Jean Baptiste Joseph Fourier meets Stephen Hawking

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My Research Focus

- Filter banks and wavelets (signal processing branch)
  - Perfect reconstruction of nonuniform filter banks.
  - Filter banks with block sampling structures.
  - Compressions with applications to scalable JPEG image coding schemes.
  - Denoising with applications to image and biomedical signal processing such as ECG signal and ultra sound image denoising.
  - Edge detection and edge linking with applications to image processing such as cancer cell image diagnosis and bad potato diagnosis.
  - Signal separations such as audio signal separations for digital audio hearing aids applications and ECG/EMG signal separations, pattern recognitions such as gait recognitions for military applications and fault analysis such as machine fault detections.

My Research Focus

- Optimization (signal processing)
  - Semi-infinite programming and functional inequality constrained optimizations with applications to filter, filter bank, wavelet kernel, sigma delta modulator and transport system designs.
  - Nonsmooth optimizations with applications to motion estimations as well as filter, filter bank, wavelet kernel and transport system designs.
  - Nonconvex optimizations with applications to spectral allocations for wireless communication networks as well as filter, filter bank, wavelet kernel and transport system designs.
  - Real-time optimizations with applications to filter, filter bank and wavelet kernel designs.

My Research Focus

- Symbolic dynamics, fractal and chaos (control branch)
  - Digital filters with two's complement arithmetic and saturation nonlinearity with applications to computer cryptography.
  - Sigma delta modulators with applications to analog-to-digital conversions.
  - Perceptron training algorithms with applications to pattern recognitions.
  - Random early detection mechanisms with applications to internet traffic control.
  - DC/DC converters with applications to industrial and consumer electronic products.
  - Road traffic light signaling with applications to road traffic system control.
  - Nano-particle quantum effect analysis with applications to nano-device fabrications.

My Research Focus

- Control theories (control branch)
  - Fuzzy control with applications to time delay feedback systems, sample data control systems and chaos synchronization systems.
  - Optimal switching control with applications to DC/DC converters and transport systems.
  - Impulsive control with applications to sigma delta modulators.
  - Chaos control with applications to TCPIP networks, HIV model systems and avian influenza model systems.

Introduction

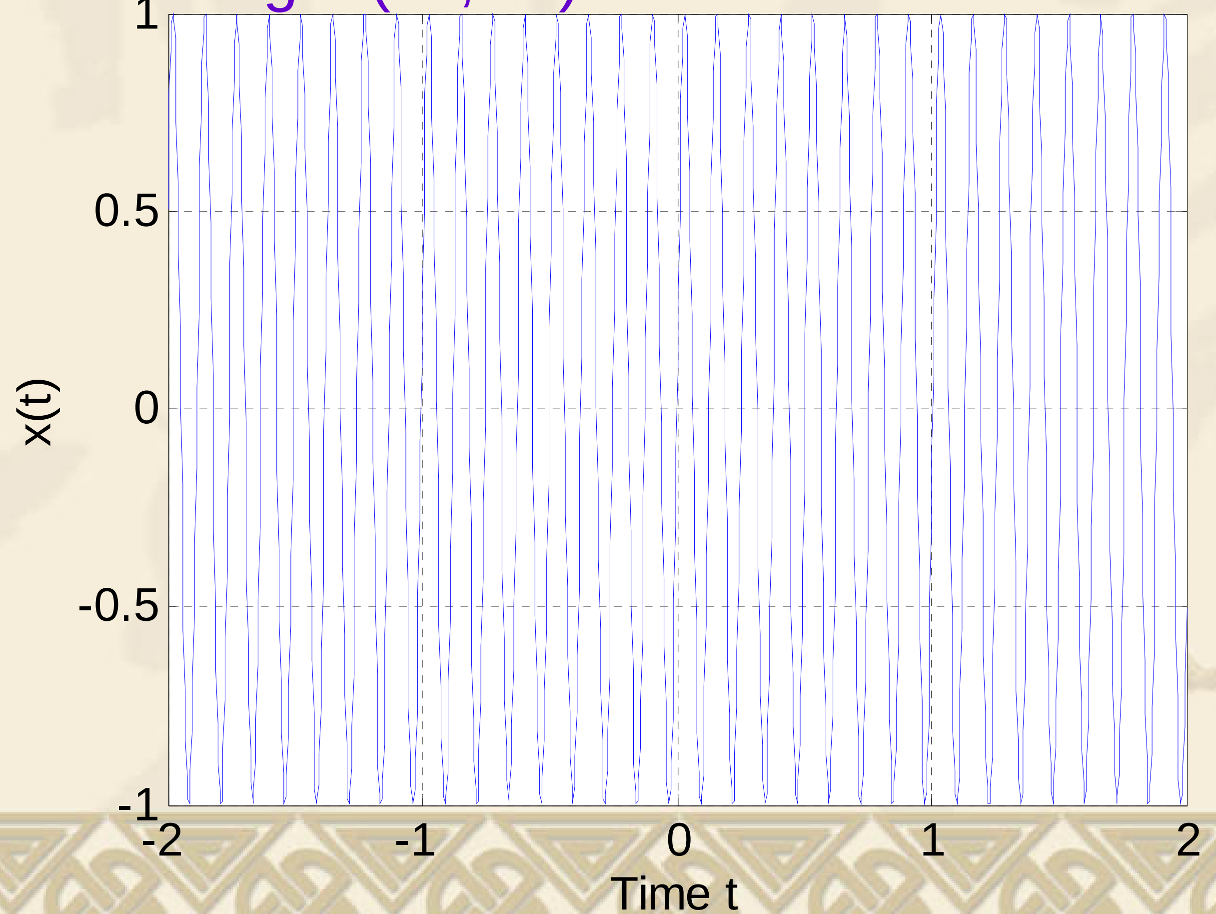
- Jean Baptiste Joseph Fourier (21 March 1768-16 May 1830)
  - Invention of the Fourier transform
  - Representation of the frequency
- Stephen Hawking (8 January 1942-now)
  - Invention of a new concept of time
  - Representation of the time
- What happens if time meets frequency?

Outline

- Frequency analysis
- Time frequency analysis
- Conclusions
- Q&A Session

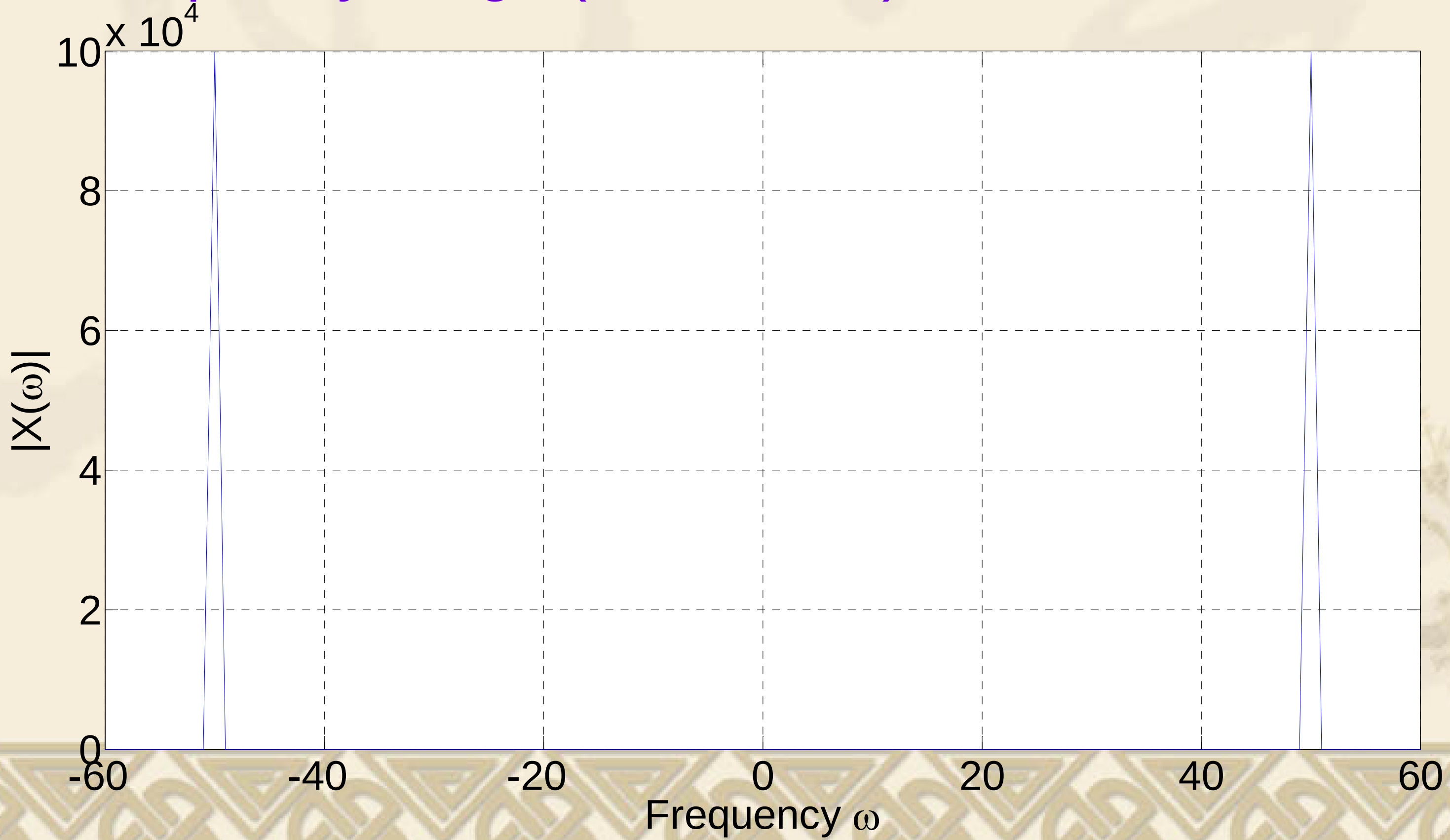
Frequency Analysis

- Understanding of time and frequency
  - Example 1: $x(t) = \sin(50t)$
    - Time range: (-∞,+∞)



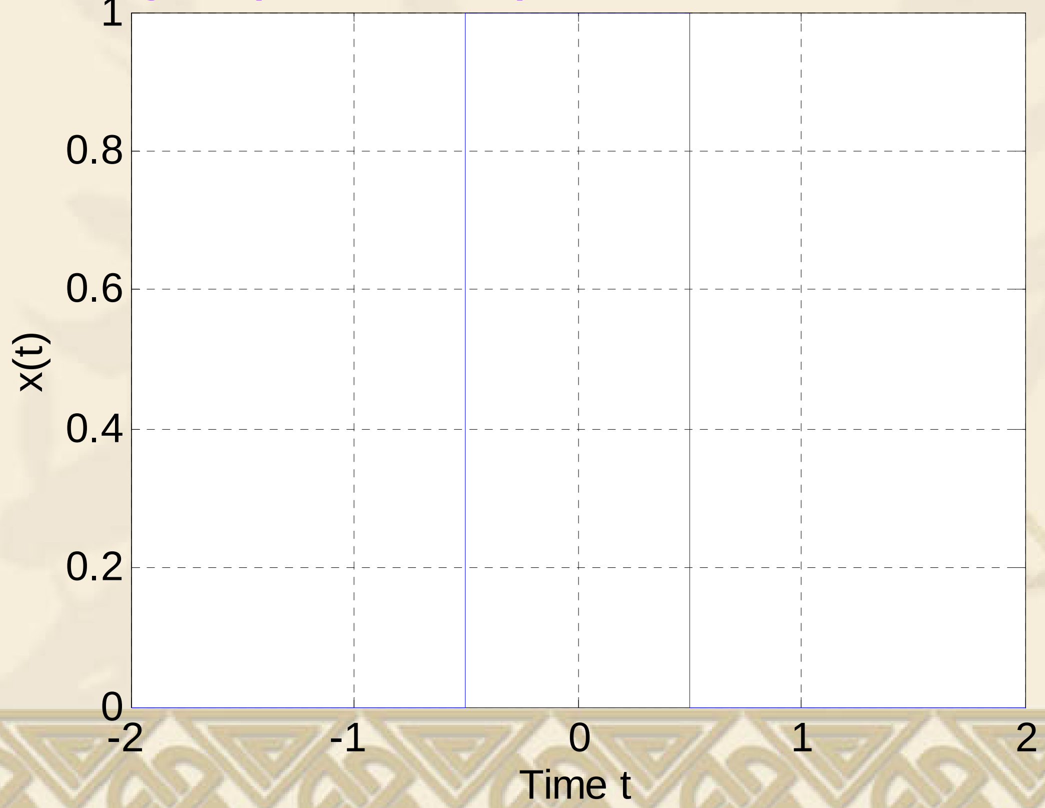
Frequency Analysis

- Understanding of time and frequency
  - Example 1: $X(\omega) = \frac{1}{2j}\left(\delta(\omega - 50) - \delta(\omega + 50)\right)$
    - Angular frequency = 50 radians per second
    - Frequency range (Bandwidth): 0



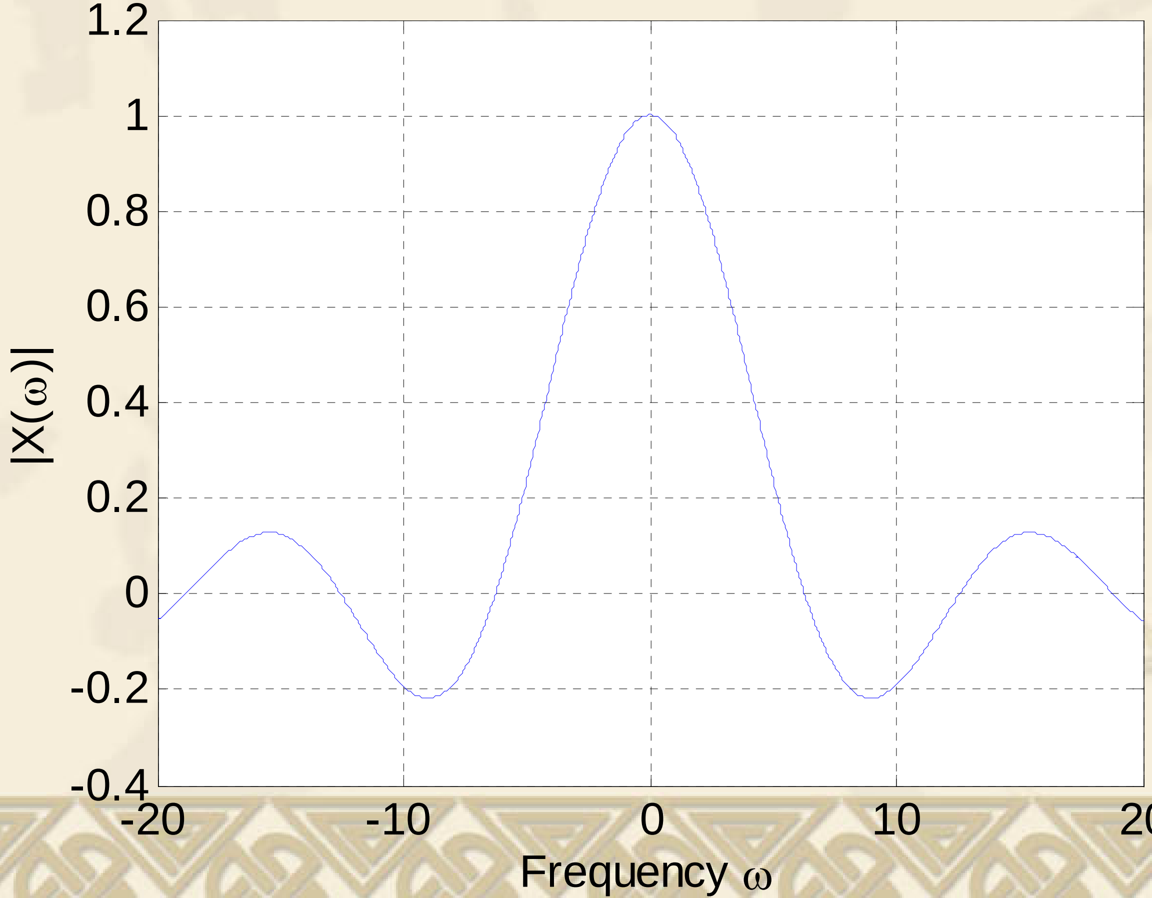
Frequency Analysis

- Understanding of time and frequency
  - Example 2: $x(t)=\begin{cases}1 & |t|\le\frac{1}{2}\\ 0 & otherwise\end{cases}$
    - Time range: (-0.5,+0.5)



Frequency Analysis

- Understanding of time and frequency
  - Example 2: $X(\omega)=\frac{2\sin\frac{\omega}{2}}{\omega}$ because $X(\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$
    - Frequency range: $(-\infty,+\infty)$



Frequency Analysis

- Understanding of time and frequency
  - Representation with a finite duration in time will result to a representation with an infinite bandwidth in frequency. That means good localization in time will result to a bad localization in frequency.
  - Representation with a finite bandwidth in frequency will result to a representation with an infinite duration in time. That means good localization in frequency will result to a bad localization in time.
  - Are there any representations that are good localizations in both time and frequency?

Frequency Analysis

- Backgrounds on linear time invariant filters
  - Definition
    - A system which is linear.

$$T\left(\sum_{i=0}^{N-1}\alpha_i x_i(t)\right)=\sum_{i=0}^{N-1}\alpha_i y_i(t) \qquad T\left(\sum_{i=0}^{N-1}\alpha_i x_i(n)\right)=\sum_{i=0}^{N-1}\alpha_i y_i(n)$$

    - A system which is time invariant.

$$T(x(t-t_0))=y(t-t_0)\ \ \forall t_0\in\Re \qquad T(x(n-n_0))=y(n-n_0)\ \forall n_0\in\mathcal{Z}$$

Frequency Analysis

- Backgrounds on linear time invariant filters
  - Systems represented by linear time invariant filters

$$\sum_{n=0}^{N-1} \frac{a_n d^n y(t)}{dt^n} = \sum_{m=0}^{M-1} \frac{b_m d^m x(t)}{dt^m}$$

$$\sum_{p=0}^{N-1} a_p y(n-p) = \sum_{q=0}^{M-1} b_q x(n-q)$$

- Many practical systems can be represented by linear time invariant filters.

Frequency Analysis

- Backgrounds on linear time invariant filters
  - Impulse response

$$h(t) \equiv T(\delta(t)) \qquad h(n) \equiv T(\delta(n))$$

- A system is linear and time invariant if and only if the input output relationship in the time domain is governed by the convolution and that in the frequency domain is governed by the multiplication. That is

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau \qquad y(n) = \sum_{n \to -\infty}^{+\infty} x(m) h(n-m)$$

$$Y(\omega) = X(\omega) H(\omega)$$

Frequency Analysis

- Backgrounds on linear time invariant filters
  - Frequency response

$$H(\omega) \equiv \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt \qquad H(\omega) \equiv \sum_{n \to -\infty}^{+\infty} h(n) e^{-j\omega n}$$

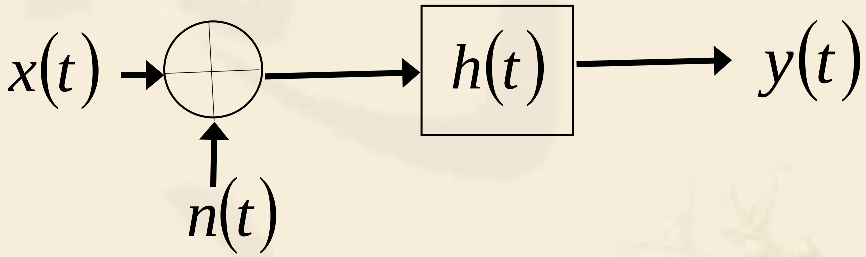
- Frequency response is the eigen function of linear time invariant filters. Hence, if the inputs of linear time invariant filters consist of monotonic frequency only, then the outputs of linear time invariant filters also consist of the same monotonic frequency only with the gains equal to $|H(\omega_0)|$ and the phase shifts equal to $\angle H(\omega_0)$.

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) e^{j\omega_0 (t-\tau)} d\tau = e^{j\omega_0 t} H(\omega_0) \qquad y(n) = \sum_{n \to -\infty}^{+\infty} h(m) e^{j\omega_0 (n-m)} = e^{j\omega_0 n} H(\omega_0)$$

$$Y(\omega) = H(\omega)\delta(\omega - \omega_0) = H(\omega_0)\delta(\omega - \omega_0)$$

Frequency Analysis

- Applications
  - Denoising
    - Apply a linear time invariant filter to a noisy signal with the bandwidth of the filter equal to the bandwidth of the uncorrupted signal.



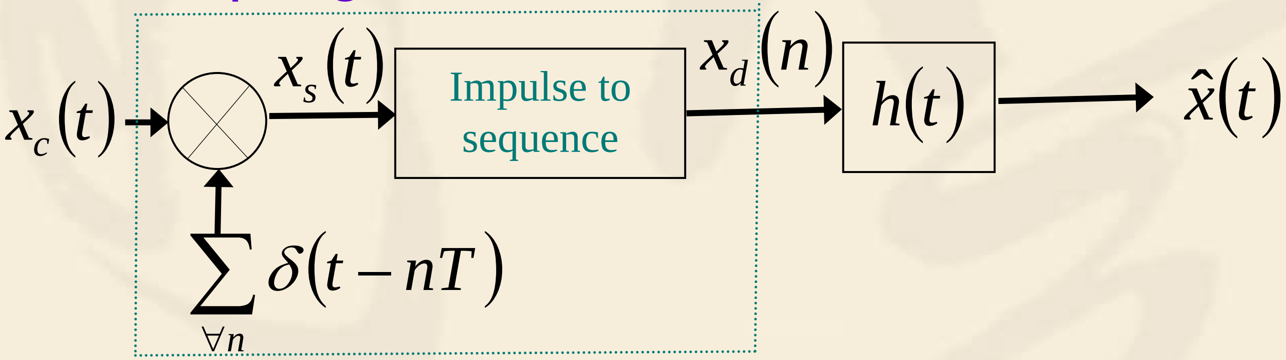
$$y(t) = \int_{-\infty}^{+\infty} \left(x(\tau) + n(\tau)\right) h(t-\tau)\, d\tau$$

$$y(p) = \sum_{m \to -\infty}^{+\infty} \left(x(m) + n(m)\right) h(p-m)$$

$$Y(\omega) = H(\omega)\left(X(\omega) + N(\omega)\right)$$

Frequency Analysis

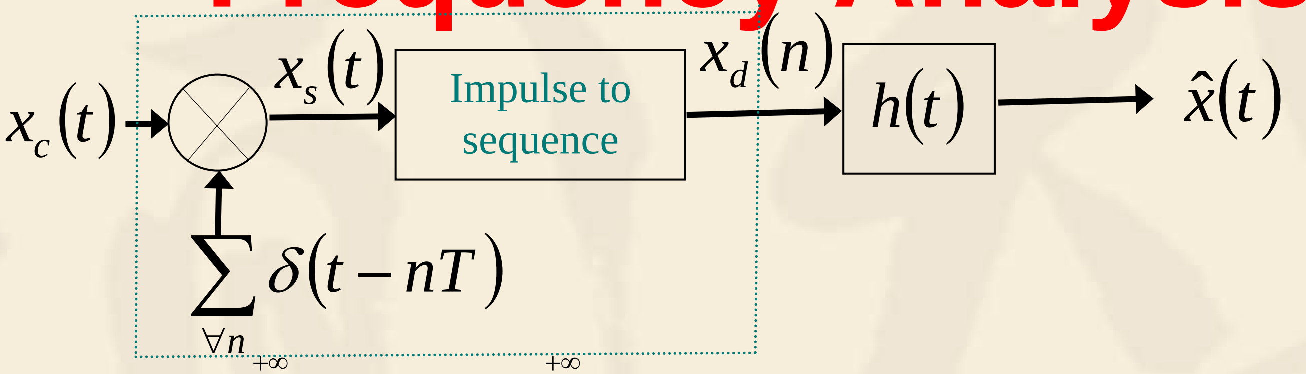
- Applications
  - Sampling
    - Shannon sampling theorem



- Suppose that $X_c(\omega)$ is bandlimited by B, that is $X_c(\omega)=0$ for $|\omega| \geq B$. The signal can be reconstructed from the sampled sequence $x_d(n)$ via an ideal lowpass filter with the impulse response $h(t)=\frac{\sin\left(\frac{\pi(t)}{T}\right)}{\frac{\pi(t)}{T}}$ if the sampling frequency is higher than or equal to twice of B. That is, if $\frac{\pi}{T} \geq B$, then

$$x_c(t)=\sum_{n\to-\infty}^{+\infty} x_d(n)\frac{\sin\left(\frac{\pi(t-nT)}{T}\right)}{\frac{\pi(t-nT)}{T}}.$$

Frequency Analysis



$$x_s(t) = x_c(t) \sum_{n \to -\infty}^{+\infty} \delta(t-nT) = \sum_{n \to -\infty}^{+\infty} x_c(nT)\delta(t-nT)$$

$$X_s(\Omega) = \int_{-\infty}^{+\infty} x_s(t) e^{-j\Omega t} dt = \int_{-\infty}^{+\infty} \left(\sum_{n \to -\infty}^{+\infty} x_c(nT)\delta(t-nT) \right) e^{-j\Omega t} dt = \sum_{n \to -\infty}^{+\infty} x_c(nT) e^{-j\Omega nT}$$

$$= \frac{1}{2\pi} X_c(\Omega) * \frac{2\pi}{T} \sum_{k \to -\infty}^{+\infty} \delta\left(\Omega - \frac{2\pi k}{T}\right) = \frac{1}{T} \sum_{k \to -\infty}^{+\infty} X_c\left(\Omega - \frac{2\pi k}{T}\right)$$

$$x_d(n) = x_c(nT)$$

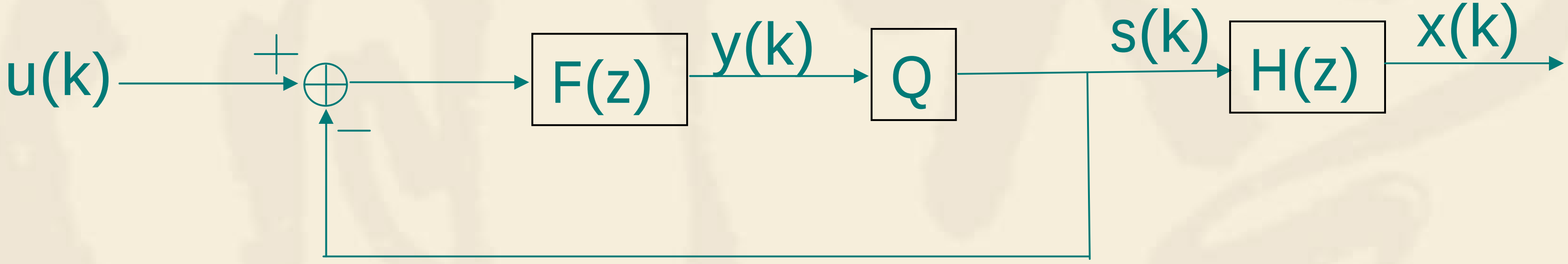
$$X_d(\omega)\Big|_{\omega=\Omega T} = \sum_{n \to -\infty}^{+\infty} x_c(nT) e^{-j\omega n} \Bigg|_{\omega=\Omega T} = X_s(\Omega) = \frac{1}{T} \sum_{k \to -\infty}^{+\infty} X_c\left(\Omega - \frac{2\pi k}{T}\right)$$

$$X_d(\omega) = \frac{1}{T} \sum_{k \to -\infty}^{+\infty} X_c\left(\frac{\omega - 2\pi k}{T}\right)$$

$$\hat{x}(t) = \sum_{n \to -\infty}^{+\infty} x_d(n) h(t-nT) = \sum_{n \to -\infty}^{+\infty} x_c(nT) \frac{\sin\left(\frac{\pi(t-nT)}{T}\right)}{\frac{\pi(t-nT)}{T}}$$

Frequency Analysis

- Applications
  - Oversampled analog-to-digital conversion



- By modeling the quantizer as an additive noise source n(k), we have

$$\frac{S(z)}{U(z)} = \frac{F(z)}{1+F(z)} \qquad \lim_{F(z)\to\pm\infty} \frac{S(z)}{U(z)} = 1 \qquad \lim_{F(z)\to 0} \frac{S(z)}{U(z)} = 0$$

$$\frac{S(z)}{N(z)} = \frac{1}{1+F(z)} \qquad \lim_{F(z)\to\pm\infty} \frac{S(z)}{N(z)} = 0 \qquad \lim_{F(z)\to 0} \frac{S(z)}{N(z)} = 1$$

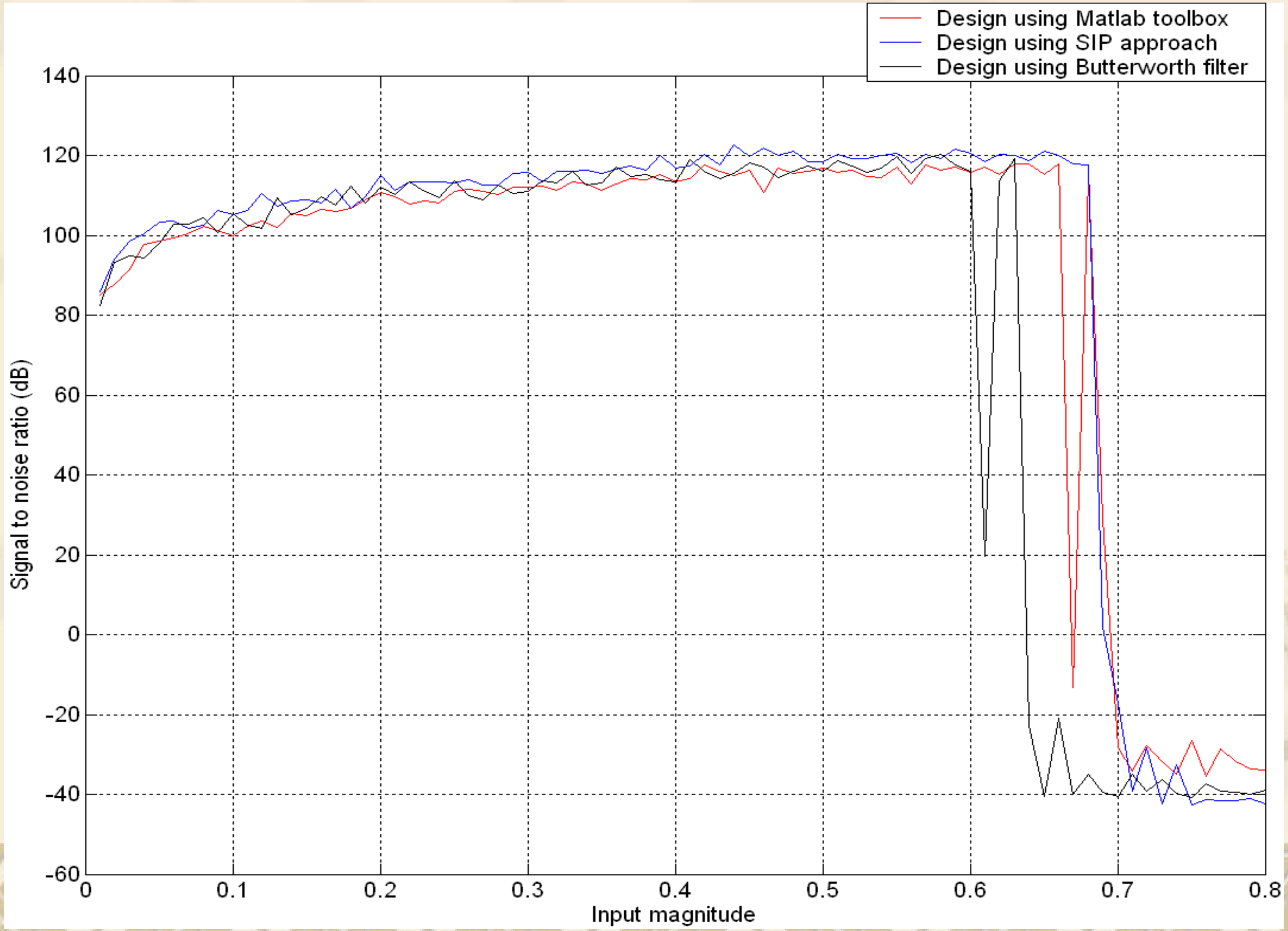
- Signal and noise can be separated.

Frequency Analysis

- Applications
  - Oversampled analog-to-digital conversion
    - Since F(z) has to be unstable, stability is an issue.
    - In order to design oversampled analog-to-digital converter, F(z) has to force the state variables to go to the infinity, but the quantizer has to force the state variables to go back to the origin.
    - Chaotic behaviours occur.
    - Linear system theory is not applied. Set theory is employed for the analysis.

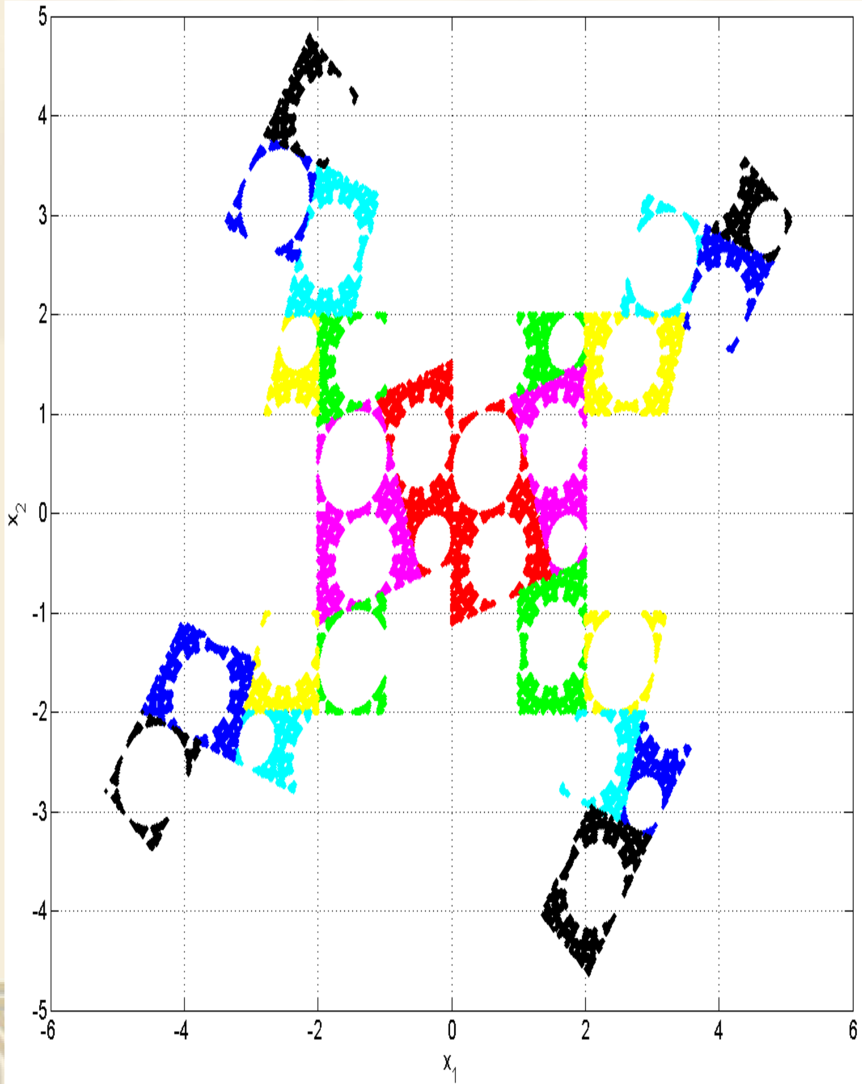
Frequency Analysis

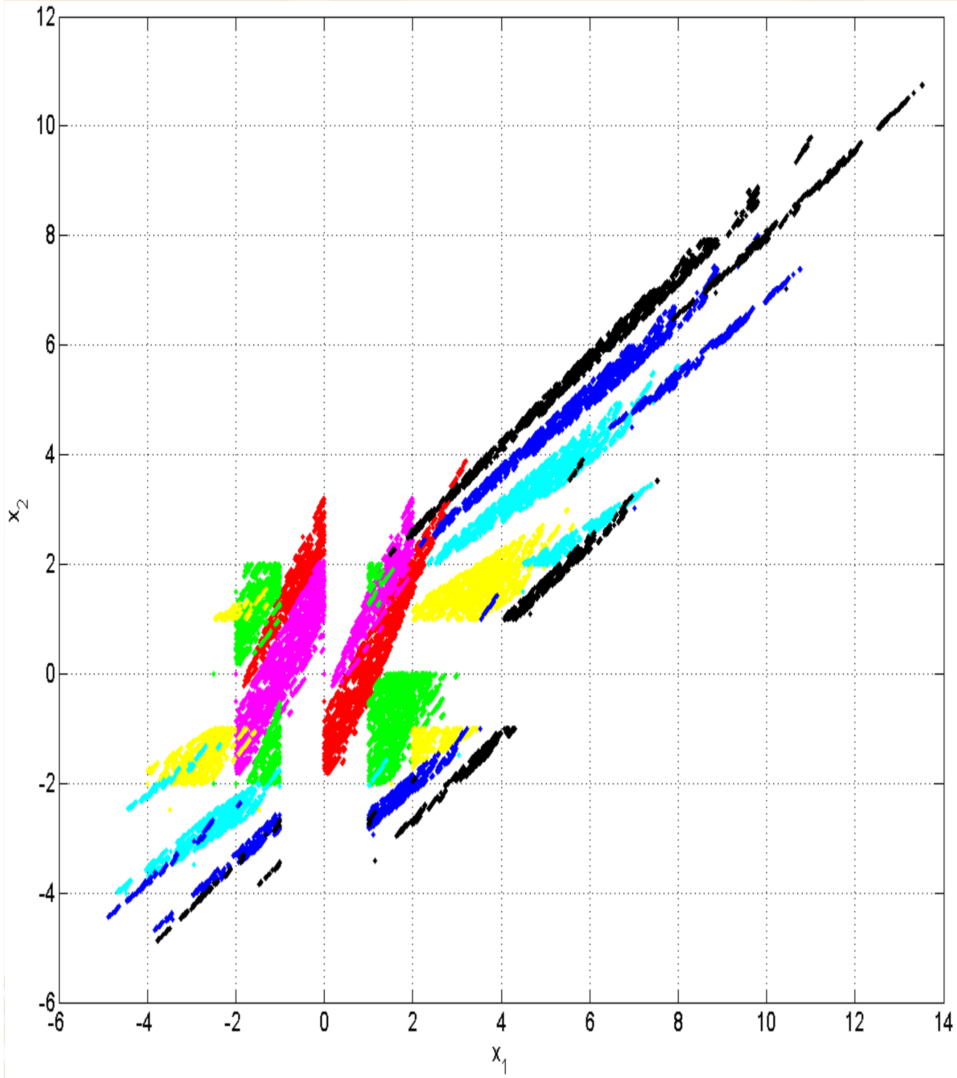
- Applications
  - Oversampled analog-to-digital conversion



Frequency Analysis

- Applications
  - Oversampled analog-to-digital conversion





Frequency Analysis

- Applications
  - Oversampled analog-to-digital conversion

$$T_0(R_0) = R_0$$

$$T_{k-1}(R_{k-1}) = R_k \text{ for } k \le 0$$

$$\bigcup_{\forall k} R_k = \Re^N$$

$$R_i \cap R_j = \varnothing \text{ for } i \ne j$$

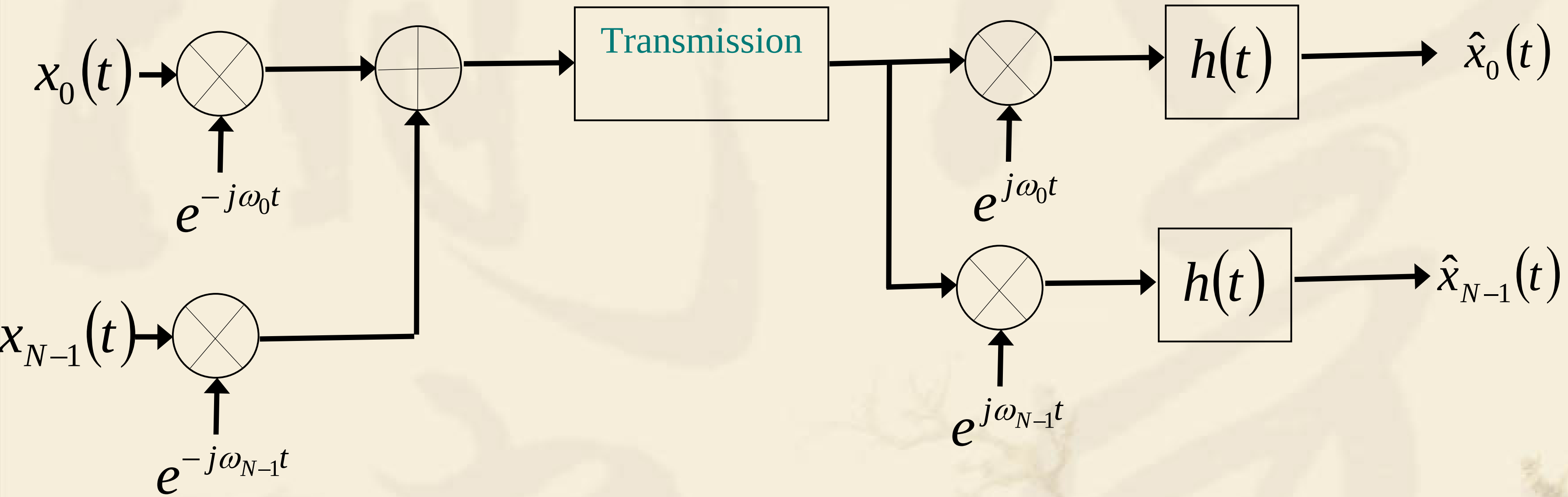
- T_0 is an invariant map and R_0 is an invariant set.
- $T:\Re^N \to \Re^N$ is surjective, but not injective. As there are two quantizer levels, we have $T_{-1}(R_{-1})=T(R_0)=R_0$. This implies that R_0 is attractive. In other words, the system is locally stable.
- To understand why the system is globally stable, as T is surjective, the set $\Re^N \cup R_k$ is also an invariant set.
- However, the invariant set has to be near the origin. Hence, $\Re^N \cup R_k$ is an empty set. In other words, R_k forms the partition of $\Re^N$ and the system is globally stable.

Frequency Analysis

- Applications
  - Oversampled analog-to-digital conversion
    - Open problems
      - Global stability of multibit high order oversampled analog-to-digital converters is general unknown.
      - Design of oversampled analog-to-digital converters subject to the global stability condition is not available yet.
      - Could the results be applied to similar symbolic dynamical systems?

Frequency Analysis

- Applications
  - Amplitude modulated radio systems



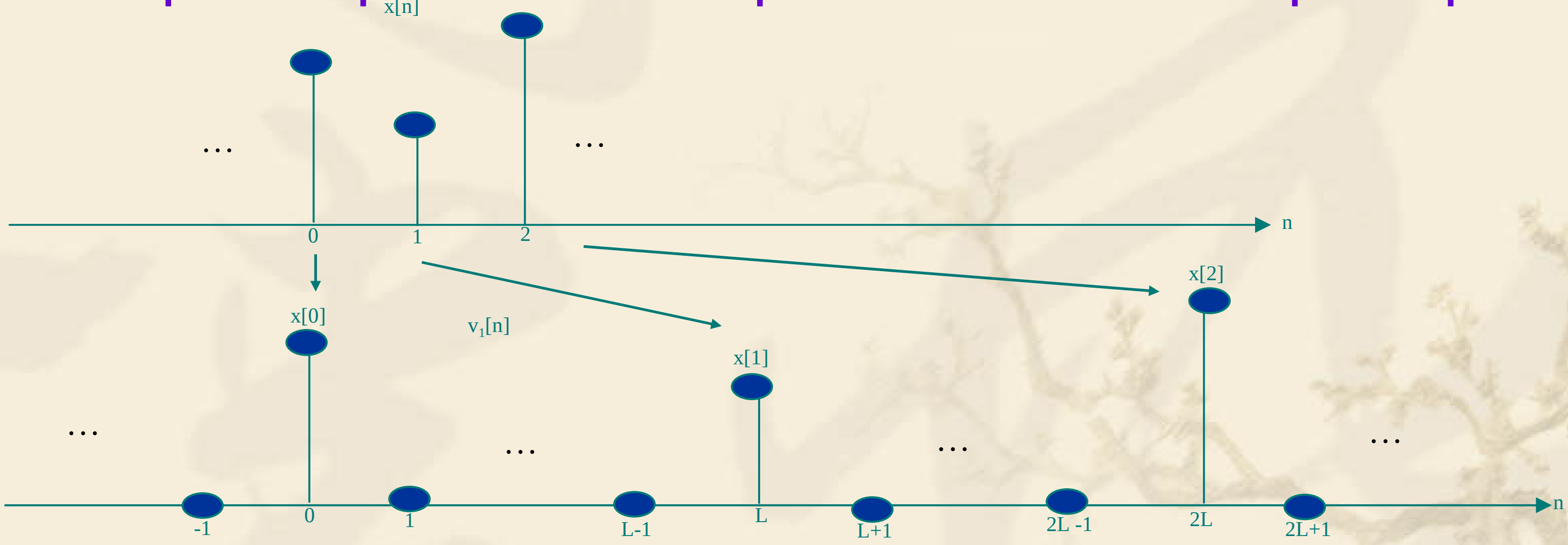
$$S(t) = \sum_{n=0}^{N-1} x_n(t) e^{-j\omega_n t}$$

$$\hat{x}_n(t) = \int_{-\infty}^{+\infty} S(\tau) e^{j\omega_n \tau} h(t-\tau) d\tau$$

Time Frequency Analysis

- Fundamental building blocks in signal processing
  - Conventional upsampler

    - Input output relationship of conventional upsampler



Time Frequency Analysis

- Fundamental building blocks in signal processing
  - Conventional upsampler
    - Input output relationship of conventional upsampler

$$y[n] = \begin{cases} x\left[\frac{n}{L}\right] & \text{n is integer multiple of L} \\ 0 & otherwise \end{cases}$$

$$Y(\omega) = \sum_{n \to -\infty}^{+\infty} y(n) e^{-j\omega n} = \sum_{n \to -\infty}^{+\infty} x(n) e^{-j\omega nL} = X(L\omega)$$

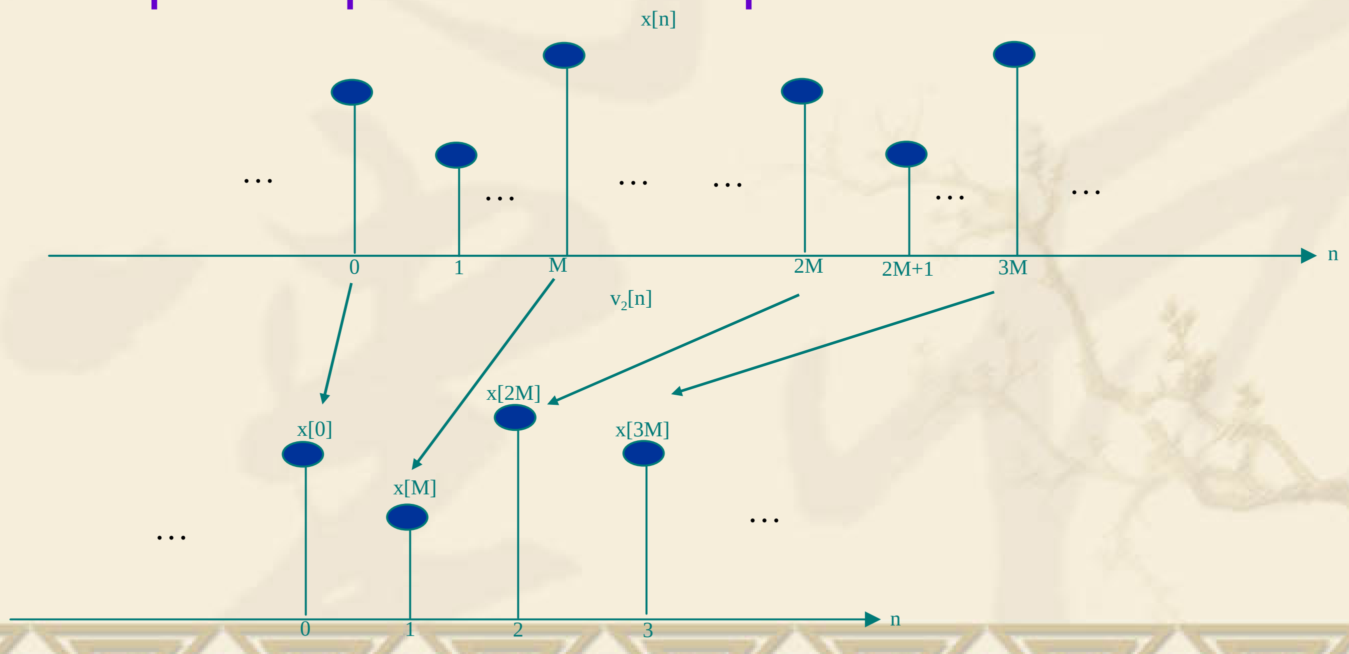
$$Y(z) = \sum_{n \to -\infty}^{+\infty} y(n) z^{-n} = \sum_{n \to -\infty}^{+\infty} x(n) z^{-nL} = X(z^{L})$$

    - No loss of information.

Time Frequency Analysis

- Fundamental building blocks in signal processing
  - Conventional downsampler

    - Input output relationship of conventional downsampler



Time Frequency Analysis

- Fundamental building blocks in signal processing
  - Conventional downsampler
    - Input output relationship of conventional downsampler

$$y[n] = x[Mn] \qquad y[n]\big|_{\uparrow M} = x[n]\sum_{k\to-\infty}^{+\infty}\delta[n-Mk]$$

$$Y(\omega)\big|_{\uparrow M} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)\frac{2\pi}{M}\sum_{k\to-\infty}^{+\infty}\delta\left(\theta-\omega-\frac{2\pi k}{M}\right)d\theta = \frac{1}{M}\sum_{k\to-\infty}^{+\infty}X\left(\omega-\frac{2\pi k}{M}\right)$$

$$Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1}X\left(\frac{\omega-2k\pi}{M}\right)$$

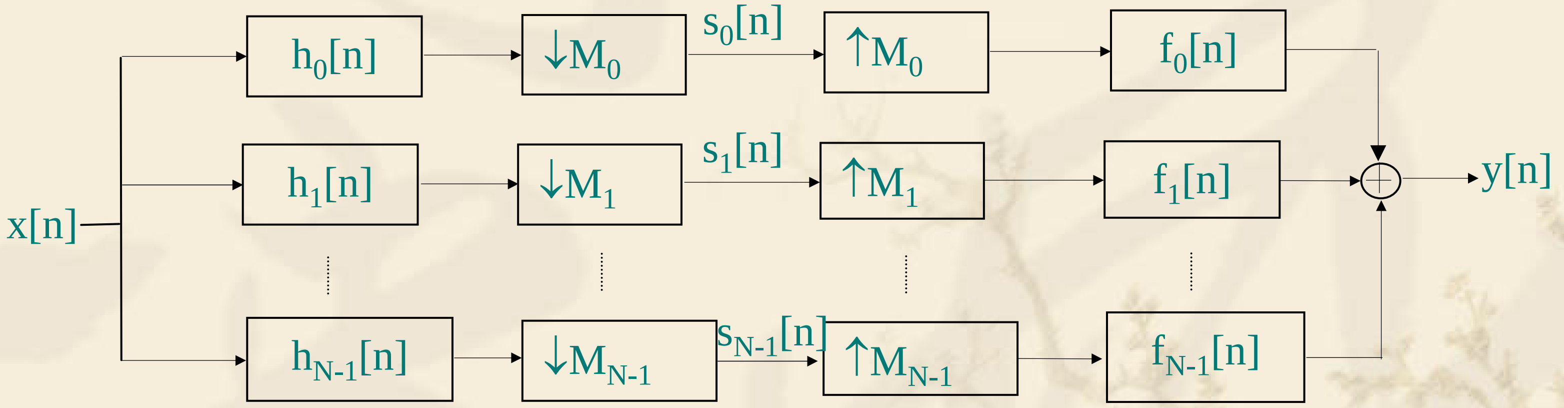
$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1}X\left(z^{\frac{1}{M}}W^{k}\right)$$

where $W = e^{-\frac{j2\pi}{M}}$

- There is a lost of information and aliasing occurs.

Time Frequency Analysis

- Filter banks
  - Filter banks are systems that contain banks of filters and conventional samplers.



Time Frequency Analysis

- Filter banks
  - Time localization of filter banks

$$s_j(n) = \sum_{m \to -\infty}^{+\infty} x(m) h_j(M_j n - m)$$

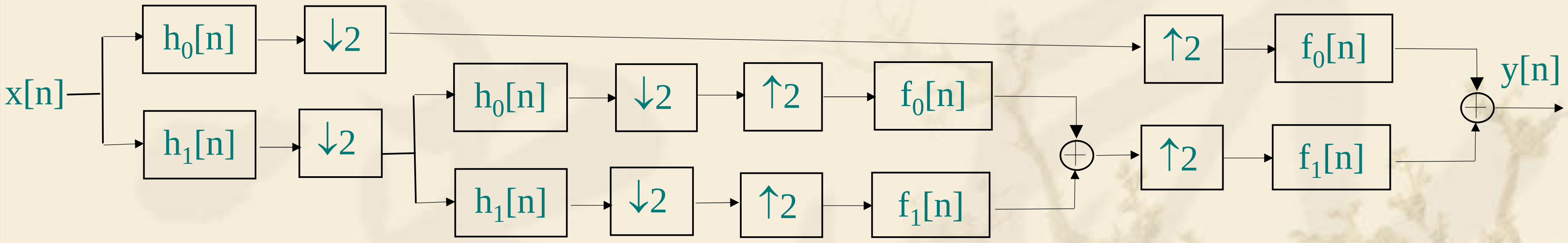
    - For finite impulse response $h_j(n)$, it becomes a finite summation of m and hence it only captures a finite time duration information of $x(n)$. That means, filter banks have a good time localization property.

Time Frequency Analysis

- Filter banks
  - Frequency localization of filter banks
    - As the filters has finite bandwidths, the filtered signals have finite bandwidths. Hence, filter banks have a good frequency localization property.

Time Frequency Analysis

- Tree structure filter banks
  - Tree structure filter banks is the most common approach for implementing discrete-time wavelet transforms if the filter bank in each tree level is paraunitary.



Time Frequency Analysis

- Tree structure filter banks
  - Paraunitary condition

$$H_i(z) \equiv \sum_{k=0}^{M-1} z^{-k} E_{i,k}\left(z^M\right)$$

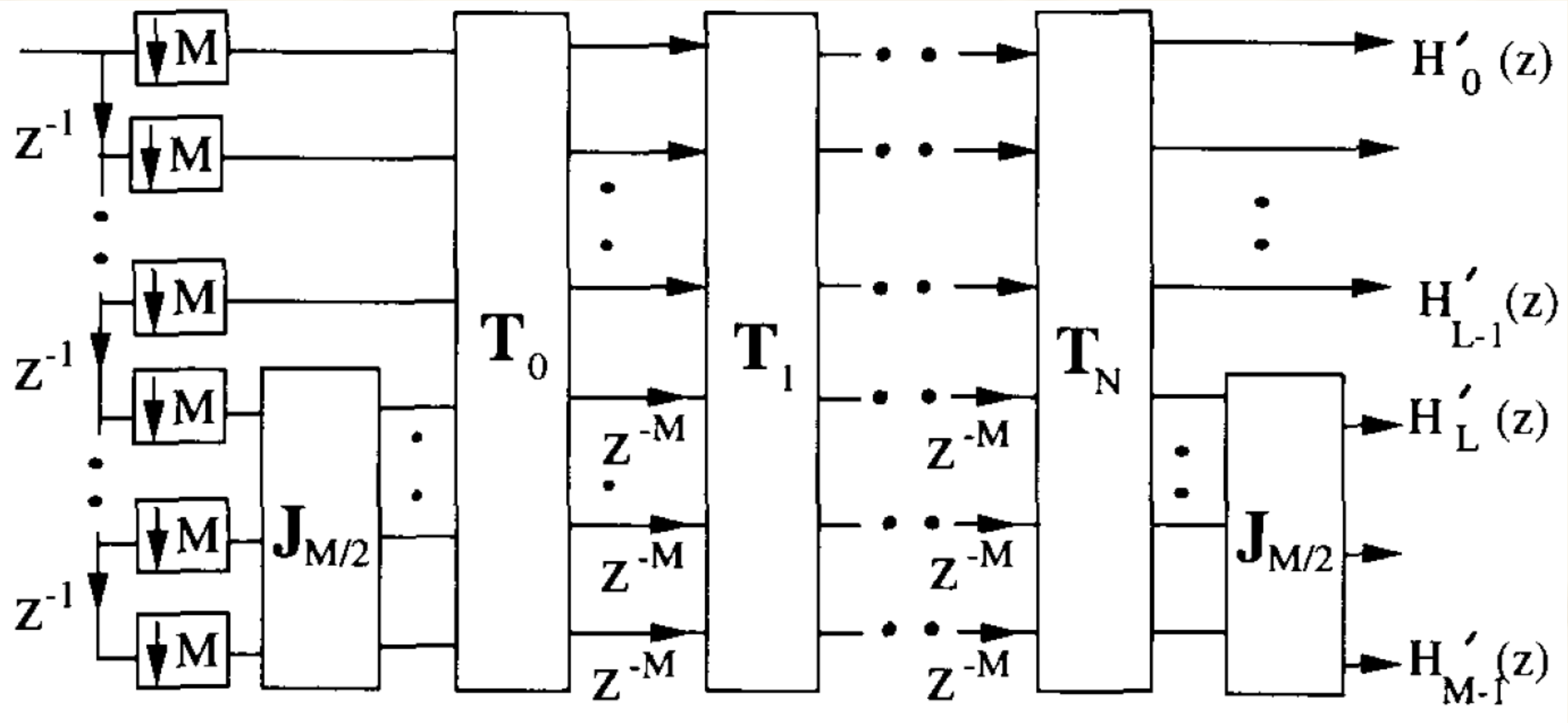
$$\mathbf{E}(z) \equiv \begin{bmatrix} E_{0,0}(z) & \cdots & E_{0,M-1}(z) \\ \vdots & \ddots & \vdots \\ E_{M-1,0}(z) & \cdots & E_{M-1,M-1}(z) \end{bmatrix}$$

$$\tilde{\mathbf{E}}(z)\mathbf{E}(z) = z^{-n_0}\mathbf{I}$$

- For each $\omega \in [-\pi,\pi]$, $\mathbf{E}\left(e^{j\omega}\right)$ is constrained to be a unitary matrix. In other words, it is constrained to be in a high dimensional ball. However, there are infinite number of ω in $[-\pi,\pi]$. Hence, this problem is an infinite number of high dimensional ball constrained problem.

Time Frequency Analysis

- Tree structure filter banks
  - Lattice structure design of paraunitary filter banks



- The problem involves products of many unitary matrices, which is a highly nonlinear and nonconvex optimization problem. It could be difficult to find the gradient vector of the objective function of the optimization problem. Hence, there is no efficient algorithm for the design.

Time Frequency Analysis

- Tree structure filter banks
  - Real-time design of paraunitary filter banks

$$\sum_{k=0}^{M-1} h_m(k) h_n(k - pM) = \delta(m-n)\delta(p)$$

$$\mathcal{H} \equiv \begin{bmatrix} h_0(0) & \cdots & h_{M-1}(0) \\ \vdots & \ddots & \vdots \\ h_0(N-1) & \cdots & h_{M-1}(N-1) \end{bmatrix}$$

$$\hat{\mathbf{I}}_p \equiv \begin{bmatrix} \mathbf{0}_{pM\times(N-pM)} & \mathbf{0}_{pM\times pM} \\ \mathbf{I}_{(N-pM)\times(N-pM)} & \mathbf{0}_{(N-pM)\times pM} \end{bmatrix}$$

$$\mathcal{H}^T \hat{\mathbf{I}}_p \mathcal{H} = \delta(p)\mathbf{I}_{M\times M}$$

Time Frequency Analysis

- Tree structure filter banks
  - Real-time design of paraunitary filter banks

$$\mathcal{H} = \mathbf{U}_H \mathbf{D}_H \mathbf{V}_H^T$$

$$\mathbf{U}_H \equiv \begin{bmatrix} \mathbf{U}_{H,1} & \mathbf{U}_{H,2} \end{bmatrix} \in \Re^{N \times N}$$

$$\mathbf{D}_H \equiv \begin{bmatrix} \mathbf{D}_{H,1} \\ \mathbf{0}_{(N-M) \times M} \end{bmatrix} \in \Re^{N \times M}$$

$$\mathcal{H} = \mathbf{U}_{H,1} \mathbf{D}_{H,1} \mathbf{V}_H^T$$

- Diagonal elements of $\mathbf{D}_{H,1}$ are either 1 or -1, $\mathbf{V}_H$ could be arbitrary unitary matrix.

  $\mathbf{D}_\Theta \equiv diag(\mathbf{U}_\Theta, \cdots, \mathbf{U}_\Theta)$ where $\mathbf{U}_\Theta$ is a unitary matrix.

  $\mathbf{D}_\Theta \mathcal{H}$ satisfies the paraunitary condition.

Time Frequency Analysis

- Tree structure filter banks
  - Real-time design of paraunitary filter banks

$$\min_{\hat{\mathbf{V}}_H} \quad tr\left(\left(\mathbf{D}_\Theta \mathbf{U}_{H,1}\hat{\mathbf{V}}_H - \hat{\mathcal{H}}\right)^T \left(\mathbf{D}_\Theta \mathbf{U}_{H,1}\hat{\mathbf{V}}_H - \hat{\mathcal{H}}\right)\right)$$

$$\text{subject to} \quad \hat{\mathbf{V}}_H^T \hat{\mathbf{V}}_H = \mathbf{I}_{M\times M}$$

- Define

$$\mathbf{B} \equiv \mathbf{U}_{H,1}^T \mathbf{D}_\Theta^T \hat{\mathcal{H}}$$

$$\mathbf{B} \equiv \mathbf{U}_B \mathbf{D}_B \mathbf{V}_B^T$$

- Then

$$\hat{\mathbf{V}}_H = \mathbf{U}_B \mathbf{V}_B^T$$

$$\overline{\boldsymbol{\lambda}} = \mathbf{V}_B \mathbf{D}_B \mathbf{V}_B^T$$

Time Frequency Analysis

- Tree structure filter banks
  - Real-time design of paraunitary filter banks

$$\min_{\mathbf{U}_\Theta} \quad tr\left(\left(\mathbf{D}_\Theta \mathbf{U}_{H,1}\hat{\mathbf{V}}_H - \hat{\mathcal{H}}\right)^T \left(\mathbf{D}_\Theta \mathbf{U}_{H,1}\hat{\mathbf{V}}_H - \hat{\mathcal{H}}\right)\right)$$

$$\text{subject to} \quad \mathbf{U}_\Theta^T \mathbf{U}_\Theta = \mathbf{I}_{M\times M}$$

- Define

$$\hat{\boldsymbol{h}}_m \equiv \begin{bmatrix} \hat{\boldsymbol{h}}_{m,0}^T & \cdots & \hat{\boldsymbol{h}}_{m,\frac{N}{M}-1}^T \end{bmatrix}^T$$

$$\hat{\boldsymbol{h}}_{m,n} \equiv \begin{bmatrix} \hbar_{m,n,0} & \cdots & \hbar_{m,n,M-1} \end{bmatrix}^T$$

$$\mathbf{U}_{H,1}\hat{\mathbf{V}}_H \equiv \boldsymbol{\Omega} \equiv \begin{bmatrix} \boldsymbol{\Omega}_0 & \cdots & \boldsymbol{\Omega}_{M-1} \end{bmatrix}$$

$$\boldsymbol{\Omega}_m \equiv \begin{bmatrix} \boldsymbol{\Omega}_{m,0}^T & \cdots & \boldsymbol{\Omega}_{m,\frac{N}{M}-1}^T \end{bmatrix}^T$$

$$\hat{\mathbf{b}}_i \equiv 2\sum_{m=0}^{M-1}\sum_{n=0}^{\frac{N}{M}-1} \hbar_{m,n,i}\boldsymbol{\Omega}_{m,n}$$

$$\hat{\mathbf{B}} \equiv \begin{bmatrix} \hat{\mathbf{b}}_0 & \cdots & \hat{\mathbf{b}}_{M-1} \end{bmatrix}$$

$$\mathbf{A} \equiv 2\sum_{m=0}^{M-1}\sum_{n=0}^{\frac{N}{M}-1} \boldsymbol{\Omega}_{m,n}\boldsymbol{\Omega}_{m,n}^T$$

$$\mathbf{A} \equiv \mathbf{U}_A \mathbf{D}_A \mathbf{V}_A^T$$

$$\hat{\mathbf{B}} \equiv \mathbf{U}_{\hat{B}} \mathbf{D}_{\hat{B}} \mathbf{V}_{\hat{B}}^T$$

Time Frequency Analysis

- Tree structure filter banks
  - Real-time design of paraunitary filter banks
    - Then

$$\mathbf{U}_{\Theta} = \mathbf{V}_{\hat{B}} \mathbf{U}_{\hat{B}}^{T}$$

$$\bar{\boldsymbol{\lambda}}^{*} = \frac{1}{2} \mathbf{V}_{\hat{B}} \left(\mathbf{D}_{\hat{B}} - \mathbf{U}_{\hat{B}}^{T} \mathbf{A} \mathbf{U}_{\hat{B}} \right) \mathbf{V}_{\hat{B}}^{T}$$

    - Implication: Numerical optimization computer aided design tools are not required to find a locally optimal solution. Hence, we could design adaptive real-time wavelet kernels.

Time Frequency Analysis

- Basis of Hilbert space
  - $\{\mathbf{x}_i\}_{i\in J}$ is a basis of Hilbert space $\mathcal{E}$ if
    - $\{\mathbf{x}_i\}_{i\in J}$ are linearly independent.
    - $\exists A, B > 0$ such that $A\|\mathbf{y}\|^2 \le \sum_{i\in J}\left|\langle \mathbf{x}_i, \mathbf{y}\rangle\right|^2 \le B\|\mathbf{y}\|^2$.
  - Example 3: $\mathcal{E} = \Re^N$, $c_i \equiv \langle \mathbf{x}_i, \mathbf{y}\rangle$ for $i = 0, \cdots, N-1$

    Let $\mathbf{c} \equiv \begin{bmatrix} c_0 & \cdots & c_{N-1}\end{bmatrix}^T$ and $\mathbf{X} \equiv \begin{bmatrix} \mathbf{x}_0 & \cdots & \mathbf{x}_{N-1}\end{bmatrix}$

$$\mathbf{c} = \mathbf{X}^T\mathbf{y}$$

$$\mathbf{y} = \left(\mathbf{X}^T\right)^{-1}\mathbf{c}$$

$$\sum_{i=0}^{N-1}\left|\langle \mathbf{x}_i, \mathbf{y}\rangle\right|^2 = \mathbf{c}^T\mathbf{X}\mathbf{X}^T\mathbf{c}$$

$$A = \min\left(eig\left(\mathbf{X}\mathbf{X}^T\right)\right)$$

$$B = \max\left(eig\left(\mathbf{X}\mathbf{X}^T\right)\right)$$

Frames **Time Frequency Analysis**

- $\{\mathbf{x}_i\}_{i\in J}$ is a frame of Hilbert space $\mathcal{E}$ if
  - $\exists A, B > 0$ such that $A\|\mathbf{y}\|^2 \le \sum_{i\in J} |\langle \mathbf{x}_i, \mathbf{y}\rangle|^2 \le B\|\mathbf{y}\|^2$.
- Example 4: $\mathbf{x}_i \in \Re^N$ for $i = 0, \cdots, M-1$, where $M > N$

  Let $\mathcal{E} = \Re^N$, $c_i \equiv \langle \mathbf{x}_i, \mathbf{y}\rangle$ for $i = 0, \cdots, M-1$

  $\mathbf{c} \equiv [c_0 \quad \cdots \quad c_{M-1}]^T$ and $\mathbf{X} \equiv [\mathbf{x}_0 \quad \cdots \quad \mathbf{x}_{M-1}]$

  Assume $rank(\mathbf{X}^T\mathbf{X}) = N$. Let $\tilde{\mathbf{X}} \equiv \mathbf{X}^T(\mathbf{X}\mathbf{X}^T)^{-1} \equiv [\tilde{\mathbf{x}}_0 \quad \cdots \quad \tilde{\mathbf{x}}_{M-1}]^T$

  and $\tilde{\mathbf{c}} \equiv \tilde{\mathbf{X}}\mathbf{y}$

$$\mathbf{X}\tilde{\mathbf{c}} = \mathbf{X}\tilde{\mathbf{X}}\mathbf{y} = \mathbf{X}\mathbf{X}^T(\mathbf{X}\mathbf{X}^T)^{-1}\mathbf{y} = \mathbf{y}$$

$$\sum_{i=0}^{M-1} |\langle \tilde{\mathbf{x}}_i, \mathbf{y}\rangle|^2 = \tilde{\mathbf{c}}^T\tilde{\mathbf{c}} = \mathbf{y}^T\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\mathbf{y}$$

$$A\|\mathbf{y}\|^2 \le \sum_{i=0}^{M-1} |\langle \tilde{\mathbf{x}}_i, \mathbf{y}\rangle|^2 \le B\|\mathbf{y}\|^2$$

$\{\tilde{\mathbf{x}}_i\}_{i\in J}$ is the dual frame to $\{\mathbf{x}_i\}_{i\in J}$

$A = \min(eig(\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}))$ and $B = \max(eig(\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}))$

Time Frequency Analysis

- Basis of filter banks
  - Define the inner product of signals as

$$\langle x(n), g(n)\rangle = \sum_{m\to-\infty}^{+\infty} x(m)g(m)$$

  - Suppose that $\{f_j(n-M_j m)\}_{j=0,\cdots,N-1,n\in Z}$ is a dual frame of $\{h_j(M_j n-m)\}_{j=0,\cdots,N-1,n\in Z}$.

Time Frequency Analysis

- Perfect reconstruction of filter banks

$$Y(z)=\sum_{j=0}^{N-1}\left(\left(X(z)H_j(z)\right)_{\downarrow M_j}\right)_{\uparrow M_j}F_j(z)$$

$$=\sum_{j=0}^{N-1}\left(\frac{1}{M_j}\sum_{k_j=0}^{M_j-1}X\left(z^{\frac{1}{M_j}}W_j^{k_j}\right)H_j\left(z^{\frac{1}{M_j}}W_j^{k_j}\right)\right)_{\uparrow M_j}F_j(z)$$

$$=\sum_{j=0}^{N-1}\left(\frac{1}{M_j}\sum_{k_j=0}^{M_j-1}X\left(zW_j^{k_j}\right)H_j\left(zW_j^{k_j}\right)\right)F_j(z)=cz^{-d}X(z)$$

$$\Rightarrow\sum_{j=0}^{N-1}\frac{F_j(z)}{M_j}\sum_{k_j=0}^{M_j-1}X\left(zW_j^{k_j}\right)H\left(zW_j^{k_j}\right)=cz^{-d}X(z)$$

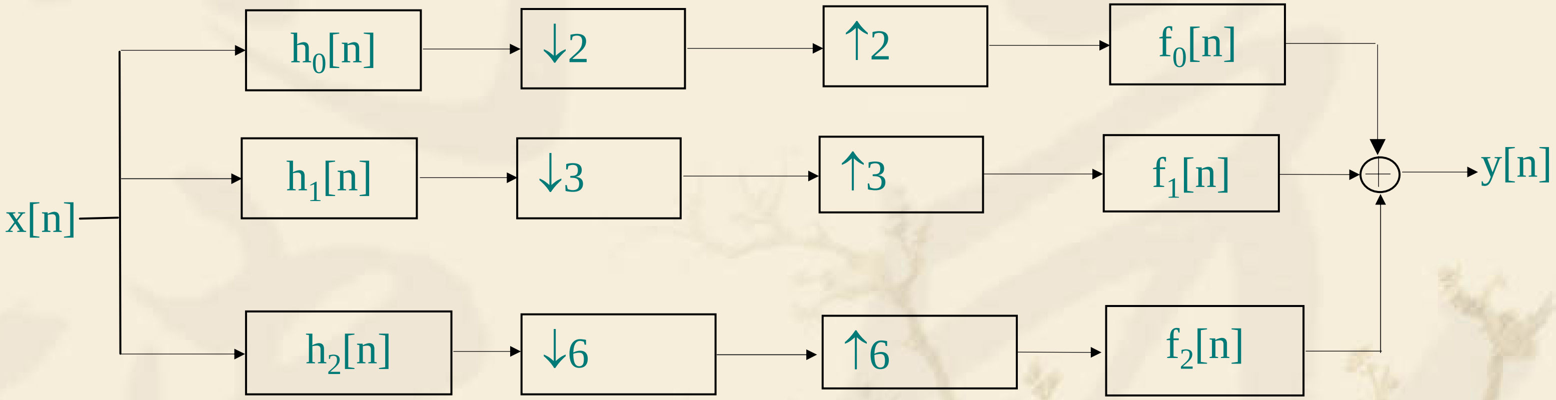
where *d* is the delay of the system, *c* is gain of system and $W_j=e^{-\frac{i2\pi}{M_j}}$

Time Frequency Analysis

- Perfect reconstruction of filter banks
  - If $\left\{f_j\left(n-M_j m\right)\right\}_{j=0,\cdots,N-1,n\in Z}$ and $\left\{h_j\left(M_j n-m\right)\right\}_{j=0,\cdots,N-1,n\in Z}$ are biorthogonal, then the filter banks achieve perfect reconstruction.

Time Frequency Analysis

- Incompatible nonuniform filter banks
  - Consider a nonuniform filter bank with $M_0=2$, $M_1=3$ and $M_2=6$.



Time Frequency Analysis

- Incompatible nonuniform filter banks
  - We *have*

$$\begin{bmatrix} X(z) & \cdots & X(zW^5) \end{bmatrix} \begin{bmatrix} \frac{1}{2}H_0(z) & \frac{1}{3}H_1(z) & \frac{1}{6}H_2(z) \\ 0 & 0 & \frac{1}{6}H_2(zW) \\ 0 & \frac{1}{3}H_1(zW^2) & \frac{1}{6}H_2(zW^2) \\ \frac{1}{2}H_0(zW^3) & 0 & \frac{1}{6}H_2(zW^3) \\ 0 & \frac{1}{3}H_1(zW^4) & \frac{1}{6}H_2(zW^4) \\ 0 & 0 & \frac{1}{6}H_2(zW^5) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = cz^{-d}X(z)$$

where $W = e^{-\frac{i2\pi}{6}}$

Time Frequency Analysis

- Incompatible nonuniform filter banks
  - If perfect reconstruction could be achieved for arbitrary bounded inputs, then

$$\begin{bmatrix} \frac{1}{2}H_0(z) & \frac{1}{3}H_1(z) & \frac{1}{6}H_2(z) \\ 0 & 0 & \frac{1}{6}H_2(zW) \\ 0 & \frac{1}{3}H_1(zW^2) & \frac{1}{6}H_2(zW^2) \\ \frac{1}{2}H_0(zW^3) & 0 & \frac{1}{6}H_2(zW^3) \\ 0 & \frac{1}{3}H_1(zW^4) & \frac{1}{6}H_2(zW^4) \\ 0 & 0 & \frac{1}{6}H_2(zW^5) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \begin{bmatrix} cz^{-d} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

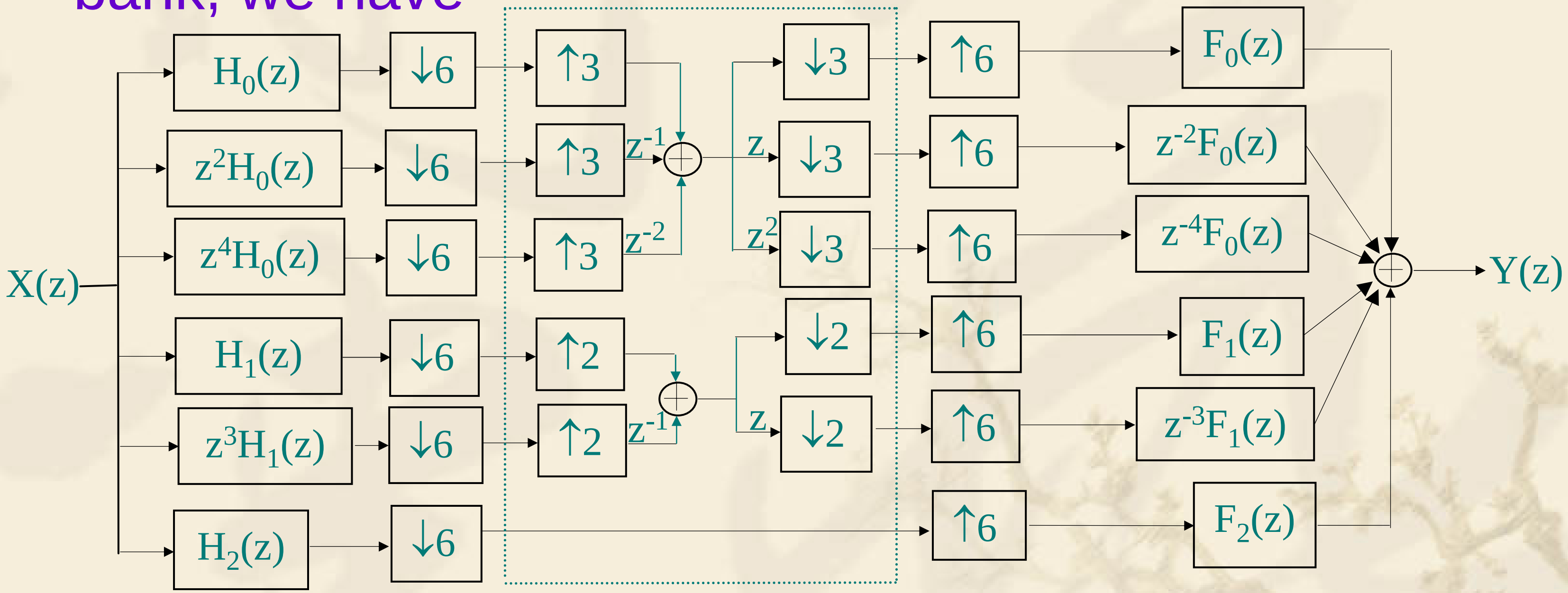
  - Impossible to achieve perfect reconstruction because aliasing cannot be cancelled.

Time Frequency Analysis

- Incompatible nonuniform filter banks
  - Open problem
    - How to design the filters and the samplers such that perfect reconstruction can be achieved?
      - P. Q. Hoang and P. P. Vaidyanathan, “Non-uniform multirate filter banks: theory and design,” International Symposium on Circuits and Systems, ISCAS, pp. 371-374, 1989. (Open problem seeking for general solutions)
      - Tongwen Chen, Li Qiu and Er-Wei Bai, “General multirate building structures with application to nonuniform filter banks,” IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing, vol. 45, no. 8, pp. 948-958, 1998. (Solution based on time varying filters)
      - Sony Akkarakaran and P. P. Vaidyanathan, “New results and open problems on nonuniform filter-banks,” International Conference on Acoustics, Speeches and Signal Processing, ICASSP, pp. 1501-1504, 1999. (Open problem seeking for time invariant solutions)
      - Charlotte Yuk-Fan Ho, Bingo Wing-Kuen Ling and Peter Kong-Shun Tam, “Representations of linear dual-rate system via single SISO LTI filter, conventional sampler and block sampler,” IEEE Transactions on Circuits and Systems-II: Express Briefs, vol. 55, no. 2, pp. 168-172, 2008. (Solution based on time invariant filters)

Time Frequency Analysis

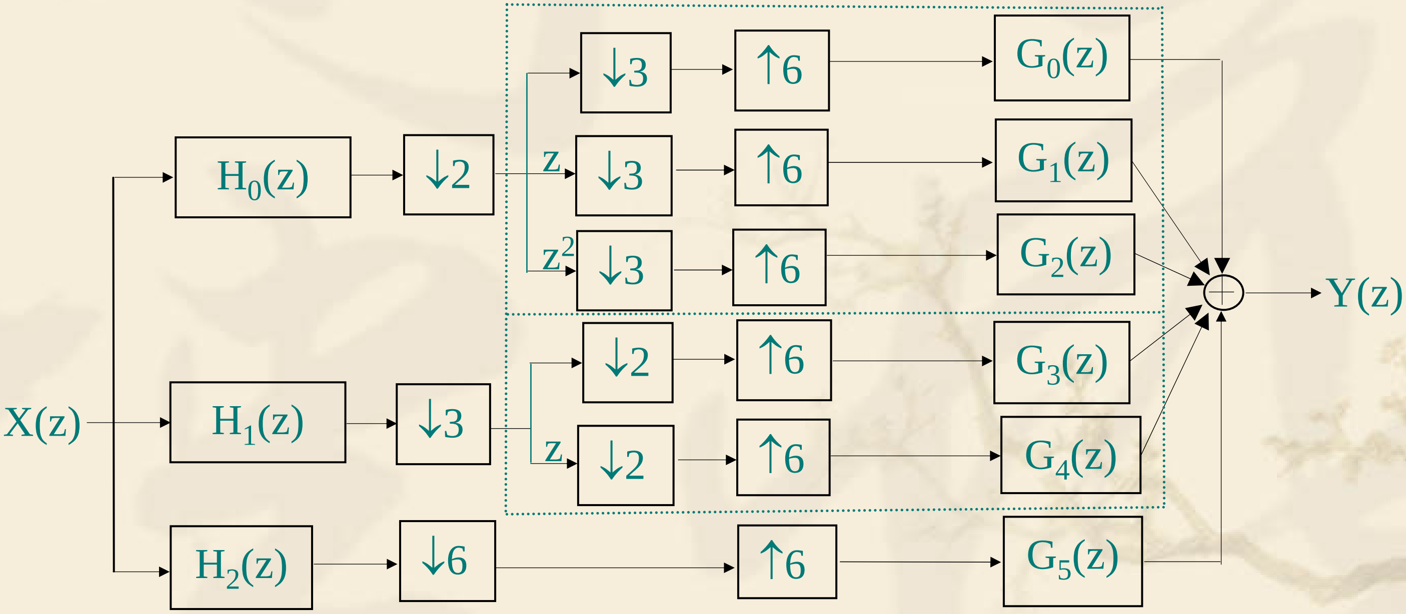
- Incompatible nonuniform filter banks
  - Time varying solution
    - By converting the nonuniform filter bank to a uniform filter bank, we have



    - The dash rectangle is an identity system, so this nonuniform filter bank becomes a uniform filter bank.

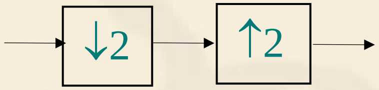
Time Frequency Analysis

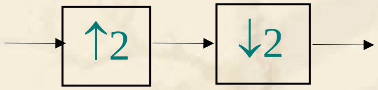
- Incompatible nonuniform filter banks
  - Time varying solution
    - The dash block is a time varying system.



Time Frequency Analysis

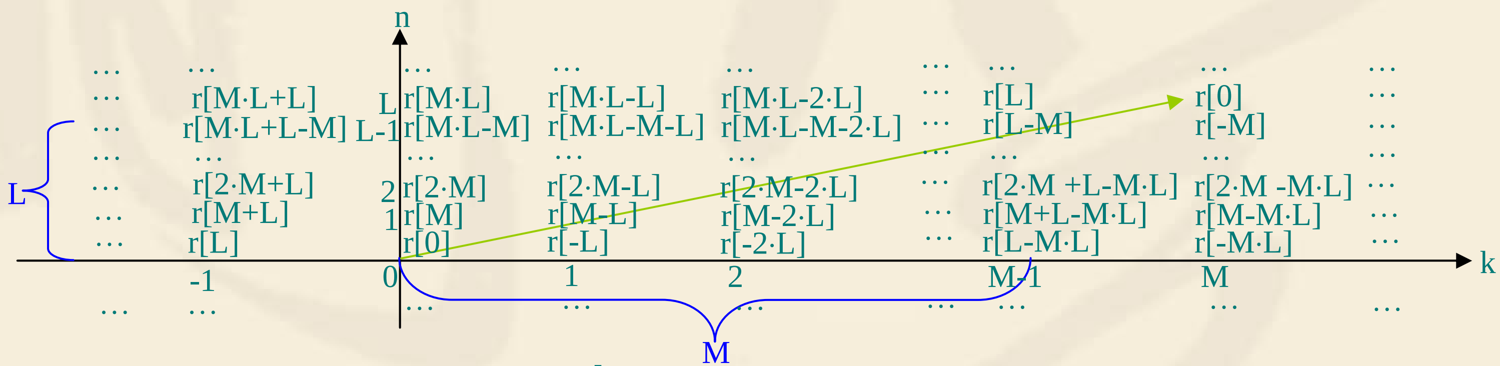
- Basic property of upsampler and downsampler
  - The upsampler and the downsampler is in general not commutative.



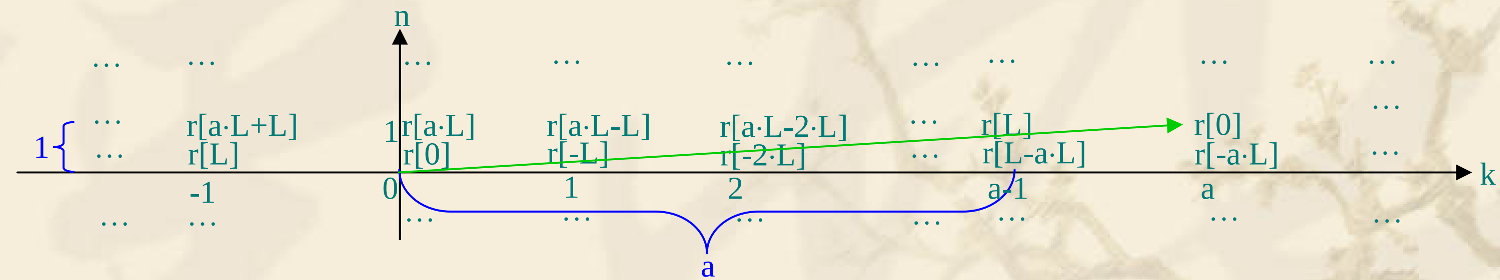


  - Upsampler (upsampled by L) and downsampler (downsampled by M) is commutative if and only if L and M is co-prime.

Time Frequency Analysis

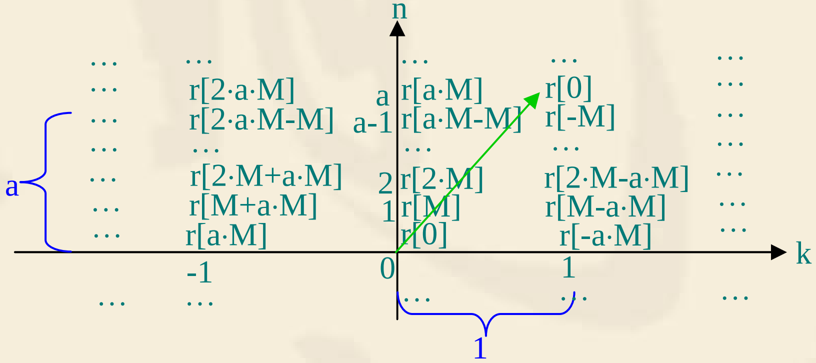


L and *M* are co-prime.

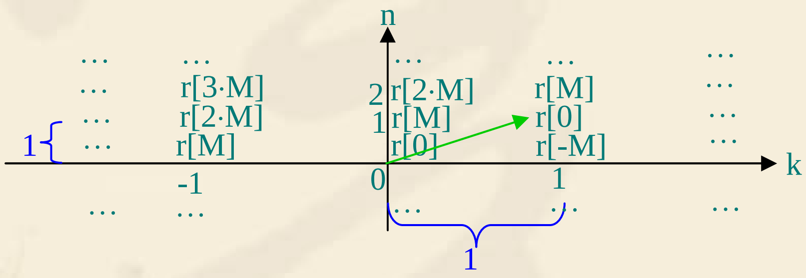


$M=a \cdot L$

Time Frequency Analysis



L=a·M

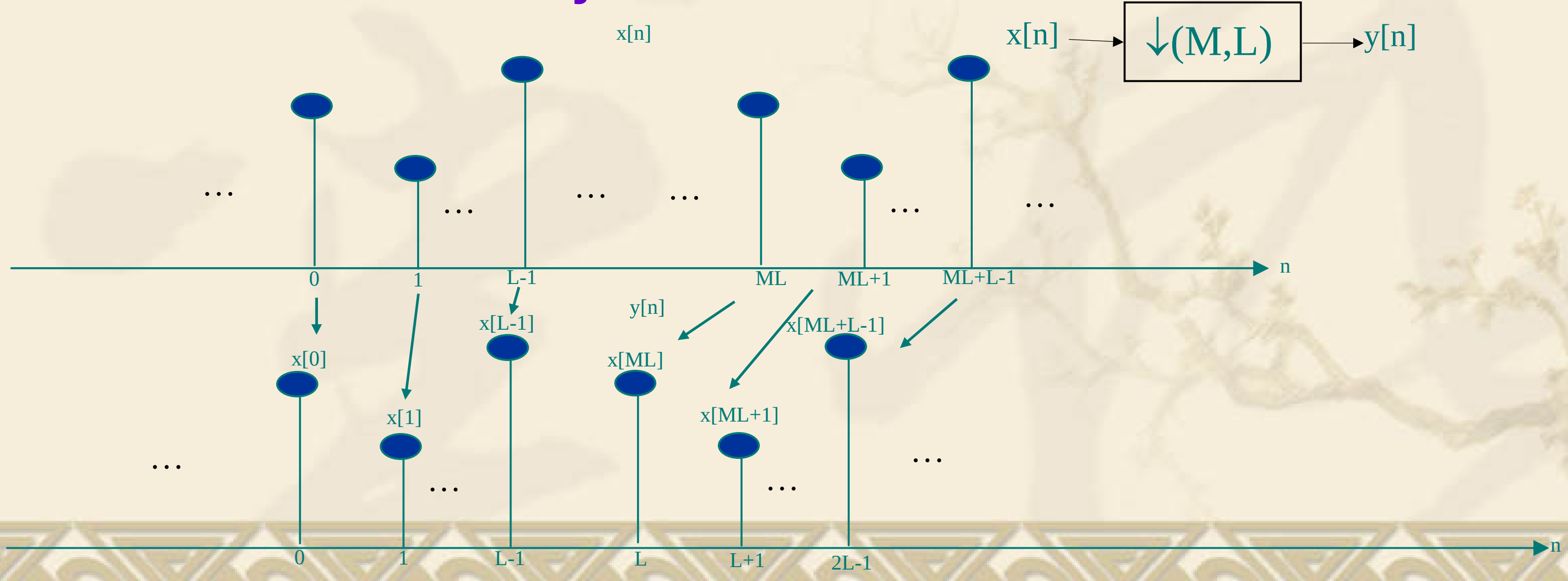


L=M

Time Frequency Analysis

- Linear time invariant solution
  - Block decimators (decimation ratio M and block length L)

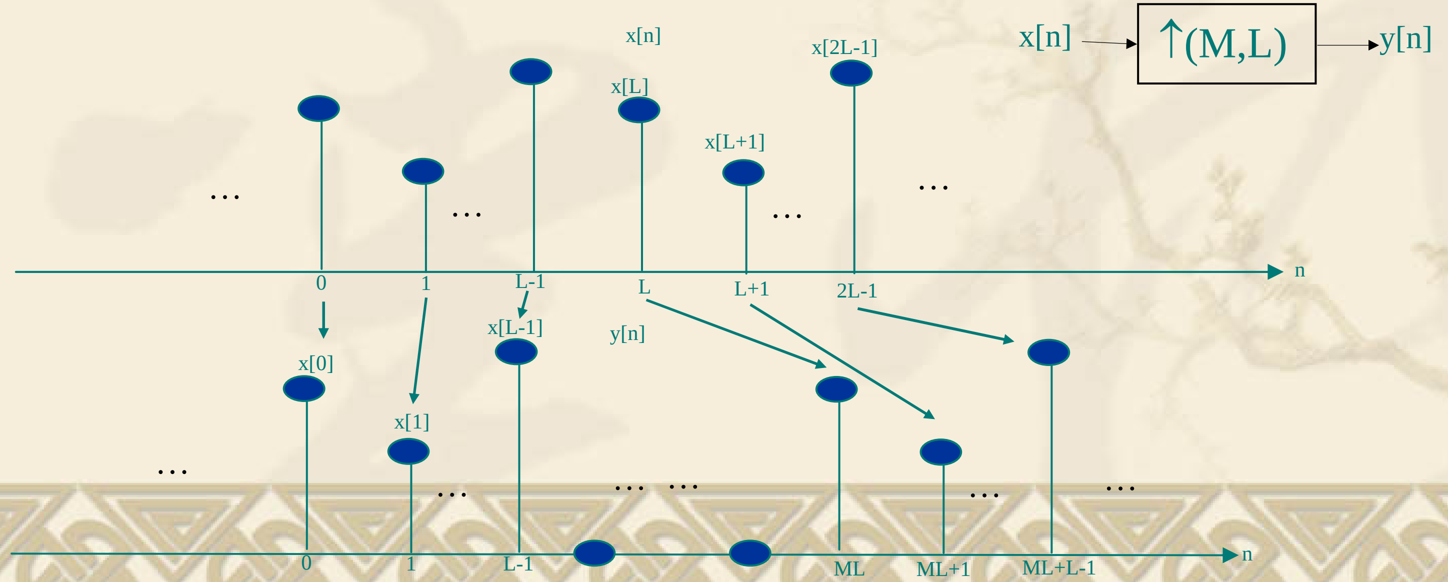
$y[Lk+j]=x[kML+j]$ for j=0,1,…,L-1 and k∈Z.



Time Frequency Analysis

- Linear time invariant solution
  - Block expanders (expansion ratio M and block length L)

$$y[k]=\begin{cases} x\left[\dfrac{k-\mathrm{mod}(k,ML)}{M}+\mathrm{mod}(k,ML)\right] & k-\mathrm{mod}(k,ML)\le k<k-\mathrm{mod}(k,ML)+L \\ 0 & k-\mathrm{mod}(k,ML)+L\le k<k+ML-\mathrm{mod}(k,ML) \end{cases}$$



Time Frequency Analysis

- Linear time invariant solution
  - $\forall m,n \in Z^+$ (no matter m and n are co-prime or not), all linear dual rate systems with shifting input by n samples resulting to shifting an output by m samples can be represented via a series cascade of $\uparrow m$, followed by a linear time invariant filter with an impulse response f[k], and then followed by $\downarrow (n,m)$.

Time Frequency Analysis

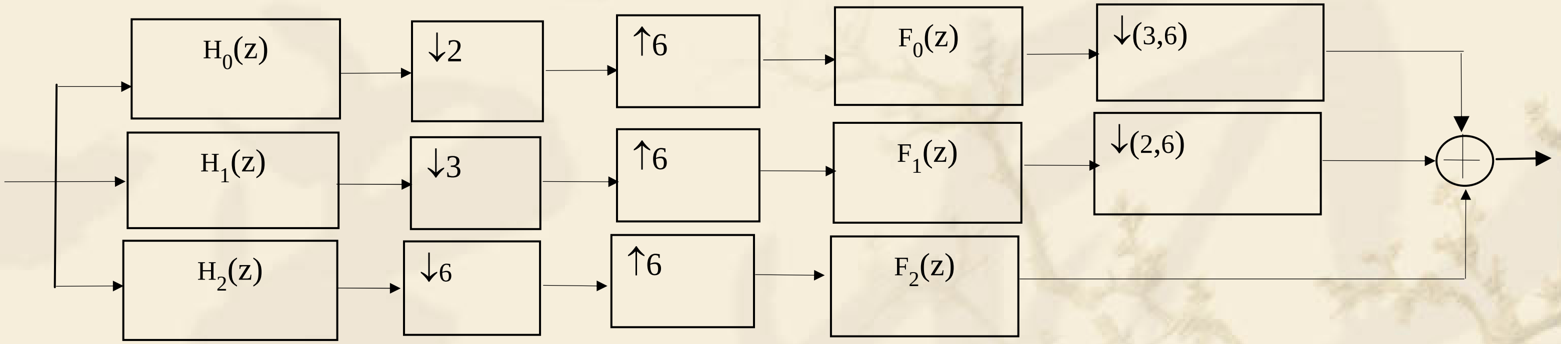
- Linear time invariant solution
  - The input output relationship of all linear dual rate systems is $y[km+i]=\sum_{l\to-\infty}^{+\infty} g[i,l-kn]u[l]$, $\forall k,l\in Z$, $\forall m,n\in Z^+$ and i=0,1,…,m-1.
  - The input output relationship of the proposed representation is $y[km+i]=\sum_{\forall l} f[kmn-ml+i]u[l]$, $\forall k,l\in Z$, $\forall m,n\in Z^+$ and i=0,1,…,m-1.
  - $\forall k,l\in Z$, $\forall m,n\in Z^+$ and i=0,1,…,m-1, the mapping from {0,1,…,m-1}xZ to Z, where $[i,l-kn]\in\{0,1,\ldots,m-1\}\times Z$ and $kmn-ml+i\in Z$ is bijective.

Time Frequency Analysis

- Linear time invariant solution
  - Hence, $\forall k,l \in Z$, $\forall m,n \in Z^+$ and i=0,1,…,m-1, there exists a unique time index kmn-ml+i corresponding to the time index [i,l-kn].
  - As a result, there exists a linear time invariant filter with an impulse response f[k] satisfying f[kmn-ml+i]=g[i,l-kn], $\forall k,l \in Z$, $\forall m,n \in Z^+$ and i=0,1,…,m-1, that the linear dual rate systems and the proposed representation are input output equivalent.

Time Frequency Analysis

- Linear time invariant solution
  - Consequently, the incompatible nonuniform filter bank can achieve perfect reconstruction via the following structure.



Time Frequency Analysis

- Linear time invariant solution
  - $\forall m,n \in Z^+$ (no matter m and n are co-prime or not), all linear dual rate systems with shifting input by n samples resulting to shifting an output by m samples can be represented via a series cascade of $\uparrow(m,n)$, followed by a linear time invariant filter with an impulse response f[k], and then followed by $\downarrow n$.

Time Frequency Analysis

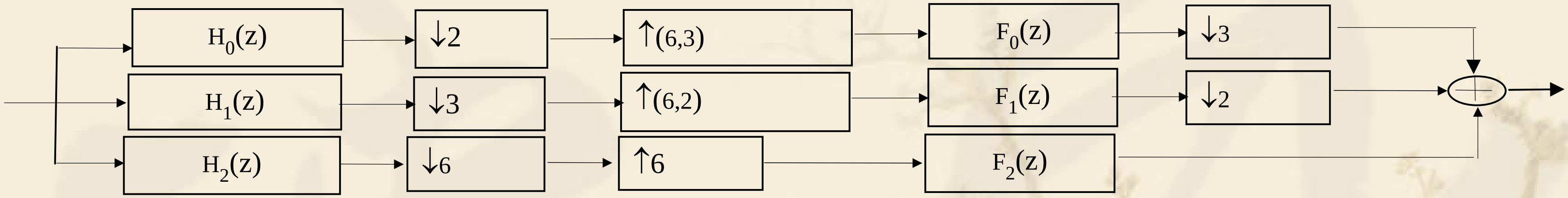
- Linear time invariant solution
  - The input output relationship of all linear dual rate systems is $y[k]=\sum_{l\to-\infty}^{+\infty}\sum_{i=0}^{n-1} g[k,nl+i]u[nl+i]$, $\forall$k,l$\in$Z, $\forall$m,n$\in$$Z^+$ and i=0,1,…,n-1.
  - The input output relationship of the proposed representation is $y[k]=\sum_{l\to-\infty}^{+\infty}\sum_{i=0}^{n-1} f[kn-mnl-i]u[nl+i]$, $\forall$k,l$\in$Z, $\forall$m,n$\in$$Z^+$ and i=0,1,…,n-1.
  - $\forall$l$\in$Z, $\forall$m,n$\in$$Z^+$, k$\in${0,1,…,m-1} and i$\in${0,1,…,n-1}, the mapping from {0,1,…,m-1}xZ to Z, where [k,nl+i]$\in${0,1,…,m-1}xZ and kn-mnl-i$\in$Z is bijective.

Time Frequency Analysis

- Linear time invariant solution
  - Hence, $\forall l \in Z$, $\forall m,n \in Z^+$, $k \in \{0,1,\ldots,m-1\}$ and $i \in \{0,1,\ldots,n-1\}$, there exists a unique time index $kn-mnl-i$ corresponding to the time index $[k,nl+i]$.
  - As a result, there exists a linear time invariant filter with an impulse response $f[k]$ satisfying $f[kn-mnl-i]=g[k,nl+i]$, $\forall k,l \in Z$, $\forall m,n \in Z^+$ and $i=0,1,\ldots,n-1$, that the linear dual rate systems and the proposed representation are input output equivalent.

Time Frequency Analysis

- Linear time invariant solution
  - Consequently, the incompatible nonuniform filter bank can achieve perfect reconstruction via the following structure.

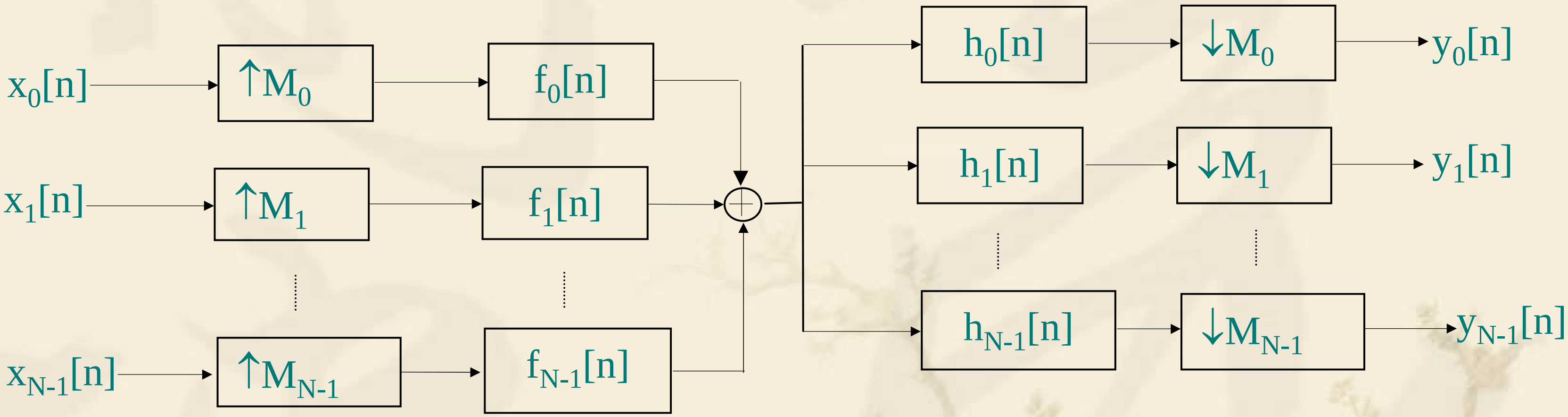


Time Frequency Analysis

- Linear time invariant solution
  - Implication: We could have arbitrarily time localization and frequency localization.

Time Frequency Analysis

- Application of nonuniform filter banks
  - Nonuniform transmultiplexers

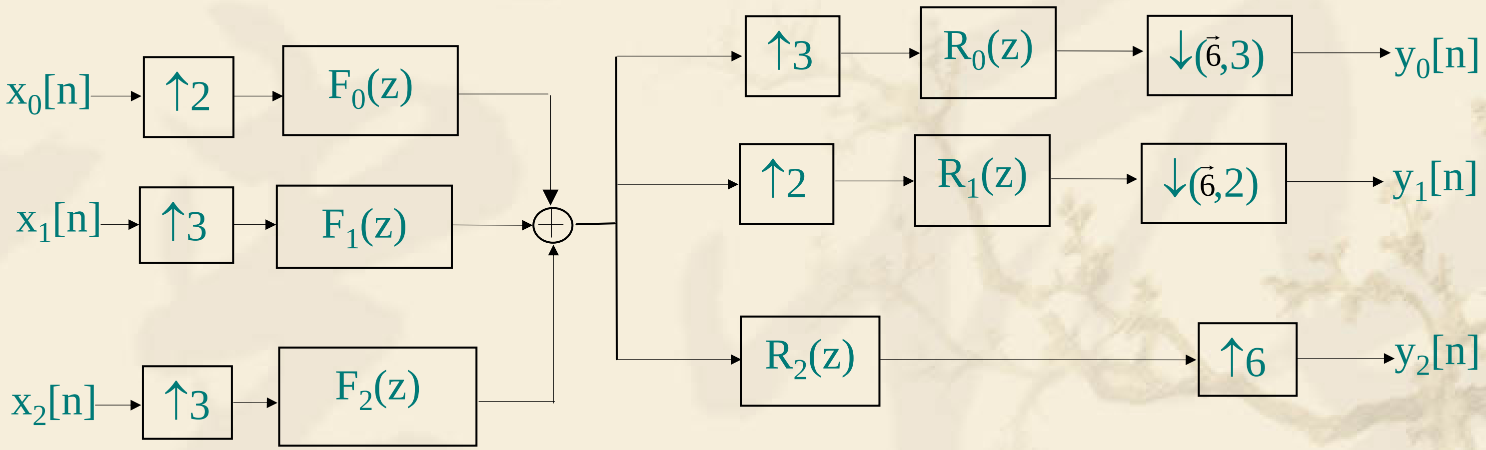


- Perfect reconstruction

$$y_i[n] = c_i x_i[n - d_i]$$

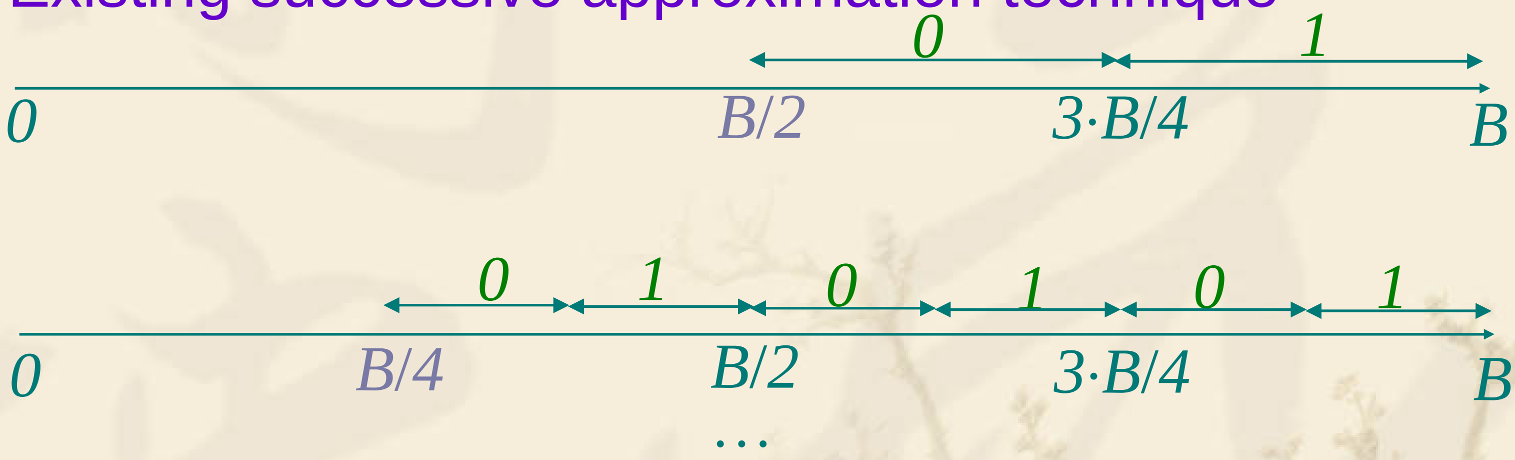
Time Frequency Analysis

- Application of nonuniform filter banks
  - Nonuniform transmultiplexers
    - Example 4: $F_0(z)=1$, $F_1(z)=z^{-4}+z^{-5}$, $F_2(z)=z^{-3}$, then $R_0(z)=1-z^2+z^5+z^{10}-z^{13}$, $R_1(z)=z^{10}+z^{13}$ and $R_2(z)=z^3$.



Time Frequency Analysis

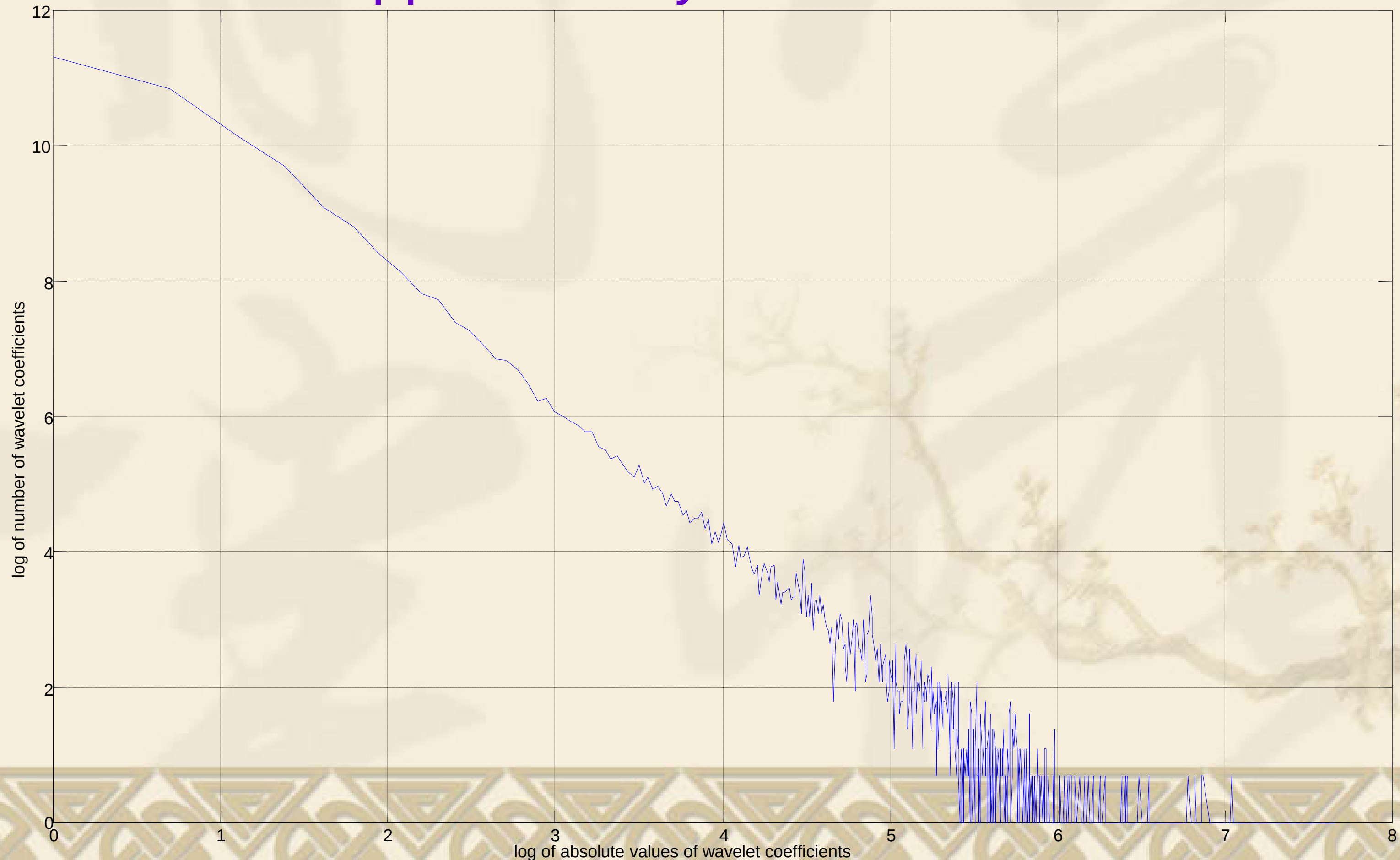
- Application of nonuniform filter banks
  - Nonuniform image coding
    - Existing successive approximation technique



where *B* is the maximum absolute value of the wavelet coefficients.

Time Frequency Analysis

- Application of nonuniform filter banks
  - Nonuniform image coding
    - Absolute values of wavelet coefficients follows Laplacian distribution approximately.

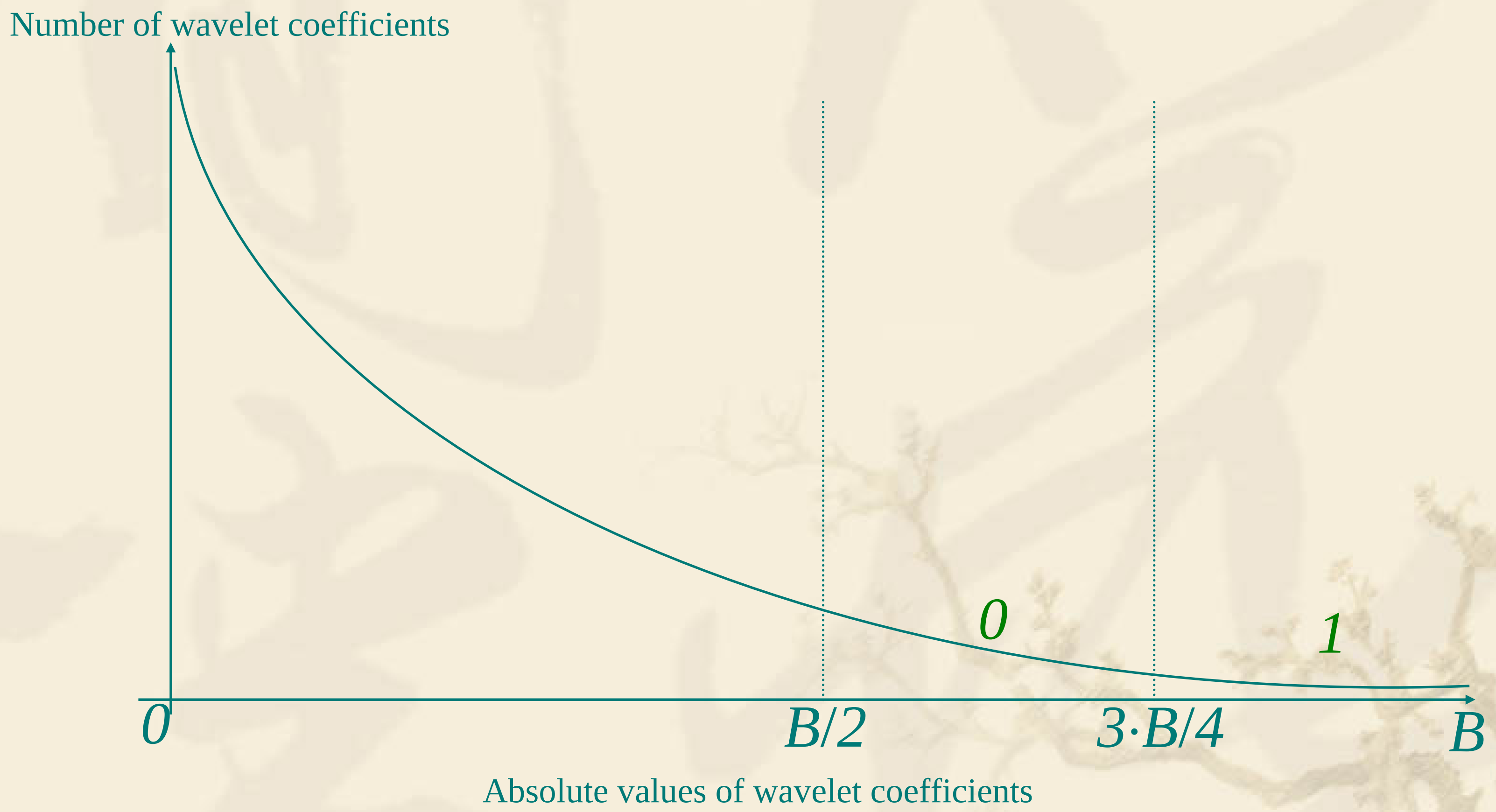


Time Frequency Analysis

- Application of nonuniform filter banks
  - Nonuniform image coding
    - Probability of assigning the symbol '*0*' is greater than that of the symbol '*1*'.
    - Uniform distribution of the symbols gives maximum entropy.
    - So the existing successive approximation technique is not optimal.
    - A higher coding gain curve may be achieved by means of non-uniform successive approximation.

Time Frequency Analysis

- Application of nonuniform filter banks
  - Nonuniform image coding



Time Frequency Analysis

- Application of nonuniform filter banks
  - Nonuniform image coding
    - Set the thresholds T_j at B/p^j, where $p>2$.
    - Let a and b are the boundaries in the region, c be the coded value and $f(x)$ be the distribution of the wavelet coefficients, then the error introduced in the quantization is:

$$E(c)=\int_a^b (x-c)^2 \cdot f(x)dx$$

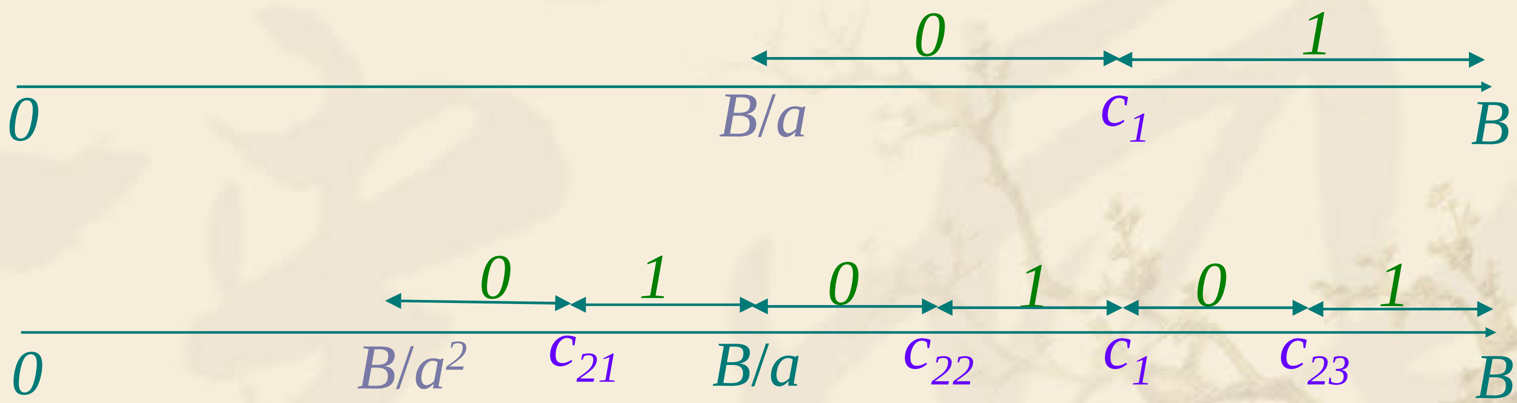
where $f(x)=A\cdot e^{k\cdot x}$

$$\frac{d}{dc}E(c)=0$$

$$\Rightarrow c=\frac{e^{b\cdot k}\cdot(b\cdot k-1)-e^{a\cdot k}\cdot(a\cdot k-1)}{k\cdot\left(e^{b\cdot k}-e^{a\cdot k}\right)}$$

Time Frequency Analysis

- Application of nonuniform filter banks
  - Nonuniform image coding



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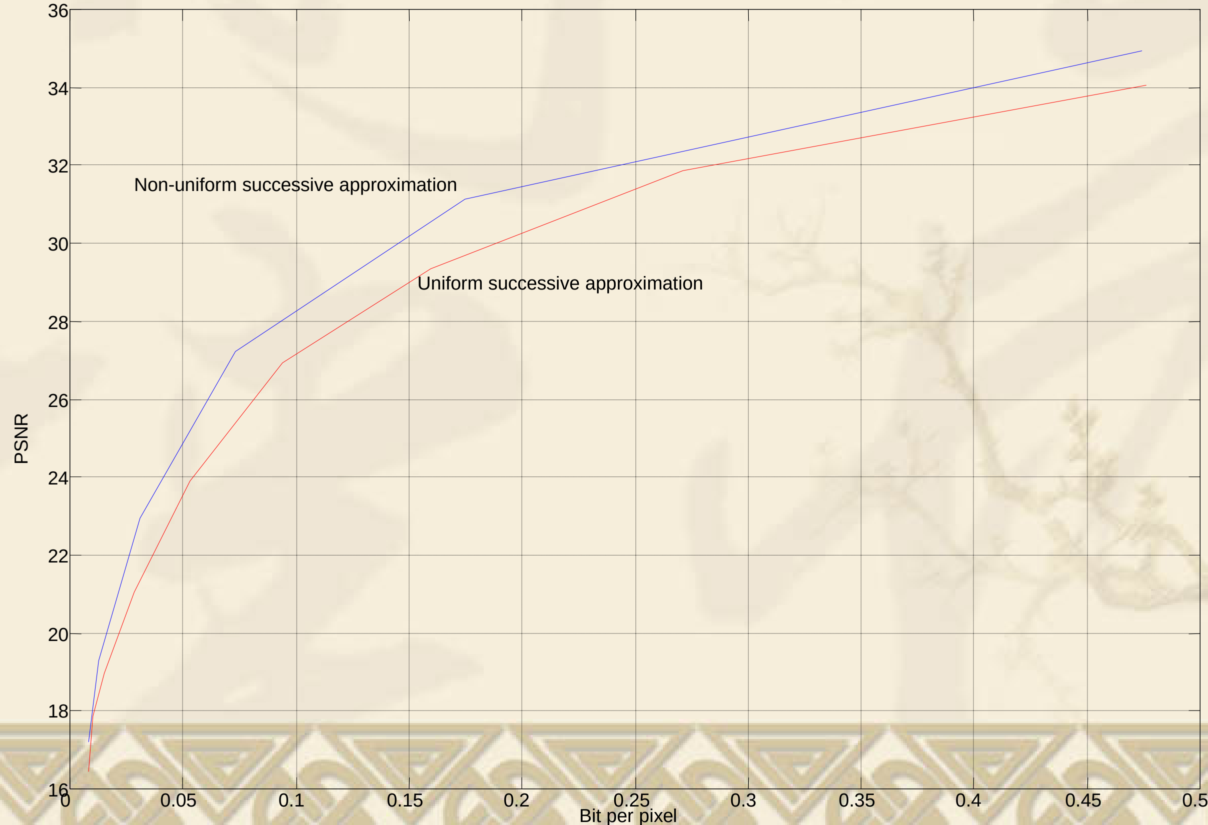
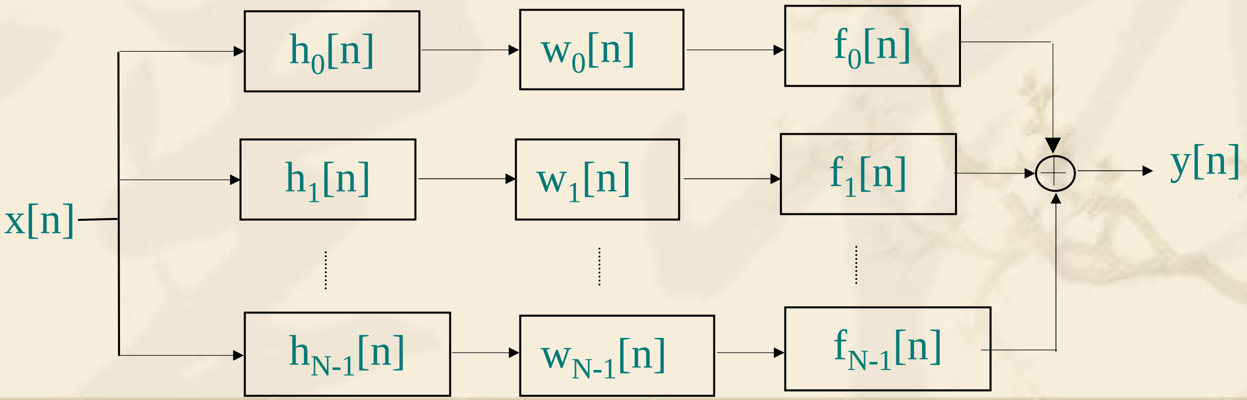
Time Frequency Analysis

- Application of nonuniform filter banks
  - Nonuniform image coding



Time Frequency Analysis

- Filter window banks and Fractional Fourier transform
  - Downsampling first and then upsampling is equivalent to a sampling window function. What happens if we have general window functions?



Time Frequency Analysis

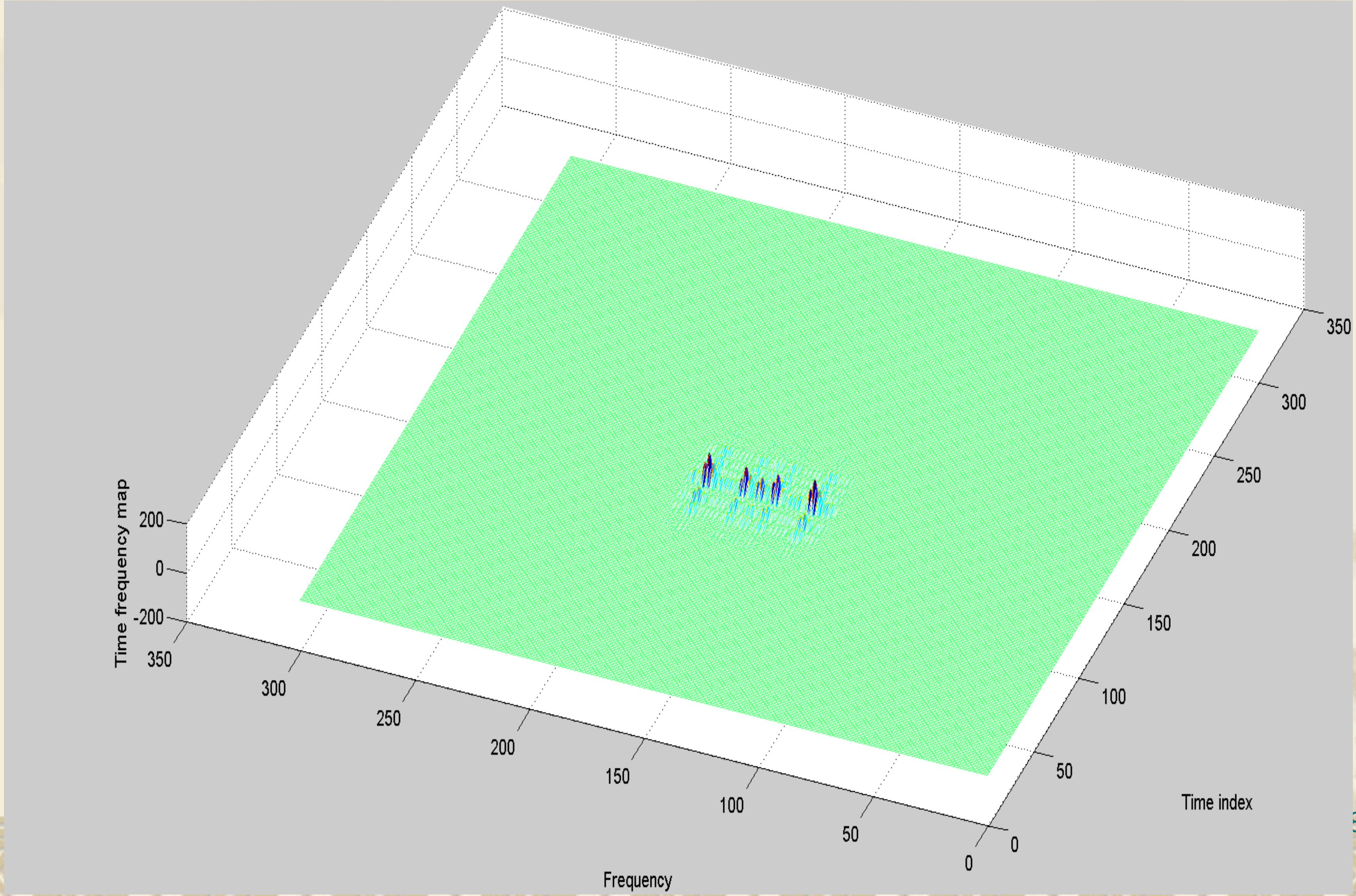
- Filter window banks and Fractional Fourier transform
  - Fractional Fourier transform is to rotate the time frequency plane

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ \omega \end{bmatrix}$$

  - By designing a set of windows and filters as well as applying the fractional Fourier transform to rotate the time frequency plane, the signals could be extracted out precisely.

Time Frequency Analysis

- Filter window banks and Fractional Fourier transform

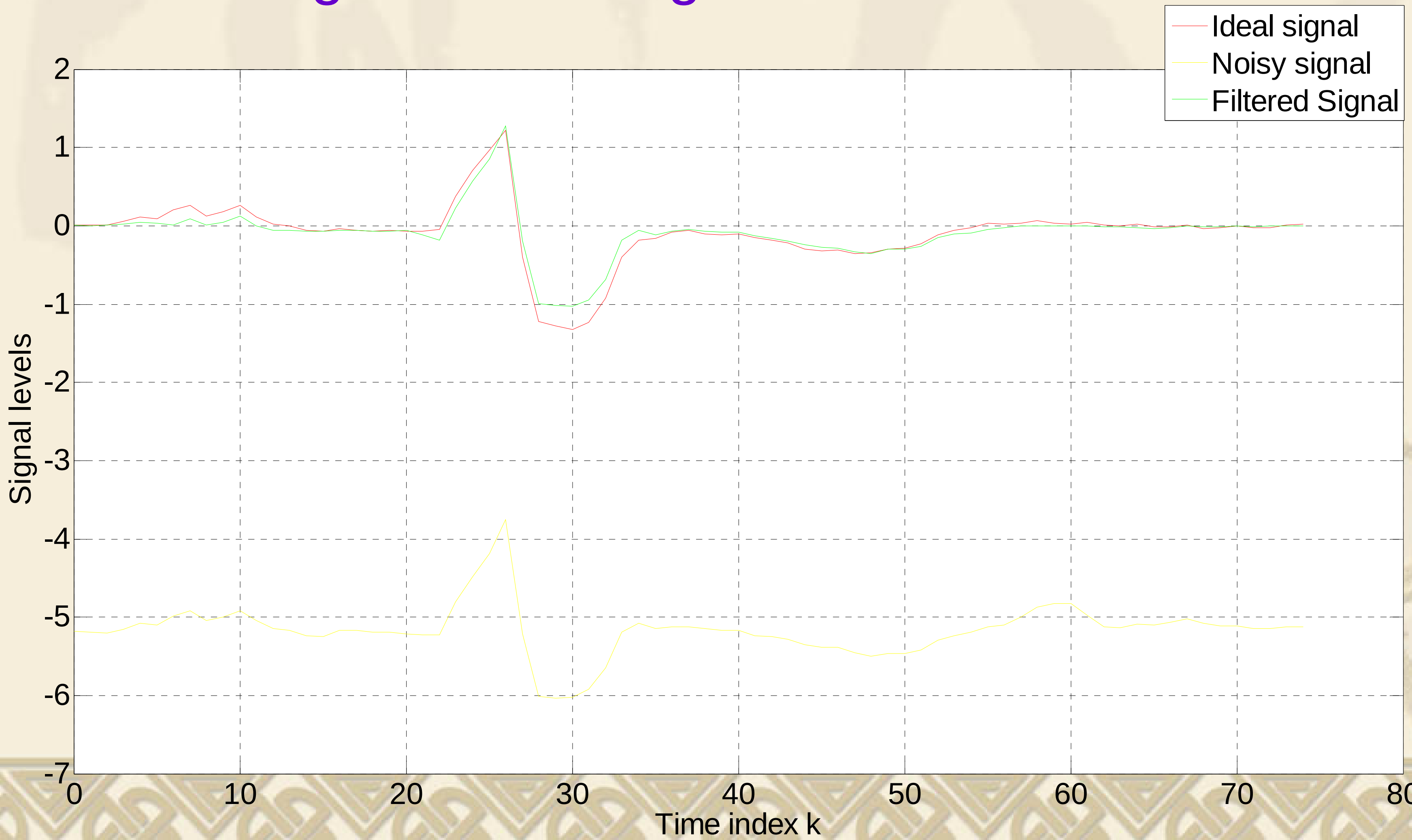


Time Frequency Analysis

- Filter window banks and Fractional Fourier transform
  - Open problems:
    - How to guarantee the perfect reconstruction?
    - How to design the globally optimal set of filters and the windows such that the filters have good frequency selectivities?
    - Filtering and windowing could be understood as the multiplication in certain particular domains, such as in the frequency domain and in the time domain. In fact, these domains are obtained by certain particular unitary transforms. For example, frequency domain is obtained by applying the DFT transform which is a unitary transform and the time domain is obtained by applying the IDFT transform which is also a unitary transform . What happens if the transform is generalized to arbitrarily unitary transforms and how to determine such optimal transform?
    - How to apply these results to some practical problems, such as denoising problems, signal separation problems, pattern recognition problems and fault detection problems?

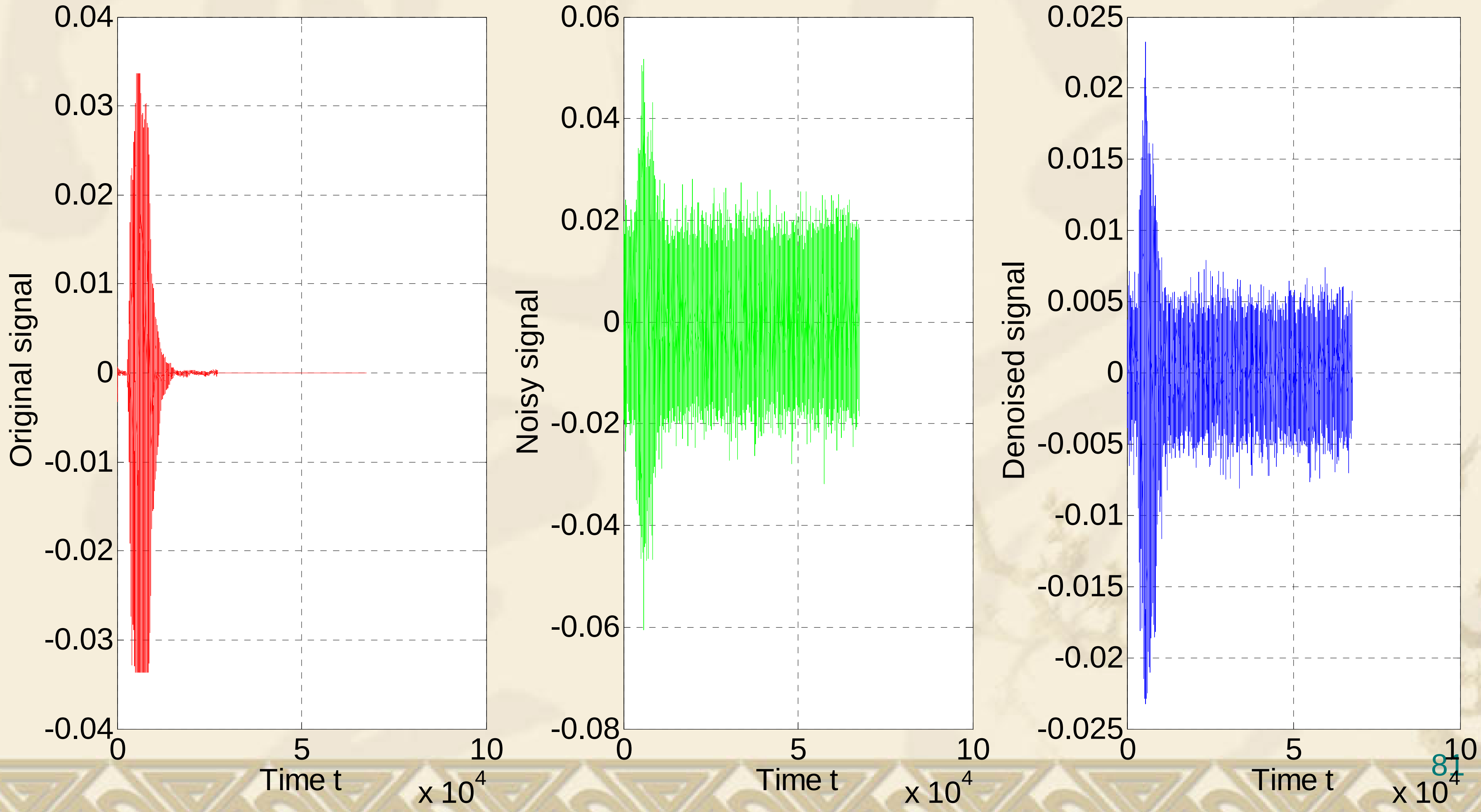
Time Frequency Analysis

- Applications
  - ECG signal denoising



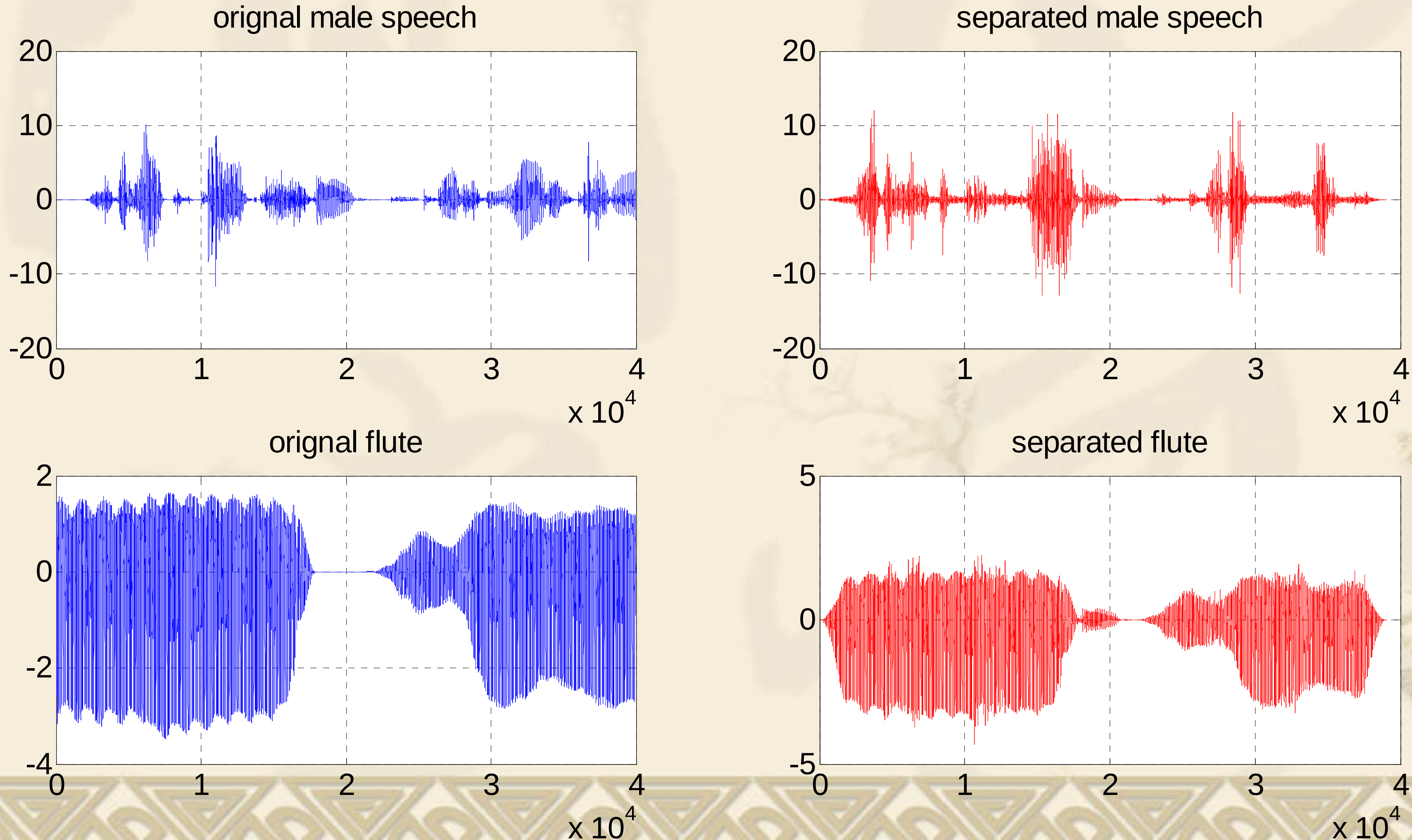
Time Frequency Analysis

- Applications
  - Audio signal denoising for digital audio hearing aids



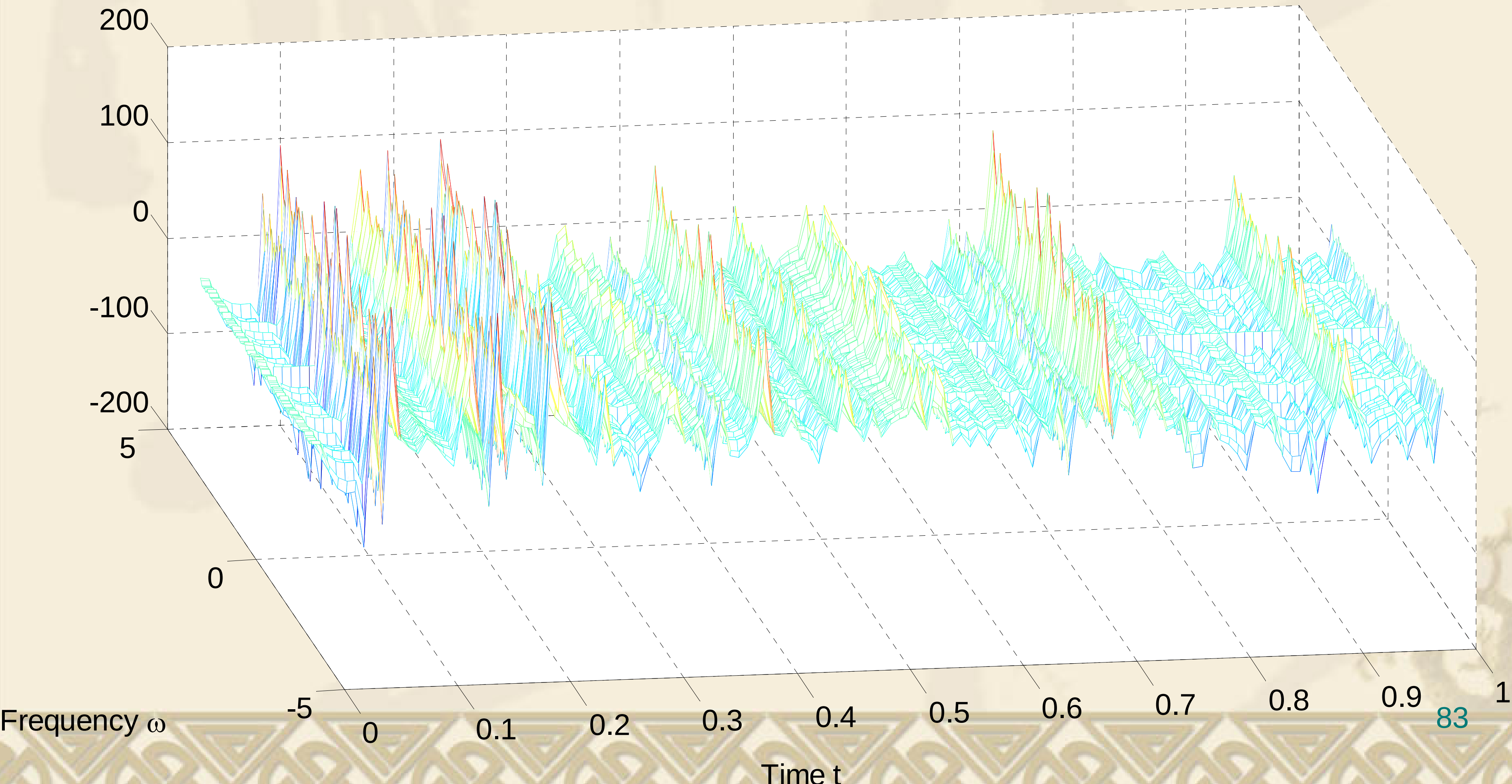
Time Frequency Analysis

- Applications
  - Signal separation



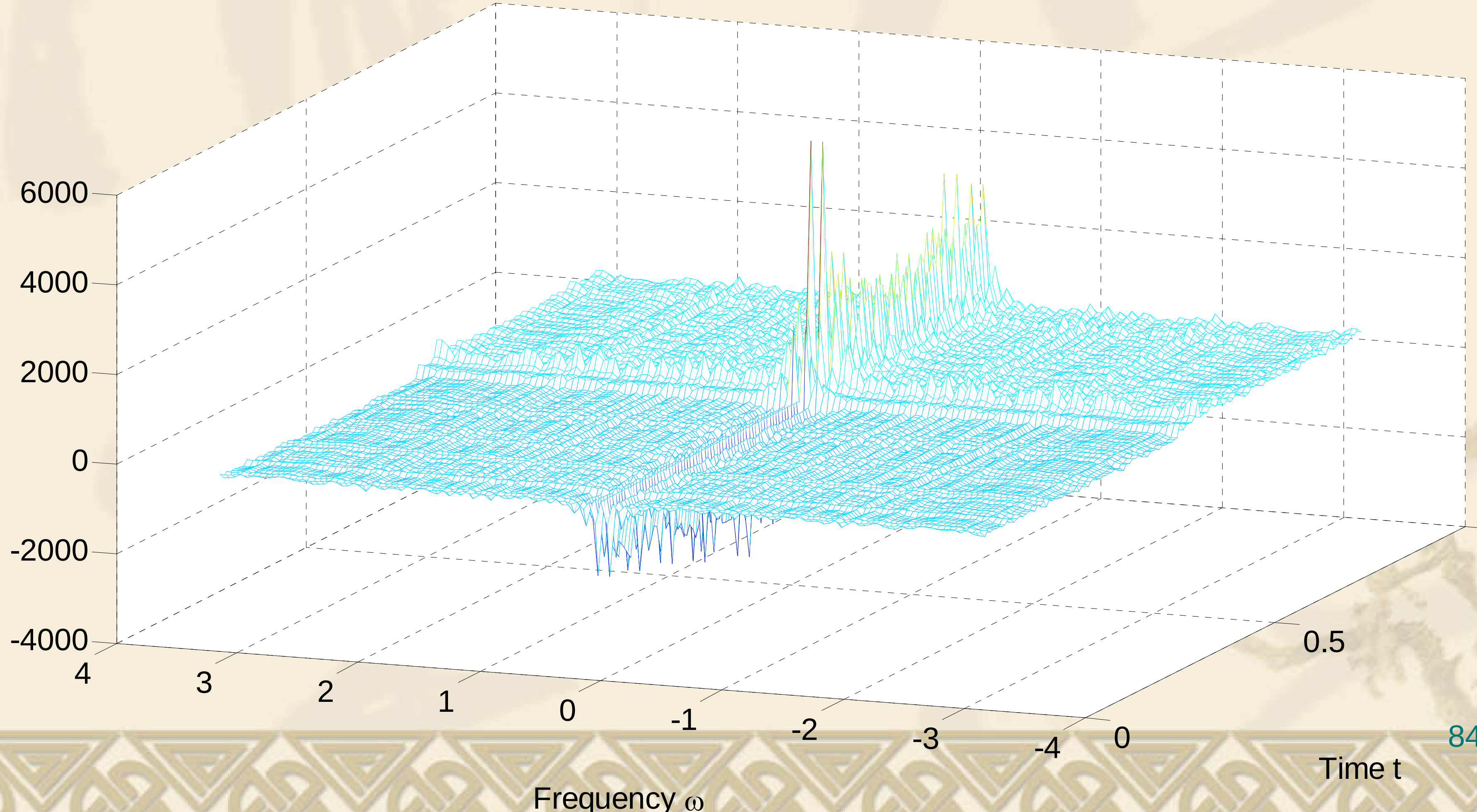
Time Frequency Analysis

- Applications
  - Machine fault analysis



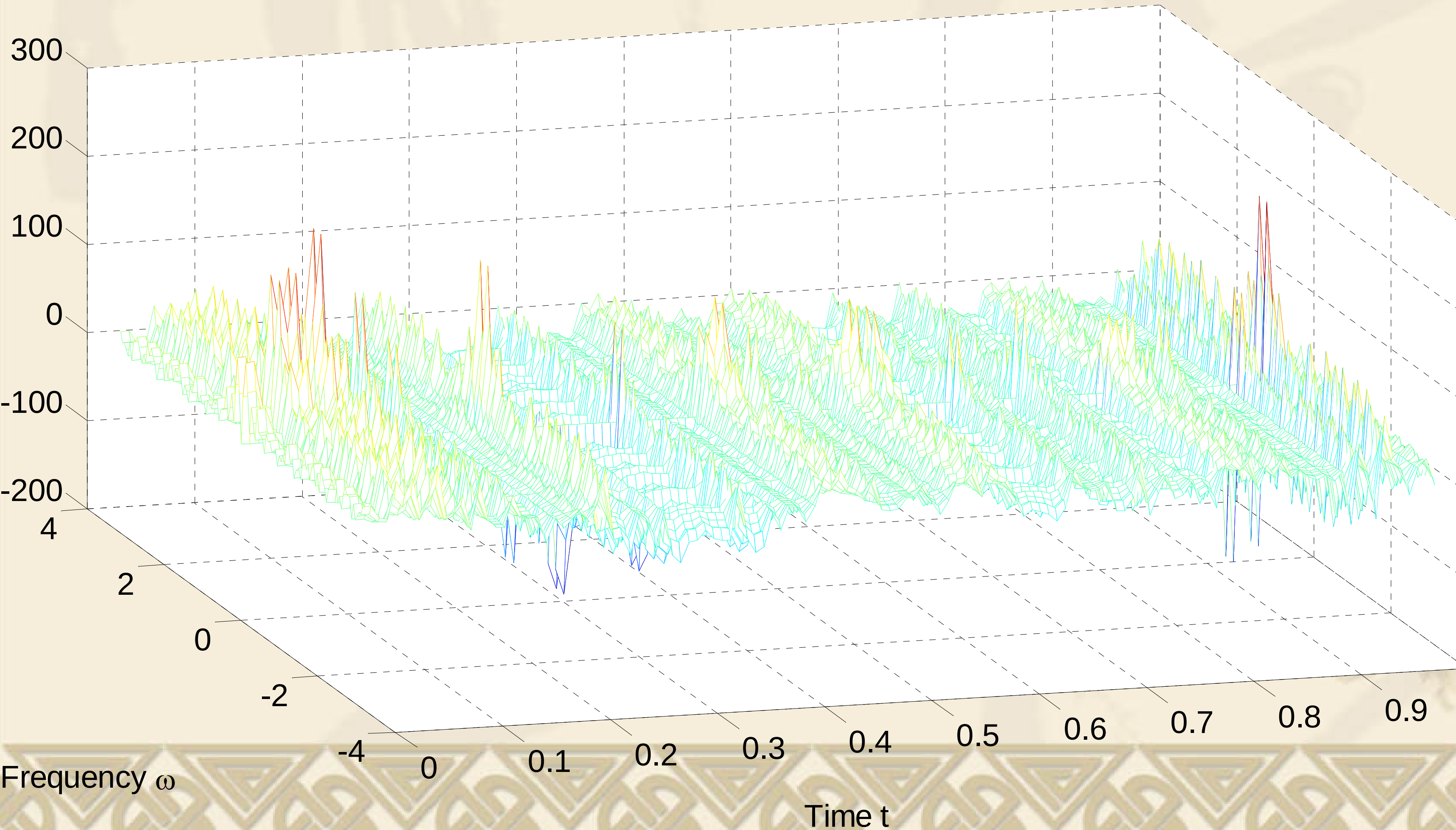
Time Frequency Analysis

- Applications
  - Machine fault analysis



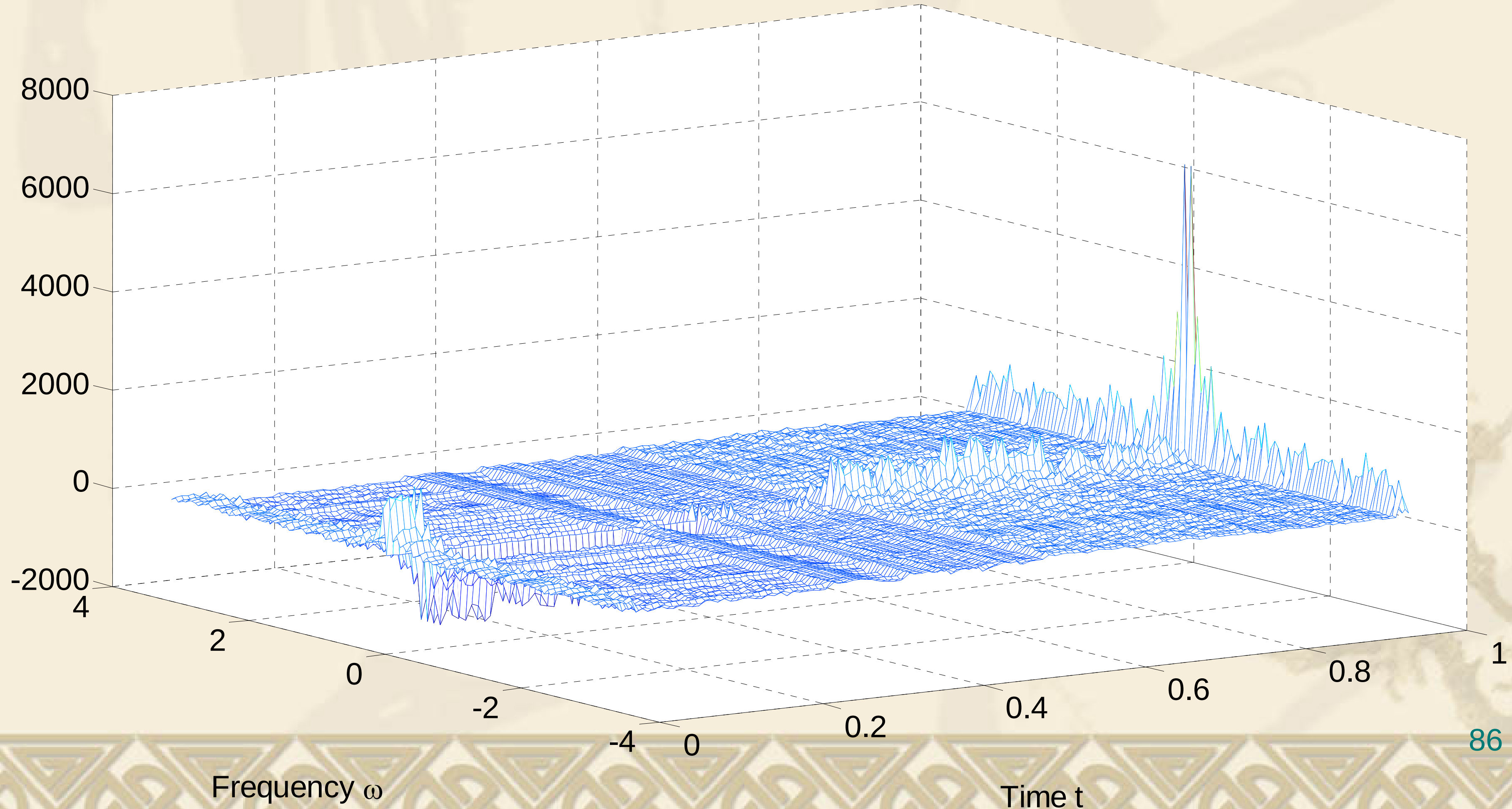
Time Frequency Analysis

- Applications
  - Machine fault analysis



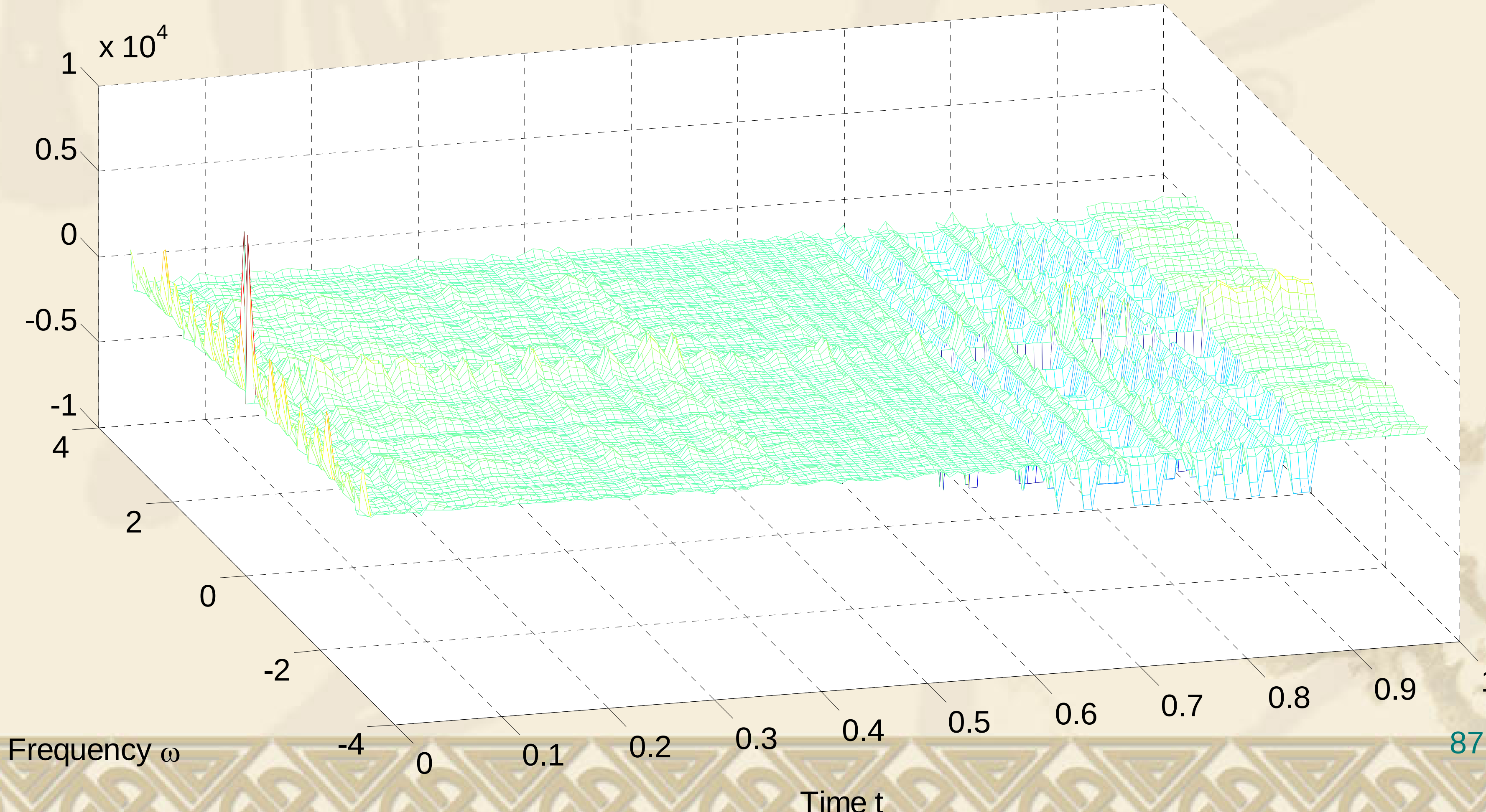
Time Frequency Analysis

- Applications
  - Machine fault analysis



Time Frequency Analysis

- Applications
  - Machine fault analysis



Conclusions

- Many applications, such as denoising, sampling, analog-to-digital conversions and amplitude modulation schemes, are derived based on frequency domain approaches.
- Further applications, such as denoising, signal separations, fault analysis, could be derived based on time frequency domain approaches.

Q&A Session

Thank you!

Let me think…