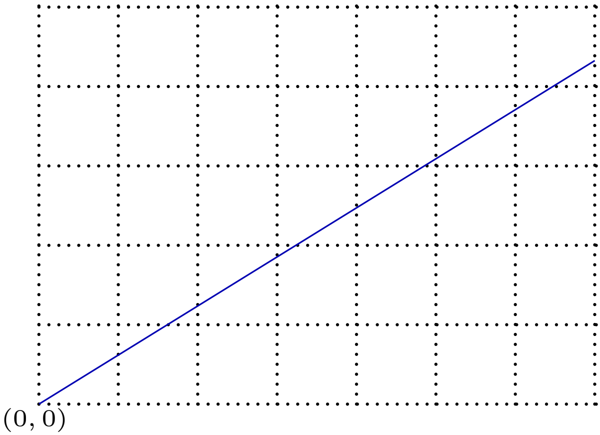

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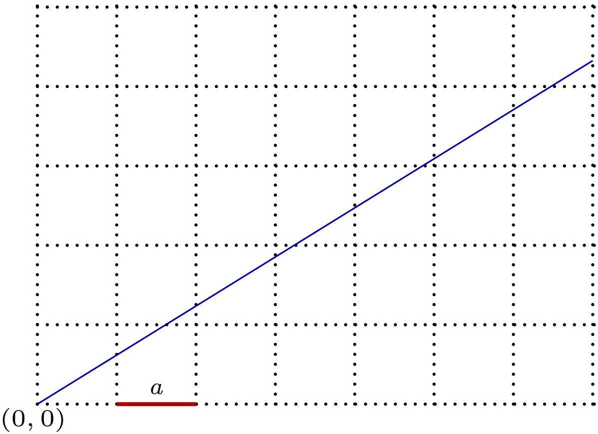

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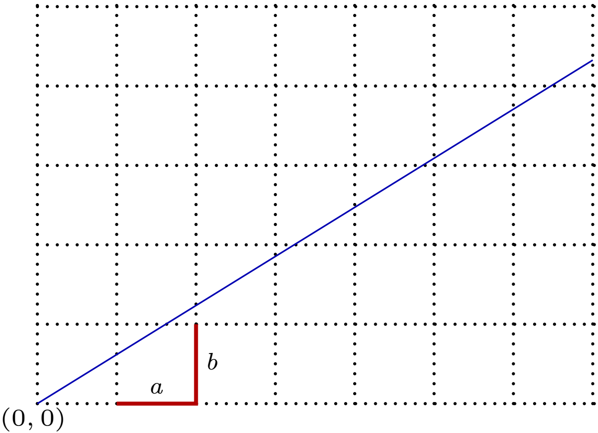

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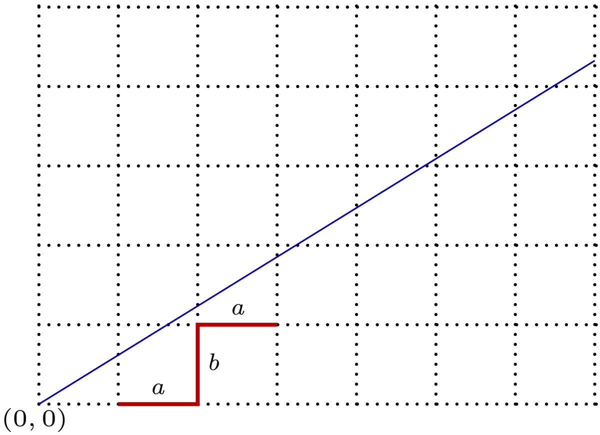

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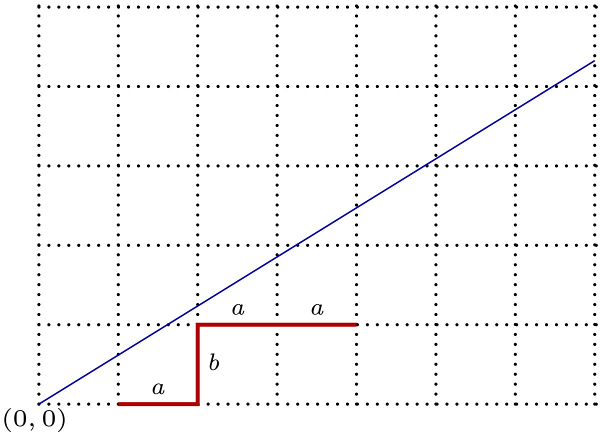

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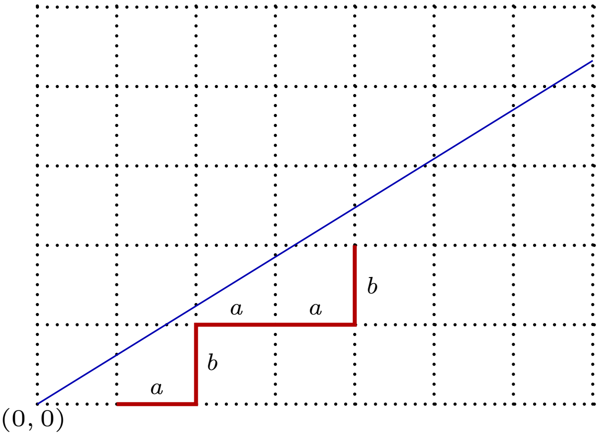

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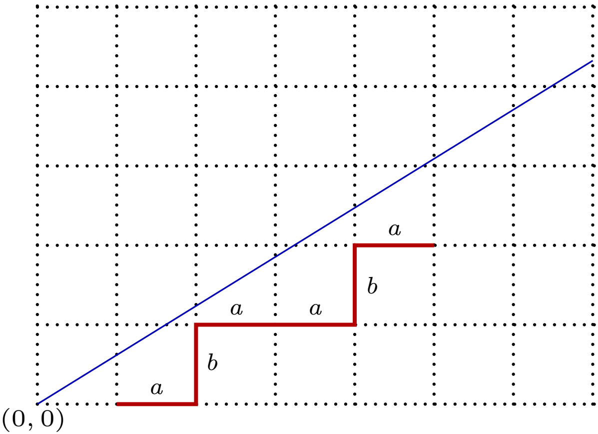

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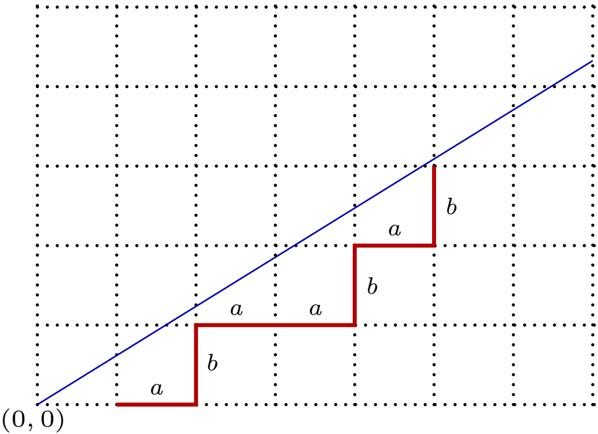

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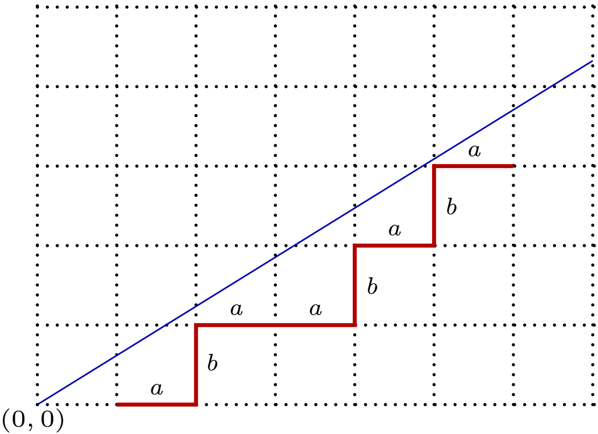

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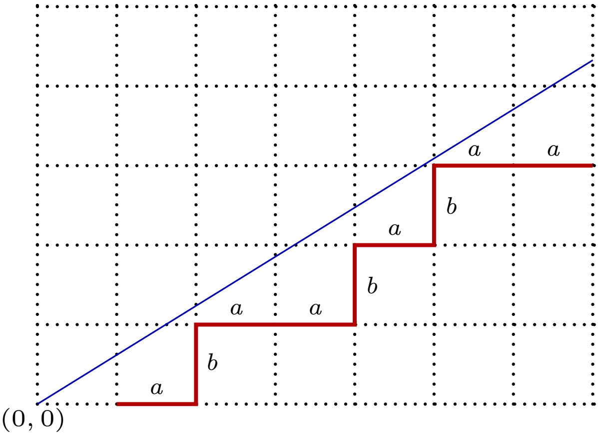

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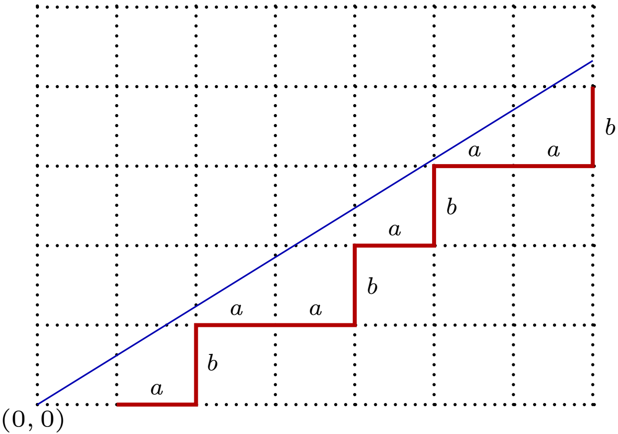

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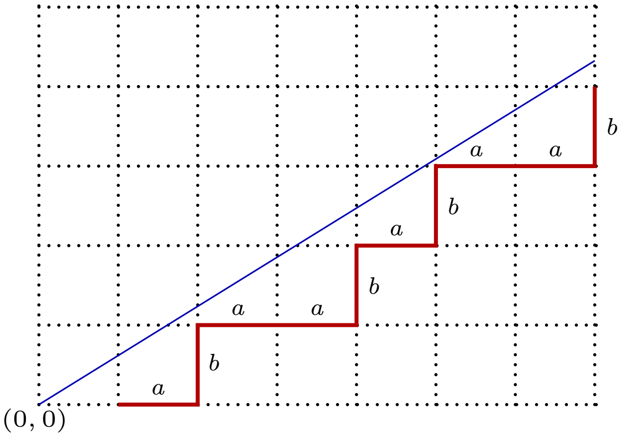

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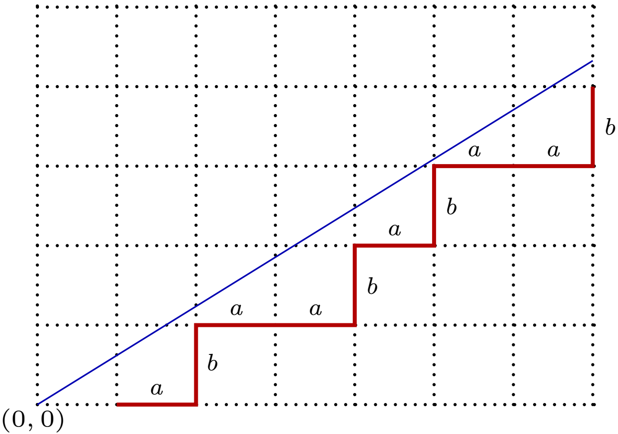

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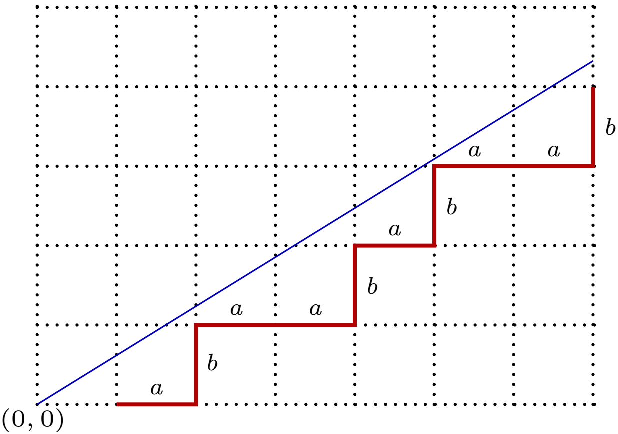

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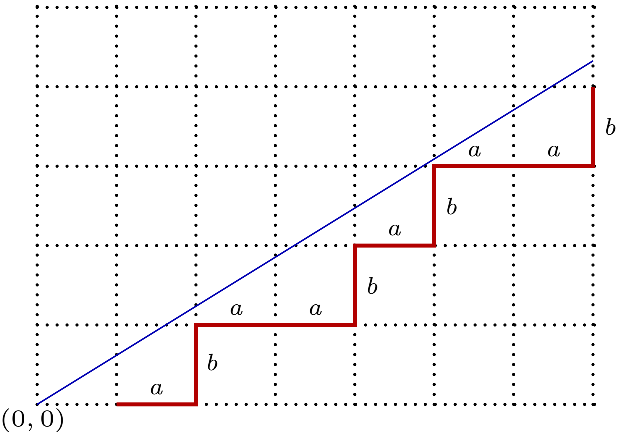

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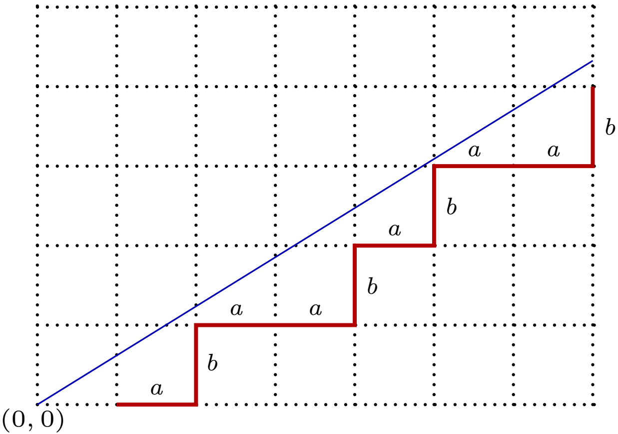

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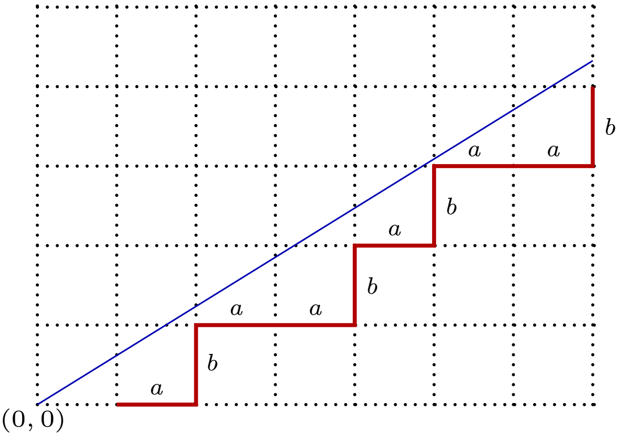

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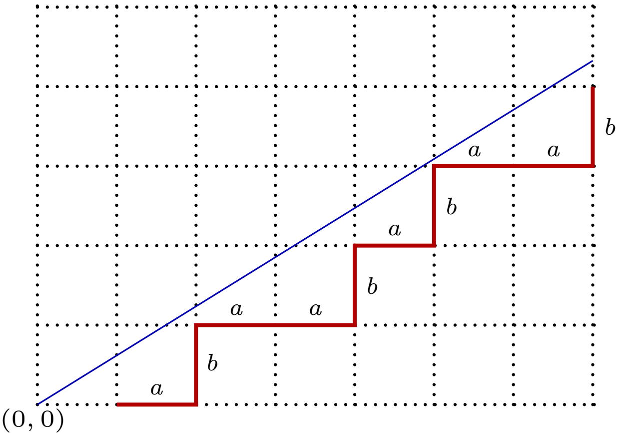

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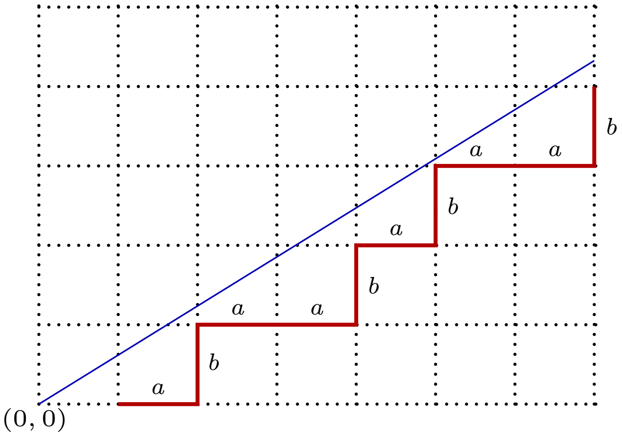

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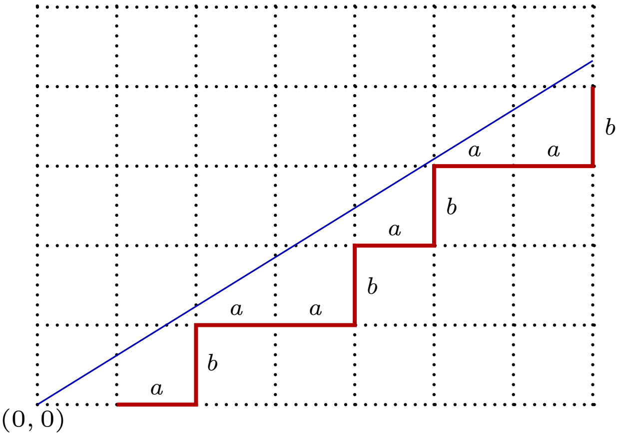

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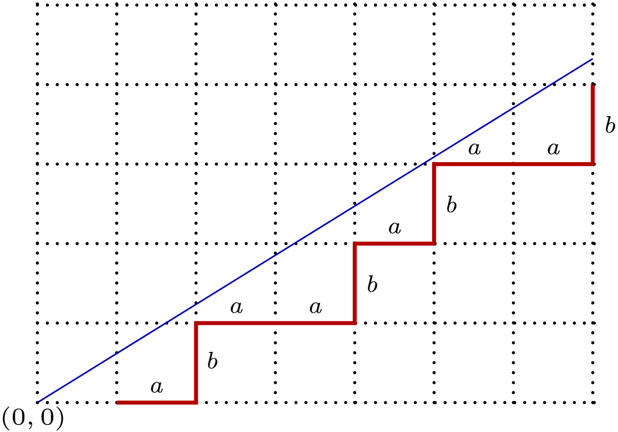

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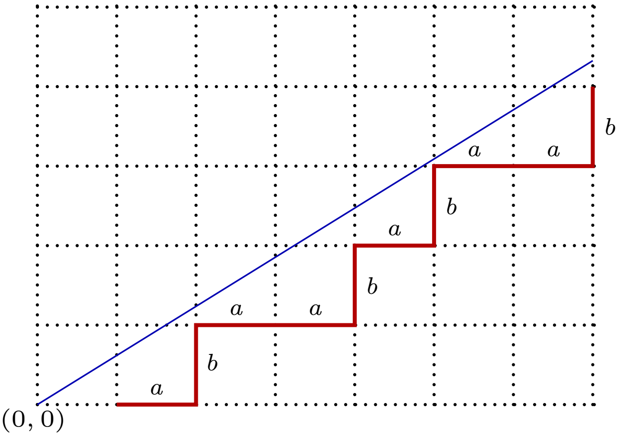

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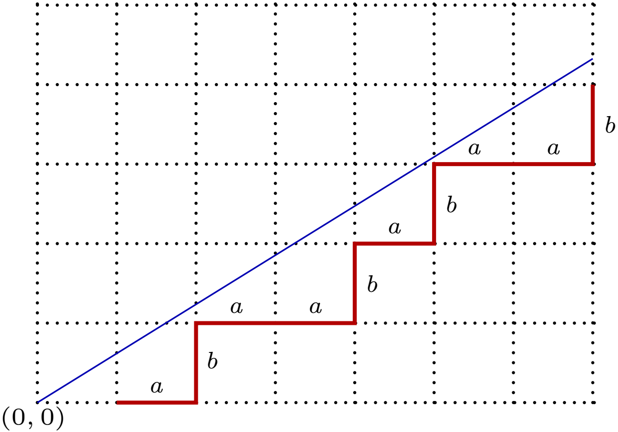

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And it can be shown that the palindromic prefixes of $\boldsymbol{f}$ have lengths

$$\{F_{n+1} - 2\}_{n\geq 1} = 0, 1, 3, 6, 11, 19, \ldots$$

where $\{F_n\}_{n\geq 0}$ is the sequence of Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, defined by: $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

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  In fact, such words have a purely combinatorial construction using the iterated palindromic closure operator . . .

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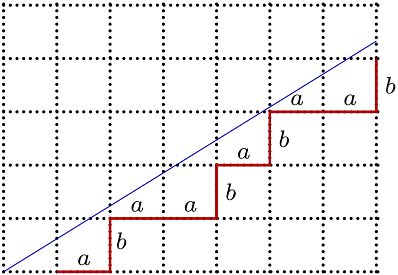

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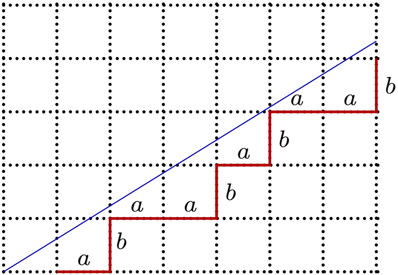

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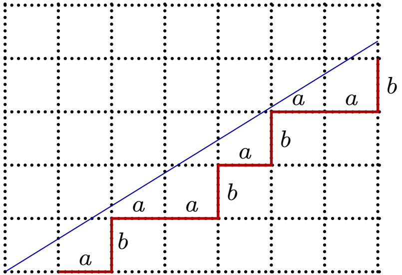

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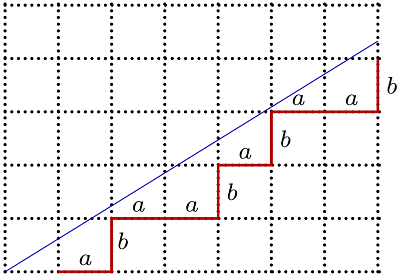

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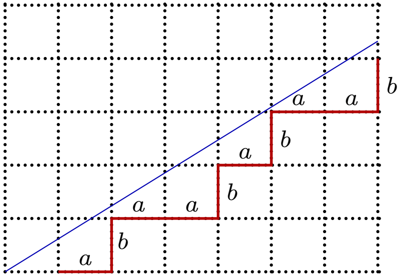

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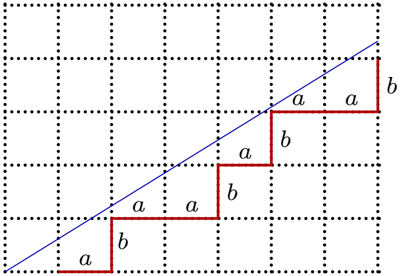

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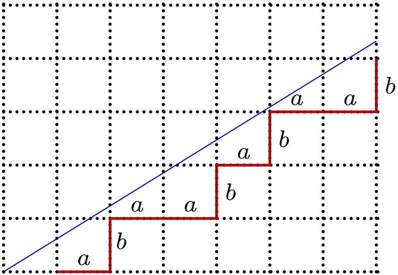

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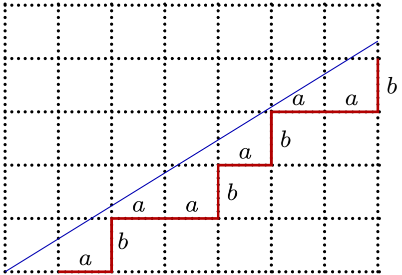

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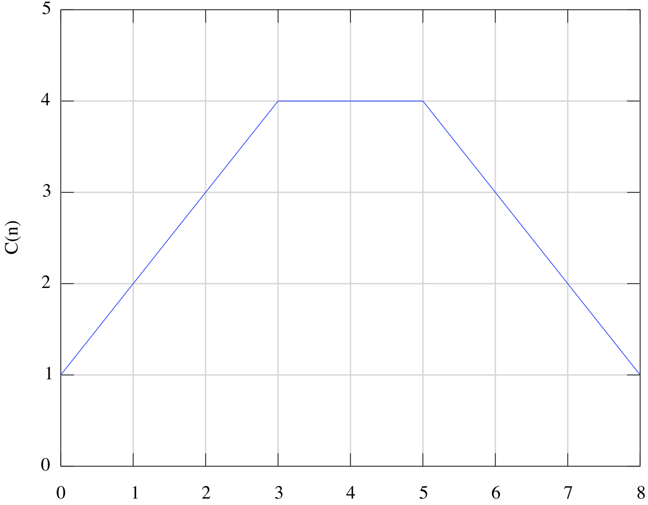
Example

Graph of the factor complexity of the finite Sturmian word $aabaabab$



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- F. D'Alessandro (2002): classified all non-Sturmian trapezoidal words.

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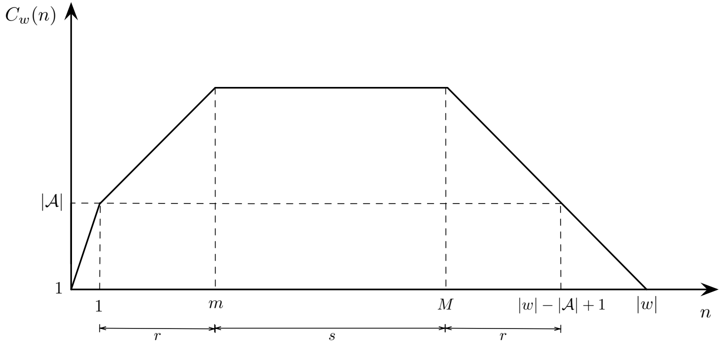

Inspired by a question of Ian Wanless at the 2010 AustMS conference, we recently introduced the following natural generalisation of trapezoidal words.

Generalised Trapezoidal Words (G.-Levé 2011)

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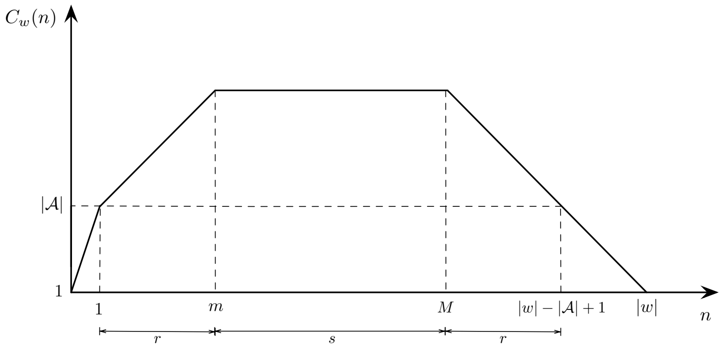

Clearly these words coincide with the (original) trapezoidal words when $|\mathcal{A}| = 2$.

Some Examples

Length 10 over $\mathcal{A} = \{a, b, c\}$

GT-word	$C(n)$ for $n = 0, 1, 2, \ldots, 10$
$aaaaaaaabc$	$1, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, 2, 1$
$abcbcbcbca$	$1, \mathbf{3}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{3}, 2, 1$
$abcbcbcbab$	$1, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{5}, \mathbf{5}, \mathbf{5}, \mathbf{4}, \mathbf{3}, 2, 1$
$abcbcabcab$	$1, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{6}, \mathbf{5}, \mathbf{4}, \mathbf{3}, 2, 1$

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Length 8 over $\mathcal{A} = \{a, b, c, d\}$

GT-word	$C(n)$ for $n = 0, 1, 2, \ldots, 8$
$aaaaabcd$	$1, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, 3, 2, 1$
$aaaabacd$	$1, \mathbf{4}, \mathbf{5}, \mathbf{5}, \mathbf{5}, \mathbf{4}, 3, 2, 1$
$aaabcdab$	$1, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{5}, \mathbf{4}, 3, 2, 1$

Some Basic Properties

The language of all GT-words is closed ...

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Moreover, the language of all GT-words is closed under reversal.

Theorem (G.-Levé 2011)

A finite word w is a GT-word if and only if its reversal is a GT-word.

Binary Case

In the case when $|\mathcal{A}| = 2$, we have proved the following.

Theorem (de Luca-G.-Zamboni 2008)

Let w be a binary palindrome. Then w is trapezoidal if and only if w is Sturmian.

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Theorem (Droubay-Justin-Pirillo 2001)

A finite word w contains at most $|w| + 1$ distinct palindromes (including ε).

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Definition (G.-Justin 2007)

A finite word w is *rich* iff w contains exactly $|w| + 1$ distinct palindromes.

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However, all palindromic GT-words are rich by the following more general result.

Theorem

Suppose w is a GT-word and let v denote the unique factor of w such that $w = bve$ where b is the longest (possibly empty) prefix of w such that $|w|_x = 1$ for each $x \in \text{Alph}(b)$ and e is the longest (possibly empty) suffix of w such that $|w|_x = 1$ for each $x \in \text{Alph}(e)$.

If v is a palindrome, then w is rich.

Examples

- The GT-word $w = abacabade$ has $v = abacaba$ (a palindrome) and w is indeed rich.

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- The converse of the theorem does not hold. For example, the GT-word $ababadac$ is rich, but the corresponding v is $ababada$ (non-palindromic).

Thank You!

Dammit, I'm mad!

U R 2 R U?

* Both phrases are (rich) palindromes! *